Describing Functions

1. **Saturation/Limiter**

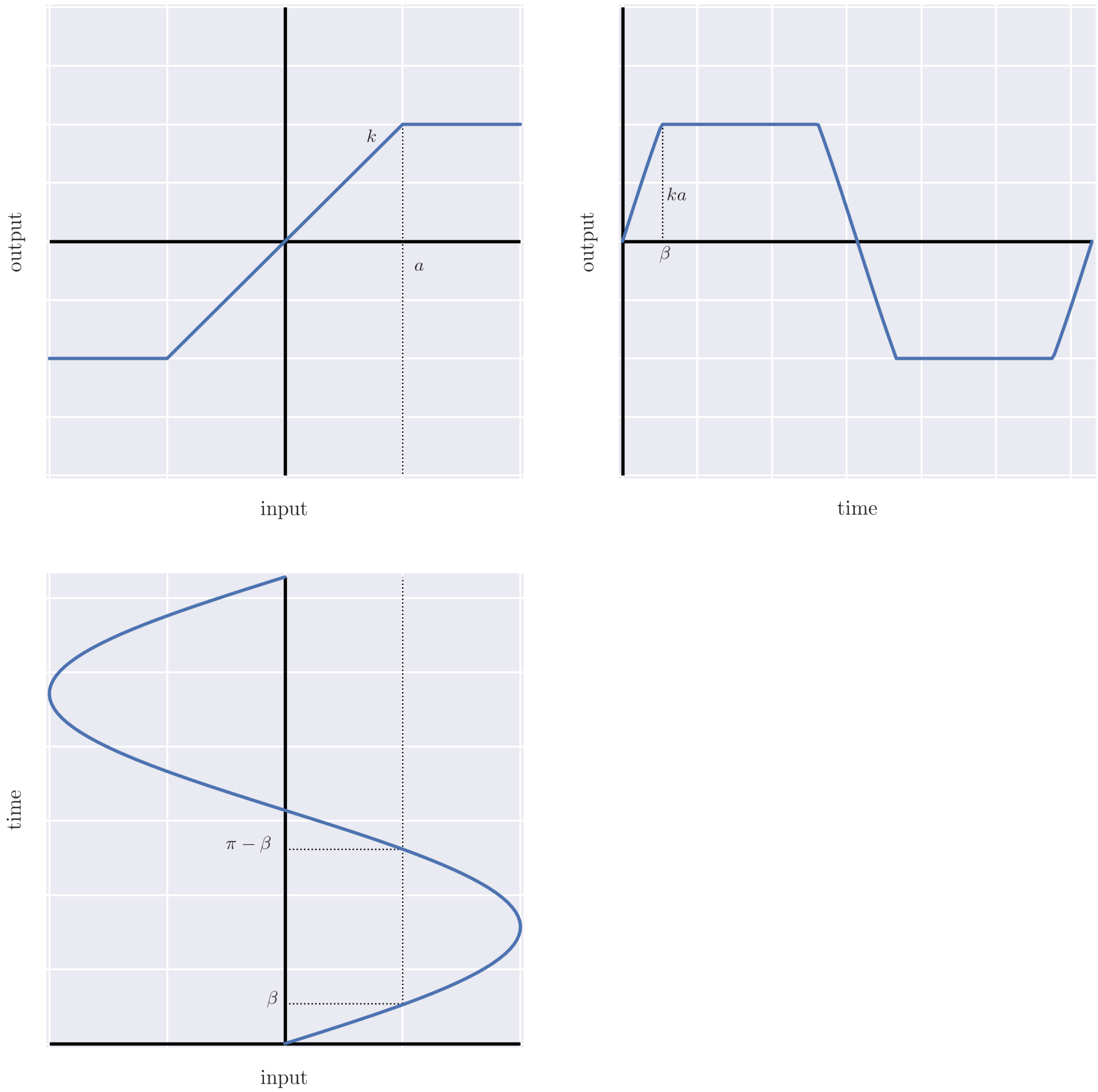

Figure 1: IO curves of a saturation function with a sinusoidal input.

