

Topic 4 | Viscosity and Fluid Flow

In Section 14.6 we briefly discussed some of the consequences of *viscosity*, the friction forces that act within a moving fluid. Let's see how to give a quantitative measure of the viscosity of a fluid. Once we have this, we'll be able to motivate an important relation called *Poiseuille's equation* that governs the flow of viscous fluids in pipes. We'll also see how viscosity affects the motion of a solid object through a fluid.

To see how to measure viscosity, let's consider the simplest example of viscous flow: the motion of a fluid between two parallel plates (Fig. T4.1). The bottom plate is stationary, and the top plate moves with constant velocity \vec{v} . The fluid in contact with each surface has the same velocity as that surface. The flow speeds of intermediate layers of fluid increase uniformly from one surface to the other, as shown by the arrows, so the fluid layers slide smoothly over one another; the flow is laminar.

A portion of the fluid that has the shape $abcd$ at some instant has the shape $abc'd'$ a moment later and becomes more and more distorted as the motion continues. That is, the fluid is in a state of continuously increasing *shear strain*. To maintain this motion, we have to apply a constant force F to the right on the upper plate to keep it moving and a force of equal magnitude toward the left on the lower plate to hold it stationary. If A is the surface area of each plate, the ratio F/A is the *shear stress* exerted on the fluid.

In Section 11.4 we defined shear strain as the ratio of the displacement dd' to the length l (see Fig T11.16 on page 419). In a solid, shear strain is proportional to shear stress. In a fluid the shear strain *increases continuously and without limit* as long as the stress is applied. The stress depends not on the shear strain, but on its *rate of change*. The rate of change of strain, also called the *strain rate*, equals the rate of change of dd' (the speed v of the moving surface) divided by l . That is,

$$\text{Rate of change of shear strain} = \text{strain rate} = \frac{v}{l}$$

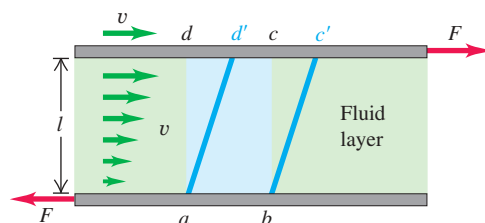
We define the **viscosity** of the fluid, denoted by η ("eta"), as the ratio of the shear stress, F/A , to the strain rate:

$$\eta = \frac{\text{Shear stress}}{\text{Strain rate}} = \frac{F/A}{v/l} \quad (\text{definition of viscosity}) \quad (\text{T4.1})$$

Rearranging Eq. (T4.1), we see that the force required for the motion in Fig. T4.1 is directly proportional to the speed:

$$F = \eta A \frac{v}{l}. \quad (\text{T4.2})$$

Fluids that flow readily, such as water or gasoline, have smaller viscosities than "thick" liquids such as honey or motor oil. Viscosities of all fluids are



T4.1 Laminar flow of a viscous fluid. The plates above and below the fluid each have area A . Viscosity is the ratio of shear stress F/A to strain rate v/l .

strongly temperature dependent, increasing for gases and decreasing for liquids as the temperature increases. An important goal in the design of oils for engine lubrication is to *reduce* the temperature variation of viscosity as much as possible.

From Eq. (T4.1) the unit of viscosity is that of force times distance, divided by area times speed. The SI unit is

$$1 \text{ N} \cdot \text{m} / [\text{m}^2 \cdot (\text{m/s})] = 1 \text{ N} \cdot \text{s} / \text{m}^2 = 1 \text{ Pa} \cdot \text{s}.$$

The corresponding cgs unit, $1 \text{ dyn} \cdot \text{s} / \text{cm}^2$, is the only viscosity unit in common use; it is called a **poise**, in honor of the French scientist Jean Louis Marie Poiseuille (pronounced “pwa-zoo-yuh”):

$$1 \text{ poise} = 1 \text{ dyn} \cdot \text{s} / \text{cm}^2 = 10^{-1} \text{ N} \cdot \text{s} / \text{m}^2.$$

The centipoise and the micropoise are also used. The viscosity of water is 1.79 centipoise at 0°C and 0.28 centipoise at 100°C . Viscosities of lubricating oils are typically 1 to 10 poise, and the viscosity of air at 20°C is 181 micropoise.

For a *Newtonian fluid* the viscosity η is independent of the speed v , and from Eq. (T4.2) the force F is directly proportional to the speed. Fluids that are suspensions or dispersions are often *non-Newtonian* in their viscous behavior. One example is blood, which is a suspension of corpuscles in a liquid. As the strain rate increases, the corpuscles deform and become preferentially oriented to facilitate flow, causing η to decrease. The fluids that lubricate human joints show similar behavior.

