

2.25**Equation of Motion for Viscous Fluids****Ain A. Sonin**

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1 Surface Stress

So far we have been dealing with quantities like density and velocity, which at a given instant have specific values at every point in the fluid or other continuously distributed material. The density $\rho(\vec{r}, t)$ is a *scalar field* in the sense that it has a scalar value at every point, while the velocity $\vec{v}(\vec{r}, t)$ is a *vector field*, since it has a direction as well as a magnitude at every point.

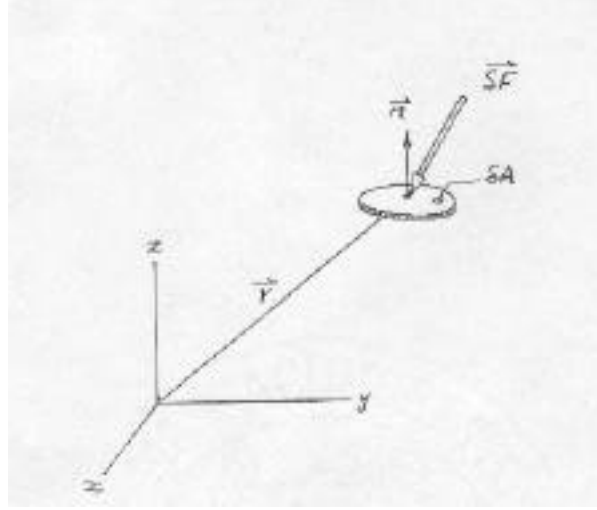


Fig. 1: A surface element at a point in a continuum.

The surface stress is a more complicated type of quantity. The reason for this is that one cannot talk of the stress at a point without first defining the particular surface through that point on which the stress acts. A small fluid surface element centered at the point \vec{r} is defined by its area δA (the prefix δ indicates an infinitesimal quantity) and by its outward unit normal vector \vec{n} . The *stress* exerted by the fluid on the side toward which \vec{n} points on the surface element is defined as

$$\vec{\sigma} = \lim_{\delta A \rightarrow 0} \frac{\delta \vec{F}}{\delta A} \quad (1)$$

where $\delta \vec{F}$ is the force exerted on the surface by the fluid on that side (only one side is involved). In the limit $\delta A \rightarrow 0$ the stress is independent of the magnitude of the area, but will in general depend on the *orientation* of the surface element, which is specified by \vec{n} . In other words,

$$\vec{\sigma} = \vec{\sigma}(\vec{x}, t, \vec{n}). \quad (2)$$

The fact that $\vec{\sigma}$ depends on \vec{n} as well as x , y , z and t appears at first sight to complicate matters considerably. One apparently has to deal with a quantity that depends on six independent variables (x , y , z , t , and the two that specify the orientation \vec{n}) rather than four. Fortunately, nature comes to our rescue. We find that because $\vec{\sigma}$ is a stress, it must depend on \vec{n} in a relatively simple way.

We have seen that, *in the absence of shear forces*, Newton's law requires that the surface stress have the particularly simple form

$$\vec{\sigma} = -p\vec{n} \quad (\text{no shear forces}) \quad (3)$$

where p , the magnitude of the normal compressive stress, is a function of \vec{r} and t only. This is *Pascal's principle*, which states that in the absence of shear forces, *at any point* in the fluid, the stress is always normal to the surface on which it acts, and its magnitude is independent of the surface orientation. In the absence of shear stresses, therefore, the stress on *any* surface, anywhere in the fluid, can be expressed in terms of a single scalar field $p(\vec{r}, t)$ provided there are no shear forces. This gives rise to the relatively simple form of the equation of motion for inviscid flow.

When shear forces are present, as they always are in practice except when the fluid is totally static in some reference frame, Newton's law imposes a somewhat more complicated constraint on the relationship between $\vec{\sigma}$ and \vec{n} . We shall see that the stress on any surface anywhere in the fluid can in general be specified in terms of six scalar functions of x , y , z , and t . These six are the independent components of a quantity called the stress tensor.

2 The Stress Tensor

The first and simplest thing that Newton's law implies about the surface stress is that, at a given point, the stress on a surface element with an orientation \vec{n} must be equal in magnitude, but opposite in direction, to that on a surface element with an opposite orientation $-\vec{n}$, that is,

$$\vec{\sigma}(\vec{r}, t, -\vec{n}) = -\vec{\sigma}(\vec{r}, t, \vec{n}) \quad (4)$$

This result can be obtained by considering a thin, disc-shaped fluid particle at \vec{r} , as shown in Fig. 2, with very small area δA and thickness δh . One side of the disc has an orientation \vec{n} and the other $-\vec{n}$. The equation of motion for this fluid particle reads

$$\rho \delta h \delta A \frac{D\vec{v}}{Dt} = \vec{\sigma}(\vec{n})\delta A + \vec{\sigma}(-\vec{n})\delta A + \rho \delta h \delta A \vec{G} \quad (5)$$

where \vec{G} is the body force per unit mass. When we let δh approach zero, so that the two faces of the disc are brought toward coincidence in space, the inertial term on the left and the body force term on the right become arbitrarily small compared with the two surface force terms, and (4) follows immediately.

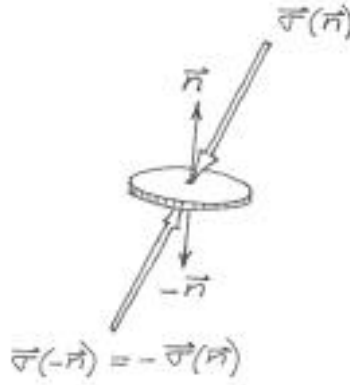


Fig. 2: Illustration for equation (4)

Newton's law also implies that the stress has a more profound attribute which leads to the concept of the stress tensor. The stress at a given point depends on the orientation of the surface element on which it acts. Let us take as "reference stresses," at a given point \vec{r} and instant t , the values of the stresses that are exerted on a surface oriented in the positive x -direction, a surface oriented in the positive y -direction, and a surface oriented in the positive z -direction (Fig. 3). We can write these three reference stresses, which of course are vectors, in terms of their components as, respectively,

$$\begin{aligned} \vec{\sigma}(\vec{i}) &= \tau_{xx}\vec{i} + \tau_{yx}\vec{j} + \tau_{zx}\vec{k} \\ \vec{\sigma}(\vec{j}) &= \tau_{xy}\vec{i} + \tau_{yy}\vec{j} + \tau_{zy}\vec{k} \\ \vec{\sigma}(\vec{k}) &= \tau_{xz}\vec{i} + \tau_{yz}\vec{j} + \tau_{zz}\vec{k} \end{aligned} \quad (6)$$

