PS 22: Problem 2.32

(a) The Legendre transform of a function $f(x)$ is given by

$$\mathcal{G}[m(x)] = f(x) - xm \tag{1}$$

where $m = f'(x)$.
Let $f = x^3$. It's Legendre transform is

$$\begin{aligned}\mathcal{G}[m(x)] &= x^3 - x(3x^2) \\ &= -2x^3\end{aligned} \tag{2}$$

But this must be expressed only in terms of m. Using $m = f'(x)$,

$$\begin{aligned} m &= 3x^2 \\ x &= \sqrt{\frac{m}{3}} \end{aligned} \tag{3}$$

Thus,

$$\boxed{\mathcal{G}\left[x^3\right] = -2\left(\frac{m}{3}\right)^{\frac{3}{2}}} \tag{4}$$

(b) Let $f(x) = x$. Its Legendre transform is

$$\begin{aligned}\mathcal{G}[m(x)] &= f(x) - xm \\ &= x - x(1) \\ \boxed{\mathcal{G}[x] = 0}\end{aligned} \tag{5}$$

Let $f(x) = \sin(x)$. Its Legendre transform is

$$\begin{aligned}\mathcal{G}[m(x)] &= \sin(x) - x\cos(x) \\ x &= \cos^{-1}(m) \\ \mathcal{G}[m(x)] &= \sin\left(\cos^{-1}(m)\right) - m\cos^{-1}(m)\end{aligned} \tag{6}$$

We can simplify this using the property

$$\sin\left(\cos^{-1}(m)\right) = \sqrt{1 - m^2} \tag{7}$$

Using this, we rewrite (6) as

$$\boxed{\mathcal{G}[\sin(x)] = \sqrt{1 - m^2} - m\cos^{-1}(m)} \tag{8}$$