Basics Notation UNIT - 4

1. The sequence $\{s_n\}$ is defined by the formula $s_n = \frac{1}{\sqrt{n}}$

$\{s_n\} = 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots, \frac{1}{\sqrt{n}}, \ldots$

2. $s_n = \frac{1}{n}$ if n is even

$= -\frac{1}{n}$ if n is odd $\{s_n\} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$

3. $s_1 = \sqrt{2}$ $s_{n+1} = \sqrt{2+s_n}$ for all $n \in \mathbb{Z}^+$

$s_2 = \sqrt{2+s_1} = \sqrt{2+\sqrt{2}}$

$s_3 = \sqrt{2+s_2} = \sqrt{2+\sqrt{2+\sqrt{2}}}$ and so on

4. $s_1 = 1$, $s_2 = 1$, $s_{n+2} = s_{n+1} + s_n$ for all $n \geq 1$

From the above formula $s_3 = s_2 + s_1 = 1+1 = 2$

$s_4 = s_3 + s_2 = 2+1 = 3$ and so on

$\{s_n\} = 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

which is called Fibonacci Sequence

Ex $s_n = 1 + \frac{(-1)^n}{n}$. (2) $s_n = \frac{n}{n+1}$, (3) $s_n = \frac{n-1}{2n}$.

(4) $s_{2n-1} = 2n-1$, $s_{2n} = \frac{1}{2}n$

(1) $s_1 = \sqrt{7}$ and $s_{n+1} = \sqrt{7+s_n}$

(2) $s_1 = \sqrt{c}$ and $s_{n+1} = \sqrt{c s_n}$

UNIT-4

Recurrence Relation

Def Recurrence relation is an equation that recursively defines a sequence based on the rule that gives the next term in the sequence as a function of previous term(s) when one or more initial terms are given.

Sequence

Def If A is a non empty set then a function $s: \mathbb{N} \to A$ called a sequence.

$$\{s_1, s_2, \ldots, s_n, \ldots\} = \{s_n\}_{n=1}^{\infty} \text{ or } \{s_n\} \text{ or } \langle s_n \rangle$$

Methods of Defining sequences

A sequence can be described in several ways

1. Listing in order

The first few elements of a sequence, till the rule for writing down different elements become clear.

eg. $\{1, 4, 9, 16, \ldots\}$ is a sequence whose nth term is n^2

2. Define a sequence by a formula for its nth term

eg. $\{1, 4, 9, 16, \ldots\}$ can be written as $\{n^2\}$

3. Define a sequence by a Recurrence formula (induction)

i.e. by a rule which express nth term in terms of (n-1) th term or (n+1) th term in terms of nth term.

// eg Let $s_1 = 1$, $s_{n+1} = 2s_n$ for all $n \in \mathbb{N}$. $n \geq 1$

These relations defines the sequence

$\{1, 2, 2^2, 2^3, \ldots\}$ its nth term is 2^{n-1}

// eg If $s_1 = 1$, $s_2 = 1$ and $s_n = s_{n-2} + s_{n-1}$ for $n > 2$

then $\{s_n\}$ is a sequence.

It is called a Fibonacci sequence.

It can be written as $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

fibonacci sequence
$$\begin{cases} s_1 = 1, \quad s_2 = 1 \\ s_n = s_{n-2} + s_{n-1} \end{cases}$$

Generating Functions

Consider a sequence of real numbers $a_0, a_1, a_2 \ldots$ Let us denote this sequence by $\langle a_r \rangle$, $r = 0, 1, 2 \ldots$ or just $\langle a_r \rangle$. Given this sequence, suppose there exists a function $f(x)$ whose expansion in a series of powers of x is as given below.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + \cdots = \sum_{r=0}^{\infty} a_r x^r \quad \text{--- (1)}$$

Then, $f(x)$ is called generating function for the sequence $a_0, a_1, \ldots, \ldots = \langle a_r \rangle$

In other words, given a sequence $\langle a_r \rangle$, if there exists a function $f(x)$ such that a_r is the coefficient of x^r in the expansion of $f(x)$ in a series of powers of x, then $f(x)$ is called a generating function of $\langle a_r \rangle$

If $f(x)$ is a generating function of the sequence $\langle a_r \rangle$, we say that $f(x)$ generates the sequence $\langle a_r \rangle$. The series of x in the R.H.S of (1) is called power series expansion of $f(x)$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots = \sum_{r\ge 0}^{\infty} x^r$$

$\therefore f(x) = (1-x)^{-1}$ is a generating function of the sequence $1, 1, 1, 1, \ldots$

Similarly $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{r\ge 0}^{\infty} (-1)^r \cdot x^r$.

$f(x) = (1+x)^{-1}$ is a generating function for the sequence

$1, -1, 1, -1, 1, \ldots$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$= \sum_{r\ge 0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \cdot x^r$$

$$1, \frac{n}{1!}, \frac{n(n-1)}{2!}, \frac{n(n-1)(n-2)}{3!}, \ldots$$

$$\binom{n}{0} \binom{n}{1} \binom{n}{2} \cdot \binom{n}{3} \ldots$$

Ex Find the sequences generated by the following function

$$(3+x)^3 = 27 \times \left(1 + \frac{x}{3}\right)^3$$

$$= \underline{27 + 27x + 9x^2 + x^3.}$$

This shows that the sequence generated by $(3+x)^3$ is

$27, 27, 9, 1, 0, 0, \ldots$

(2) $2x^2(1-x)^{-1} = 2x^2(1+x+x^2+x^3+\cdots)$

$= 0 + 0x + 2x^2 + 2x^3 + 2x^4 + \cdots$

$= 0, 0, 2, 2, 2, \ldots$

(3) $\frac{1}{1-x} + 2x^3 = (1-x)^{-1} + 2x^3.$

$= 1 + x + x^2 + x^3 + x^4 + \cdots + 2x^3$

$= 1, 1, 1, 3, 1, \ldots$

(4) $(1+3x)^{-1/3} =$

(5) $3x^3 + e^{2x} = 3x^3 + 1 + \left(\frac{2x}{1!}\right) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$

Find the generating functions of the following

(1) $1, 2, 3, 4, \ldots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

(2) $1, -2, 3, -4, \ldots$

$(1+x)^{-2} =$

(3) $0, 1, 2, 3, 4, \ldots$

$0 + 1\cdot x + 2x^2 + 3x^3 + 4x^4 + \cdots$

$= x(1 + 2x + 3x^2 + 4x^3 + \cdots)$

$= x(1-x)^{-2}$

(4) $0, 1, -2, 3, -4, \ldots$

Find the generating functions for the following sequences

$1^2, 2^2, 3^2, \ldots$

$$0 + 1x + 2x^2 + 3x^3 + \ldots = x(1-x)^{-2}.$$

Differentiating this, we get:

$$1 + 2(2x) + 3(3x^2) + 4(4x^3) + \ldots \to \frac{d}{dx}\left\{\frac{x}{(1-x)^2}\right\}$$

$$= \frac{(1+x)}{(1-x)^3}.$$

$$\therefore f(x) = \frac{1+x}{(1-x)^3}$$

Ex $1, 1, 0, 1, 1, 1, \ldots$

$a_0 = 1$, $a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$ and so on.

