Numerical Methods

<u>Algebraic equation</u>: Contains only powers of x.

$x^3 - x - 1 = 0$.

<u>Transcedental equation</u>: Eg: $xe^x = 1$

Most of engineer problems involves roots of higher order equations But we know the direct formula for finding quadratic equation only in such cases we can use numerical methods

There are three methods they are

1. Bisection Method
2. Regular Falsi Method
   (or)
   Method of false position.
3. Newton Raphson Method.

<u>Bisection method working rule</u>

1. choose any two real numbers a and b such that $f(a)f(b) < 0$. Assume that $f(a) > 0$ and $f(b) < 0$ Then root lies b/w a and b.
2. Take the initial approximation $x_1 = \frac{a+b}{2}$ and find $f(x_1)$
3. Case-i, If $f(x_1) = 0$ then x_1 is root of eq.
   Case-ii, If $f(x_1) < 0$ then root lies b/w a and x_1.
   Now find $x_2 = \frac{a+x_1}{2}$ and find $f(x_2)$
   Case-iii, If $f(x_1) > 0$ then root lies b/w x_1 & b
   Now find $x_2 = \frac{x_1+b}{2}$ and find $f(x_2)$

you can repeat this process until the difference b/w two consecutive approximations are negligible.

Ex. Find root of equation $f(x) = x^3 - x - 1 = 0$ using bisection method correct upto 2 decimal points.

Sol: $f(x) = x^3 - x - 1 = 0$

$f(0) = -1 < 0$

$f(1) = -1 < 0$

$f(2) = 5 > 0$

Root lies b/w 1 and 2.

$$x_1 = \frac{1+2}{2} = 1.5$$

$$f(x_1) = 0.875 > 0$$

Root lies b/w 1.5 and 1

$$x_2 = \frac{1+1.5}{2} = 1.25$$

$$f(1.25) = -0.296 < 0$$

Root lies b/w 1.25 and 1.5

$$x_3 = \frac{1.5+1.25}{2} = 1.375$$

$$f(1.375) = 0.224 > 0$$

Root lies b/w 1.3725 and 1.375

$$x_4 = \frac{1.25+1.375}{2} = 1.3125$$

$$f(1.3125) = -0.051 < 0$$

Root lies b/w 1.3125 and 1.375

$$x_5 = \frac{1.3125+1.375}{2} = 1.34375$$

$$f(1.34375) = 0.082 > 0$$

Root lies b/w 1.34375 & 1.3125

$$x_6 = \frac{1.3475+1.3125}{2} = 1.328,$$

$f(1.3281) = 0.014 > 0$

Root lies b/w 1.3125 and 1.3281

$$x_1 = \frac{1.3125 + 1.3281}{2} = 1.3203$$

The root of eq is 1.32.

Ex: Find +ve root of equation $xe^x = 1$ which lies b/w 0 and 1.

Sol: $f(x): xe^x = 1 \Rightarrow xe^x - 1 = 0$

$f(0) = -1 < 0$

$f(1) = 1.718 > 0.$

Root lies b/w 0 and 1

$$x_1 = \frac{0+1}{2} = 1.5$$

$f(0.5) = -0.175 < 0$

Root lies b/w 0.5 and 1

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$f(0.75) = 0.5877 > 0$

Root lies b/w 0.5 and 0.75

$$x_3 = \frac{0.5+0.75}{2} = 0.625$$

$f(0.625) = 0.1676 > 0.$

Root lies b/w 0.5 and 0.625

$$x_4 = \frac{0.5+0.625}{2} = 0.5625$$

$f(0.5625) = -0.01278 < 0$

Root lies b/w 0.5625 & 0.625

$$x_5 = \frac{0.5625+0.625}{2} = 0.59375$$

Root lies b/w 0.5625 and 0.59375

Ans: 0.56

Q. Find the value of $\sqrt[4]{21}$ using bisection method upto 2 decimals.

Sol: Let $\sqrt[4]{21} = x$

$$x^4 = 21$$

$$x^4 - 21 = 0.$$

$$f(x) : x^4 - 21 = 0$$

$$f(0) = -21 < 0$$

$$f(1) = -20 < 0$$

$$f(2) = -5 < 0$$

$$f(3) = 60 > 0.$$

Root lies b/w 2 and 3

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(2.5) = 18.06 > 0$$

Root lies b/w 2 and 2.5

$$x_2 = \frac{2+2.5}{2} = 2.25$$

$$f(2.25) = 4.628 > 0$$

Root lies b/w 2 and 2.25

$$x_3 = \frac{2+2.25}{2} = 2.125$$

$$f(2.125) = -0.609 < 0$$

Root lies b/w 2.125 and 2.25

$$x_4 = \frac{2.125+2.25}{2} = 2.1875$$

$$f(2.1875) = 1.98 > 0$$

Root lies b/w 2.125 and 2.1875

$$x_5 = \frac{2.125+2.1875}{2} = 2.15625$$

$$f(2.15625) = 0.6180$$

Root lies b/w 2.125 and 2.15625

$$x_6 = \frac{2.125 + 2.15625}{2}$$

$$= 2.140625$$

$$f(2.140625) = -2.75 \times 10^{-3} < 0.$$

Root lies b/w 2.14062 and 2.15625

$$x_7 = \frac{2.14602 + 2.15625}{2}$$

$$= 2.151135$$

$$f(2.151135) = 0.412 > 0$$

Root lies b/w 2.14062 and 2.151135

$$x_8 = \frac{2.14062 + 2.151135}{2}$$

$$= 2.1458 \qquad f(2.1458) > 0.$$

~~f~~ Ans: 2.14.