Consider an input of the form $V\sin(\omega t)$. The output function is described as

$$n(t) = \begin{cases} kV\sin(\omega t) & , \quad 0 \leq \omega t \leq \beta \\ ka & , \quad \beta \leq \omega t \leq (\pi - \beta) \\ kV\sin(\omega t) & , \quad (\pi - \beta) \leq \omega t \leq \pi \end{cases} \tag{1}$$

We cascade this nonlinear system with a some linear system $G(s)$, typically a low-pass filter and equate it to the first terms of the Fourier series:

$$n(t) = A_1 \cos(\omega t) + B_1 \sin(\omega t) \tag{2}$$

Let $T \equiv 2\pi/\omega$ and calculate the coefficients A_1 and B_1:

$$A_1 = \frac{2}{T}\int_{t_0}^{t_0+T} n(t)\cos(\omega t)\,\mathrm{d}t \tag{3}$$

$$B_1 = \frac{2}{T}\int_{t_0}^{t_0+T} n(t)\sin(\omega t)\,\mathrm{d}t \tag{4}$$

We evaluate the integral over one full period from initial time t_0. For simplicity, we let $t_0 = 0$. Graphically, to get A_1, we multiply $n(t)$ with $\cos(\omega t)$ then get the area under the curve over one full period, as shown in Fig. 2.

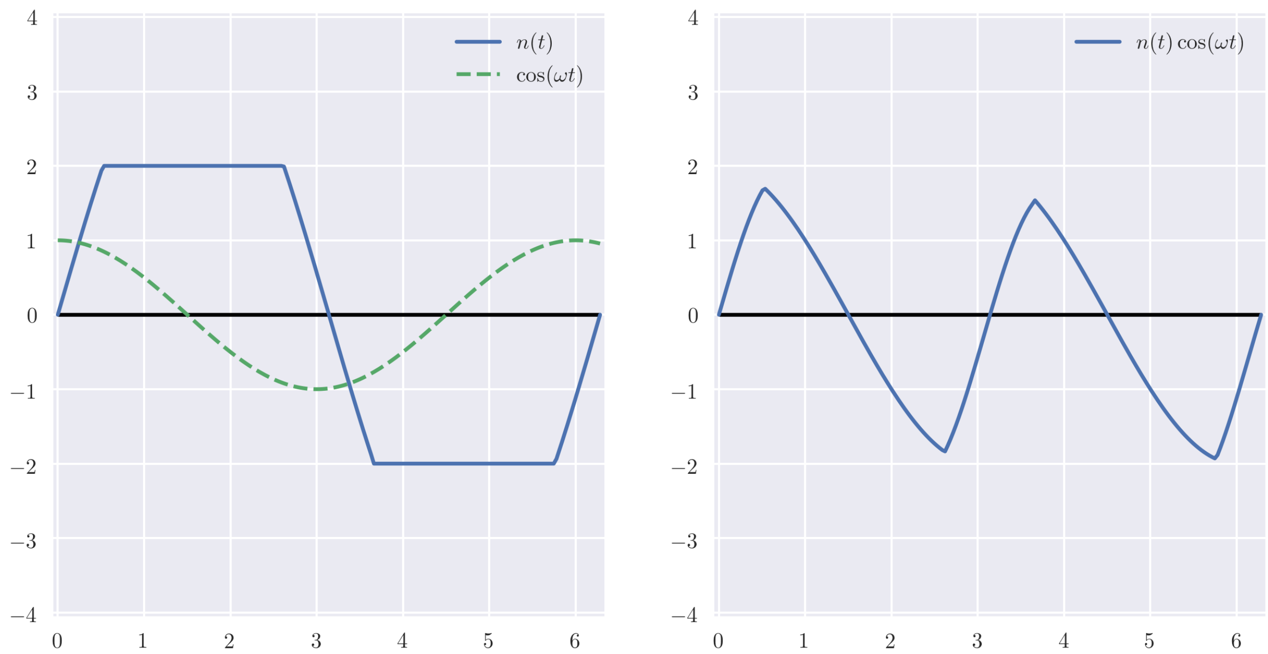

Figure 2: Product of $n(t)$ and $\cos(\omega t)$, with $|V| = 2$.

Extending $n(t)$ in the negative direction shows that it is an odd function, and cos is an even function, so the resulting product is even. The integral of their product over one full period is zero. Thus, $A_1 = 0$. We apply the same graphical analysis to B_1 as in Fig. 3.