A generating function for this sequence is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$= 1 + 1 \cdot x + 0 \cdot x^2 + x^3 + \ldots$$

$$= (1 + x + x^2 + x^3 + \ldots) - x^2$$

$$= (1-x)^{-1} - x^2$$

Ex (1) $0, 2, 6, 12, 20, 30, 42, \ldots$

(2) $8, 26, 54, 92, \ldots$

Finding the coefficients in generating functions

Ex: Determine the coefficient of

(i) x^{12} in $x^3(1-2x)^{10}$ $\qquad \sum_{r=0}^{\infty} {}^{m}C_r \, x^r$.

$$(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)(m-2)\cdots(m-r+1)}{r!}x^r + \cdots$$

$$= 1 + \sum_{r=1}^{\infty} \frac{m(m-1)\cdots(m-r+1)}{r!}x^r.$$

m is a rational number (but not a +ve integer) and x is a real number such that $|x|<1$ i.e $-1<x<1$, then

If m be a non-ve integer, say $m=-n$ (n is a +ve integer) and $|x|<1$. Then

$$(1+x)^{-n} = 1 + \frac{(-n)}{1!}x + \frac{(-n)(-n-1)}{2!}x^2 + \cdots + \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!}x^r + \cdots$$

$$= 1 - \frac{n!}{1!}x + \frac{(n)(n+1)}{2!}x^2 + \cdots + (-1)^r \frac{n(n+1)\cdots(n+r-1)}{r!}x^r + \cdots$$

$$\therefore (1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r . {}^{(n+r-1)}C_r \, x^r.$$

on replacing x by $-x$ in the above we get

$$(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{(n+r-1)}C_r . x^r$$

Sol $\quad x^3(1-2x)^{10} = x^3 \times \sum_{r=0}^{10} \binom{10}{r}(-2x)^r$.

$$= \sum_{r=0}^{\infty} \binom{10}{r} . (-2)^r . x^{r+3}$$

The coefficient of x^{12} occurs is $\quad r+3=12 \Rightarrow r=9$

$$C_{12} = (-2)^9 . \binom{10}{9} = -(10 \times 2^9) = \underline{-5120}$$

Ex. Find the coefficient of x^0 in $\left(3x^2-(2/x)\right)^{15}$.

$$\left(3x^2-(2/x)\right)^{15} = (3x^2)^{15}\left(1-\frac{2}{3x^3}\right)^{15}$$

$$= 3^{15}x^{30}\left(\sum_{r=0}^{15}\binom{15}{r}\left(\frac{-2}{3x^3}\right)^r\right)$$

$$= 3^{15}\cdot\sum_{r=0}^{15}\binom{15}{r}\left(\frac{-2}{3}\right)^r\cdot x^{30-3r}$$

Evidently, the coefficient of x^0 (namely the constant term) in R.H.S is

$30-3r=0 \Rightarrow r=10$

$$C_0 = 3^{15}\times\binom{15}{10}\left(\frac{-2}{3}\right)^{10} = \underline{3^5\times 2^{10}\times\binom{15}{10}}$$

Ex Find the coefficient of x^5 in $(1-2x)^{-7}$

Sol. If n is a +ve integer then $\underline{(1-x)^{-n} = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r}$

$$(1-2x)^{-7} = \sum_{r=0}^{\infty}\binom{7+r-1}{r}(2x)^r = \sum_{r=0}^{\infty}\binom{6+r}{r}(2x)^r.$$

Therefore, the coefficient of x^5 in the RHS is

$$C_5 = 2^5\binom{11}{5} = 2^5\times\frac{11!}{5!\,6!} = 14,784.$$

Ex. Find the coefficient of x^{10} in $\dfrac{(x^3-5x)}{(1-x)^3}$

$$\frac{(x^3-5x)}{(1-x)^3} = (x^3-5x)(1-x)^{-3} = (x^3-5x)\sum_{r=0}^{\infty}\binom{3+r-1}{r}x^r.$$

$$= (x^3-5x)\times\sum_{r=0}^{\infty}\binom{2+r}{r}x^r.$$

Therefore, the coefficient of x^{10} in the R.H.S is

$x^{3+r}=x^{10}$

$r=7$

$x^{r+1}=x^{10}$

$r=9$

$$C_{10} = \binom{9}{7} - 5\binom{11}{9} =$$

$$\frac{9!}{7!\times 2!} - 5\times\frac{11!}{9!\,2!} = \underline{-239}$$

Ex find the coefficient of x^{15} in $(1+x)^4/(1-x)^4$.

$$\frac{(1+x)^4}{(1-x)^4} = (1+4x+6x^2+4x^3+x^4)(1-x)^{-4}$$
$$= (1+4x+6x^2+4x^3+x^4) \times \sum_{r\geq 0}^{\infty} \binom{3+r}{r} x^r.$$

$\therefore$ Therefore the coefficient of x^{15} in RHS is

$$C_{15} = \binom{18}{15} + 4\binom{17}{14} + 6\binom{16}{13} + 4\times\binom{15}{12} + \binom{14}{11}$$

Ex find the coefficient of x^8 in $\frac{1}{(x-3)(x-2)^2}$.