Regular Falsi Method (or) Method of false position

The steps in Regular Falsi Method and Bisection Method are same but the main formula is different

Formula

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

It reduces the no of iterations Compared to bisection Method.

Disadvantage

The calculation is difficult.

Interpolation

$f(x)$

2/10/23

Q. Find the root of equation $x^3 - x - 1 = 0$ using Regular Falsi Method.

Sol: $f(x) : x^3 - x - 1 = 0$.

Root lies b/w 1 and 2.

$$x_1 = \frac{1 f(2) - 2 f(1)}{f(2) - f(1)}$$

$$= \frac{1(5) - 2(-1)}{5 - (-1)}$$

$$= \frac{7}{6} = 1.166 \quad f(1.166) = -0.58 < 0$$

Root lies b/w 1.166 and 2.

$$x_2 = \frac{1.166 f(2) - 2 f(1.166)}{f(2) - f(1.166)}$$

$$= \frac{1.166(5) - 2(-0.58)}{5 - (-0.58)}$$

$$= \frac{6.99}{5.58} = 1.252 \quad f(1.252) = -0.289 < 0$$

Root lies b/w 1.252 and 2.

$$x_3 = \frac{1.252 f(2) - 2 f(1.252)}{f(2) - f(1.252)}$$

$$= \frac{1.252(5) - 2(-0.289)}{5 - (-0.289)}$$

$$= \frac{6.838}{5.289}$$

$$= f(1.292) = -0.135 < 0$$

Root lies b/w 1.292 and 2.

$$x_4 = \frac{1.292 f(2) - 2 f(1.292)}{f(2) - f(1.292)}$$

$$= \frac{1.292(5) - 2(-0.135)}{5 - (-0.135)}$$

$$= \frac{6.13}{5.135} = 1.31$$

$$= f(1.31) = -0.061 < 0$$

Root lies b/w 1.31 and 2.

$$x_5 = \frac{1.31 f(2) - 2 f(1.31)}{f(2) - f(1.31)}$$

$$= \frac{1.31(5) - 2(-0.061)}{5 - (-0.061)}$$

$$= \frac{6.55 + 0.121}{5.061} = 1.32.$$

$$f(1.32) = -0.02 < 0$$

Root lies b/w 1.32 and 2.

$$x_6 = \frac{1.32 f(2) - 2 f(1.32)}{f(2) - f(1.32)}$$

$$= \frac{1.32(5) - 2(-0.02)}{5 - (-0.02)}$$

$$= \frac{1.32(5) - 2(-0.02)}{5.02} = 1.322.$$

Ans: 1.32

Newton Raphson Method

The steps in / iterations in Newton Raphson Method are minimum compared to that of Bisection Method and Regular Falsi Method.

Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Disadvantage

Have to find derivative.

Q. $x^3 - x - 1 = 0$

Sol: $f(x): x^3 - x - 1 = 0$

$f'(x) = 3x^2 - 1 = 0$

$f(1) = -1 < 0$

$f(2) = 5 > 0$

Root lies b/w 1 and 2.

Let $x_0 = 1.4$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.4 - \frac{f(1.4)}{f'(1.4)}$$

$$= 1.4 - \frac{0.344}{4.88}$$

$$= 1.4 - 0.0704$$

$$= 1.3296.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.3296 - \frac{0.0209}{4.3035}$$

$$= 1.3296 - 0.0048$$

$$= 1.3248.$$

$\therefore$ Root of eq $x^3 - x - 1 = 0$ is 1.32.

Q. Find the value of $\sqrt[4]{21}$.

Sol: $f(x): x^4 - 21 = 0$

Root lies b/w 2 and 3.

$f'(x) = 4x^3$

Let $x_0 = 2.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.5 - \frac{18.0625}{62.5}$$

$$= 2.5 - 0.289$$

$$= 2.211$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.211 - \frac{f(2.211)}{f'(2.211)}$$

$$= 2.211 - \frac{2.8976}{43.23}$$

$$= 2.211 - 0.067$$

$$x_2 = 2.144$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.144 - \frac{0.13}{39.42}$$

$$= 2.144 - 0.0033$$

$$= 2.1407$$

∴ The value of $\sqrt[4]{21} = 2.14$

problems

Q. Find real root of $f(x) = x^3 + x^2 + x + 7 = 0$ to 3 decimal places.

Sol: $f(-1) = 6$ $f(-2) = 1$ $f(-3) = -14$.

