Units – 3: Estimation and Test of Hypothesis of Means

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3.1 Estimation

3.1.1 Basics

Estimation: Estimating a population parameter using sample data is called Estimation.
It is two types (i) Point Estimation (ii) Interval Estimation
Point Estimation: Estimating a population parameter in terms of a single numerical value is called Point Estimation.
Interval Estimation: Estimating a population parameter in terms of an interval is called Interval Estimation.
Estimator: If a statistic θ is used to estimate a population parameter λ, then θ is called an estimator for λ
Unbiased Estimator: A statistic θ is called an unbiased estimator for a population parameter λ if the mean of the sampling distribution the statistic θ is λ; that is $\mu_\theta = \lambda$ or $E[\theta] = \lambda$
For example,
(i) We know that $\mu_{\overline{X}} = \mu$ or $E[\overline{X}] = \mu$
Therefore, the statistic $\overline{X}$ is an unbiased estimator for the population parameter μ
In other words, **sample mean is an unbiased estimator for the population mean**
(ii) Similarly, $\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ is an unbiased estimator for the population variance σ^2
(iii) If X is Binomial random variable with parameters n and p, then we have $E\left[\frac{X}{n}\right] = \frac{np}{n} = p$
Therefore, the statistic $\frac{X}{n}$ is an unbiased estimator for the population parameter p
Here p is called proportion and $\frac{X}{n}$ is called sample proportion

More efficient unbiased estimator: Let λ be a population parameter and θ_1, θ_2 be two statistics such that
(i) θ_1 and θ_2 are unbiased estimators for λ; that is, $E[\theta_1] = \lambda$ and $E[\theta_2] = \lambda$
(ii) The variance of the sampling distribution of the statistic θ_1 is less than that of the statistic θ_2; that is, $V[\theta_1] < V[\theta_2]$
Then θ_1 called more efficient unbiased estimator than θ_2 for λ

3.1.2 Maximum Error of the Mean

Case (i): For large samples $(n \geq 30)$ or σ known,

$E = z_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{n}}\right)$ is called the maximum error of the Mean with the probability $1-\alpha$

Case (ii): For small samples $(n < 30)$ and σ unknown,

$E = t_{\frac{\alpha}{2}}\left(\frac{s}{\sqrt{n}}\right)$ is called the maximum error of the Mean with the probability $1-\alpha$

Note: For a large sample of size n, verify that the probability is $1-\alpha$ that the mean of a random sample from an infinite population with standard deviation σ differ from μ by less than the maximum error E;
That is, $P\left(\left|\overline{X}-\mu\right| < E\right) = 1-\alpha$

Consider, $P\left(\left|\overline{X}-\mu\right| < E\right) = P\left(\left|\overline{X}-\mu\right| < z_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{n}}\right)\right) \quad = P\left(\left|\frac{\overline{X}-\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}\right| < z_{\frac{\alpha}{2}}\right) = P\left(|Z| < z_{\frac{\alpha}{2}}\right)$

$$= P\left(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}\right) \quad = F\left(z_{\frac{\alpha}{2}}\right) - F\left(-z_{\frac{\alpha}{2}}\right) = F\left(z_{\frac{\alpha}{2}}\right) - \left[1 - F\left(z_{\frac{\alpha}{2}}\right)\right]$$

$$= \left(1-\frac{\alpha}{2}\right) - \left[1 - \left(1-\frac{\alpha}{2}\right)\right] \qquad = 1-\alpha$$

3.1.3 Confidence Interval - One Mean:

Let $\bar{x}$ be the mean of a random sample of size n, taken from a population having mean μ and variance σ^2 and E be the maximum error of the mean. Then

$\bar{x} \pm E$ are called the **Upper and Lower confidence limits** of the Mean μ with the probability $1-\alpha$ or $(1-\alpha)100\%$ confidence. And the interval $(\bar{x}-E, \bar{x}+E)$ is called the **Confidence Interval** of the Mean μ with the probability $1-\alpha$ or $(1-\alpha)100\%$ confidence.

