PS 40: Problem 4.16

Given the following:

$$\ln\Gamma(E,V,N) = N\ln\frac{V}{N} + \frac{3}{2}N\ln\frac{mE}{3N\pi\hbar^2} + \frac{5}{2}N \tag{1}$$

The density of states is given by

$$g(E,V,N) \approx \frac{\mathrm{d}\Gamma(E)}{\mathrm{d}E} \tag{2}$$

We take the exponential of both sides of (1):

$$\begin{aligned}
\ln\Gamma(E,V,N) &= \ln\left(\frac{V}{N}\right)^N + \ln\left(\frac{mE}{3N\pi\hbar^2}\right)^{3N/2} + \frac{5}{2}N \\
\ln\Gamma(E,V,N) &= \ln\left[\left(\frac{V}{N}\right)^N\left(\frac{mE}{3N\pi\hbar^2}\right)^{3N/2}\right] + \frac{5}{2}N \\
\Gamma(E,V,N) &= \left(\frac{V}{N}\right)^N\left(\frac{mE}{3N\pi\hbar^2}\right)^{3N/2} e^{5N/2} \qquad (3)
\end{aligned}$$

Since $\Gamma \propto V^N$, $\Gamma \propto N^{-5N/2}e^{5N/2}$, and $\Gamma \propto E^{3N/2}$, Γ is a rapidly increasing function of E and V. Note that because $\mathrm{d}^2_N\left(e^{5N/2}\right) > \mathrm{d}^2_N\left(N^{-5N/2}\right)$, the exponential term dominates and thus, Γ is also a rapidly increasing function of N.

From (2), we differentiate (3) w.r.t. E:

$$\boxed{g(E,V,N) = \left(\frac{V}{N}\right)^N\left(\frac{m}{3N\pi\hbar^2}\right)^{3N/2} e^{5N/2}\frac{3N}{2}E^{N/2}} \tag{4}$$

Now, $g \propto V^N$, $g \propto e^{5N/2}$, and $g \propto E^{N/2}$, so g is also a rapidly increasing function of E, V, and N.