A paraxial cloak with four lenses of different focal lengths

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M Revilla, J C Lorenzo and N Hermosa

National Institute of Physics, University of the Philippines, Diliman, Quezon City, Philippines 1101

E-mail: mrevilla@nip.upd.edu.ph

Abstract. We demonstrate cloaking with four lenses of different focal lengths. To achieve this, we compute the separation distances between lenses such that the effective ABCD matrix is equal to just a propagation in free space. Previously, calculations using this method are restricted to 2 pairs of lenses with equal focal lengths. Our computations, on the other hand, had no such restrictions: we generalize even with an asymmetric case. We derive expressions that show the dependence of the distances on the lenses and their focal lengths. Furthermore, we propose a way to minimize the length of this cloak. Our equations reduce to the Rochester cloak when its restriction is imposed. A general paraxial cloak could be useful in advanced undergraduate experiments in optics because: 1) the limit imposed by the focal length has been lifted; and 2) there is a pedagogical benefit of doing experiments with lenses beyond the usual imaging experiments such as reinforcing concepts in Gaussian optics.

A paraxial cloak with four lenses of different focal lengths 2

1. Introduction

Invisibility has been a popular idea in science fiction and in photonics research. The general idea behind the perfect cloak is to bend electromagnetic waves around a region, called the cloaking region, while keeping the amplitude and phase of outgoing light preserved. Perfect cloaks have to be invisible in all directions and it should be invisible itself. Although complete omnidirectional invisibility has been ruled out [1], many have not dismissed the possibility of making a near-perfect cloak. As a matter of fact, different methods have been studied in the last 20 years to achieve a good invisibility cloak.

A common method employed in making a cloak is with the use of transformation optics, a technique wherein the coordinates are transformed to control the path of light [2]. Applying this technique, numerous researchers have simulated and fabricated materials that can direct electromagnetic waves around a cloaking region [3–7]. Cloaks with multiple cloaking regions have been realized [8]. Complex structures, such as split ring resonators, have also been made from existing materials that can act as wave guides and replicate cloaking [9–11]. Some of these nano-structures have even succeeded in making polarization-independent cloaks [12]. Moreover, materials with graded index of refraction have been employed to mimic cloaking [13, 14]. This transformation technique has been implemented with terahertz [15] and radar [16, 17], under diffuse background lighting [18], and with unique and arbitrary cloaking regions [19, 20]. The use of transformation optics has been extended to other similar phenomena such as acoustics cloaking [21], heat redirection [22], and magnetic field manipulation [23].

Transformation optics thru geometric optics is perhaps the simplest approach to make a cloak. Over the years, cloaking has been achieved with various optical elements such as mirrors and Fresnel and diverging lenses [24], as well as using the refraction of light by using L-shaped glass containers filled with water [25] and with the concept of mirage [26]. Interestingly, cloaks using geometrical optics have also been shown to preserve both wave phase and amplitude [27]. There have also been efforts to look into the spectrum of the incident light and to preserve it by shifting energies to regions that the cloaked object transmits [28].

By far the most popular among cloaks using geometrical optics is the Rochester cloak made by Howell and Choi, [29, 30]. Their method makes use of four converging lenses to achieve a perfect paraxial cloak. A *paraxial perfect cloak* preserves, more or less, the incoming field, satisfying some characteristics of a perfect cloak but has only a certain region to hide an object. They are not exempt from edge effects and aberrations. Moreover, the observation is limited to small angles from the optical axis.

In the work of Howell and Choi, they were able to show that one, two, and three lens systems cannot achieve perfect paraxial cloak. They did this by setting the effective ray transfer matrix of the system to correspond to that of free space. They found that with less than 4 lenses, no solutions can be obtained for the separation distances that will satisfy the free space correspondence and that are physically possible. With four lenses, however, solutions for the separation distances were obtained. They were able to

A paraxial cloak with four lenses of different focal lengths   3

check that the cloak indeed has a cloaking region. The limitation in their calculations, however, is that they set a condition that the focal lengths of first and fourth lenses be equal and the other two lenses also have equal focal lengths. They also set the separation distance between first and second lenses equal to that of the third and fourth. With these conditions, their system effectively became two $4f$ systems, which preserved the input wavefront. Imposing these conditions on the focal length also led to a simpler effective ray transfer matrix, in which the remaining separation distance can be easily solved.

