Unit - 3

Interpolation

The process of finding the values of $y = f(x)$ for some value of x is called Interpolation.

2 Types

i. Interpolation with equal intervals
  - Newton forward interpolation
  - Newton backward interpolation

ii. Interpolation with unequal intervals
  - Lagrange's interpolation

Newton forward interpolation

Let $y_0, y_1, y_2 \dots y_n$ be set of values of the function $y = f(x)$ Corresponding to $x_0, x_1, x_2 \dots x_n$ respectively where x values are equally spaced. Then Newton forward interpolation formula is

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

where $p = \dfrac{x - x_0}{h}$

Newton Backward interpolation

Let $y_0, y_1, y_2 \dots y_n$ be set of values of the function $y = f(x)$ Corresponding to $x_0, x_1, x_2 \dots x_n$ respectively where x values are equally spaced. Then Newton Backward interpolation formula is

$$f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \dots$$

$p = \cancel{x+}\, x - x_n$

Problems

1. Given $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$, $\sin 55^\circ = 0.8192$ and $\sin 60^\circ = 0.8660$ find $\sin 52^\circ$ using Newton Forward Interpolation formula and Estimate the error.

2. The population of a town in decimal census was given below

Year (x)	1891	1901	1911	1921	1931
population (y) in Thousands	46	66	81	93	101

Estimate population of the year 1925

3. Find the cubic polynomial which takes the following values
$y(0) = 1$, $y(1) = 0$, $y(2) = 1$, $y(3) = 10$ then find $y(4)$

1 Sol:- Forward interpolation table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

$$P = \frac{x - x_0}{h} = \frac{52 - 45}{5} = \frac{7}{5} = 1.4$$

$$f(52) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

$$= 0.7071 + (1.4)(0.0589) + \frac{(1.4)(0.4)(-0.0057)}{2!} + \frac{(1.4)(0.4)(-0.6)}{3!}(-0.0007)$$

$$= 0.7071 + 0.08246 - \cancel{0.009192} - 0.001596 + 3.92\times10^{-5}$$

$$= 0.7880032$$

$$\text{Error} = \frac{0.7880032 - 0.7880}{0.7880}\times 100$$

$$= 4.06\times10^{-4}$$

2 Sol:- Newton Backward interpolation table.

year (x)	population (y)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

$$p = \frac{x - x_n}{h} = \frac{1925 - 1931}{10} = \frac{-6}{10} = -0.6$$

$$f(1925) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^3 y_n$$

$$= 101 + (-0.6)(8) + \frac{(-0.6)(0.4)}{2}(-4) + \frac{(-0.6)(0.4)(1.4)}{6}(-1)$$

$$+ \frac{(-0.6)(0.4)(1.4)(2.4)}{24}(-3)$$

$$= 96.8368 \text{ Thousad}$$

3Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

$$p = \frac{x-0}{1} = x.$$

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0$$

$$= 1 + x(-1) + \frac{x(x-1)}{\cancel{2}}(\cancel{2}) + \frac{x(x-1)(x-2)}{\cancel{6}}\cancel{6}$$

$$= 1 - x + x^2 - x + (x^2 - x)(x-2)$$

$$= 1 - x + x^2 - x + x^3 \underbrace{- 2x^2 - x^2} + 2x$$

$$= x^3 - 2x^2 + 1$$

$$f(x) = x^3 - 2x^2 + 1$$

$$f(4) = 64 - 32 + 1$$

$$= 33.$$

Lagrange's Interpolation

If x values are unequally spaced in that case we find interpolation and extrapolation values using Lagrange's Interpolation

Let $y_0, y_1, \ldots y_n$ be set of values of function $y = f(x)$ corresponding to $x_0, x_1 \ldots x_n$ then Lagrange's interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1$$

$$+ \ldots \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})} y_n .$$

1. Using Newton forward interpolation find the value of $f(1.7)$ & $f(5.6)$

Q.

x	1	2	3	4	5	6
y	24	37	94	62	45	67

Sol:- Forward differences table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	24					
		13				
2	37		44	-133		
		57		15		
3	94		-89		237	-317 .
		-32		104		
4	62		15		-080	
		-17		24		
5	45		39			
		22				
6	67					

$$p = \frac{x - x_0}{h} = \frac{1.7 - 1}{1} = 0.7$$

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 24 + (0.7)(13) + \frac{(0.7)(-0.3)}{2}44 + \frac{(0.7)(-0.3)(-1.3)}{6}(-132) +$$

$$\frac{(0.7)(-0.3)(-1.3)(-2.3)}{24}(237) + \frac{(0.7)(-0.3)(-1.3)(-2.3)(-3.3)}{120}(\ldots)$$

$$= 24 + 9.1 - 4.62 - 6.0515 - 6.2 = 10.75.$$

2 Sol: $f(5.6)$; $p = \frac{x - x_n}{h} = \frac{5.6 - 6}{1} = -0.4$.

$$f(5.6) = y_n + p\nabla y_n + \frac{p(p+1)}{2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{6}\nabla^3 y_n +$$