Root lies b/w -3 and -2.

$$x_1 = \frac{-2-3}{2} = -2.5$$

$$f(-2.5) = -4.875 < 0.$$

x_1 x_2 x_3

Root lies b/w -2 and -2.5

$$x_2 = \frac{-2-2.5}{2} = -2.25$$

$$f(x_2) = -1.5781 < 0$$

Root lies b/w -2 and -2.25

$$x_3 = \frac{-2-2.25}{2} = -2.125$$

$$f(x_3) = -0.20507 < 0$$

Root lies b/w -2 and -2.125

$$x_4 = \frac{-2-2.125}{2} = -2.0625$$

$$f(x_4) > 0.$$

Root lies b/w -2.0625 and -2.125

$$x_5 = \frac{-2.0625-2.125}{2} = -2.0938$$

$$f(x_5) > 0.$$

Root lies b/w -2.0938 and -2.125

$$x_6 = \frac{-2.0938-2.125}{2} = -2.1094.$$

$$f(x_6) = -0.04 < 0$$

Root lies b/w -2.0938 and -2.1094

$$x_7 = \frac{-2.0938-2.1094}{2}$$

$$= -2.1016.$$

$$f(-2.1016) = 0.032 > 0$$

Root lies b/w -2.1016 and -2.1094

$$x_8 = \frac{-2.1016-2.1094}{2} = -2.1055$$

$$f(-2.1055) = \underline{-6.32 \times 10^{-3}} < 0.$$

Root lies b/w -2.1016 and -2.1055

$$x_9 = \frac{-2.1016 - 2.1055}{2} = -2.10355$$

$$f(2.10355) = 0.0133 > 0.$$

Root lies b/w -2.10355 and -2.1055

$$x_{10} = \frac{-2.10355 - 2.1055}{2} = -2.1045.$$

$$f(-2.1045) = 3.75 \times 10^{-3} > 0$$

Root lies b/w -2.1055 and -2.1045

$$x_{11} = \frac{-2.1045 - 2.1055}{2} = -2.105.$$

$$f(-2.105) = -1.28 < 0.$$

Root lies b/w -2.105 and -2.1055

Ans: -2.105.

Q. Find root correct to three decimal places lying between 0 and 0.5 of equation

$$4e^{-x}\sin x - 1 = 0$$

Sol:- $f(x) = 4e^{-x}\sin x - 1$

$f(0) = -1$ $\quad f(0.5) = 0.163145.$

Root lies b/w 0 and 0.5

$$x_1 = \frac{0 + 0.5}{2} = 0.25$$

$$f(x_1) = -0.22929 < 0$$

Root lies b/w 0.25 and 0.5

$$x_2 = \frac{0.25 + 0.5}{2} = 0.375$$

$$f(x_2) = 6.90 \times 10^{-3} > 0.$$

Root lies b/w 0.25 and 0.375

$$x_3 = \frac{0.25+0.375}{2} = 0.3125$$

$$f(x_3) = -0.1002 < 0.$$

Root lies b/w 0.3125 and 0.375

$$x_4 = \frac{0.3125+0.375}{2} = 0.3438.$$

$$f(x_4) = -0.0439 < 0$$

Root lies b/w 0.3438 and 0.375

$$x_5 = \frac{0.3438+0.375}{2} = 0.3594$$

$$f(x_5) = -0.0178 < 0$$

Root lies b/w 0.3594 and 0.375

$$x_6 = \frac{0.3594+0.375}{2} = 0.3672.$$

$$f(x_6) = \cancel{0.3672} -0.5314\times10^{-4} < 0$$

Root lies b/w 0.3692 and 0.375

$$x_7 = \frac{0.3672+0.375}{2} = 0.3711$$

$$f(x_7) = 8.52\times10^{-4} > 0$$

Root lies b/w 0.3692 and 0.3711

$$x_8 = \frac{0.3692+0.3711}{2} = 0.3692.$$

$$f(x_8) = -2.1423\times10^{-3} < 0$$

Root lies b/w 0.3692 and 0.3711

$$x_9 = \frac{0.3692+0.3711}{2} = 0.3702$$

$$f(x_9) = -5.63\times10^{-4} < 0$$

Root lies b/w 0.3702 and 0.3711

$x_{10} = 0.3706.$ $f(x_{10}) = 6.59\times10^{-5} > 0.$

Root lies b/w 0.3702 and 0.3706

$x_{11} = 0.3704$.

$f(x_{11}) = -2.48 \times 10^{-4} < 0$.

$\therefore$ Root of $4e^{-x}\sin x - 1$ is $\underline{0.370}$.

Exercise

using bisection method.

1. $x^3 - 4x - 9 = 0$

Sol:- $f(0) = -9$ $f(1) = -12$ $f(2) = -9$ $f(3) = 6$.

Root lies b/w 2 and 3.

$x_1 = \frac{2+3}{2} = 2.5$ $f(x_1) = -3.375$

Root lies b/w 2.5 and 3.

$x_2 = \frac{2.5+3}{2} = 2.75$ $f(x_2) = 0.79 > 0$.

Root lies b/w 2.5 and 2.75

$x_3 = \frac{2.5+2.75}{2} = 2.625$ $f(x_3) = -1.412 < 0$.