Without imposing conditions on the focal lengths and separation distances, this paper addresses the question: *can cloaking still occur on the four lens system using separation distances solved directly from its effective ray transfer matrix?* We also optimize the system by calculating possible placement of lenses which produce the least total length. With this generalized four lens system, we also test its solution when Rochester cloak conditions are imposed. Lastly, we comment on the applicability of this setup as an undergraduate experiment in optics.

2. Method

In making the Rochester cloak, Howell and Choi utilized two ray transfer matrices: that of free space and of a thin lens. The transformations caused by the optical elements under paraxial approximation can be best summarized in terms of ABCD or ray transfer matrices. A system composed of n thin lens and free space units has an effective ABCD matrix that can be expressed as

$$\mathbf{M}_{\text{eff}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_n} & 1 \end{pmatrix} \prod_{i=1}^{n-1} \begin{pmatrix} 1 - \frac{d_{n-i}}{f_{n-i}} & d_{n-i} \\ -\frac{1}{f_{n-i}} & 1 \end{pmatrix} \tag{1}$$

where n is the number of lens, f_n is the last lens, d_i is the ith separation distance between lens i and $i+1$ and f_i is the focal length of lens i. Considering the location of the observer, the nth distance, is not necessary in the effective ABCD matrix because the output wavefront is assumed to be preserved after passing through the system. This is also the case for the distance from the source to the first lens. The matrices in the product in equation 1 are the product of a free space and lens ABCD matrices given by [31]

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \tag{2}$$

for a length of free space d, and

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}, \tag{3}$$

A paraxial cloak with four lenses of different focal lengths   4

for a lens with focal length f, respectively. The matrix in equation 1 can be reduced to a 2x2 matrix with elements

$$\mathbf{M}_{\text{eff}} = \begin{pmatrix} A_{\text{eff}} & B_{\text{eff}} \\ C_{\text{eff}} & D_{\text{eff}} \end{pmatrix} \qquad (4)$$

where the eff subscript means effective and A, B, C, and D are the elements of the matrices.

Imposing the free space condition on equation 4 means that each elements would have to be equal to that of free space. Hence,

$$A_{\text{eff}}(d_i, d_{i+1}, d_{i+2}) = 1 \qquad (5)$$

$$B_{\text{eff}}(d_i, d_{i+1}, d_{i+2}) = \sum_{i=1}^{n-1} d_i = d_{\text{total}} \qquad (6)$$

$$C_{\text{eff}}(d_i, d_{i+1}, d_{i+2}) = 0 \qquad (7)$$

$$D_{\text{eff}}(d_i, d_{i+1}, d_{i+2}) = 1. \qquad (8)$$

These constraints are satisfied by the appropriate lens separation distances which can be solved with any combination of the equations above. Equation 1 can be expanded as,

