Designing and Estimating Transfer Functions from Bode Plots of Basic Factors of G(s)

Any transfer function G(s) can be expressed as a combinations of 4 basic factors:

1. Constant gain K
2. Sinusoidal or integral factor $\left(\frac{1}{s}\right)^{\pm 1}$
3. First order factor $\left(\frac{a}{s+a}\right)^{\pm 1}$
4. Second order factor $\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)^{\pm 1}$

The frequency response of any linear system is obtained by substituting jω for s in the transfer function.

Positive phase angle is called PHASE LEAD.
Negative phase angle is called PHASE LAG.

Bode Plots of Basic Factors

1. Constant Gain K

Log magnitude
= 20 log KdB

Effect of varying K raises or lowers the log magnitude curve of the transfer function by the corresponding amount but does not change the phase (imaginary part is zero).

20 log (1/K) = -20 log K (Reciprocal in log space produces a sign change)
20 log (K × 10) = 20 log K + 20
20 log (K × 10^n) = 20 log K + 20n

2. Integral and Derivative Factors $(j\omega)^{\pm 1}$

Let's consider first G(s) = 1/s. Separate G(jω) into real and imaginary parts

$$G(j\omega)=\frac{1}{j\omega}=\frac{1}{j\omega}\times\frac{-j\omega}{-j\omega}=\frac{-j\omega}{\omega^2} \tag{1}$$

Log magnitude

$$20\log\left|\frac{1}{j\omega}\right|=-20\log\omega\, dB \tag{2}$$

Phase

$$\phi = \arctan\frac{0}{-1/\omega} = -90^0 \tag{3}$$

Decade: Frequency band from ω_1 to $10\omega_1$
0dB happens when $\omega = 1$.
At $\omega = 10\omega_1$
$-20 \log (10\omega_1)$ dB $= -20 \log \omega_1 - 20$ dB

Therefore, slope of log magnitude curve = -20dB/decade as shown in Figure (1).

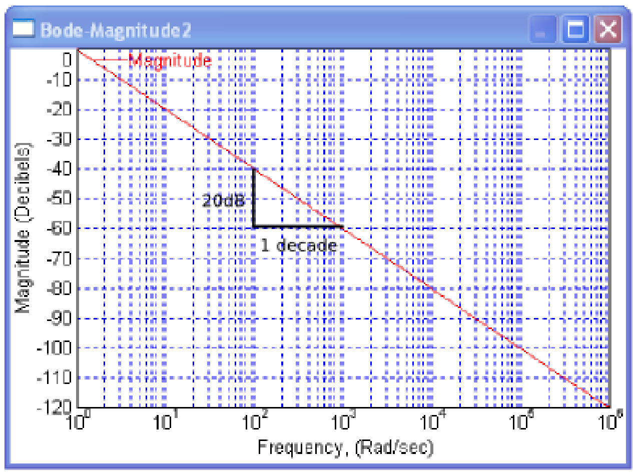

Figure 1: Bode log magnitude plot for G(s) = 1/s.

For G(s) = s , the graph is just inverted, the slope is 20 dB/decade.

3. First order factors $\left(\frac{a}{s+a}\right)^{\mp 1}$

Separate G into real and imaginary parts

$$G(j\omega) = \frac{a}{j\omega + a} = \frac{a}{j\omega + a} \times \frac{-j\omega + a}{-j\omega + a} = \frac{a^2 - ja\omega}{\omega^2 + a^2} = \frac{a^2}{\omega^2 + a^2} - j\frac{a\omega}{\omega^2 + a^2} \tag{4}$$

Log magnitude

$$|G(j\omega)| = \sqrt{Real^2 + Imag^2} = a\sqrt{\frac{a^2 + \omega^2}{(\omega^2 + a^2)^2}} = \frac{a}{\sqrt{\omega^2 + a^2}} \tag{5}$$

$$20\log\left|\frac{a}{\sqrt{j\omega + a}}\right| = -20\log a\sqrt{\omega^2 + a^2} = -20\log\sqrt{\omega^2/a^2 + 1} \tag{6}$$

At low ω, log magnitude becomes

$$20\log\left|\frac{a}{j\omega+a}\right|\approx -20\log 1\, db=0. \tag{7}$$

At high ω, log magnitude becomes

$$20\log\left|\frac{a}{j\omega+a}\right|\approx -20\log(\omega/a)\, dB \tag{8}$$

With $\omega_1 = \omega/a$ Equation (8) becomes the same as Equation 2 and within a decade band, from ω_1 to $10\omega_1$ the slope is again -20dB / decade.

The Bode plot of first order transfer functions is then composed of **two asymptotic lines**, one is horizontal at 0 dB and another inclined downwards with a slope of -20dB/decade as shown in Figure (2). The plot shows the Bode magnitude plot of G(s) = 1000/(s + 1000). The two asymptotes intersect at ω= 1000 rad/sec. This is called the **corner frequency**. (Label of x-axis should be radians/ sec not time.)