$$\frac{1}{(x-3)(x-2)^2} = \frac{1}{(-3)(1-x/3)(-2)^2(1-x/2)^2}$$
$$= \frac{1}{-12}\cdot(1-x/3)^{-1} \times (1-x/2)^{-2}$$
$$= -\frac{1}{12} \times \sum_{r\geq 0}^{\infty} \left(\frac{x}{3}\right)^r \cdot \times \sum_{r\geq 0}^{\infty} \binom{1+r}{r} x^r.$$

from ~~this~~ we find that the coefficient of x^8 is

$$C_8 = -\frac{1}{12}\left[\left(\frac{1}{3}\right)^0 \cdot \left(\frac{1}{\right.\right.$$

$$= -\frac{1}{12} \cdot \sum_{r\geq 0}^{\infty} \left(\frac{1}{3}\right)^r \cdot x^r \times \sum_{r\geq 0}^{\infty} \left(\frac{1}{2}\right)^r \cdot \binom{1+r}{r} x^r.$$

from this, we find that the coefficient of x^8 is

$$C_8 = -\frac{1}{12}\left[\left(\frac{1}{3}\right)^0\left(\frac{1}{2}\right)^8 \cdot \binom{9}{8} + \left(\frac{1}{3}\right)^1\left(\frac{1}{2}\right)^7\binom{8}{7} + \cdots \right.$$
$$\left. \cdots + \left(\frac{1}{3}\right)^8 \cdot \left(\frac{1}{2}\right)^0 \binom{1}{0}\right\}$$

$$= -\frac{1}{12} \sum_{K\geq 0}^{\infty} \left(\frac{1}{3}\right)^K \cdot \left(\frac{1}{2}\right)^{8-K} \binom{9-K}{8-K}$$

Ex 7 find the Coefficient x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$

$$(x^2+x^3+x^4+x^5+x^6)^5 = x^{10}\left[1+x+x^2+x^3+x^4\right]^5$$

$$= x^{10}\left(\frac{1-x^5}{1-x}\right)^5$$

$$= x^{10}(1-x^5)^5(1-x)^{-5}$$

$$= x^{10}\sum_{r=0}^{5}\binom{5}{r}(-x^5)^r \times \sum_{s=0}^{\infty}\binom{4+s}{s}x^s.$$

~~Similar thus~~ from this we find the coefficient of x^{20} is

$$C_{20} = \binom{5}{0}\binom{4+10}{10} - \binom{5}{1}\binom{4+5}{5} + \binom{5}{2}\binom{4+0}{0}$$

$5r+s=20$

$r=0.\ s=$

$$= \binom{14}{10} - 5\binom{9}{5} + \binom{5}{2}.$$

Note $$\sum_{r=0}^{n} a_r x^r = 1+x+x^2+\cdots+x^n = \frac{1-x^{n+1}}{1-x}.$$

Ex find the generating functions of for the following sequency

1.. 0, 1, 2, 3, ... ——— (1)

$0x^0 + 1x + 2x^2 + 3x^3 + \cdots$

$x(1 + 2x + 3x^2 + 4x^3 + \cdots) = x(1-x)^{-2}$.

This shows that $f(x) = x(1-x)^{-2}$ is a generating function for the sequence 0, 1, 2, 3, ..

Ex 2. 0, 1, −2, 3, −4, ...

$0.x^0 + 1.x - 2x^2 + 3x^3 - 4x^4 + \cdots$

$x(1 - 2x + 3x^2 - 4x^3 + \cdots) = x(1+x)^{-2}$

Accordingly $f(x) = x(1+x)^{-2}$ is generating function for the sequence 0, 1, −2, 3, −4, ..

Ex 3 $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$ ——— (2)

1, 2, 3, 4, ... ——— (1)

Compare (1) and (2)

$a_0 = 1,\ a_1 = 2,\ a_3 = 3,\ a_4 = 4. \ldots$

Substitute the values in eq (1)

$f(x) = 1.x^0 + 2.x^1 + 3x^2 + 4x^3 + \cdots$

$f(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$f(x) = (1-x)^{-2}$

Ex 4 1, −2, 3, −4, ... ——— (1)

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ ——— (2)

~~f(x)~~ $= a_0 = 1,\ a_1 = -2,\ a_3 = 3,\ a_4 = -4, \ldots$

Substitute these values in (2) we get

$f(x) = 1.x^0 - 2x^1 + 3x^2 - 4x^3 + \cdots$

$= 1 - 2x + 3x^2 - 4x^3 + \cdots = (1+x)^{-2}$.

Ex: $1^2, 2^2, 3^2, \ldots$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$a_0 = 1^2, \ a_1 = 2^2, \ a_2 = 3^2, \ a_3 = 4^2$

$f(x) = 1^2 \cdot x^0 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots$

$f(x) = 1 + 2(2x) + 3(3x^2) + 4(4x^3) + \ldots$

W.K.T. $x(1-x)^{-2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

Differentiating both sides we get

$$\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = 1 + 2(2x) + 3(3x^2) + 4(4x^3) + \ldots$$

$$\Rightarrow \frac{1(1-x)^2 - x\,2(1-x)(-1)}{(1-x)^4} = 1 + 2(2x) + 3(3x^2) + 4(4x^3) + \ldots$$

$$\Rightarrow \frac{(1-x)^2 - x(1-x)(-2)}{(1-x)^4}$$

$$\Rightarrow \frac{(1-x)(1-x+2x)}{(1-x)^4}$$

$$\Rightarrow \frac{(1+x)}{(1-x)^3} = 1 + 2(2x) + 3(3x^2) + 4(4x^3) + \ldots$$

$\rightarrow$ generating function of Given sequence

Ex. $0^2, 1^2, 2^2, 3^2, 4^2, \ldots$

$a_0 = 0^2,\ a_1 = 1^2,\ a_2 = 2^2,\ a_3 = 3^2,\ a_4 = 4^2, \ldots$

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$
$$= 0^2 x^0 + 1^2 x + 2^2 x^2 + 3^2 x^3 + 4^2 x^4$$
$$= 0 + 1x + 2^2 x^2 + 3^2 x^3 + 4^2 x^4 + \cdots$$
$$= x\left[1 + 2x + 3x^2 + 4x^3 + \cdots\right)$$
$$= \frac{x(1+x)}{(1-x)^3} \rightarrow \text{generating function}$$

Ex. $1^2 x + 2^2 x^2 + 3^2 x^3 + \cdots = \frac{x(1+x)}{(1-x)^3}$

Differentiating both sides of the expression

$$1^3 + 2^3 x + 3^3 x^2 + \cdots = \frac{d}{dx}\left[\frac{x(1+x)}{(1-x)^3}\right\} = \frac{x^2+4x+1}{(1-x)^4}.$$

$$\therefore f(x) = \frac{x^2+4x+1}{(1-x)^4}.$$

is a generating function for the sequence $1^3, 2^3, 3^3, \ldots$

Ex. we have $0^3 + 1^3 x + 2^3 x^2 + 3^3 x^3 + \cdots$

$$= x(1^3 + 2^3 x + 3^3 x^2 + \cdots)$$
$$= \frac{x(x^2+4x+1)}{(1-x)^4}$$

Thus $f(x) = \frac{x(x^2+4x+1)}{(1-x)^4}$ is the generating function for the sequence $0^3, 1^3, 2^3, 3^3, \ldots$

