PS 31: Problem 3.39

The random variable x has the probability density

$$p(x) = \begin{cases} Ae^{-\lambda x} & 0 \leq x \leq \infty \\ 0 & x < 0 \end{cases} \tag{1}$$

(a) To get the normalization constant A of (1), we take its integral over all space. Since the function is zero if $x < 0$, we consider only the area where it is non-zero and equate it to unity:

$$\begin{aligned} 1 &= \int_0^\infty Ae^{-\lambda x}\,\mathrm{d}x &(2) \\ 1 &= A\int_0^\infty e^{-\lambda x}\,\mathrm{d}x \\ 1 &= A\left.\frac{e^{-\lambda x}}{-\lambda}\right|_0^\infty \\ 1 &= \frac{A}{-\lambda}\left[e^{-\lambda\infty} - e^0\right] \\ 1 &= \frac{A}{-\lambda}[-1] \\ &\boxed{A = \lambda} &(3) \end{aligned}$$

(1) now becomes

$$p(x) = \begin{cases} \lambda e^{-\lambda x} & 0 \leq x \leq \infty \\ 0 & x < 0 \end{cases} \tag{4}$$

(b) The mean value of x is

$$\begin{aligned} \langle x \rangle &= \int_0^\infty xp(x)\,\mathrm{d}x &(5) \\ &= \int_0^\infty x\lambda e^{-\lambda x}\,\mathrm{d}x \\ &= \lambda\int_0^\infty xe^{-\lambda x}\,\mathrm{d}x &(6) \end{aligned}$$

Using integration by parts, let $u \equiv x$ and $\mathrm{d}v \equiv e^{-\lambda x}\,\mathrm{d}x$. Consequently, $\mathrm{d}u \equiv \mathrm{d}x$ and $v \equiv -e^{-\lambda x}/a$. We have

$$\langle x \rangle = \lambda\left[-\left.\frac{xe^{-\lambda x}}{\lambda}\right|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\,\mathrm{d}x\right] \tag{7}$$

The integral in (7) follows the same solution as in part (a), and we end up with

$$\begin{aligned} \langle x \rangle &= \lambda\left[-\frac{xe^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right|_0^\infty \\ &= -\left[\frac{\lambda xe^{-\lambda x} + e^{-\lambda x}}{\lambda}\right|_0^\infty &(8) \end{aligned}$$

Substituting ∞ directly into the first term causes it to blow up. However, upon examining the terms involved we see that $\frac{\mathrm{d}^2}{\mathrm{d}x^2}x < \frac{\mathrm{d}^2}{\mathrm{d}x^2}e^{-\lambda x}$. Thus, we can say that the exponential term approaches zero faster than x approaches infinity, so the exponential term dominates. Therefore,

$$\begin{aligned} \langle x \rangle &= -\left[\frac{e^{-\lambda\infty} - e^0}{\lambda}\right] \\ &= -\frac{0-1}{\lambda} \\ &\boxed{\langle x \rangle = \frac{1}{\lambda}} &(9) \end{aligned}$$

The most probable value of $\boxed{x = 0}$.

(c) The mean value of x^2 is

$$\langle x^2 \rangle = \lambda\int_0^\infty x^2e^{-\lambda x}\,\mathrm{d}x \tag{10}$$

Once again, integrate by parts, letting $u \equiv x^2$ and $\mathrm{d}v \equiv e^{\lambda x}\,\mathrm{d}x$. So $\mathrm{d}u \equiv 2x\,\mathrm{d}x$ and $v \equiv -e^{-\lambda x}/a$. We have

$$\langle x^2 \rangle = \lambda\left[-\left.\frac{x^2e^{-\lambda x}}{\lambda}\right|_0^\infty + \frac{2}{\lambda}\int_0^\infty xe^{-\lambda x}\,\mathrm{d}x\right] \tag{11}$$

Following the steps in part (b), the integral can be simplified as

$$\begin{aligned}\left\langle x^2\right\rangle &= \lambda\left[-\frac{2xe^{-\lambda x}}{\lambda^2} - \frac{x^2e^{-\lambda x}}{\lambda}\bigg|_0^\infty + \frac{2}{\lambda}\int_0^\infty e^{-\lambda x}\,\mathrm{d}x\right] \\ &= \lambda\left[-\frac{2e^{-\lambda x}}{\lambda^3} - \frac{2xe^{-\lambda x}}{\lambda^2} - \frac{x^2e^{-\lambda x}}{\lambda}\bigg|_0^\infty\right. \\ &= \lambda\left[\frac{2}{\lambda^3} + 0 + 0\right]\end{aligned}$$

$$\boxed{\left\langle x^2\right\rangle = \frac{2}{\lambda^2}} \tag{12}$$

(d) For $\lambda = 1$, the probability that a measurement of x yields a value $\in (1, 2)$ is

$$\begin{aligned}\Pr(1 < x < 2) &= \int_1^2 p(x)\,\mathrm{d}x \qquad (13) \\ &= \int_1^2 e^{-x}\,\mathrm{d}x \\ &= -e^{-x}\Big|_1^2 \\ &= -e^{-2} + e^{-1}\end{aligned}$$

$$\boxed{\Pr(1 < x < 2) = \frac{1}{e} - \frac{1}{e^2} \approx 0.23} \tag{14}$$

(e) For $\lambda = 1$, the probability that a measurement of x yields a value < 0.3 is

$$\begin{aligned}\Pr(x < 0.3) &= \int_0^{0.3} p(x)\,\mathrm{d}x \qquad (15) \\ &= \int_0^{0.3} e^{-x}\,\mathrm{d}x \\ &= -e^{-x}\Big|_0^{0.3} \\ &= -e^{-0.3} + e^{-0}\end{aligned}$$

$$\boxed{\Pr(1 < x < 2) = 1 - \frac{1}{e^{0.3}} \approx 0.26} \tag{16}$$