The Method of Isoclines

Consider a 2nd-order nonlinear system described by

$$\dot{x}_1 = f_1(x_{1,} x_2) = x_2 \quad (1)$$

$$\dot{x}_2 = f_2(x_{1,} x_2) \quad (2)$$

If $x_2 > 0$, then x_1 is increasing while if $x_2 < 0$, x_1 is decreasing.

Define M as

$$M = \frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2(x_{1,} x_2)}{f_1(x_{1,} x_2)} = \frac{f_2(x_{1,} x_2)}{x_2} \quad . \quad (3)$$

We can rewrite Equation (3) as

$$x_2 = \frac{1}{M} f_2(x_1, x_2) \quad . \quad (4)$$

For a constant M value, Equation (4) describes a curve called an *isocline* since a state trajectory will always have a slope M as it crosses this curve.

By plotting several isoclines in the state plane, the phase portrait can be sketched.

Steps in sketching the phase portrait using the method of isoclines

Example:
Sketch the phase portrait of a nonlinear system given by:

$$\ddot{y} + y^2 = 1$$

1. Set up the state equation

$$x_1 = y$$
$$x_2 = \dot{x}_1 = \dot{y}$$
$$\dot{x}_2 = \ddot{y} = 1 - y^2$$

2. Form the equation $x_2 = \frac{1}{M} f_2(x_{1,} x_2)$

$$x_2 = \frac{1}{M}(1 - x_1^2)$$

3. Tabulate different M values and the corresponding isocline equation

M	***Isocline Equation***
0	Rewrite as $Mx_2 = 0 = (1 - x_1^2)$ or $x_1 = \pm 1$ (line)

M	***Isocline Equation***
± 1	$x_2 = \pm(1 - x_1^2)$
$\pm ½$	$x_2 = \pm 2(1 - x_1^2)$
± 2	$x_2 = \pm ½(1 - x_1^2)$
∞	$x_2 = 0$ (line)

4. Draw the isoclines with representative slope lines across them.

5. Start sketching trajectories using the isoclines as a guide.

Watch a demonstration on https://youtu.be/vEeN9vwSg6o

Exercise:

14.27 Sketch the phase plot of the nonlinear system described by $\ddot{y} + y^2 = 1$.

14.28.A nonlinear system is described by the equation

$$\ddot{\theta} + \dot{\theta} + |\theta| = 0$$

where

$$|\theta| = \begin{cases} \theta, \theta \geq 0 \\ -\theta, \theta < 0 \end{cases} .$$

Sketch some representative phase trajectories.