3.6 Application-Forced Spring Mass Systems and Resonance

In this section we introduce an external force that acts on the mass of the spring in addition to the other forces that we have been considering. For example, suppose that the mass of a spring/mass system is being pushed (or pulled) by an additional force (perhaps the spring is mechanically driven or is being acted upon by magnetic forces). We will call this net external force $F_{external}$ and allow it to vary over time, that is $F_{external} = F(t)$.

As before,

$$F_{total} = -by' - ky + F_{external}$$

and by Newton's Law

$$my'' = -by' - ky + F_{external}.$$

So we obtain the nonhomogeneous ODE:

$$my'' + by' + ky = F(t).$$

As we saw earlier, this homogeneous can be solved by using the principle of superposition, where $y_{homo}(t)$ is the general solution to the associated homogeneous and $y_P(t)$ is any one solution to the nonhomogeneous DE.

From the previous section, so long as $b > 0$, then $\lim_{t\to\infty} y_{homo}(t) = 0$, so the long-term behavior of any solution to the nonhomogeneous will be determined by the behavior of $y_P(t)$. In such problems, the associated homogeneous solution $y_{homo}(t) = 0$ is called transient part of the solution (since it dies away) and $y_P(t)$ is called the *steady state solution* since it determines the long-term behavior.

Example 3.22 *A spring with spring constant $4N/m$ is attached to a 1kg mass with friction constant 4Ns/m is forced to the right by a constant force of $2N$. Find the steady state solution.*

Solution: In light of the discussion above, we need only find $y_P(t)$ which we can obtain by undetermined coefficients on the non homogeneous ODE

$$y'' + 4y' + 4y = 2$$

to obtain $y_P(t) = \frac{1}{2}$ meter. So no matter what the initial conditions are, the mass will limit to a displacement $\frac{1}{2}$ meter to the right. □

Example 3.23 *A spring with spring constant* $4N/m$ *is attached to a 1kg mass with friction constant* $4Ns/m$ *is forced periodically by a constant force of* $2\cos(t)N$*. (a) Find the steady state solution and express it in phase/angle notation.*

(b) Find the particular solution that satisfies $y(0) = 1$ *and* $y'(0) = 2$*, and verify that the graph limits to the steady state solution.*

Solution: In light of the discussion above, we need only find $y_P(t)$ which we can obtain by undetermined coefficients on the non homogeneous ODE

$$y'' + 4y' + 4y = 2\cos t.$$

We use undetermined coefficients on the form: $y_P(t) = A\cos t + B\sin t$ and obtain

$$y'' + 4y' + 4y = -A\cos t - B\sin t - 4A\sin t + 4B\cos t + 4A\cos t + 4B\sin t$$

$$= (3A + 4B)\cos t + (3B - 4A)\sin t$$

which we set equal to $2\cos t + 0\sin t$ so

$$3A + 4B = 2 \qquad and \qquad 3B - 4A = 0,$$

so

$$12A + 16B = 8 \qquad and \qquad 9B - 12A = 0,$$

which, when added, yields

$$25B = 8$$

so $B = \frac{8}{25}$ and $A =$ 625.

So

$$y_P(t) = \frac{1}{25}(6\cos t + 8\sin t)$$

which can be expressed as

$$y_P(t) = \frac{\sqrt{36 + 64}}{25}(\cos(t - \arctan(\frac{4}{3})) = \frac{2}{5}(\cos(t - \arctan(\frac{4}{3})).$$

To solve part (b), notice that the general solution to the DE is given by

$$y(t) = y_{homo}(t) + y_P(t) = c_1e^{-2t} + c_2te^{-2t} + \frac{1}{25}(6\cos t + 8\sin t)$$

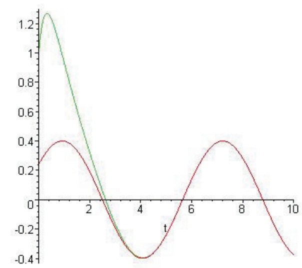

Figure 3.7: **Plots of the solution and the steady state solution from Example (3.23)**

so the particular solution that we seek satisfies $y(0) = c_1 + \frac{6}{25} = 1$ or $c_1 = \frac{19}{25}$ and

$$y'(t) = -2(\frac{19}{25})e^{-2t} - 2c_2te^{-2t} + c_2e^{-2t} + \frac{1}{25}(8\cos t - 6\sin t)$$

so since $y'(0) = 2$ we obtain $2 = -\frac{38}{25} + c_2 + \frac{8}{25}\ c_2 = \frac{80}{25} = \frac{16}{5}$. So the particular solution we seek is

$$y(t) = y_{homo}(t) + y_P(t) = \frac{19}{25}e^{-2t} + \frac{16}{5}te^{-2t} + \frac{1}{25}(6\cos t + 8\sin t).$$

