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Moment of Inertia Measurement with a Bifilar Pendulum

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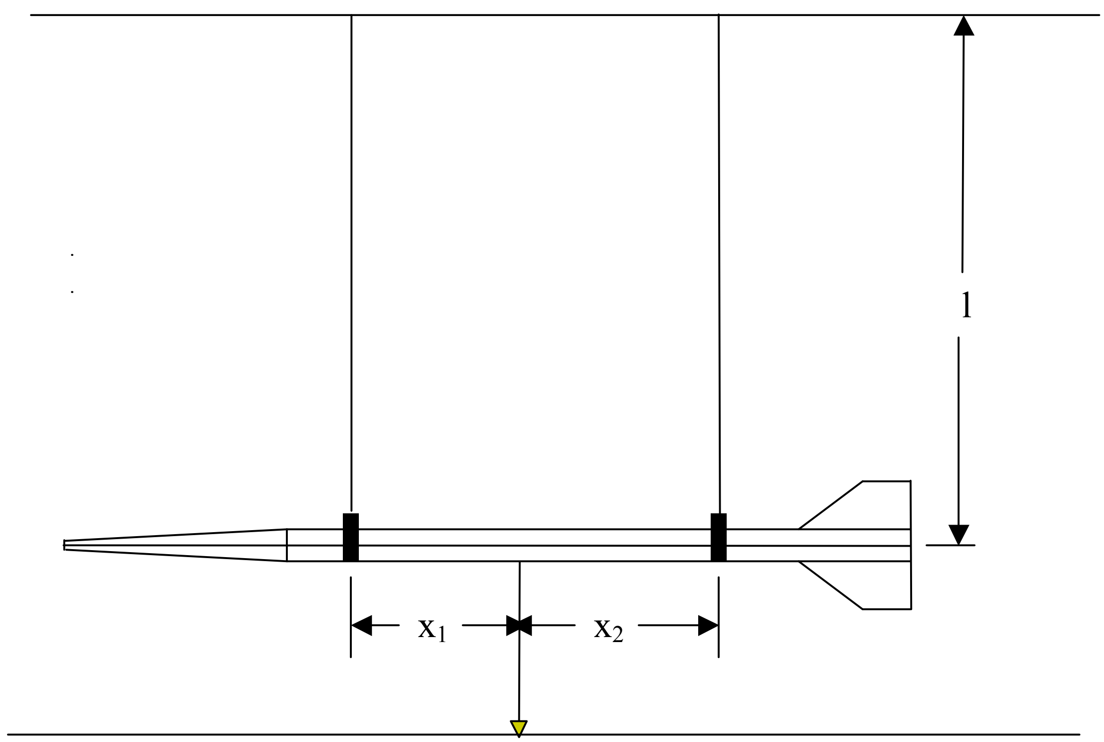
Introduction

The pitch/yaw moment of inertia is an important parameter affecting rocket short period dynamics. It can, of course, be computed analytically by summing the contributions of all the individual elements. This would be analogous to estimating mass and center of gravity from design data and layout drawings.

This note, however, describes a little known experimental technique for measuring the as-built moment of inertia. The bifilar pendulum technique dates to the earliest days of aviation, and is analogous to weighing the flight hardware to measure its mass.

Experiment Arrangement

A sketch of the set-up is shown below. The rocket is suspended by two wires of equal length that hold it level. It is necessary for the rocket center of gravity to lie between the suspensory wires. A plumb bob is attached to the rocket at its center of gravity, and an indicial line is marked on the floor below the position of the rocket center line when it is at rest.



Plumb Bob

In operation, the rocket is yawed by a crewman at the nose and another at its tail, taking care to ensure the plumb bob is not displaced. After a short count down the rocket is released, and its yaw period measured with a stopwatch. It's good experimental practice to measure the period of many yaw oscillations to be able to average the error of estimating when the nose appears to cross the indicial line on the floor.

