Oscillation modes of a rod

8

Introduction

A cylindrical rod suitably prepared may be made to oscillate around many points and axes, thereby revealing a surprising variety of physical phenomena, ranging from a regular physical pendulum through bifilar twisting motion and torsional oscillations, all the way to possible chaotic behavior in a double pendulum [1–3]. The construction of the various suspension and support points, the period measuring device, and the theoretical calculations of the appropriate periods all require ingenuity and increasing depths of understanding of the rigid body dynamics involved, and provide plenty of scope for playing around with the physics.

Theoretical ideas

Let us suppose that we have prepared some means of supporting the rod in a series of positions illustrated by the diagrams that follow, Figs 8.1–8.12. In each case we shall describe the modes of oscillation and provide the formula for its period – proving them is in some cases a non-trivial exercise and may be considered part of the project. Proofs are given in the more complex cases.

Figure 8.1 shows a rod of radius R, length L, and mass M having a series of holes drilled symmetrically on either side of the center of mass (*com*). The holes are of radius r_h. A pin of radius $r_p < r_h$ can be used to suspend the rod from any of the holes, or strings can be attached to the rod through them.

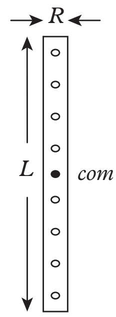

Figure 8.1 *Drilled rod.*

Vertical pendulum

The rod can oscillate in a vertical plane by suspending it from a pin of radius r_p through one of the holes, as in Fig. 8.2. Figure 8.3 shows that the top of the hole will roll on the pin without slipping, keeping the point of contact at rest. If the distance from the *com* to the top of the hole is d, and if one neglects the finite diameter of the pin, the period for small oscillations is given by

$$T_0 = 2\pi \sqrt{\frac{(L^2/12) + d^2}{gd}}. \tag{8.1}$$

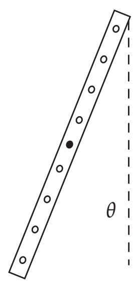

Figure 8.2 *Vertical oscillations.*

T as a function of d will exhibit a well-known minimum.

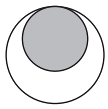

Figure 8.3 *Finite radius.*

A good exercise is to extend this formula in two ways: First, to large amplitudes, in which case the period will be given by an elliptic integral [1] which can be evaluated numerically for any amplitude; secondly, to take into account the finite radius of the pin, shown in Fig. 8.3.

Horizontal pendulum

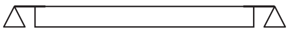

Figure 8.4 *Horizontal pendulum.*

Pins, attached to the ends of the rod, are supported horizontally, as shown in Fig. 8.4. The rod oscillates about an axis parallel to its long axis. Its period is given by

$$T = 2\pi\sqrt{3R/2g}. \tag{8.2}$$

Bifilar suspension

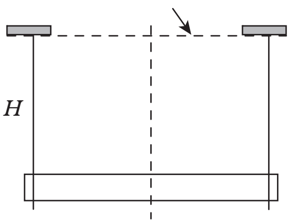

Figure 8.5 *Bifilar suspension.*

Here, the rod is suspended horizontally by two vertical parallel strings which may be of equal length H, as in Fig. 8.5 (H is measured from the point of suspension to the rod axis), or of different lengths, as in Fig. 8.7.

Consider first equal length strings. Several motions are possible.

(a) In and out of its equilibrium plane, like a swing. The axis of rotation is parallel to the rod, passing through the suspension points of the strings. The period is given by

$$T = 2\pi\sqrt{(0.5R^2 + H^2)/gH}. \tag{8.3}$$

(b) Laterally, parallel to the rod length, in its equilibrium plane. The period is given by

$$T = 2\pi\sqrt{(H - R)/g}. \tag{8.4}$$

(c) Torsional oscillations about a vertical axis through the center of mass (see Fig. 8.5). In this case, while the left end of the rod moves out of the paper the right end moves into the paper, and vice versa. The period for small oscillations will be

$$T = 2\pi\sqrt{\frac{H'}{g}\frac{L/d}{\sqrt{12}}}, \tag{8.5}$$

where $H' = H - R$.

