LE3 Problem 2

a) **Sketch of output**

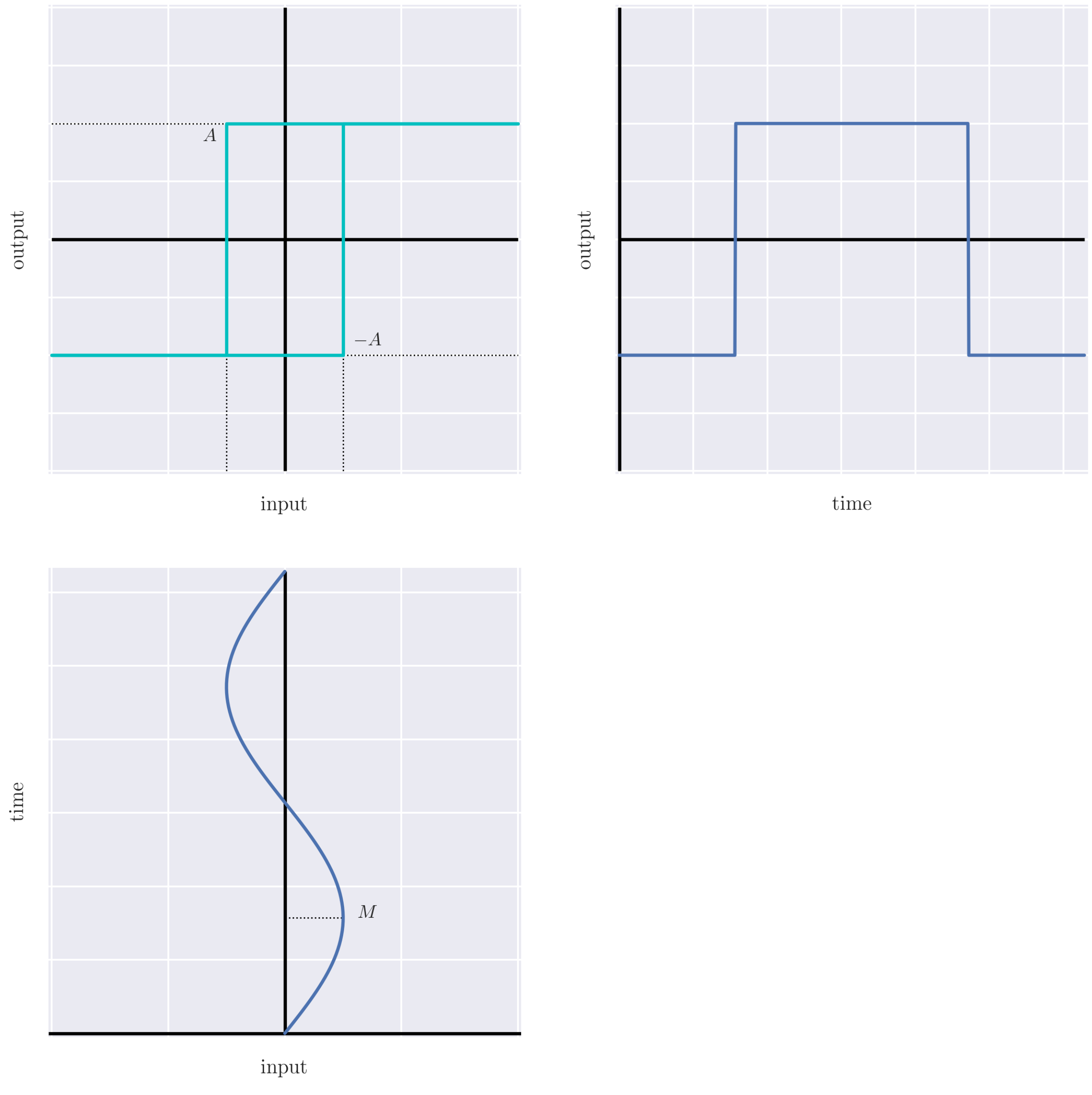

Figure 1: IO curves of an ideal relay with hysteresis, with a sinusoidal input.

