Describing Functions

The describing function of a nonlinear element is its approximate transfer function if the nonlinearity is in cascade with a linear process that acts as a low pass filter. It is useful for finding limit cycles in the system.

Let's continue with detailed examples of computing the describing function of some nonlinearities.

Example: Compute the describing function of an ideal relay:
The input-output curve of the ideal relay is shown below. Note that it is an odd function. Here in detail are the steps toward the solution:
1. **Draw the output**. First we need to get the output given a sinusoidal input. Shown below is the result.

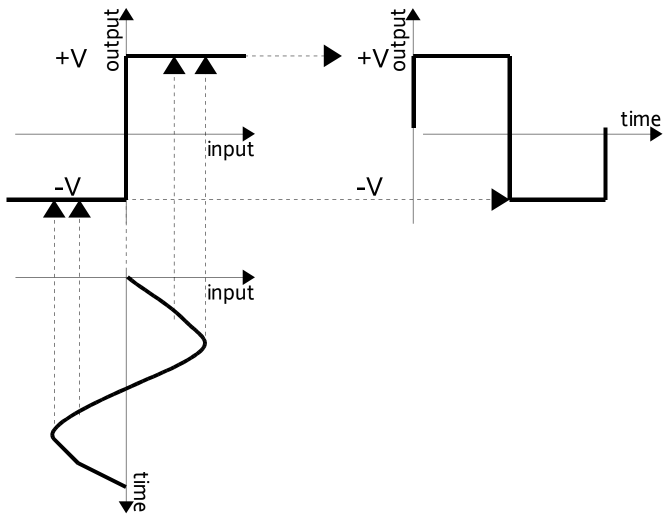

Illustration 1 Output of an Ideal Relay.

2. **Equate n(t) to the first terms of the Fourier expansion**. Remember that with G(s) acting as low pass the higher order terms vanish. That is

$$n(t) = A_1 \cos(\omega t) + B_1 \sin(\omega t) \quad . \tag{1}$$

3. **Calculate coefficients** A_1 **and** B_1 .
From definition

$$A_1 = \frac{2}{T} \int_{t_0}^{t_0+T} n(t) \cos(\omega t) dt \tag{2}$$

$$B_1 = \frac{2}{T} \int_{t_0}^{t_0+T} n(t) \sin(\omega t) dt \tag{3}$$

where $T = \dfrac{2\pi}{\omega}$

The integral is evaluated over one full period from the starting time t_0 (which can be any time).

Graphically, to get A_1 we first multiply n(t) to $\cos(\omega t)$ then get the area under the curve over one full period. Shown below is the $n(t)\cos(\omega t)$.

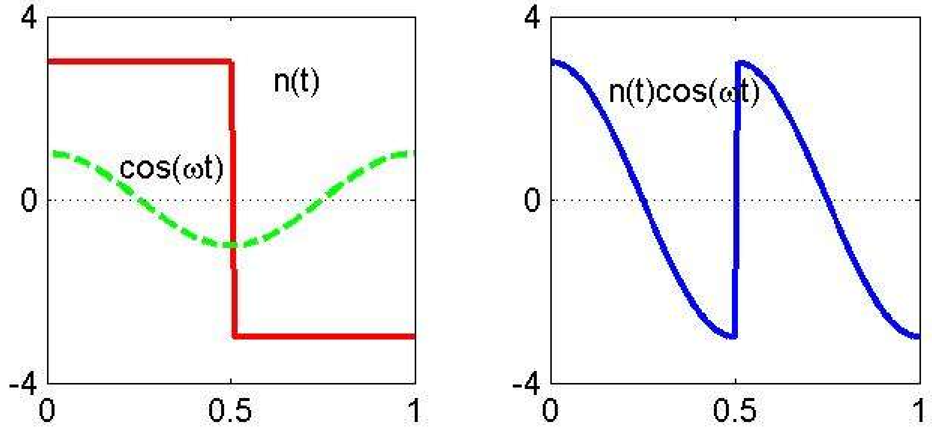

Illustration 2 Product of n(t) and cos(w t). In this example |V| = 3

Since n(t) is an odd function and $\cos(\omega t)$ is an even function, the integral of their product over one full period is zero. Therefore, A_1 in this case is zero.

If we apply the same analysis to B_1 we see that area under the curve is not zero because n (t) and $\sin(\omega t)$ are both odd.

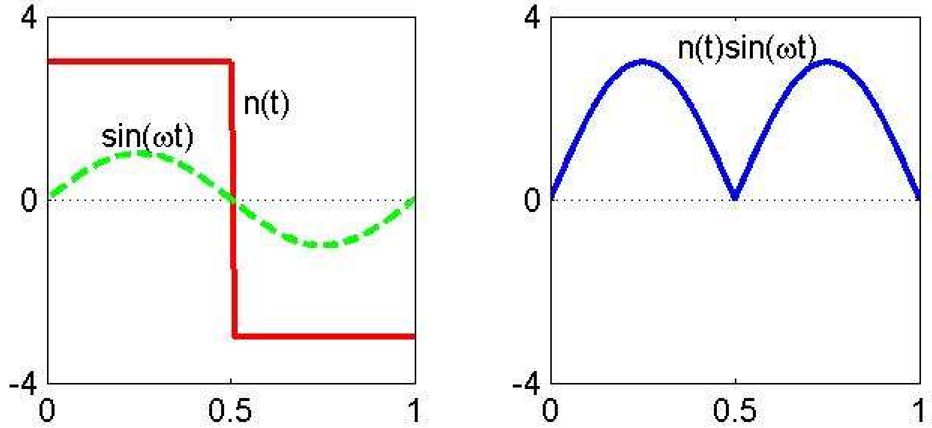

Illustration 3 Product of n(t) and sin(wt).