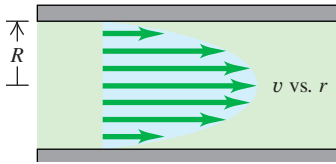
Figure T4.2 shows the flow speed profile for laminar flow of a viscous fluid in a long cylindrical pipe. The speed is greatest along the axis and zero at the pipe walls. The motion is like a lot of concentric tubes sliding relative to one another, with the central tube moving fastest and the outermost tube at rest. By applying Eq. (T4.1) to a cylindrical fluid element, we could derive an equation describing the speed profile. We’ll omit the details; the flow speed v at a distance r from the axis of a pipe with radius R is

$$v = \frac{p_1 - p_2}{4\eta L} (R^2 - r^2), \quad (\text{T4.3})$$

where p_1 and p_2 are the pressures at the two ends of a pipe with length L . The speed at any point is proportional to the change of pressure per unit length, $(p_2 - p_1)/L$ or dp/dx , called the *pressure gradient*. (The flow is always in the direction of decreasing pressure.) To find the total volume flow rate through the pipe, we consider a ring with inner radius r , outer radius $r + dr$, and cross-section area $dA = 2\pi r dr$. The volume flow rate through this element is $v dA$; the total volume flow rate is found by integrating from $r = 0$ to $r = R$. The result is

$$\frac{dV}{dt} = \frac{\pi}{8} \left(\frac{R^4}{\eta} \right) \left(\frac{p_1 - p_2}{L} \right) \quad (\text{Poiseuille's equation}) \quad (\text{T4.4})$$

This relation was first derived by Poiseuille and is called **Poiseuille’s equation**. The volume flow rate is inversely proportional to viscosity, as we would expect. It is also proportional to the pressure gradient $(p_2 - p_1)/L$, and it varies as the *fourth power* of the radius R . If we double R , the flow rate increases by a factor of 16. This relation is important for the design of plumbing systems and hypodermic needles. Needle size is much more important than thumb pressure in determining the flow rate from the needle; doubling the needle diameter has the same effect as increasing the thumb force sixteenfold. Similarly, blood flow in arteries and veins can be controlled over a wide range by relatively small changes



T4.2 Velocity profile for a viscous fluid in a cylindrical pipe.

in diameter, an important temperature-control mechanism in warm-blooded animals. Relatively slight narrowing of arteries due to arteriosclerosis can result in elevated blood pressure and added strain on the heart muscle.

One more useful relation in viscous fluid flow is the expression for the force F exerted on a sphere of radius r moving with speed v through a fluid with viscosity η . When the flow is laminar, the relationship is simple:

$$F = 6\pi\eta rv \quad (\text{Stokes's law}) \quad (\text{T4.5})$$

We encountered this kind of velocity-proportional force in Section 5.3. Equation (T4.5) is called **Stokes's law**.

Example T4.1

Terminal speed in a viscous fluid

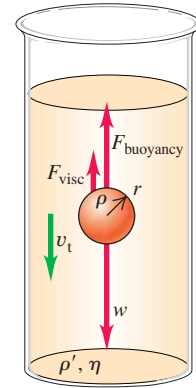
Derive an expression for the terminal speed v_t of a sphere falling in a viscous fluid in terms of the sphere's radius r and density ρ and the fluid viscosity η , assuming that the flow is laminar so that Stokes's law is valid.

SOLUTION

As in Example 5.20 in Section 5.3, the terminal speed is reached when the total force is zero, including the weight of the sphere, the viscous retarding force, and the buoyant force (Fig. T4.3). Let ρ and ρ' be the densities of the sphere and of the fluid, respectively. The weight of the sphere is then $\frac{4}{3}\pi r^3 \rho g$, and the buoyant force is $\frac{4}{3}\pi r^3 \rho' g$. At terminal speed,

$$\begin{aligned} \sum F_y &= F_{\text{buoyancy}} + F_{\text{visc}} + (-mg) \\ &= \frac{4}{3}\pi r^3 \rho' g + 6\pi\eta r v_t - \frac{4}{3}\pi r^3 \rho g = 0, \\ v_t &= \frac{2}{9} \frac{r^2 g}{\eta} (\rho - \rho'). \end{aligned}$$

We can use this formula to *measure* the viscosity of a fluid. Or, if we know the viscosity, we can determine the radius of the sphere (or the approximate sizes of other small particles) by measuring the terminal speed. Robert Millikan used this method to determine the radii of very small electrically charged oil drops by observing their



T4.3 A sphere falling in a viscous fluid reaches a terminal speed when the sum of the forces acting on it is zero.

terminal speeds in air. He used these drops to measure the charge of an individual electron.

The terminal speed for a water droplet of radius 10^{-5} m in air is of the order of 1 cm/s; the Stokes's-law force is what keeps clouds in the air. At higher speeds the flow often becomes turbulent, and Stokes's law is no longer valid. In air the drag force at highway speeds is approximately proportional to v^2 . We discussed the applications of this relation to skydiving in Section 5.3 and to air drag of a moving automobile in Topic 2 (Automotive Power).