Modeling radioactive decay with dice

The process of radioactive decay, of isotopes or particles, is fundamental to the universe and to particle physics. The characteristic exponential decay (and the related exponential growth) is found in lots of places in nature, anywhere the rate of change of something is proportional to the amount of that something. Radioactive isotopes, bacteria populations, investments in a bank account. The other characteristic of radioactive decay is its inherent randomness, which leads to some interesting issues in observing and measuring the decay, problems shared with other inherently random things such as political polling and the and its ubiquitous ±3% error and separately with (!) rolling dice. These activities use the dice process to give you a tactile, physical experience with what is really happening in these other processes that are nanoscopic or abstract.

There are two related activities here. Rolling one hundred dice several times, and rolling a single die multiple times, then repeating that a hundred times. Here, we've assumed you are able to do both, possibly saving the second one as an out-of-lab homework activity. It is reasonable that, due to time or material constraints, you choose to do only the second one or only the first. If you picked this up with the MINERvA muon decay activities, the second one is the most like what the students will experience looking at muon decay.

The two ways are first summarized here, together. The pages that follow break them into two separate activities, with specific instructions and questions for the group doing the activities.

First way: roll one hundred dice (multiple times) to represent a sample of a hundred radioactive isotopes decaying over time. This does a good job of illustrating the idea of "half-life" and the exponential decrease in the activity of a radioactive sample over time. The question you are asking is "how many remain after some time". Such samples might be medical isotopes, waste from your local nuclear reactor, or the uranium and thorium atoms that remain on earth long after they were formed ten billion years ago in supernovae.

Second way: roll a single die (multiple times) to represent how long a single muon lives, then do it again for the next muon, and the next, one hundred times. This is a good way to illustrate the range of lifetimes you will observe if you have a sample of unstable particles such as muons. The question you are quantifying is "how long (what range of times) does each particle live". Such samples are common in particle and nuclear physics.

In fact, these two illustrations are just two ways of looking at the same phenomena. After doing one or both activities, it should be clear that you are observing a random process, observing many at once, or one at a time doesn't matter, and they both lead to the exponential decay that is characteristic of the random process. This will also help overcome the problem that you can't "see" radioactive isotopes or muons; it makes these real but hidden details plain. As a bonus, you will also observe how the randomness also means you will get somewhat different results when you repeat a measurement with otherwise identical initial conditions, such as you might have noticed applies to political polling, and maybe baseball, among other everyday phenomena.

First way of doing the dice activity: one hundred at once.

You need: a large supply of dice (about one hundred), a cup or bucket large enough for them, a sheet of paper to record data, and some graph paper for graphing your results.

Imagine: you have a supply of some radioactive isotope that has a 1/6 chance of decaying in the next minute. How much of that isotope remains after six minutes? How about after 20 minutes? How much time does it take for approximately half the sample to decay?

Put all the dice in your cup or jar and roll them on the table.
Separate all the dice that turned up "1", these are the particles that decayed.
Count and record those that decayed and those that remain, measured after "one minute" passed.
Separate the decayed "ones" into a pile that you can look at later.
Consider: is the pile of decayed dice about as many as you expect? Exactly as many?
Put the others (2-6) back into the cup.

Repeat all these steps and record a measurement of what decayed during and what remains after two minutes.   Separate this next batch of dice that turned up "1" into its own pile next to the first – you will save the progression of decays so you can see the process visually, in addition to the numbers you have recorded.

Repeat again for the third minute.
Repeat again and again until they are all decayed.

You now have a sequence of piles you can see (and numbers recorded that you can graph) that represent how many decays happened during each minute. Describe you can see both these seemingly contradictory properties: a) the number of decays is proportional to the number available to decay and b) the number that decayed is random.

Lets represent the situation with a couple graphs. Graph the number that decayed in each minute, using a histogram or bar chart. Separately, graph the number that remain undecayed in the sample, using a histogram or bar chart.

Estimate from your graph, by counting, how much time it takes for approximately half the sample to decay. Draw a mark or an arrow on the horizontal axis of each graph indicating where this time is.

Look up the exponential decay function; if you have a graphing calculator or similar program, plot it with a constant of (0.16666666 = 1/6), in other words, plot: $e^{-(1/6)x}$. Does that function describe the data you graphed? If you have access to the right software, you might even be able to plot your data and ask the software to fit an exponential, but sketching this one by hand would be good enough. Does the inherent randomness of this process make it difficult to see the exponential nature, and if so, can you think of a change in your procedure

How do you feel about the idea that there was likely a die that remained and didn't decay for fifteen to twenty rolls? Is it possible that one could remain for a hundred or more rolls?

Second way of doing the dice activity: simulating one hundred separate muons

(If you did the first one as part of a group in class or lab, you might do the second one as homework.)

