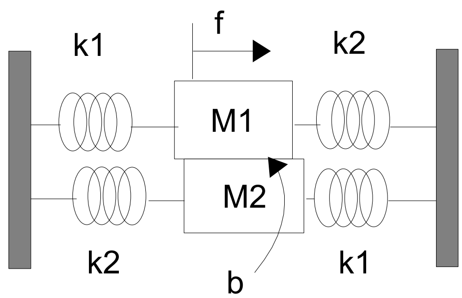
2nd Long Exam Applied Physics 183

26 February 2008 8:30-10am
CSRC

1. Two masses M1 and M2 are each attached to two springs with force constants k1 and k2 and are in contact as shown below. Frictional force between the surfaces of M1 and M2 produce damping characterized by the coefficient b. Find the transfer function of each mass, X_1/F and X_2/F when an external force f is applied to mass 1. (10 pts)



Answer:

Compute the contact forces on each mass.

Equation of motion for mass m1 (2 pts):

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2 x_1 - b\dot{x}_1 + b\dot{x}_2 + f \tag{1}$$

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 + b\dot{x}_1 - b\dot{x}_2 = f \tag{2}$$

Equation of motion for mass 2 (2 pts):

$$\begin{gathered} m_2 \ddot{x}_2 = -(k_1 + k_2) x_2 - b\dot{x}_2 + b\dot{x}_1 = 0 \\ m_2 \ddot{x}_2 + (k_1 + k_2) x_2 + b\dot{x}_2 - b\dot{x}_1 = 0 \end{gathered} \tag{3}$$

Take the Laplace transform of Eq. 2 and 3.

$$m_1 s^2 X_1 + (k_1 + k_2) X_1 + bs\, X_1 - bs\, X_2 = F \tag{4}$$

$$m_2 s^2 X_2 + (k_1 + k_2) X_2 + bs\, X_2 - bs\, X_1 = 0 \tag{5}$$

From Eq. 5, express X_2 in terms of X_1. (2 pts)

$$X_2 = \frac{bs}{m_2 s^2 + (k_1 + k_2) + bs} X_1 \tag{6}$$

Replace X_2 in Eq. 4.

$$\left[m_1 s^2+(k_1+k_2)+bs\right]X_1-\frac{(bs)^2}{m_2 s^2+bs+(k_1+k_2)}X_1=F \tag{7}$$