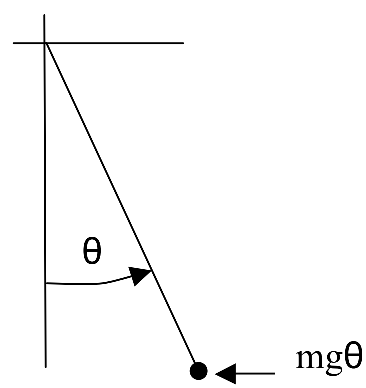
The description above glosses over the trap in all this. The author knows; he fell into it the first time he tried this technique. The thing is, a rocket suspended as described can oscillate in *two* distinct modes. In fact, the basic differential equations for the bifilar pendulum couple the two modes. If care is not exercised, we will measure the period of the wrong mode, and therefore get the wrong answer.

The Two Modes

The first mode is that of a simple Galilean pendulum in which both the nose and the tail are on the same side of the indicial line at the same time. This is *not* the desired mode because it can tell us nothing about the moment of inertia! The second mode is a yawing mode with the nose and tail always on different sides of the indicial line. This is the one we want. The problem is that it's difficult to excite purely one or the other if their periods are close. Unfortunately, unless care is taken during design, this is likely to be what we get.

Analysis

Although it could be done, a coupled two mode analysis will not be presented here. In stead, consider the sketch below of a Galilean pendulum:



Now, consider the mass **m** suspended by a wire of length **l**. The wire tension must be, to first order, just **mg**. Taking the pendulum as a piece of rotating machinery, the torque due to gravity is just $-mg\theta l$. Then the dynamics are

$$m\, l^2\, d^2\theta/dt^2 = -m\, g\, \theta\, l,$$

or

$$\omega_G^2 = g / l.$$

Thus the natural frequency ω_G of a Galilean pendulum does not depend on its mass, but only on its length.

Next, consider the yawing motion of this two wire pendulum. The first step is to find the tension forces in the two wires. From ordinary statics,

$$\mathbf{T_1 = m\,g\,x_2 / (x_1 + x_2)} \text{ and } \mathbf{T_2 = m\,g\,x_1 / (x_1 + x_2)}$$

Now, assume an initial yaw displacement through an angle $\boldsymbol{\psi}$. At the forward wire, the lateral displacement is $\mathbf{x_1\,\psi}$. But, this is also $\mathbf{l\,\theta_1}$. Note that the angles $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$ are not the same unless $\mathbf{x_1 = x_2}$. Then,

$$\boldsymbol{\theta_1 = x_1\,\psi / l} \text{ and } \boldsymbol{\theta_2 = x_2\,\psi / l}.$$

The yaw torque is

$$\mathbf{T_1\,x_1\,\theta_1 + T_2\,x_2\,\theta_2 = m\,g\,\psi\,x_1\,x_2 / l}$$

Then, if $\mathbf{I}$ is the yaw moment of inertia, the natural frequency $\boldsymbol{\omega_Y}$ of the yawing mode is

$$\boldsymbol{\omega_Y^2 = m\,g\,x_1\,x_2 / I\,l},$$

and if $\mathbf{P_Y = 2\,\pi / \omega_Y}$ is the measured yaw mode period, the moment of inertia $\mathbf{I}$ is found to be

$$\mathbf{I = m\,x_1\,x_2\,P_Y^2 / P_G^2},$$

where $\mathbf{P_G}$ is the measured period of the rocket swung as a Galilean pendulum.

<u>Experiment Design</u>
The ratio of the two periods squared appears explicitly in the result above. As already noted, it is important that this ratio not be to close to unity. Recall that the radius of gyration $\mathbf{k}$ is defined by

$$\mathbf{I = m\,k^2}.$$

Then,

$$\mathbf{P_Y^2 / P_G^2 = k^2 / x_1\,x_2}.$$

For a uniform rod of length $\mathbf{L}$,

$$\mathbf{P_Y^2 / P_G^2 = L^2 / 12\,x_1\,x_2}$$

Now, suppose the wires were placed at the ends of the rod. Then, $\mathbf{P_Y / P_G = \sqrt{1/3} = 0.577}$. If the wires were half way between the centroid and the rod ends, $\mathbf{P_Y / P_G = 1.155}$. It's easy to see how a resonant condition can inadvertently occur. It's also clear that good experiment design implies that the wires be close to the rocket center of gravity.