PS 34: Problem 3.45

(a) The uniform distribution is given by

$$p(s) = \frac{1}{b-a} \tag{1}$$

where a and b are the bounds of concern. For a variable s_i uniformly distributed $\in [0,1]$, the mean can be obtained by calculating the first moment of (1):

$$\begin{aligned} \bar{s} = \left\langle s^1 \right\rangle &= \int_0^1 s \, \mathrm{d}s \qquad (2) \\ &= \frac{1}{2} s^2 \Big|_0^1 \\ \boxed{\langle s \rangle = \frac{1}{2}} & \qquad (3) \end{aligned}$$

The standard deviation can be obtained by calculating the square root of the difference of the second moment and the square of the first moment:

$$\begin{aligned} \sigma &= \sqrt{\left\langle s^2 \right\rangle - \langle s \rangle^2} \qquad (4) \\ \left\langle s^2 \right\rangle &= \int_0^1 s^2 \, \mathrm{d}s \qquad (5) \\ &= \frac{1}{3} s^2 \Big|_0^1 \\ &= \frac{1}{3} \qquad (6) \\ \sigma &= \sqrt{\frac{1}{3} - \left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{1}{12}} \\ \boxed{\sigma = \frac{1}{\sqrt{12}} \approx 0.29} & \qquad (7) \end{aligned}$$

(b) As the number of measurements S is increased, the standard deviation σ becomes smaller and the expectation value becomes more defined, i.e.

$$\boxed{\lim_{S \to \infty} p(S) = \delta(S)} \tag{8}$$

(c) For $N = 12$, running the program `CentralLimitTheorem` for 40,000 trials yields a variance $\sigma^2 = 0.007$. Its width can be obtained by doubling the standard deviation, given by $\sqrt{\sigma^2}$. Thus, the $\boxed{\text{width} = 1.6}$.

(d) Consider the probability density

$$f(s) = e^{-s}, \quad s \geq 0 \tag{9}$$

Its mean is calculated as

$$\langle s \rangle = \int_0^\infty s e^{-s} \, \mathrm{d}s \tag{10}$$

which can be evaluated using integration by parts. Let $u \equiv s$ and $\mathrm{d}v \equiv e^{-s} \, \mathrm{d}s$. Consequently, $\mathrm{d}u \equiv \mathrm{d}s$ and $v \equiv -e^{-s}$. We have

$$\begin{aligned} \langle s \rangle &= -s e^{-s} \Big|_0^\infty + \int_0^\infty e^{-s} \, \mathrm{d}s \qquad (11) \\ &= \Big[-s e^{-s} - e^{-s} \Big|_0^\infty \end{aligned}$$

Since $\frac{\mathrm{d}^2}{\mathrm{d}s^2} s < \frac{\mathrm{d}^2}{\mathrm{d}s^2} e^{-s}$, the exponential term dominates. Thus,

$$\begin{aligned} \langle s \rangle &= 0 \cdot e^0 + e^0 \\ \boxed{\langle s \rangle = 1} & \qquad (12) \end{aligned}$$

Using $N = 12$, the distribution appears similar to a Poisson distribution. As the number of trials S is increased, the distribution first approaches that of a Gaussian, then of a Dirac delta, similar to (8).

Running the program once again for $N = 12$ and 40,000 trials, we have $\sigma^2 = 0.021$, which corresponds to a $\boxed{\text{width} = 0.3}$.

(e) Consider the distribution

$$f(s) = \frac{1}{\pi} \frac{1}{s^2 + 1}, \quad -\infty \leq s \leq \infty \tag{13}$$

Its mean value is

$$\langle s \rangle = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{s}{s^2+1} \, \mathrm{d}s \tag{14}$$

Let $u \equiv s^2 + 1$, $\mathrm{d}u \equiv 2s \, \mathrm{d}s$,

$$\begin{aligned} \langle s \rangle &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{d}u}{u} \\ &= \frac{1}{2\pi} \ln u \Big|_{-\infty}^{+\infty} \\ \Aboxed{\langle s \rangle &= \text{undefined}} \end{aligned} \tag{15}$$

From (4), we know that the variance is dependent on the mean. Consequently, the variance of s is undefined.