Math 500: Graduate Algebra. Instructor: Walton

Homework #1 on an Introduction to Groups

Due: Wednesday, September 4, 2019 at the beginning of class.

Include full statements of problems in your solution set. See syllabus for grading guide.

Let G denote a group.

(1) Dummit-Foote §1.1 #9

(2) Dummit-Foote §1.1 #25

(3) Dummit-Foote §1.2 #7

(4) Dummit-Foote §1.3 #15, 19

(5) Dummit-Foote §1.4 #1, 2, 7

(6) Let X be a finite nonempty set with associative binary operation $*$ so that X is closed under $*$, and further,

$$x * y = x * z \;\Rightarrow\; y = z \qquad \text{and} \qquad y * x = z * x \;\Rightarrow\; y = z$$

for all $x, y, z \in X$.

(a) Prove that $(X, *)$ admits the structure of a finite group.

(b) Show that $(X, *)$ does not necessarily admit the structure of a group if X is an infinite set.

(7) Determine if the following subsets G' of the groups G below are subgroups.

(a) $G = S_n$ for $n \geq 3$, and G' is the set of transpositions in S_n.

(b) $G = D_8$, and $G' = \{1_G, r^2, sr, sr^3\}$.

(c) $G = GL_n(\mathbb{C})$, and $G' = SL_n(\mathbb{C})$.

(d) $G = GL_n(\mathbb{C})$, and $G' = \{(a_{ij}) \in GL_n(\mathbb{C}) \mid a_{ij} = 0, \forall i > j\}$ (upper triangular matrices).

(e) $G = GL_n(\mathbb{C})$, and $G' = \{(a_{ij}) \in GL_n(\mathbb{C}) \mid a_{ii} = 0\}$.

(8) Prove that the following pairs of groups are not isomorphic:

(a) $\mathbb{R}^\times$ and $\mathbb{Q}^\times$; (b) $(\mathbb{R}, +)$ and $(\mathbb{Q}, +)$; (c) D_8 and Q_8; (d) D_{24} and S_4.

(9) Give an example of an epimorphism from $\mathbb{R}^\times$ onto $(\mathbb{R}, +)$, and show that homomorphisms from $\mathbb{R}^\times$ onto $(\mathbb{R}, +)$ cannot be injective.

(10) (a) Prove that G acts faithfully on a set X if and only if the kernel of the action is $\langle 1_G \rangle$.

(b) Find the kernel of the left regular action of G on itself.

(c) Show that if G is non-abelian, then the maps $g \cdot x = xg$ for all $g, x \in G$ do <u>not</u> yield a left action of the group G on itself. Determine if the maps yield a *right* action of the group G on itself. If so, find the kernel of the action.

(d) Verify that the maps $g \cdot x = xg^{-1}$ for all $g, x \in G$ <u>do</u> yield a left action of the group G on itself. Find the kernel of the action.

(e) Verify that the maps $g \cdot x = gxg^{-1}$ for all $g, x \in G$ <u>do</u> yield a left action of the group G on itself. (This is called *conjugation*.) Find the kernel of the action.