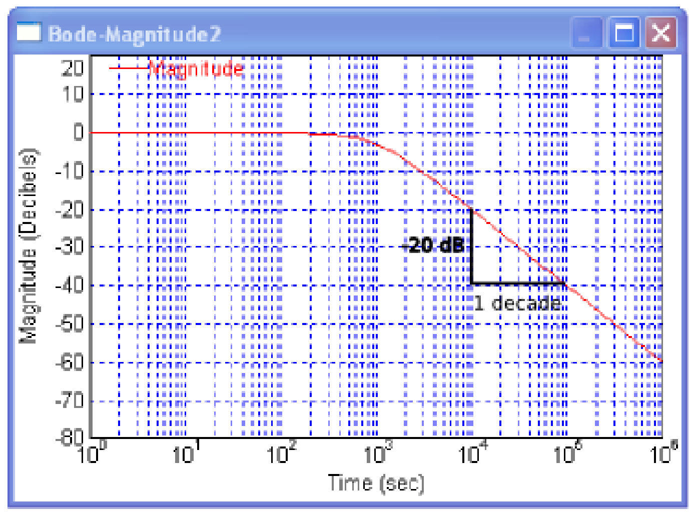

Figure 2: Bode plot of 1000/(s + 1000)

For G(s) = (s + a)/ a the Bode log magnitude plot is inverted, the slope is 20dB/decade.

Phase is

$$\phi=\arctan\frac{Imag}{Real}=\arctan\frac{-a\omega}{a^2}=-\arctan(\omega/a) \tag{9}$$

The Bode phase plot of G(s) = 1000/(s + 1000) is shown in Figure (3). At ω = a = 1000Hz, the phase is -45° as is expected from Equation (9).

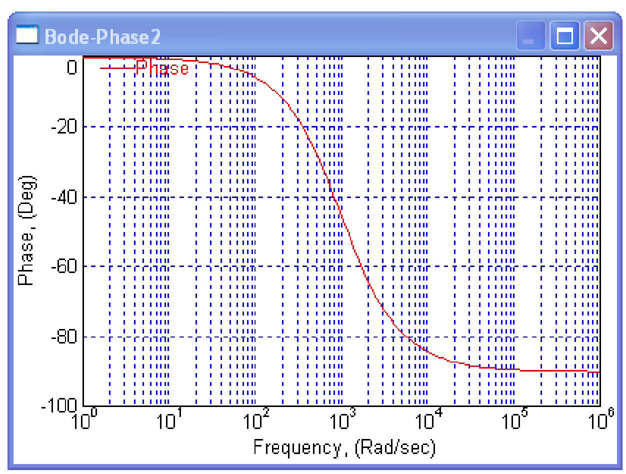

Figure 3: Bode phase plot of G(s) = 1000/(s + 1000)

4. Quadratic Factors $\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)^{\pm 1}$

First, rewrite G(s) as

$$G(s)=\frac{1}{s^2/\omega_n^2+2\zeta\, s/\omega_n+1} \tag{10}$$

then separate into real and imaginary parts

$$G(j\omega)=\frac{1}{\left(\frac{-\omega^2}{\omega_n^2}\right)+\left(\frac{2\zeta\, j\omega}{\omega_n}\right)+1}=\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)+j\left(\frac{2\zeta\omega}{\omega_n}\right)}\times\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)-j\left(\frac{2\zeta\omega}{\omega_n}\right)}{\left(1-\frac{\omega^2}{\omega_n^2}\right)-j\left(\frac{2\zeta\omega}{\omega_n}\right)} \tag{11}$$

$$G(j\omega)=\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)-j\left(\frac{2\zeta\omega}{\omega_n}\right)}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2} \tag{12}$$

Log Magnitude

$$|G(j\omega)|=\sqrt{\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}{\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^2}}=\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \tag{13}$$

$$20\log|G(j\omega)|=-20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}\,dB=-20\log\sqrt{1-2\frac{\omega^2}{\omega_n^2}(1-2\zeta^2)+\frac{\omega^4}{\omega_n^4}}\,dB \tag{14}$$

At low frequencies, $20\log|G(j\omega)|=0$.
At frequencies $\omega \gg \omega_n$

$$20\log|G(j\omega)| \approx -20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n} \tag{15}$$

Thus, the Bode log magnitude plot will have two asymptotes, one horizontal line at 0dB and another sloping downward. This time, the slope of the asymptotic line is -40dB/decade.