Root lies b/w 2.625 and 2.75

$x_4 = \frac{2.625+2.75}{2} = 2.6875$ $f(x_4) = -0.339 < 0$

Root lies b/w 2.6875 and 2.75

$x_5 = \frac{2.6875+2.75}{2} = 2.7187$ $f(x_5) = 0.22 > 0$

Root lies b/w 2.7187 and 2.75

$x_6 = \frac{2.7187+2.75}{2} = 2.70825$ $f(x_6) = -0.066$

Root lies b/w 2.70285 and 2.7187

$x_7 = \frac{2.70285+2.7187}{2} = 2.710775$ $f(x_7) = 0.0794$.

[illegible] and 0.7109

$x = \frac{[illegible] + 0.7109}{2} = 0.707$

$f(x) = 0.009$

Ans [illegible]

ii. $x^3 + x^2 - 1 = 0$

$f(0) = -1$

$f(1) = 1$

Root lies b/w 0 and 1

$x_1 = \frac{0+1}{2} = 0.5$

$f(x_1) = -0.0156 < 0$

Root lies b/w 0.5 and 1

$x_2 = \frac{0.5+1}{2} = 0.75$

$f(x_2) = 0.875 > 0$

Root lies b/w 0.75 and 1

$x_3 = \frac{0.75+1}{2} = 0.875$

$f(x_3) = 0.[illegible] > 0$

Root lies b/w 0.75 and 0.875

$x_4 = \frac{0.75+0.875}{2} = 0.8125$

$f(x_4) = 0.1965 > 0$.

Root lies b/w 0.75 and 0.8125.

$x_5 = \frac{0.75+0.8125}{2} = 0.7812$.

$f(x_5) = 0.0872 > 0$

Root lies b/w 0.75 and 0.7812.

$x_6 = \frac{0.75 + 0.78}{2} = 0.7656$

$f(x_6) = 0.035 > 0$

Root lies b/w 0.75 and 0.7656

$x_7 = \frac{0.75 + 0.7656}{2} = 0.7578$

$f(x_7) = 0.0095$

Root lies b/w 0.75 and 0.7578.

Ans: 0.75

iii $5x \log_{10} x - 6 = 0$

Sol: $f(2) = -2.9897$ $f(3) = 1.1563$

Root lies b/w 2 and 3.

$x_0 = \frac{2+3}{2} = 2.5$

$f(x_0) = -1.0257$

Root lies b/w 2.5 and 3

$x_1 = \frac{2.5+3}{2} = 2.75$

$f(x_1) = 0.0408 > 0$

Root lies b/w 2.5 and 2.75

$x_2 = \frac{2.5 + 2.75}{2} = 2.625$

$f(x_2) = -0.4938 < 0$

Root lies b/w 2.625 and 2.75

$x_3 = \frac{2.625 + 2.75}{2} = 2.6875$

$f(x_3) = -0.2306 < 0$

Root lies b/w 2.6875 and 2.75

$x_4 = \frac{2.6875 + 2.75}{2} = 2.7188$ $f(x_4) = -0.0953$

Root lies b/w 2.7188 and 2.75

$$x_5 = \frac{2.7188 + 2.75}{2} = 2.7344$$

$f(x_5) = -0.0273$

Root lies b/w 2.7344 and 2.75

$$x_6 = \frac{2.7344 + 2.75}{2} = 2.7422$$

$f(x_6) = 0.0067$.

Root lies b/w 2.7344 and 2.7422

$$x_7 = \frac{2.7402 + 2.7422}{2} = 2.7412$$

$f(x_7) = -0.0018$

Ans: 2.74

Q. Solve $x^3 - x - 4 = 0$ by using Regular Falsi Method & Newton Raphson Method

Sol: $f(1) = -4$

$f(2) = 2$

Root lies b/w 1 and 2.

$x_0 = 1$

$x_1 = 2$.

$$x_2 = \frac{2f(1) - 1f(2)}{f(1) - f(2)} = \frac{1(-4) - 2(-1)}{2-(-4)}$$

$= 1.6667$

$f(x_2) = -1.037$

Root lies b/w 1.6676 and 2.

