A Fast Noniterative Algorithm for Compressive Sensing Using Binary Measurement Matrices

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Abstract

In this paper we present a new algorithm for compressive sensing that makes use of *binary* measurement matrices and achieves *exact* recovery of ultra sparse vectors, in a single pass and without any iterations. Due to its noniterative nature, our algorithm is hundreds of times faster than ℓ_1-norm minimization, and methods based on expander graphs, both of which require multiple iterations. Our algorithm can accommodate nearly sparse vectors, in which case it recovers index set of the largest components, and can also accommodate burst noise measurements. Compared to compressive sensing methods that are guaranteed to achieve exact recovery of *all* sparse vectors, our method requires fewer measurements. However, methods that achieve *statistical* recovery, that is, recovery of *almost* all but not all sparse vectors, can require fewer measurements than our method.

Keywords: Compressive Sensing, Ultra Sparse Vector Recovery, Deterministic Methods, Expander Graphs, Restricted Isometry Property, Basis Pursuit

1   Introduction

1.1   Definition of Compressive Sensing

Compressive sensing refers to the recovery of high-dimensional but low-complexity objects using a very small number of measurements. Some examples are the recovery of high-dimensional vectors that are sparse (with very few nonzero components) or nearly sparse, or high-dimensional matrices of low rank. The focus in this paper is on vector recovery. In order to make the discussion precise, we give an exact formulation of what "compressive sensing" means. Our terminology more or less follows that in [1, 2]. Throughout, n denotes

the dimension of the unknown vector, and $[n]$ denotes the set $\{1, \ldots, n\}$. The symbol $\operatorname{supp}(x) \subseteq [n]$ denotes the "support" of a vector $x \in \mathbb{R}^n$; that is

$$\operatorname{supp}(x) := \{i \in [n] : x_i \neq 0\}.$$

If $k < n$ is a specified integer, then $\Sigma_k \subseteq \mathbb{R}^n$ denotes the set of k-**sparse** vectors, that is

$$\Sigma_k := \{x \in \mathbb{R}^n : |\operatorname{supp}(x)| \leq k\}.$$

Suppose $\| \cdot \|$ is some specified norm on $\mathbb{R}^n$, and $k < n$ is a specified integer. Then the **sparsity index** $\sigma_k(x, \| \cdot \|)$ is defined by

$$\sigma_k(x, \| \cdot \|) := \min_{z \in \Sigma_k} \|x - z\|. \tag{1}$$

For a given $x \in \mathbb{R}^n$ and an integer $k < n$, the symbols $x_d \in \mathbb{R}^n$ and $x_r \in \mathbb{R}^n$ denote respectively the **dominant** part and the **residual** part of x. Thus x_d is the vector consisting of the k largest components by magnitude of x with the remaining components set equal to zero, and $x_r = x - x_d$. Note that, strictly speaking, we should write $x_{d,k}$ because the dominant part depends on the specified integer k, but we do not do this in the interests of less cluttered notation. It is obvious that, for any $p \in [1, \infty]$, we have that $\sigma_k(x, \| \cdot \|_p) = \|x_r\|_p$. Also, in case of "ties," when two or more components of x have the same magnitude, the symbol x_d can be defined in any consistent fashion.

Following the notation in [2], we view compressive sensing as consisting of two maps: A **measurement matrix** $A \in \mathbb{R}^{mn}$ where $m < n$ is the number of measurements, and a **decoding map** $\Delta : \mathbb{R}^m \to \mathbb{R}^n$.

Definition 1. *A pair (A, Δ) is said to achieve **exact sparse recovery** of order k if*

$$\Delta(Ax) = x, \ \forall x \in \Sigma_k. \tag{2}$$

*A pair (A, Δ) is said to achieve **stable sparse recovery** of order k if there exists a constant C such that*

$$\|\Delta(Ax) - x\|_2 \leq C\sigma_k(x, \| \cdot \|_1), \ \forall x \in \mathbb{R}^n. \tag{3}$$

*A pair (A, Δ) is said to achieve **robust sparse recovery** of order k if there exist constants C, D such that, whenever $\eta \in \mathbb{R}^m$ satisfies $\|\eta\|_2 \leq \epsilon$, we have that*

$$\|\Delta(Ax + \eta) - x\|_2 \leq C\sigma_k(x, \| \cdot \|_1) + D\epsilon, \ \forall x \in \mathbb{R}^n. \tag{4}$$

It is easy to verify that robust recovery implies stable recovery, which in turn implies exact recovery.

Note that in Definition 1, the various inequalities are required to hold for *all* vectors x. For want of a better phrase, this could be thought of as "guaranteed" sparse recovery. An alternate and weaker requirement would be "statistical" sparse recovery, in which the pair (A, Δ) recovers almost all, but not necessarily all, sparse vectors, with respect to a predefined probability measure on the set of vectors. In such a case the number of measurements m can be significantly reduced. We will describe this alternative in greater detail in Section 2.3.

1.2 Overview and Our Contributions

By now there are several approaches to the recovery of sparse vectors. If $x \in \mathbb{R}^n$ is an unknown sparse vector and $y = Ax \in \mathbb{R}^n$ is the measured vector, then the most "logical" approach to finding x would be to solve

$$\hat{x} = \underset{z}{\operatorname{argmin}} \|z\|_0 \text{ s.t. } Az = y,$$

where $\|z\|_0 = |\operatorname{supp}(z)|$ denotes the number of nonzero components of z. Unfortunately this problem is NP-hard, as shown in [3]. Therefore one can think of replacing the function $\|\cdot\|_0$ by its "convex envelope," which is the largest convex function that is dominated by $\|\cdot\|_0$. Using methods similar to those in [4], it can be shown that the convex envelope of $\|\cdot\|_0$ over the unit ball in $\|\cdot\|_\infty$ is $\|\cdot\|_1$. Consequently the most popular algorithm is ℓ_1-norm minimization with a measurement matrix A chosen to satisfy the so-called restricted isometry property (RIP), which is defined precisely in Section 2.1. The methods used for choosing the matrix A so as to satisfy the RIP can either be deterministic or probabilistic. Probabilistic methods lead to A matrices that do not have any structure; moreover, verifying whether such a probabilistically generated matrix satisfies the RIP condition is NP-hard [5]. In contrast, deterministically constructed matrices are often binary and thus readily verified to have the required properties, easy to implement, and faster than using random matrices. However, even with binary measurement matrices, convex optimization is far slower than greedy methods such as matching pursuit [6], orthogonal matching pursuit [7, 8], and CoSaMP [9]. The main disadvantage of greedy methods is that the known sufficient conditions for them to work are more stringent than those for convex relaxation methods to work. Another recent innovation, that combines the advantages of greedy algorithms with weak sufficient conditions of convex optimization, is based on the use of expander graphs [10, 11]. The measurement matrices in this approach are always binary and thus easy to implement. Further, the measurement matrices in this approach are expected to satisfy an analog of the RIP condition, known as ℓ_1-RIP. There is reason to believe that, at least in principle, the number of measurements in this approach can be order-optimal [12, 13].