For large amplitude oscillations the period will once again be given by an elliptic integral, to be evaluated numerically.

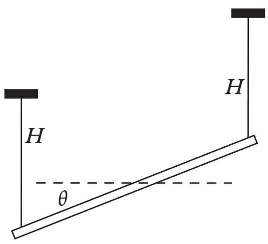

Figure 8.6 *Inclined pendulum.*

(d) Inclined bifilar pendulum. Let the system execute torsional oscillations about a vertical axis, making an angle θ with the rod at the *com* of the rod – see Fig. 8.6. The moment of inertia about the vertical axis is given by

$$I = M\{L^2/12 + R^2/4\}\cos^2\theta + M\{R^2/2\}\sin^2\theta. \tag{8.6}$$

For small angles, say $\theta < \pi/6$, the period is expected to be the result of (8.5) multiplied by $\cos\theta$.

For swing-like oscillations, the result of (8.3) is still valid.

(e) Horizontal rod suspended from unequal-length strings. Let d be the distance of the points of attachments of the strings to the center of mass, as in Fig. 8.7. In a torsional displacement of the rod the attachment points of the string move out of the equilibrium plane by x_1 and x_2, the center of mass moves by $x_{\rm cm}$, while the rod swings by an angle θ, as in Fig. 8.8. These parameters are related by $x_{\rm cm} = (x_1 + x_2)/2$ and $\theta = (x_1 - x_2)/2d$.

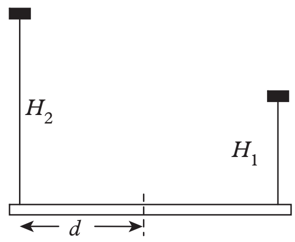

Figure 8.7 *Unequal suspensions.*

The kinetic and potential energies of the system in terms of these parameters are given by

$$E_{\rm k} = \frac{1}{2} I_{\rm cm}\dot\theta^2 + \frac{1}{2} M x_{\rm cm}^2, \tag{8.7}$$

$$U = Mgy_{\rm cm} = \frac{1}{2} Mg\left(H_1 + H_2 - \sqrt{H_1^2 - x_1} - \sqrt{H_2^2 - x_2}\right). \tag{8.8}$$

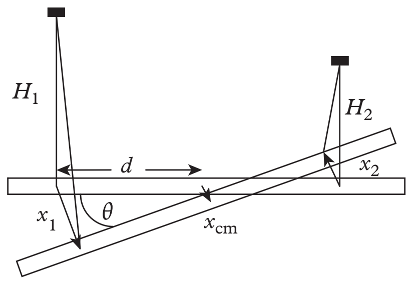

Figure 8.8 *Coordinates for unequal suspensions.*

The Lagrangian is $L = E_{\rm k} - U$. In the small-angle approximation one obtains the coupled Euler–Lagrange equations for the two variables θ and $x_{\rm cm}$:

$$\ddot x_{\rm cm} = -\omega_0^2 (x_{\rm cm} - \varepsilon d\theta), \tag{8.9}$$

$$\ddot\theta = -\omega_0^2 \lambda(\theta - \varepsilon x_{\rm cm}/d), \tag{8.10}$$

in which

$$\omega_0^2 = g/H_{\rm eff},\ \lambda = md^2/I_{\rm cm},\ \varepsilon = (H_1 - H_2)/2H_{\rm eff},\ H_{\rm eff} = 2H_1H_2/(H_1 + H_2).$$

From (8.9) and (8.10) one gets the normal mode frequencies

$$\omega_\pm^2 = \omega_0^2\left\{\frac{1}{2}(1+\lambda) \pm \sqrt{4(1-\lambda)^2 + \lambda\varepsilon^2}\right\} \tag{8.11}$$

and the periods $T_\pm = 2\pi/\omega_\pm$.

Double pendulum

Here the rod is hung from a string passing through one of its holes, as in Fig. 8.9. String and rod can either swing together or in opposite directions. The method of deriving the normal mode frequencies is similar to that for case (e) above; this time the dynamical variables are θ_1 and θ_2, and the frequencies are given by

$$\omega_\pm^2 = \frac{\omega_0^2}{2(1-d/s)}\left\{(H/s+1) \pm \sqrt{(H/s+1)^2 - 4(H/s)(1-d/s)}\right\}, \tag{8.12}$$

where $s = [L^2/12 + d^2]/d$, $\omega_0^2 = g/H$.