In this case, $n(t)$ and $\sin(\omega t)$ are both odd functions, so the integral of their product is non-zero. We now have

$$n(t) = B_1 \sin(\omega t) \tag{5}$$

Solving for the coefficient B_1:

$$B_1(t) = \frac{2}{T}\int_0^T n(t)\sin(\omega t)\,\mathrm{d}t \tag{6}$$

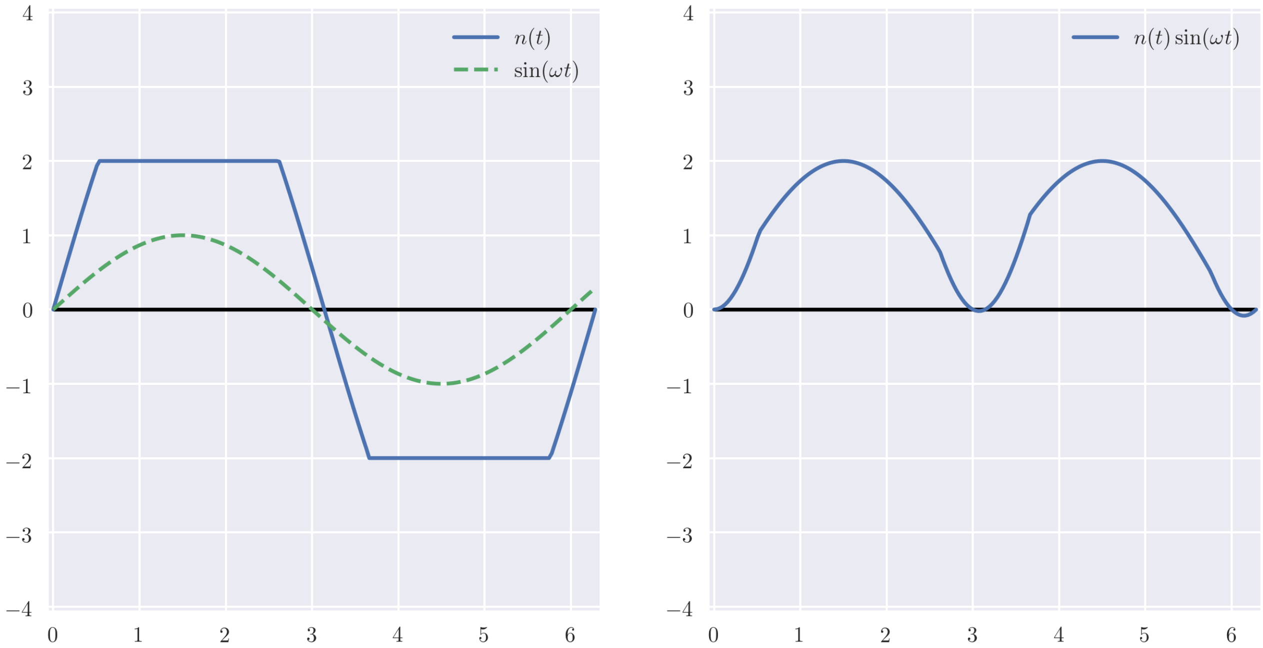

Figure 3: Product of $n(t)$ and $\sin(\omega t)$, with $|V| = 2$.

Notice that one wave period is made up of two symmetric patterns, both of which are in turn, symmetric about their respective centers. Thus, we need only to integrate 1/4 of the area under the curve and multiply it by 4:

$$B_1(t) = \frac{8}{T}\int_0^{T/4} n(t)\sin(\omega t)\,\mathrm{d}t \tag{7}$$

$$= \frac{8}{T}\left[\int_0^{\beta} kV\sin^2(\omega t)\,\mathrm{d}t + \int_{\beta}^{\pi/2} ka\sin(\omega t)\,\mathrm{d}t\right] \tag{8}$$

$$= \frac{8kV}{\omega T}\left[\beta + \frac{a}{V}\sqrt{1-\frac{a^2}{V^2}}\right] \tag{9}$$