The above discussion pertains to methods that lead to the recovery of *all* sufficiently sparse vectors. By weakening the requirement, and asking only that the algorithm be able to recover *almost* all sufficiently sparse vectors, the number of measurements drops drastically. Among such methods, approximate message passing (AMP) introduced in [14] is among the most thoroughly studied. The analysis in [14] shows that the recovery algorithm displays a *phase transition* whereby the performance of the algorithm undergoes an abrupt change. A readable survey of these results is given in [15]. In [16], a very general model is studied, wherein the unknown vector obeys a known probability distribution, the encoder is allowed to be nonlinear (as opposed to a linear map A), and the decoder is Lipschitz-continuous and may make use of the known probability distribution of the unknown vector. It is shown that, in the case of i.i.d. input processes, the phase-transition threshold for optimal encoding is given in terms of the Rényi information dimension of the input distribution. More details are given in Section 2.3. In [17], the procedure in [16] is incorporated into the approximate message passing (AMP) framework, along with the idea of spatial coupling taken from [18], and the phase transition properties of this algorithm are studied. In [19], a general theory

of phase transitions is developed for convex regularizers, which includes the widely used technique of ℓ_1-norm minimization, and a very sharp characterization of the phase transition boundary is given. Thus there is a fairly complete theory for the case of "statistical" signal recovery, which is only briefly touched upon here.

Against this backdrop, we now describe the contributions of the present paper and place them in perspective. In this paper we present a new *noniterative* algorithm for the recovery of vectors that are extremely sparse. Due to its noniterative nature, it is hundreds of times faster than both ℓ_1-norm minimization using binary measurement matrices, and methods based on expander graphs. Our algorithm works also for vectors that are nearly sparse but not exactly sparse, and/or measurements that are corrupted by noise. Our method requires slightly fewer measurements than either the RIP condition or the expander graph construction. However, this claim should be tempered by the results based on approximate message passing and the Rényi information dimension, which show that if one is ready to settle for the recovery of *almost all* sparse vectors, the required number of measurements is far smaller than those required by the RIP condition, and by inference, far smaller than those required by our method.

2 Literature Review

2.1 Compressive Sensing via ℓ_1-Norm Minimization

One of the most popular approaches to compressive sensing is ℓ_1-norm minimization. This approach is originally introduced in [20, 21] as a heuristic and is called "basis pursuit." The method consists of defining the decoding map Δ as

$$\Delta(y) = \hat{x} := \underset{z}{\operatorname{argmin}} \|z\|_1 \text{ s.t. } y = Az \tag{5}$$

in the case where $y = Ax$, and

$$\Delta(y) = \hat{x} := \underset{z}{\operatorname{argmin}} \|z\|_1 \text{ s.t. } \|y - Az\|_2 \le \epsilon \tag{6}$$

in the case where $y = Ax + \eta$ with $\|\eta\|_2 \le \epsilon$. In several papers over the years, beginning with [22, 23], it is shown that ℓ_1-norm minimization achieves robust sparse recovery in the sense of Definition 1 if the matrix A satisfies suitable conditions. At present, the two most popular sufficient conditions for ℓ_1-norm minimization to achieve robust sparse recovery are the restricted isometry property (RIP), and the robust null space property (RNSP). It is shown in [24] that RIP implies the RNSP, so we restrict our attention to the RIP.

Definition 2. *A matrix $A \in \mathbb{R}^{mn}$ is said to satisfy the **restricted isometry property (RIP)** of order k with constant δ if*

$$(1 - \delta)\|u\|_2^2 \le \|Au\|_2^2 \le (1 + \delta)\|u\|_2^2, \ \forall u \in \Sigma_k. \tag{7}$$

Available results show that ℓ_1-norm minimization achieves robust sparse recovery provided the measurement matrix A satisfies the RIP with a sufficiently small constant. The definitive results in this direction are derived in [25].

Theorem 1. *(See Theorems 1.1 and 2.1 of [25].) Suppose that, for some number $t > 1$, the matrix A satisfies the RIP of order $\lceil tk \rceil$ with constant $\delta < \sqrt{(t-1)/t}$. Then ℓ_1-norm minimization as in* (6) *achieves robust sparse recovery.*

Theorem 2. *(See [25, Theorem 2.2].) Suppose $t \geq 4/3$. Then for all $\xi > 0$ and all $k \geq 5/\xi$, there exists a matrix A that satisfies the RIP of order tk with constant $\delta_{tk} < \sqrt{(t-1)/t} + \xi$, and a vector $x \in \Sigma_k$ such that*

1. *With the noise-free measurement $y = Ax$, the decoder map Δ defined in* (5) *fails to recover x.*

2. *With a noisy measurement $y = Ax + \eta$ where $\|\eta\|_2 \leq \epsilon$, the decoder map Δ defined in* (6) *fails to recover x.*

This raises the question as to how one may construct measurement matrices that satisfy the RIP. There are two distinct approaches to this, namely probabilistic, and deterministic. In the probabilistic approach, A is chosen to equal $(1/\sqrt{m})\Phi$, where Φ is an $m \times n$ matrix consisting of independent samples of a sub-Gaussian random variable, such as Bernoulli or any finite-valued random variable, or a normal random variable. Such a construction leads to a matrix A that satisfies the RIP *with high probability* which can be made close to, but not equal to, one. Moreover, as shown in [5], verifying whether *a particular* randomly generated matrix A satisfies the RIP is NP-hard. An alternative that is gathering interest in recent times is the use of deterministic methods. The paper [26] is apparently the first to provide a deterministic method for constructing matrices that satisfy the RIP. This construction results in a binary matrix that is well-suited for implementation.

2.2 Compressive Sensing Using Expander Graphs

A recent development is the application of ideas from algebraic coding theory to compressive sensing. In [27], a method called "sudo-codes" is proposed, which is based on low density parity check (LDPC) codes, which are well-established in coding theory. The sudo-codes method can recover sparse signals with high probability. Motivated by this method, Xu and Hassibi in [10] proposed a method based on expander graphs, which are a special type of bipartite graph. For the convenience of the reader, the definition of an expander graph is recalled next.

The object under study is an *undirected bipartite graph*, consisting of a set $\mathcal{V}_I$ of input vertices, a set $\mathcal{V}_O$ of output vertices, and an edge set $\mathcal{E} \subseteq \mathcal{V}_O \times \mathcal{V}_I$, where $(i, j) \in \mathcal{E}$ if and only if there is an edge between node $i \in \mathcal{V}_O$ and node $j \in \mathcal{V}_I$. The corresponding matrix $A \in \{0, 1\}^{|\mathcal{V}_O| \times |\mathcal{V}_I|}$ is called the **bi-adjacency matrix** of the bipartite graph. The graph is said to be **left-regular** of degree D, or D-**left regular**, if every input node has degree D. This is equivalent to requiring that every column of the bi-adjacency matrix A has exactly D elements equal to 1. Given an input vertex $j \in \mathcal{V}_I$, let $\mathcal{N}(i) \subseteq \mathcal{V}_O$ denote the set of its neighbors, defined as

$$\mathcal{N}(j) := \{i \in \mathcal{V}_O : (i, j) \in \mathcal{E}\}.$$

Given set of input vertices $S \subseteq \mathcal{V}_I$, the set of its neighbors $\mathcal{N}(S) \subseteq \mathcal{V}_O$ is defined as

$$\mathcal{N}(S) := \bigcup_{j \in S} \mathcal{N}(j) = \{i \in \mathcal{V}_O : \exists j \in S \text{ s.t. } (i, j) \in \mathcal{E}\}.$$

Definition 3. *A D-left regular bipartite graph $(\mathcal{V}_I, \mathcal{V}_O, \mathcal{E})$ is said to be a $(K, 1-\beta)$-**expander** for some integer K and some number $\beta \in (0, 1)$ if, for every $S \subseteq V_I$ with $|S| \leq K$, we have that $|N(S)| \geq (1 - \beta)D|S|$.*

It can be shown [28] that randomly generated left-regular graphs are expanders. The next theorem is a paraphrase of [1, Theorem 13.6] in the current notation; note that this theorem is based on [28].

