Nonlinear Systems

We have discussed at length linear systems and ways of analyzing their response to input and their stability to changing system parameters. Linear systems are actually more of exceptions than the rule. The real world is inherently nonlinear. When we model a system to be linear, we are actually looking at a limited range of that system's response. Linear systems are also convenient to analyze. We can predict with certainty when a linear system will be stable or not. We can confidently forecast the output of a linear system. Things are not so well-behaved with nonlinear systems. In fact, they have a rich variety of “special effects” and we begin by enumerating the most common ones.

Properties of Nonlinear Systems

1. **Limit cycles** – These are periodic oscillation in a nonlinear system which are in general, non-sinusoidal. In a closed loop and with the slightest disturbance, a nonlinear system can be driven to oscillation .
2. **Subharmonic & harmonic response under periodic input** – upon a sinusoidal excitation, nonlinear systems may output a periodic output whose frequency is either a subharmonic or harmonic of the input frequency. Unlike a linear system, nonlinear systems can produce multiple periodic outputs.

   eg. 10 Hz input -> 5 Hz output (subharmonic) or 30 Hz (harmonic)

3. **Jump Phenomenon and Hysteresis** – Figure below shows jump phenomenon. As input value increases, system output goes from A to B to C but jumps to D if input is increased beyond C. Output signal continues on to E. When input signal from E is decreased, output goes to F but suddenly jumps to B if input is decreased further. In Hysteresis, points C and D coincide and so do points F and B.

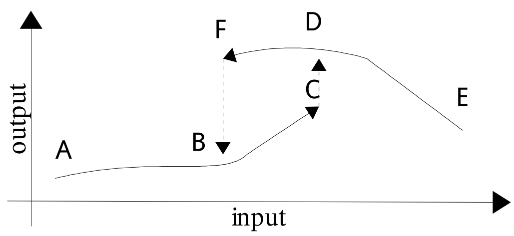


Figure 1 A system with jump phenomenon

4. **Multiple Equilibrium states** – a linear system output settles to zero when input is zero. For nonlinear stable systems there may be different states other than x = 0 that the system can settle into as $t \rightarrow \infty$. Where the system settles depends critically on the initial condition. On the other hand, a linear system, whatever the initial condition, will always settle to the same steady state.

Some Nonlinear Systems in Actual Use

Nonlinearities are extremely useful. Much of analog and digital electronics exploit nonlinearities.

1. **Transistors** are the building blocks of computers. Logic states 0 and 1 are generated by driving a transistor to saturation or cutoff.
2. **Diodes** allow the passage of current through one direction only. They are used as rectifiers in power supplies, and reverse current protection in common appliances.
3. UJT's or **unijunction transistors** are used for creating oscillations necessary for timing.

Ways of Describing Nonlinear Systems

1. Differential Equations

A nonlinear system may be described analytically through a differential equation. Let t be the independent variable, x(t) the input to the system and y(t) the dependent variable. A nonlinear differential equation results if the coefficients of the differential equation are functions of the dependent variable y(t). The general form is

$$A_n \frac{d^n y(t)}{dt^n} + A_n \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + A_1 \frac{dy(t)}{dt} + A_0 y(t) + \epsilon f\left(y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{n-1} y(t)}{dt^{n-1}} \right) = x(t)$$

where f is a nonlinear function of y(t) and its derivatives, and ε is a constant. When ε is zero, the equation reduces to a linear system.

2. Input-Output Relations of Common Nonlinearities

The response of a nonlinear system to any input may be obtained through its input vs output curve. Figure 2 below shows the input-output relation for an **Ideal Relay**. Shown are the steps in obtaining the output.

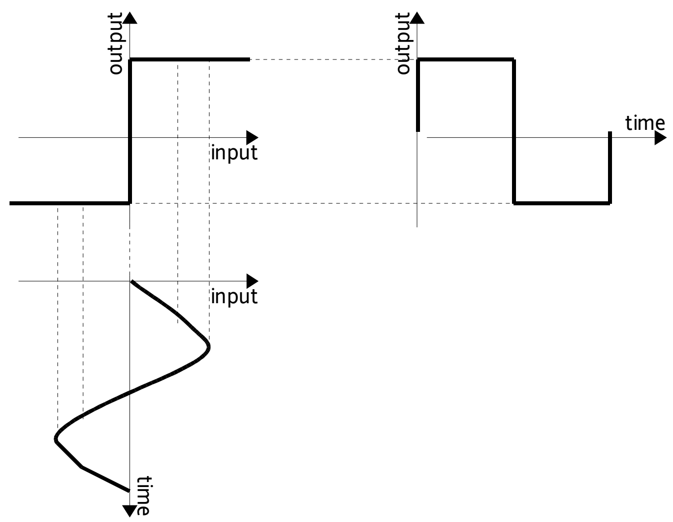


Figure 2 Steps in deriving the output of a nonlinear system from its input-output curve

The steps are summarized as follows:
Step 1. Draw the input-output relation.
Step 2. Draw the input signal with its amplitude aligned to the input axis.
Step 3. Project each point on the input to the input-output curve.
Step 4. Project the output to a graph whose amplitude is aligned to the output axis.