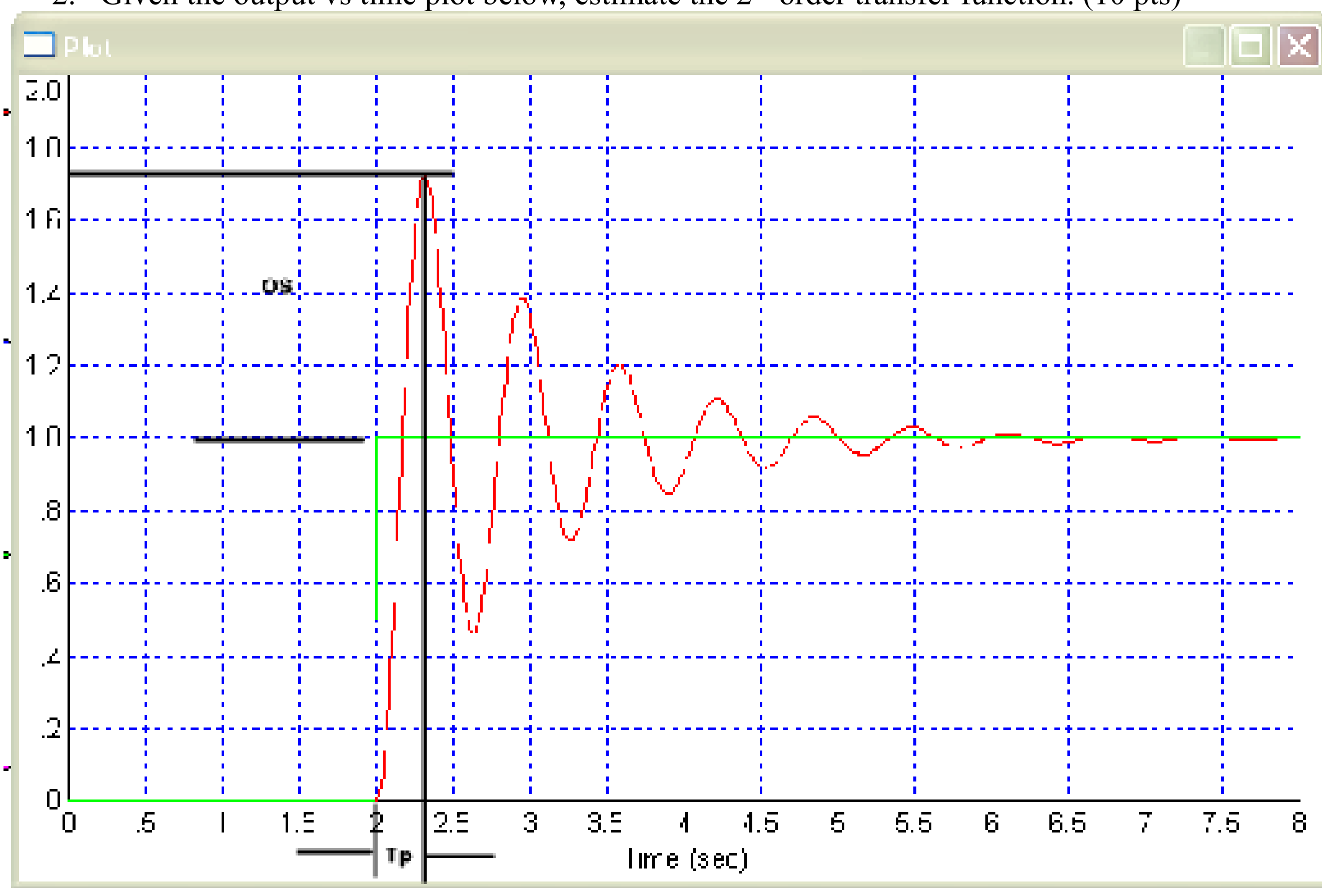
Solve for X_1/F. (2 pts)

$$\boxed{\frac{X_1}{F}=\frac{m_2 s^2+bs+(k_1+k_2)}{\left(m_1 s^2+bs+(k_1+k_2)\right)\left(m_2 s^2+bs+(k_1+k_2)\right)-(bs)^2}} \tag{8}$$

Solve for X_2/F by using Eq. 8 to replace X_1 in Eq. 6. (2 pts)

$$\boxed{\frac{X_2}{F}=\frac{bs}{\left(m_1 s^2+bs+(k_1+k_2)\right)\left(m_2 s^2+bs+(k_1+k_2)\right)-(bs)^2}} \tag{9}$$

2. Given the output vs time plot below, estimate the 2nd order transfer function. (10 pts)



Measure OS% (1 pt)

$$\%OS=\frac{c_{max}-c_{final}}{c_{final}}\times 100\%=\frac{1.73-1}{1}\times 100\% \tag{10}$$

$$\boxed{\%OS=73\%}$$

Estimate ζ (1 pt)

$$\zeta=\frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}=\frac{-\ln(0.73)}{\sqrt{(\pi^2+\ln^2(0.73))}} \tag{11}$$

$$\boxed{\zeta=0.1}$$

Use expression for peak time T_p to estimate natural frequency ω_n (2 pts)

$$\boxed{T_p=(2.3-2.0)=0.3}$$

$$T_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad \text{which leads to} \quad \omega_n=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}=\frac{\pi}{0.3\sqrt{1-0.1^2}} \tag{12}$$

