PS 33: Problem 3.43

(a) The Lorentz distribution is given by

$$p(x) = \frac{1}{\pi} \frac{\gamma}{(x-a)^2 + \gamma^2} \qquad (1)$$

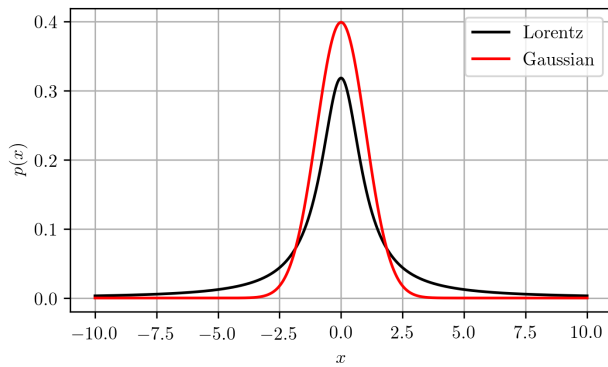Comparing it with the Gaussian distribution:

Figure 1: Comparison of Lorentzian and Gaussian distributions.

This shows that for the same set of parameters, the Gaussian has a higher peak, and quickly falls off and approaches zero as one moves away from the mean/central value.

(b) For $a = 0$ and $\gamma = 1$, the first moment of the Lorentzian is given by

$$\langle x^1 \rangle = \int_{-\infty}^{+\infty} x p(x)\, dx \qquad (2)$$
$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x}{x^2 + 1}\, dx$$

Let $u \equiv x^2 + 1$, $du \equiv 2x\, dx$,

$$\langle x^1 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{u}\, du$$
$$= \frac{1}{2\pi} \left[\ln(u) \Big|_{-\infty}^{+\infty} \right]$$
$$= \frac{1}{2\pi} \left[\ln\left(x^2 + 1\right) \Big|_{-\infty}^{+\infty} \right]$$
$$= \infty - \infty$$
$$\boxed{\langle x \rangle = \text{undefined}} \qquad (3)$$

(c) The second moment is given by

$$\langle x^2 \rangle = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x^2}{x^2 + 1}\, dx \qquad (4)$$

By long division of the integrand,

$$
\begin{array}{r}
1 \\
x^2 + 1 \,\overline{\big)\, x^2 \ +0x+ \ 0} \\
\underline{x^2 \ +0x+ \ 1} \\
-1
\end{array}
$$

From this, (4) can be rewritten as

$$\langle x^2 \rangle = \frac{1}{\pi} \int_{-\infty}^{+\infty} 1 - \frac{1}{x^2 + 1}\, dx$$
$$= \frac{1}{\pi} \left[\int_{-\infty}^{+\infty} dx - \int_{-\infty}^{+\infty} \frac{dx}{x^2 + 1} \right]$$

The second term is an even function about zero, and can be rewritten as

$$\langle x^2 \rangle = \frac{1}{\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - 2\int_{0}^{+\infty} \frac{\mathrm{d}x}{x^2+1}\right]$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - 2\arctan(x)\Big|_{0}^{\infty}\right]$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - 2(\arctan(\infty) - \arctan(0))\right]$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - 2\left(\frac{\pi}{2} - 0\right)\right]$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{+\infty} \mathrm{d}x - \pi\right]$$

$$= \frac{1}{\pi}[\infty - \pi]$$

$$\boxed{\langle x^2 \rangle = \infty} \tag{5}$$

The second moment of the Lorentz distribution exists and has a value of infinity.