Adaptive time-frequency decompositions

Geoffrey Davis
Stéphane Mallat
Zhifeng Zhang
New York University
Courant Institute
251 Mercer Street
New York, New York 10012
E-mail: davisg@cims.nyu.edu

Abstract. Computing the optimal expansion of a signal in a redundant dictionary of waveforms is an NP-hard problem. We introduce a greedy algorithm, called a matching pursuit, which computes a suboptimal expansion. The dictionary waveforms that best match a signal's structures are chosen iteratively. An orthogonalized version of the matching pursuit is also developed. Matching pursuits are general procedures for computing adaptive signal representations. With a dictionary of Gabor functions, a matching pursuit defines an adaptive time-frequency transform. Matching pursuits are chaotic maps whose attractors define a generic noise with respect to the dictionary. We derive an algorithm that isolates the coherent structures of a signal and describe an application to pattern extraction from noisy signals.

Subject terms: adaptive wavelet transforms; matching pursuits; adaptive approximation; greedy algorithms; signal denoising.

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1 Introduction

Flexible decompositions are particularly important for representing signal components whose localizations in time and frequency vary widely. The goal is to expand a signal into waveforms whose time-frequency properties are adapted to the signal's local structures. The waveforms we use for these expansions are called time-frequency atoms. For example, impulses need to be decomposed into functions well localized in time, while spectral lines are better represented by waveforms that have a narrow frequency support. When the signal includes both of these elements, the time-frequency atoms must be adapted accordingly. We must use a procedure that selects from all the time-frequency atoms of a large dictionary the waveforms that are best adapted to decomposing the signal structures.

Computing the optimal approximation of a signal in a redundant dictionary is an NP-hard problem.[1] We therefore introduce a suboptimal greedy algorithm, called a matching pursuit, which decomposes any signal into a linear expansion of waveforms belonging to a redundant collection called a dictionary. These waveforms are selected to best match the signal's structures. Although matching pursuits are nonlinear, they possess an energy conservation relation similar to that of an orthogonal expansion that guarantees their convergence. We also introduce an orthogonalized version of a matching pursuit.

The application of matching pursuits to adaptive time-frequency decompositions is described in Sec. 6. The signal is decomposed into a selected set of time-frequency atoms, the dilations, translations, and modulations of a single window function. We derive a time-frequency energy distribution by adding the Wigner distributions of the selected time-frequency atoms. Unlike the Wigner distribution or Cohen's class distributions, this energy distribution does not include interference terms and thus provides a clear picture in the time-frequency plane.

Matching pursuits are chaotic maps whose properties are studied. The approximation error of the pursuit converges to an attractor that corresponds to a class of signals that are not efficiently represented by the waveforms of the dictionary. Measurement of this convergence to the attractor allows us to separate our decomposition into portions that are coherent and incoherent with respect to the dictionary. Isolating the coherent part of a signal enables us to perform denoising.

The space $L^2(R)$ is the Hilbert space of complex valued functions such that

$$\|f\|^2 = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt < +\infty \ . \tag{1}$$

The inner product of $f(t), g(t) \in L^2(R)$ is defined by

$$\langle f,g \rangle = \int_{-\infty}^{+\infty} f(t)\overline{g}(t) \, dt \ , \tag{2}$$

where $\overline{g}(t)$ is the complex conjugate of $g(t)$. The Fourier transform of $f(t) \in L^2(R)$ is written $\hat{f}(\omega)$ and defined by

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) \exp(-i\omega t) \, dt \ . \tag{3}$$

2 Time-Frequency Atoms

A general family of time-frequency atoms can be generated by scaling, translating, and modulating a single window function $g(t) \in L^2(R)$. We suppose that $g(t)$ is real and centered at

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0. We also impose that $\|g\|=1$, that the integral of $g(t)$ is nonzero, and that $g(0)\neq 0$. For any scale $s>0$, frequency modulation ξ, and translation u, we denote $\gamma=(s,u,\xi)$ and define

$$g_\gamma(t)=\frac{1}{\sqrt{s}}g\left(\frac{t-u}{s}\right)\exp(i\xi t) \ . \tag{4}$$

The index γ is an element of the set $\Gamma=R^+\times R^2$. The factor $1/\sqrt{s}$ normalizes $\|g_\gamma(t)\|$ to 1. The function $g_\gamma(t)$ is centered at the abscissa u and its energy is concentrated in a neighborhood of u of size proportional to s. Its Fourier transform is centered at the frequency $\omega=\xi$ and has its energy concentrated in a neighborhood of ξ, of size proportional to $1/s$. For our numerical examples we use the Gaussian window $g(t)=2^{1/4}\exp(-\pi t^2)$.

The dictionary of time-frequency atoms $D=[g_\gamma(t)]_{\gamma\in\Gamma}$ is a very redundant set of functions that includes window Fourier frames and wavelet frames.[2] When the signals include time-frequency structures of very different types, one cannot choose *a priori* a frame that is well adapted to performing the expansion. Instead, we need to find the atoms in the dictionary that best match each given signal's structures to perform a compact decomposition. In the next section we develop an algorithm for computing such adaptive decompositions in redundant dictionaries.

3 Matching Pursuits

Let H be a signal space. A dictionary for H is a family $D=(g_\gamma)_{\gamma\in\Gamma}$ of vectors in H, such that linear combinations of the g_γ are dense in H and for which $\|g_\gamma\|=1$. The smallest possible dictionary is a basis of H; general dictionaries are redundant families of vectors. A signal does not have a unique representation as a sum of elements of a redundant dictionary. Unlike the case of a basis, we have some degree of freedom in choosing a signal's particular representation. This freedom allows us to choose a subset of the dictionary that is tailored to the signal in question and which provides the most compact representation. We can choose a subset of the dictionary for which the signal energy is concentrated in as few terms as possible. The chosen vectors highlight the predominant signal features.

