PS 32: Problem 3.40

Consider the probability density function

$$p(v_x) = \left(\frac{a}{\pi}\right)^{\frac{3}{2}} e^{-av_x^2} \tag{1}$$

for the velocity of a particle in the x-direction. The probability densities for v_y and v_z have the same form. Each of the three velocity components can range from $-\infty$ to $+\infty$, and a is a constant.

(a) The velocity vector probability density is given by

$$\begin{aligned} p(\mathbf{v}) &= p(v_x + v_y + v_z) && (2) \\ &= \iiint_{-\infty}^{+\infty} \left(\frac{a}{\pi}\right)^{\frac{3}{2}} e^{-a(v_x^2+v_y^2+v_z^2)}\,\mathrm{d}\mathbf{v} && (3) \\ 1 &= \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left(\int_{-\infty}^{+\infty} e^{-av_x^2}\,\mathrm{d}v_x\right)\cdot \\ &\quad \left(\int_{-\infty}^{+\infty} e^{-av_y^2}\,\mathrm{d}v_y\right)\cdot && (4) \\ &\quad \left(\int_{-\infty}^{+\infty} e^{-av_z^2}\,\mathrm{d}v_z\right) \end{aligned}$$

where (4) follows from the normalization condition. From (GT 3.123),

$$\int_0^{\infty} e^{-au^2}\,\mathrm{d}u = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{5}$$

The multiple integrals defined in (4) can be expressed in a form similar to (5) by noting that they are symmetric about zero, so that they can be recast in the form

$$\int_{-\infty}^{+\infty} e^{-av_x^2}\,\mathrm{d}v_x = 2\int_0^{\infty} e^{-av_x^2}\,\mathrm{d}v_x \tag{6}$$

(4) now becomes

$$\begin{aligned} 1 &= \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left[\left(\sqrt{\frac{\pi}{a}}\right)\left(\sqrt{\frac{\pi}{a}}\right)\left(\sqrt{\frac{\pi}{a}}\right)\right] \\ 1 &= \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left(\sqrt{\frac{\pi}{a}}\right)^3 \\ 1 &= \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left(\frac{\pi}{a}\right)^{\frac{3}{2}} \\ &\boxed{1 = 1} && (7) \end{aligned}$$

Therefore, $p(\mathbf{v})$ is already normalized.

(b) To get the probability that a particle has a velocity $\in (v_x, v_x + \mathrm{d}v_x) \cup (v_y, v_y + \mathrm{d}y) \cup (v_z, v_z + \mathrm{d}z)$, we evaluate the indefinite form of (3).

$$\begin{aligned} &\Pr(\mathbf{v} \le \mathbf{v}' \le \mathbf{v} + \mathrm{d}\mathbf{v}) = \\ &\quad \iiint \left(\frac{a}{\pi}\right)^{\frac{3}{2}} e^{-a\cdot\mathbf{v}^2}\,\mathrm{d}\mathbf{v} && (8) \\ &= \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left(\int e^{-av_x^2}\,\mathrm{d}v_x\right)\cdot \\ &\quad \left(\int e^{-av_y^2}\,\mathrm{d}v_y\right)\cdot && (9) \\ &\quad \left(\int e^{-av_z^2}\,\mathrm{d}v_z\right) \end{aligned}$$

$$\boxed{\begin{aligned} &\Pr(\mathbf{v} \le \mathbf{v}' \le \mathbf{v} + \mathrm{d}\mathbf{v}) = \\ &\left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left(\frac{\pi}{a}\right)^{\frac{3}{2}} \operatorname{erf}(\sqrt{a}v_x)\cdot \\ &\quad \operatorname{erf}(\sqrt{a}v_y)\cdot\operatorname{erf}(\sqrt{a}v_z) \end{aligned}} \tag{10a}$$

where the error function $\operatorname{erf}(x)$ is defined as

$$\operatorname{erf}(x) \equiv \frac{1}{\pi}\int_{-x}^{x} e^{-t^2}\,\mathrm{d}t\,. \tag{11}$$

(c) To get the probability of all the components of $\mathbf{v} \ge 0$ simultaneously, we evaluate (3) over the non-negative range:

$$\Pr(\mathbf{v} \geq 0) = \iiint_0^\infty \left(\frac{a}{\pi}\right)^{\frac{3}{2}} e^{-a\cdot\mathbf{v}}\,\mathrm{d}\mathbf{v} \tag{12}$$

$$= \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left(\int_0^{+\infty} e^{-av_x^2}\,\mathrm{d}v_x\right)\cdot \left(\int_0^{+\infty} e^{-av_y^2}\,\mathrm{d}v_y\right)\cdot \left(\int_0^{+\infty} e^{-av_z^2}\,\mathrm{d}v_z\right)$$

$$= \left(\frac{a}{\pi}\right)^{\frac{3}{2}} \left[\frac{1}{2}\left(\sqrt{\frac{\pi}{a}}\right)\right]^3$$

$$\boxed{\Pr(\mathbf{v} \geq 0) = \frac{1}{8}} \tag{13}$$