PS 30: Problem 3.37

With the given

$$\ln A = \ln P_n(n = \bar{n}) \tag{1}$$

$$P_N(n) = Ae^{-\frac{1}{2\sigma^2}(n-\bar{n})^2} \tag{2}$$

$$P_N(n) = \frac{N!}{n!(N-n)!}p^N(1-p)^{N-n} \tag{3}$$

$$\ln N! = N\ln N - N + \frac{1}{2}\ln(2\pi N) \tag{4}$$

Take the ln of both sides of (3)

$$\begin{aligned}\ln P_N(n) &= \ln\left[\frac{N!}{n!(N-n)!}p^N(1-p)^{N-n}\right]\\ \ln P_N(n) &= \ln N! - \ln n! - \ln(N-n)!\\ &\quad + N\ln p + (N-n)\ln q\end{aligned} \tag{5}$$

Since n is constant, we can write it interchangeably with $\bar{n}$:

$$\begin{aligned}\ln P_N(n) &= \ln N! - \ln \bar{n}! - \ln(N-\bar{n})!\\ &\quad + N\ln p + (N-\bar{n})\ln q\end{aligned} \tag{6}$$

Recall that $\bar{n} = Np$. We can write

$$\begin{aligned}\ln P_N(n) &= \ln N! - \ln(Np)! - \ln(N-Np)!\\ &\quad + N\ln p + (N-Np)\ln q\\ \ln P_N(n) &= \ln N! - \ln(Np)! - \ln(N-Np)!\\ &\quad + N\ln p + [N(1-p)]\ln q\end{aligned} \tag{7}$$

Using Stirling's approximation in (4),

$$\begin{aligned}\ln P_N(n) &= N\ln N - N + \frac{1}{2}\ln(2\pi N)\\ &\quad - Np\ln(Np) + Np - \frac{1}{2}(2\pi Np)\\ &\quad - N(1-p)\ln[N(1-p)] + N(1-p)\\ &\quad - \frac{1}{2}\ln[2\pi N(1-p)]\\ &\quad + Np\ln p + N(1-p)\ln q\end{aligned} \tag{8}$$

$$\begin{aligned}\ln P_N(n) &= N\ln N - N + \frac{1}{2}\ln(2\pi N)\\ &\quad - Np\ln(Np) + Np - \frac{1}{2}(2\pi Np)\\ &\quad - Nq\ln(Nq) + Nq\\ &\quad - \frac{1}{2}\ln[2\pi Nq]\\ &\quad + Np\ln p + Nq\ln q\end{aligned} \tag{9}$$

$$\begin{aligned}\ln P_N(n) &= N\ln N - N\frac{1}{2}\ln(2\pi N)\\ &\quad - Np\ln N - Np\ln p - \frac{1}{2}\ln p\\ &\quad - \frac{1}{2}\ln(2\pi N) - Nq\ln N - Nq\ln q\\ &\quad + Nq - \frac{1}{2}\ln(2\pi Nq) + Np\ln p + Nq\ln q\end{aligned} \tag{10}$$

$$\ln P_N(n) = -\frac{1}{2}\ln(2\pi Npq) \tag{11}$$

But $\ln P_N(n) = \ln A$

$$\ln A = -\frac{1}{2}\ln(2\pi Npq) \tag{12}$$

$$A = \frac{1}{\sqrt{2\pi Npq}} \tag{13}$$

But $Npq = \sigma^2$, so

$$\boxed{A = \frac{1}{\sqrt{2\pi\sigma^2}}} \tag{14}$$