Robustness of compressive Fourier-domain sampling against rounding-off errors and noise

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Abstract
We investigate the robustness of compressive Fourier-domain sampling against the effect of rounding-off errors and ambient noise. Compressive sampling (CS) is accomplished with a two-dimensional line mask that samples the low frequency components of a signal at the prescribed Nyquist rate while undersampling its associated high-frequency components. Generally, the signal energy and details are encoded mostly in the low and high-frequency components, respectively. Rounding-off errors arise when an analog signal is digitized with an *n*-bit analog-to-digital converter (ADC) that limits the dynamic range of the digital-signal amplitude representation to 2^n different possible values. We show that reducing the dynamic range from ADC bit-number n = 64 to n = 14, does not compromise the CS reconstruction performance even at an effective sampling rates that is only a fraction (1/10, 1/5 and 3/10) of the Nyquist rate in the presence of additive Fourier domain noise. Lowering the dynamic range from n = 14 to n = 1 produces a reconstruction error that increases linearly with decreasing n. This behavior is observed for different images and noise strengths.

Keywords: Compressive sampling, analog-to-digital conversion, rounding-off errors in signal recovery

1 Introduction

To capture accurately the features of a signal the Shannon-Nyquist sampling theorem requires that it is sampled at a rate that is at least twice its cut-off frequency or bandwidth. The Nyquist criterion is increasingly difficult to satisfy in cases that involve rapid kinetics and ultrafast phenomena [1]. This technical difficulty has led to the development of alternative sampling techniques such as the non-uniform sampling through interpolation [2] and compressive sampling (CS) [3].

CS is a non-adaptive method that is employed to directly acquire a signal at rates lower than the Nyquist value. It is based on the concept that real-world signals often have sparse representations in some basis and if sampled by a largely incoherent measurement process, they can be reconstructed accurately via optimization procedures that search for the sparsest solution even if sampled at rates that are below Nyquist [4]. L1 minimization is a common optimization procedure for finding the desired solution *x* and it is defined as:

$$x = \arg\min \|x'\|_1 \qquad Ax' = y \tag{1}$$

where *A* is the measurement matrix. The L1 norm of a vector x is the sum of the absolute values of its elements with:

$$\|x'\|_1 = \sum_0^{N-1} |x| \tag{2}$$

For two-dimensional (2D) signals, we use a revised optimization procedure where the reconstruction assumes that the gradient is sparse and the total variation (TV) of the reconstructed image is the one minimized where:

$$TV(x) = \sum_{ij} \sqrt{(D_{h;ij}x)^2 + (D_{v;ij}x)^2}; \text{ where } D_{h;ij}x = \begin{cases} x_{i+1,j} - x_{i,j} if i < N \\ 0 if i = N \end{cases} \text{ and } D_{v;ij}x = \begin{cases} x_{i,j+1} - x_{i,j} if j < N \\ 0 if j = N \end{cases} \tag{3}$$

We consider a 2D image where *x* denotes the ith and jth column of an *N* x *N* image [5]. Instead of Eqn (1), the constraint that is used for recovery is:

$$TV_1(x) \min TV(x)\, such that Ax' = y \tag{6}$$

We have shown earlier [6] that undersampling in the Fourier domain is robust against noise and that accurate reconstruction is possible for noise-free images even at only 3/10 of the Nyquist rate. The said findings assume access to complete information of the analog signal. In real-world applications however, the analog signal is first converted into its corresponding digital representation for storage and post-detection processing [7]. This is done by inputting the analog signal to an *n*-bit analog-to-digital converter (ADC) that limits the dynamic range of the digital-signal amplitude representation to 2^n different possible values where n is called the ADC bit-number. Choosing the ADC specifications is usually a trade-off between AD conversion speed and dynamic range resolution which is proportional to $(2^n)^{-1}$ [8]. Improving the sampling resolution is usually achieved at the expense of slower conversion time that limits the attainable signal sampling rate. CS is a promising alternative for observing transient events. Here we study the robustness of CS against the effect of rounding-off errors and ambient noise in the Fourier domain. Rounding-off errors are inevitable during ADC conversion that produces a 2^n digital representation of finite dynamic range.

2 Methodology

Figure 1a presents the grayscale Lenna test image used in our numerical experiments. Sampling is done in the Fourier domain by multiplying the complex-valued Fourier transform of the image with a line mask (Fig. 1b). The mask architecture features a number of equally spaced lines passing through the Fourier domain origin where the zero frequency is located. Increasing the line density of the mask corresponds to increasing the sampling rate and therefore the number of sampled points in the Fourier domain. The mask introduces a non-uniform sampling of the Fourier transform where the low-frequency components are sampled approximately at the prescribed Nyquist rate while undersampling the corresponding high frequency components. Normally, the signal energy and details are concentrated in the low and high-frequency components respectively.

The Fast Fourier Transform (FFT) of a 512 x 512 pixel image contains 512 x 512 real and imaginary Fourier transform components and mask-sampling is applied separately on the two components. The resulting sampled transforms are then projected back to the real domain and CS is utilized to minimize the total variation of the reconstructed image. For this work, Nyquist rate sampling means utilizing all the possible pixel values. Increasing the number of lines in the mask increases the area sampled in the Fourier domain. We consider mask line density values that yield equivalent sampling rates that are only fractions (1/10, 1/5 and 3/10) of the Nyquist rate.