Therefore from (1) n(t) becomes

$$n(t)=B_1\sin(\omega t) \tag{4}$$

In solving for B_1 we note that the integration can be carried over a half period (from t_0 = 0 to T/2) so long as we multiply the integral by 2. With that

$$B_1(t)=\frac{4}{T}\int_0^{T/2} V\sin(\omega t)dt \tag{5}$$

$$B_1(t) = \frac{4V}{T}\left[\frac{-\cos(\omega t)}{\omega}\right], t=0 \text{ to } T/2 \quad (6)$$

$$= \frac{4V}{T\omega}\left[-\cos(\omega T/2) + \cos(0)\right] \quad (7)$$

Substituting

$$T = \frac{2\pi}{\omega}$$

$$B_1 = \frac{4V}{2\pi}\left[-\cos(\pi) + \cos(0)\right] = \frac{4V}{\pi} \quad (8)$$

4. **Solve for** $N(M, \omega)$.

$$N(M, \omega) = \frac{B_1 + jA_i}{M} = \frac{B_1}{M} \quad (9)$$

$$\boxed{N(M, \omega) = \frac{4V}{\pi M}} \quad (10)$$

Try it.

Exercise:
1. Compute the describing function of a Saturation (also known as a “Limiter”).
2. Compute the describing function of a Dead Zone.

Finding Limit Cycles

A nonlinear system in a feedback loop will have a limit cycle with the output approximately sinusoidal if the sinusoid at the input regenerates itself in the loop.

Consider the unity feedback loop below with a nonlinearity.

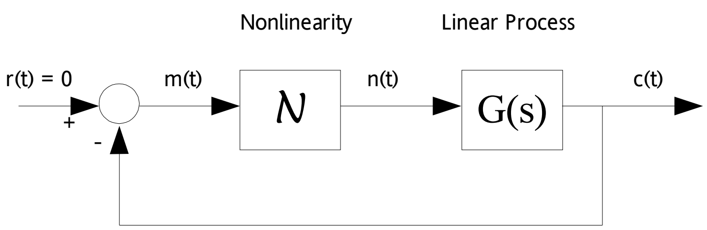

Illustration 4 A unity feedback loop with nonlinearity.

We can replace the nonlinearity with a describing function. The condition for the existence of a limit cycle means that m(t) = -c(t).

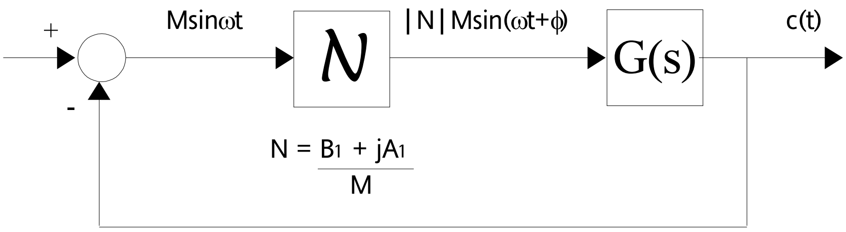

Illustration 5 Nonlinearity replaced with describing function in finding limit cycles.

This means

$$M\sin(\omega t) = -|G(j\omega)||N(M,\omega)|M\sin(\omega t+\phi+\theta) \tag{11}$$

The negative sign is due to the negative sign in the summing junction.
For sustained oscillation, equate the phases.

$$M = -G(j\omega)N(M,\omega)M \tag{12}$$

$$1 = -G(j\omega)N(M,\omega) \tag{13}$$

$$1+N(M,\omega)G(j\omega)=0 \tag{14}$$

Or

$$G(j\omega) = \frac{-1}{N(M,\omega)} \tag{15}$$

Any M (input sinusoid amplitude) and ω (sinusoid frequency) which satisfies Eq. (15) will produce a limit cycle.

Example

In Illustration 5, let $G(s) = \frac{4}{s(1+s)^2}$ and the nonlinearity an ideal relay. Will there be a limit cycle?

1. **Express G(s) as a frequency response,** $G(j\omega)$.

$$G(j\omega) = \frac{4}{j\omega(1+j\omega)^2} \tag{16}$$

$$= \frac{4}{j(\omega-\omega^3)-2\omega}\,\frac{(-j(\omega-\omega^3)-2\omega)}{(-j(\omega-\omega^3)-2\omega)} \tag{17}$$

$$= \frac{-8\omega+4j(\omega-\omega^3)}{4\omega^2+(\omega-\omega^3)^2} \tag{18}$$

2. **Remove possible phase differences by zeroing the imaginary component**

$$\omega-\omega^3=0 \tag{19}$$

$$\omega=1 \tag{20}$$

Replacing $\omega=1$ in Eq. 18 We get

$$G(j1)=\frac{8\omega}{4\omega^2}=-2 \tag{21}$$

3. **Solve for** $G=\frac{-1}{N}$

For an ideal relay $N=\frac{4V}{\pi M}$

$$G=\frac{-1}{N}$$

$$2=\frac{-\pi M}{4V} \tag{22}$$

$$\boxed{\frac{8V}{\pi}=M} \tag{23}$$

Conclusion: There's a limit cycle when $\omega=1$ and $M=8V/\pi$.