Let D be a dictionary of vectors in an N-dimensional Hilbert space. For any given $\beta\in(0,1)$, we define an optimal approximation of $f\in H$ to be an expansion

$$\tilde{f}=\sum_{n=1}^{\beta N}a_n g_{\gamma_n} \ ,$$

where the a_n and $g_{\gamma_n}\in D$ are chosen to minimize

$$\|f-\tilde{f}\| \ .$$

When the dictionary is redundant, we can show that finding an optimal solution is a fundamentally intractable problem. If we restrict the number of bits of a_n to $\Theta(N^j)$ and the number of vectors in D to $\Theta(N^k)$, for fixed j,k, then we can prove[1] that finding an optimal expansion is NP-hard.[3]

Because of the difficulty of finding optimal solutions, we instead develop a greedy algorithm that computes a good suboptimal approximation. Let $f\in H$. We want to compute a linear expansion of f over a set of vectors selected from D that best matches the inner structures of f. A matching pursuit is a greedy algorithm that successively approximates f with orthogonal projections onto elements of D. Let $g_{\gamma 0}\in D$. The vector f can be decomposed into

$$f=\langle f,g_{\gamma 0}\rangle g_{\gamma 0}+Rf \ , \tag{5}$$

where Rf is the residual vector after approximating f in the direction of $g_{\gamma 0}$. Clearly, $g_{\gamma 0}$ is orthogonal to Rf, hence

$$\|f\|^2=|\langle f,g_{\gamma 0}\rangle|^2+\|Rf\|^2 \ . \tag{6}$$

To minimize $\|Rf\|$, we must choose $g_{\gamma 0}\in D$ such that $|\langle f,g_{\gamma 0}\rangle|$ is maximal. In some cases, it is only possible to find a vector $g_{\gamma 0}$ that is close to the maximum in the sense that

$$|\langle f,g_{\gamma 0}\rangle|\geq\alpha\sup_{\gamma\in\Gamma}|\langle f,g_\gamma\rangle| \ , \tag{7}$$

where $\alpha\in(0,1]$ is an optimality factor.

We subdecompose the residue Rf by projecting it onto the vector of D that best matches Rf, as was done for f. This projection of Rf generates a second residue R^2f, which we again decompose to obtain a third residue, and so on.

We describe the algorithm inductively. Let $R^0f=f$. We suppose that we have computed the n'th order residue R^nf, for $n\geq 0$. We choose with a choice function C an element $g_{\gamma_n}\in D$ that closely matches the residue R^nf in the sense that

$$|\langle R^nf,g_{\gamma_n}\rangle|\geq\alpha\sup_{\gamma\in\Gamma}|\langle R^nf,g_\gamma\rangle| \ . \tag{8}$$

The residue R^nf is subdecomposed into

$$R^nf=\langle R^nf,g_{\gamma_n}\rangle g_{\gamma_n}+R^{n+1}f \ , \tag{9}$$

which defines the residue at order $n+1$. Since $R^{n+1}f$ is orthogonal to g_{γ_n}, we have

$$\|R^nf\|^2=|\langle R^nf,g_{\gamma_n}\rangle|^2+\|R^{n+1}f\|^2 \ . \tag{10}$$

Let us carry this decomposition up to order m. We decompose f into the telescoping sum

$$f=\sum_{n=0}^{m-1}(R^nf-R^{n+1}f)+R^mf \ . \tag{11}$$

Equation (9) yields

$$f=\sum_{n=0}^{m-1}\langle R^nf,g_{\gamma_n}\rangle g_{\gamma_n}+R^mf \ . \tag{12}$$

Similarly, we write $\|f\|^2$ as a telescoping sum,

$$\|f\|^2=\sum_{n=0}^{m-1}(\|R^nf\|^2-\|R^{n+1}f\|^2)+\|R^mf\|^2 \ , \tag{13}$$

which we combine with Eq. (10) to obtain an energy conservation equation,

$$\|f\|^2=\sum_{n=0}^{m-1}|\langle R^nf,g_{\gamma_n}\rangle|^2+\|R^mf\|^2 \ . \tag{14}$$

Thus, the original vector f is decomposed into a sum of dictionary elements that are chosen to best match its residues. Although this decomposition is nonlinear, we maintain an energy conservation as though it were a linear, orthogonal decomposition. An important issue is to understand the behavior of the residue $R^m f$ when m increases. By adapting a result proved by Jones[4] for projection pursuit algorithms,[5,6] one can prove[7] that the matching pursuit algorithm converges, even in infinite dimensional spaces.

Theorem 1. Let $f \in H$. The residue $R^m f$ defined by the induction equation (9) satisfies

$$\lim_{m \to +\infty} \|R^m f\| = 0 \ . \tag{15}$$

Hence,

$$f = \sum_{n=0}^{+\infty} \langle R^n f, g_{\gamma_n} \rangle g_{\gamma_n} \ , \tag{16}$$

and

$$\|f\|^2 = \sum_{n=0}^{+\infty} |\langle R^n f, g_{\gamma_n} \rangle|^2 \ . \tag{17}$$

When H is of finite dimension, $\|R^m f\|$ decays exponentially to zero.

4 Implementation of Matching Pursuits

When the dictionary is very redundant, the search for the vectors that best match the signal residues can be limited to a subdictionary $D_\alpha = (g_\gamma)_{\gamma \in \Gamma_\alpha} \subset D$. We suppose that Γ_α is a finite set of indices from Γ such that for any $f \in H$,

$$\sup_{\gamma \in \Gamma_\alpha} |\langle f, g_\gamma \rangle| \geq \alpha \sup_{\gamma \in \Gamma} |\langle f, g_\gamma \rangle| \ . \tag{18}$$

Depending upon α and the dictionary redundancy, the set Γ_α can be much smaller than Γ. The matching pursuit is initialized by computing the inner products $(\langle f, g_\gamma \rangle)_{\gamma \in \Gamma_\alpha}$, and continues by induction as follows. Suppose that we have already computed $(\langle R^n f, g_\gamma \rangle)_{\gamma \in \Gamma_\alpha}$, for $n \geq 0$. We search in D_α for an element $g_{\tilde{\gamma}_n}$ for which

$$|\langle R^n f, g_{\tilde{\gamma}_n} \rangle| = \max_{\gamma \in \Gamma_\alpha} |\langle R^n f, g_\gamma \rangle| \ . \tag{19}$$

We can find a dictionary element that matches f even better than $g_{\tilde{\gamma}_n}$ by using Newton's method to maximize $|\langle f, g_\gamma \rangle|$ for $\gamma \in \Gamma$ in a neighborhood of $\tilde{\gamma}_n$. We then have

$$|\langle R^n f, g_{\gamma_n} \rangle| \geq |\langle R^n f, g_{\tilde{\gamma}_n} \rangle| \geq \alpha \sup_{\gamma \in \Gamma} |\langle R^n f, g_\gamma \rangle| \ . \tag{20}$$

The choice function mentioned in Sec. 3 is defined indirectly by this double search strategy. Once the vector g_{γ_n} is selected, we compute the inner product of the new residue $R^{n+1} f$ with any $g_\gamma \in D_\alpha$, with an updating formula derived from Eq. (9),

$$\langle R^{n+1} f, g_\gamma \rangle = \langle R^n f, g_\gamma \rangle - \langle R^n f, g_{\gamma_n} \rangle \langle g_{\gamma_n}, g_\gamma \rangle \ . \tag{21}$$

Since we have already computed $\langle R^n f, g_\gamma \rangle$ and $\langle R^n f, g_{\gamma_n} \rangle$, this update requires only that we compute $\langle g_{\gamma_n}, g_\gamma \rangle$. Dictionaries are generally built so that this inner product is computed with a small number of operations. We describe in Ref. 5 how to compute efficiently the inner product of two discrete Gabor atoms, in $O(1)$ operations.