A peak will occur at a frequency value that will minimize the value inside the square root in Equation (14). Log of a small number is a large negative number and the negative sign makes it positive. Minimizing leads to

$$-4\frac{\omega}{\omega_n^2}(1-2\zeta^2)+4\frac{\omega^3}{\omega_n^4}=0 \quad . \tag{16}$$

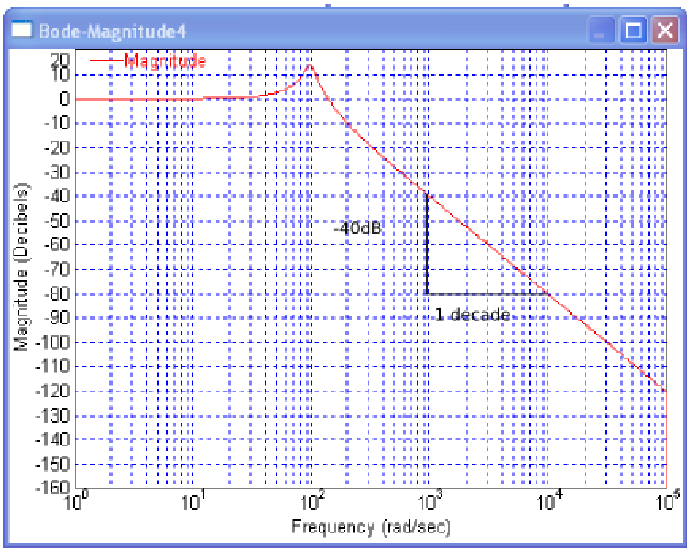

Figure 4: Bode log magnitude plot for G(s) = 10000/ (s^2 + 20s + 10000).

Solving for ω we get

$$\omega_r=\omega_n\sqrt{1-2\zeta^2} \quad . \tag{17}$$

The frequency ω_r is called the Resonance Frequency and is the corner frequency of the two asymptotic lines. Figure 4 shows the plot for ω_n = 100 rad/sec, ζ = 0.1, thus, G(s) = 10000/ (s^2 + 20s + 10000).

A peak will occur when $0 \le \zeta \le 0.707$. When $\zeta > 0.707$ the system is overdamped and no resonance peak occurs. The magnitude of the resonant peak Mr can be found by substituting Equation 17 to Equation 13.

$$M_r=|G(j\omega)|_{max}=|G(j\omega_r)|=\frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{18}$$

The phase is given by

$$\phi=-\arctan\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \tag{19}$$

At low frequencies Φ = -arctan 0 = 0. At $\omega=\omega_n$, Φ = -arctan ∞ = -90°. At ω = ∞, Φ = -180° as shown in Figure 5.

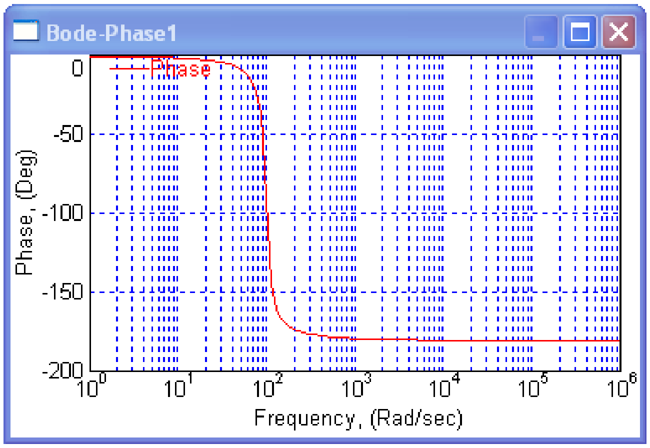

Figure 5: Quadratic order Bode phase plot for $G(s) = 10000/(s^2 + 20s + 10000)$.

Procedure for sketching Bode Diagrams

1. Rewrite G(s) as a product of basic factors.
2. Identify corner frequencies.
3. Draw asymptotic log magnitude curves with proper slopes.
4. Add the curves for each basic factor.

Procedure for estimating transfer functions from Bode diagrams.

1. Identify elements of basic factors in graph.
2. Look for corner frequencies
3. Find slopes. +- 20 dB/decade are first order factors, +-40 dB/decade are quadratic factors.
4. If there are quadratic factors, find resonance peaks and estimate damping factor.
5. Plot Bode diagram of estimated transfer function and see if it resembles given diagram.

Activity:

To be submitted:
1. Hand-sketched plots with self-evaluation.
2. *.doc, *.ods or *.pdf file showing VisSim of estimated transfer functions.

1. Sketch the bode plots of the following BY HAND. Then whenever possible, use VisSim to verify if you got the sketch correctly. DO THIS VERIFICATION STEP ONLY AFTER YOU'VE SKETCHED THE DIAGRAM. For each item, comment if you got it right or wrong. If wrong indicate your mistake. Use the attached log scale graphing paper for your sketches.
   1. G(s) = 10/s
   2. G(s) = 10s

3. $G(s) = 5/(s + 5)$
4. $G(s) = 1/(s + 5)$
5. $G(s) = 10/(s+2)(s+500)$
6. $G(s) = (s+2)/(s+500)$
7. $G(s) = 1/s(s+5)$
8. $G(s) = 5s/(s+5)$
9. $G(s) = 2/(s^2 + 2s + 100)$

2. Given the Bode diagram below, estimate the transfer function. Explain your steps. Use VisSim to compare your estimated transfer function Bode diagram to the given diagram. Submit a document showing the actual versus the Bode plot from estimated transfer function. The plot is also attached as a png file.