$$\mathbf{M}_{\text{eff}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_4} & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_3}{f_3} & d_3 \\ -\frac{1}{f_3} & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_2}{f_2} & d_2 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{d_1}{f_1} & d_1 \\ -\frac{1}{f_1} & 1 \end{pmatrix} \quad (9)$$
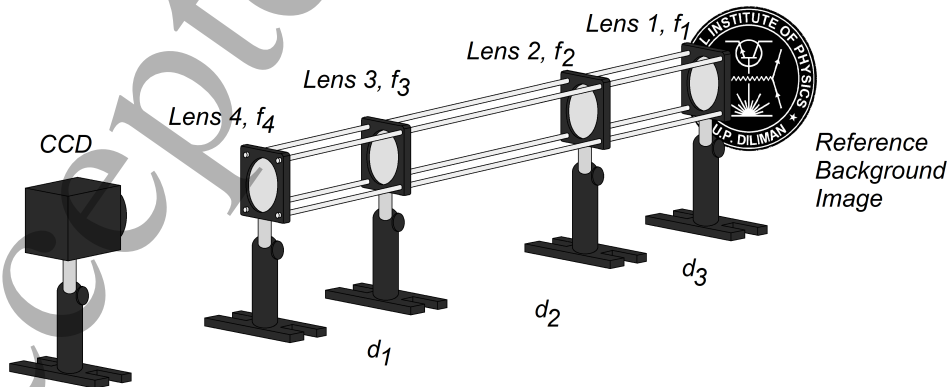
with the lens system shown in figure 1.

Figure 1. The four-lens system. Distances between the lenses are labelled as d_1, d_2, and d_3. The focal lengths of lenses 1 to 4 are labelled as f_1, f_2, f_3, and f_4 correspondingly.

A paraxial cloak with four lenses of different focal lengths   5

After imposing free space propagation, Eqs. 5, 6, 7 and 8 for the 4-lens system will be,

$$A_{\text{eff}} = \prod_{i=1}^{3} \left(1 - \frac{d_i}{f_i}\right) - \frac{d_2}{f_1}\left(1 - \frac{d_3}{f_3}\right) - \frac{d_3}{f_2}\left(1 - \frac{d_1}{f_1}\right) - \frac{d_3}{f_1} = 1 \tag{10}$$

$$B_{\text{eff}} = d_1 \prod_{i=1}^{2}\left(1 - \frac{d_{i+1}}{f_{i+1}}\right) + d_2\left(1 - \frac{d_3}{f_3}\right) + d_3\left(1 - \frac{d_1}{f_2}\right) = d_{\text{total}} \tag{11}$$

$$C_{\text{eff}} = -\frac{1}{f_4}A_{\text{eff}} - \frac{1}{f_3}\prod_{i=1}^{2}\left(1 - \frac{d_i}{f_i}\right) + \frac{d_2}{f_3 f_1} - \frac{1}{f_2}\left(1 - \frac{d_1}{f_1}\right) - \frac{1}{f_1} = 0 \tag{12}$$

$$D_{\text{eff}} = -\frac{1}{f_4}B_{\text{eff}} - \frac{d_1}{f_3}\left(1 - \frac{d_2}{f_2}\right) + \left(1 - \frac{d_1}{f_2}\right) - \frac{d_2}{f_3} = 1 \tag{13}$$

Any 3 of these equations and knowing that $d_{\text{total}} = d_1 + d_2 + d_3$, the distances d_i's can be solved.

Technically, the number of lenses beyond 4 may be used for cloaking. However, additional constraints must be imposed to match the number of unknowns. Moreover, realizing a more than 4-lens system in experiments can be challenging as it is very sensitive on the alignment of the lenses.

3. Results and Discussion

The ensuing system of equations, Eqs. 10 to 12 and d_{total}, solutions for d_1, d_2, and d_3 are given by,

$$d_1 = \alpha \frac{1 - \sqrt{\frac{f_1 f_2}{f_3 f_4}}}{f_3 f_4 - f_1 f_2} \tag{14}$$

$$d_2 = \alpha \frac{1 + \sqrt{\frac{f_2 f_3}{f_1 f_4}}}{f_1 f_4 - f_2 f_3} \tag{15}$$

$$d_3 = \alpha \frac{1 - \sqrt{\frac{f_3 f_4}{f_1 f_2}}}{f_1 f_2 - f_3 f_4} \tag{16}$$

where

$$\alpha = f_2 f_3 \left(f_1 + f_4\right) + f_1 f_4 \left(f_2 + f_3\right) \tag{17}$$

These separation distances, Eqs. 14 to 17, are the main results of this paper. The d_i's are functions of the focal lengths only. These illustrate that relaxing most of the conditions set in the Rochester cloak can still yield viable, or realizable separation distances given appropriate choice of focal lengths. These equations, however, must still hold true if ever focal length restrictions of the Rochester cloak were to be imposed. We show that it is indeed so, later in our discussion.