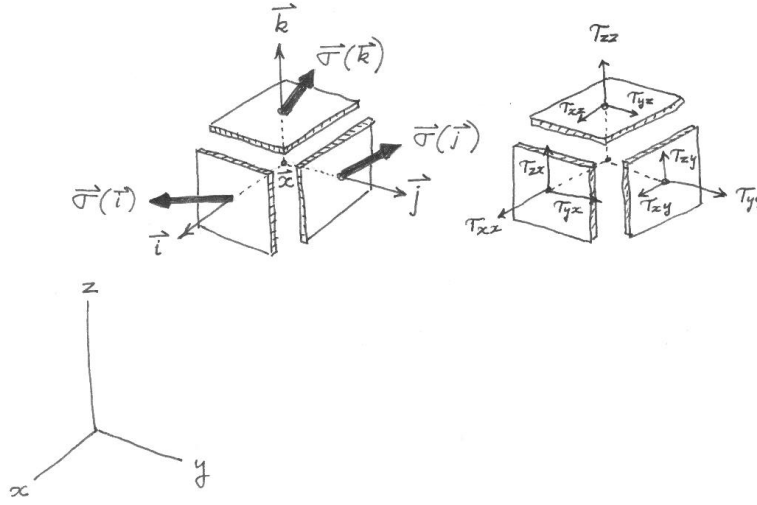


Figure 3: Reference stresses at a point in the continuum.

Thus, τ_{xx} , τ_{yx} and τ_{zx} represent the x , y , and z components of the stress acting on the surface whose outward normal is oriented in the positive x -direction, etc. (Fig. 3). The first subscript on τ_{ij} identifies the direction of the stress, and the second indicates the outward normal of the surface on which it acts. In (6) the τ_{ij} 's are of course functions of position x , y , z , and time t , and the reference stresses themselves also depend on x , y , z , and t ; we have simply not indicated this dependence.

We shall now show, again by using Newton's law, that the stress on a surface having *any* orientation \vec{n} at the point \vec{r} can be expressed in terms of the reference stresses $\vec{\sigma}(\vec{i})$, $\vec{\sigma}(\vec{j})$, and $\vec{\sigma}(\vec{k})$ or, more specifically, in terms of their nine components τ_{xx} , τ_{yx} , ..., τ_{zz} .

Consider a fluid particle which at time t has the shape of a small tetrahedron centered at x , y , z . One of its four faces has an area δA and an arbitrary outward normal \vec{n} , as shown in Fig. 4, and the other three faces have outward normals in the negative x , y and z directions, respectively. The areas of the three orthogonal faces are related to δA by

$$\delta A_x = \cos \theta_{nx} \delta A = n_x \delta A$$

$$\delta A_y = \cos \theta_{ny} \delta A = n_y \delta A \quad (7)$$

$$\delta A_z = \cos \theta_{nz} \delta A = n_z \delta A$$

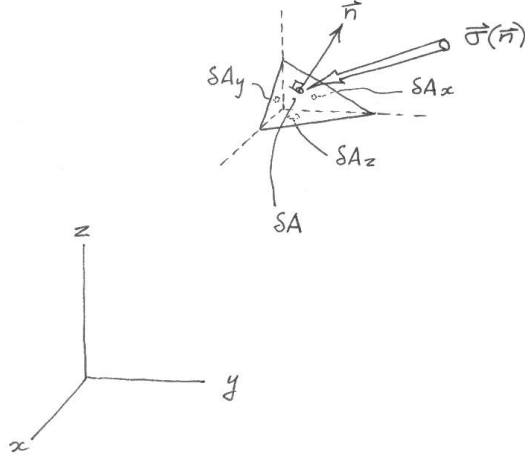


Fig. 4: Tetrahedron-shaped fluid particle at (x, y, z) .

where δA_x represents the area of the surface whose outward normal is in the negative x -direction, θ_{nx} is the angle between \vec{n} and the x -axis and n_x is the x -component of \vec{n} , and so on.

Consider what Newton's law tells us about the forces acting on the tetrahedron as we let it shrink in size toward the point \vec{r} around which it is centered. Since the ratio of the mass of the tetrahedron to the area of any one of its faces is proportional to the length of any one of the sides, both the mass times acceleration and the body force become arbitrarily small compared with the surface force as the tetrahedron is shrunk to a point (c.f. (5) and the paragraph that follows it). Hence, in the limit as the tetrahedron is shrunk to a point, the surface forces on the four faces must balance, that is,

$$\vec{\sigma}(\vec{n})\delta A + \vec{\sigma}(\vec{j})\delta A_x + \vec{\sigma}(\vec{k})\delta A_z = 0. \quad (8)$$

Now we know from (4) that the stress on a surface pointing in the $-\vec{i}$ direction is the negative of the stress on a surface in the $+\vec{i}$ direction, etc. Using this result and (7) for the areas, (8) becomes

$$\vec{\sigma}(\vec{n}) = \vec{\sigma}(\vec{i})n_x + \vec{\sigma}(\vec{j})n_y + \vec{\sigma}(\vec{k})n_z. \quad (9)$$

Alternatively, if we use (6) to write the reference stresses in terms of their components, we obtain the components of $\vec{\sigma}(\vec{n})$ as

$$\begin{aligned}
\sigma_x(\bar{n}) &= \tau_{xx}n_x + \tau_{xy}n_y + \tau_{xz}n_z \\
\sigma_y(\bar{n}) &= \tau_{yx}n_x + \tau_{yy}n_y + \tau_{yz}n_z \\
\sigma_z(\bar{n}) &= \tau_{zx}n_x + \tau_{zy}n_y + \tau_{zz}n_z .
\end{aligned} \tag{10}$$

Thus the stress $\bar{\sigma}(\bar{n})$ acting at \bar{x}, t on a surface with any arbitrary orientation \bar{n} can be expressed in terms of the nine reference stress components

$$\begin{array}{ccc}
\tau_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \tau_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \tau_{zz} .
\end{array}$$

These nine quantities, each of which depends on position and time, are the *stress tensor components*. Once the stress tensor components are known at a given point, one can compute the surface stress acting on *any* surface drawn through that point: one simply determines the components of the outward unit normal \bar{n} of the surface involved, and uses (10).