Theorem 3. *Given integers d, m, n with $d < m < n$, let $\mathcal{B}(m, n, d)$ denote the set of d-left-regular bipartite graphs. Suppose an integer $K < m/d$ and real numbers $\beta \in (0, 1)$, $\epsilon \in (0, 0.5)$ are specified. Define*

$$d = \left\lceil \frac{1}{\beta} \ln \left(\frac{en}{2\epsilon} \right) \right\rceil, m = \lceil \exp(2/\beta)dK \rceil, \tag{8}$$

where e denotes the base of the natural logarithm. Then the fraction of graphs in $\mathcal{B}(m, n, d)$ that are $(K, 1 - \beta)$ expanders exceeds $1 - \epsilon$.

Note that the above theorem is not very useful, because testing whether *a given* randomly generated left-regular graph is a $(K, 1 - \beta)$-expander or not would require us to compute the neighbors of $\binom{n}{K}$ sets of vertices. While this number is "polynomial" in n, it would be impractically large.

In [10], Xu and Hassibi introduce a new signal recovery algorithm in which the bi-adjacency matrix of an expander graph with $\beta \leq 1/4$ is used as the measurement matrix. It is referred to here as the "Expander Recovery Algorithm." Xu and Hassibi show that their algorithm recovers an unknown k-sparse vector x *exactly* in $O(k \log n)$ iterations. Subsequently, their method was updated in [11] by increasing the expansion factor from $1 - 1/4 = 3/4$ to $1 - \epsilon$ in which $\epsilon < 1/4$. With this change, it is shown that the number of recovery iterations required is $O(k)$. However, the number of measurements is more than in the Xu-Hassibi algorithm.

Expander Recovery Algorithm

1: Initialize $x - 0_{n \times 1}$
2: **if** $Y = Ax$ **then return** output x and exit
3: **else**
4: find a variable x_j such that at least $(1 - 2\epsilon)$D of the measurements it participates in have identical gap g
5: $x_j \leftarrow x_j + g$ and go to step 2
6: **end if**

In the algorithm above, the term g is called the *gap* and it determines the amount of information of the unknown signal that is missing in the estimate. The gap is defined as following:

$$g_i = y_i - \sum_{j=1}^{n} A_{ij} x_j$$

in which x, y and A are the unknown signal, the measurement vector and the measurement matrix, respectively.

2.3 Statistical Recovery and Phase Transitions

The preceding two subsections were devoted to two methods for *guaranteed* recovery of all sparse vectors. For such methods, the restricted isometry property (RIP) is the most popular sufficient condition. It is known that, if the measurement matrix A is generated in a probabilistic fashion, the RIP holds with high probability with $m = O(k \ln(n/k))$. In contrast, with deterministic methods, the number of measurements m is typically $O(\max\{k^2, n^{1/2}\})$. However, in practice deterministic methods often require fewer measurements. Further details can be found in the Appendix.

If the requirement is relaxed from *guaranteed* recovery to recovery with high probability, or *statistical* recovery, then there is a parallel body of research showing that the number of measurements m can be reduced quite substantially. In fact $m = O(k)$ measurements suffice. In this subsection, we highlight just a few of the many papers in this area of research. To streamline the presentation, the papers are not always cited in chronological order.

In [16], the underlying assumption is that the unknown vector x is generated according to a known probability distribution p_X, which can in fact be used by the decoder. Three different dimensions of the probability distribution p_X are introduced, namely the Rényi information dimension, the MMSE dimension, and the Minkowski dimension. The encoder is permitted to be nonlinear, in contrast to earlier cases where the encoding consisted of multiplication by a measurement matrix. The decoder is also permitted to be nonlinear but is assumed to be Lipschitz continuous. The optimal performance in this setting is analyzed. A central result in this paper states that, asymptotically as the vector dimension n and the number of measurements m both approach infinity, statistical recovery is possible *if and only if*

$$m \geq n\bar{d}(p_X) + o(n),$$

where $\bar{d}(p_X)$ denotes the Rényi information dimension of p_X. Since the Rényi information dimension is comparable to the ratio k/n, the above result states that $O(k)$ measurements are sufficient. However, no procedure is given to construct an encoder-decoder pair.

In a series of papers [29, 14, 30], Donoho and various co-workers studied "phase transitions" in the performance of various recovery algorithms. A readable survey of these results is given in [15]. The unknown n-vector is assumed to be k-sparse, and the measurement vector $y \in \mathbb{R}^m$ equals Ax, where A consists of samples of normal random variables, scaled by the normalization factor $1/\sqrt{m}$. Two quantities are relevant here, namely the "undersampling rate" $\delta = m/n$, and the sparsity $\rho = k/m$. In all of these papers, the aim is to show that for each algorithm there exists a sharp threshold $\rho_\theta(\delta)$ such that, if $\rho > \rho_\theta(\delta)$, then the

unknown vector is recovered with probability approaching one, whereas if $\rho < \rho_\theta(\delta)$, then the algorithm *fails* with probability approaching one.

Specifically in [14] an algorithm known as "approximate message passing" (AMP) is analyzed. AMP is a simple thresholding type of algorithm that is much faster than minimizing the ℓ_1-norm. Specifically, suppose $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth "threshold" function, and extend it to a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ by applying it component-wise. The AMP algorithm begins with an initial guess $x^0 = 0$, and then one sets

$$x^{t+1} = \phi(A^\top w^t + x^t),$$

$$w^t = y - Ax^t + \frac{1}{\delta} w^{t-1}(\phi'(A^\top w^{t-1} + x^{t-1})),$$

where ϕ' denotes the derivative of ϕ. It is clear that AMP is much faster than ℓ_1-norm minimization. Despite this, it is shown in [14] that the phase transition behavior of AMP is comparable to that of ℓ_1-norm minimization. In [17], the AMP algorithm is modified to incorporate the results of [16], and phase transition results are derived. In this paper, the authors also introduce the idea of "spatial coupling" introduced in [18].

Finally, in [19], the authors study a very general class of algorithms. Suppose as before that $y \in \mathbb{R}^m$ equals Ax, where A consists of samples of normal random variables, scaled by the normalization factor $a/\sqrt{m}$. The decoding algorithm is

$$\hat{x} = \underset{z}{\operatorname{argmin}} f(z) \text{ s.t. } y = Az,$$

where the "regularizer" $f(\cdot)$ is a convex function satisfying some technical conditions. So this theory applies to ℓ_1-norm minimization. In this paper, a central role is played by the "descent cone" of f at a point x, which is defined as

$$\mathcal{D}(f, x) := \bigcup_{\tau > 0} \{h \in \mathbb{R}^n : f(x + \tau h) \le f(x)\}.$$

It is clear that $\mathcal{D}(f, x)$ is indeed a cone. Next, for each cone, a quantity called the "statistical dimension," denoted by δ, is defined; see [19, Section 2.2] for a precise definition. With all these items in place, a central result is established; see [19, Theorem II].