$$\boxed{\omega_n=10.52}$$

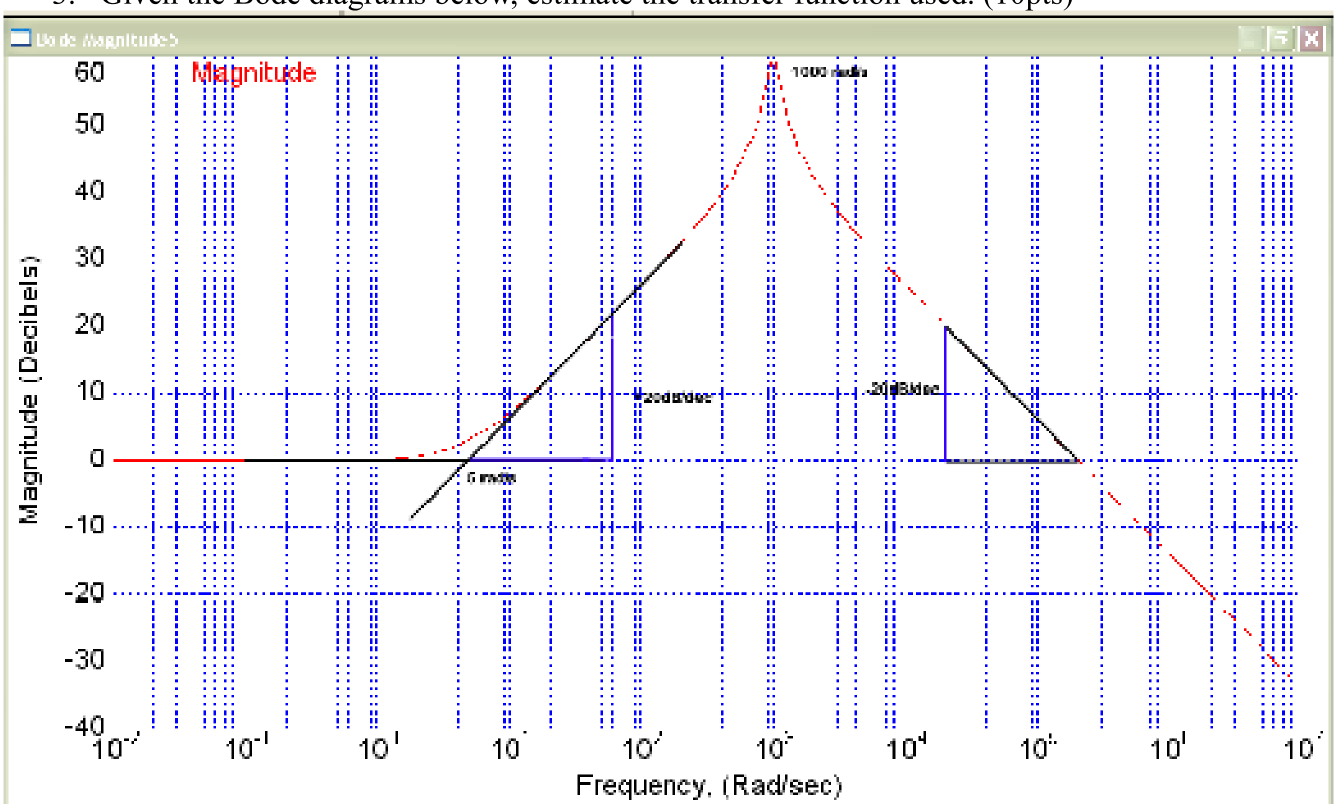
Substitute constants into 2nd order transfer function. (6 pts)