Ex: 0, 2, 6, 12, 20, 30, 42, ----

$a_0 = 0 = 0 + 0$

$a_1 = 2 = 1 + 1$

$a_2 = 6 = 2 + 2^2$

$a_3 = 12 = 3 + 3^2$

$a_4 = 20 = 4 + 4^2$

So that $a_r = r + r^2$ for $r = 0, 1, 2, \dots$

We recall that a generating function for the sequence $\langle r \rangle = 0, 1, 2, 3, \dots$

$$f(x) = \frac{x}{(1-x)^2}$$

and a generating function for the sequence $\langle r^2 \rangle = 0^2, 1^2, 2^2, 3^2, \dots$ is

$$g(x) = \frac{x(1+x)}{(1-x)^3}.$$

$\therefore$ A generating function for the given sequence is

$$f(x) + g(x) = \frac{x}{(1-x)^2} + \frac{x(1+x)}{(1-x)^3} = \frac{2x}{(1-x)^3}$$

Recurrence Relations:

Sequences are generally defined by specifying their general terms. Alternatively, a sequence may be defined by indicating a relation connecting its general term with one or more of the preceding terms.

In other words, a sequence $\langle a_r \rangle$ may be denoted by indicating a ~~general~~ relation connecting its general term a_n with a_{n-1} and a_{n-2} etc. Such a relation is called a recurrence relation for the sequence.

The process of determining a_n from a recurrence relation is called "solving" of relation.

The value of a_n that satisfies a recurrence relation is called its "general solution".

If the values of some particular terms of the sequence are specified, then by making use of these values in general solution we obtain the "particular solution" that uniquely determines the sequence.

First order ~~linear (or) Homogeneous~~ Recurrence relation

Recurrence Relations

Homo. Recurrence Relations / ~~Linear~~ RR.

1. First order H.R.R
2. 2nd order H.R.R
3. 3rd order H.R.R
4. Higher order H.R.R

Non Homogeneous Recurrence Relation

1. 1st order
2. 2nd order
3. 3rd order
4. Higher order Non H.R.R

1. ~~The first order homogeneous R.R~~ / linear recurrence relation of 1st order with constant coefficient is

$$a_n = c\,a_{n-1} + f(n), \text{ for } n \geq 1. \quad \text{—(1)}$$

where c is constant

$f(n)$ is known function.

In eq (1) if $f(n) = 0$ then the relation is called Homogeneous recurrence relation otherwise it is called a non-homogeneous recurrence relation (in homogeneous)

Eq (1) can be solved by substituting n by $n+1$ in eq (1)

$$a_{n+1} = c\,a_{n+1-1} + f(n+1) \quad \text{for } n \geq 0$$

$$= c \cdot a_n + f(n+1) \quad \text{—(2)}$$

Substituting $n = 0, 1, 2, 3, \ldots$ in eq (2)

put $n = 0 \Rightarrow \quad a_{0+1} = c \cdot a_0 + f(0+1)$

$$a_1 = c \cdot a_0 + f(1)$$

Put $n=1 \Rightarrow a_2 = c \cdot a_1 + f(2)$

$$= c\left[c a_0 + f(1)\right] + f(2)$$

$$= c^2 a_0 + c f(1) + f(2)$$

Put $n=2 \Rightarrow a_3 = c a_2 + f(3)$

$$= c\left[c^2 a_0 + c f(1) + f(2)\right] + f(3)$$

$$= c^3 a_0 + c^2 f(1) + c f(2) + f(3)$$

$$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \cdots + c f(n-1) + f(n)$$

Examine these, we obtain, by induction

$$a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k) \quad \text{for } n \geq 1 \quad \text{—— (3)}$$

This is the general solution of recurrence relation (2). which is equivalent to the relation (1).

If the recurrence relation is homogeneous i.e. if $f(n) = 0$ the solution (3) becomes

$$a_n = c^n a_0 \quad \text{for } n \geq 1 \quad \text{—— (4)}$$

The solution (3) and (4) yield ~~particular~~ particular solution if a_0 is specified. The specified value of a_0 is called the initial condition.

Ex Solve the recurrence relation $a_{n+1} = 4a_n$ for $n \geq 0$ given that $a_0 = 3$.

$a_n = 4a_{n-1}$ for $n \geq 1$

Sol The given relation is homogeneous

Its general sol is

$$a_n = 4^n . a_0 \text{ for } n \geq 1 \quad \text{——— (1)}$$

It is given that $a_0 = 3$. putting this in (1) we get

$$a_n = 3 \times 4^n . \text{ for } n \geq 1$$

This is a particular Solution of the given relation satisfying the initial condition $a_0 = 3$

$a_0 = 3$

$a_1 = 4a_0 = 4.3 = 12$

$a_2 = 4a_1 = 4.12 = 48$

$a_3 = 4a_2 = 4.48 = 192$

- - - - - - -

$a_1 = 3 \times 4^1 = 12$

$a_2 = 3 \times 4^2 = 3 \times 16 = 48$

$a_3 = 3 \times 4^3 = 3 \times 64 = 192$

⋮

Ex Find the recurrence relation and the initial condition for the sequence

$$2, 10, 50, 250, \ldots$$

Hence find the general form of the sequence

Sol The given sequence is $\langle a_r \rangle$

where $a_0 = 2$, $a_1 = 10$, $a_2 = 50$, $a_3 = 250$, ...

$a_1 = 5a_0$

$a_2 = 5a_1$

$a_3 = 5a_2$ ... and so on

From these, we readily note that the recurrence relation for the given sequence is $a_n = 5a_{n-1}$ for $n \geq 1$ with $a_0 = 2$ as the initial condition.

The solution of this relation is

$$a_n = 5^n a_0 = 5^n \times 2$$

This is the general term of the given sequence.

Second-order linear Homogeneous Recurrence Relations

We now consider a method of solving recurrence relations of the form

$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0$ for $n \geq 2$ ——— (1)

where C_n, C_{n-1} and C_{n-2} are real constants with $C_n \neq 0$.

A relation of this types is called a second-order linear homogeneous recurrence relation with constant coefficients.

We seek a solution of relation (1) in the form of $a_n = ck^n$ where $c \neq 0$ and $k \neq 0$. Putting $a_n = ck^n$ in (1) we get

$$C_n c.k^n + C_{n-1} ck^{n-1} + C_{n-2} ck^{n-2} = 0.$$

$$C_n k^2 + C_{n-1} k + C_{n-2} = 0 \text{ ——— (2)}$$

Thus $a_n = ck^n$ is a solution of (1) if k satisfies the quadratic equation (2). This quadratic equation is called the auxiliary equation or the characteristic equation for the relation (1).

