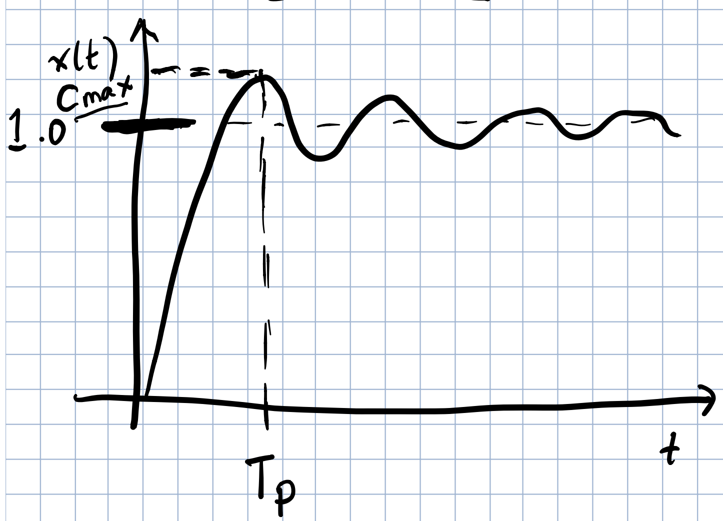
UNDERDAMPED 2nd-ORDER RESPONSE

$$x(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos\left(\omega_n\sqrt{1-\zeta^2}\,t - \theta\right)$$

$$\theta = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

$x(t)$ is the response of a 2nd-order system to a unit step.

Today's Objective: Find the <u>peak time</u> and <u>overshoot</u>.



Let's use

$C(t)$: controlled variable for traditional reasons.

$$c(t) = x(t)$$

To get peak we do

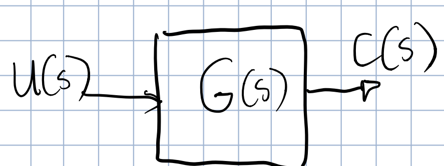
$$\frac{dc(t)}{dt} = 0$$

It's easier to solve it in Laplace Space:

$$\mathcal{L}\left\{\frac{dc(t)}{dt}\right\} = s\,C(s)$$ ← derivative property of L.T

$$= s\,U(s)G(s)$$

$$= \cancel{s}\frac{1}{\cancel{s}}G(s)$$



$$C(s) = U(s)G(s)$$

$$\mathcal{L}^{-1}\left\{\frac{d\,c(s)}{dt} = \underline{G(s)}\right\}$$

$$\mathcal{L}^{-1}\{G(s)\} = g(t)$$

So getting the peak (maxima) is the same as getting the inverse LT of $G(s)$.

NOTE: Valid only for step input!

$$\frac{d\,c(t)}{dt} = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} = 0$$

$$\Rightarrow \frac{\omega_n^2}{\underline{(s+\zeta\omega_n)^2} + \underline{\omega_n^2(1-\zeta^2)}} =$$

reminds you of $\mathcal{L}\left\{\frac{\omega}{s^2+\omega^2}\right\}$ ✓

$$\mathcal{L}\{\sin\omega t\}$$

$$\rightarrow s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 - \zeta^2\omega_n^2 + \omega_n^2$$

$$G(s) = \left(\frac{\omega_n}{\sqrt{1-\zeta^2}}\right)\frac{\omega_n\sqrt{(1-\zeta^2)}}{\left[(\underbrace{s+\zeta\omega_n}_{s \text{ w/a shift}})^2 + \omega_n^2(1-\zeta^2)\right]}$$

$$\mathcal{L}^{-1}\{G(s)\} = \boxed{g(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)}$$

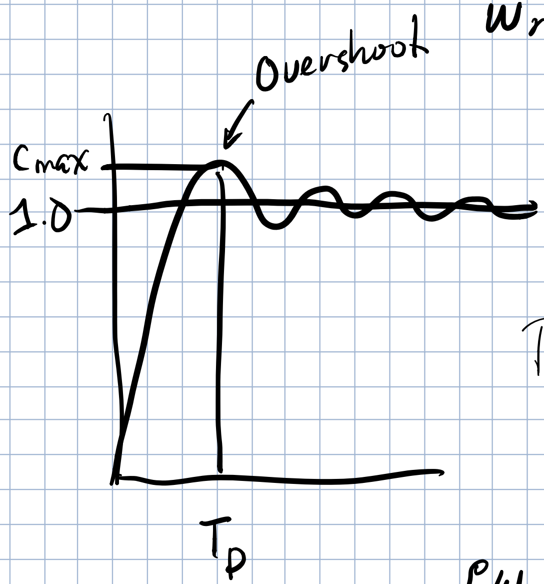
This is already $\frac{dc(t)}{dt}$. To be equal to zero let sin( ) = 0. And that happens when:

$$\omega_n\sqrt{1-\zeta^2}\,t = n\pi$$

$$\omega_n\sqrt{1-\zeta^2}\,T_p = \pi \quad (n=1 \text{ first peak})$$

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

↑ peak time!



