unit-I

Bisection Method:

i) $f(a)\cdot f(b) < 0$; $f(a) > 0$ & $f(b) < 0$

ii) $x_1 = \frac{a+b}{2}$; find $f(x_1)$

iii) If $f(x_1) = 0 \rightarrow x_1$ is root

$f(x_1) < 0 \rightarrow x_2 = \frac{x_1 + a}{2}$

$f(x_1) > 0 \rightarrow x_2 = \frac{x_1 + b}{2}$

Regula Falsi method:

i) $f(a) > 0$; $f(b) < 0$ & $f(a)\cdot f(b) < 0$

ii) $$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

iii) If $f(x_1) < 0$ $a = x_1$.

$f(x_1) > 0$ $b = x_1$

Newton Raphson method:

i) $f(a)\cdot f(b) < 0$ $f(a) > 0$; $f(b) < 0$

ii) $x_0 = \frac{a+b}{2}$

iii) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Forward difference operator:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\cancel{\Delta^4 y}$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		
		Δy_2			
x_3	y_3				

find till $N-n$ iterations

N = total no. of values
n = total " " missing values.

Note: $\boxed{\Delta y_{n-1} = y_n - y_{n-1}}$

Formulas:

i) Forward difference operator : $\Delta f(x) = f(x+h) - f(x)$

ii) Backward " " : $\nabla f(x) = f(x) - f(x-h)$

iii) Central " " : $\delta f(x) = f(x+h/2) - f(x-h/2)$

iv) Shifting operator : $E^n f(x) = f(x+nh)$

v) Averaging operator : $\mu f(x) = \dfrac{f(x+h/2) + f(x-h/2)}{2}$

Relations:

i) $\Delta = E - 1$

ii) $\nabla = 1 - E^{-1}$

(iii) $\delta = E^{1/2} - E^{-1/2}$

(iv) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

(v) $\Delta = E\nabla = \delta E^{1/2}$

(vi) $E = e^{hD}$ (D = derivative)

Unit - II

Newton Forward Interpolation:

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

where $p = \frac{x - x_0}{h}$

Newton Backward Interpolation:

$$f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$p = \frac{x - x_n}{h}$

Newton's Divide difference interpolation:-

$$f(x) = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2]$$
$$+ (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] + \ldots\ldots$$

for finding $\Delta y = \dfrac{b-a}{x_2 - x_1} = [x_0, x_1]$

$$\Delta^2 y = [x_0, x_1, x_2].$$

Numerical Differentiation:-

(i) using forward difference:

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

(ii) using backward difference:-

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

Lagrange's interpolation:

If $y = f(x)$ with values $y_0, y_1, \ldots y_n$ corresponding to $x_0, x_1, \ldots x_n$ then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

Inverse interpolation:

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1 + \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n$$

Unit-III

i) Trapezoidal rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

$h = \frac{b-a}{n}$ $\quad$ n = no. of intervals (or) step size

ii) Simpsons $\frac{1}{3}$rd rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4 + y_6 + \cdots)\right]$$

iii) Simpsons $\frac{3}{8}$th rule:

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + \cdots) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots)\right]$$

Eulers Method:

i) consider first differential eq^n $\frac{dy}{dx} = f(x,y)$

initial condition $y(x_0) = y_0$

ii) approx value of y at $x = x_1$

$\Rightarrow y_1 = y_0 + hf(x_0, y_0)$

general recursive relation:

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

Modified Euler's method:

(i) $\frac{dy}{dx} = f(x,y)$ initial condition $y(x_0) = y_0$

(ii) $y_1^{(0)} = y_0 + hf(x_0, y_0)$

$y_1^{(1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(0)})\right)$

$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(1)})\right) \ldots$

similarly

$$\boxed{y_n^{(n)} = y_{n-1} + \frac{h}{2}\left(f(x_{n-1}, y_{n-1}) + f(x_{n-1}, y_{n-1}^{(n-1)})\right)}$$

Runge Kutta method of fourth order:

(i) $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + k_1/2)$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + k_2/2)$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$\boxed{k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$ then $y_1 = y_0 + k$.

