UNIT-3

Part A: Electromagnetic Waves

Syllabus:

Divergence and Curl of Electric and Magnetic Fields -Maxwell's Equations-Electromagnetic wave propagation in free space- Poynting Theorem.

Divergence of vector field :

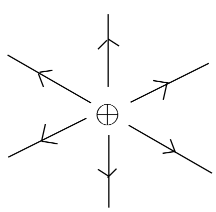
The divergence of a vector field at any point is defined as the amount of flux per unit volume diverging from that point. If $\overline{V}$ is a vector function, divergence is expressed

$$\nabla.\bar{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right). V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

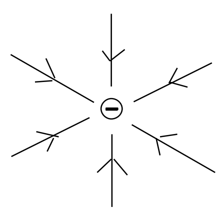
$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Divergence of a vector function is a scalar.

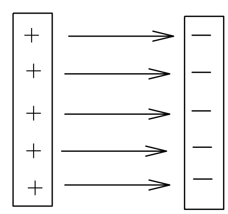
$\nabla.\bar{V}$ is a measure of how much the vector $\overline{V}$ spreads out from the point in question.



$Fig.(a)$



$Fig.(b)$



$Fig.(c)$

The vector function in Fig(a) has large (positive) divergence. Fig(b) indicates large negative divergence. Fig (c) indicates zero divergence.

The point of large positive divergence is a source and the point of large negative divergence is a sink.

Curl of a vector field :

The curl of a vector field is defined as the maximum line integral of vector per unit area. It is a vector quantity. The direction is normal to area.

$$\text{Curl } \bar{A} = \nabla \times \bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Curl is a measure of how much the vector $\bar{A}$ curls around the point in the question. A point of large curl is a whirlpool.

Gauss divergence theorem (Green's Theorem)

$$\int_v (\nabla.\bar{V})dv = \oint_s \bar{V}.da$$

Stoke's Theorem

$$\int_s (\nabla \times \bar{V})da = \oint_\ell \bar{V}.dl$$

Maxwell's equations:

	Integral form	**Differential form**	**Physical Significance**
1.	$\oint_s \bar{E}\,.\,d\bar{a} = \frac{q}{\varepsilon_o}$ (Gauss Law)	$\nabla.\bar{E} = \frac{\rho}{\varepsilon_o}$	The electric flux density through a surface enclosing a volume is equal to charge density ρ within the volume.
2.	$\oint_S \bar{B}.d\bar{a} = 0$	$\nabla.\bar{B} = 0$	Magnetic monopoles do not exist.
3.	$\oint \bar{E}.d\bar{\ell} = -\frac{d\phi}{dt}$ (Faraday's Law)	$\nabla \times \bar{E} = \frac{-\partial B}{\partial t}$	The electric field can also be generated by time varying magnetic fields.
4.	$\oint \bar{B}.d\bar{\ell} = \mu_o(i + i_d)$	$(\nabla \times \bar{B}) = \mu_o(\bar{J} + \epsilon_0 \frac{\partial \bar{E}}{\partial t})$	A magnetic field is generated by time varying electric field.

Maxwell's equations in free space

In free space where there is no charge or current, ρ=0 and J=0 and so the Maxwell's equations read

(a) $\nabla.\bar{E} = 0$ (c) $\nabla \times \bar{E} = \frac{-\partial B}{\partial t}$

(b) $\nabla.\bar{B} = 0$ (d) $\nabla \times \bar{B} = \mu_o \epsilon_0 . \frac{\partial \bar{E}}{\partial t}$

These are coupled, first order, partial differential equations for $\bar{E}$ and $\bar{B}$

Velocity of electromagnetic wave in free space:

The equations (c) and (d) can be decoupled by applying curl.

$$\nabla \times (\nabla \times \bar{E}) = \nabla(\nabla.\bar{E}) - \nabla^2 E = \nabla \times (-\frac{\partial \bar{B}}{\partial t})$$

$$= -\frac{\partial(\nabla \times \bar{B})}{\partial t} = -\mu_o \epsilon_0 . \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\nabla \times (\nabla \times \bar{B}) = \nabla(\nabla.\bar{B}) - \nabla^2 B = \nabla \times (\mu_o \epsilon_0 . \frac{\partial \bar{E}}{\partial t})$$

$$= \mu_o \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \bar{E}) = -\mu_o \epsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2}$$

Since $\nabla.\bar{E} = 0$ and $\nabla.\bar{B} = 0$ $\nabla^2 E = \mu_o \epsilon_0 . \frac{\partial^2 \bar{E}}{\partial t^2}$ and $\nabla^2 B = \mu_o \epsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2}$

Comparing the above with classical wave equation $\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ the speed of electromagnetic wave in free space can be expressed as $v = \frac{1}{\sqrt{\mu_o \epsilon_0}}$ = $3\times 10^8 m/s$

Poynting's Theorem:

Poynting's Theorem says that the work done on the charges by the electromagnetic force is equal to decrease in the energy stored in the field, less the energy which flowed out through the surface.

Suppose we have some charge and current configuration which at time 't' produces fields $\bar{E}$ and $\bar{B}$. In the next instant dt, the charges move around a bit. Let dW is the work done by electromagnetic forces on these charges in the interval dt. According to Lorentz force law the work done on an element of charge dq is

$$\bar{F}.d\bar{l} = dq(\bar{E} + \bar{v} \times \bar{B}).\bar{v}\,dt = \bar{E}.\bar{v}\,dq\,dt$$

Now dq= ρ dV and $\rho\bar{v} = \bar{J}$

So the rate at which work is done on all the charges in some volume V is given by

$$\frac{dW}{dt} = \int_V \bar{E}.\bar{J}\,dV \quad ....(1)$$

We have $\bar{E}.\bar{J} = \frac{1}{\mu_0}\bar{E}.(\nabla \times \bar{B}) - \varepsilon_0 \bar{E}.\frac{\partial \bar{E}}{\partial t}$

From product rule we have $\nabla.(\bar{E} \times \bar{B}) = \bar{B}.(\nabla \times \bar{E}) - \bar{E}.(\nabla \times \bar{B})$

According to Faraday's law $\nabla \times \bar{E} = \frac{-\partial B}{\partial t}$

$$\bar{E}.(\nabla \times \bar{B}) = -\bar{B}.\frac{\partial \bar{B}}{\partial t} - \nabla.(\bar{E} \times \bar{B})$$

Also

$$\bar{B}.\frac{\partial \bar{B}}{\partial t} = \frac{1}{2}\frac{\partial B^2}{\partial t} \quad \text{and} \quad \bar{E}.\frac{\partial E}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t}$$

So $\bar{E}.\bar{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla.(\bar{E} \times \bar{B})$ ………………..(2)

Substituting (2) into (1) and applying divergence theorem to second term,

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}(\varepsilon_0 \bar{E}^2 + \frac{1}{\mu_0}\bar{B}^2)\,dV - \frac{1}{\mu_0}\oint_S (\bar{E} \times \bar{B}).da$$

This is Poynting's theorem.

The first integral on the right is the total energy stored in the fields and the second term indicates rate at which energy is carried out of the volume across its boundary surface by electromagnetic fields.

The energy per unit time, per unit area, transported by the fields is called the Poynting vector $\bar{\boldsymbol{S}}$.

$$\bar{\boldsymbol{S}} = \frac{1}{\mu_0}(\bar{E} \times \bar{B}).$$