Theorem 4. *Define* $a(\epsilon) := \sqrt{8 \log(4/\epsilon)}$. *With all other symbols as above, if*

$$m \le \delta(\mathcal{D}(f, x)) - a(\epsilon)\sqrt{n},$$

then the decoding algorithm fails with probability $\ge 1 - \epsilon$. *If*

$$m \ge \delta(\mathcal{D}(f, x)) + a(\epsilon)\sqrt{n},$$

then the decoding algorithm succeeds with probability $\ge 1 - \epsilon$.

3 The New Algorithm

Now we present our new algorithm, and show that it can exactly recover sparse signals in a single pass, without any iterations. Then we analyze the performance of the algorithm when the true but unknown vector is not exactly sparse, and/or the measurement is corrupted by noise. The performance of our algorithm is compared with those of ℓ_1-norm minimization and expander graph algorithms in the next section.

3.1 The New Algorithm

Suppose a matrix $A \in \{0,1\}^{m \times n}$ has the following properties, referred to as the **main assumption**:

1. Every column a_j of A has precisely q entries of 1 and $m-q$ entries of 0.

2. If a_j, a_t are distinct columns of A, then $\langle a_j, a_t \rangle \leq r-1$.

Suppose $x \in \Sigma_k$ is a k-sparse n-dimensional vector, and define $y = Ax$ to be the measurement vector. For a given index $j \in [n]$, let $\{v_1(j), \ldots, v_q(j)\} \subseteq [m]$ denote the q rows such that $a_{ij} = 1$. For an index $j \in [n]$, the **reduced measurement vector** $\bar{y}_j \in \mathbb{R}^q$ is defined as

$$\bar{y}_j := [y_{v_1(j)} \ldots y_{v_q(j)}]^\top.$$

Note that $\bar{y}_j$ is the vector consisting of the q measurements in which the component x_j participates.

The main result is given next. Recall that $\|v\|_0$ denotes the number of nonzero components of a vector v.

Theorem 5. *Suppose $x \in \Sigma_k$, $y = Ax$. Then:*

1. *If $j \notin supp(x)$, then $\|\bar{y}_j\|_0 \leq k(r-1)$.*

2. *If $j \in supp(x)$, then $\bar{y}_j$ contains at least $q - (k-1)(r-1)$ components that are all equal to x_j.*

Proof. For $t \in [n]$, let $\mathbf{e}_t \in \mathbb{R}^n$ denote the t-th canonical basis vector, which has a 1 as its t-th element, and zeros elsewhere, and let $\mathbf{1}_q \in \mathbb{R}^q$ denote the column vector consisting of all ones. Then we can write:

$$x = \sum_{t \in \text{supp}(x)} x_t \mathbf{e}_t,$$

$$y = Ax = \sum_{t \in \text{supp}(x)} x_t A\mathbf{e}_t = \sum_{t \in \text{supp}(x)} x_t a_t,$$

where a_t denotes the t-th column of A. Therefore, for a fixed $j \in [n]$ and $l \in [q]$, we have that

$$y_{v_l(j)} = \sum_{t \in \text{supp}(x)} x_t (a_t)_{v_l(j)}.$$

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Letting l range over $[q]$ shows that

$$\bar{y}_j = \sum_{t \in \text{supp}(x)} x_t(\overline{a_t})_j, \tag{9}$$

where $(\overline{a_t})_j$ is the reduced vector of a_t consisting of $(a_t)_{v_1(j)}, \ldots, (a_t)_{v_q(j)}$.

Proof of (1): Suppose $j \notin \text{supp}(x)$. Then $j \neq t$ for all $t \in \text{supp}(x)$. Therefore, according to item (ii) of the main assumption, we have that $\langle a_j, a_t \rangle \leq r-1$. Recall that $v_1(j), \ldots, v_q(j)$ are the row indices of column j that contain a 1. Therefore, for a fixed index $t \neq j$, the number of 1's in the set $\{(a_t)_{v_1(j)}, \ldots, (a_t)_{v_q(j)}\}$ equals the inner product $\langle a_j, a_t \rangle$ and thus cannot exceed $r - 1$. Therefore, for a fixed index $t \in \text{supp}(x)$, the vector $x_t(\overline{a_t})_j$ contains no more than $r - 1$ nonzero entries. Substituting this fact into (9) shows that $\bar{y}_j$ is the sum of at most k vectors (because x is k-sparse), each of which has no more than $r - 1$ nonzero entries. Therefore $\|\bar{y}_j\|_0 \leq k(r-1)$.

Proof of (2): Suppose $j \in \text{supp}(x)$. Then we can write

$$\bar{y}_j = \sum_{t \in \text{supp}(x)} x_t(\overline{a_t})_j \tag{10}$$

$$= x_j \mathbf{1}_q + \sum_{t \in \text{supp}(x) \setminus \{j\}} x_t(\overline{a_t})_j, \tag{11}$$

because the "reduced vector" $(\overline{a_j})_j$ consists of q 1's, as denoted by $\mathbf{1}_q$. By the same reasoning as in the proof of (1), it follows that

$$\left\| \sum_{t \in \text{supp}(x) \setminus \{j\}} x_t(\overline{a_t})_j \right\|_0 \leq (k-1)(r-1).$$

Therefore at least $q - (k-1)(r-1)$ terms in $\bar{y}_j$ equal x_j. $\square$

In view of Theorem 5, we can formulate an algorithm for the recovery of k-sparse vectors, as follows:

Note that there is no iterative process involved in the recovery – the estimate $\hat{x}$ is generated after *a single pass* through all n indices.

Theorem 6. *If x is k-sparse, and A satisfies the main assumption with $q > 2k(r-1)$, then $\hat{x} = x$.*

Proof. Note $q > 2k(r-1)$ implies that

$$k(r-1) < q/2, q - (k-1)(r-1) > q - k(r-1) > q/2.$$

Therefore, by Statement 1 of Theorem 5, it follows that if $j \notin \text{supp}(x)$, then $\|\bar{y}_j\|_0 \leq k(r-1) < q/2$. Taking the contrapositive shows that if $\|\bar{y}_j\|_0 \geq q/2$, then $j \in \text{supp}(x)$. Therefore, by Statement 2 of Theorem 5, it follows that at least $q - (k-1)(r-1) > q - k(r-1) > q/2$ elements of $\bar{y}_j$ equal x_j. $\square$

Next we present the extension of our basic algorithm to the cases of a sparse signal with measurement noise, and a nearly sparse signal.

New Recovery Algorithm

1: **for** $j \in [n]$ **do**
2:     Construct the *reduced measurement vector* $\bar{y}_j$.
3:     Find the number of the elements of $\bar{y}_j$ that are nonzero; call it ν.   ▷ (In implementation, we find the number of elements that are greater than some tolerance δ.)
4:     **if** $\nu > q/2$ **then**
5:         Find a group of $q/2$ elements in $\bar{y}_j$ that are equal; call this value θ_j.   ▷ (In implementation, we allow some tolerance here.)
6:         $\hat{x}_j = \theta_j$.
7:     **else**
8:         $\hat{x}_j = 0$
9:     **end**
10: **end**

3.2 Recovery of Sparse Signals with Measurement Noise

In previous work, the model for noisy measurements is that $y = Ax + \eta$ where there is a prior bound of the form $\|\eta\|_2 \leq \epsilon$. If $x \in \Sigma_k$, then $\sigma_k(x, \|\cdot\|_1) = 0$. Therefore, if robust sparse recovery is achieved, then the bound in (4) becomes $\|\hat{x} - x\|_2 \leq D\epsilon$. However, our approach draws its inspiration from coding theory, wherein it is possible to recover a transmitted signal correctly provided the transmission is not corrupted in too many places. Therefore our noise model is that $\|\eta\|_0 \leq M$. In other words, it is assumed that a maximum of M components of the "true" measurement Ax are corrupted by additive noise, but there are no assumptions regarding the *magnitude* of the error signal η. In this case it is shown that, by increasing the number of measurements, it is possible to recover the true sparse vector x *perfectly*.