Consider an input of the form $M\sin(\omega t)$ being controlled by an ideal relay with hysteresis. The output for one cycle of the input can be described as

$$n(t) = \begin{cases} -A & , \quad 0 \leq t \leq \frac{\pi}{2} \\ A & , \quad \frac{\pi}{2} < t \leq \frac{3\pi}{2} \\ -A & , \quad \frac{3\pi}{2} < t \leq T \end{cases} \tag{1}$$

b) **Fourier coefficients**

We cascade $n(t)$ with some linear system $G(s)$, which we take in this case to be a low-pass filter, and equate it to the first-order terms of the Fourier series:

$$n(t) = A_1 \cos(\omega t) + B_1 \sin(\omega t) \tag{2}$$

Figs. 2 and 3 show the product of $n(t)$ with $\cos(\omega t)$ and $\sin(\omega t)$, respectively.

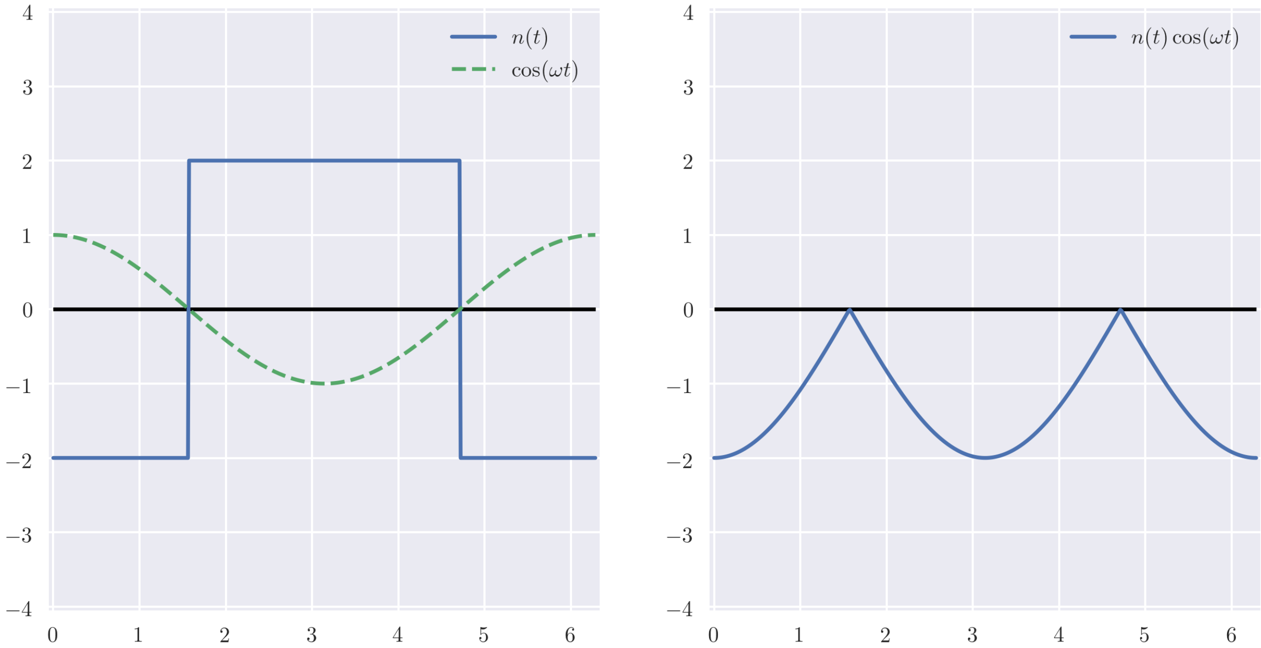

Figure 2: Product of $n(t)$ and $\cos(\omega t)$, with $|M| = 1$, $|A| = 2$.

