

PS 31: Problem 3.39

The random variable x has the probability density

$$p(x) = \begin{cases} Ae^{-\lambda x} & 0 \leq x \leq \infty \\ 0 & x < 0 \end{cases} \quad (1)$$

- (a) To get the normalization constant A of (1), we take its integral over all space. Since the function is zero if $x < 0$, we consider only the area where it is non-zero and equate it to unity:

$$1 = \int_0^{\infty} Ae^{-\lambda x} dx \quad (2)$$

$$1 = A \int_0^{\infty} e^{-\lambda x} dx$$

$$1 = A \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty}$$

$$1 = \frac{A}{-\lambda} [e^{-\lambda \infty} - e^0]$$

$$1 = \frac{A}{-\lambda} [-1]$$

$$\boxed{A = \lambda} \quad (3)$$

(1) now becomes

$$p(x) = \begin{cases} \lambda e^{-\lambda x} & 0 \leq x \leq \infty \\ 0 & x < 0 \end{cases} \quad (4)$$

- (b) The mean value of x is

$$\langle x \rangle = \int_0^{\infty} xp(x) dx \quad (5)$$

$$= \int_0^{\infty} x\lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} xe^{-\lambda x} dx \quad (6)$$

Using integration by parts, let $u \equiv x$ and $dv \equiv e^{-\lambda x} dx$. Consequently, $du \equiv dx$ and

$v \equiv -e^{-\lambda x}/\lambda$. We have

$$\langle x \rangle = \lambda \left[-\frac{xe^{-\lambda x}}{\lambda} \right]_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx \quad (7)$$

The integral in (7) follows the same solution as in part (a), and we end up with

$$\begin{aligned} \langle x \rangle &= \lambda \left[-\frac{xe^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty} \\ &= - \left[\frac{\lambda xe^{-\lambda x} + e^{-\lambda x}}{\lambda} \right]_0^{\infty} \end{aligned} \quad (8)$$

Substituting ∞ directly into the first term causes it to blow up. However, upon examining the terms involved we see that $\frac{d^2}{dx^2}x < \frac{d^2}{dx^2}e^{-\lambda x}$. Thus, we can say that the exponential term approaches zero faster than x approaches infinity, so the exponential term dominates. Therefore,

$$\begin{aligned} \langle x \rangle &= - \left[\frac{e^{-\lambda \infty} - e^0}{\lambda} \right] \\ &= - \frac{0 - 1}{\lambda} \end{aligned}$$

$$\boxed{\langle x \rangle = \frac{1}{\lambda}} \quad (9)$$

The most probable value of $\boxed{x = 0}$.

- (c) The mean value of x^2 is

$$\langle x^2 \rangle = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx \quad (10)$$

Once again, integrate by parts, letting $u \equiv x^2$ and $dv \equiv e^{-\lambda x} dx$. So $du \equiv 2x dx$ and $v \equiv -e^{-\lambda x}/\lambda$. We have

$$\langle x^2 \rangle = \lambda \left[-\frac{x^2 e^{-\lambda x}}{\lambda} \right]_0^{\infty} + \frac{2}{\lambda} \int_0^{\infty} x e^{-\lambda x} dx \quad (11)$$

Following the steps in part (b), the integral can be simplified as

$$\begin{aligned}
 \langle x^2 \rangle &= \lambda \left[-\frac{2xe^{-\lambda x}}{\lambda^2} - \frac{x^2 e^{-\lambda x}}{\lambda} \right]_0^\infty + \frac{2}{\lambda} \int_0^\infty e^{-\lambda x} dx \\
 &= \lambda \left[-\frac{2e^{-\lambda x}}{\lambda^3} - \frac{2xe^{-\lambda x}}{\lambda^2} - \frac{x^2 e^{-\lambda x}}{\lambda} \right]_0^\infty \\
 &= \lambda \left[\frac{2}{\lambda^3} + 0 + 0 \right] \\
 \boxed{\langle x^2 \rangle} &= \frac{2}{\lambda^2} \tag{12}
 \end{aligned}$$

- (d) For $\lambda = 1$, the probability that a measurement of x yields a value $\in (1, 2)$ is

$$\begin{aligned}
 \Pr(1 < x < 2) &= \int_1^2 p(x) dx \tag{13} \\
 &= \int_1^2 e^{-x} dx \\
 &= -e^{-x} \Big|_1^2 \\
 &= -e^{-2} + e^{-1}
 \end{aligned}$$

$$\boxed{\Pr(1 < x < 2) = \frac{1}{e} - \frac{1}{e^2} \approx 0.23} \tag{14}$$

- (e) For $\lambda = 1$, the probability that a measurement of x yields a value < 0.3 is

$$\begin{aligned}
 \Pr(x < 0.3) &= \int_0^{0.3} p(x) dx \tag{15} \\
 &= \int_0^{0.3} e^{-x} dx \\
 &= -e^{-x} \Big|_0^{0.3} \\
 &= -e^{-0.3} + e^{-0}
 \end{aligned}$$

$$\boxed{\Pr(x < 0.3) = 1 - \frac{1}{e^{0.3}} \approx 0.26} \tag{16}$$