Note: (i) For large samples $(n \geq 30)$ or σ known,
Upper and Lower confidence limits: $\bar{x} \pm E$

Confidence Interval: $(\bar{x}-E, \bar{x}+E)$ where, $E = z_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{n}}\right)$

(ii) For small samples $(n < 30)$ and σ unknown,

Upper and Lower confidence limits: $\bar{x} \pm E$

Confidence Interval: $(\bar{x}-E, \bar{x}+E)$ where, $E = t_{\frac{\alpha}{2}}\left(\frac{s}{\sqrt{n}}\right)$

Maximum Error & Confidence Interval for ONE MEAN

S.No.		Maximum Error	Confidence Interval	Probability/ Confidence
1	One Mean Large Samples $(n \geq 30)$ or σ known	$E = z_{\frac{\alpha}{2}}\left(\frac{\sigma}{\sqrt{n}}\right)$	$(\bar{x} - E, \bar{x} + E)$	$1-\alpha$ / $(1-\alpha)100\%$
2	One Mean Small Samples $(n < 30)$ and σ unknown	$E = t_{\frac{\alpha}{2}}\left(\frac{s}{\sqrt{n}}\right)$	$(\bar{x} - E, \bar{x} + E)$	$1-\alpha$ / $(1-\alpha)100\%$

Note:

(1)* $z_{0.01} = 2.33$ **(2)*** $z_{0.05} = 1.645$ **(3)*** $z_{0.005} = 2.575$ **(4)*** $z_{0.025} = 1.96$

3.2 Test of Hypothesis

3.2.1 Basics

Hypothesis: A definite statement about a population parameter is called a Hypothesis

Examples:

1. Average height of the students in a Class is 5.8ft; that is $\mu = 5.8$
2. Average height of the students in a Class is at most 6.2ft; that is $\mu \leq 6.2$
3. Average height of the students in a Class is at least 5.1ft; that is $\mu \geq 5.1$
4. $\mu > 15$
5. $\mu \geq 20$
6. Variance of the marks obtained by the students of SSC in Mathematics is 21; that is $\sigma^2 = 18$
7. $\sigma^2 < 8$
8. $\sigma^2 \geq 8$

Classification of Hypothesis: It is classified as two types (i) Simple Hypothesis (ii) Composite Hypothesis

Simple Hypothesis: A hypothesis which gives complete information about a population parameter is called a simple hypothesis. In other words, a hypothesis which contains the symbol $'='$ is called a simple hypothesis.

Composite Hypothesis: A hypothesis which is not simple is called a composite hypothesis.

Examples:

1. $\mu > 15$ is composite
2. $\mu = 15$ is simple
3. $\mu \geq 20$ is composite
4. $\sigma^2 = 18$ is simple
5. $\sigma^2 < 8$ is composite

Testing of Hypothesis: Verifying the validity of a hypothesis using a given sample data is called a Testing of Hypothesis

Errors in Testing of Hypothesis: If a hypothesis is true and accepted or a hypothesis is false and rejected, then in either case there is no error in the decision. Otherwise, there is an error in the decision and this error is in types. (i) Type I Error (ii) Type II Error
Type I Error: If a hypothesis is true but rejected, then there is an error in rejecting is called Type I error. The probability of obtaining Type I error is called **Level of Significance (LoS)** and it is denoted by α
Type II Error: If a hypothesis is false but accepted, then there is an error in accepting is called Type II error. The probability of obtaining Type II error is denoted by β
Null Hypothesis (NH): In the testing of a hypothesis, a hypothesis which is formulated for the sake rejection under the assumption that it is true. Null hypothesis is denoted by H_0
Note: Usually a null hypothesis is simple
Alternative Hypothesis (AH): In the testing of a hypothesis, a hypothesis which is not the null hypothesis is called Alternative hypothesis and it is denoted by H_1
Note: Usually Alternative hypothesis is composite
Examples:

1. Null hypothesis $H_0 : \mu = 15$, Alternative hypothesis $H_1 : \mu < 15$
2. Null hypothesis $H_0 : \mu = 15$, Alternative hypothesis $H_1 : \mu > 15$
3. Null hypothesis $H_0 : \mu = 15$, Alternative hypothesis $H_1 : \mu \neq 15$
4. Null hypothesis $H_0 : \mu \geq 15$, Alternative hypothesis $H_1 : \mu < 15$
5. Null hypothesis $H_0 : \mu \leq 15$, Alternative hypothesis $H_1 : \mu > 15$