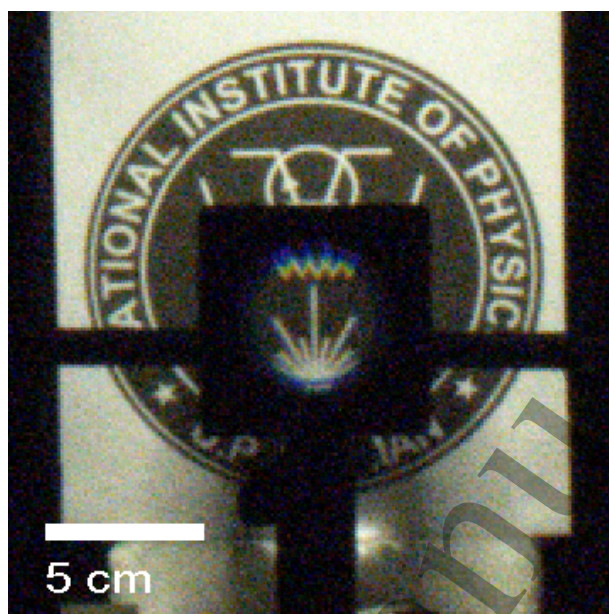*A paraxial cloak with four lenses of different focal lengths* 6

Figure 2. The cloak in action as viewed on-axis. A metal rod is placed horizontally inside the cage system. The background, the National Institute of Physics logo, can still be viewed even though an obstruction is placed inside. Lenses 1-4 for this system had focal lengths of 20.0 cm, 6.0 cm, 7.5 cm, and 17.5 cm, respectively. Chromatic aberrations can be seen at the edge of the lens.

As proof of principle, a cloak was designed using four 2" uncoated N-BK7 bi-convex lenses from ThorLabs. The focal length of these lenses were 20.0 cm (lens 1), 6.0 cm (lens 2), 7.5 cm (lens 3), and 17.5 cm (lens 4). Using equations 14 to 17, the separation distances given these focal lengths were $d_1 = 25.0$ cm, $d_2 = 28.6$ cm, and $d_3 = 26.1$ cm, for a total length of 79.7 cm. A logo of the institute was used as background image. The object was placed between lenses 3 and 4 and its position inside was adjusted to observe the cloaking phenomenon. The images were taken by a PointGrey Grasshopper 3 4.1 megapixel CCD camera with a Tamron 1/1.8" imaging lens. Although, any camera with a lens will do.

Similar to the Rochester cloak, cloaking happens when the object is placed off axis from the centre: it does not happen when the object is placed at the centre of the lens system. Moreover, cloaking can be perceived only at a small angle with respect to the optical axis similar to the Rochester cloak. Since we have used uncoated non-achromatic lenses, chromatic aberrations are present at the edges of the lens. These aberrations are also seen in the image as some brightly coloured bands at the edge of the cloak. Furthermore some magnification is observed in the background image. This magnification and aberration are further highlighted in figure 3. However, *we believe that we clearly demonstrate cloaking with our setup.*