Now the following three cases arise

Case (1) The two roots k_1 and k_2 of eq (2) are real and distinct.

then we take $a_n = Ak_1^n + Bk_2^n$ ——— (3)

where A and B are arbitrary real constants, as the general solution of the relation (1).

Case 2 The two roots k_1 and k_2 of equation (2) are real and equal with k as the common value. Then we take

$$a_n = (A + Bn) k^n.$$

where A and B are arbitrary real constants as the general solution

Case(3) The two roots k_1 and k_2 of eq(2) are complex

Then k_1 and k_2 are complex conjugates of each other, so that if $k_1 = p + iq$ then $k_2 = p - iq$ and we take

$$a_n = r^n (A \cos n\theta + B \sin n\theta)$$

where A and B are arbitrary complex constants.

$$r = |k_1| = |k_2| = \sqrt{p^2 + q^2} \text{ and}$$

$$\theta = \tan^{-1}(q/p).$$ as the general solution of the relation (1)

Ex 1. Solve the recurrence relation

$$a_n + a_{n-1} - 6a_{n-2} = 0 \quad \text{for } n \geq 2$$

given that $a_0 = -1$ and $a_1 = 8$

Sol: Here the coefficients of a_n, a_{n-1} and a_{n-2} are

$C_n = 1$, $C_{n-1} = 1$ and $C_{n-2} = -6$ respectively.

Therefore the characteristic eq is

$$k^2 + k - 6 = 0 \quad \text{or} \quad (k+3)(k-2) = 0.$$

Evidently, the roots of this equation are $k_1 = -3$ and $k_2 = 2$. which are real and distinct.

Therefore the general solution of given relation is

$$a_n = A \times (-3)^n + B \times 2^n \quad \text{——— (1)}$$

where A and B are arbitrary constants.

For this solution we get $a_0 = A + B$.

$$a_1 = -3A + 2B.$$

Using the given values of a_0 and a_1, these become

$$-1 = A + B.$$

$$8 = -3A + 2B$$

Solving these we get $A = -2$ and $B = 1$ putting these into (1)

We get $a_n = -2 \times (-3)^n + 2^n$

This is the solution of the given relation under the given initial conditions $a_0 = -1$ and $a_1 = 8$

Ex Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$

Ex 3 Solve the recurrence relation

$a_n - 6 a_{n-1} + 9 a_{n-2} = 0$ for $n \geq 2$

given that $a_0 = 5$, $a_1 = 12$

Sol: The Characteristic equation for the given relation is

$k^2 - 6k + 9 = 0$ or $(k-3)^2 = 0$

whose roots are $k_1 = k_2 = 3$.

Therefore, the general solution is a_n is

$$a_n = (A + Bn) 3^n \quad \text{———— (1)}$$

where A and B are arbitrary real constants.

Using the given initial conditions $a_0 = 5$ and $a_1 = 12$ in (1)

we get $5 = A$ and $12 = 3(A+B)$

Solving these we get $A = 5$ and $B = -1$. Putting these

Values in (1) we get

$$a_n = (5 - n) 3^n \quad \text{———— (2)}$$

This is the general sol of the given relation, under the

given initial conditions $a_0 = 5$ and $a_1 = 12$.

Ex Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2}) \text{ for } n \geq 2$$

given that $a_0 = 1$ and $a_1 = 2$

Sol For the given relation the characteristic equation is $k^2 - 2k + 2 = 0$ whose roots are

$$k = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Therefore, the general solution for a_n is

$$a_n = r^n (A \cos n\theta + B \sin n\theta) \quad \text{--- (1)}$$

where A and B are arbitrary constants

$$r = |1 \pm i| = \sqrt{2}$$

$$\theta = \tan^{-1}(1/1) = \pi/4.$$

$$a_n = (\sqrt{2})^n \left(A \cos \frac{n\pi}{4} + B \sin \frac{n\pi}{4} \right) \quad \text{--- (2)}$$

Using the initial conditions

$a_0 = 1$ and $a_1 = 2$ we get -

$1 = A$. and $2 = \sqrt{2} \left[A \cos \frac{\pi}{4} + B \sin \frac{\pi}{4} \right] = A + B.$

which yield $A = 1$ and $B = 1$

putting these values of A and B in (2) we get

$$a_n = (\sqrt{2})^n \cdot \left[\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4} \right] \quad \text{--- (3)}$$

This is the solution of the given relation under the initial conditions $a_0 = 1$ and $a_1 = 2$.

Third and higher-order linear Homogeneous Recurrence Relations

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} + C_{n-3} a_{n-3} + \cdots + C_{n-k} a_{n-k} = 0.$$

for $n \geq k \geq 3$. ——— (1)

where $C_n, C_{n-1}, C_{n-2}, \ldots C_{n-k}$ are real constants with $C_n \neq 0$.

The method of solution is analogous to that out lined in the 2nd order linear homogeneous Recurrence Relation

Ex: Solve the recurrence relation

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n \quad \text{for } n \geq 0$$

with $a_0 = 0,\ a_1 = 1,\ a_2 = 2$

Sol: The given relation is same as

$$2a_n - a_{n-1} - 2a_{n-2} + a_{n-3} = 0 \quad \text{for } n \geq 3$$

This is 3rd order relation of the form with

$C_n = 2 \quad C_{n-1} = -1,\ C_{n-2} = -2,\ C_{n-3} = 1.$

For this relation, we consider the characteristic equation

$$2k^3 - k^2 - 2k + 1 = 0 \quad \text{i.e.} \quad (2k-1)(k^2-1) = 0$$

The roots of this equation are $k_1 = 1/2 \quad k_2 = 1 \quad k_3 = -1.$ which are real and distinct.