Classification of Tests Hypothesis: Tests of hypothesis are classified as two types (i) One Tailed Test (OTT) (ii) Two Tailed Test (TTT)
And One Tailed Tests are classified as two types (i) Left One Tailed Test (LOTT) (ii) Right One Tailed Test (ROTT)
Left One Tailed Test (LOTT): In the testing of a hypothesis, if the alternative hypothesis H_1 contains the symbol $'<'$, then the test is called Left One Tailed Test
Right One Tailed Test (ROTT): In the testing of a hypothesis, if the alternative hypothesis H_1 contains the symbol $'>'$, then the test is called Right One Tailed Test
Two Tailed Test (TTT): In the testing of a hypothesis, if the alternative hypothesis H_1 contains the symbol $'\neq'$, then the test is called Two Tailed Test

Examples:

1. In a testing of a hypothesis, if
   Null hypothesis $H_0 : \mu = 15$, Alternative hypothesis $H_1 : \mu < 15$
   Then it is One Tailed Test or Left One Tailed Test

2. In a testing of a hypothesis, if
   Null hypothesis $H_0 : \mu = 15$, Alternative hypothesis $H_1 : \mu > 15$
   Then it is One Tailed Test or Right One Tailed Test

3. In a testing of a hypothesis, if
   Null hypothesis $H_0 : \mu = 15$
   Alternative hypothesis $H_1 : \mu \neq 15$
   Then it is Two Tailed Test

Critical region: The region under a probability curve in which H_0 is rejected, is called Critical region

Guidelines for formulating H_0 **and** H_1**:** When the goal of an experiment is to establish an assertion, the negation of the assertion should be taken as the Null hypothesis H_0. The assertion becomes the Alternative hypothesis H_1. In detail we have the following.

S.No.	Claim / Assertion	H_0	H_1	Type of test
1	$\mu \le \mu_0$ or $\mu < \mu_0$	$\mu = \mu_0$	$\mu < \mu_0$	LOTT
2	$\mu \ge \mu_0$ or $\mu > \mu_0$	$\mu = \mu_0$	$\mu > \mu_0$	ROTT
3	$\mu = \mu_0$ or $\mu \ne \mu_0$	$\mu = \mu_0$	$\mu \ne \mu_0$	TTT

Procedure for testing a hypothesis: To test a hypothesis we follow the steps.

(1) **Null Hypothesis:** Formulate Null Hypothesis H_0

(2) **Alternative Hypothesis:** Formulate Alternative Hypothesis H_1

(3) **Level of Significance:** Specify the Level of Significance (LoS) α

(4) **Test statistic:** Specify the test statistic or test formula

(5) **Criterion:** Specify the criterion for testing H_0 against H_1

(6) **Calculation:** Calculate the test statistic value using sample data

(7) **Decision:** Decide whether H_0 reject or not

Note:

(1)* $z_{0.01} = 2.33$ **(2)*** $z_{0.05} = 1.645$ **(3)*** $z_{0.005} = 2.575$ **(4)*** $z_{0.025} = 1.96$

3.2.2 Test of Hypothesis – One Mean:

Case (i): For large samples $(n \ge 30)$,

(1) **Null Hypothesis** H_0 : $\mu = \mu_0$

(2) **Alternative Hypothesis** H_1 : Any one of the following $\mu < \mu_0$, $\mu > \mu_0$, $\mu \ne \mu_0$

(3) **Level of Significance** : α

(4) **Test statistic** : $Z = \dfrac{\overline{X} - \mu}{\left(\dfrac{\sigma}{\sqrt{n}}\right)}$

(5) **Criterion** :

H_1	Reject H_0 if
$\mu < \mu_0$	$Z < -z_\alpha$
$\mu > \mu_0$	$Z > z_\alpha$
$\mu \ne \mu_0$	$\lvert Z \rvert > z_{\frac{\alpha}{2}}$

Note: If $n \ge 30$ and σ is not known, then we can use s in place of σ

Case (ii): For small samples $(n < 30)$,

(1) Null Hypothesis H_0 : $\mu = \mu_0$

(2) Alternative Hypothesis H_1 : Any one of the following $\mu < \mu_0$, $\mu > \mu_0$, $\mu \neq \mu_0$

(3) Level of Significance : α

(4) Test statistic : $t = \dfrac{\overline{X} - \mu}{\left(\dfrac{s}{\sqrt{n}}\right)}$, $\nu = n - 1$

(5) Criterion :