Theorem 7. *Suppose $x \in \Sigma_k$, and that $y = Ax + \eta$ where $\|\eta\|_0 \leq M$. Suppose further that the matrix A satisfies the main assumption. Then*

    1. If $j \notin supp(x)$, then $\bar{y}_j$ contains no more than $k(r-1) + M$ nonzero components.

    2. If $j \in supp(x)$, then $\bar{y}_j$ contains at least $q - [(k-1)(r-1) + M]$ components that are all equal to x_j.

    3. Suppose the new recovery algorithm is applied with a measurement matrix A that satisfies the main assumption with $q > 2[k(r-1) + M]$. Then $\hat{x} = x$.

Proof. Suppose $x \in \Sigma_k$ and let $y = Ax + \eta$ where A satisfies the main assumption and $\|\eta\|_0 \leq M$. Let $u = Ax$ denote the uncorrupted measurement. For a fixed index $j \in [n]$, let $\bar{y}_j \in \mathbb{R}^q$ denote the reduced measurement vector, consisting of the components $y_{v_1(j)}$ through $y_{v_q(j)}$, and define $\bar{u}_j \in \mathbb{R}^q$ and $\bar{\eta}_j \in \mathbb{R}^q$ analogously.

First suppose $j \notin supp(x)$. Then it follows from Item (1) of Theorem 5 that $\|\bar{u}_j\|_0 \leq k(r-1)$. Moreover, because η has no more than M nonzero components and $\bar{\eta}_j$ is a sub-vector of η, it follows that $\|\bar{\eta}_j\|_0 \leq M$. Therefore

$$\|\bar{y}_j\|_0 = \|\bar{u}_j + \bar{\eta}_j\|_0 \leq \|\bar{u}_j\|_0 + \|\bar{\eta}_j\|_0 \leq k(r-1) + M.$$

This is Item (1) above. Next, suppose that $j \in \text{supp}(x)$. Then it follows from Item (1) of Theorem 5 that at least $q - (k - 1)(r - 1)$ elements of $\bar{u}_j$ equal x_j. Because $\|\bar{\eta}_j\|_0 \leq M$, it follows that at least $q - (k - 1)(r - 1) - M$ components of $\bar{y}_j$ equal x_j. This is Item (2) above. Finally, if $q > 2k(r-1) + 2M$, it follows as in the proof of Theorem 6 that $\hat{x} = x$. □

Note that the assumption on the noise signal η can be modified to $\|\bar{\eta}_j\|_0 \leq M$ for each $j \in [n]$. In other words, instead of assuming that η has no more than M nonzero components, one can assume that every reduced vector $\bar{\eta}_j$ has no more than M nonzero components.

3.3 Recovery of Nearly Sparse Signals

As before, if $x \notin \Sigma_k$, then let $x_d \in \mathbb{R}^n$ denote the projection of x onto its k largest components, and let $x_r = x - x_d$. We refer to x_d, x_r as the dominant part and the residual respectively. Note that, for any $p \in [1, \infty]$, we have that the sparsity index $\sigma_k(x, \|\cdot\|_p)$ equals $\|x_r\|_p$. To (nearly) recover such a vector, we modify the New Recovery Algorithm slightly. Let δ be a specified threshold.

Modified Recovery Algorithm

1: **for** $j \in [n]$ **do**
2:   Construct the *reduced measurement vector* $\bar{y}_j$.
3:   Find the number of the elements of $\bar{y}_j$ that are greater than δ in magnitude; call it ν.
4:   **if** $\nu > q/2$ **then**
5:     Find a group of $q/2$ elements in $\bar{y}_j$ such that the difference between the largest and smallest elements is no larger than 2δ; Let θ_j denote the average of these numbers.
6:     $\hat{x}_j = \theta_j$.
7:   **else**
8:     $\hat{x}_j = 0$
9:   **end**
10: **end**

Theorem 8. *Suppose $x \in \mathbb{R}^n$ and that $\sigma_k(x, \|\cdot\|_1) \leq \delta$. Write $x = x_d + x_r$ where x_d is the dominant part of x consisting of its k largest components, and $x_r = x - x_d$ is the residual. Let $y = Ax$ where A satisfies the main assumption with $q > 2k(r-1)$, and apply the modified recovery algorithm. Then (i) $\text{supp}(\hat{x}) = \text{supp}(x_d)$, and (ii) $\|\hat{x} - x_d\|_\infty \leq \delta$.*

Remark: If ℓ_1-norm minimization is used to recover a *nearly sparse* vector using (5), then the resulting estimate $\hat{x}$ need not be sparse, and second, the support set of the dominant part of $\hat{x}$ need not equal the support set of the dominant part of x.

Proof. Write $x = x_d + x_r$ where x_d consists of the dominant part of x and x_r consists of the residual part. By assumption, $\|x_r\|_1 \leq \delta$. Note that the measurement y equals

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$Ax = Ax_d + Ax_r$. Let $u = Ax_d$ and observe that $x_d \in \Sigma_k$. Further, observe that, because the matrix A is binary, we have that the induced matrix norm

$$\|A\|_{1\to\infty} := \sup_{v\neq 0} \frac{\|Av\|_\infty}{\|v\|_1} = \max_{i,j} |a_{ij}| = 1.$$

Therefore $\|Ax_r\|_\infty \leq \|x_r\|_1 \leq \delta$. Now, by Item (1) of Theorem 5, we know that if $j \notin \text{supp}(x_d)$, then no more than $k(r-1)$ components of the reduced vector $\bar{u}_j$ are nonzero. Therefore then no more than $k(r-1)$ components of the reduced vector $\bar{y}_j$ have magnitude more than δ. By Item (2) of Theorem 5, we know that if $j \in \text{supp}(x_d)$, then at least $q - (k-1)(r-1)$ components of $\bar{u}_j$ equal x_j. Therefore at least $q - (k-1)(r-1)$ components of $\bar{y}_j$ lie in the interval $[x_j - \delta, x_j + \delta]$. Finally, if $q > 2k(r-1)$, then there is only one collection of $q - (k-1)(r-1) > q/2$ components of the reduced vector $\bar{y}_j$ that lie in an interval of width 2δ. The true x_j lies somewhere within this interval, and we can set $\hat{x}_j$ equal to the midpoint of the interval containing all of these components. In this case $|\hat{x}_j - x_j| \leq \delta$. Because this is true for all $j \in \text{supp}(x_d)$, it follows that (i) $\text{supp}(\hat{x}) = \text{supp}(x_d)$, and (ii) $\|\hat{x} - x_d\|_\infty \leq \delta$. $\qquad\square$

Finally, it is easy to combine the two proof techniques and to establish the following theorem for the case where x is not exactly sparse and the measurements are noisy.