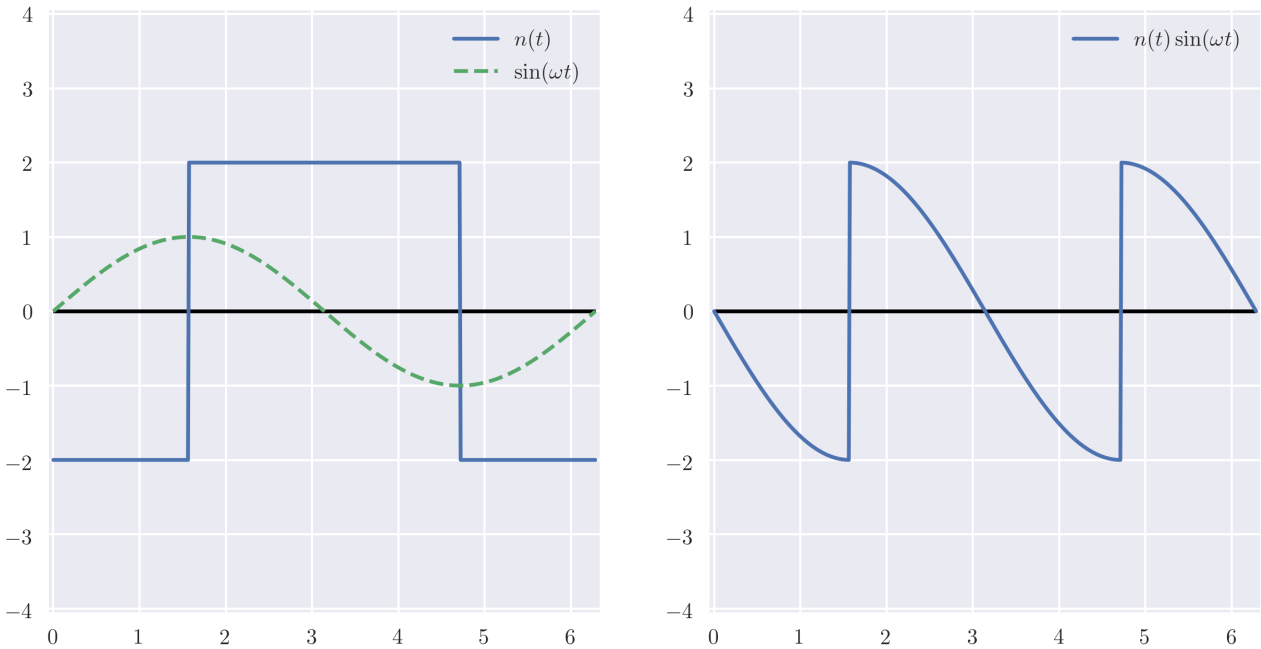

Figure 3: Product of $n(t)$ and $\sin(\omega t)$, with $|M| = 1$, $|A| = 2$.

By graphical analysis, we see that the output is even when multiplied by cos, and odd when multiplied by sin. Therefore, the integral of the sin term vanishes and $\boxed{B_1 = 0}$. Letting $T \equiv 2\pi/\omega$ (one wave period), the coefficient A_1 can be calculated by

$$A_1 = \frac{2}{T}\int_{t_0}^{t_0+T} n(t)\cos(\omega t)\,\mathrm{d}t \tag{3}$$

For simplicity, we take $t_0 = 0$ and integrate w.r.t. ωt over one cycle s.t. $T = 2\pi$. Evaluating A_1:

$$\begin{align}
A_1 &= \frac{1}{\pi}\int_0^{2\pi} n(t)\cos(\omega t)\,\mathrm{d}(\omega t) \tag{4}\\
&= \frac{1}{\pi}\left[\int_0^{\pi/2} -A\cos(\omega t)\,\mathrm{d}(\omega t) + \int_{\pi/2}^{3\pi/2} A\cos(\omega t)\,\mathrm{d}(\omega t) + \int_{3\pi/2}^{2\pi} -A\cos(\omega t)\,\mathrm{d}(\omega t)\right] \tag{5}\\
&= \frac{A}{\pi}\left[\int_0^{\pi/2} -\cos(\omega t)\,\mathrm{d}(\omega t) + \int_{\pi/2}^{3\pi/2} \cos(\omega t)\,\mathrm{d}(\omega t) + \int_{3\pi/2}^{2\pi} -\cos(\omega t)\,\mathrm{d}(\omega t)\right] \tag{6}\\
&= \frac{A}{\pi}\left[-\sin(\omega t)\Big|_{\omega t=0}^{\pi/2} + \sin(\omega t)\Big|_{\omega t=\pi/2}^{3\pi/2} - \sin(\omega t)\Big|_{\omega t=3\pi/2}^{2\pi}\right] \tag{7}\\
&= \frac{A}{\pi}[-1-2-1] \tag{8}\\
&\boxed{A_1 = -\frac{4A}{\pi}} \tag{9}
\end{align}$$

c) **Describing function**

The describing function is calculated as:

$$\begin{align}
N(M,\omega) &= \frac{B_1 + jA_1}{M} \tag{10}\\
&= -\frac{jA_1}{M} \tag{11}\\
&\boxed{N(M,\omega) = -\frac{4jA}{\pi M}} \tag{12}
\end{align}$$