The plots of the steady state solution and the particular solution are given in figure (3.5.3), notice how the solution limits to the steady state. □

3.6.1 Resonance

In this section we look at a particular phenomenon called resonance. In principle, anyone who has ever pushed a child on a swing is familiar with this concept. A child swinging on a swing will oscillate back and forth with a given frequency. In order to push the child effectively on the swing, the frequency of the pushes needs to coincide with the frequency of the swing otherwise, you will be pushing when the child is swinging toward you. This principle is also what is responsible for the ability of an opera singer to shatter a champagne glass, the oscillatory forces that are capable of destroying a bridge, or exciting molecules at their natural frequency in microwave ovens. More formally, a spring/mass system exhibits *resonance* if the steady state solution obtained by forcing the system with amplitude F_0 has a greater maximum displacement than the steady state solution obtained by forcing the system with a constant force F_0.

Example 3.24 *Show that a spring/mass system with spring constant* $6N/m$ *attached to a 1kg mass with friction constant* $1Ns/m$ *exhibits resonance by comparing the steady-state solutions for*

(a) $F_{force} = 2$

(b) $F_{force} = 2\sin 3t$

Solution: (a) The steady state solution to

$$y'' + y' + 6y = 2$$

is $y(t) = \frac{1}{3}$. (Use undetermined coefficients on $y = A$, and solve for A.)

(b) The steady state solution to

$$y'' + y' + 6y = 2\sin 3t$$

is obtained by plugging $y = A\cos 3t + B\sin 3t$ into the left side of the DE, we obtain

$$-9A\cos 3t - 9B\sin 3t - 3A\sin 3t + 3B\cos 3t + 6A\cos 3t + 6B\sin 3t$$
$$= (-3A + 3B)\cos 6t + (3A + 3B)\sin 6t$$

so

$$-3A + 3B = 0 \quad \text{and} \quad -3A - 3B) = 2$$

solving, we get

$$-6A = 2$$

so

So $A = \frac{-}{1}3$ and $B = -\frac{1}{3}$ which implies that the steady state solution is

$y(t) = -\frac{1}{3}\sin 3t - \frac{1}{3}\cos 3t$ which can be rewritten in phase/angle form as $y(t) = \frac{\sqrt{2}}{3}\cos(3t - (\frac{\pi}{4} + \pi))$

Clearly, the amplitude obtained in (b) is larger than the one obtained in (a) as Figure (3.6.1) demonstrates, so the system exhibits resonance. □

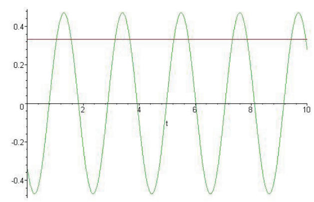

Figure 3.8: **Plots of the two steady state solutions from Example (3.24)**

3.6.2 Sinusoidal Forcing

Suppose that a spring/mass system with spring constant $k > 0$ attached to a mass of $m > 0$ kilograms with with friction constant $b > 0$. We wish to examine when a sinusoidal forcing function of the form $F_0 \cos(\omega t - \phi)$ produces a steady state solution with a larger amplitude than the steady state solution obtained by forcing with constant force of $F_0 > 0$.

As before, by the method of undetermined coefficients,

$$y'' + by' + ky = F_0$$

has a steady state solution of $y(t) = F_0/k$.

Next, we use undetermined coefficients to solve

$$y'' + by' + ky = F_0 \cos(\omega t - \phi)$$

with

$$y = A\cos(\omega t - \phi) + B\sin(\omega t - \phi)$$

to obtain

$$\begin{aligned} my'' &= -\omega^2 Am\cos(\omega t - \phi) - Bm\omega^2 \sin(\omega t - \phi) \\ by' &= \omega Bb\cos(\omega t - \phi) - Ab\omega\sin(\omega t - \phi) \\ ky &= Ak\cos(\omega t - \phi) + Bk\sin(\omega t - \phi) \end{aligned}$$

So matching coefficients:

$$-\omega^2 Am + Bb\omega + Ak = F_0 \quad \text{and} \quad - Bm\omega^2 - Ab\omega + Bk = 0$$

or

$$(k-\omega^2 m)A + Bb\omega = F_0 \quad \text{and} \quad -Ab\omega + B(k-m\omega^2) = 0$$

$$(k-\omega^2 m)^2 A + Bb\omega(k-m\omega^2) = F_0(k-m\omega^2) \quad \text{and} \quad Ab^2\omega - B(k-m\omega^2)b = 0$$