Equation (10) can be written more succinctly in conventional tensor notation, where i and j can represent x, y , or z and where it is understood that any term which contains the same index twice actually represents the sum of all such terms with all possible values of the repeated index (for example, $\sigma_{ii} \equiv \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$). In this notation (10) reads simply

$$\sigma_i(\bar{r}, t, \bar{n}) = \tau_{ij}(\bar{r}, t) n_j . \tag{11}$$

The importance of the stress tensor concept in continuum theory is this: It allows us to describe the state of stress in a continuum in terms of quantities that depend on position and time, but *not* on the orientation of the surface on which the stress acts. Admittedly, nine such quantities are needed (actually only six are independent, as we shall see shortly). Still, it is far easier to deal with them than with a single quantity which, at any given position and time, has a doubly infinite set of values corresponding to different surface orientations \bar{n} .

Physically, the stress tensor represents the nine components of the three reference stresses at the point \bar{r} and time t in question. The reference stresses are by custom chosen as the stresses on the three surface elements that have outward normals in the direction of the positive axes of the coordinate system being used. Thus in our Cartesian coordinates,

the reference stresses are the stresses on the surfaces pointing in the positive x , y , and z directions, and the stress tensor is made up of the nine components of these three stresses, τ_{ij} being the i -component of the stress on the surface whose normal points in the j -direction. In a cylindrical coordinate system, the stress tensor would be comprised of the components of the stresses acting on the three surfaces having outward normals in the positive r , θ and z directions.

Why are the quantities τ_{ij} "tensor components," and not just an arbitrary bunch of nine scalar quantities? The answer lies in the special way the values of these nine quantities transform when one changes one's reference frame from one coordinate system to another. Equation (10) tells us that when a coordinate change is made, the three sums $\tau_{ij}n_j$ must transform as components of a vector. A set of nine quantities τ_{ij} that transform in this manner is by definition a tensor of second rank. (A tensor of first rank is a vector, whose three components transform so that the magnitude and direction of the vector remain invariant; a tensor of zeroth rank is a scalar, a single quantity whose magnitude remains invariant with coordinate changes.)

3 Symmetry of the Stress Tensor

One further piece of information emerges from applying Newton's law to an infinitesimal fluid particle. This is that the stress tensor is in most cases symmetric, that is, $\tau_{ij} = \tau_{ji}$ for $i \neq j$.

The proof follows from considering the angular acceleration of a little fluid particle at x , y , z . For convenience, we let it be shaped like a little cube with infinitesimal sides δx , δy , and δz (Fig. 5). Since we shall be taking the limit where δx , δy , $\delta z \rightarrow 0$, where the fluid particle is reduced to a point, we can safely assume that the values of the density, velocity, stress tensor components, etc. are almost uniform throughout the cube. What is more, if the cube rotates by an infinitesimal amount, it does so almost as a solid body (i.e. at essentially zero angular distortion), since in the limit δx , δy , $\delta z \rightarrow 0$, a finite angular distortion would require infinite shear in a viscous fluid. If the cube has an angular velocity $\dot{\theta}_z$ in the z -direction, say, and rotates like a solid body, we can derive from Newton's law

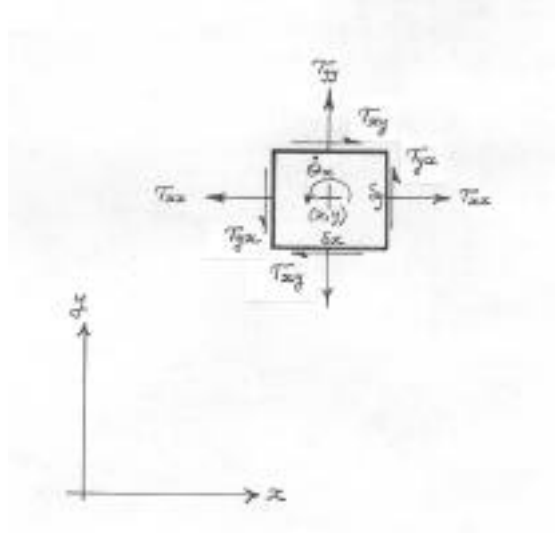


Fig. 5: Illustration of the reason for the stress tensor's symmetry.

written in angular momentum form for a material volume, that at any given instant its angular velocity increases according to

$$I_z \frac{d\dot{\theta}_z}{dt} = T_z, \quad (12)$$

where

$$I_z = \int_{-\delta x/2}^{+\delta x/2} \int_{-\delta y/2}^{+\delta y/2} \int_{-\delta z/2}^{+\delta z/2} \rho (\bar{x}^2 + \bar{y}^2) d\bar{x} d\bar{y} d\bar{z}$$

$$= \frac{\rho [(\delta x)^2 + (\delta y)^2]}{12} \delta x \delta y \delta z \quad (13)$$

is the moment of inertia of the cube and T_z is the net torque acting on the cube, relative to an axis running through the center of the cube parallel to the z -axis. Equation (13) is obtained by writing the cube's angular velocity as $\dot{\nu}_\theta = \dot{\theta}(t)r$, where $r^2 = \bar{x}^2 + \bar{y}^2$, \bar{x} and \bar{y} being the Cartesian coordinates fixed in the rotating cube.

The torque in (12) is obtained by considering the stresses acting on the cube (Fig. 5). On the face with $\bar{n} = \bar{i}$, for example, there is by definition a stress τ_{xx} in the positive x -direction and a stress τ_{yx} in the positive y -direction. On the face with $\bar{n} = -\bar{i}$, the corresponding stresses have the same magnitudes but opposite directions [see (10) or (4)].

The net torque about an axis through the cube's center, parallel to the z -axis, is caused by the shear forces (the pressure forces act through the cube's center) and by any volumetric torque exerted by the external body force field. A body force field like gravity acts through the cube's center of mass and exerts no torque about that point. Let us assume for the sake of generality, however, that the external body force may exert *a torque \bar{t} per unit volume* at the particle's location. The net torque in the z -direction around the particle's center would then be

$$\begin{aligned} T_z &= 2 \frac{\delta x}{2} \tau_{yx} \delta y \delta z - 2 \frac{\delta x}{2} \tau_{xy} \delta y \delta z + t_z \delta x \delta y \delta z \\ &= (\tau_{yx} - \tau_{xy} + t_z) \delta x \delta y \delta z \end{aligned} \quad (14)$$

From (12) - (14) we see that

$$\tau_{yx} - \tau_{xy} + t_z = \frac{\rho}{12} \frac{d\dot{\theta}_z}{dt} [(\delta x)^2 + (\delta y)^2] \quad (15)$$

As we approach a *point* in the fluid by letting $\delta x, \delta y \rightarrow 0$, this reduces to

$$\tau_{yx} = \tau_{xy} - t_z ,$$

or, more generally, the result that the off-diagonal stress tensor components must satisfy

$$\tau_{ji} = \tau_{ij} + t_k , \quad (16)$$

where i, j, k form a right-hand triad (e.g. in Cartesian coordinates they are in the order x, y, z , or y, z, x , or z, x, y).