Let I be the number of operations required to compute $\langle g_{\gamma_n}, g_\gamma \rangle$ and let Z be the average number of g_γ's in D_α for which $\langle g_{\gamma_n}, g_\gamma \rangle$ is nonzero. In Ref. 1 we show that p iterations of a pursuit requires $O(pIZ)$ operations.

The number of times we subdecompose the residues of a given signal f depends upon the desired precision of the approximation ε. The number of iterations we require is the minimum p for which

$$\|R^p f\| = \|f - \sum_{n=0}^{p-1} \langle R^n f, g_{\gamma_n} \rangle g_{\gamma_n}\| \leq \varepsilon \|f\| \ . \tag{22}$$

From the energy conservation relation (14), we see that Eq. (22) is equivalent to

$$\|f\|^2 - \sum_{n=0}^{p-1} |\langle R^n f, g_{\gamma_n} \rangle|^2 \leq \varepsilon^2 \|f\|^2 \ . \tag{23}$$

Since we do not compute the residue $R^n f$ at each iteration, we test condition (23) to determine when to stop the decomposition. Software implementing matching pursuits with a dictionary of time-frequency atoms is available via anonymous ftp at the address cs.nyu.edu in the directory \pub\wave\software.

5 Back-Projection and Orthogonal Pursuits

After m iterations, a matching pursuit decomposes a signal f into a sum of m dictionary vectors and an error term. We have

$$f = \sum_{n=0}^{m-1} \langle R^n f, g_{\gamma_n} \rangle g_{\gamma_n} + R^m f \ . \tag{24}$$

Suppose that $\langle g_{\gamma_n}, g_{\gamma_{n+1}} \rangle = \beta \neq 0$; $R^{n+1} f$ is obtained by removing the component of $R^n f$ in the direction of g_{γ_n},

$$R^{n+1} f = R^n f - \langle R^n f, g_{\gamma_n} \rangle g_{\gamma_n} \ . \tag{25}$$

Similarly, $R^{n+2} f$ is obtained from $R^{n+1} f$ by removing the component of $R^{n+1} f$ in the direction of $g_{\gamma_{n+1}}$. Because $\langle g_{\gamma_n}, g_{\gamma_{n+1}} \rangle \neq 0$, we find that the component of $R^{n+2} f$ in the g_{γ_n} direction is no longer zero:

$$\begin{aligned} \langle R^{n+2} f, g_{\gamma_n} \rangle &= \langle R^{n+1} f, g_{\gamma_n} \rangle - \langle R^{n+1} f, g_{\gamma_{n+1}} \rangle \langle g_{\gamma_{n+1}}, g_{\gamma_n} \rangle \\ &= -\beta \langle R^{n+1} f, g_{\gamma_{n+1}} \rangle \ . \end{aligned} \tag{26}$$

In removing $g_{\gamma_{n+1}}$, we have replaced a small portion of the g_{γ_n} component that we previously removed.

Let V_m be the space generated by $(g_{\gamma_n})_{0 \leq n < m}$ and P_{V_m} be the orthogonal projector onto V_m. For any $f \in H$, $P_{V_m} f$ is the closest vector to f that can be written as linear expansion of the m vectors $(g_{\gamma_n})_{0 \leq n < m}$. We obtain from Eq. (24) that

$$P_{V_m} f = \sum_{n=0}^{m-1} \langle R^n f, g_{\gamma_n} \rangle g_{\gamma_n} + P_{V_m} R^m f \ . \tag{27}$$

When the family of vectors $(g_{\gamma_n})_{0 \le n < m}$ is not orthogonal, which is generally the case, then $P_{V_m} R^m f \neq 0$. The computation of

$$P_{V_m} R^m f = \sum_{n=0}^{m-1} x_n g_{\gamma_n} \tag{28}$$

is called a back-projection. The values x_n give corrections to the coefficients $\langle R^n f, g_{\gamma_n} \rangle$ that improve the linear expansion approximation to f. The approximation error for the corrected sum,

$$P_{W_m} f = f - P_{V_m} f \ , \tag{29}$$

is the orthogonal projection of f on the space W_m, the orthogonal complement of V_m in H.

The calculation of the coefficients $(x_n)_{0 \le n < m}$ requires that we solve the following linear system. For any $g_{\gamma k}$, $0 \le k < m$,

$$\langle P_{V_m} R^m f, g_{\gamma k} \rangle = \langle R^m f, g_{\gamma k} \rangle = \sum_{n=0}^{m-1} x_n \langle g_{\gamma_n}, g_{\gamma k} \rangle \ . \tag{30}$$

Let us denote $X = (x_n)_{0 \le n < m}$ and $Y = (\langle R^m f, g_{\gamma k} \rangle)_{0 \le k < m}$. Let $G = (\langle g_{\gamma_n}, g_{\gamma k} \rangle)_{0 \le k < m, 0 \le n < m}$ be the Gram matrix of the selected vectors. The linear system of Eq. (30) can be written $Y = GX$. A solution of this system is computed efficiently with a conjugate gradient algorithm.[7] If H is of finite dimension N, there are many classes of dictionaries for which any collection of N distinct dictionary vectors is a basis of H. This is the case for the Gabor dictionary used for time-frequency decompositions. Hence, after selecting N different vectors with a matching pursuit, the back-projection reduces to 0 the remaining residue.