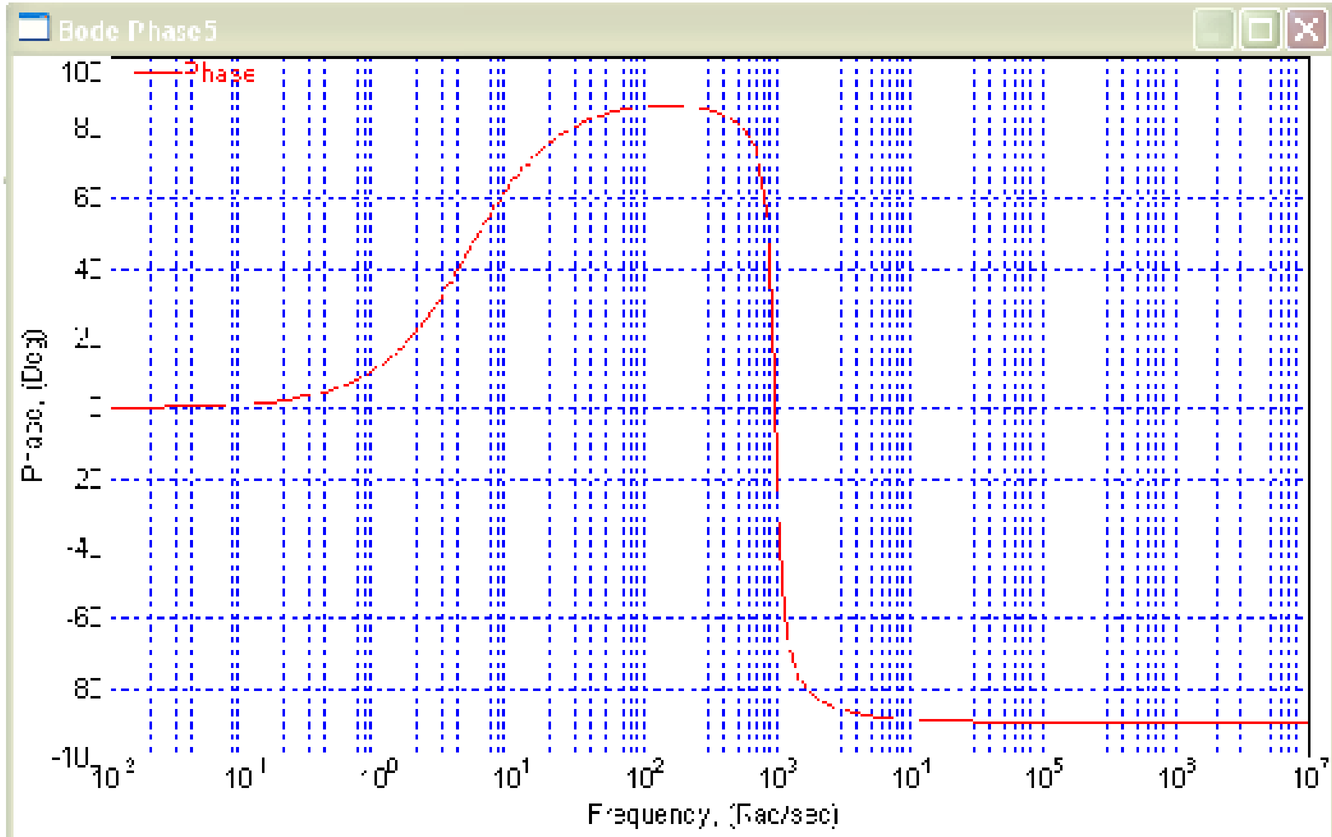
$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}=\frac{(10.52)^2}{s^2+2(0.1)(10.52)+(10.52)^2}$$

$$\boxed{G(s)=\frac{110.7}{s^2+2.1s+110.7}} \qquad 13)$$

In comparison, the actual transfer function I used in generating the graph is $\boxed{G(s)=\frac{100}{s^2+2s+100}}$.

3. Given the Bode diagrams below, estimate the transfer function used. (10pts)





Observations:

1. There's + 20dB/dec slope intersecting with zero asymptote at ω = 5 rad/s. Possibly a first order (s+a)/a.
2. There's a -20dB/dec slope after 1000 rad/s. Could it be an b/(s+b) form?
3. But there's a sharp peak at 1000 rad/s which looks like that coming from a second order.
4. No way could that peak at 1000 rad/s come from addition of Bode diagrams from (s + a)/ a and b/(s + b). Must then be addition of (s + a) / a and $\omega_n^2/(s^2 + 2\zeta\omega_n + \omega_n^2)$.
5. Also, phase diagram shows signature of 2nd order transfer function after 1000 rad/s.
6. Check : Slope of (s+a)/a = + 20 dB/dec ; Slope of $\omega_n^2/(s^2 + 2\zeta\omega_n + \omega_n^2)$ = - 40 dB/dec ; combining graphs will result in a line with slope (+20dB/dec – 40dB/dec = - 20dB/dec)!!

Estimate: (s + a)/a.
Since asymptote is at 5 rad/sec, must be G1(s) = (s+5)/5.

Estimate : $\omega_n^2/(s^2 + 2\zeta\omega_n + \omega_n^2)$
ω_n = approx 1000 rad/s
ζ = Here, you can foster a guess. Rule of thumb is, if the resonance peak looks sharp and high, zeta is less than 0.5. So suppose you assume 0.2.
Then second order estimate may be

$$G2(s)=\frac{10^6}{(s^2+400+10^6)}$$

Thus $G(s)=\dfrac{2\times10^5(s+5)}{s^2+400+10^6}$. (The actual zeta I used is 0.1.)