Volumetric body torque can exist in magnetic fluids, for example (e.g. see R. E. Rosensweig, *Ferrohydrodynamics*, 1985, Chapter 8). In what follows we shall *assume that volumetric body torque is absent*, in which case (16) shows that the off-diagonal or shear terms in the stress tensor are *symmetric*,

$$\tau_{ji} = \tau_{ij} \quad (i \neq j) . \quad (17)$$

This means that three of the nine components of the stress tensor can be derived from the remaining ones; that is, *the stress tensor has only six independent components*.

4 Equation of Motion in Terms of the Stress Tensor

A general equation of motion in differential form may be derived by applying Newton's law to a small but finite fluid particle. Consider again a particle which at time t has the shape of a cube centered about (x, y, z) as in Fig. 6, with sides δx , δy , and δz parallel to the x , y , and z axes at time t . Although the sides are small, they are not zero and the components of the stress tensor will have slightly different values on the faces of the cube than at the center of the cube. For example, if the stress tensor components τ_{ij} are specified at (x, y, z) , the center of the cube, then their values will be

$$\tau_{ij} + \frac{\partial \tau_{ij}}{\partial x} \frac{\delta x}{2}$$

at the face whose outward normal is in the positive x -direction, and

$$\tau_{ij} - \frac{\partial \tau_{ij}}{\partial x} \frac{\delta x}{2}$$

at the opposite face.

Figure 6 shows all those stresses which act on the cube in the x -direction, expressed in terms of the stress tensor. The arrows indicate the directions of the stresses for positive values of τ_{ij} [see (10)]. The net x -component of surface force on the cube is obtained by multiplying the stresses by the areas on which they act and summing:

$$\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \delta x \delta y \delta z . \quad (18)$$

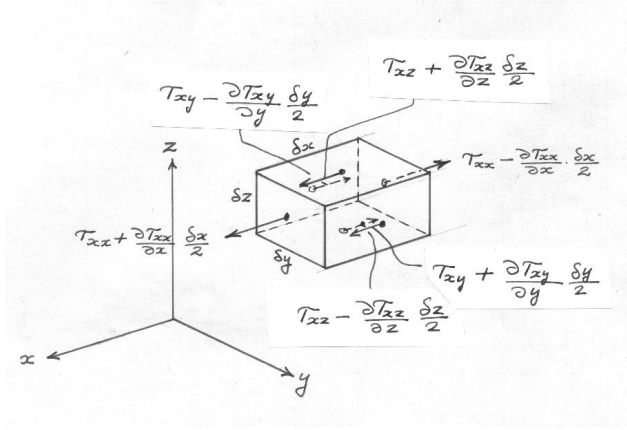


Fig. 6: x-direction surface stresses acting on a fluid particle.

Since $\delta x \delta y \delta z$ is the particle's volume, we identify the quantity within the brackets as the *net x-component of surface force per unit volume at a point in a fluid*. The expressions for the y and z components are similar, except that the first subscript x is replaced by y and z, respectively.

The equation of motion can now be written down directly for the cubical fluid particle in Fig. 6. The x-component of the equation states that the mass times the acceleration equals the net surface force plus the body force acting on the particle:

$$\rho \delta x \delta y \delta z \frac{Dv_x}{Dt} = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \delta x \delta y \delta z + \rho \delta x \delta y \delta z G_x$$

Here, D/Dt represents the substantial derivative, which is defined elsewhere, and G_x is the x-component of the external body force per unit mass. This yields

$$\rho \frac{Dv_x}{Dt} = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) + \rho G_x. \quad (20a)$$

For the y and z components we obtain similarly

$$\rho \frac{Dv_y}{Dt} = \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) + \rho G_y \quad (20b)$$

$$\rho \frac{Dv_z}{Dt} = \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \rho G_z \quad (20c)$$

or, more succinctly,

$$\rho \frac{Dv_i}{Dt} = \frac{\partial \tau_{ij}}{\partial x_j} + \rho G_i \quad (20)$$

where a summation over $j=x, y, \text{ and } z$ is implied. Equation (20) states that at a given point and time, the mass per unit volume times the acceleration in the i -direction (the left-hand term) equals the the net surface force per unit volume in the i -direction (the first term on the right) plus the body force per unit volume in the i -direction (the second term on the right). The equation applies quite generally to *any* continuous distribution of matter, whether fluid or solid, and is not based on any assumption other than that the continuum hypothesis applies.¹ Eq. (20) is, however, incomplete as it stands. To complete it, one must specify the stress tensor components and the body force components, just as one must define the forces acting on a solid particle before one can derive its motion. The specification of the body force is straightforward. In a gravitational field, for example, the force \bar{G} per unit mass is well known and is of the same form for all substances. The form of the stress tensor is different for different classes of materials.

5 Stress Tensor for Newtonian Fluids

There remains the task of specifying the relationship between the stress tensor components and the flow or deformation field. The simplest model of a *solid* continuum is the well-known elastic one, where stresses and strains are linearly related. The defining attribute of a simple fluid, however, is that it keeps deforming, or straining, as long as any shear stress, no matter how small, is applied to it. Obviously, no unique relation can exist between the shear stresses and the shear strains if strain can increase indefinitely at constant shear. It is observed, however, that a fluid tends to resist the *rate* of deformation: the higher the applied shear stress, the faster the *rate* of shear deformation. In many fluids the relation between stress and *rate* of strain in a fluid particle is linear under normal conditions.

The *Newtonian model* of fluid response is based on three assumptions:

¹In solid mechanics the acceleration term in Eq. (20) is not included if one is concerned with static or quasi-static situations, and the body force term is neglected when the gravitational loads induced by the weight of the structure itself are small. In such cases one is left with the simple statement that the net surface stress per unit volume is zero at every point in the medium.

- (a) shear stress is proportional to the *rate* of shear strain in a fluid particle;
- (b) shear stress is zero when the rate of shear strain is zero;
- (c) the stress to rate-of-strain relation is isotropic—that is, there is no preferred orientation in the fluid.

A Newtonian fluid is the simplest type of viscous fluid, just like an elastic solid (where stresses are proportional to strains) is the simplest type of deformable solid.