Recall $T \equiv 2\pi/\omega$. We have,

$$B_1(t) = \frac{4kV}{\pi}\left[\beta + \frac{a}{V}\sqrt{1-\frac{a^2}{V^2}}\right] \tag{10}$$

Solving for $N(M,\omega)$:

$$N(M,\omega) = \frac{B_1 + jA_1}{M} \tag{11}$$

$$= \frac{B_1}{M} \tag{12}$$

$$\boxed{N(M,\omega) = \frac{4kV}{M\pi}\left[\beta + \frac{a}{V}\sqrt{1 - \frac{a^2}{V^2}}\right]} \tag{13}$$

2. **Dead Zone**

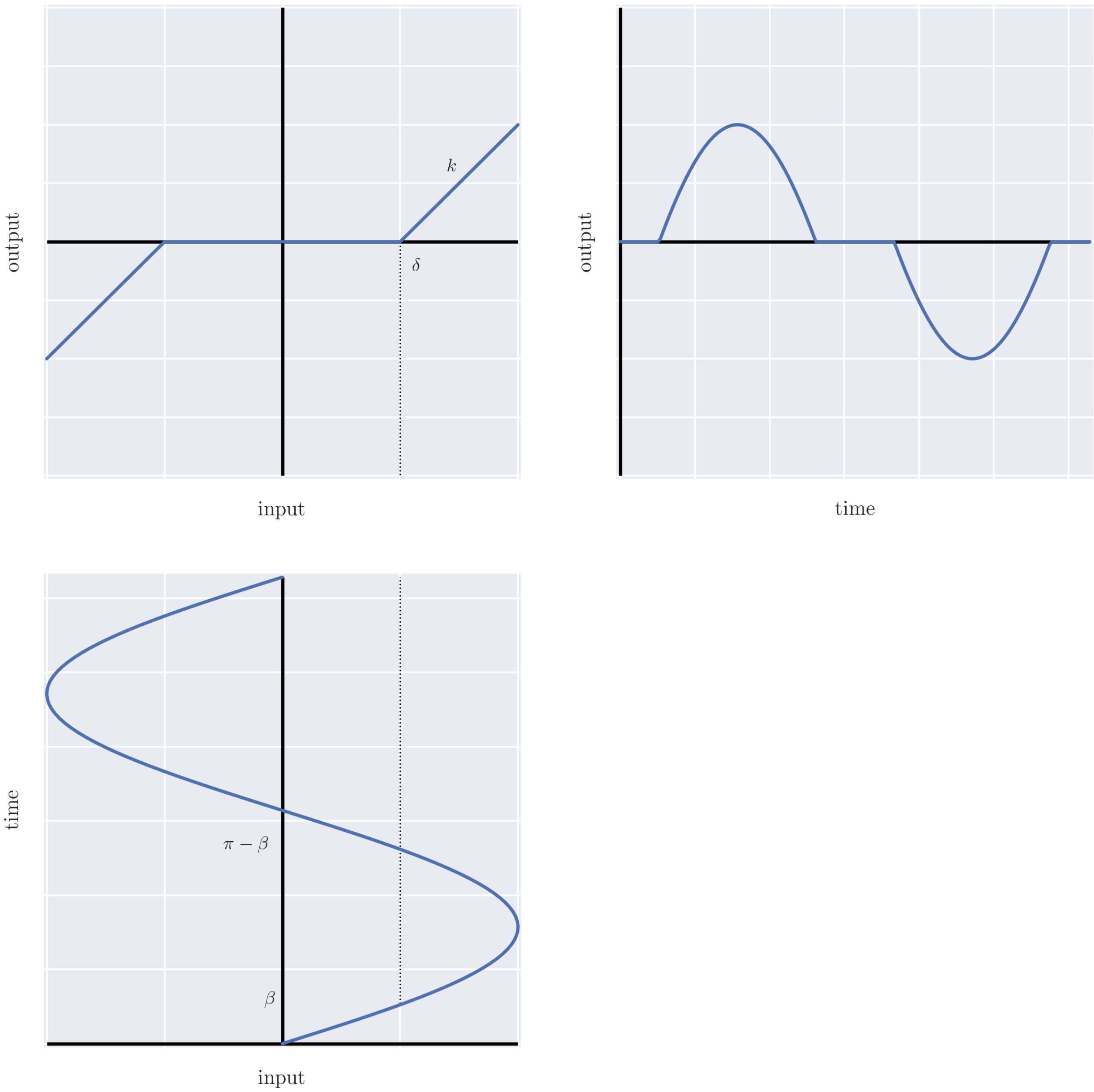

Figure 4: IO curves of a ramp with a dead zone and a sinusoidal input.