Theorem 9. *Suppose $x \in \mathbb{R}^n$ and that $\sigma_k(x, \|\cdot\|_1) \leq \delta$. Write $x = x_d + x_r$ where x_d is the dominant part of x consisting of its k largest components, and $x_r = x - x_d$ is the residual. Let $y = Ax + \eta$ where $\|\eta\|_0 \leq M$, and A satisfies the main assumption with $q > 2k(r-1) + 2M$. Apply the modified recovery algorithm. Then (i) $\text{supp}(\hat{x}) = \text{supp}(x_d)$, and (ii) $\|\hat{x} - x_d\|_\infty \leq \delta$.*

3.4   Construction of a Binary Measurement Matrix

The results presented until now show that the key to the procedure is the construction of a binary matrix A that satisfies the main assumption. In this subsection, it is shown that previous work by DeVore [26] provides a simple recipe for constructing a binary matrix with the desired properties. Note that [26] was the first paper to propose a completely deterministic procedure for constructing a matrix that satisfies the restricted isometry property. It is shown in this section that DeVore's matrix is also a special case of the bi-adjacency matrix of an expander graph. Therefore the DeVore matrix acts as a bridge between two distinct compressive sensing algorithms.

We now describe the construction in [26]. Suppose q is a prime number or a power of a prime number, and let $\mathbb{F}_q$ denote the finite field with q elements. Suppose a is a polynomial of degree $r - 1$ or less with coefficients in $\mathbb{F}_q$, and define its "graph" as the set of all pairs $(x, a(x))$ as x varies over $\mathbb{F}_q$. Now construct a vector $u_a \in \{0,1\}^{q^2 \times 1}$ by setting the entry in row (i, j) to 1 if $j = a(i)$, and to zero otherwise. To illustrate, suppose $q = 3$, so that $\mathbb{F}_q = \{0, 1, 2\}$ with arithmetic modulo 3. Let $r = 4$, and let $a(x) = 1 + 2x + x^2 + x^3$. With this choice, we have that $a(0) = 1$, $a(1) = 2$, and $a(2) = 2$. The corresponding 9×1 column vector has 1's in positions $(0, 1), (1, 2), (2, 2)$ and zeros elsewhere. This construction results in a $q^2 \times 1$ column vector u_a that consists of q blocks of size $q \times 1$, each of which contains a single 1 and $q - 1$ zeros. Therefore u_a contains q elements of 1 and the rest equal to zero.

Now let $\Pi_{r-1}(\mathbb{F}_q)$ denote the set of all polynomials of degree $r-1$ or less with coefficients in $\mathbb{F}_q$. In other words,

$$\Pi_{r-1}(\mathbb{F}_q) := \left\{ a(x) = \sum_{i=0}^{r-1} a_i x^i, a_i \in \mathbb{F}_q \right\}.$$

Note that $\Pi_{r-1}(\mathbb{F}_q)$ contains precisely q^r polynomials, because each of the r coefficients can assume q different values.[1] Now define

$$A := [u_a, a \in \Pi_{r-1}(\mathbb{F}_q)] \in \{0,1\}^{q^2 \times q^r}. \tag{12}$$

The following theorem from [26] shows that the matrix A constructed as above satisfies the main assumption, and also the RIP with appropriately chosen constants.

Theorem 10. *(See [26, Theorem 3.1]) For the matrix $A \in \{0,1\}^{q^2 \times q^r}$ defined in (12), we have that*

$$\langle u_a, u_b \rangle \leq r - 1 \tag{13}$$

whenever a, b are distinct polynomials in $\Pi_{r-1}(\mathbb{F}_q)$. Consequently, if we define the column-normalized matrix $A' = (1/\sqrt{q})A$, then A satisfies the RIP of order k with constant $\delta_k \leq ((k-1)(r-1))/q$.

In Theorem 3 it is shown that randomly generated left-regular graphs are expanders, but this result is not particularly useful. It is therefore of interest to have available methods that are *guaranteed* to generate expander graphs, even if the number of output vertices is larger than with random constructions. One such procedure is given in [31]. We now describe this construction, and then show that the DeVore construction is a special case of it. The construction in [31] is as follows: Let $h \geq 2$ be any integer. Then the map $\Gamma : \mathbb{F}_q^r \times \mathbb{F}_q \to \mathbb{F}^{s+1}$ is defined as

$$\Gamma(f, y) := [y, f(y), f^h(y), f^{h^2}(y), \ldots, f^{h^{s-1}}(y)]. \tag{14}$$

An alternate way to express the function Γ is:

$$\Gamma(f, y) = [y, (f^{h^i}(y), i = 0, \ldots, s-1)].$$

In the definition of the function Γ, y ranges over $\mathbb{F}_q$ as the "counter," and the above graph is left-regular with degree q. The set of input vertices is $\mathbb{F}_q^r$, consisting of polynomials in some indeterminate Y with coefficients in $\mathbb{F}_q$ of degree no larger than $r-1$. The set of input vertices has cardinality q^r. The set of output vertices is $\mathbb{F}^{s+1}$ and each output vertex is an $(s+1)$-tuple consisting of elements from $\mathbb{F}_q$. The set of output vertices has cardinality q^{s+1}. Note that the graph is q-left regular in that every input vertex has exactly q outgoing edges.

Theorem 11. *(See [31, Theorem 3.3].) For every pair of integers h, s, the bipartite graph defined in (14) is a $(h^s, 1-\beta)$-expander with*

$$\beta = \frac{(r-1)(h-1)s}{q} \tag{15}$$

whenever

$$h < \frac{q}{s(r-1)} + 1.$$

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[1] If the leading coefficient of a polynomial is zero, then the degree would be less than r.

Note that the inequality simply ensures that $\beta > 0$.

Now we relate the construction of DeVore with that in [31].

Theorem 12. *The matrix A constructed in [26] is a special case of the graph in Theorem 11 with $s = 1$, and any value for h. Therefore a bipartite graph with the biadjacency matrix of [26] is a $(h, 1 - \beta)$-expander with*

$$\beta = \frac{(r-1)(h-1)}{q} \tag{16}$$

whenever

$$h < \frac{q}{r-1} + 1.$$

Proof. Suppose that $s = 1$ and that h is any integer. In this case each polynomial f with coefficients in $\mathbb{F}_q$ of degree $r - 1$ or less gets mapped into the pair $(y, f(y))$ as y ranges over $\mathbb{F}_q$. This is precisely what was called the "graph" of the polynomial f in [26]. $\square$

4   Computational Results

Theorems 10 and 12 show that the measurement matrix construction proposed in [26] falls within the ambit of both the restricted isometry property as well as expander graphs. In other words, the binary DeVore's measurement matrix satisfies both RIP-2 and RIP-1 due to its construction and expander graph nature, respectively. Hence this matrix can be used together with ℓ_1-norm minimization, the expander graph algorithm of Xu-Hassibi, as well as our proposed algorithm. In this section we compare the performance of all three algorithms using the DeVore construction. Note however that the number of rows of the matrix (or equivalently, the number of measurements) will vary from one method to another. This is discussed next.

4.1   Number of Measurements Required by Various Methods

In this subsection we compare the number of measurements required by ℓ_1-norm minimization, expander graphs, and our method.