$$x_3 = 1.6667 - (-1.037). \frac{2 - 1.6667}{2 - (-1.037)}$$

$= 1.7805$

$f(x_3) = -0.1361 < 0$

Root lies b/w 1.7805 and 2

$$x_4 = 1.7805 - (-0.1361) \cdot \frac{2-1.7805}{2-(-0.1361)} = 1.7945$$

$f(x_4) = -0.016$

Root lies b/w 1.7945 and 2

$$x_5 = 1.7945 - (-0.016) \cdot \frac{2-1.7945}{2-(-0.016)} = 1.7961$$

$f(x_5) = -0.0019$

Q. $x\tan x + 1 = 0$

Sol:- $f(1) = 2.5574$

$f(2) = -3.3701$

Root lies b/w 1 and 2

$$x_2 = 1 - 2.5574 \cdot \frac{2-1}{-3.3701-2.3574} = 1.4314$$

$f(x_2) = 11.2059$

Root lies b/w 1.4314 and 2

$$x_3 = 1.4314 - 11.2059 \cdot \frac{2-1.4314}{-3.3701-11.2059} = 1.8685$$

$f(x_3) = -5.089$

Root lies b/w 1.4314 and 1.8685

$$x_4 = 1.4314 - 11.2059 \cdot \frac{1.8685-1.4314}{-5.089-11.2059} = 1.732$$

Root lies b/w 1.4314 and 1.732

$$x_5 = 1.4314 - 11.2059 \cdot \frac{1.732 - 1.4354}{-9.6486 - 11.2059}$$

$$= 1.593$$

$$f(x_5) = -70.836.$$

Q. Solve $e^x = 3x$ by Newton Raphson Method.

Sol: $f(x) = e^x - 3x = 0$

$$f'(x) = e^x - 3$$

$$f(0) = 1$$

$$f(1) = -0.2817$$

$$x_0 = \frac{0+1}{2} = 0.5$$

$$f(x_0) = 0.1487$$

$$f'(x_0) = -1.3513$$

$$x_1 = 0.5 - \frac{0.1487}{-1.3513}$$

$$= 0.61061$$

$$f(x_1) = 0.0104$$

$$f'(x_1) = -1.1595$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.619$$

$$f(x_2) = 0.0001$$

$$f'(x_2) = -1.1429$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6191$$

Ans: 0.619.

Q. $x = \frac{1}{(x+1)^2}$

Sol: $x(x+1)^2 = 1$

$x(x^2+2x+1) = 1$

$f(x) \cdot x^3 + 2x^2 + x - 1 = 0$

$f'(x) = 3x^2 + 4x + 1 = 0$

$f(0) = -1$

$f(1) = 3$

$x_0 = \frac{0+1}{2} = 0.5$

$f(x_0) = 0.125$

$f'(x_0) = 3.75$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0.5 - \frac{0.125}{3.75}$

$= 0.4667$

$f(x_1) = 0.0039$

$f'(x_1) = 3.52$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.4667 - \frac{0.0039}{3.52}$

$= 0.4656$

$f(x_2) = 0$

$f'(x_2) = 3.5126$

Ans: 0.46

8.27

Q. $x^3 - 2x^2 + 3x = 5$ b/w 1 and 2 using

a) False position

b) Newton Raphson

Sol:- False position Method

$f(1) = -3$

$f(2) = 1$

$x_0 = 1$

$x_1 = 2$.

$$x_2 = 1 - (-3) \cdot \frac{2-1}{1-(-3)} = 1.75$$

$f(x_2) = -0.5156$

Root lies b/w 1.75 and 2.

$$x_3 = 1.75 - (-0.5156) \cdot \frac{2-1.75}{1-(-0.5156)}$$

$= 1.8351$

$f(x_3) = -0.0503$

Root lies b/w 1.8351 and 2

$$x_4 = 1.8351 - (-0.0503) \cdot \frac{2-1.8351}{1-(-0.0503)} = 1.843$$

$f(x_4) = -0.0046$.

Ans: 1.843

Newton Raphson Method

$f'(x) = 3x^2 - 4x + 3$

$x_0 = \frac{1+2}{2} = 1.5$

$f(x_0) = 1.625$

$f'(x_0) = 3.75$

$x_1 = 1.5 - \frac{(-1.625)}{3.75}$

$= 1.9333$

$f(x_1) = 0.5508$

$f'(x_1) = 6.48$

$x_2 = 1.9333 - \frac{0.5508}{6.48}$

$= 1.8483$

$f(x_2) = 0.0268$

$f'(x_2) = 58557$

$x_3 = 1.8483 - \frac{0.0268}{5.8357}$

$= 1.8437$

Ans: 1.843

Q. $\sin x + x - 1 = 0$

Sol: $f'(x) = \cos x + 1$

$f(0) = -1$

$f(1) = 0.8415$

$x_0 = \frac{0+1}{2} = 0.5$

$f(x_0) = -0.0206$

$f'(x_0) = 1.8776$

$x_1 = 0.5 - \frac{-0.0206}{1.8776}$

$\doteq 0.511$

$f(x_1) = 0$

Q) $x\, e^{x} - \cos x = 0$

$f(x) = x e^{x} - \cos x$

$f'(x) = e^{x} + x e^{x} + \sin x$

$f(0) = -1$

$f(1) = 2.178$

$x_0 = \frac{0+1}{2} = 0.5$

$f(x_0) = -0.0532$

$f'(x_0) = 2.9525$

$$x_1 = 0.5 - \frac{(-0.0532)}{2.9525}$$

$= 0.518$

$f(x_1) = 0.0008$

$f'(x_1) = 3.0435$

$$x_2 = 0.518 - \frac{0.0008}{3.0435}$$

$= 0.5178$

$f(x_2) = 0.$

Q. $x^3 - 4x^2 + 5x - 2 = 0$

Sol: $f'(x) = 3x^2 - 8x + 5$

$f(0) = -2$

$f(1) = 0.$

Root is 1

Q. $x^3 - 2x^2 - 5x + 6 = 0$

Sol: $f'(x) = 3x^2 - 4x - 5$

$f(0) = 6$

$f(1) = 0$

One of roots of eq $x^3 - 2x^2 - 5x + 6$ is 1

Q. $f(x) = x^3 - 6x^2 + 11x - 6 = 0$

Sol: $f'(x) = 3x^2 - 12x + 11$

$f(0) = -6$

$f(1) = 0$

One of roots of eq $x^3 - 6x^2 + 11x - 6 = 0$ is 1.