The shear stresses and the ordinary viscosity

To implement the Newtonian assumptions we consider first a typical shear term in the tensor, e.g. τ_{xy} . Fig. 7 depicts the deformation of a fluid particle as it moves between time t and time $t+dt$. In this interval the shear stress τ_{xy} produces in the fluid particle an incremental angular strain $d\gamma_{xy}$

$$d\gamma_{xy} = \frac{\left(\frac{\partial v_x}{\partial y} \delta y dt\right)}{\delta y} + \frac{\left(\frac{\partial v_y}{\partial x} \delta x dt\right)}{\delta x} .$$

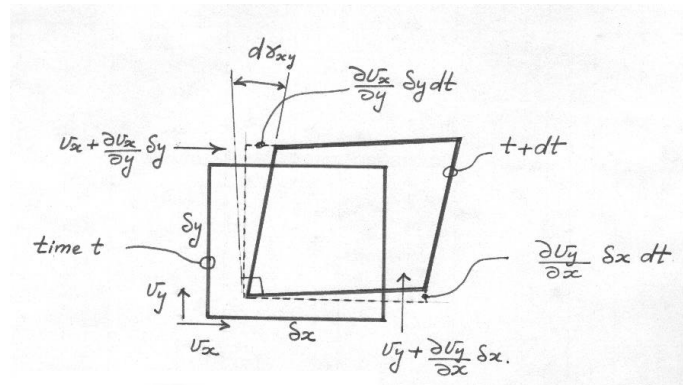


Fig. 7: Shear deformations in a fluid particle.

The *rate* of angular (or *shear*) strain in the fluid particle as seen by an observer sitting on it is therefore

$$\frac{D\gamma_{xy}}{Dt} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} . \quad (21)$$

The Newtonian assumptions (a) and (b) thus require that

$$\tau_{xy} = \mu \frac{D\gamma_{xy}}{Dt} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right). \quad (22a)$$

where the coefficient of proportionality μ is called the shear, or "ordinary", *viscosity coefficient*, and is a property of the fluid. Similarly,

$$\tau_{xz} = \mu \frac{D\gamma_{xz}}{Dt} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \quad (22b)$$

$$\tau_{yz} = \mu \frac{D\gamma_{yz}}{Dt} = \mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \quad (22c)$$

or in general,

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (i \neq j) . \quad (22)$$

The coefficient of proportionality is the same in all three shear stresses because a Newtonian fluid is *isotropic*.

The normal stresses

Next consider a typical normal stress, that is, one of the stress tensor's diagonal terms, say τ_{xx} . The derivation of such a term's form is not as simple as that of the shear terms, but can nevertheless be done in fairly physical terms by noting that linear and shear deformations generally occur hand in hand. The trick is to find how the linear stresses and deformations are related to the shear stresses and deformations.

Consider a small fluid particle which at time t is a small cube with sides of length h parallel to the x , y and z axes. We will again be considering the limit of a particle "at a point", that is, the limit $h \rightarrow 0$. At time t , its corner A is at (x, y, z) . Between t and $t+dt$, it moves and deforms as in Fig. 8. The sides AB and AD will in general rotate by unequal amounts. This will result in a shear deformation of the particle. The shear deformation will cause one of the diagonals AC and BD to expand and the other to contract, that is, it will

give rise to *linear* deformations in the x' and y' directions which are rotated 45° relative to the x and y axes.

Now, we know the relationship between the shear stress and the rate of angular strain of the particle in the (x, y) frame. If we can connect the shear stresses in this frame and the stresses in the rotated (x', y') frame, and the shear strain rates in the (x, y) frame and the

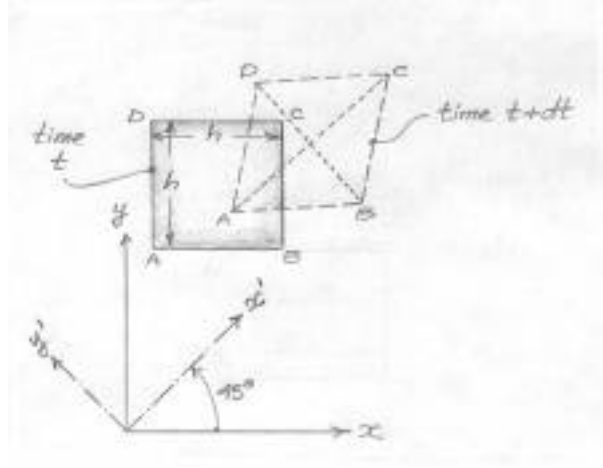


Fig. 8: Why shear and linear deformations are related.

strain rates on the (x', y') frame, we will arrive at a relation between the stresses and the strain rates in the (x', y') frame. Since the reference frames are arbitrary, the relationship between stresses and rates of strain for the (x', y') frame must be general in form.

We start by considering the forces acting on one half of the fluid particle in Fig. 8: the triangular fluid particle ABD as shown in Fig. 9. Since we are considering the limit $\delta h \rightarrow 0$, where the ratio of volume to area vanishes, the equation of motion for the particle will reduce to the statement that the surface forces must be in balance. Figure 9 shows the surface forces on particle ABD, expressed in terms of the stress tensor components in the original and the rotated reference frames. A force balance in the x' -direction requires that

$$\tau'_{xx} = \frac{\tau_{xx} + \tau_{yy}}{2} + \tau_{yx} . \quad (23)$$

Similarly, a force balance in the y' -direction on the triangular particle ACD requires that

$$\tau'_{yy} = \frac{\tau_{xx} + \tau_{yy}}{2} - \tau_{yx} . \quad (24)$$

Adding (23) and (24) we obtain

$$\tau'_{xx} - \tau'_{yy} = 2\tau_{yx} . \quad (25)$$

Using the relation (22a) between the shear stress and the rate of strain, this becomes

$$\tau'_{xx} - \tau'_{yy} = 2\mu \frac{D\gamma_{xy}}{Dt} \quad (26)$$

which relates the diagonal stress tensor terms in the (x', y') frame to the angular strain rate in the (x, y) frame.

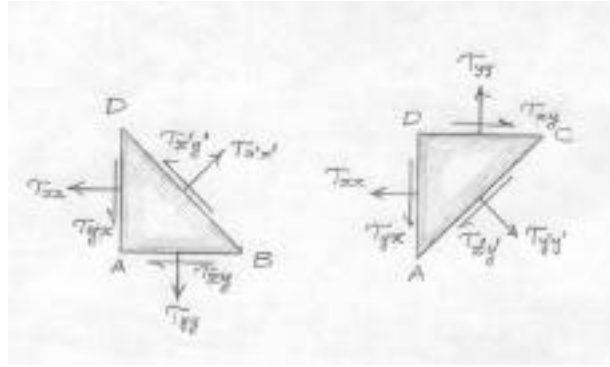


Fig. 9: Stresses on two halves of the particle in Fig. 8.