$\therefore$ The general sol for a_n as

$$a_n = A \times (1/2)^n + B(1)^n + C(-1)^n \quad \text{——— (2)}$$

where A, B, C are arbitrary constants

To determine A, B, C, we use the given values (initial condition)

$a_0 = 0 \quad a_1 = 1 \quad a_2 = 2.$

putting these values into the solution (2) we get.

$$0 = a_0 = A \times \left(\tfrac{1}{2}\right)^0 + B \times 1^0 + C \times (-1)^0$$

$$1 = a_1 = A \times \left(\tfrac{1}{2}\right)^1 + B \times 1^1 + C \times (-1)^1$$

$$2 = a_2 = A \times \left(\tfrac{1}{2}\right)^2 + B \times 1^2 + C \times (-1)^2$$

These can be written as

$$A + B + C = 0, \quad A + 2B - 2C = 2 \quad A + 4B + 4C = 8$$

Solving these we get $A = -8/3$, $B = 5/2$ and $C = 1/6$

putting these into (2), we get

$$a_n = -\frac{8}{3} \times \left(\frac{1}{2}\right)^n + \frac{1}{6}(-1)^n + \frac{5}{2}.$$

This is required solution

Ex. Solve the recurrence relation

$$a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0 \quad n \geq 3$$

with $a_0 = 1$, $a_1 = 5$, $a_2 = 1$

Sol The characteristic equation for the given relation is

$$k^3 + k^2 - 8k - 12 = 0 \quad \text{i.e.} \quad (k+2)^2 (k-3) = 0$$

whose roots are $k_1 = k_2 = -2$ and $k_3 = 3$

Accordingly, we take the general solution for a_n as

$$a_n = (A + Bn)(-2)^n + C \times 3^n \quad \text{—— (1)}$$

where A, B, C are arbitrary constants.

Using the initial conditions $a_0 = 1$, $a_1 = 5$, $a_2 = 1$ in this, we get

$$1 = a_0 = A + C$$
$$5 = a_1 = -2(A + B) + 3C$$
$$1 = a_2 = 4(A + 2B) + 9C$$

Solving these we get $A = 0$, $B = -1$, $C = 1$ putting these into (1) we get

$$a_n = (-n) \times (-2)^n + 3^n$$

as the solution for a_n with $a_0 = 1$, $a_1 = 5$; $a_2 = 1$

Ex Solve the recurrence relation

$$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} \text{ given } a_0 = 1, a_1 = 4, a_2 = 28$$

Sol: The characteristic equation for the given relation is

$$k^3 - 6k^2 + 12k - 8 = 0 \quad \text{i.e. } (k-2)^3 = 0$$

whose roots are $k_1 = k_2 = k_3 = 2$

Consequently, we take the general solution for a_n, as

$$a_n = (A + Bn + Cn^2) \times 2^n \quad \text{———— (1)}$$

where A, B, C are arbitrary constants

Using the initial conditions $a_0 = 1$, $a_1 = 4$, $a_2 = 28$ in this we get

$$1 = A, \quad 4 = 2(A+B+C), \quad 28 = 4(A+2B+4C)$$

Solving these we get $A = 1$, $B = -1$, $C = 2$, putting these values into (1) we get

$$a_n = (1 - n + 2n^2) \times 2^n$$

This is the solution for the given recurrence relation with $a_0 = 1$, $a_1 = 4$, $a_2 = 28$

Ex: Find the general solution of the recurrence relation

$$a_n + a_{n-3} = 0 \quad n \geq 3.$$

The characteristic eq for the given relation is .

$$(k^3 + 1) = 0 \quad (k+1)(k^2 - k + 1) = 0$$

whose roots are $k_1 = -1$ $k_2 = \frac{1}{2}(1 + i\sqrt{3})$ and $k_3 = \frac{1}{2}(1 - i\sqrt{3})$

Accordingly we take the general sol for a_n as

$$a_n = A(-1)^n + r^n [c_1 \cos n\theta + c_2 \sin n\theta]$$

where A, c_1, c_2 are arbitrary constants and

$$r = |k_2| = |k_3| = \frac{1}{2}\sqrt{1^2 + (\sqrt{3})^2} = 1. \quad \tan\theta = \frac{(\sqrt{3}/2)}{(1/2)} = \sqrt{3}$$

So that $\theta = \pi/3$.

$$a_n = A(-1)^n + \left\{ c_1 \cos \frac{n\pi}{3} + c_2 \sin \frac{n\pi}{3} \right\}$$

is the general solution of the given recurrence relation, Ex Find the general solution of the recurrence relation

$$a_n - 7a_{n-2} + 10a_{n-4} = 0 \quad n \geq 4.$$

Sol. The given relation is a fourth order one.
Its characteristic eq is

$$K^4 + 0K^3 - 7K^2 + 0K + 10 = 0$$

i.e. $$K^4 - 7K^2 + 10 = 0$$

This yields.

$$K^2 = \frac{7 \pm \sqrt{49-40}}{2} = \frac{1}{2}(7 \pm 3)$$

i.e. $$K_1^2 = 5 \quad K_2^2 = 2$$

So that $K_1 = \pm\sqrt{5}$. and $K_2 = \pm\sqrt{2}$.
Thus $\sqrt{5}, -\sqrt{5}, \sqrt{2}, -\sqrt{2}$ are the distinct real roots of the characteristic eq. Accordingly, the general solution for a_n is

$$a_n = A(\sqrt{5})^n + B(-\sqrt{5})^n + C(\sqrt{2})^n + D(-\sqrt{2})^n$$

where A, B, C, D are arbitrary constants

Linear Non-Homogeneous Recurrence Relations of 2nd and Higher order

An eq of the form $C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} + \cdots + C_{n-k} a_{n-k} = f(n)$
for $n \geq k \geq 2$ — (1)

where $C_n, C_{n-1}, \ldots C_{n-k}$ are real constants with $C_n \neq 0$.
and $f(n)$ is a given real valued function of 'n'.
higher order linear non-homogeneous recurrence relations with constant coefficients.

A general solution of the recurrence relation (1) is given by

$$a_n = a_n^{(h)} + a_n^{(p)} \quad \text{——— (2)}$$

where $a_n^{(h)}$ is the general solution of homogeneous part of relation (1) namely relation (1) with $f(n) = 0$

$a_n^{(p)}$ is particular solution of the relation (1)

The part $a_n^{(h)}$ of the solution (2) can be obtained by the methods discussed previously.

But, the determination of $a_n^{(p)}$ for arbitrary $f(n)$ is a tedious task
It is only in some special cases that can be find $a_n^{(p)}$ in a straight-forward way.

The following are some of these special cases -

(1) Suppose $f(n)$ is a polynomial of degree q and 1 is not a root of characteristic equation of the homogeneous part of the relation (1).

In this case $a_n^{(p)}$ is taken in the form

$$a_n^{(p)} = A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q \quad \text{———} (3)$$

where $A_0, A_1, A_2, \ldots A_q$ are constants to be evaluated by using the fact that $a_n = a_n^{(p)}$ satisfies the relation (1).