Difference operators

1. Forward difference operator (Δ)
2. Backward difference operator (∇) (nabla or del
3. Shift operator (E)
4. Central difference operator (δ)
5. Averaging operator (μ)

Difference opero

Forward Difference operator.

Let $y_0, y_1, \cdots y_n$ denotes a set of values of y then $y_1 - y_0, y_2 - y_1, \cdots y_n - y_{n-1}$ are called the differenc of y denoting these differences by $\Delta y_0, \Delta y_1, \cdots$ Δy_{n-1} respectively we have

$\Delta y_0 = y_1 - y_0$

$\Delta y_1 = y_2 - y_1$

$\vdots$

$\Delta y_{n-1} = y_{n} - y_{n-1}$

where Δ is forward difference operator.

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$

In general we can write

$$\boxed{\Delta^k y_n = \Delta^{k-1} y_{n+1} - \Delta^{k-1} y_n}$$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		Δy_0		
x_1	y_1		$\Delta^2 y_0$	
		Δy_1		$\Delta^3 y_0$
x_2	y_2		$\Delta^2 y_1$	
		Δy_2		
x_3	y_3			

Backward differences

The differences $y_1 - y_0$, $y_2 - y_1$, ... $y_n - y_{n-1}$ are called first backward differences if they are denoted by $\nabla y_1, \nabla y_2, \dots \nabla y_n$. So that

$\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_0$

$\vdots$

$\nabla y_n = y_n - y_{n-1}$

∇ is del / nabla

Backward differenc table

x	y	∇y	$\Delta^2 y$	$\delta^2 y$
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^2 y_3$.
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

Central differenc operator (δ)

$$\boxed{\delta y_n = y_{n+1/2} - y_{n-1/2}}$$

$$\delta y_{1/2} = y_1 - y_0$$

$$\delta y_{3/2} = y_2 - y_1$$

$$\delta y_{5/2} = y_3 - y_2 .$$

Shift operator (E)

$$\boxed{E y_r = y_{r+1}}$$

$$E^2 y_r = E y_{r+2} .$$

and in general

$$\boxed{E^n y_r = y_{r+n}}$$

Averaging operator (μ)

$$\boxed{\mu y_n = \frac{y_{n+1/2} + y_{n-1/2}}{2} .}$$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\nabla f(x) = f(x) - f(x-h)$$

$$Ef(x) = f(x+h)$$

Q) Show that

$$\Delta\left[\frac{1}{f(x)}\right] = \frac{-\Delta f(x)}{f(x) \cdot f(x+1)}$$

Sol:
$$\Delta\left[\frac{1}{f(x)}\right] = \frac{1}{f(x+h)} - \frac{1}{f(x)}$$

$$= \frac{f(x) - f(x+h)}{f(x) f(x+h)}$$ put $h=1$

$$= \frac{-[f(x+1) - f(x)]}{f(x) f(x+1)}$$

$$= \frac{-\Delta f(x)}{f(x) f(x+1)}.$$

Hence proved.

Q. show that

$$\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right].$$

Sol:
$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log f(x+1) - \log f(x)$$

$$= \log\left[\frac{f(x+1)}{f(x)}\right]$$

$$= \log\left[\frac{f(x) + f(x+1) - f(x)}{f(x)}\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Hence proved

Q. show that $\Delta^n a^x = (a-1)^n a^x$.

Sol.

$$\Delta a^x = a^{x+h} - a$$

Let $h=1$ then

$$\Delta a^x = a^x(a-1)$$

$$\Delta^2 a^x = \Delta(\Delta a^x)$$

$$= \Delta(a^{x+1} - a^x a)$$

$$= \Delta(a^x(a-1))$$

$$= (a-1)a^{x+1} - (a-1)a^x$$

$$= (a-1)a^x[a-1]$$

$$= (a-1)^2 a^x.$$

$$\Delta^n a^x = (a-1)^n a^x$$

Hence proved.

Q. $\Delta^2\left(\frac{1}{x^2+5x+6}\right)$

$$= \Delta\left(\Delta\left(\frac{1}{(x+2)(x+3)}\right)\right)$$

$$= \Delta\left[\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right]$$

$$= \Delta\left[\frac{x+2-x-4}{(x+2)(x+3)(x+4)}\right]$$

$$= \Delta\left[\frac{-2}{(x+2)(x+3)(x+4)}\right]$$

$$= \frac{-2}{(x+3)(x+4)(x+5)} + \frac{2}{(x+2)(x+3)(x+4)}$$

$$= \frac{-2(x+2)+2(x+5)}{(x+2)(x+3)(x+4)(x+5)}$$

$$= \frac{10-4}{(x+2)(x+3)(x+4)(x+5}$$

$$= \frac{6}{(x+2)(x+3)(x+4)(x+5)}$$

Q. $\Delta\left(\frac{1}{x(x+1)(x+4)}\right)$

Sol: $\Delta\left(\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+4}\right)$

$$\frac{1}{x(x+1)(x+4)} = \frac{A(x+1)(x+4)+Bx(x+4)+Cx(x+1)}{x(x+1)(x+4)}$$

put $x = -0$.