Instead of recovering the orthogonal projection $P_{V_m} f$ at the end of the matching pursuit, we can modify the pursuit algorithm to prevent the replacement of components that were previously removed, which we saw in Eq. (26). We accomplish this by orthogonalizing the set of dictionary vectors as we proceed with the decomposition. This iterative orthogonalization is equivalent to performing the back-projection described earlier at each step of the decomposition, but is much more efficient. This type of algorithm was first introduced for control applications[8] and also studied independently from this work by Pati, Rezaiifar, and Krishnaprasad.[9] It has the advantage of providing better approximations than the matching pursuit algorithm, but it requires much more computation and can introduce numerical instabilities into the expansions. We describe by induction this orthogonal pursuit.

For $n = 0$, we set $R^0 f = f$. As in a matching pursuit, we define an optimality factor α, with $0 < \alpha \le 1$, and choose $g_{\gamma 0} \in D$, which satisfies

$$|\langle f, g_{\gamma 0} \rangle| \ge \alpha \sup_{\gamma \in \Gamma} |\langle f, g_\gamma \rangle| \ . \tag{31}$$

The space V_1 is generated by the single vector $g_{\gamma 0}$. We use a Gram-Schmidt orthogonalization to generate a basis for V_1. The first orthogonal basis vector for V_1 is $u_0 = g_{\gamma 0}$. The next residue is defined by

$$Rf = f - P_{V_1} f = f - \langle f, g_{\gamma 0} \rangle g_{\gamma 0} \ . \tag{32}$$

We explain by induction how to compute the orthogonal residue $R^{n+1} f$ from $R^n f$. We suppose that we have already selected n vectors $(g_{\gamma p})_{0 \le p < n}$ that are linearly independent and that we have computed the corresponding Gram-Schmidt orthogonal basis $(u_p)_{0 \le p < n}$ of the space V_n spanned by these n vectors. We have

$$R^n f = f - P_{V_n} f \ . \tag{33}$$

We choose a vector $g_{\gamma_n} \in D$ that satisfies

$$|\langle R^n f, g_{\gamma_n} \rangle| \ge \alpha \sup_{\gamma \in \Gamma} |\langle R^n f, g_\gamma \rangle| \ . \tag{34}$$

If $\langle R^n f, g_{\gamma_n} \rangle \neq 0$, then the vector g_{γ_n} cannot belong to the space V_n since $R^n f$ is orthogonal to V_n. Hence the vectors $(g_{\gamma p})_{0 \le p \le n}$ are linearly independent. The next vector u_n of the Gram-Schmidt basis is obtained by subtracting from g_{γ_n} its projection on the space V_n:

$$u_n = g_{\gamma_n} - \sum_{p=0}^{n-1} \frac{\langle g_{\gamma_n}, u_p \rangle}{\|u_p\|^2} u_p \ . \tag{35}$$

The family $(u_p)_{0 \le p \le n}$ is an orthogonal basis of V_{n+1}. The residue $R^{n+1} f$ is defined by

$$R^{n+1} f = f - P_{V_{n+1}} f = f - \sum_{p=0}^{n} \frac{\langle f, u_p \rangle}{\|u_p\|^2} u_p \ . \tag{36}$$

This can also be rewritten

$$R^{n+1} f = R^n f - \frac{\langle R^n f, u_n \rangle}{\|u_n\|^2} u_n \ . \tag{37}$$

Since $R^n f$ is orthogonal to the vectors $(g_{\gamma p})_{0 \le p < n}$, Eq. (35) implies that $\langle R^n f, u_n \rangle = \langle R^n f, g_{\gamma_n} \rangle$ and thus

$$R^{n+1} f = R^n f - \frac{\langle R^n f, g_{\gamma_n} \rangle}{\|u_n\|^2} u_n \ . \tag{38}$$

This equation is similar to the residue updating equation (9) of a matching pursuit, but instead of subtracting a vector in the direction of g_{γ_n}, we subtract a component in a direction orthogonal to all vectors previously selected. Since $R^{n+1} f$ and u_n are orthogonal,

$$\|R^{n+1} f\|^2 = \|R^n f\|^2 - \frac{|\langle R^n f, g_{\gamma_n} \rangle|^2}{\|u_n\|^2} \ . \tag{39}$$

An orthogonal pursuit guarantees that the selected vectors $(g_{\gamma_n})_{0 \le n \le m}$ are linearly independent, and computes the best possible approximation of f from these vectors. Since $R^0 f = f$, we derive from Eq. (38) and (39) that for any $m > 0$

$$f = \sum_{0 \le n < m} \frac{\langle R^n f, g_{\gamma_n} \rangle}{\|u_n\|^2} u_n + R^m f \ , \tag{40}$$

and

$$\|f\|^2 = \sum_{0 \le n < m} \frac{|\langle R^n f, g_{\gamma_n} \rangle|^2}{\|u_n\|^2} + \|R^m f\|^2 \ . \tag{41}$$

The derivations are similar to those for Eqs. (12) and (14). The next theorem is similar to theorem 1 and guarantees the convergence of the orthogonal pursuit.[1]

Theorem 2. Let $f \in H$. Let N be the dimension of H (N may be infinite). The orthogonal matching pursuit converges in $M \leq N$ iterations (M may be infinite if N is infinite). The residue $R^n f$ defined inductively by equation (38) satisfies

$$\lim_{n \to M} \|R^n f\| = 0 \ , \tag{42}$$

$$f = \sum_{0 \leq n < M} \frac{\langle R^n f, g_{\gamma_n} \rangle}{\|u_n\|^2} u_n \ , \tag{43}$$

and

$$\|f\|^2 = \sum_{0 \leq n < M} \frac{|\langle R^n f, g_{\gamma_n} \rangle|^2}{\|u_n\|^2} \ . \tag{44}$$

If H is of finite dimension, the orthogonal pursuit converges in a finite number of iterations.

Although the orthogonalized version of the pursuit has better convergence properties, it has several disadvantages. Because the updating relation (38) involves the orthogonalized dictionary vectors u_n instead of the g_{γ_n}, updating the inner products $\langle R^n f, g_\gamma \rangle$ requires much more work. In Ref. 2 we show that computing p iterations of an orthogonalized pursuit requires $O(p^2 IZ)$ operations, as compared to $O(pIZ)$ for a nonorthogonal pursuit. Here I is the number of operations required to compute $\langle g_{\gamma_n}, g_\gamma \rangle$ and Z is the average number of g_γ's in D_α for which $\langle g_{\gamma_n}, g_\gamma \rangle$ is nonzero. So for p iterations, the orthogonalized pursuit is $O(p)$ times slower. Unless we are interested in decomposing signals into a very small number of elements, the orthogonalized pursuit will be much slower.

