PS 33: Problem 3.43

(a) The Lorentz distribution is given by

$$p(x) = \frac{1}{\pi}\frac{\gamma}{(x-a)^2+\gamma^2} \tag{1}$$

Comparing it with the Gaussian distribution:

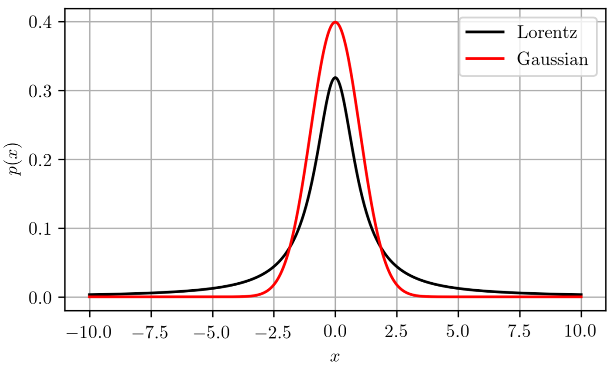

Figure 1: Comparison of Lorentzian and Gaussian distributions.

This shows that for the same set of parameters, the Gaussian has a higher peak, and quickly falls off and approaches zero as one moves away from the mean/central value.

(b) For $a = 0$ and $\gamma = 1$, the first moment of the Lorentzian is given by

$$\begin{aligned}\langle x^1 \rangle &= \int_{-\infty}^{+\infty} x p(x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x}{x^2+1}\,\mathrm{d}x\end{aligned} \tag{2}$$

Let $u \equiv x^2+1$, $\mathrm{d}u \equiv 2x\,\mathrm{d}x$,

$$\begin{aligned}\langle x^1 \rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{u}\,\mathrm{d}u \\ &= \frac{1}{2\pi}\left[\ln(u)\Big|_{-\infty}^{+\infty}\right] \\ &= \frac{1}{2\pi}\left[\ln\left(x^2+1\right)\Big|_{-\infty}^{+\infty}\right] \\ &= \infty - \infty\end{aligned}$$

$$\boxed{\langle x \rangle = \text{undefined}} \tag{3}$$

(c) The second moment is given by

$$\langle x^2 \rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{x^2}{x^2+1}\,\mathrm{d}x \tag{4}$$

By long division of the integrand,

$$\begin{array}{r|lll} & & & 1 \\ \hline x^2+1 & x^2 & +0x+ & 0 \\ & x^2 & +0x+ & 1 \\ \hline & & & -1 \end{array}$$

From this, (4) can be rewritten as

$$\begin{aligned}\langle x^2 \rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 1 - \frac{1}{x^2+1}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty}\mathrm{d}x - \int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{x^2+1}\right]\end{aligned}$$

The second term is an even function about zero, and can be rewritten as

$$
\begin{aligned}
\left\langle x^2 \right\rangle &= \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - 2\int_0^{+\infty} \frac{\mathrm{d}x}{x^2+1}\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - 2 \arctan(x)\Big|_0^{\infty}\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - 2(\arctan(\infty) - \arctan(0))\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - 2\left(\frac{\pi}{2} - 0\right)\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - \pi\right] \\
&= \frac{1}{2\pi}[\infty - \pi]
\end{aligned}
$$

$$\boxed{\left\langle x^2 \right\rangle = \infty} \tag{5}$$

The second moment of the Lorentz distribution exists and has a value of infinity.