$4A = 1 \Rightarrow A = 1/4$.

put $x = -1$

$-B(\cancel{x}\,3) = 1$

$B = -1/3$.

put $x = -4$

$-4C(-3) = 1$

$C = 1/12$.

$\frac{1}{4}\Delta(1/x) - \frac{1}{3}\Delta\left(\frac{1}{x+1}\right) + \frac{1}{12}\Delta\left(\frac{1}{x+4}\right)$

$\frac{1}{4}\left[\frac{1}{x+1} - \frac{1}{x}\right] - \frac{1}{3}\left[\frac{1}{x+2} - \frac{1}{x+1}\right] + \frac{1}{12}\left[\frac{1}{x+5} - \frac{1}{x+4}\right].$

Relations b/w difference operators

i, $\Delta = E - 1$

ii, $\nabla = 1 - E^{-1}$

iii, $\delta = E^{1/2} - E^{-1/2}$

iv, $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

v, $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

vi, $E = e^{hD}$

proof 1 $\Delta y_n = y_{n+1} - y_n$

$= Ey_n - y_n$

$\Delta y_n = (E-1)\, y_n$

$\boxed{\Delta = E - 1}$

proof 2 $\nabla y_n = y_n - y_{n-1}$

$= y_n - E^{-1} y_n$

$\nabla y_n = y_n (1 - E^{-1})$

$\boxed{\nabla = 1 - E^{-1}}$

(iii) $\delta y_n = y_{n+h/2} - y_{n-h/2}$

put $h = 1$.

$\delta y_n = E^{1/2} y_n - E^{-1/2} y_n$

$\delta y_n = y_n (E^{1/2} - E^{-1/2})$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

proof. iv.

$\mu y_n = \frac{1}{2}[y_{n+1/2} + y_{n-1/2}]$

$= \frac{1}{2}[E^{1/2} y_n + E^{-1/2} y_n]$

$\mu y_n = \frac{1}{2}[E^{1/2} + E^{-1/2}] y_n$

$\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$.

proof - v.

Consider $E\nabla y_n$

$E\nabla y_n = E[y_n - y_{n-1}]$

$= y_{n+1} - y_n$

$E\nabla y_n = \Delta y_n$

$$\boxed{E\nabla = \Delta}$$

$\nabla E y_n = \nabla(y_{n+1})$

$= y_{n+1} - y_n = \Delta y_n$.

$$\boxed{\nabla E = \Delta}$$

$$\delta E^{1/2} y_n = \delta\left(y_{n+1/2}\right)$$

$$= y_{n+1/2+1/2} - y_{n+1/2-1/2}$$

$$= y_{n+1} - y_n$$

$$\delta E^{1/2} y_n = \Delta y_n$$

$$\boxed{\delta E^{1/2} = \Delta}$$

<u>proof - vi</u>

$$E = e^{hD}$$

$$Ef(x) = Ef(x+h)$$

$$= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \cdots$$

$$= f(x)\left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} \cdots\right]$$

$$Ef(x) = f(x)\, e^{hD}$$

$$\boxed{E = e^{hD}}$$

PT

i. $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

Sol: We know that

$$e^{hD} = E = 1+\Delta$$

$$hD = \log(1+\Delta)$$

Also $hD = \log E$

$$= -\log E^{-1}$$

$$= -\log(1-\nabla)$$

$$\mu\delta = \frac{E^{1/2}+E^{-1/2}}{2} \quad E^{1/2}-E^{-1/2}$$

$$= \frac{E-E^{-1}}{2}.$$

$$= \frac{1}{2}(e^{hD}-e^{-hD})$$

$$= \sinh(hD)$$

$$hD = \sinh^{-1}(\mu\delta).$$

ii,
P.T

$$(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta.$$

LHS: $(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2}$

$$(E^{1/2}+E^{-1/2})(E^{1/2})$$

$$= E+1$$

$$= \Delta+1+1 = \Delta+2.$$

Imp
iii

$$\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\delta^2/4}$$

Sol: RHS $= \frac{1}{2}\delta^2 + \delta\sqrt{1+\delta^2/4}$

$$= \frac{1}{2}\left[E^{1/2}-E^{-1/2}\right]^2 + E^{1/2}-E^{-1/2}\sqrt{1+\frac{(E^{1/2}-E^{-1/2})^2}{4}}$$

$$= \frac{1}{2}\left[E+E^{-1}-2\right] + E^{1/2}-E^{-1/2}\sqrt{1+\frac{E+E^{-1}-2}{4}}$$

$$= \frac{1}{2}\left[E+E^{-1}-2\right] + E^{1/2}-E^{-1/2}\sqrt{\frac{E+E^{-1}+2}{\cancel{2}4}}$$

$$= \frac{1}{2}\left[E+E^{-1}-2\right] + (E^{1/2}-E^{-1/2})\,\frac{E^{1/2}+E^{-1/2}}{2}$$

