PS 24: Problem 2.54

(a) The radius of a black hole depends on the following:

$$F_G = G\frac{M}{R^2} = \left[\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}\right]\left[\frac{\mathrm{kg}}{\mathrm{m}^2}\right] \tag{1}$$

$$c = \text{constant} = \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \tag{2}$$

$$M = \text{constant} = [\mathrm{kg}] \tag{3}$$

We can estimate the radius R of the black hole by dimensional analysis of (1)-(3). We have

$$R \approx \frac{GM}{c^2} = \frac{\left[\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}\right][\mathrm{kg}]}{\left[\frac{\mathrm{m}}{\mathrm{s}}\right]^2} = [\mathrm{m}] \tag{4}$$

Therefore, the radius of a black hole is

$$\boxed{R \approx \frac{GM}{c^2}} \tag{5}$$

(b) Taking photons with wavelength λ

$$\lambda = 2R = 2\frac{GM}{c^2} \tag{6}$$

whose energy E_γ is

$$E_\gamma = \frac{hc}{\lambda} \tag{7}$$

$$= \frac{hc^3}{2GM} \tag{8}$$

and the black hole's total energy E

$$E = Mc^2 \tag{9}$$

The momentum of a photon is given by

$$p_\gamma = \frac{h}{\lambda} \tag{10}$$

If we consider the system to behave classically and non-relativistically, then we can recall the classical momentum

$$p = mv \tag{11}$$

and from this, we see that we can divide (10) by the photon's velocity c to get its mass

$$m_\gamma = \frac{h}{c\lambda} \tag{12}$$

Recall the third law of thermodynamics

$$S = k_B \log \Omega \tag{13}$$

If we assume that S is of the order Nk, where N is the number of particles in the black hole, and that all the particles are photons, (13) has the form

$$S = k_B \frac{M}{m_\gamma} \tag{14}$$

where k_B is Boltzmann's constant. The black hole's entropy is

$$S = k_B \frac{Mc\lambda}{h} \tag{15}$$

Plugging (6) into this,

$$\boxed{S = k_B \frac{2GM^2}{hc}} \tag{16}$$

The entropy for a black hole of one solar mass is $\boxed{S \approx 4 \times 10^{52}\ \mathrm{J}\cdot\mathrm{K}}$.

(c) Recall the surface area of a sphere:

$$A = 4\pi R^2 \tag{17}$$

Plugging in (5) into this,

$$A = \frac{4\pi M^2}{c^4} \tag{18}$$

Plugging this into (16),

$$\boxed{S = k_B \frac{\pi G c^3}{2h} A} \tag{19}$$

Thus, entropy increases when black holes coalesce.

(d) Using (9) to express (16) in terms of E,

$$S = k_B \frac{2GE^2}{hc^5} \tag{20}$$

From the fundamental thermodynamic relation

$$\mathrm{d}E = T\,\mathrm{d}S - P\,\mathrm{d}V + \mu\,\mathrm{d}N \tag{21}$$

we see that the natural variable T can be expressed as

$$T = \left(\frac{\partial S}{\partial E}\right)^{-1} \tag{22}$$

Performing the differentiation on (20),

$$T = \left(k_B \frac{4GE}{hc^5}\right)^{-1} \tag{23}$$

We express (23) once again in terms of M so that

$$\boxed{T = \frac{hc^3}{4k_B G M}} \tag{24}$$

So the temperature for a black hole of one solar mass is $\boxed{T \approx 2 \times 10^{-6}\ \mathrm{K}}$.
Expressing (24) in terms of R in (5),

$$\boxed{T = \frac{hc}{4k_B R}} \tag{25}$$