(2) Suppose $f(n)$ is a polynomial of degree q and 1 is a root of multiplicity m of characteristic equation of the homogeneous part of the equation (1). In this case $a_n^{(p)}$ is taken in the form

$$a_n^{(p)} = n^m \left[A_0 + A_1 n + A_2 n^2 + \cdots + A_q n^q \right]. \quad \text{———} (4)$$

where $A_0, A_1, \ldots, A_q$ are constants to be evaluated by using the fact that $a_n = a_n^{(p)}$ satisfies the relation (1).

(3). Suppose $f(n) = \alpha b^n$, where α is a constant and b is not a root of the characteristic equation of the homogeneous part then $a_n^{(p)}$ is taken in the form

$$a_n^{(p)} = A_0 b^n.$$

where A_0 is a constant to be evaluated by using the fact that $a_n = a_n^{(p)}$ satisfies the relation

(4) Suppose $f(n) = \alpha b^n$, where α is a constant and b is a root of multiplicity m of the characteristic equation of the homogeneous part of the relation (1), then $a_n^{(p)}$ is taken in the form

$$a_n^{(p)} = A_0 n^m b^n$$

where A_0 is constant to be evaluated by using the fact that $a_n = a_n^{(p)}$ satisfies the relation (1)

Ex: Solve the recurrence relation

$$a_n + 4a_{n-1} + 4a_{n-2} = 8 \text{ for } n \geq 2$$

and $a_0 = 1$, $a_1 = 2$.

Sol: For the homogeneous part of the given relation the characteristic eq is

$$k^2 + 4k + 4 = 0 \Rightarrow (k+2)^2 = 0$$

whose roots are -2 and -2

$$\therefore \quad a_n^{(h)} = (A + Bn)(-2)^n \quad \text{——— (1)}$$

where A and B are arbitrary constants.

Keeping the RHS of the given relation in mind, we seek $a_n^{(p)}$ in the form

$$a_n^{(p)} = A_0 \quad \text{——— (2)}$$

putting this $a_n^{(p)}$ for a_n in the given relation, we get

$$A_0 + 4A_0 + 4A_0 = 8$$

which yields $A_0 = 8/9$ putting this into (2) we get

$$a_n^{(p)} = \frac{8}{9} \quad \text{——— (2)}$$

$\therefore$ The general solution for a_n is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = (A + Bn)(-2)^n + \frac{8}{9} \quad \text{——— (4)}$$

It is given that $a_0 = 1$ and $a_2 = 2$ Using these initial conditions in (4) we get

$$1 = A + \frac{8}{9} \quad \text{and} \quad 2 = (A+B)(-2) + \frac{8}{9}$$

These give $A = 1/9$ and $B = -\frac{2}{3}$. Putting these into (4) we get

$$a_n = \left(\frac{1}{9} - \frac{2}{3}n\right)(-2)^n + \frac{8}{9}$$

is the required solution

Ex Solve the recurrence relation

$$a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - 2. \quad n \geq 0.$$

Sol: For the homogeneous part of the given recurrence relation the characteristic eqn of is

$$K^2 - 10K + 21 = 0 \quad \text{——— (1)}$$

[The given relation may be written as

$$a_n - 10a_{n-1} + 21a_{n+2} = 3(n-2)^2 - 2 \quad n \geq 2$$

replace n by (n−2)]

whose roots are 3 and 7

Therefore $a_n^{(h)} = A \times 3^n + B \times 7^n$ ——— (2)

where A and B are constants.

Since the RHS of the given relation is a polynomial of degree 2 we seek $a_n^{(p)}$ in the form

$$a_n^{(p)} = A_0 + A_1 n + A_2 n^2 \quad \text{——— (3)}$$

Putting this $a_n^{(p)}$ for a_n in the given relation; we obtain

$$\{A_0 + A_1(n+2) + A_2(n+2)^2\} - 10\{A_0 + A_1(n+1) + A_2(n+1)^2\}$$
$$+ 21\{A_0 + A_1 n + A_2 n^2\} = 3n^2 - 2$$

or $\{A_2 n^2 + (4A_2 + A_1)n + (4A_2 + 2A_1 + A_1)\}$
$- 10\{A_2 n^2 + (2A_2 + A_1)n + (A_2 + A_1 + A_1)\} + 21\{A_2 n^2 + A_1 n + A_0\} = 3n^2 + 2$

Equating the corresponding terms on both sides we get

$12A_2 = 3$, $-16A_2 + 12A_1 = 0$, $-6A_2 - 8A_1 + 12A_0 = -2$

This gives $A_2 = 4$, $A_1 = 16/3$, $A_0 = 47/9$. Putting these in (2) we get

$$a_n^{(p)} = 4n^2 + \frac{16}{3}n + \frac{47}{9}. \quad \text{——— (3)}$$

∴ The G.S is $a_n = a_n^{(h)} + a_n^{(p)} = A \times 3^n + B \times 7^n + \left(4n^2 + \frac{16}{3}n + \frac{47}{9}\right)$

Ex Solve the recurrence relation

$$a_{n+2}^2 - 5a_{n+1}^2 + 6a_n^2 = 7n \quad \text{for } n \geq 0$$

given $a_0 = a_1 = 1$.

Sol. put $b_n = a_n^2$ then the given relation reads

$$b_{n+2} - 5b_{n+1} + 6b_n = 7n \quad \text{——— (1)}$$

For homogeneous part of this relation

The characteristic equation is $K^2 - 5K + 6K = 0$

$$\Rightarrow (K-3)(K-2) = 0$$

$$\therefore b_n^{(h)} = A \times 3^n + B\,2^n \quad \text{——— (2)}$$

where ~~A and~~ B are constants.

Keeping the RHS of (1) in mind we take

$$b_n^{(p)} = A_0 + A_1 n. \quad \text{——— (3)}$$

Putting this for b_n in (1), we get

$$\{A_0 + A_1(n+2)\} - 5[A_0 + A_1(n+1)] + 6[A_0 + A_1 n] = 7n$$

Equating the corresponding terms on both sides, we get

$$A_1 - 5A_1 + 6A_1 = 7 \text{ and } (2A_1 + A_0) - 5(A_1 + A_0) + 6A_0 = 0.$$

$\Rightarrow A_1 = 7/2$ and $A_0 = 21/4$. putting these into (3) we get

$$b_n^{(p)} = \frac{7}{2}n + \frac{21}{4} \quad \text{——— (4)}$$

Therefore, the general solution for b_n is

$$b_n = b_n^{(h)} + b_n^{(p)}.$$

$$= \cancel{A \times B}\; A \times 3^n + B \times 2^n + \frac{7}{2}n + \frac{21}{4} \quad \text{——— (5)}$$

It is given that $a_0 = a_1 = 1$.

These give $b_0 = a_0^2 = 1$ $b_1 = a_1^2 = 1$.

