State Plane Analysis

Once a nonlinear system is linearized, state space analysis can aid in describing the behaviour of the system in the vicinity of its equilibrium points.

Consider a general linear second-order differential equation,

$$\ddot{x}+a\dot{x}+bx=0 \tag{1}$$

Let

$$x_1=x \tag{2}$$

$$x_2=\dot{x}_1 \tag{3}$$

$$\dot{x}_2=\ddot{x}_1=\ddot{x}=-a\dot{x}-bx \tag{4}$$

The state equations are

$$\dot{X}_1=X_2 \tag{5}$$

$$\dot{x}_2=-a\dot{x}_2-bx_1 \tag{6}$$

Taking the characteristic equation

$$A=\begin{bmatrix}0 & 1\\ -b & a\end{bmatrix} \tag{7}$$

$$|\lambda\mathbf{I}-\mathbf{A}|=0 \tag{8}$$

$$\begin{vmatrix}\lambda & -1\\ -b & \lambda+a\end{vmatrix}=0 \tag{9}$$

$$\lambda^2+a\lambda+b=0 \tag{10}$$

which yields two roots, λ_1,λ_2

The response for $\lambda_1\neq\lambda_2$ is of the form

$$x(t)=k_1e^{\lambda_1 t}+k_2e^{\lambda_2 t} \tag{11}$$

and if $\lambda_1=\lambda_2$

$$x(t)=k_1e^{\lambda_1 t}+t\,k_2e^{\lambda_2 t} \tag{12}$$

Thus, if $\lambda_1\neq\lambda_2$ then from Equation (2) and (3)

$$x_1(t)=x(t)=k_1e^{\lambda_1 t}+k_2e^{\lambda_2 t} \tag{13}$$

$$x_2(t)=\dot{x}(t)=k_1\lambda_1e^{\lambda_1 t}+k_2\lambda_2e^{\lambda_2 t} \tag{14}$$

Note that the system has only one equilibrium point which is at the origin $\mathbf{x=0}$. With nonlinear systems, when we expand about an equilibrium point it becomes the new origin in state space.

Phase Portraits

Activity

Generate the phase plots for the following cases of λ .

Case A

When λ_1 and λ_2 are real and of the same sign.

$\lambda_{1,}\lambda_2<0$: **ATTRACTOR**

$\lambda_{1,}\lambda_2>0$:**REPELLOR**

Case B

When λ_1 and λ_2 are complex with nonzero real parts

Real$\{\lambda_{1,}\lambda_1\}<0$: **STABLE FOCUS**

Real$\{\lambda_{1,}\lambda_1\}>0$: **UNSTABLE FOCUS**

Case C

When λ_1 and λ_2 are imaginary : **LIMIT CYCLE**

Case D

When λ_1 and λ_2 are real, with $\lambda_1 > 0$ and $\lambda_2 < 0$: **SADDLE NODE**