PS 44: Problem 4.22

Given the following:

$$E = \left(Q + \frac{1}{2}N\right)\hbar\omega \tag{1}$$

Solving for Q:

$$\frac{E}{\hbar\omega} = Q + \frac{1}{2}N \tag{2}$$

$$Q = \frac{E}{\hbar\omega} - \frac{1}{2}N \tag{3}$$

The number of accessible microstates in which N distinguishable oscillators can share Q indistinguishable quanta is given by

$$\Omega = \frac{(Q+N-1)!}{Q!(N-1!)} \tag{4}$$

Substituting (3) into (4):

$$\Omega = \frac{\left(\frac{E}{\hbar\omega} - \frac{1}{2}N + N - 1\right)!}{\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)!(N-1)!} \tag{5}$$

$$= \frac{\left(\frac{E}{\hbar\omega} + \frac{1}{2}N - 1\right)!}{\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)!(N-1)!} \tag{6}$$

The thermodynamic entropy S is given by

$$S = k \ln \Omega \tag{7}$$

Substituting (6) into (7):

$$S = k \ln \left[\frac{\left(\frac{E}{\hbar\omega} + \frac{1}{2}N - 1\right)!}{\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)!(N-1)!}\right] \tag{8}$$

$$S = k\left[\ln\left(\frac{E}{\hbar\omega} + \frac{1}{2}N - 1\right)! - \ln\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)! - \ln(N-1)!\right] \tag{9}$$

Consider a collection of N distinguishable harmonic oscillators. If we let $N \gg 1$, then (9) becomes:

$$S = k\left[\ln\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)! - \ln\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)! - \ln N!\right] \tag{10}$$

By Stirling's (weak) approximation,

$$\ln N! = N \ln N - N \tag{11}$$

$$\ln\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)! = \left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)\ln\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right) - \left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right) \tag{12}$$

$$\ln\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)! = \left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)\ln\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right) - \left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right) \tag{13}$$

The thermodynamic temperature T is given by:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N \tag{14}$$

Applying (14) to (9)

$$\frac{\partial}{\partial E}(\ln N!) = 0 \tag{15}$$

$$\frac{\partial}{\partial E}\left[\ln\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)!\right] = \left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)^{-1}\left(\frac{1}{\hbar\omega}\right) + \left(\frac{1}{\hbar\omega}\right)\ln\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right) - \left(\frac{1}{\hbar\omega}\right) \tag{16}$$

$$= \left(\frac{1}{\hbar\omega}\right)\ln\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right) \tag{17}$$

$$= \left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)^{-1}\left(\frac{1}{\hbar\omega}\right) + \left(\frac{1}{\hbar\omega}\right)\ln\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right) - \left(\frac{1}{\hbar\omega}\right) \tag{18}$$

$$= \left(\frac{1}{\hbar\omega}\right)\ln\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right) \tag{19}$$

We then have, for the thermodynamic temperature:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N \tag{20}$$

$$= k\left[\left(\frac{1}{\hbar\omega}\right)\ln\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right) - \left(\frac{1}{\hbar\omega}\right)\ln\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)\right] \tag{21}$$

$$\boxed{\frac{1}{T} = \frac{k}{\hbar\omega}\ln\left[\frac{\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)}{\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)}\right]} \tag{22}$$

Solving for E:

$$\frac{1}{T} = \frac{k}{\hbar\omega} \ln\left[\frac{\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)}{\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)}\right]$$

$$\frac{\hbar\omega}{kT} = \ln\left[\frac{\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)}{\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)}\right] \tag{23}$$

$$e^{\hbar\omega/kT} = \frac{\left(\frac{E}{\hbar\omega} + \frac{1}{2}N\right)}{\left(\frac{E}{\hbar\omega} - \frac{1}{2}N\right)} \tag{24}$$

$$\frac{E}{\hbar\omega}e^{\hbar\omega/kT} - \frac{1}{2}Ne^{\hbar\omega/kT} = \frac{E}{\hbar\omega} - \frac{1}{2}N \tag{25}$$

$$\frac{E}{\hbar\omega}e^{\hbar\omega/kT} - \frac{E}{\hbar\omega} = \frac{1}{2}Ne^{\hbar\omega/kT} + \frac{1}{2}N \tag{26}$$

$$\frac{E}{\hbar\omega}\left(e^{\hbar\omega/kT} - 1\right) = \frac{N}{2}\left(e^{\hbar\omega/kT} + 1\right) \tag{27}$$

$$\boxed{E(T) = \frac{N\hbar\omega}{2}\left(\frac{e^{\hbar\omega/kT} + 1}{e^{\hbar\omega/kT} - 1}\right)} \tag{28}$$