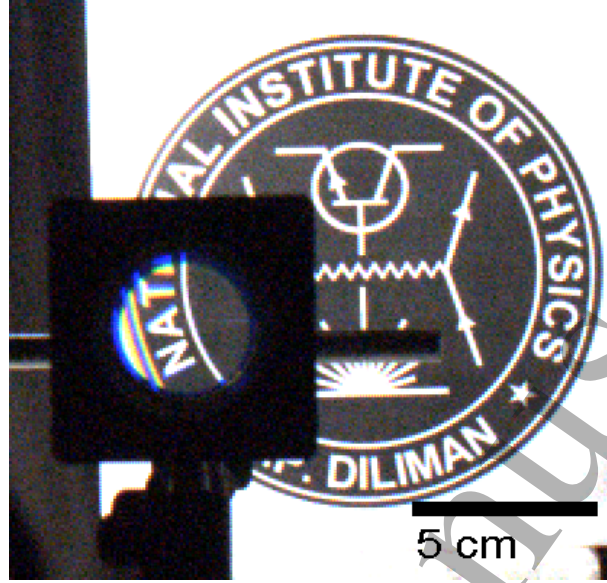*A paraxial cloak with four lenses of different focal lengths* 7

Figure 3. The cloak with the background image off-centred. By placing the background as such, magnification and image distortion can easily be checked by looking at the circular edge of the logo. The lenses used in this system were the same as those in figure 2.

One particular interest in this system is how to minimize its total length d_{total}. With Eqs. 14 to 17, d_{total} can be expressed as,

$$d_{\text{total}} = \alpha \left(\frac{1 - \sqrt{\dfrac{f_1 f_2}{f_3 f_4}}}{f_3 f_4 - f_1 f_2} + \frac{1 + \sqrt{\dfrac{f_2 f_3}{f_1 f_4}}}{f_1 f_4 - f_2 f_3} - \frac{1 - \sqrt{\dfrac{f_3 f_4}{f_1 f_2}}}{f_3 f_4 - f_1 f_2} \right) \quad (18)$$

$$= \alpha \left(\frac{\sqrt{\dfrac{f_3 f_4}{f_1 f_2}} - \sqrt{\dfrac{f_1 f_2}{f_3 f_4}}}{f_3 f_4 - f_1 f_2} + \frac{1 + \sqrt{\dfrac{f_2 f_3}{f_1 f_4}}}{f_1 f_4 - f_2 f_3} \right) \quad (19)$$

Looking at Eq.19, the first term has a difference for its numerator, meanwhile, the second term has a sum. A way to minimize d_{total} is to make the second term of the sum smaller while keeping the first term relatively suppressed. This can be achieved by greatly increasing the difference $f_1 f_4 - f_2 f_3$, denoted as Δ_{1423}. By making values of f_1 and f_4 close and considerably larger than those of f_2 and f_3, Δ_{1423} is increased while keeping the first term roughly the same.

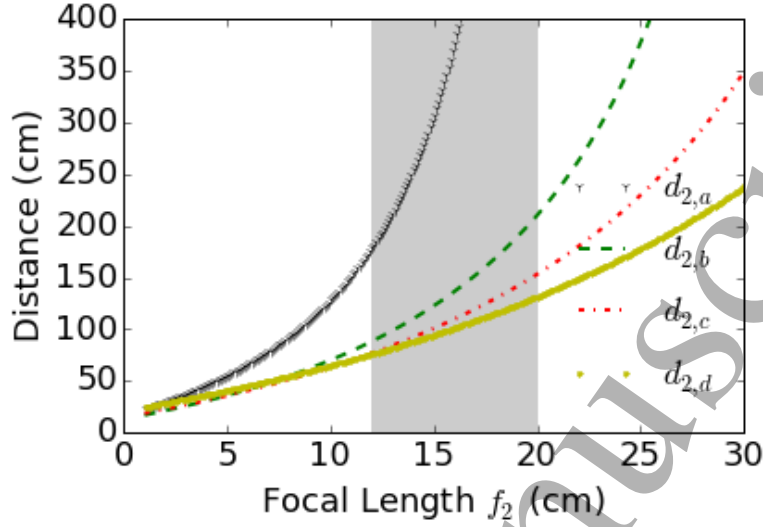*A paraxial cloak with four lenses of different focal lengths*   8

Figure 4. Plot of d_2 values for increasing f_2, effectively decreasing Δ_{1423}. In $d_{2,a}$, the values of f_1, f_3, and f_4 were 18 cm, 16 cm, 20 cm, respectively. As for $d_{2,b}$: 20 cm, 12 cm, and 22 cm; for d_{2_c}: 25 cm, 14 cm, and 27.5 cm; and for d_{2_d}: 35 cm, 20 cm, 40 cm. The shaded region in the figure points where f_2 ranges closely with its corresponding f_3 value.

