PS 29: Problem 3.35

A random walker takes n right steps and n' left steps for a total of N steps, so $N = n + n'$. Each step is a units long, and the probability of a right step is p. The net displacement after N steps is

$$x = a(n - n') \tag{1}$$

The mean displacement is

$$\begin{aligned}
\bar{x} &= a\overline{(n - n')} \\
&= a\bar{n} - a\bar{n'} \\
&= a\bar{n} - a\overline{N - n} \\
&= a\bar{n} - a\bar{N} + a\bar{n} \\
&= 2a\bar{n} - a\bar{N} \\
&= a(2\bar{n} - N)
\end{aligned} \tag{2}$$

Recall that the mean can also be expressed as $\bar{n} = pN$. So,

$$\begin{aligned}
\bar{x} &= a(2pN - N) \\
&\boxed{\bar{x} = aN(2p - 1)}
\end{aligned} \tag{3}$$

If the probabilites of n and n' are equal, (3) becomes zero, as expected. The variance is

$$\begin{aligned}
\overline{(\Delta x)^2} &= \overline{x^2} - \bar{x}^2 \\
&= a^2\overline{(n - n')^2} - [aN(2p - 1)]^2 \\
&= a^2\overline{(n^2 + n'^2 - 2nn')} - [aN(2p - 1)]^2 \\
&= a^2\overline{\left[n^2 + (N - n)^2 - 2n(N - n)\right]} - [aN(2p - 1)]^2 \\
&= a^2\overline{[n^2 + N^2 + n^2 - 2Nn - 2Nn + 2n^2]} - [aN(2p - 1)]^2 \\
&= a^2\overline{[4n^2 - 4Nn + N^2]} - [aN(2p - 1)]^2 \\
&= a^2\overline{[4n^2 - 4Nn + N^2]} - \left[a^2N^2(2p - 1)^2\right]
\end{aligned} \tag{4}$$

Recall that $\overline{n^2} = Npq + \bar{n}^2$. We write

$$\begin{aligned}
\overline{(\Delta x)^2} &= a^2\left[4Npq + 4\bar{n}^2 - 4N\bar{n} + N^2\right] - \left[a^2N^2(2p - 1)^2\right] \\
&= a^2\left[4Npq + (2p - 1)^2N^2\right] - \left[a^2N^2(2p - 1)^2\right] \\
&= 4a^2Npq + a^2(2p - 1)^2N^2 - \left[a^2N^2(2p - 1)^2\right] \\
&\boxed{\overline{(\Delta x)^2} = 4a^2}
\end{aligned} \tag{5}$$

The variance has no direct dependence on N.