A second disadvantage of the orthogonalized pursuit is that we obtain an expansion of our signal in the orthogonalized basis u_n rather than in g_{γ_n}'s. The process of transforming the sum of the u_n's into a sum of g_{γ_n}'s can introduce instabilities into the expansion. We are looking for coefficients $(\beta_n)_{0 \leq n < M}$ such that

$$f = \sum_{0 \leq n < M} \beta_n g_{\gamma_n} \ . \tag{45}$$

Since $u_n \in V_n$ and $(g_{\gamma p})_{0 \leq p \leq n}$ is a basis of V_n, we can decompose u_n into

$$u_n = \sum_{p=0}^{n} b_{p,n} g_{\gamma p} \ . \tag{46}$$

These coefficients are computed during the pursuit. Inserting Eq. (46) into Eq. (43) yields

$$f = \sum_{0 \leq n < M} \frac{\langle R^n f, g_{\gamma_n} \rangle}{\|u_n\|^2} \sum_{p=0}^{n} b_{p,n} g_{\gamma p} \ . \tag{47}$$

One could naively try to rearrange the terms of this double summation to obtain

$$f = \sum_{0 \leq p < M} g_{\gamma p} \sum_{p \leq n < M} b_{p,n} \frac{\langle R^n f, g_{\gamma_n} \rangle}{\|u_n\|^2} \ . \tag{48}$$

However, when $M = +\infty$ the infinite sum over n that defines each coefficient β_p may not converge. Such a situation arises when the family $(g_{\gamma_n})_{0 \leq n < M}$ is not a Riesz basis of the closed space V_M that it generates. For such a case, we cannot obtain an expansion of the form of Eq. (45) from the orthogonal matching pursuit. If the signal space H has a finite dimension N, then M is finite, so we can always invert the two sums of Eq. (47) to obtain Eq. (48). The basis $(g_{\gamma_n})_{0 \leq n < M}$ may, however, be very badly conditioned, in which case we can have numerical instabilities, where

$$\sum_{0 \leq n < M} |\beta_n|^2 \gg \|f\|^2 \ . \tag{49}$$

We find that the orthogonalized pursuit converges more quickly, but is much harder to compute and potentially less numerically stable than the nonorthogonalized pursuit. Figure 1 compares the convergence rates for an orthogonal matching pursuit and a nonorthogonal matching pursuit for a synthetic signal of 512 samples. The benefits of the orthogonalization do not become evident until after roughly 150 iterations, but in the tail of the expansion the convergence of the orthogonalized pursuit is much faster. We are uninterested in the tail of the expansion because it corresponds to noncoherent signal components, as explained in Sec. 7.

6 Matching Pursuits with Time-Frequency Dictionaries

For dictionaries of time-frequency atoms, a matching pursuit yields an adaptive time-frequency transform. It decomposes any function $f(t) \in L^2(R)$ into a sum of complex time-frequency atoms that best match its residues. This section studies the properties of this particular matching pursuit decomposition. We derive a new type of time-frequency energy distribution by summing the Wigner distributions of each time-frequency atom.

Since a time-frequency atom dictionary is complete in $L^2(R)$, theorem 1 implies that a matching pursuit decomposes

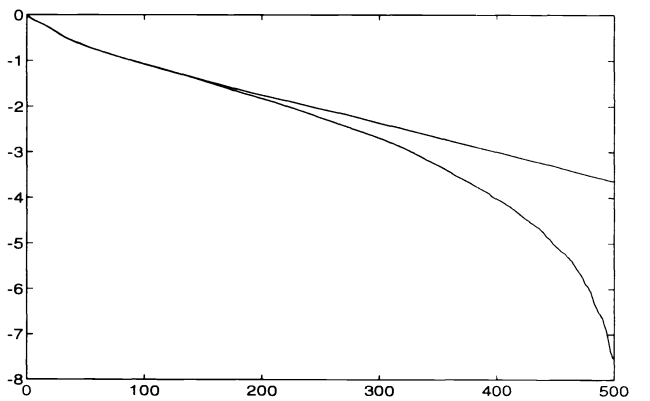

Fig. 1 Shown is $\log_{10}(\|R^n f\|/\|f\|)$ as a function of n for a synthetic signal with 512 samples. The top curve is for a nonorthogonal matching pursuit and the bottom is for an orthogonal matching pursuit.

any function $f \in L^2(R)$ into

$$f = \sum_{n=0}^{+\infty} \langle R^n f, g_{\gamma_n} \rangle g_{\gamma_n} \ , \tag{50}$$

where $\gamma_n = (s_n, u_n, \xi_n)$ and

$$g_{\gamma_n}(t) = \frac{1}{\sqrt{s_n}} \, g\left(\frac{t-u_n}{s_n}\right) \exp(i\xi_n t) \ . \tag{51}$$

These atoms are chosen to best match the residues of f.

We derive a new time-frequency energy distribution from the decomposition of a function $f(t)$ within a time-frequency dictionary by adding the Wigner distributions of each selected atom. The cross Wigner distribution of two functions $f(t)$ and $h(t)$ is defined by

$$W[f,h](t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f\left(t+\frac{\tau}{2}\right) \overline{h}\left(t-\frac{\tau}{2}\right) \exp(-i\omega t) \, d\tau \ . \tag{52}$$

The Wigner distribution of $f(t)$ is $Wf(t,\omega) = W[f,f](t,\omega)$. Since the Wigner distribution is quadratic, we derive from the atomic decomposition (50) of $f(t)$ that

$$\begin{aligned} Wf(t,\omega) = & \sum_{n=0}^{+\infty} |\langle R^n f, g_{\gamma_n} \rangle|^2 \, Wg_{\gamma_n}(t,\omega) \\ & + \sum_{n=0}^{+\infty} \sum_{m=0, m \neq n}^{+\infty} \langle R^n f, g_{\gamma_n} \rangle \\ & \times \overline{\langle R^m f, g_{\gamma m} \rangle} \, W[g_{\gamma_n}, g_{\gamma m}](t,\omega) \ . \end{aligned} \tag{53}$$