Do- Little's method:

i) $AX = B$

ii) $A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

(iii) $AX = B \Rightarrow LUX = B$

Consider $UX = Y \Rightarrow LY = B$

$$\Rightarrow L \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix}$$

(iv) From $UX = Y \Rightarrow U \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Crout's method:

(i) $AX = B$

(ii) $A = LU$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left.\begin{array}{ll} \text{(iii)} & LUX = B \\ \text{(iv)} & UX = Y \\ \text{(v)} & LY = B \end{array}\right\} \text{Same as Do-Little's}$$

Cholesky decomposition:-

(i) $A = LL^T$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

(ii) $AX = B \Rightarrow LL^T X = B$

let $L^T X = Y$

(iii) $LY = B$

(iv) Solve $L^T X = Y$.

Singular Value Decomposition:

(i) Find AA^T

(ii) Find Characteritic eq^n $|AA^T - \lambda| = 0$

ex: $\begin{vmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{vmatrix} = 0$ $\lambda_1 = \sigma_1$

$\lambda_2 = \sigma_2$

(iii) $\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$

(iv) Find eigen vectors for λ_1, λ_2.

i.e $(AA^T - \lambda I)X = 0$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ (or) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

ex: $\begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(v) $u = \begin{bmatrix} \frac{\text{eigen vector1 value1}}{\sqrt{(\text{eigen vector 1})^2 + (\text{value2})^2}} & \frac{\text{eigen vector 2 value 1}}{\sqrt{(\text{value1})^2 + (\text{value2})^2}} \\ \frac{\text{eigen vector1 value2}}{\sqrt{(\text{value1})^2 + (\text{value2})^2}} & \frac{\text{eigen vector 2 value 2}}{\sqrt{(\text{value1})^2 + (\text{value 2})^2}} \end{bmatrix}$

(vi) $V_1 = \frac{1}{\sigma_1} A^T e_1$ e_1 = eigen vector1 values divided by $\sqrt{(\text{val1})^2 + (\text{val2})^2}$

$V_2 = \frac{1}{\sigma_2} A^T e_2$

(vii) $V = \begin{bmatrix} \begin{bmatrix} v_1 \\ \\ \end{bmatrix} & \begin{bmatrix} v_2 \\ \\ \end{bmatrix} \end{bmatrix}$

find V^T.

(viii) find $U \Sigma V^T$

Verification $U \Sigma V^T = A$ //

Gram schmidt orthogonalization:

(i) The vectors $\beta_1, \beta_2, \beta_3$ are given in the Question.

(ii) $\beta_1 = \gamma_1$ | $\alpha_1 = \dfrac{\gamma_1}{\|\gamma_1\|}$

(iii) $\gamma_2 = \beta_2 - \underbrace{(\beta_2, \alpha_1)}_{\text{dot product}}\alpha_1$ | $\alpha_2 = \dfrac{\gamma_2}{\|\gamma_2\|}$

(iii) $\gamma_3 = \beta_3 - (\beta_3, \alpha_1)\alpha_1 - (\beta_3, \alpha_2)\alpha_2$ | $\alpha_3 = \dfrac{\gamma_3}{\|\gamma_3\|}$

Verification: dot product of any two alpha vectors should be zero.

QR factorisation:-

(i) If matrix is given in the Question like this $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ then

$\beta_1 = (1,0,0)$ $\beta_2 = (1,0,1)$ $\beta_3 = (1,1,0)$

(ii) find the values of $\alpha_1, \alpha_2, \alpha_3$ like in the gram schmidt orthogonalization:-

(iii) $R = \begin{bmatrix} \beta_1\alpha_1 & \beta_2\alpha_1 & \beta_3\alpha_1 \\ 0 & \beta_2\alpha_2 & \beta_3\alpha_2 \\ 0 & 0 & \beta_3\alpha_3 \end{bmatrix}$

(iv) $Q = \left[\begin{bmatrix} \alpha_1 \\ \\ \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \\ \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \\ \end{bmatrix} \right] \cdot \begin{bmatrix} R \text{ matrix} \\ \\ \end{bmatrix}$

Verification

$\boxed{Q = A}$,,

Guass Elimination:

(i) $AX = B$

(ii) Transform the matrix to echelon form

i.e $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ and perform the same operations to "B" matrix & Solve the equations.

ex - 28.9

Jackobi Iterative method:

Q) $10x + y - z = 11.19$ $\quad x + 10y + z = 28.08$

$-x + y + 10z = 35.61$

sol: $x = \frac{1}{10}(11.19 - y + z)$ $\quad y = \frac{1}{10}(28.08 - x - z)$

$z = \frac{1}{10}(35.61 + x - y)$ $\quad$ let $x_0 = y_0 = z_0 = 0$

$x_1 = \frac{11.19}{10} = 1.119$ $\quad y_1 = \frac{28.08}{10}$ $\quad z_1 = \frac{35.61}{10} = 3.561$