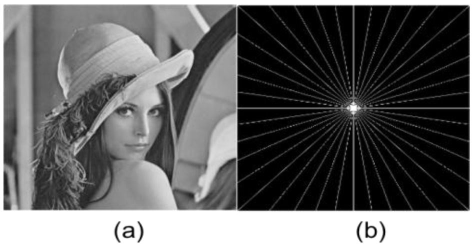

Figure 1: (a) Lenna test image (512 x 512 pixels) and b) Line mask (22 projections) that samples at 1/20 (5%) of Nyquist rate.

We note that sampling the image by low-pass filtering would totally exclude the image details that are mostly concentrated in the high-frequency regions. On the other hand, the line-mask samples the low frequency region at approximately the Nyquist rate while only undersampling the high-frequency region. The details that are missed during undersampling are then recovered via CS. Before projecting the masked Fourier transform into the real domain, we bin each real and imaginary component in order to simulate the dynamic range reduction due to n-bit AD conversion.

The minimum resolution of each bin is determined by dividing the maximum amplitude value by 2^n with n = 0, 2, 4, …, 64. Sampling the masked Fourier transform with an 8-bit ADC corresponds to binning at n = 8. To simulate a ladder type AD conversion with a dynamic range resolution that is proportional to 2^n, the quotient is rounded off to the nearest low integer value before multiplying it with the minimum resolution of each bin to get the new binned value.

$$x_{binned} = \frac{\lfloor x_{analog} \rfloor}{resolution} * resolution \text{ where } resolution = \frac{MaxFTAmplitude}{2^n} \tag{7}$$

We evaluate the CS reconstructions that correspond to equivalent sampling rates that are lower (1/10, 1/5 and 3/10) than Nyquist at different binning (2^n) values. The normalized root mean squared error (NRMSE) is used to compare the reconstruction quality with the original Lenna image:

$$NRMSE = \frac{1}{\max(K(i,j) - \min(K(i,j)}\sqrt{\frac{1}{mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[I(i,j) - K(i,j)]^2} \tag{8}$$

where indices i and j correspond to ith and jth column of an m x n image, K is the original image and I is the reconstructed image. Perfect reconstruction corresponds to: NRMSE = zero. Finally, we investigate the effect of uniform Gaussian white noise in the Fourier plane.

3. Results and Discussion

We first determine the effects of sampling resolution from n = 64 (analog signal equivalent) to n = 1. CS is applied on the Lenna image that is undersampled at an equivalent rate that is only 10% (1/10), 20% (1/5) and 30% (3/10) of the Nyquist value. Figure 2 plots the NRMSE as a function of the bit number n together with the NRMSE that results when the Fourier domain is sampled at the Nyquist rate.

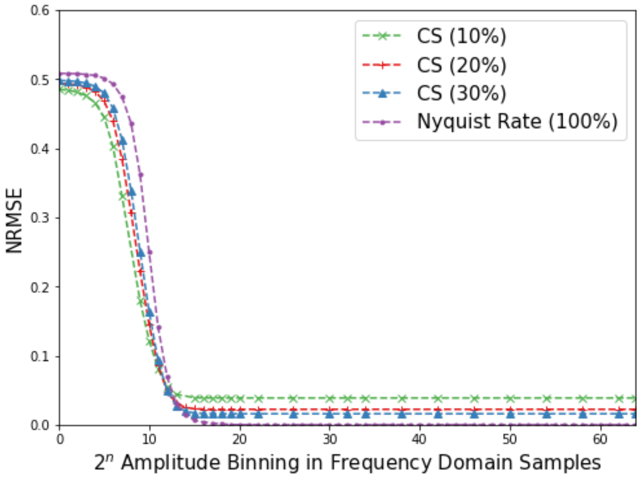

Figure 2: Reconstruction NRMSE versus ADC bit number n at 1/10, 1/5 and 3/10 of Nyquist rate.

We observe that AD conversion in the range: 14 ≤ n ≤ 64, produces only small variations in the image reconstruction NRMSE. At 10%, 20%, 30% and 100% sampling, the NRMSE standard deviation values are: 4.76 x 10^{-5}, 7.46 x 10^{-5}, 1.05 x 10^{-4} and 6.75 x 10^{-4}, respectively while the corresponding average NRMSEs are: 3.9325 x 10^{-2}, 2.2953 x 10^{-2}, 1.6524 x 10^{-2} and 7.24 x 10^{-4}. Reducing the sampling rate increases the NRMSE although the increase is negligible from full Nyquist to 1/10 Nyquist sampling rate. The CS image reconstruction obtained at 1/10 the Nyquist rate and binned at n = 64 is no different from the one that is directly sampled at Nyquist rate.

