PS 23: Problem 2.33

From the fundamental thermodynamic relation,

$$\mathrm{d}E = T\,\mathrm{d}S - P\,\mathrm{d}V + \mu\,\mathrm{d}N \tag{1}$$

Applying Legendre transform on (1) to eliminate S in favor of T,

$$\mathcal{L}[E(S,V,N)]_S = E - S\frac{\partial E}{\partial S} \tag{2}$$

From (1), we know that $\frac{\partial E}{\partial S}$ is the expression for the natural variable T. Thus we have

$$\mathcal{L}[E(S,V,N)]_S = E - TS \tag{3}$$

We know this expression to be the Helmholtz free energy

$$F = E - TS \tag{4}$$

Therefore,

$$\boxed{\mathcal{L}[E(S,V,N)] = F(T,V,N)} \quad \square \tag{5}$$