In ℓ_1-norm minimization, as shown in Theorem 10, the matrix A, after column normalization dividing each column by $\sqrt{q}$, satisfies the RIP with constant $\delta_k = (k-1)(r-1)/q$. Combined with Theorem 1, we conclude that ℓ_1-norm minimization with the DeVore construction achieves robust k-sparse recovery whenever

$$\frac{(\lceil tk \rceil - 1)(r-1)}{q} < \sqrt{\frac{t-1}{t}}. \tag{17}$$

To maximize the value of k for which the above inequality holds, we set r to its minimum permissible value, which is $r = 3$. Also, we replace $\lceil tk \rceil - 1$ by its upper bound tk, which leads to

$$\frac{2tk}{q} < \sqrt{\frac{t-1}{t}}, \text{ or } \frac{2k}{q} < \sqrt{\frac{t-1}{t^3}}.$$

Method	q	m
ℓ_1-norm min.	$\lceil \max\{6k, n^{2/3}\}\rceil_p$	q^2
Expander Graph	$\lceil \max\{8(2k-1), n^{2/3}\}\rceil_p$	q^2
New Algorithm	$\lceil \max\{4k, n^{2/3}\}\rceil_p$	q^2

Table 1: Number of Measurements for Various Approaches

Elementary calculus shows that the right side is maximized when $t = 1.5$. So the RIP constant of the measurement matrix must satisfy

$$\delta_{tk} < \sqrt{(t-1)/t} = 1/\sqrt{3} \approx 0.577.$$

Let us choose a value of 0.5 for δ_{tk} to give some "cushion." Substituting the values $t = 1.5, r = 3$ in (17) and ignoring the rounding operations finally leads to the condition

$$\frac{3k}{q} < 0.5, \text{ or } q > 6k. \tag{18}$$

For expander graphs, we can calculate the expansion factor $1 - \beta$ from Theorem 11. This gives

$$\beta = \frac{(r-1)(h-1)s}{q}.$$

Since we wish the expansion factor $1 - \beta$ to be as close to one as possible, or equivalently, β to be as small as possible, we choose s to be its minimum value, namely $s = 1$. Now we substitute $r = 3, h = 2k$ (following [10]), and set $1 - \epsilon \geq 3/4$, or equivalently $\epsilon \leq 1/4$. This leads to

$$\frac{2(2k-1)}{q} \leq 1/4, \text{ or } q \geq 8(2k-1).$$

Finally, for the new algorithm, it has already been shown that $q \geq 2(r-1)k = 4k$. Note that, since the matrix A has q^3 columns, we must also have that $n \leq q^3$.

The required number of measurements for each of the three algorithms are as shown in Table 1. To facilitate the presentation, we introduce the notation $\lceil x \rceil_p$ to denote the **smallest prime number** that is no smaller than x.

4.2 Computational Complexity

Because the new algorithm does not involve any iteration, it is very fast. In this subsection we analyze the number of arithmetic operations involved in implementing it. For each index $j \in [n]$, there are in essence two steps: First, to determine whether j belongs to the support of the unknown vector, and second, if j does belong to supp(x), to determine the value of x_j. This is achieved as follows: For each index $j \in [n]$, the reduced vector $\bar{y}_j$ is computed; then $\bar{y}$ is sorted in decreasing order of magnitude. If $\bar{y}_{(q+1)/2} = 0$, then $j \notin$ supp(x). If $\bar{y}_{(q+1)/2} \neq 0$, then the sorted vector is scanned over a window of width $(q+1)/2$, and an index a is chosen such that $\bar{y}_a = \bar{y}_{a+(q-1)/2}$. This is the value of x_j. Thus, for each index

Method	q	m
ℓ_1-norm min.	37	1,369
Expander graph	89	7,921
New algorithm	29	841

Table 2: Number of measurements required for the numerical examples with $n = 20,000$ and $k = 6$.

j, the most time-consuming step is to sort $\bar{y}$. Since $\bar{y}$ has q components, the complexity is $O(q \log q)$, and since $q = O(k)$, the complexity is $O(k \log k)$. Since this has to be done n times, the overall complexity is $O(nk \log k)$. Note that the algorithm is fully parallelizable, in that each index j can be processed separately and independently of the rest.

4.3   Numerical Examples

In this section we present a numerical example to compare the three methods. We chose $n = 20,000$ to be the dimension of the unknown vector x. Since all three methods produce a measurement matrix with $m = q^2$ rows, we must have $q < 141 \approx \sqrt{20000}$, because otherwise the number of measurements would exceed the dimension of the vector! Since the expander graph method requires the most measurements, the sparsity count k must satisfy $8(2k-1) < 141$, which gives $k \leq 9$. However, if we try to recover k-sparse vectors with $k = 9$ using the expander graph method, the number of measurements m would be essentially equal to the dimension of the vector n. Hence we chose value of $k = 6$. With this choice, the values of q and the number of measurements are shown in Table 2. Note that q must be chosen as a prime number.

Having chosen the values of n and k, we generated 100 different k-sparse n-dimensional vectors, with both the support set of size k and the nonzero values of x generated at random.[2] As expected, both the expander graph method and the new algorithm recovered the unknown vector x *exactly* in all 100 cases. The ℓ_1-norm minimization method recovers x with very small error. However, there was a substantial variation in the average time over the 100 runs. Our algorithm took an average of 0.0951 seconds, or about 95 milliseconds, ℓ_1-norm minimization took 21.09 seconds, and the expander-graph algorithm took 76.75 seconds. Thus our algorithm was about 200 times faster than ℓ_1-norm minimization and about 800 times faster than the expander-graph algorithm.

As a final example, we introduced measurement noise into the output. As per Theorem 7, if $y = Ax + \eta$ where $\|\eta\|_0 \leq M$, then it is still possible to recover x exactly by increasing the prime number q. (Note that it is also possible to retain the same value of q by reducing the sparsity count k so that $k + M$ is the same as before.) Note that the only thing that matters here is the number of nonzero components of the noise η, and not their magnitudes. One would expect that, if the norm of the noise gets larger and larger, our algorithm would continue to recover the unknown sparse vector exactly, while ℓ_1-norm minimization would not be able to. In other words, our algorithm is tolerant to "shot" noise whereas ℓ_1-norm

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[2]Matlab codes are available from the authors.

	New Algorithm		ℓ_1-norm minimization	
Alpha	Err.	Time	Err.	Time
10^{-5}	0	0.1335	3.2887e-06	26.8822
10^{-4}	0	0.1325	3.2975e-05	26.6398
10^{-3}	0	0.1336	3.3641e-04	28.1876
10^{-2}	0	0.1357	0.0033	23.1727
10^{-1}	0	0.1571	0.033	28.9145
10	0	0.1409	1.3742	26.6362
20	0	0.1494	1.3967	26.5336

Table 3: Performance of new algorithm and ℓ_1-norm minimization with additive shot noise

minimization is not. The computational results bear this out. We choose $n = 20,000$ and $k = 6$ as before, and $M = 6$, so that we perturb the true measurement Ax in six locations. Specifically we chose $\eta = \alpha v$ where each component of v is normally distributed, and then increased the scale factor α. Each experiment was repeated with 100 randomly generated sparse vectors and shot noise. The results are shown in Table 3.