$$\frac{p(p+1)(p+2)(p+3)}{24}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{120}\nabla^5 y_n$$

$$= 67 + (-0.4)22 + \frac{(-0.4)(0.6)}{2}39 + \frac{(-0.4)(0.6)(1.6)}{6}24 +$$

$$\frac{(-0.4)(0.6)(1.6)(2.6)}{24}(-80) + \frac{(-0.4)(0.6)(1.6)(2.6)(3.6)}{120}(-317)$$

$$= 64.806.$$

Q. From the following table estimate the no of students obtained marks between 40 and 45.

marks	30-40	40-50	50-60	60-70	70-80
No of students	31	42	51	35	31

Sol:- Forward difference table

marks less than x	No of stud y	Δy	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
40	31	42	9	-25	37
50	73	51	-16	12	
60	124	35	-4		
70	159	31			
80	190				

$$p = \frac{x - x_0}{h}$$

$x = 45$

$x_0 = 40$

$h = 10$

$$= \frac{45 - 40}{10}$$

$$= 5/10 = 0.5$$

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 +$$

$$\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 31 + (0.5)42 + \frac{(0.5)(-0.5)9}{2} + \frac{(0.5)(-0.5)(-1.5)}{6}(-25)$$

$$+ \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(37)$$

$$= 48.27$$

$$\approx 48.$$

No of students obtained marks b/w 40 and 45

$= 48 - 31 = 17$.

Q3 Find the polynomial $f(x)$ from the given data

x	1	2	3	4
y	74	97	34	69

Q4. Find the value of $f(10)$

x	4	9	13	15
y	24	38	47	75

Q5. Find $f(x)$ from given data

x	0	3	12	17
y	14	64	32	91

Q7
Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	74			
		23		
2	97		-86	
		-63		184
3	34		98	
		35		
4	69			

$$p = \frac{x - x_0}{h} = \frac{x-1}{1} = x-1$$

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0$$

$$= 74 + (x-1)23 + \frac{(x-1)(x-2)}{\cancel{2}}(-\overset{43}{\cancel{86}}) + \frac{(x-1)(x-2)(x-3)}{6}184$$

$$= 74 + 23x - 23 - 43x^2 - 86 + 129x + \frac{(x^2-3x+2)(x-3)}{6}184$$

$$= 74 + 23x - 23 - 43x^2 - 86 + 129x + (x^3 - 6x^2 + 11x - 6)(30.67)$$

$$= 30.67x^3 - 227x^2 + 489.37x - 293.$$

Q: From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46.

Age	45	50	55	60	65
premium (in rupees)	114.84	90.16	83.32	74.48	68.48

2. Estimate the value of f(22) and f(42) from the following data

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

pb3. Find the polynomial interpolating the data

x	0	1	2
$f(x)$	0	5	2

pb4. The table gives the distances in nautical miles of visible horizon for the given heights in feets above the earth's surface

x (height)	100	150	200	250	300	350	400
y (distance)	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the value of y when $x = 160$ ft

pb1 Forward Difference table

Sol:

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-3	45	114.84				
-1			-18.68			
	50	96.16		5.84		
			-12.84		-1.84	
	55	83.32		4		0.68
			-8.84		-1.16	
	60	74.48		2.84		
			-6.0			
	65	68.48				

$$P = \frac{x - x_0}{h}$$

Here $h = 5$. $x_0 = 45$ $x = 46$.

$$P = \frac{46-45}{5} = 1/5 = 0.2.$$

$$f(46) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{24}\Delta^4 y_0$$

$$= 114.84 + (0.2)(-18.68) + \frac{0.2(-0.8)}{2}(5.84) + \frac{(0.2)(-0.8)(-1.8)}{6}(-1.84)$$

$$+ \frac{(0.2)(-0.8)(-1.8)(-2.8)}{24}(0.68)$$

$$= 110.994.$$

pb3

Sol:

x	y	Δy	$\Delta^2 y$
0	0	5	-8
1	5	-3	
2	2		

$$p = \frac{x-x_0}{h} = \frac{x-0}{1} = x.$$

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 \cancel{+ \frac{p(p-1)(p-2)}{6}}$$

$$= 0 + x(5) + \frac{x(x-1)}{2}(-8)$$

$$= 5x - 4x^2 + 4x.$$

$$f(x) = 9x - 4x^2$$

pb2.