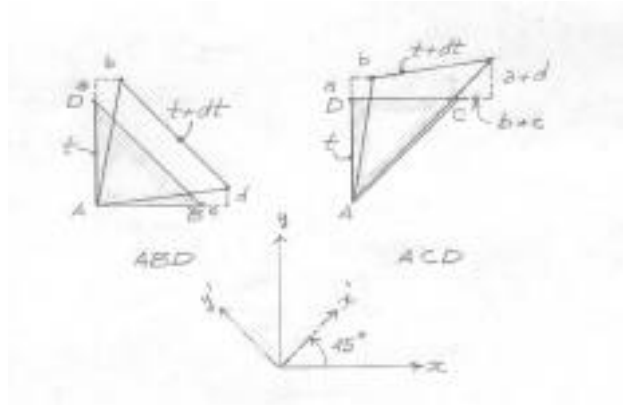


Fig. 10: Deformations of the two triangular particles in Fig. 9.

To close the loop we must relate the angular strain rate in the (x, y) frame to the strain rates in the (x', y') frame. Figure 10 shows the deformations of the triangular particles ABD and ACD between t and $t+dt$. The deformations a , b , c , and d in the figure are related to the incremental linear strains $d\epsilon$ and angular strains $d\gamma$ in the (x, y) frame by

$$\begin{aligned}
d\epsilon_x &= \frac{c}{h} \\
d\epsilon_y &= \frac{a}{h} \\
d\gamma_{xy} &= \frac{c+d}{h} .
\end{aligned} \tag{27}$$

Here, $d\epsilon_x$ is the linear strain (increase in length divided by length) of the particle in the x-direction, and $d\epsilon_y$ is its linear strain in the y-direction. The linear strain in the x' direction can be computed in terms of these quantities from the fractional stretching of the diagonal AC , which is oriented in the x' direction. Recalling that ACD is an equilateral triangle at time t , and that the deformations between t and $t+dt$ are infinitesimally small, we obtain

$$d\epsilon_{x'} = \frac{d(AC)}{(AC)} = \frac{\frac{a+d}{\sqrt{2}} + \frac{b+d}{\sqrt{2}}}{h\sqrt{2}} = \frac{1}{2} \left(\frac{c}{h} + \frac{a}{h} + \frac{b+d}{h} \right) = \frac{1}{2} (d\epsilon_x + d\epsilon_y + d\gamma_{xy}) . \tag{28}$$

The linear strain in the y' direction is obtained similarly from the fractional stretching of the diagonal BD of the triangular particle ABD as

$$d\epsilon_{y'} = \frac{d(BD)}{(BD)} = \frac{1}{2} (d\epsilon_x + d\epsilon_y - d\gamma_{xy}) . \tag{29}$$

The sum of the last two equations shows that the difference of the linear strains in the x' and y' directions is equal to the angular strain in the (x, y) plane:

$$d\epsilon_{x'} - d\epsilon_{y'} = d\gamma_{xy} . \tag{30}$$

The differentials refer to changes *following the fluid particle*. The *rates* of strain following the fluid motion are therefore related by

$$\frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{D\gamma_{xy}}{Dt} . \tag{31}$$

If we now eliminate the reference to the (x, y) frame by using (26), we obtain

$$\tau_{x'x'} - \tau_{y'y'} = 2\mu \left(\frac{D\varepsilon_{x'}}{Dt} - \frac{D\varepsilon_{y'}}{Dt} \right) . \quad (32)$$

The linear strain rates can be evaluated in terms of the velocity gradients by referring to Fig. 11. Between t and $t+dt$, the linear strain suffered by the fluid particle's side parallel to the x' axis is

$$d\varepsilon_{x'} = \frac{\frac{\partial v_{x'}}{\partial x'} \delta x dt}{\delta x} = \frac{\partial v_{x'}}{\partial x'} dt$$

so that

$$\frac{D\varepsilon_{x'}}{Dt} = \frac{\partial v_{x'}}{\partial x'} . \quad (33)$$

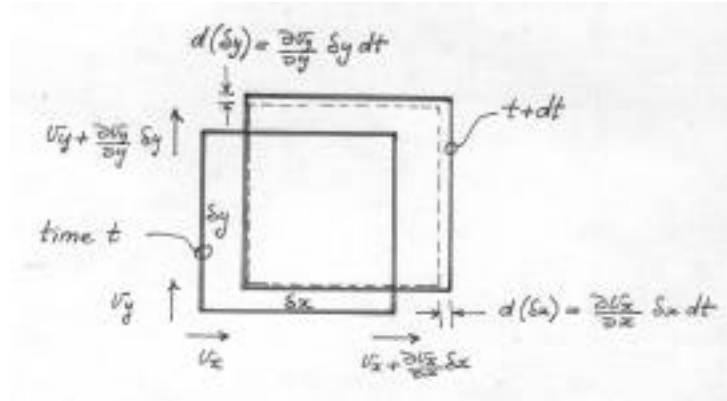


Fig. 11: Linear deformations of a fluid particle.

A similar equation is obtained for the linear strain rate in the y' direction. Using these relations in (32), we now obtain

$$\tau_{x'x'} - \tau_{y'y'} = 2\mu \left(\frac{\partial v_{x'}}{\partial x'} - \frac{\partial v_{y'}}{\partial y'} \right) . \quad (34)$$

Similarly we obtain, by viewing the particle in the (x', z') plane,

$$\tau_{x'x'} - \tau_{z'z'} = 2\mu \left(\frac{\partial v_{x'}}{\partial x'} - \frac{\partial v_{z'}}{\partial z'} \right). \quad (35)$$

Adding equations (34) and (35) we get

$$\tau_{x'x'} = \frac{\tau_{x'x'} + \tau_{y'y'} + \tau_{z'z'}}{3} + 2\mu \frac{\partial v_{x'}}{\partial x'} - \frac{2}{3}\mu \left(\frac{\partial v_{x'}}{\partial x'} + \frac{\partial v_{y'}}{\partial y'} + \frac{\partial v_{z'}}{\partial z'} \right) \quad (36)$$

Since the coordinate system (x', y') is arbitrary, this relationship must apply in any coordinate system. We thus have our final result:

$$\tau_{xx} = -p_m + 2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu \nabla \cdot \vec{v} \quad (37)$$

where the quantity

$$p_m = -\frac{(\tau_{xx} + \tau_{yy} + \tau_{zz})}{3} = -\frac{\tau_{ii}}{3} \quad (38)$$

is the "mechanical" pressure, to be distinguished from the "thermodynamic" pressure which is discussed below. The mechanical pressure is the negative of the average value of the three diagonal terms of the stress tensor, and serves as a measure of local normal compressive stress in viscous flows where that stress is not the same in all directions. The mechanical pressure is a well defined physical quantity, and is a true scalar since the trace of a tensor remains invariant under coordinate transformations. Note that although the definition is phrased in terms of the normal stresses on surfaces pointing in the x , y and z directions, it can be shown that p_m as defined in (38) is in fact equal to the average normal compressive stress on the surface of a sphere centered on the point in question, in the limit as the sphere's radius approaches zero (see G. K. Batchelor, *An Introduction to Fluid Mechanics*, Cambridge University Press, 1967, p.141 ff).