Figure 4 illustrates this idea of increasing Δ_{1423} to diminish d_2, and in turn d_{total}, to workable distances. Shown in the figure are different set-ups of a general four-lens cloak, labelled from a to d. Comparing the behaviour of d_2 in set a (where f_1, f_3, and f_4 are 18 cm, 16 cm, and 20 cm) to its value in set d (f_1, f_3 and f_4: 35 cm, 20 cm, 40 cm), there is a noticeable difference between their d_2 values: ≈ 200 cm for set a and ≈ 140 cm for set d.

Lastly, we looked into equations 14 to 17 under Rochester cloak-like conditions: $f_1 = f_4$ and $f_2 = f_3$. The equations for the separation distances transform into

$$d_1^{\text{R}} = \lim_{f_1 \to f_4, f_2 \to f_3} \alpha \frac{1 - \sqrt{\dfrac{f_1 f_2}{f_3 f_4}}}{f_3 f_4 \left(1 - \frac{f_1 f_2}{f_3 f_4}\right)} = f_1 + f_2 \tag{20}$$

$$d_2^{\text{R}} = \lim_{f_1 \to f_4, f_2 \to f_3} \alpha \frac{1 + \sqrt{\dfrac{f_2 f_3}{f_1 f_4}}}{f_1 f_4 - f_2 f_3} = \frac{2 f_2 (f_1 + f_2)}{f_1 - f_2} \tag{21}$$

$$d_3^{\text{R}} = \lim_{f_2 \to f_3, f_4 \to f_1} \alpha \frac{1 - \sqrt{\dfrac{f_3 f_4}{f_1 f_2}}}{f_1 f_2 \left(1 - \frac{f_1 f_2}{f_3 f_4}\right)} = f_3 + f_4 = f_1 + f_2 \tag{22}$$

With Eqs. 20 and 22, d_1 reduces to $f_1 + f_2$ and d_3 to $f_3 + f_4$ which is equal to $f_1 + f_2$ under Rochester cloak conditions. Such result agrees with the findings of Howell and

Choi. In addition, d_2 simplifies into $2f_2(f_1 + f_2)/(f_1 - f_2)$, which is also consistent with the Rochester cloak. Equations for separation distances in the Rochester cloak can be recovered upon application of the necessary f to the formulas we presented. This fact provides proof that our equations are consistent with earlier findings.

4. Conclusions

We made an optical cloak with four lenses of different focal lengths by determining appropriate separation distances in terms of the lenses' focal lengths. In addition, we propose a criterion for minimizing the total length by making both f_2 and f_3 smaller compared to the values of f_1 and f_4, given that the values of f_2 and f_3 as well as those of f_1 and f_4 are close together. The equations for the separation distances we presented also reduce to the Rochester cloak calculations with the appropriate focal length conditions.

Given the cloak's straightforward calculations and method, it can be included in undergraduate optics experiments. Students may be asked to setup and solve the solution for the distances with Eqns. 14 to 17 together with the expression for d_{total}. This can serve as an introduction for physics undergraduates to analytical ray tracing method via the ABCD matrices. This is beyond the usual image formation computations that are currently used in undergraduate laboratory classes. Furthermore with focal length constraints relaxed, the cloaking phenomenon is now more easily replicable. Teachers can readily obtain distances between accessible lenses. This way, cloaking can be displayed on optics exhibits and in-class demonstrations effortlessly. A setup that cloaks is usually a major draw in exhibits. It can be an opportunity to educate and excite young minds.

Acknowledgements

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