PS 43: Problem 4.21

Given the following:

$$\frac{1}{T} = -k\frac{1}{2\mu B}\ln\left(\frac{N-n}{n}\right) \tag{1}$$
$$E = -(2n-N)\mu B \tag{2}$$

Solving for n/N in (1):

$$-\frac{2\mu B}{kT} = \ln\left(\frac{N-n}{n}\right) \tag{3}$$
$$e^{-2\mu B/kT} = \frac{N-n}{n} \tag{4}$$
$$e^{-2\mu B/kT} = \frac{N}{n} - 1 \tag{5}$$
$$1 + e^{-2\mu B/kT} = \frac{N}{n} \tag{6}$$
$$\boxed{\frac{n}{N} = \frac{1}{1+e^{-2\mu B/kT}}} \tag{7}$$

Let $\beta \equiv 1/kT$. With some further manipulation:

$$\frac{n}{N} = \frac{1}{1+e^{-2\beta\mu B}} \cdot \frac{e^{\beta\mu B}}{e^{\beta\mu B}} \tag{8}$$
$$\boxed{\frac{n}{N} = \frac{e^{\beta\mu B}}{e^{\beta\mu B}+e^{-\beta\mu B}}} \tag{9}$$

which is the same form obtained in GT (4.74b). Solving for n/N in (2):

$$E = -(2n-N)\mu B \cdot \frac{N}{N} \tag{10}$$
$$= -\frac{2n-N}{N}N\mu B \tag{11}$$
$$= -\left(\frac{2n}{N} - 1\right)N\mu B \tag{12}$$

Substitute (7) into this:

$$E = -\left(2\frac{e^{\beta\mu B}}{e^{\beta\mu B}+e^{-\beta\mu B}} - 1\right)N\mu B \tag{13}$$
$$= -\left(\frac{2e^{\beta\mu B} - e^{\beta\mu B} - e^{-\beta\mu B}}{e^{\beta\mu B}+e^{-\beta\mu B}}\right)N\mu B \tag{14}$$
$$= -\left(\frac{e^{\beta\mu B} - e^{-\beta\mu B}}{e^{\beta\mu B}+e^{-\beta\mu B}}\right)N\mu B \tag{15}$$

which can be simplified using the relation $\tanh(x) \equiv \frac{e^x-e^{-x}}{e^x+e^{-x}}$:

$$E = -\tanh(\beta\mu B)N\mu B \tag{16}$$
$$\boxed{E = -N\mu B\tanh\left(\frac{\mu B}{kT}\right)} \tag{17}$$

which is the same form as that in GT (4.73).