The double sum corresponds to the cross terms of the Wigner distribution. It contains the terms that one usually tries to remove to obtain a clear picture of the energy distribution of $f(t)$ in the time-frequency plane. We therefore only keep the first sum and define

$$Ef(t,\omega) = \sum_{n=0}^{+\infty} |\langle R^n f, g_{\gamma_n} \rangle|^2 \, Wg_{\gamma_n}(t,\omega) \ . \tag{54}$$

A similar decomposition algorithm over time-frequency atoms was derived independently by Qian and Chen[10] to define this energy distribution in the time-frequency plane. From the dilation and translation properties of the Wigner distribution and the expression (51) of a time-frequency atom, we derive that for $\gamma = (s, \xi, u)$,

$$Wg_{\gamma}(t,\omega) = Wg\left[\frac{t-u}{s}, s(\omega-\xi)\right] \ , \tag{55}$$

and hence

$$Ef(t,\omega) = \sum_{n=0}^{+\infty} |\langle R^n f, g_{\gamma_n} \rangle|^2 \, Wg\left[\frac{t-u_n}{s_n}, s_n(\omega-\xi_n)\right] \ . \tag{56}$$

The Wigner distribution also satisfies

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Wg(t,\omega) \, dt \, d\omega = \|g\|^2 = 1 \ , \tag{57}$$

so the energy conservation equation (17) implies that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Ef(t,\omega) \, dt \, d\omega = \|f\|^2 \ . \tag{58}$$

We can thus interpret $Ef(t,\omega)$ as an energy density of f in the time-frequency plane (t,ω). Unlike the Wigner and the Cohen class distributions, it does not include cross terms. It also remains positive if $Wg(t,\omega)$ is positive, which is the case when $g(t)$ is Gaussian. On the other hand, the energy density $Ef(t,\omega)$ does not satisfy marginal properties, as opposed to certain Cohen class distributions.[11] However, the importance of these marginal properties for signal processing is not clear. If $g(t)$ is the Gaussian window

$$g(t) = 2^{1/4} \exp(-\pi t^2) \ , \tag{59}$$

then

$$Wg(t,\omega) = 2 \exp\left\{-2\pi\left[t^2 + \left(\frac{\omega}{2\pi}\right)^2\right]\right\} \ . \tag{60}$$

The time-frequency atoms $g_{\gamma}(t)$ are Gabor functions, and the time-frequency energy distribution $Ef(t,\omega)$ is a sum of Gaussian blobs whose locations and variances along the time and frequency axes depend upon the parameters (s_n, u_n, ξ_n).

Figure 2 is a signal f of 512 samples that has been built by adding waveforms of different time-frequency localizations. It is the sum of $\cos\{[1-\cos(ax)]bx\}$, two truncated sinusoids, two dirac functions, and $\cos(cx)$. Figure 3 shows the time-frequency energy distribution $Ef(t,\omega)$. Since $Ef(t,\omega) = Ef(t,-\omega)$, we only display its values for $\omega \geq 0$. Each Gabor time-frequency atom selected by the matching pursuit is an elongated Gaussian blob in the time-frequency plane. We see clearly the arch of the $\cos\{[1-\cos(ax)]bx\}$. The truncated sinusoids are in the center and upper left-hand corner of the graph. The horizontal line corresponds to $\cos(cx)$ and the two vertical lines are the Dirac functions.

Figure 4 is the graph of a digitized recording of a female speaker pronouncing the word ''wavelets'' sampled at 11.125

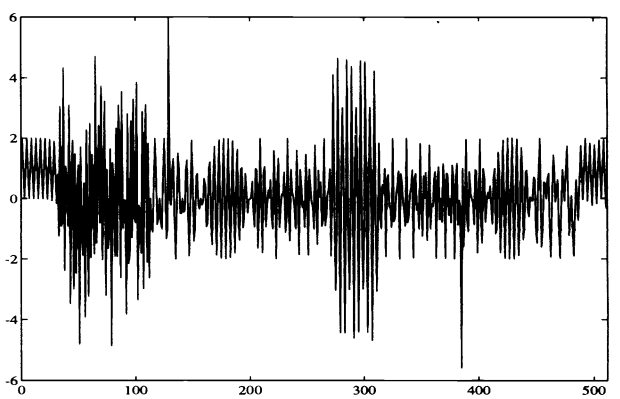

Fig. 2 Signal of 512 samples built by adding $\cos\{[1-\cos(ax)]bx\}$, two truncated sinusoids, two dirac functions, and $\cos(cx)$.

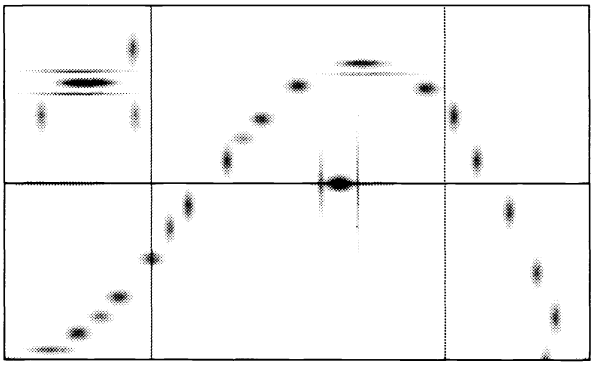

Fig. 3 Time-frequency energy distribution $Ef(t,\omega)$ of the signal shown in Fig. 2. The horizontal axis is time and the vertical axis is frequency. The darkness of the image increases with $Ef(t,\omega)$.

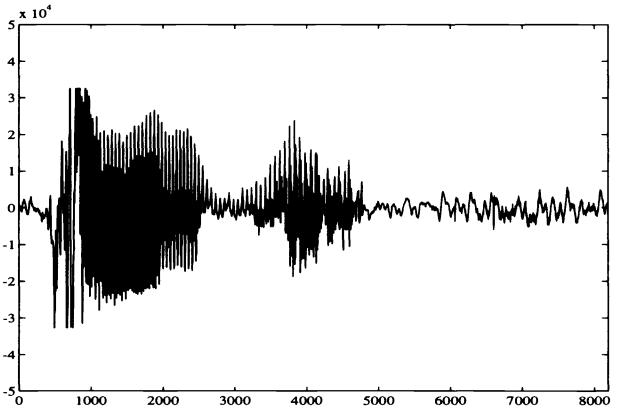

Fig. 4 Digitized recording of a female speaker pronouncing the word "wavelets." Sampling is at 11 kHz.

kHz. From the time-frequency energy density shown in Fig. 5, we can see the initial low-frequency onset of the "w" followed by the long "a" and its harmonics. The central cluster of energy corresponds to the "l" and the short "e" of the second syllable. The last third of the time-frequency plane is the "s," which resembles a bandlimited white noise. Most of the signal energy is characterized by few time-frequency atoms. For $n=300$ atoms, $\|R^n f\|/\|f\|=0.073$, although the signal has 8192 samples, and the sound recovered from these atoms is of excellent quality.