Figure 2 also reveals a threshold point (n = 14) where the NRMSE increases linearly with decreasing n. The linear dependence is observed at 20%, 30% and 100% Nyquist sampling with respective curve-fit R^2 values of 0.8601, 0.9736, 0.9802, and 0.9675. The plots also show that the deleterious effect of undersampling is uniformly reduced starting at at n = 13. AD conversion at n ≤ 5, produces a negligible difference in NRMSE values.

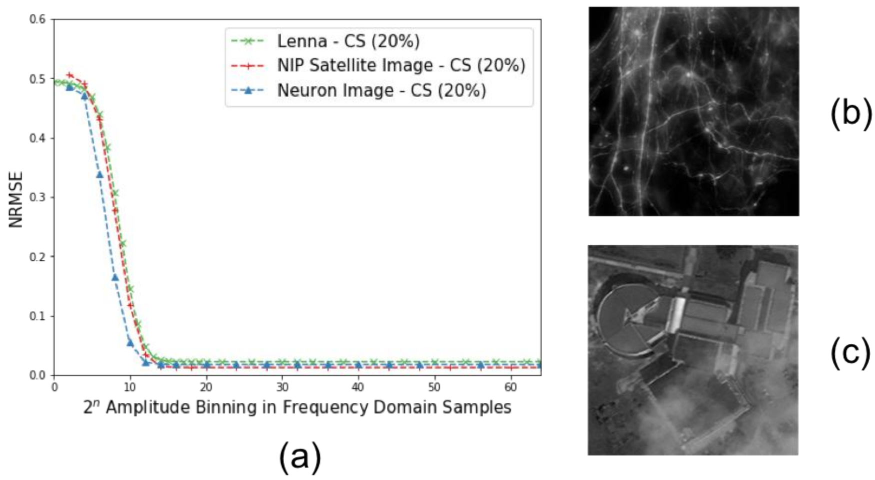

Figure 3: (a) NRMSE of CS reconstructions at 1/5 (20%) the Nyquist rate, (b) Neuron test image and (c) satellite image of the National Institute of Physics (source: zoom.earth).

Different images produce the same NRMSE profile for their CS reconstructions. We also tested a microscopic neuron and a satellite (Google Map) image (Fig. 3b - 3c) and observed stable reconstruction NRMSE value for n ≥ 14.

Lastly, we investigate the effects of additive zero-mean Gaussian noise [standard deviations = 0.1 (weak), 0.5 and 1.0 (strong)] in the Fourier plane. Lenna image reconstructions obtained at Nyquist rate with n = 64, produce NRMSEs of 0.099, 0.351, and 0.441 for low, medium and strong noise strengths, respectively. Consistent with previous results [6], CS is able to reduce the reconstruction NRMSE at 3/10 the Nyquist rate - the reconstruction NRMSEs are 0.0679, 0.239 and 0.341, respectively. The CS reconstruction of very noisy images that are undersampled at 3/10 the Nyquist rate yields a lower reconstruction NRMSE than that of a fully-sampled image with medium noise strength since the line mask undersamples regions in the Fourier plane other than those in the low frequency domain. For reconstructions of noisy images, AD conversion at $n \geq 14$ produces NRMSEs that converge to a single value as $n \rightarrow 1$ where the NRMSE standard deviation approaches zero. At low n values, noise is insufficient to affect the outputs of low dynamic-range detectors operating in the Fourier plane.

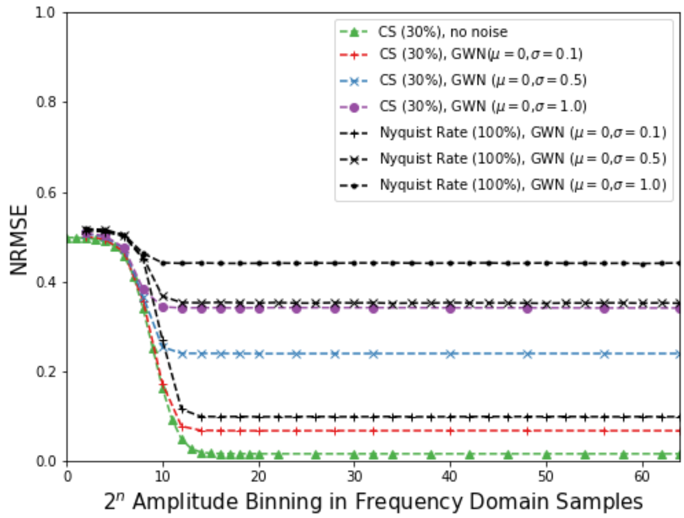

Figure 4: NRMSE of CS reconstructions of Lenna image sampled at 30% and 100% of the Nyquist rate at different additive noise strengths in the Fourier domain. Each point represents the average of five trials (standard deviation < 10^{-4}).

4 Conclusions

We have demonstrated that decreasing the ADC bit number from n = 64 to n = 14 yields no deleterious effect on CS reconstruction quality. In the range: $13 \geq n \geq 5$, the NRMSE increases linearly with decreasing n. This behavior is consistently observed in different types of images. The same NRMSE characteristics are found in the presence of noise in the Fourier plane. Hence CS produces reconstructions with lower NRMSE values than those obtained through direct sampling at Nyquist rate.

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