Consider again an input of the form $V\sin(\omega t)$. The output function is described by

$$n(t) = \begin{cases} 0 \quad , & 0 \leq \omega t \leq \beta \\ k[V\sin(\omega t) - \delta] \quad , & \beta \leq \omega t \leq \frac{\pi}{2} \end{cases} \tag{14}$$

Following the same graphical analysis as before, we obtain Figs. 5 and 6.

We observe once again that the cos product is even, while the sin product is odd. Therefore,

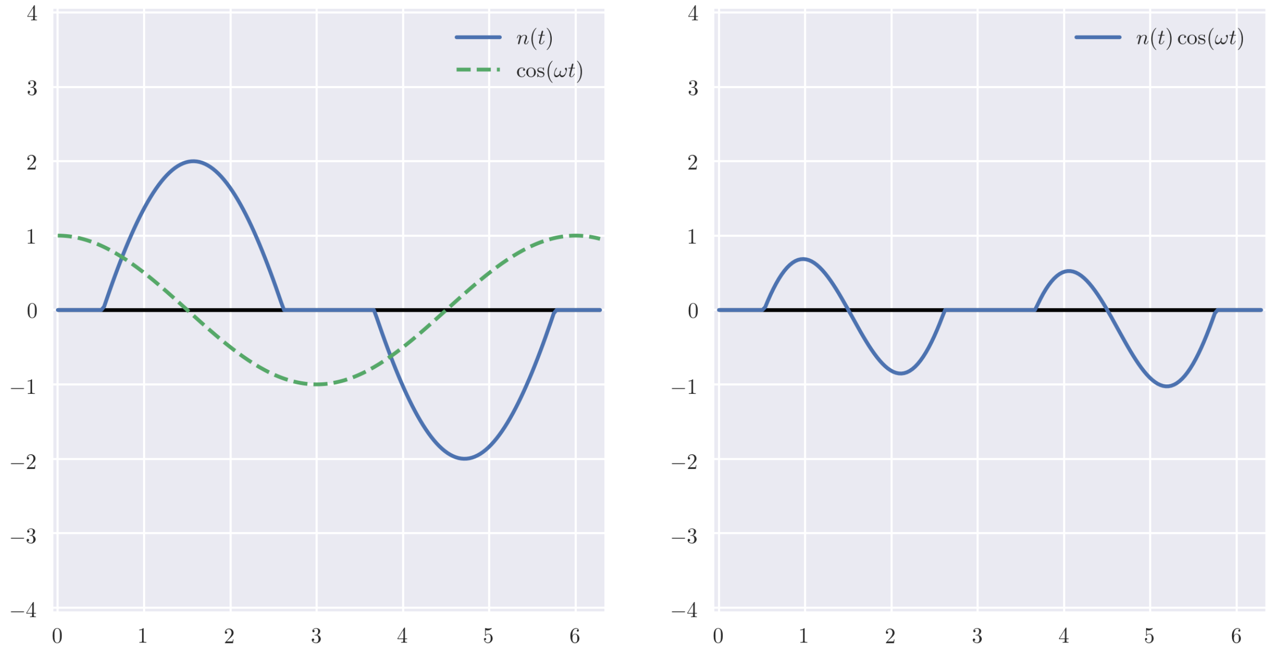

Figure 5: Product of $n(t)$ and $\cos(\omega t)$, with $|V| = 2$.