Methods Available for Analysing Nonlinear Functions

- Describing functions
- Linearization
- For piecewise linear systems - Phase Plane method
- Lyapunov Stability Criterion

Describing Function

- Equivalent to transfer functions in nonlinear systems.
- Good for finding limit cycles.
- Assumption : there's only one nonlinear in a feedback loop. Excitation is zero r(t) = 0 but input to nonlinearity is sinusoidal. The system is "autonomous" but can generate a sinusoid by itself.
  Assume the system looks like Figure 3 below:

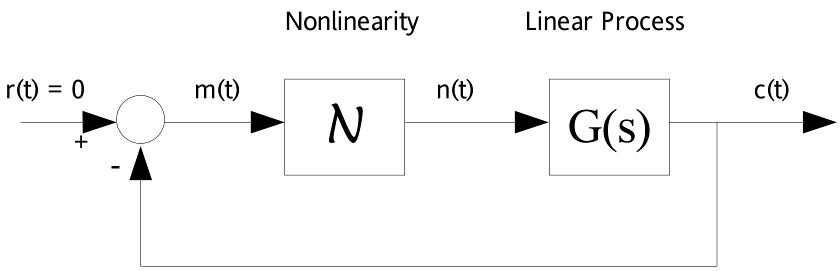


Figure 3 A nonlinearity inside a closed loop

Consider that at feedback, the input to the nonlinearity, m(t) is sinusoidal, that is,

$$m(t) = M \sin \omega t$$

In general, the output n(t) is periodic but may not be sinusoidal. Since the output is periodic just the same, it may be represented as a Fourier series:

$$n(t) = \frac{A_o}{2} + \sum_{k=1}^{\infty} A_k \cos k \omega t + \sum_{k=1}^{\infty} B_k \sin k \omega t$$

Coefficients of the Fourier series are given by:

$$A_k = \frac{2}{T}\int_{t_0}^{t_0+T} n(t)\cos k\omega t$$

$$B_k = \frac{2}{T}\int_{t_0}^{t_0+T} n(t)\sin k\omega t$$

T is the period of the input sinusoid m(t) and $T = 2\pi/\omega$, t_o is at any time and for convenience, $A_0 = 0$. This is alright to assume if nonlinearity is symmetric with respect to the amplitude of the input.

Describing function method can find limit cycles if G(s) acts as a low pass filter to exclude higher harmonics of n(t), since nonlinear systems can output harmonics and subharmonics.

(But how about subharmonics? They can still pass through!) Assume the system does not produce subharmonics.

This means $[G(j\omega)]$ small for all other components of n(t). Then the output c(t) can be expressed as

$$c(t) \approx C\sin(\omega t + \theta)$$

And output from the nonlinearity can be expressed as

$$\begin{aligned} n(t) &\approx A_1\cos(\omega t) + B_1\sin(\omega t) \\ &= A_1(\sin\omega t + 90^0) + B_1\sin(\omega t) \\ &= N_1\sin(\omega t + \phi) \end{aligned}$$

In terms of phasors

$$N_1 < \phi = B_1 + jA_1$$

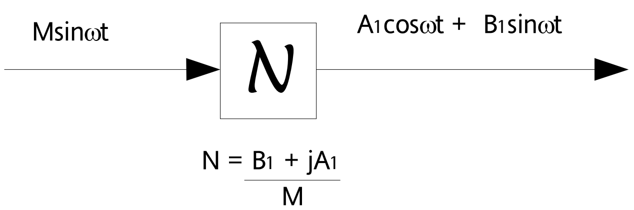
Since the input is in phase with B1 term and input is real, B1 is real term.

Nonlinearity can be replaced by complex gain

Gain of nonlinearity

$$N(m,\omega) = \frac{B_1 + jA_1}{M} = \sqrt{(B_1/M)^2 + (A_1/M)^2}\exp j(\tan^{-1} A_1/B_1)$$

This is the **Describing Function** . In general, it is a function of both the amplitude and frequency of the input. In effect the nonlinearity appears to perform:



The describing function is an equivalent complex gain under 2 very restrictive conditions:

1. G(s) acts as a low-pass filter.
2. Input to the nonlinearity is a sinusoid.

Example:

1. Find the describing function of the nonlinearity

$$n(t)=m^3(t)$$

Begin with

$$m(t)=M\sin(\omega t) \quad .$$

Substitute

$$n(t)=M^3\sin^3(\omega t)=M^3\sin(\omega t)\sin^2(\omega t)$$

Use

$$\sin^2(\omega t)=½-1/2\cos(2\omega t)$$

$$n(t)=M^3\sin(\omega t)\left[\frac{1-\cos(2\omega t)}{2}\right]$$

$$=\frac{M^3}{2}\sin(\omega t)-\frac{M^3}{2}\sin(\omega t)\cos(2\omega t)$$

From

$$\sin A\cos B=½\left[\sin(A-B)+\sin(A+B)\right]$$

$$\sin(\omega t)\cos(2\omega t)=\frac{\sin(-\omega t)+\sin(3\omega t)}{2}$$

$$n(t)=\frac{M^3}{2}\sin(\omega t)+\frac{M^3}{4}\sin(\omega t)-\frac{M^3}{4}\sin(3\omega t)$$

Last term is third harmonic and is ignored following assumption that higher harmonics are filtered by G(s) which is low pass.

$$n(t)=\frac{3M^3}{4}\sin(\omega t)$$

$$=A_1\cos(\omega t)+B_1\sin(\omega t)$$

Thus the describing function for this nonlinearity is

$$N(M,\omega)=\frac{B_1+jA_1}{4}=\frac{3M^3/4+j0}{M}=\frac{3M^2}{4}$$

2. Find the describing function of an ideal relay

Show that it is $N(M,\omega)=\dfrac{4V}{\pi M}$

where +/- V is the amplitude of the step.

References

C. Phillips, R. Harbor, Chapter 14 *Feedback Control Systems 2nd Ed.* Prentice-Hall

S. Shinners, Chapter 8 *Modern Control System Theory and Design* John Wiley & Sons 1992