General form of the stress tensor and the second viscosity

Expressions similar to (37) are obtained for τ_{yy} and τ_{zz} , except that $\partial v_x/\partial x$ is replaced by $\partial v_y/\partial y$ and $\partial v_z/\partial z$, respectively. From these expressions and (22) for the off-

diagonal terms, it is evident that all the terms of the Newtonian stress tensor can be represented by the equation

$$\tau_{ij} = -\left(p_m + \frac{2}{3}\mu\nabla\cdot\vec{v}\right)\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad (39)$$

where

$$\begin{aligned} \delta_{ij} &= 1 & \text{if } i=j \\ &= 0 & \text{if } i\neq j \end{aligned}$$

is the Kronecker delta. Note that (39) represents any single component of the tensor, and *no sum is implied* in this equation when one writes down the general form of the diagonal terms by setting $j=i$.

The mechanism whereby stress is exerted by one fluid region against another is actually a molecular one. An individual molecule in a fluid executes a random thermal motion, bouncing against other molecules, which is superposed on the mean drift motion associated with flow. Normal stress on a surface arises from average momentum transfer by the fluid molecules executing their random thermal motion, each molecule imparting an impulse as it collides with the surface and rebounds. Normal stress is exerted even in a static, non-deforming fluid. Shear stress arises when there is a mean velocity gradient in the direction transverse to the flow. Molecules which move by random thermal motion transverse to the flow from a higher mean velocity region toward a lower mean velocity region carry more streamwise momentum than those moving in the opposite direction, and the net transfer of the streamwise molecular momentum manifests itself as a shear stress on the macroscopic level at which we view the fluid.

The molecular theory of the shear viscosity coefficient is quite different for gases and liquids. In gases the molecules are sparsely distributed and spend most of their time in free flight rather than in collisions with each other. In liquids, on the other hand, the molecules spend most of their time in the short-range force fields of their neighbors (see for example J. O. Hirschfelder, C. F. Curtiss and R. B. Bird, *Molecular Theory of Liquids Gases and Liquids*). The shear viscosity is mainly a function of temperature for both gases and liquids, the dependence on pressure being relatively weak. There is, however, one big difference between gases and liquids: the viscosity of gases increases with temperature, while the viscosity of liquids decreases, usually at a rate much faster than the increase in gases. The viscosity of air, for example, increases by 20% when temperature increases

from 18°C to 100°C. The viscosity of water, on the other hand, decreases by almost a *factor* of four over the same temperature range.

Equation (39) contains only a single empirical coefficient, the shear or ordinary coefficient of viscosity μ . A second coefficient is, however, introduced in our quest for a complete set of flow equations when we invoke the fluid's equation of state and are forced to ask how the "thermodynamic" pressure which appears in that equation is related to the mechanical pressure p_m . The equation of state expresses the fluid's density as a function of temperature and pressure under *equilibrium* conditions. The "thermodynamic" pressure which appears in that equation is therefore the hypothetical pressure that would exist if the fluid were in static equilibrium at the local density and temperature. Arguments derived from statistical thermodynamics suggest that this equilibrium pressure may differ from the mechanical pressure when the fluid is composed of complex molecules with internal degrees of freedom, and that the difference should depend on the rate at which the fluid density or pressure is changing with time. The quantity which provides the simplest measure of rate of density change is the divergence of the velocity vector, $\nabla \cdot \vec{v}$, which represents the rate of change of fluid volume per unit volume, as seen by an observer moving with the fluid. It is customary to assume a simple linear relationship which may be thought of as being in the same spirit as the original Newtonian postulates, but in fact rests on much more tenuous experimental grounds:

$$p_m = p - \lambda \nabla \cdot \vec{v} . \quad (40)$$

Here, λ is an empirical coefficient which happens to have the same dimension as the shear viscosity μ , and is called the *expansion viscosity* (Batchelor, *An Introduction to Fluid Dynamics*; alternative terms are "second coefficient of viscosity" and "bulk viscosity"). Thermodynamic second-law arguments show that λ must be positive. This implies that the thermodynamic pressure tends to be higher than the mechanical pressure when the mechanical pressure is decreasing (volume increasing, $\nabla \cdot \vec{v} > 0$), and lower than the mechanical pressure when the pressure is increasing (volume decreasing, $\nabla \cdot \vec{v} < 0$). In other words, the thermodynamic pressure always tends to "lag behind" the mechanical pressure when a change is occurring. The difference depends, however, on both the rate of expansion ($\nabla \cdot \vec{v}$) and the molecular composition of the fluid (via λ : see below).

Written in terms of the thermodynamic pressure p , the Newtonian stress tensor reads

$$\tau_{ij} = -\left[p + \left(\frac{2}{3}\mu - \lambda\right)\nabla \cdot \vec{v}\right]\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \quad (41)$$

The $\nabla \cdot \vec{v}$ term is associated with the dilation of the fluid particles. The physical interpretation of $\nabla \cdot \vec{v}$ is that it represents the rate of change of a fluid particle's volume recorded by an observer sitting on the particle, divided by the particle's instantaneous volume.

It can be shown rigorously that $\lambda=0$ for dilute monatomic gases. For water λ is about three times larger than μ , and for complex liquids like benzene it can be over 100 times larger. Nevertheless, the effect on the flow of the term which involves $\nabla \cdot \vec{v}$ and the expansion viscosity is usually very small even in compressible flows, except in very special and difficult-to-achieve circumstances. Only when density changes are induced either over extremely small distances (e.g. in the interior of shock waves, where they occur over a molecular scale) or over very short time scales (e.g. in high-intensity ultrasound) will the term involving $\nabla \cdot \vec{v}$ be large enough to have a noticeable effect on the equation of motion. Indeed, attempts to study the expansion viscosity are hampered by the difficulty of devising experiments where its effect is significant enough to be accurately measured. For most flows, therefore, including most compressible flows where the fluid's density is changing, we can approximate the stress tensor by

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad (42)$$

or

$$\begin{aligned} \tau_{xx} &= -p + 2\mu \frac{\partial v_x}{\partial x} \\ \tau_{yy} &= -p + 2\mu \frac{\partial v_y}{\partial y} \\ \tau_{zz} &= -p + 2\mu \frac{\partial v_z}{\partial z} \\ \tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \end{aligned} \quad (43)$$

$$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) .$$

Equations (42) and (43) are rigorously valid in the limit of incompressible flow ($\nabla \cdot \bar{v} \approx 0$).