So adding,

$$Ab^2\omega + (k-\omega^2 m)^2 A = F_0(k-m\omega^2)$$

or

$$A = \frac{F_0(k-m\omega^2)}{b^2\omega + (k-m\omega^2)^2}$$

$$B = \frac{Ab\omega}{k-m\omega^2}$$

so

$$B = \frac{bF_0}{b^2 + (k-m\omega^2)^2}$$

So the steady state solution will have amplitude

$$\sqrt{A^2+B^2} = \sqrt{A^2 + \frac{A^2b^2\omega^2}{(k-m\omega^2)^2}} = \sqrt{A^2(1+\frac{b^2\omega^2}{(k-m\omega^2)^2})}$$

$$= \sqrt{A^2(\frac{(k-m\omega^2)^2 + b^2\omega^2}{(k-m\omega^2)^2})}$$

which simplifies

$$= \sqrt{\frac{F_0^2}{b^2\omega^2 + (k-m\omega^2)^2}}.$$

This exceeds $\frac{F_0}{k}$ precisely when

$$k^2 > b^2\omega^2 + (k-m\omega^2)^2.$$

or

$$\omega^2\left(-m^2\omega^2 + (2mk-b^2)\right) > 0$$

or

$$(2mk-b^2) > m^2\omega^2.$$

The above inequality implies that $b^2 - 2mk < 0$, since the term on the right hand side is clearly positive. Moreover, so long as

$$0 < \omega < \frac{\sqrt{2km-b^2}}{m}$$

then the system exhibits resonance. The interval $(0, \frac{\sqrt{2km-b^2}}{m})$ is called the **interval of resonance**.

To derive the **optimal** forcing frequency, we minimize $g(\omega) = b^2\omega^2 + (k - m\omega^2)^2$ with respect to ω (thereby, maximizing $= \sqrt{\frac{F_0^2}{b^2\omega^2+(k-m\omega^2)^2}}$).

Taking a derivative, we obtain

$$g'(\omega) = 2b^2\omega + 2(k - m\omega^2)(-2m\omega)$$

and obtain

$$2b^2 - 4m(k - m\omega^2) = 0$$

or

$$\omega^2 = \frac{4mk - 2b^2}{4m^2}$$

or

$$\omega = \frac{\sqrt{4mk - 2b^2}}{2m} = \frac{\sqrt{2mk - b^2}}{\sqrt{2}m}.$$

We summarize our work below:

Resonance in Sinusoidally Forced Spring/Mass Systems

The sinusoidally forced spring mass system

$$my'' + by' + ky = F_0 \cos(\omega t - \phi) \tag{3.16}$$

exhibits resonance when $b^2 - 2mk < 0$ (we call such a system lightly damped). In particular,

ω inside the interval of resonance,

$$0 < \omega < \frac{\sqrt{2km - b^2}}{m},$$

the steady state solution of the periodically forced system exceeds the steady state solution of the system forced by a constant. The optimal forcing frequency (called **the resonance frequency**) is

$$\omega = \frac{\sqrt{2mk - b^2}}{\sqrt{2}m} \tag{3.17}$$

Not all systems exhibit resonance. In particular, for a spring mass system that is forced by a sinusoidal function to exhibit resonance, it must be lightly damped, meaning $b^2 - 2mk < 0$. Note that lightly damped implies that the system is underdamped since $b^2 - 4mk < b^2 - 2mk$ we have $b^2 - 2mk < 0$ implies that $b^2 - 4mk < 0$.

Example 3.25 *Compute the frequencies ω for which*

$$y'' + y' + 6y = \cos(\omega t - \phi)$$

produces resonance. Also, find the resonance frequency and plot the steady state solution when the system is forced at that frequency.

Solution: Note also that we are forcing the system sinusoidally with $F_0 = 1$. Since $m = 1$, $b = 1$ and $k = 6$, we see that $b^2 - 2mk = 1 - 12 < 0$ so the system is lightly damped and exhibits resonance.

In particular, for frequencies ω satisfying

$$0 < \omega < \sqrt{11},$$

the steady state solution of the periodically forced system exceeds the steady state solution of the system forced by a constant. (Note in example (3.24) since $0 < 3 < \sqrt{11}$, the steady state solution of the periodically driven system exceeds that of the system driven by a constant.)

