PS 38: Problem 4.14

From GT 4.42, the number of microstates accessible to a gas molecule in a 1-liter box is

$$\Gamma(E) = \frac{4\pi}{3}\frac{V}{h^3}(2mE)^{3/2} \tag{1}$$

and from GT 4.17, the number of microstates in the energy interval $[E, E+\Delta E]$ is

$$g(E)\Delta E = \Gamma(E+\Delta E) - \Gamma(E) \approx \frac{\mathrm{d}\Gamma(E)}{\mathrm{d}E}\Delta E \tag{2}$$

The mean energy of a gas molecule is given by

$$E = \frac{3}{2}kT \tag{3}$$

If we consider room temperature to be $T = 25°C = 298K$, then (3) becomes

$$\begin{aligned} E &= \frac{3}{2}\left(1.38\times 10^{-23}\mathrm{m}^2\cdot\mathrm{kg}\cdot\mathrm{s}^{-2}\cdot\mathrm{K}^{-1}\right)(298\mathrm{K}) \\ &\approx 6.17\times 10^{-21}\mathrm{J} \end{aligned} \tag{4}$$

We consider diatomic nitrogen since it comprises around 3/4 of the atmosphere. Its molar mass is 28.02 g$\cdot$mol^{-1}. The mass for a nitrogen molecule is then given by

$$\begin{aligned} m &= \frac{\nu}{N_A} \\ &= \frac{28.02\mathrm{g}}{6.022\times 10^{23}} \\ &= 4.65\times 10^{-26}\mathrm{kg} \end{aligned} \tag{5}$$

From (2), we differentiate (1) w.r.t. E:

$$\begin{aligned} \frac{\mathrm{d}\Gamma(E)}{\mathrm{d}E}\Delta E &= \frac{4\pi V}{3h^3}(2m)^{3/2}\frac{3}{2}E^{1/2}\Delta E \\ &= \frac{2\pi V}{h^3}(2m)^{3/2}\sqrt{E}\Delta E \end{aligned}$$

If we consider an energy interval $\Delta E = 10^{-27}$ J, this becomes

$$g(E)\Delta E = \frac{2\pi(1\mathrm{L})[2(4.65\times 10^{-26}\mathrm{kg})]^{3/2}\sqrt{6.17\times 10^{-21}\mathrm{J}}}{\left(6.63\times 10^{-34}\mathrm{m}^2\cdot\mathrm{kg}\cdot\mathrm{s}^{-1}\right)^3}\left(10^{-27}\mathrm{J}\right) \tag{6}$$

$$\boxed{g(E)\Delta E \approx 4.80\times 10^{22}} \tag{7}$$