Figure 6 shows a signal obtained by adding a Gaussian white noise to the speech recording given in Fig. 4, with a SNR of 2 dB. Figure 7 is the time-frequency energy distribution of this noisy signal. The white noise generates time-frequency atoms spread across the whole time-frequency plane, but we can still distinguish the time-frequency structures of the original signal because their energy is more concentrated than the noise.

7 Chaos in Matching Pursuits

For signal spaces with finite dimension, the energy of the residue converges exponentially to zero. We renormalize the signal residues to study their properties when the number of iterations increases, and define

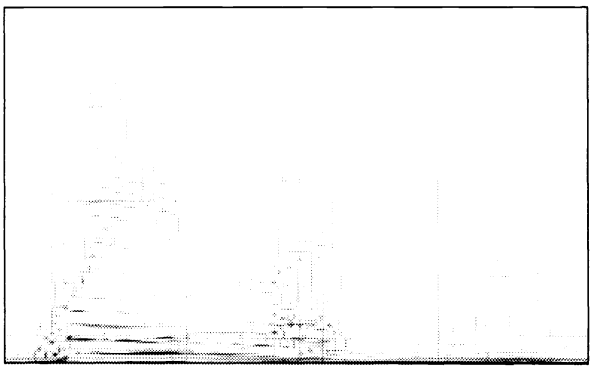

Fig. 5 The time-frequency energy distribution of the speech recording shown in Fig. 4. The initial cluster contains the low-frequency "w" and the harmonics of the long "a." The second cluster is the "le." The final portion of the signal is the "s," which resembles a bandlimited noise.

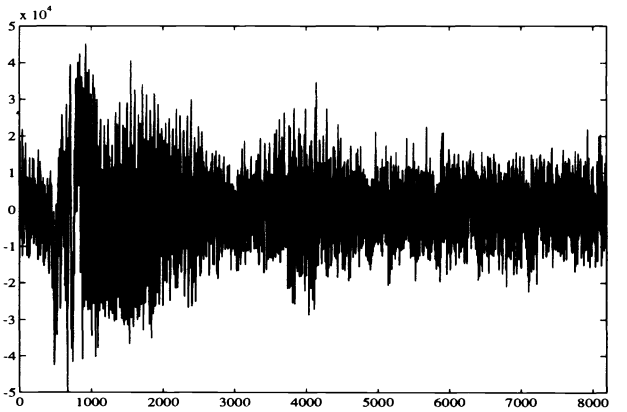

Fig. 6 Signal obtained by adding a Gaussian white noise to the digitized speech signal in Fig. 4. The SNR is 2 dB.

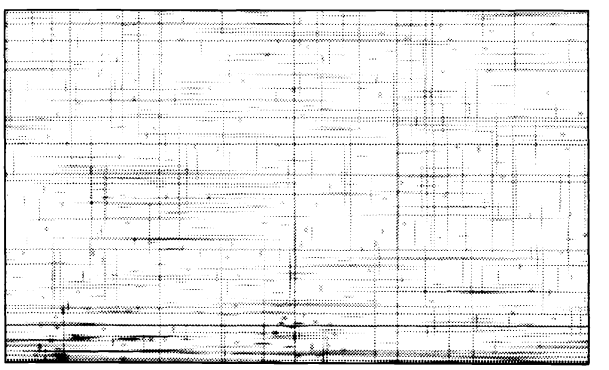

Fig. 7 The time-frequency energy distribution of the noisy speech signal. The energy of the white noise is spread across the entire time-frequency plane.

$$\tilde{R}^n f=\frac{R^n f}{\|R^n f\|} \quad . \tag{61}$$

The renormalized matching pursuit map M maps the n'th renormalized residue of a matching pursuit to the $n+1$,

$$M(\tilde{R}^n f) = \tilde{R}^{n+1} f \ . \tag{62}$$

At each iteration the renormalized matching pursuit map removes the largest dictionary component of the residue and renormalizes the new residue. This action is much like that of a left-shift operator acting on a base-N decimal: The shift operator removes the most significant digit of the expansion and then multiplies the decimal by N, which is analogous to a renormalization. Let Σ_N be the set of all base-N decimals. The left-shift map $L_N: \Sigma_N \rightarrow \Sigma_N$ is formally defined by

$$L_N(0.s_1 s_2 s_3 ...) = 0.s_2 s_3 s_4 ... \ , \tag{63}$$

where $0.s_1 s_2 ...$ is the base-N decimal $\Sigma_{k=1}^{\infty} s_k / N^k$. The left-shift map is known to be a chaotic map. The topological properties of the renormalized matching pursuit map are similar to those of the left-shift map, at least locally. We proved[1] that for a particular dictionary in R^3, the renormalized matching pursuit map is topologically equivalent to a shift map, which proves that this renormalized matching pursuit map is a chaotic map with well-understood properties.

Experimental data suggest that the residues of a normalized matching pursuit converge to a chaotic attractor in high-dimensional spaces as well. This is shown[1] for a simple dictionary composed of Diracs and complex exponentials, and a similar behavior is observed for more complicated dictionaries such as the one composed of Gabor functions. The residues converge to realizations of a specific process that we call dictionary noise. If the dictionary is invariant when we translate its elements or multiply them by a complex exponential, one can then prove[1] that this process is white and stationary. This is the case for a Gabor dictionary. Realizations of a dictionary noise are signals whose inner products with elements of the dictionary are uniformly small. In other words, such signals have no structure that is particularly coherent with respect to the dictionary.

The correlation ratio, defined by

$$\lambda(\tilde{R}^m f) = \sup_{\gamma \in \Gamma} |\langle \tilde{R}^m f, g_\gamma \rangle| \ , \tag{64}$$

is an important measure of the degree to which structures in the residue $\tilde{R}^m f$ resemble dictionary elements. A signal f that possesses structures that are well represented by dictionary elements will have large values of $\lambda(f)$. As the matching pursuit proceeds, these structures are removed, and $\lambda(R^m f)$ decreases. As the residues approach the attractor, we find that $\lambda(\tilde{R}^m f)$ approaches a dictionary-dependent constant λ_e.