The optimal frequency to force the system (i.e. the resonance frequency) occurs at $\omega = \frac{\sqrt{11}}{\sqrt{2}}$ and the maximum amplitude is given by

$$\text{Max Amplitude} = \sqrt{\frac{1}{11/2 + (6 - 11/2)^2}} = \frac{1}{\sqrt{\frac{23}{4}}} = \frac{2}{\sqrt{23}} \approx 0.417028828. \quad \square$$

3.6.3 Pure Mathematical Resonance

An interesting phenomenon occurs in a spring mass system that has no damping, i.e. $b = 0$. In particular, the steady state solution obtained by forcing at the resonance frequency is unbounded.

From the formula for the optimal resonance frequency above, we see that $\omega = \sqrt{\frac{k}{m}}$.

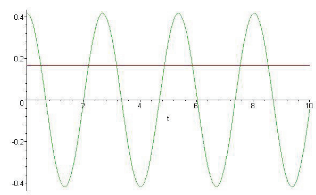

Figure 3.9: **Steady state solutions of a spring mass system forced at its resonant frequency as well as being forced by a constant force of 1.**

If we solve

$$my'' + ky = F_0 \cos(\sqrt{\frac{k}{m}}t - \phi) = F_0 \cos\phi \cos(\sqrt{\frac{k}{m}}t) - F_0 \sin\phi \sin(\sqrt{\frac{k}{m}}t)$$

then we see that the steady state solution (due to interaction with the homogenous solution) will be of the form

$$y(t) = At\cos(\sqrt{\frac{k}{m}}t) + Bt\sin(\sqrt{\frac{k}{m}}t)$$

which can, in turn, be written in phase/angle form as

$$y(t) = t\sqrt{A^2 + B^2}\cos(\sqrt{\frac{k}{m}}t - \phi).$$

This solution is clearly unbounded as $t \to \infty$ and results in (wild) oscillations with amplitude going to infinity. This occurrence is called **pure mathematical resonance,** and although it cannot occur in an actual spring/mass system, the concept is relevant to systems with extremely light damping.

Example 3.26 *Plot the steady state solution to*

$$y'' + 4y = \cos(\omega t),$$

where ω is the resonance frequency.

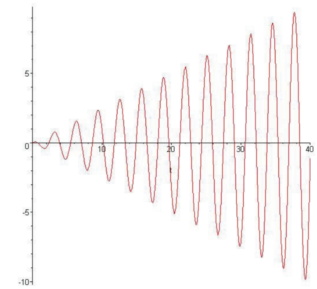

Figure 3.10: **Steady state solutions associated with pure mathematical resonance.**

Solution: As above the resonance frequency by Equation (3.6.2) is $\omega = \sqrt{\frac{k}{m}}$. So $\omega = 2$. Applying undetermined coefficients to:

$$y'' + 4y = \cos(2t),$$

we obtain $y(t) = At\cos(2t) + Bt\sin(2t)$ yielding:

$$y' = A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t)$$

$$y'' = -2A\sin(2t) - 2A\sin(2t) - 4At\cos(2t) + 2B\cos(2t) + 2B\cos(2t) - 4Bt\sin(2t)$$

So $y'' + 4y = 4B\cos(2t) - 4A\sin(2t)$
and so $A = 0$ and $B = \frac{1}{4}$.
Thus the steady state solution is $y(t) = \frac{1}{4}t\cos(2t)$ which is plotted in Figure (3.6.3) □

Exercises

1. Plot the steady state solutions to

$$y'' + 2y' + 10y = 4$$

and

$$y'' + 2y' + 10y = 4\cos(2t).$$

Does the associated system exhibit resonance?

2. Plot the steady state solutions to

$$y'' + 2y' + 1y = 4$$

and

$$y'' + 2y' + 1y = 4\cos(2t).$$

Can you use this example to conclude that the associated system does not exhibit resonance?

3. For

$$y'' + 2y' + 10y = 4\cos(\omega t),$$

plot the steady state solutions for specific values of ω where ω is:

(a) the optimal resonance frequency,

(b) inside the interval of resonance but not the optimal frequency,

(c) outside the interval of resonance.

4. The spring/mass system

$$y'' + \frac{1}{100}y' + 10y = 4\cos(\omega t)$$

has the mass initially at equilibrium and at rest and is forced at the optimal resonance frequency. If the spring can tolerate a displacement of at most $y = 20$ units, when will the spring break?

5. For

$$y'' + 10y = 4\cos(\omega t),$$

plot the steady state solutions for specific values of ω where ω is:

(a) the optimal resonance frequency,

(b) inside the interval of resonance but not the optimal frequency,

(c) outside the interval of resonance.