That the term which involves λ is usually negligible is fortunate, for experiments have shown that the assumed linear relation between the mechanical and thermodynamic pressures, (40), is suspect. The value of λ , when it is large enough to be measured accurately, usually turns out to be not a fluid property but dependent on the rate of expansion, i.e. on $\nabla \cdot \bar{v}$ and thus on the particular flow field. By contrast, the Newtonian assumption of linearity between the shear stresses and rates of shear strain is very accurately obeyed in a large class of fluids under a wide range of flow conditions. All gases at normal conditions are Newtonian, as are most liquids with relatively simple molecular structure. For further discussion of the expansion viscosity, see for example G. K. Batchelor, *An Introduction to Fluid Mechanics*, pp. 153-156, Y. B. Zeldovich and Y. P. Razier, *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, Vol. I, pp. 73-74, or L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, pp. 304-309. The theory of the expansion viscosity is discussed in J. O. Hirschfelder, C. F. Curtiss and R. B. Bird's *Molecular Theory of Gases and Liquids*; some experimental values can be found for example in the paper by L. N. Lieberman, *Physical Review*, Vol. 75, pp 1415-1422, (1949). For the expansion viscosity in gases, see also the editorial footnote by Hayes and Probst in Y. B. Zel'dovich and Y. P. Raizer's *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, Vol. II, pp 469-470.

6 The Navier-Stokes Equation

The Navier-Stokes equation is the equation which results when the Newtonian stress tensor, (41), is inserted into the general equation of motion, (20):

$$\rho \frac{Dv_i}{Dt} = -\frac{\partial}{\partial x_i} \left[p + \left(\frac{2}{3} \mu - \lambda \right) \nabla \cdot \vec{v} \right] + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \right] + \rho G_i \quad (44)$$

For *constant* μ and λ , this equation reduces to a form which can be written in vector notation as

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \left(\frac{1}{3} \mu + \lambda \right) \nabla (\nabla \cdot \vec{v}) + \mu \nabla^2 \vec{v} + \rho \vec{G} \quad (45)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (46)$$

is a scalar operator, operating in (45) on the vector \vec{v} , just like D/Dt on the left side is the well-known scalar operator, operating on \vec{v} .

For *incompressible* flows with constant viscosity,

$$\frac{\partial v_j}{\partial x_j} = \nabla \cdot \vec{v} = 0 ,$$

and one obtains from (44) or (45)

$$\rho \frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \rho G_i , \quad (47)$$

or, in vector form,

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{G} . \quad (48)$$

As mentioned above, (47) or (48) are in many cases very good approximations even when the flow is compressible. Written out fully in Cartesian coordinates, (47) reads

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho G_x \quad (49a)$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho G_y \quad (49b)$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho G_z \quad (49c)$$

Appendix A gives the equations in cylindrical coordinates.

The Navier-Stokes equation of motion was derived by Claude-Louis-Marie Navier in 1827, and independently by Siméon-Denis Poisson in 1831. Their motivations of the stress tensor were based on what amounts to a molecular view of how stresses are exerted by one fluid particle against another. Later, Barré de Saint Venant (in 1843) and George Gabriel Stokes (in 1845) derived the equation starting with the linear stress vs. rate-of-strain argument.

Boundary conditions

A particular flow problem may in principle be solved by integrating the Navier-Stokes equation, together with the mass conservation equation plus whatever other equations are required to form a complete set, with the boundary conditions appropriate to the particular problem at hand. A solution yields the velocity components and pressure at the boundaries, from which one obtains the stress tensor components via equation (42) [or (43)] and the stress vector from equation (11).

In the absence of surface tension, the boundary conditions consistent with the continuum hypothesis are that (a) *the velocity components* and (b) *the stress tensor components must be everywhere continuous*, including across phase interfaces like the boundaries between the fluid and a solid and between two immiscible fluids. That this must be so can be proved by applying mass conservation and the equation of motion to a small disc-shaped control volume at a point in space, similar to the disc depicted in Fig. 1, and considering the limit where the thickness of the disc go to zero. The proof for the continuity

of τ_{ij} is essentially the same as the one for equation (4), with the requirement that the equation of motion must be satisfied at every point for *any* orientation \vec{n} of the surface.

Surface tension gives rise to a discontinuity in the normal stress at the interface between two immiscible fluids.

Appendix A

The Navier-Stokes Equations and Mass Conservation Equation for Cylindrical Coordinates (r, θ, z) in Incompressible Flow

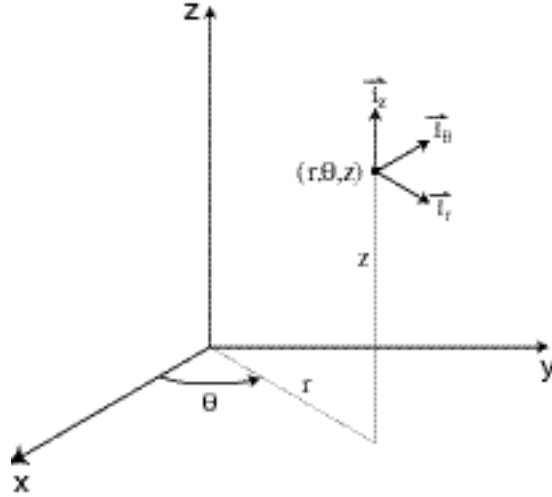


Fig. A.1: Cylindrical coordinate system

Navier-Stokes equation of motion

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] + \rho G_r \quad (\text{A.1})$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] + \rho G_\theta \quad (\text{A.2})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho G_z \quad (\text{A.3})$$

Stress tensor components

$$\tau_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}$$

$$\tau_{\theta\theta} = -p + 2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)$$

$$\tau_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}$$

(A.4)

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right)$$

$$\tau_{rz} = \tau_{zr} = \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)$$

Mass conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) + \frac{1}{r} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0 \quad (\text{A.5})$$