$x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$ $\quad y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.24$

$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$

$x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$ | $y_3 = \frac{1}{10}(28.03 - x_2 - z_2)$

$= 2.35$

$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$

$x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$ $\quad y_4 = \frac{1}{10}(28.03 - x_3 - z_3)$

$= 2.34$

$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$

$x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$ | $y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$

$z_5 = \frac{1}{10}(35.61 + x_4 + y_4) = 3.45$

on solving $\boxed{x = 1.23}$ $\boxed{y = 2.34}$ $\boxed{z = 3.45}$

Guass Saidal

Q) $2x + y + 6z = 9$ $8x + 3y + 2z = 13$ $x + 5y + z = 7$

$$x_{k+1} = \frac{1}{8}(13 - 3y_k - 2z_k)$$

$$y_{k+1} = \frac{1}{5}(7 - x_{k+1} - z_k)$$

$$z_{k+1} = 1/6\,(9 - 2x_{k+1} - y_{k+1})$$

(i) $x_1 = \frac{1}{8}(13) = 1.625$ $\quad y_1 = \frac{(5-3.75)}{5} = 1.075$

$z_1 = \frac{(4.675)}{6} = 0.779$

(ii) $x_2 = \frac{1}{8}\left(13 - 3(1.075) - 2(0.779)\right) = 1.027$

$y_2 = \frac{1}{5}\left(7 - (1.027) - (0.779)\right) = 1.039$

$z_2 = \frac{1}{6}\left(9 - 2(1.027) - (1.039)\right] = 0.985$

(iii) $x_3 = \frac{1}{8}\left(13 - 3(1.039) - 2(0.985)\right) = 0.989$

$y_3 = \frac{1}{5}\left(7 - (0.989) - (0.985)\right] = 1.005$

$z_3 = \frac{1}{6}\left(9 - 2(0.989) - (1.005)\right] = 1.003$

(iv) $x_4 = \frac{1}{8}\left[13 - 3(1.005) - 2(1.003)\right] = 0.997$

$y_4 = \frac{1}{5}\left[7 - 0.997 - 1.003\right) = 1$

$z_4 = 1/6\left[9 - 2(0.997) - 1\right] = 1.001$

(v) $x_5 = \frac{1}{8}\left[13 - 3(1) - 2(1.001)\right] = 1$

$y_5 = \frac{1}{5}\left[7 - (1) - (1.001)\right] = 1$

$z_5 = \frac{1}{6}\left[9 - 2(1) - (1)\right] = 1$

(vi) $x_6 = \frac{1}{8}\left[13 - 3(1) - 2(1)\right] = 1$

$y_6 = \frac{1}{5}\left[7 - (1) - (1)\right] = 1$

$z_6 = \frac{1}{6}\left[9 - 2(1) - 1\right] = 1$

$\therefore \boxed{x = 1}\ \boxed{y = 1}\ \boxed{z = 1}$

unit - V

Q)

i) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \Rightarrow (5-\lambda)(2-\lambda)-4=0$

$\Rightarrow \lambda = 1, 6$

$(AA^T - 6I)\cancel{X} = 0 \quad \Rightarrow \left(\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 35 & 13 \\ 13 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Q) using Power method

$X^0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad AX^0 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$

$AX^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.2857 \end{bmatrix}$

$AX^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2857 \end{bmatrix} = \begin{bmatrix} 6.1428 \\ 1.5714 \end{bmatrix} = 6.1428 \begin{bmatrix} 1 \\ 0.2555 \end{bmatrix}$

$AX^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2558 \end{bmatrix} = \begin{bmatrix} 6.0232 \\ 1.5116 \end{bmatrix} = 6.0232 \begin{bmatrix} 1 \\ 0.2504 \end{bmatrix}$

$AX^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2509 \end{bmatrix} = \begin{bmatrix} 6.0036 \\ 1.5018 \end{bmatrix} \Rightarrow 6.0036 \begin{bmatrix} 1 \\ 0.2501 \end{bmatrix}$

$AX^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2501 \end{bmatrix} = 6.0004 \begin{bmatrix} 1 \\ 0.2500 \end{bmatrix}$