Sol:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
20	354	-22	-19	29	-37	45
25	332	-41	10	-8	8	
30	291	-31	2	0		
35	260	-29	2			
40	231	-27				
45	204					

$$p = \frac{x-x_0}{h}$$

Here $x = 22$; $x_0 = 20$; $h = 5$

$$\therefore p = \frac{22-20}{5} = \frac{2}{5} = 0.4$$

$f(22) =$

Inverse interpolation :

Let $x_0, x_1, x_2 \ldots x_n$ be set of x values and $y_0, y_1, y_2 \ldots y_n$ be set of y values then the inverse interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_1 + \ldots$$

$$+ \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n$$

Eg: Find x for $y = 14$ from the given table

x	1	4	9	
y	3	7	6	14

Sol: Let $x_0 = 1 \quad x_1 = 4 \quad x_2 = 9$

$y_0 = 3 \quad y_1 = 7 \quad y_2 = 6$

when $y = 14$ and $x = ?$

1

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}x_2$$

$$= \frac{(14-7)(14-6)}{(3-7)(3-6)}(1) + \frac{(14-3)(14-6)}{(7-3)(7-6)}(4) + \frac{(14-3)(14-7)}{(6-3)(6-7)}(6)$$

$$= \frac{7\times 8}{4(3)} + \frac{11(8)}{4}(4) + \frac{(11)(7)}{3(-1)}(6)$$

$= -138.33$

Eg: Find x when $f(x) = 15$

x	5	6	9	11	—
$f(x)$	12	13	14	16	15

Sol:

$$\cancel{x = \frac{(15-5)(15-6)(15-9)(15-11)}{(5-12)}}$$

$$x = \frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)} 5 + \frac{(15-12)(15-14)(15-16)}{(13-12)(13-14)(13-16)} 6$$

$$+ \frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)} 9 + \frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)}$$

$$= 1.25 - 6 + 13.5 + 2.75.$$

$$= 11.5$$

Newton's Divide Difference Interpolation

Let $x_0, x_1, \dots x_n$ be set of x values and $y_0, y_1, \dots y_n$ be set of y values then the Newton Divide difference interpolation formula is

$$f(x) = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2]$$

$$+ (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] + \dots$$

Ex: Find $f(9)$ using Newton's divide diff Interpolation

x	5	7	11	13	17
y	150	392	1452	2366	5202.

Sol:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
5	150			
		$\frac{392-150}{7-5} = 121$		
7	392		$\frac{265-121}{11-5} = 24$	
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$
11	1452		$\frac{457-265}{13-7} = 32$	
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$
13	2366		$\frac{709-457}{17-11} = 42$	
		$\frac{5202-2366}{17-13} = 709$		
17	5202			

$$f(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0$$

$$= 150 + (9-5)121 + (9-5)(9-7)(24) + (9-5)(9-11)(9-7)(1)$$

$$= 810.$$

Ex: Find $f(5.5)$ using Newton's divide difference Interpolation table

x	0	1	4	5	6
y	1	14	15	6	3

Sol: x y Δy $\Delta^2 y$ $\Delta^3 y$ $\Delta^4 y$

$$
\begin{array}{cccccc}
0 & 1 & & & & \\
 & & \frac{14-1}{1-0} = 13 & & & \\
1 & 14 & & -3.16 & & \\
 & & \frac{15-14}{4-1} = 0.33 & & 0.166 & \\
4 & 15 & & -2.33 & & 0.15 \\
 & & \frac{6-15}{5-4} = -9 & & 1.066 & \\
5 & 6 & & 3 & & \\
 & & \frac{3-6}{6-5} = -3 & & & \\
6 & 3 & & & &
\end{array}
$$

$$f(5.5) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^4 y$$

$$= 1 + (5.5-0)(13) + (5.5-0)(5.5-1)(-3.16) + (5.5-0)(5.5-1)(5.5-4)(0.166) + (5.5)(5.5-1)(5.5-4)(5.5-5)(0.15)$$

$$= 1 + 71.5 - 78.21 + 6.16275 + 2.784375$$

$$= 3.23$$

Q. using Newton's divide difference find the polynomial in x.

x	5	6	9	11
y	12	13	14	16

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
5	12			
		1		
6	13		-0.17	
		0.33		0.05
9	14		0.13	
		1		
11	16			

$$f(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0$$

$$= 12 + (x-5)(1) + (x-5)(x-6)(-0.17) + (x-5)(x-6)(x-9)(0.05)$$

$$= 12 + x - 5 - 0.17x^2 - 5.1 + 1.87x + 0.05x^3 - x^2 + 6.45x - 13.5$$

$$= 0.05x^3 - 1.17x^2 + 9.32x - 11.6$$

Numerical Differentiation

Derivatives using Forward differences

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \dots\right]$$

Derivatives using Backward difference

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

Q. Find $\frac{dy}{dx}$ at 1.1

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	7.989	0.414	-0.036	0.006	-0.002
1.1	8.403	0.378	-0.03	0.004	-0.001
1.2	8.781	0.348	-0.026	0.003	0.002
1.3	9.129	0.322	-0.023	0.005	
1.4	9.451	0.299	-0.018		
1.5	9.750	0.281			
1.6	10.031				

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right]$$

$$\cancel{= \frac{1}{0.1}\left[8.403 - \frac{1}{2}(0.378) + \frac{1}{3}(-0.03) - \frac{1}{4}(0.004)\right]}$$

$$= 10\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001)\right]$$

$$= 10\left[0.378 + 0.015 + 0.0013 + 0.00025\right]$$

$$= 3.9455$$