5 Discussion and Conclusions

In this paper we have presented a new algorithm for compressive sensing that makes use of *binary* measurement matrices and achieves *exact* recovery of sparse vectors, *without any iterations*. Exact recovery continues to hold even when the measurements are corrupted by a noise vector with a sufficiently small support set; this noise model is reminiscent of the model used in algebraic coding. When the unknown vector is not exactly sparse, but is nearly sparse with a sufficiently small residual, our algorithm exactly recovers the support set of the dominant components, and finds an approximation for the dominant part of the unknown vector. Because our algorithm is non-iterative, it executes orders of magnitude faster than algorithms based on ℓ_1-norm minimization and methods based on expander graphs (both of which require multiple iterations). Moreover, our method requires a smaller number of measurements in comparison to these two approaches when the measurement matrix is binary. On test examples of k-sparse n-dimensional vectors with $k = 6$ and $n = 20,000$, our algorithm executes roughly 1,000 times faster than the Xu-Hassibi algorithm [10] based on expander graphs, and roughly 200 times faster than ℓ_1-norm minimization.

On the other hand, these two methods do have their own advantages over the algorithm proposed here. The Xu-Hassibi algorithm [10] and its extension in [11] can be used with *any* expander graph with an expansion factor that is sufficiently close to one. In contrast, our algorithm makes use of *a particular* family of expander graphs whose bi-adjacency matrix satisfies the "main assumption." Similarly, if ℓ_1-norm minimization is used to reconstruct a vector, then a bound of the form (4) holds no matter what the unknown vector is. In contrast, our error bounds require that the residual part of the unknown vector must be sufficiently small compared to the dominant part of the vector. This might not be a serious

drawback however, because the objective of compressive sensing is to recover nearly sparse vectors, and not arbitrary vectors.

Appendix

In this appendix, we compare the number of measurements used by probabilistic as well as deterministic methods to guarantee that the corresponding measurement matrix A satisfies the restricted isometry property (RIP), as stated in Theorem 1. Note that the number of measurements is computed from the best available sufficient condition. In principle it is possible that matrices with fewer rows might also satisfy the RIP. But there would not be any theoretical justification for using such matrices.

In probabilistic methods, the number of measurements m is $O(k \log(n/k))$. However, in reality the O symbol hides a huge constant. It is possible to replace the O symbol by carefully collating the relevant theorems in [1]. This leads to the following explicit bounds.

Theorem 13. *Suppose X is a random variable with zero mean, unit variance, and suppose in addition that there exists a constant c such that[3]*

$$E[\exp(\theta X)] \leq \exp(c\theta^2), \ \forall \theta \in \mathbb{R}. \tag{19}$$

Define

$$\gamma = 2, \zeta = 1/(4c), \alpha = \gamma e^{-\zeta} + e^{\zeta}, \beta = \zeta, \tag{20}$$

$$\tilde{c} := \frac{\beta^2}{2(2\alpha + \beta)}. \tag{21}$$

Suppose an integer k and real numbers $\delta, \xi \in (0,1)$ are specified, and that $A = (1/\sqrt{m})\Phi$, where $\Phi \in \mathbb{R}^{m \times n}$ consists of independent samples of X. Then A satisfies the RIP of order k with constant δ with probability $\geq 1 - \xi$ provided

$$m \geq \frac{1}{\tilde{c}\delta^2} \left(\frac{4}{3} k \ln \frac{en}{k} + \frac{14k}{3} + \frac{4}{3} \ln \frac{2}{\xi} \right). \tag{22}$$

In (22), the number of measurements m is indeed $O(k \log(n/k))$. However for realistic values of n and k, the number of measurements n would be comparable to, or even to exceed, n, which would render "compressive" sensing meaningless.[4] For "pure" Gaussian variables it is possible to find improved bounds for m (see Also, for binary random variables where X equals ± 1 with equal probability, another set of bounds is available [32]. While all of these bounds are $O(k \log(n/k))$, in practical situations the bounds are not useful.

This suggests that it is worthwhile to study *deterministic* methods for generating measurement matrices that satisfy the RIP. There are very few such methods. Indeed, the

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[3]Such a random variable is said to be **sub-Gaussian**. A normal random variable satisfies (19) with $c = 1/2$.

[4]In many papers on compressive sensing, especially those using Gaussian measurement matrices, the number of measurements m is *not* chosen in accordance with any theory, but simply picked out of the air.

authors are aware of only three methods. The paper [26] uses a finite field method to construct a binary matrix, and this method is used in the present paper. The paper [33] gives a procedure for choosing rows from a unitary Fourier matrix such that the resulting matrix satisfies the RIP. This method leads to the same values for the number of measurements m as that in [26]. Constructing partial Fourier matrices is an important part of reconstructing time-domain sparse signals from a limited number of frequency measurements (or vice versa). Therefore the results of [33] can be used in this situation. In both of these methods, m equals q^2 where q is appropriately chosen prime number. Finally, in [34] a method is given based on chirp matrices. In this case m equals a prime number q. Note that the partial Fourier matrix and the chirp matrix are complex, whereas the method in [26] leads to a binary matrix. In all three methods, $m = O(n^{1/2})$, which grows faster than $O(k \log(n/k))$. However, the constant under this O symbol is quite small. Therefore for realistic values of k and n, the bounds for m from these methods are much smaller than those derived using probabilistic methods.

Table 4 gives the values of m for various values of n and k. Also, while the chirp matrix has fewer measurements than the binary matrix, ℓ_1-norm minimization with the binary matrix runs much faster than with the chirp matrix, due to the sparsity of the binary matrix. In view of these numbers, in the present paper we used DeVore's construction as the benchmark for the recovery of sparse vectors.

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n	k	m_G	m_{SG}	m_A	m_D	m_C
10^4	5	5,333	28,973	3,492	841	197
10^4	6	5,785	31,780	3,830	1,369	257
10^4	7	6,674	37,308	4,496	1,681	401
10^4	8	7,111	40,035	4,825	2,209	487
10^4	9	7,972	45,424	5,474	2,809	677
10^4	10	8,396	48,089	5,796	3,481	787
10^5	10	10,025	57,260	6,901	3,481	787
10^5	12	11,620	66,988	8,073	5,041	1,163
10^5	14	13,190	76,582	9,229	6,889	1,601
10^5	16	14,739	86,061	10,372	9,409	2,129
10^5	18	16,268	95,441	11,502	11,449	2,707
10^5	20	17,781	104,733	12,622	16,129	3,371
10^6	5	7,009	38,756	4,671	10,201	1,009
10^6	10	11,639	66,431	8,006	10,201	1,009
10^6	15	16,730	96,976	11,687	10,201	1,949
10^6	20	21,069	123,076	14,832	16,129	3,371
10^6	25	25,931	152,373	18,363	22,201	5,477
10^6	30	30,116	177,635	21,407	32,041	7,753
10^6	50	47,527	283,042	34,110	94,249	21,911
10^6	60	55,993	334,440	40,304	128,881	31,687
10^6	70	64,335	385,171	46,417	175,561	43,271
10^6	80	72,573	435,331	52,462	229,441	56,659
10^6	90	80,718	484,992	58,447	292,681	71,837
10^6	100	88,781	534,210	64,378	358,801	88,807

Table 4: Best available bounds for the number of measurements for various choices of n and k using both probabilistic and deterministic constructions. For probabilistic constructions, the failure probability is $\xi = 10^{-9}$. m_G, m_{SG}, m_A denote respectively the bounds on the number of measurements using a normal Gaussian, a sub-Gaussian with $c = 1/2$, and a bipolar random variable and the bound of Achlioptas. For deterministic methods m_D denotes the number of measurements using DeVore's construction, while m_C denotes the number of measurements using chirp matrices.

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