To find cmax:

$$c_{max} = c(T_p)$$

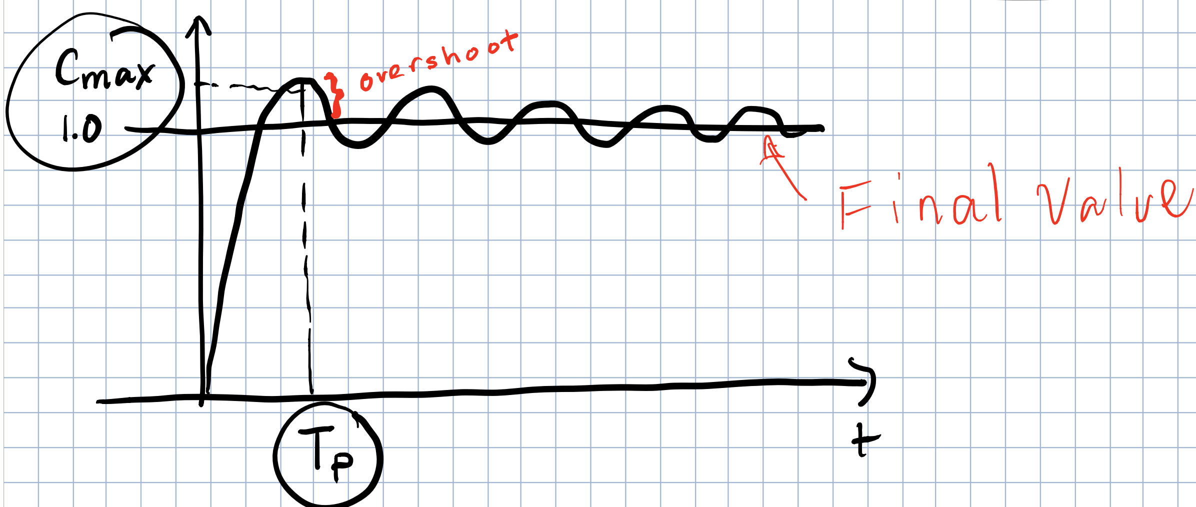
$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t - \theta\right)$$

rewrite cos( ) this way ?

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + \zeta\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right]$$

$$c(T_p) = 1 - \frac{e^{-\frac{\zeta\omega_n\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\cancel{\sqrt{1-\zeta^2}}}\left[\cancel{\sqrt{1-\zeta^2}}\,\overset{-1}{\cancel{\cos(\pi)}} + \zeta\,\overset{0}{\cancel{\sin(\pi)}}\right]$$

$$c(t) = \boxed{1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = C_{max}}$$



$$\underset{\text{(in decimal)}}{\%\ \text{OVERSHOOT}} = \frac{C_{max} - C_{final}}{C_{final}} \quad *$$

$$\boxed{\underset{\text{(in decimal)}}{\%\ \text{OVERSHOOT}} = e^{\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}$$

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

* Note that depending on the system, $C_{final} \neq 1$ sometimes. So observe the graph to get C_{final} and properly get (.OS) decimal overshoot.

$$(.OS) = e^{\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

Take ln of both sides and square.

$$\ln(.05) = \frac{-\zeta\pi}{\sqrt{1-\zeta^2}}$$

$$\left(\sqrt{1-\zeta^2} = \frac{-\zeta\pi}{\ln(.05)}\right)^2$$

$$1-\zeta^2 = \frac{+\zeta^2\pi^2}{\ln^2(.05)}$$

$$\ln^2(.05) - \ln^2(.05)\zeta^2 = \zeta^2\pi^2$$

$$\ln^2(.05) = \zeta^2\left(\ln^2(.05) + \pi^2\right)$$

$$\boxed{\zeta = \frac{\ln(.05)}{\sqrt{\ln^2(.05)+\pi^2}}}$$

and Together w/

$$\boxed{\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}}}$$

getting T_p and $(.05)$ decimal overshoot allows us to estimate the 2nd-order model of the system.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$