H_1	Reject H_0 if
$\mu < \mu_0$	$t < -t_\alpha$
$\mu > \mu_0$	$t > t_\alpha$
$\mu \neq \mu_0$	$\lvert t \rvert > t_{\frac{\alpha}{2}}$

Test statistics and Critical regions for tests of Hypotheses for ONE MEAN

S.No	Test of Hypothesis	Test Statistic	H_1	Reject H_0 if
1	One Mean Large Samples	$Z = \dfrac{\overline{X} - \mu}{\left(\dfrac{\sigma}{\sqrt{n}}\right)}$	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$Z < -z_\alpha$ $Z > z_\alpha$ $\lvert Z \rvert > z_{\frac{\alpha}{2}}$
2	One Mean Small Samples	$t = \dfrac{\overline{X} - \mu}{\left(\dfrac{s}{\sqrt{n}}\right)}$, $\nu = n - 1$	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$t < -t_\alpha$ $t > t_\alpha$ $\lvert t \rvert > t_{\frac{\alpha}{2}}$

3.2.3 Test of Hypothesis – Two Means:

Case (i): For large samples $(n_1 \geq 30, n_2 \geq 30)$,

(i) **Null Hypothesis** H_0 : $\mu_1 - \mu_2 = \delta$

(ii) **Alternative Hypothesis** H_1 : Any one of the following $\mu_1 - \mu_2 < \delta$, $\mu_1 - \mu_2 > \delta$, $\mu_1 - \mu_2 \neq \delta$

(iii) **Level of Significance** : α

(iv) **Test statistic** : $Z = \dfrac{\left(\overline{X}_1 - \overline{X}_2\right) - \delta}{\sqrt{\left(\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)}}$

(v) **Criterion** :

H_1	Reject H_0 if
$\mu_1 - \mu_2 < \delta$	$Z < -z_\alpha$
$\mu_1 - \mu_2 > \delta$	$Z > z_\alpha$
$\mu_1 - \mu_2 \neq \delta$	$\lvert Z \rvert > z_{\frac{\alpha}{2}}$

Note: If $n_1 \geq 30, n_2 \geq 30$ and σ_1, σ_2 are not known, then we can use s_1, s_2 in place of σ_1, σ_2 respectively

Case (ii): For small samples $(n_1 < 30, n_2 < 30)$ and σ_1, σ_2 unknown,

(1) Null Hypothesis H_0 : $\mu_1 - \mu_2 = \delta$

(2) Alternative Hypothesis H_1 : Any one of the following $\mu_1 - \mu_2 < \delta$, $\mu_1 - \mu_2 > \delta$, $\mu_1 - \mu_2 \neq \delta$

(3) Level of Significance : α

(4) Test statistic : $t = \dfrac{\left(\overline{X}_1 - \overline{X}_2\right) - \delta}{\sqrt{\left(\dfrac{\sigma_p^2}{n_1} + \dfrac{\sigma_p^2}{n_2}\right)}}$ with $\nu = n_1 + n_2 - 2$

where $\sigma_p^2 = \dfrac{(n_1 - 1)\, s_1{}^2 + (n_2 - 1)\, s_2{}^2}{n_1 + n_2 - 2}$

or $\sigma_p^2 = \dfrac{\sum\left[(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2\right]}{n_1 + n_2 - 2}$

(5) Criterion :

H_1	Reject H_0 if
$\mu_1 - \mu_2 < \delta$	$t < -t_\alpha$
$\mu_1 - \mu_2 > \delta$	$t > t_\alpha$
$\mu_1 - \mu_2 \neq \delta$	$\lvert t \rvert > t_{\frac{\alpha}{2}}$

Note: (i) σ_p^2 is known as the **variance by pooling**

(ii) If $n_1 + n_2 - 2 \geq 30$, then we can use $\nu = \infty$

3.2.4 Confidence Interval - Two Means:

Case (i): For large samples $(n_1 \geq 30, n_2 \geq 30)$, with the probability $1-\alpha$ or $(1-\alpha)100\%$ confidence,

Upper and Lower confidence limits: $(\bar{x}_1 - \bar{x}_2) \pm E$

Confidence Interval: $\left((\bar{x}_1 - \bar{x}_2) - E,\ (\bar{x}_1 - \bar{x}_2) + E\right)$ where, $E = z_{\frac{\alpha}{2}} \sqrt{\left(\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)}$