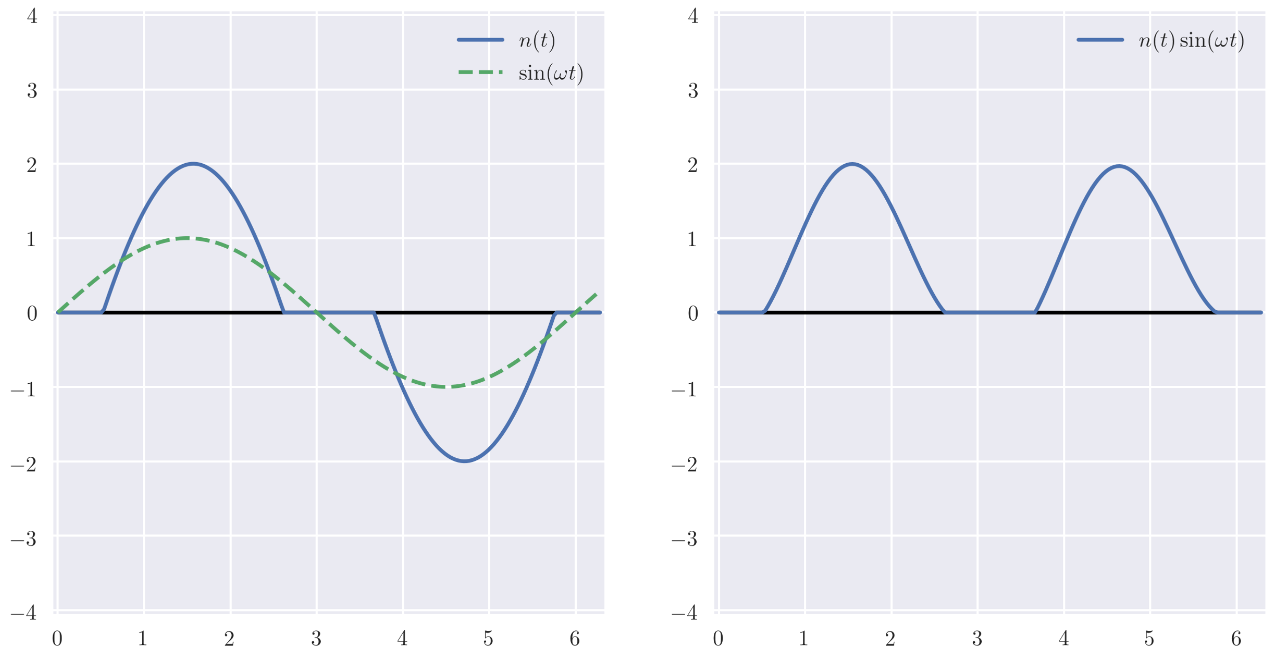

Figure 6: Product of $n(t)$ and $\sin(\omega t)$, with $|V| = 2$.

the A_1 term vanishes again and we are left to solve for B_1:

$$B_1(t) = \frac{2}{T}\int_0^T n(t)\sin(\omega t)\,\mathrm{d}t \tag{15}$$

Following the same arguments as in the previous case,

$$\begin{aligned} B_1(t) &= \frac{8}{T}\int_0^{T/4} n(t)\sin(\omega t)\,\mathrm{d}t && (16)\\ &= \frac{8}{T}\left[\int_\beta^{\pi/2} k(V\sin(\omega t)-\delta)\sin(\omega t)\,\mathrm{d}t\right] && (17)\\ &= \frac{8V}{\omega T}\left[\frac{\pi}{2}-\beta-\frac{\delta}{V}\sqrt{1-\frac{\delta^2}{V^2}}\right] && (18) \end{aligned}$$

By trigonometry, $\beta \equiv \arcsin(\delta/V)$, so that

$$\begin{aligned} B_1(t) &= \frac{8V}{\omega T}\left[\frac{\pi}{2}-\arcsin\left(\frac{\delta}{V}\right)-\frac{\delta}{V}\sqrt{1-\frac{\delta^2}{V^2}}\right] && (19)\\ &= \frac{4V}{\pi}\left[\frac{\pi}{2}-\arcsin\left(\frac{\delta}{V}\right)-\frac{\delta}{V}\sqrt{1-\frac{\delta^2}{V^2}}\right] && (20) \end{aligned}$$

$$\boxed{N(M,\omega) = \frac{4V}{\pi M}\left[\frac{\pi}{2}-\arcsin\left(\frac{\delta}{V}\right)-\frac{\delta}{V}\sqrt{1-\frac{\delta^2}{V^2}}\right]} \qquad (21)$$