You need: one (or a few) dice, a bit of time, a few pieces of paper for recording your data, and a couple more for graphing.

Imagine: you are creating, and then observing an unstable particle like a muon or a neutron, and can measure how long each one of them lives, one by one. Each one has a 1/6 chance of decaying in the next one nanosecond, you will count the number of nanoseconds until each one decays.   Do you expect short lifetimes to be more common, or long lifetimes, or something in the middle? What is the longest you expect to see a particle live? Can you guess what the average (mean) lifetime will be, when you are done and have analyzed all the decays?

Take your die and count the number of rolls until you roll a "1", and record it on your piece of paper.

Repeat about 100 times, each time recording the number of rolls until you roll the number "1". You have some quality time during this activity to consider the questions mentioned in the "imagine" paragraph above, and what you learned from activity one. (You could divide the effort among three or four people and get it done in 1/3 or 1/4 the amount of time, if you are in a hurry.)

Lets graph the results. Look at the data, and count the number of times where the muon lasted up to one nanosecond. Record that on a graph in the 0-1 interval. Count the number of times it took two rolls, and record that in the 1-2 interval. You are making a histogram of the decay time. Continue until you have filled all the intervals; toward the end many of them will be filled with zeros, with occasional intervals where a few long-lived particles fall.

Now compute the average life, in nanoseconds. Clearly you can average all 100 numbers you recorded by adding them and diving by 100. You might see that you can do this more quickly from your graph than you can from the sheet with 100 numbers on it, but either way works. Draw a mark on the horizontal axis of your graph to represent the average lifetime.

Look up the exponential decay function; if you have a graphing calculator or similar program, plot it with a constant of (0.16666666 = 1/6), in other words, plot: $e^{-(1/6)x}$. Does that function describe the data you graphed?

Does the inherent randomness of this process make it difficult to see the exponential nature, and where is it the most difficult? Why?

How do you feel about the idea that there was likely a muon that remained and didn't decay for fifteen to twenty rolls? Is it possible that one could remain for a hundred or more rolls?

Now what are your answers to these questions: do you expect short lifetimes to be more common, or long lifetimes, or something in the middle? What is the longest you expect to see a particle live?

If you have done both exercises, your one graph from the second exercise is most like which graph from the first exercise? And it should look approximately like the other graph from the first exercise, but 6 times smaller (!).   In these cases, they should look similar enough that you can recognize the similarity, but they certainly won't be identical.

How would you explain why they should look so similar but not identical?

What would happen if you repeated one or the other exercise again, would the results be identical?   For both exercises, you had one die that lasted the longest, but might you ever expect to see a particle last ten-times that long?   What would happen if you had the luxury of doing 1,000 trials, or 1,000,000 trials, or an Avagadro's number (6.02 x 1023) number of trials?

The function that best describes these graphs is a decaying exponential $Ne^{-t/\tau}$, (sometimes also written $Ne^{-\lambda t}$) where our situation is such that t is time, N is either the initial number of particles that might decay, or the initial activity which is 1/6 of the number of particles, and τ (tau) is the mean lifetime which is (!) 6 seconds or nanoseconds depending on which activity you are doing or λ which is the probability that a particle will decay in the next bit of time. This function occurs a lot for interesting random and non-random situations, but random processes often give rise to behavior that can be described mathematically like this.

Notice, in the second activity you directly measured the the value of this parameter τ tau = mean lifetime. If you recognized how the second activity is the same as the first, then you have again measured it. Go back to your first data set and calculate the mean lifetime there. Is it the same? Should it be? What is the accuracy of your measurement and the role of random fluctuations in your measurement? If you had a need to make a more accurate measurement, how would you do it?

Advanced questions.

You might notice that the spot where you marked "half-life" in the first activity is not at all in the same location as the spot you marked "mean-life" in the second activity. These are different. Think carefully about how you knew where to mark those spots and try to describe the subtle difference between them.

Political polling has been mentioned as another random process that we suppose you have encountered. Though it doesn't obviously give rise to an exponential distribution, but it clearly does give rise to fluctuations between repeated measurements of apparently the same thing, such as the public's support one or the other political candidate. If you don't expect two polling firms to get the same answer if they poll on the same night, how can you still draw conclusions from their results?

I teased that the same issues involving random processes might also apply to the game baseball. Think on that.

If you are comfortable enough with your calculus, you can derive the connection between the probability to decay and the exponential function. An internet search or textbook will get you started.

If you are comfortable enough with programming, and can find a pseudo-random number generator to use, you can explore further and faster than you can with dice. Using the same 1/6 probability, code the procedure into a program. Use a programming platform like C++, Java, Matlab, Mathematica, or possibly even a spreadsheet. When its ready, play with the mean lifetime and compare the exponential to high statistics histograms. Then play with changing the time step or probability.