$$= \frac{1}{2}\left[E+E^{-1}-2\right] + \frac{E-E^{-1}}{2}$$

$$= \frac{E+E^{-1}}{2} - 1 + \frac{E-E^{-1}}{2}$$

$$= E-1$$

$$= \Delta$$

Q)
P.T

$$\Delta^3 y_2 = \nabla^3 y_5$$

Sol: LHS $\Delta^3 y_2 = (E-1)^3 y_2$

$$= (E^3-3E^2+3E-1)\,y_2$$

$$= y_5 - 3y_4 + 3y_3 - y_2$$

$$\nabla^3 y_5 = (1-E^{-1})^3 y_5$$

$$= (1-3E^{-1}+3E^{-2}-E^{-3})\,y_5$$

$= y_5 - 3y_4 + 3y_3$

$\Delta^3 y_2 = \nabla^3 y_5$

Hence proved

Q) prove that

$$\mu^2 = 1 + \frac{\delta^2}{4}$$

Sol: $RHS = 1 + \frac{\delta^2}{4}$

$$= 1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}$$

$$= 1 + \frac{E + E^{-1} - 2}{4}$$

$$= \frac{4 + E + E^{-1} - 2}{4}$$

$$= \frac{E + E^{-1} + 2}{4}$$

$$= \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)^2$$

$$= \mu^2$$

$RHS = LHS$

Q) prove that

$$\nabla^r f_k = \Delta^r f_{k-r}$$

Sol: $RHS = \Delta^r f_{k-r}$

$$= (E-1)^r f_{k-r}$$

$$= E^r (1 - E^{-1})^r f_k$$

$= (1-E^{-1})' f_K$

$= \nabla' f_K$

RHS = LHS

Q) $\Delta f_K^2 = (f_K + f_{K+1}) \Delta f_K$

Sol: RHS: $(f_K + f_{K+1}) \Delta f_K$

$= (f_K + f_{K+1})(f_{K+1} - f_K)$

$= f_{(K+1)^2} - f_K^2$

$= \Delta f_K^2$

RHS = LHS

Q. Find the missing values from the following data

x	0	1	2	3	4	5
y	125	-	454	229	324	576

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	125	a-125	579-2a	3a-1258	2257-4a	5a-3419
1	a	454-a	a-679	999-a	a-1162	
2	454	-225	320	-163		
3	229	95	157			
4	324	252				
5	576					

$5a - 3419 = 0 \Rightarrow a = \frac{3419}{5} = 683.8$

x	1	3	5	7	9	11	13
y	47	64	78	-	91	-	68

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	47	17	-3	$a-89$	$350-4a$	$10a+b-962$
3	64	14	$a-92$	$261-3a$	$6a+b-612$	$1304-10a-5b$
5	78	$a-78$	$169-2a$	$3a+b-351$	$692-4a-4b$	
7	a	$91-a$	$a+b-182$	$341-a-3b$		
9	91	$b-91$	$159-2b$			
11	b	$68-b$				
13	68					

$$10a+b-962=0 \rightarrow (1)$$

$$10a+5b-1304=0 \rightarrow (2)$$

$$-4b = -342$$

$$b = 85.5$$

$$10a-876.5=0$$

$$a = 87.65$$

Q. Evaluate $\Delta \tan^{-1}x$.

Sol: $\Delta \tan^{-1}x = \tan^{-1}(x+1) - \tan^{-1}x$

$$= \tan^{-1}\left(\frac{x+1-x}{1+(x+1)x}\right)$$

$$= \tan^{-1}\left(\frac{1}{1+x+x^2}\right).$$

Q. Evaluate $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$.

Sol:- $\frac{\Delta^2 - \nabla^2}{\nabla\Delta}$

$\Delta = E - 1$

$\nabla = 1 - E^{-1}$

$$= \frac{(E-1)^2 - (1-E^{-1})^2}{(E-1)(1-E^{-1})}$$

$$= \frac{E^2+1-2E-(1+E^{-2}-2E^{-1})}{E-1-1+E^{-1}}$$

$$= \frac{E^2+1-2E-1-E^{-2}+2E^{-1}}{E+E^{-1}-2}$$

$$= \frac{E^2-\frac{1}{E^2}-2E+\frac{2}{E}}{E+\frac{1}{E}-2}$$

$$= \frac{\frac{E^4-1-2E^3+2E}{E^2}}{\frac{E^2+1-2E}{E}}$$

$$= \frac{E^4-2E^3+2E-1}{E^2-2E+1}$$

$$\frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$$

$$\frac{E-1}{1-\frac{1}{E}} - \frac{1-\frac{1}{E}}{E-1}$$

$$\frac{E-1}{\frac{E-1}{E}} - \frac{\frac{E-1}{E}}{E-1}$$

$$E - \frac{1}{E}$$

$$\therefore \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = E - \frac{1}{E}$$

Q)

x	1	3	5	7	9	11	13	15	17
y	494	-	687	-	432	-	964	-	794

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	Δ^4
1	494				
		a-494			
3	a		1181-2a		
		687-a		3a+b-2555	
5	687		a+b-1374		5048-4a-4b
		b-687		2493-a-3b	
7	b		1119-2b		a+6b+c-4476
		432-b		3b+c-1983	
9	432		b+c-864		4243-4b-4c
		c-432		2260-b-3c	
11	c		1396-2c		b+6d+d-5584
		964-c		3c+d-3324	
13	964		c+d-1928		7010-4c-4d
		d-964		3686-2c-3d	
15	d		1758-2d		
		794-d			
17	794				