To determine whether a given residue $R^m f$ is close to the attractor, we compute $\lambda(\tilde{R}^m f)$ and compare it to λ_e. If $\lambda(\tilde{R}^m f) \leq \lambda_e$, then $R^m f$ does not include any more coherent components with respect to the dictionary. The coherent components of f, then, are the first m selected dictionary vectors $(g_{\gamma_n})_{0 \leq n < m}$.

The expected value λ_e has been measured numerically for a Gabor dictionary.[7] The values of $\lambda(\tilde{R}^m f)$ as a function of m for the "wavelets" signal and the noisy "wavelets" signal are shown in Fig. 8. The coherence ratio for the noisy signal converges much more quickly to λ_e because the noise has diluted the coherent structures. We find that the noisy speech signal in Fig. 6 has $m = 119$ coherent structures, whose time-frequency distributions are shown in Fig. 9. Figure 10 is the signal reconstructed from these time-frequency atoms. The SNR of the reconstructed signal is 8.7 dB. The white noise has been removed and this signal has a good auditory quality because the main time-frequency structures of the original speech signal have been retained.

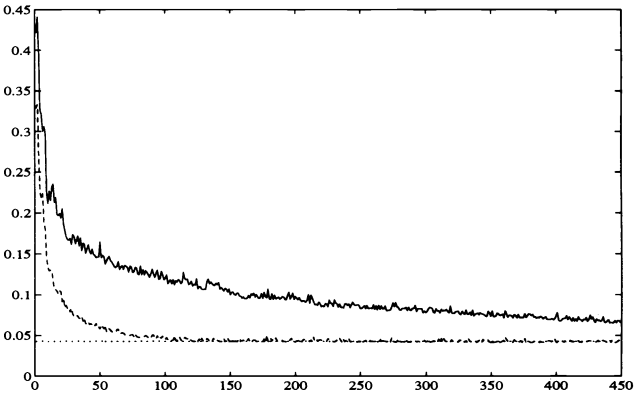

Fig. 8 Shown is $\lambda(R^n f)$ as a function of the number of iterations n for the speech signal in Fig. 4 (solid line) and the noisy speech signal in Fig. 6 (dashed line). The dotted line is λ_e. The first 119 elements of the noisy speech signal are coherent.

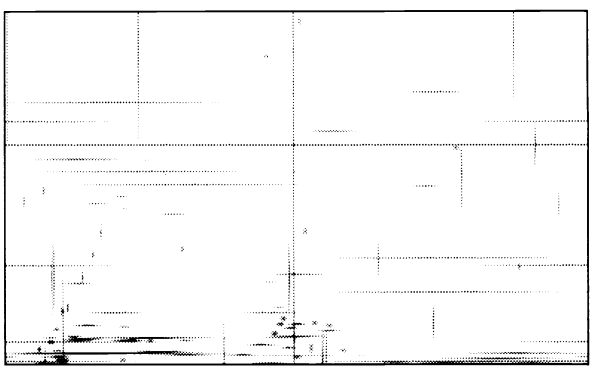

Fig. 9 The time-frequency energy distribution of the 119 coherent structures of the noisy speech signal.

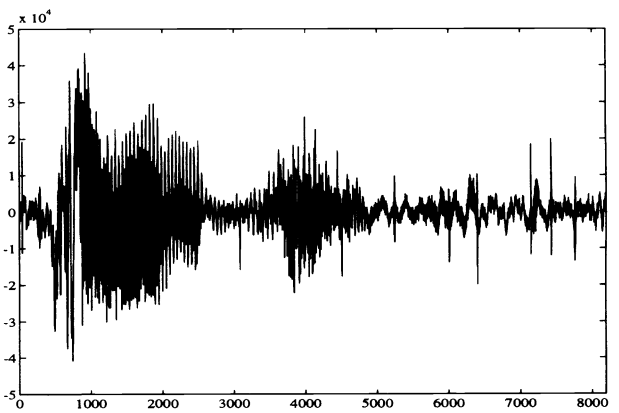

Fig. 10 Signal reconstructed from the 119 coherent structures from the matching pursuit decomposition of the noisy speech signal in Fig. 6. The white noise has been removed, and the new SNR is 8.7 dB.

8 Conclusion

Matching pursuits provide extremely flexible signal representations because the decomposition is adapted to the structures of the signal, and because the set of waveforms over which we decompose is not limited to any single basis. By using a dictionary of time-frequency atoms, we have obtained an adaptive time-frequency energy distribution for signals. The convergence properties of the pursuit enable us to define a notion of coherence with respect to a dictionary, and this enables us to perform denoising of signals.

Acknowledgments

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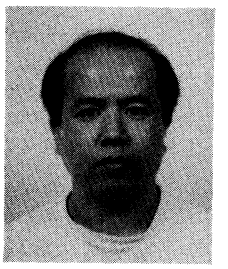
Geoffrey Davis received a BS in mathematics and physics in 1988 from Duke University, Durham, North Carolina, and a PhD in applied mathematics in 1994 from the Courant Institute of Mathematical Sciences at New York University. He received an Office of Naval Research graduate fellowship in 1989. As of September 1994, he will be a John Wesley Young research instructor at Dartmouth College, Hanover, New Hampshire. His research interests include image and signal processing.

Stéphane Mallat graduated from Ecole Polytechnique, Paris, in 1984 and from Ecole Nationale Superieure des Telecommunications, Paris, in 1985. He received a PhD degree in electrical engineering from the University of Pennsylvania in Philadelphia. In 1988, Dr. Mallat joined the Courant institute of Mathematical Sciences at New York University, where he is currently an associate professor of computer science. He received the 1990 IEEE Signal Processing Society's paper award and the Alfred Sloan Fellowship in mathematics in 1993. His research interests include computer vision signal processing, and applied mathematics.



Zhifeng Zhang received a BS in chemical engineering from the South China Institute of Technology in 1982, an MS in applied mathematics from the New Jersey Institute of Technology in 1987, and a PhD in mathematics from the Courant Institute of Mathematical Sciences at New York University in 1993. His dissertation received the Wilhelm Magnus award for an outstanding scientific contribution. He is currently a visiting postdoctoral fellow at Stanford University.