Bifilar, trifilar pendulum theory.   July 31, 2006

SIDE VIEW

L

2r

A bifilar (2-filament) pendulum is shown in the sketch. A mass of overall length 2r is suspended by two filaments (light cords) each of length L. Normally, one wishes the filament length L to be longer than 2r, maybe 3 or 4 times 2r. For a symmetric object (like a uniform bar) it is set into motion in and out of the page (the left end goes into the page as the right end comes out of the page).

To find out the frequency of oscillation we refer to a top view, and apply torque = I α, where α is the angular acceration. *The z-axis is up out of the page in the top view shown at the right.* The torque is calculated by r x F. At equibrium, the net torque on the body about its cm is zero. The force on each end is given by ½ mg in the +z direction, and when r from the cm is crossed into the force at the end, we obtain a torque at each end. At the right end the torque is down (-y direction) and at the left end the torque is up (+y-direction). These two add vectorially to zero, just as they should when we are at equilibrium.

TOP
VIEW

r

y

x   z (out of page)
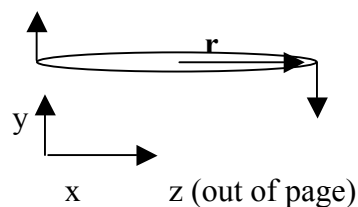
When the object is rotated through a small angle, the force continues to be mostly upward due to the tension in the filament, but there is an additional tension component.

The object is now shown as a bar rotated through a small clockwise angle. The additional tension components are shown at each end of the bar. When r from the cm is crossed into the extra tension at the right end of the bar, we find a torque component in the +z direction. When r from the cm to the other end of the bar is crossed into the extra tension at the left end, we find a torque component in the +z direction also. These torques add instead of cancelling, and we have a net torque due to the twisting of the bar before we release it to start rotating.

F_d

d

The small displacement at either end of the bar is d, and length of the bar is 2r, so the angle α through which the bar is rotated is given by tan $\theta \cong$ d/r . Since at small angles tan x $\cong$ x, we have $\theta \cong$ d/r.

The extra force exerted by each filament is F_d when we displace the rod by d. The extra torque is $r \times F_d$, a magnitude of $r\, F_d \sin(90^\circ - \theta)$. This is approximately equal to $r\, F_d$. Since r = length/2, and because there are two torques adding, the total torque is length $F_d = 2r\, F_d$.

To obtain an expression for F_d we need to refer to the sketch at the right, which shows a side view of the forces after the bar has been twisted at each end by a displacement d.
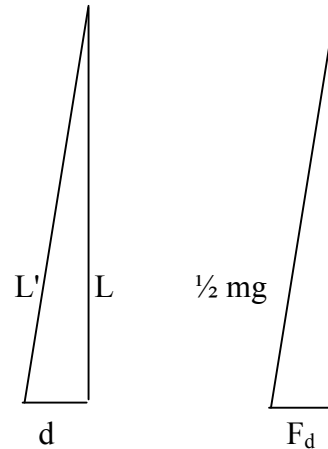
The left-hand triangle shows the distances L and d involved, while the right-hand trangle shows the forces involved in one filament, ½ mg and F_d. Because the triangles are similar, we can say $d/L' = \frac{1}{2}\, mg / F_d$. For small d, L is nearly equal to L' (try d = 0.03 m, L = 1 m and see how big L' is)

$$F_d = \frac{1}{2}\, mg\, d/L .$$

Then the total torque magnitude is

$$|\text{torque}| = \frac{1}{2}\, 2r\, mg\, d/L .$$

From the triangles in the horizontal plane we found out that $d = r\theta$, so we can write the torque as

$$\text{torque} = -(mg\, r^2/L)\, \theta .$$

(We put in the minus sign since the torque tends to decrease the angle θ.) Since torque $= I\,\alpha$, we now write

$$\text{torque} = I\,\alpha, \text{ or}$$

$$-mg\, r^2/L\ \theta = I\, d^2\theta/dt^2 ,$$

where I is the rotational inertia of the object with respect to its cm. This equation is just like that of the mass-spring system

$$-k\, x = m\, d^2x/dt^2 ,$$

with $k = mg\, r^2/L$, $m = I$, and θ playing the role of x. For a mass-spring the oscillation frequency is

$$\omega_{\text{mass-spring}} = \sqrt{(k/m)},$$

so for the bifilar pendulum we finally have (with T = pendulum period, 2r = distance between supports, L = vertical length from ceiling to cm)

$$\omega_{\text{bifilar}} = \sqrt{\{mgr^2/(IL)\}} = 2\pi/T .$$