PS 24: Problem 2.54

(a) The radius of a black hole depends on the following:

$$F_G = G\frac{M^2}{R^2} = \left[\frac{m^3}{kg \cdot s^2}\right]\left[\frac{kg^2}{m^2}\right] \quad (1)$$

$$c = \text{constant} = \left[\frac{m}{s}\right] \quad (2)$$

$$M = \text{constant} = [kg] \quad (3)$$

We can estimate the radius R of the black hole dimensional analysis of (1)-(3). We have

$$R \approx \frac{GM}{c^2} = \frac{\left[\frac{m^3}{kg \cdot s^2}\right][kg]}{\left[\frac{m}{s}\right]^2} = [m] \quad (4)$$

Therefore, the radius of a black hole is

$$\boxed{R \approx \frac{GM}{c^2}} \quad (5)$$

(b) Taking photons with wavelength λ

$$\lambda = 2R = 2\frac{GM}{c^2} \quad (6)$$

whose energy E_γ is

$$E_\gamma = \frac{hc}{\lambda} \quad (7)$$

$$= \frac{hc^3}{2GM} \quad (8)$$

and the black hole's total energy E

$$E = Mc^2 \quad (9)$$

The momentum of a photon is given by

$$p_\gamma = \frac{h}{\lambda} \quad (10)$$

If we consider the system to behave classically and non-relativistically, then we can recall the classical momentum

$$p = mv \quad (11)$$

and from this, we see that we can divide (10) by the photon's velocity c to get its mass

$$m_\gamma = \frac{h}{c\lambda} \quad (12)$$

Recall the third law of thermodynamics

$$S = k_B \log \Omega \quad (13)$$

If we assume that S is of the order Nk, where N is the number of particles in the black hole, and that all the particles are photons, (13) has the form

$$S = k_B \frac{M}{m_\gamma} \quad (14)$$

where k_B is Boltzmann's constant. The black hole's entropy is

$$S = k_B \frac{Mc\lambda}{h} \quad (15)$$

Plugging (6) into this,

$$\boxed{S = k_B \frac{2GM^2}{hc}} \quad (16)$$

The entropy for a black hole of one solar mass is $\boxed{S \approx 4 \times 10^{52} \text{ J} \cdot \text{K}}$.

(c) Recall the surface area of a sphere:

$$A = 4\pi R^2 \quad (17)$$

Plugging in (5) into this,

$$A = \frac{4\pi M^2}{c^4} \quad (18)$$

Plugging this into (16),

$$S = k_B \frac{\pi G c^3}{2h} A \qquad (19)$$

Thus, entropy increases when black holes coalesce.

(d) Using (9) to express (16) in terms of E,

$$S = k_B \frac{2GE^2}{hc^5} \qquad (20)$$

From the fundamental thermodynamic relation

$$dE = T\,dS - P\,dV + \mu\,dN \qquad (21)$$

we see that the natural variable T can be expressed as

$$T = \left(\frac{\partial S}{\partial E}\right)^{-1} \qquad (22)$$

Performing the differentiation on (20),

$$T = \left(k_B \frac{4GE}{hc^5}\right)^{-1} \qquad (23)$$

We express (23) once again in terms of M so that

$$T = \frac{hc^3}{4k_B GM} \qquad (24)$$

So the temperature for a black hole of one solar mass is $\boxed{T \approx 2 \times 10^{-6} \text{ K}}$.

Expressing (24) in terms of R in (5),

$$T = \frac{hc}{4k_B R} \qquad (25)$$