Case (ii): For small samples $(n < 30, n_2 < 30)$, with the probability $1-\alpha$ or $(1-\alpha)100\%$ confidence,

Upper and Lower confidence limits: $(\bar{x}_1 - \bar{x}_2) \pm E$

Confidence Interval: $\left((\bar{x}_1 - \bar{x}_2) - E,\ (\bar{x}_1 - \bar{x}_2) + E\right)$ where, $E = t_{\frac{\alpha}{2}} \sqrt{\left(\dfrac{\sigma_p^2}{n_1} + \dfrac{\sigma_p^2}{n_2}\right)}$ with $\nu = n_1 + n_2 - 2$

and $\sigma_p^2 = \dfrac{(n_1 - 1)\, s_1{}^2 + (n_2 - 1)\, s_2{}^2}{n_1 + n_2 - 2}$

Or $\sigma_p^2 = \dfrac{\sum\left[(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2\right]}{n_1 + n_2 - 2}$

Test statistics and Critical regions for tests of Hypotheses for ONE & TWO means

S.No	Test of Hypothesis	Test Statistic	H_1	Reject H_0 if
1	One Mean Large Samples	$Z = \dfrac{\overline{X} - \mu}{\left(\dfrac{\sigma}{\sqrt{n}}\right)}$	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$Z < -z_\alpha$ $Z > z_\alpha$ $\lvert Z \rvert > z_{\frac{\alpha}{2}}$
2	One Mean Small Samples	$t = \dfrac{\overline{X} - \mu}{\left(\dfrac{s}{\sqrt{n}}\right)}, \qquad \nu = n - 1$	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$t < -t_\alpha$ $t > t_\alpha$ $\lvert t \rvert > t_{\frac{\alpha}{2}}$
3	Two Means Large Samples	$Z = \dfrac{\left(\overline{X}_1 - \overline{X}_2\right) - \delta}{\sqrt{\left(\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)}}$	$\mu_1 - \mu_2 < \delta$ $\mu_1 - \mu_2 > \delta$ $\mu_1 - \mu_2 \neq \delta$	$Z < -z_\alpha$ $Z > z_\alpha$ $\lvert Z \rvert > z_{\frac{\alpha}{2}}$
4	Two Means Small Samples	$t = \dfrac{\left(\overline{X}_1 - \overline{X}_2\right) - \delta}{\sqrt{\left(\dfrac{\sigma_p^2}{n_1} + \dfrac{\sigma_p^2}{n_2}\right)}}, \quad \nu = n_1 + n_2 - 2$ where, $\sigma_p^2 = \dfrac{(n_1 - 1)\, s_1{}^2 + (n_2 - 1)\, s_2{}^2}{n_1 + n_2 - 2}$ or $\sigma_p^2 = \dfrac{\sum \left[(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2\right]}{n_1 + n_2 - 2}$	$\mu_1 - \mu_2 < \delta$ $\mu_1 - \mu_2 > \delta$ $\mu_1 - \mu_2 \neq \delta$	$t < -t_\alpha$ $t > t_\alpha$ $\lvert t \rvert > t_{\frac{\alpha}{2}}$

Maximum Error & Confidence Interval

S.No.		Maximum Error	Confidence Interval	Probability/ Confidence
1	Two Means Large Samples	$E = z_{\frac{\alpha}{2}} \sqrt{\left(\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)}$	$\left((\bar{x}_1 - \bar{x}_2) - E,\ (\bar{x}_1 - \bar{x}_2) + E\right)$	$1 - \alpha$ / $(1-\alpha)100\%$
2	Two Means Small Samples	$E = t_{\frac{\alpha}{2}} \sqrt{\left(\dfrac{\sigma_p^2}{n_1} + \dfrac{\sigma_p^2}{n_2}\right)}$ where $\sigma_p^2 = \dfrac{(n_1 - 1)\, s_1{}^2 + (n_2 - 1)\, s_2{}^2}{n_1 + n_2 - 2}$ or $\sigma_p^2 = \dfrac{\sum \left[(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2\right]}{n_1 + n_2 - 2}$	$\left((\bar{x}_1 - \bar{x}_2) - E,\ (\bar{x}_1 - \bar{x}_2) + E\right)$	$1 - \alpha$ / $(1-\alpha)100\%$