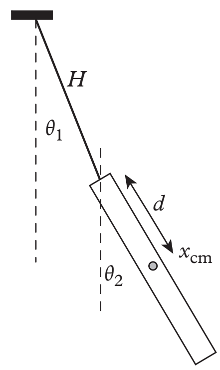

Figure 8.9 *Double pendulum.*

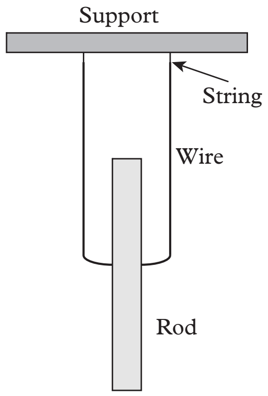

Figure 8.10 *Chaotic pendulum.*

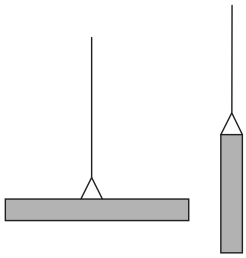

Figure 8.11 *Torsional pendulums.*

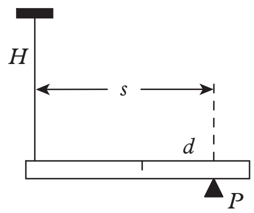

Figure 8.12 *Horizontal pendulum.*

Chaotic pendulum

As shown in Fig. 8.10, a steel wire passing through a hole near the *com* and suspended by very short strings to a support allows the rod to swing freely about its axis of support. Releasing the upper part of the pendulum from a large angle will result in chaotic oscillations of the rod, swinging unpredictably in one direction or the other. Numerical solution of the Euler–Lagrange equation for the system will show extreme sensitivity to the initial conditions.

Torsional pendulum

The torque in a wire of length H and radius r when twisted through an angle θ is given by

$$\tau = -\frac{G\pi r^4}{2H}\theta, \tag{8.13}$$

where G is the shear modulus. The period is the well-known expression

$$T = 2\pi\sqrt{2HI/G\pi r^4}. \tag{8.14}$$

The moment of inertia I depends on whether the rod is suspended horizontally or vertically, shown for both cases in Fig. 8.11.

Horizontal pendulum

Suspend the rod at one end by a string, while a second point on the rod, P, rests on a support which is a horizontal distance d from the *com*, and a distance s from the point of attachment of the string, as in Fig. 8.12. Let the rod swing through an angle φ about P, then, out of the plane, the string moves through an angle θ such that $H\theta = s\varphi$. In the small-angle approximation, the torque exerted by the string is given by $\tau = F\theta s = Fs^2\varphi/H$, where the tension in the string is $F = Mgd/s$. Hence the period is given by

$$T = 2\pi\sqrt{\frac{HI}{Mgds}} = 2\pi\sqrt{\frac{H(L^2/12 + d^2)}{gds}}. \tag{8.15}$$

Experimental suggestions

- Periods can be timed manually, but preferably with suitably positioned photogates.
- For the vertical pendulum, for large angle oscillations it is best to use the outermost hole in the rod.

- In case (e) for bifilar suspension, release the rod after rotating it through a small angle about a string.
- For the double pendulum, the normal modes are obtained by starting with the string and rod displaced in the same direction, or in the opposite direction.
- Experiment with different suspension lengths.
- Invent oscillations that have not been described here; many variants are possible.
- If you have a device that can follow and record two- or three-dimensional motion, such as Alberti's windows [4] (which uses two coupled video cameras), the displacement versus time of the pendulums may be sampled and compared with numerical solutions of the differential equations of motion.

REFERENCES

1. A. Cromer, Many oscillations of a rigid rod, *Am. J. Phys.* **63**, 112 (1995).
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3. A. Cromer, C. Zahopoulos, and M. Silevich, Chaos in the corridor, *Phys. Teach.* **30**, 382 (1992).
4. Alberti's Window, *Alberti's window motion visualizer*, n.d. <http://www.albertiswindow.com>.