Using these in (5) we get

$$1 = A + B + \frac{21}{4}, \quad 1 = 3A + 2B + \frac{7}{2} + \frac{21}{4}$$

Solving $\Rightarrow A = 3/4,\ B = -5$. $\therefore$ from (5) $b_n = \frac{3}{4} \times 3^n - 5\,2^n + \frac{7}{2}n + \frac{21}{4}$

$$a_n = \sqrt{b_n}$$

$$= \pm\left\{\frac{3}{4}3^n - 5 \times 2^n + \frac{7}{2}n + \frac{21}{4}\right\}^{1/2}$$

This is required sol

Ex: Solve the recurrence relation

$$a_{n+2} - 4a_{n+1} + 3a_n = -200 \quad n \geq 0$$

and $a_0 = 3000$, $a_1 = 3300$

Sol For the homogeneous part of the given relation, the characteristic equation is

$$k^2 - 4k + 3 = 0 \text{ i.e. } (k-1)(k-3) = 0$$

whose roots are 1 and 3. Therefore

$$a_n^{(h)} = A \times 1^n + B \times 3^n \quad \text{--- (1)}$$

where A and B are arbitrary constants

Since 1 is a (simple) root of characteristic equation and the RHS of the given relation is a polynomial of degree '0'.

we seek $a_n^{(p)}$ in the form

$$a_n^{(p)} = A_0 n. \quad \text{--- (2)}$$

put $a_n^{(p)}$ in the given ~~equation~~ relation, we get

$$A_0(n+2) - 4A_0(n+1) + 3A_0 n = -200$$

or $-2A_0 = -200$. or $A_0 = 100$

putting this into (2) $a_n^{(p)} = 100 \cdot n$ --- (3)

Therefore, the general sol for a_n is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A + B \cdot 3^n + 100n \quad \text{--- (4)}$$

It is given that $a_0 = 3000$ and $a_1 = 3300$

Using these in (4), we get

$$3000 = A + B. \quad 3300 = A + 3B + 100$$

$\rightarrow A = 2900,\ B = 100.$ Putting these in eq (4)

$$a_n = 2900 + 100 \cdot 3^n + 100n$$

This is the required solution.

T $a_n - 4a_{n-1} + 3a_{n-2} = -200$ for $n \geq 2$

Ex: Solve the recurrence relation

$$a_n - 2a_{n-1} + a_{n-2} = 5n \quad \text{--- (1)}$$

Sol For the homogeneous part of the given relation, the characteristic equation is

$$k^2 - 2k + 1 = 0. \text{ i.e. } (k-1)^2 = 0 \Rightarrow k = 1, 1.$$

$$\therefore \quad a_n^{(h)} = (A + Bn)(1)^n \quad \text{--- (2)}$$

where A and B are arbitrary constants.

Since 1 is a root (of multiplicity 2) of the characteristic eq and RHS of the given relation is a polynomial of degree 1 we seek $a_n^{(p)}$ in the form

$$a_n^{(p)} = n^2(A_0 + A_1 n) = A_0 n^2 + A_1 n^3 \quad \text{--- (3)}$$

Putting this for a_n in the given relation, we get

$$(A_0 n^2 + A_1 n^3) - 2\{A_0(n-1)^2 + A_1(n-1)^3\} + [A_0(n-2)^2 + A_1(n-2)^3] = 5n$$

i.e. $$(A_0 n^2 + A_1 n^3) - 2[A_0(n^2 - 2n + 1) + A_1(n^3 - 3n^2 + 3n - 1)\} + \{A_0(n^2 - 4n + 4) + A_1(n^3 - 6n^2 + 12n - 8)\} = 5n$$

Equating the corresponding coefficients on the two sides, we get

$$A_1 - 2A_1 + A_1 = 0$$

$$A_0 - 2A_0 + 6A_1 + A_0 - 6A_1 = 0$$

$$4A_0 - 6A_1 - 4A_0 + 12A_1 = 5$$

$$-2A_0 + 2A_1 + 4A_0 - 8A_1 = 0$$

Solving these (last two) $A_1 = 5/6$, $A_0 = 5/2$.

Putting these into (3) $a_n^{(p)} = n^2\left(\frac{5}{2}\right) + \frac{5}{6} n^3$ --- (4)

$\therefore$ The general sol for a_n is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$= A + Bn + \frac{5}{6} n^2(n+3)$$

Ex. Solve the recurrence relation

$$a_n + 2a_{n-1} - 3a_{n-2} = 4n^2 - 5 \quad \text{for } n \geq 2$$

Sol: ~~For~~ homogeneous part of the given relation
The characteristic eq is

$$k^2 + 2k - 3 = 0 \Rightarrow (k-1)(k+3) = 0$$

whose roots are 1 and −3. Therefore

$$a_n^{(h)} = A \times (1)^n + B \times (-3)^n \quad \text{——— (1)}$$

where A and B are arbitrary constants
Since 1 is a simple root of the characteristic equation and RHS of the given relation is a polynomial of degree '2'.
we take $a_n^{(p)}$ is in the form

$$a_n^{(p)} = n^1 (A_0 + A_1 n + A_2 n^2) = A_0 n + A_1 n^2 + A_2 n^3 \quad \text{—— (2)}$$

where A_0, A_1, A_2 are constants
putting (2) for a_n in the given relation, we get

$$(A_0 n + A_1 n^2 + A_2 n^3) + 2[A_0(n-1) + A_1(n-1)^2 + A_2(n-1)^3]$$
$$-3\{A_0(n-2) + A_1(n-2)^2 + A_2(n-2)^3\} = 4n^2 - 5$$

Equating the corresponding terms on the two sides, we get

$$\cancel{A_0 n + A_1 n^2 +} A_2 + 2A_2 - 3A_2 = 0$$

$$A_1 + 2A_1 - 6A_2 - 3A_1 + 18A_2 = 4.$$

$$A_0 + 2A_0 - 4A_1 + 6A_2 - 3A_0 + 2A_1 - 36A_2 = 0$$

$$-2A_0 + 2A_1 - 2A_2 + 6A_0 - 12A_1 + 24A_2 = -5$$

These give

$$A_2 = 1/3 \quad A_1 = 5/6, \quad A_0 = 1/4.$$

putting these values in (2), we get

$$a_n^{(p)} = n\left(\frac{1}{4} + \frac{5}{6} n + \frac{1}{3} n^2\right)$$

∴ The G.S for a_n is

$$a_n = a_n^{(h)} + a_n^{(p)} = A + B \times (-3)^n + n\left(\frac{1}{4} + \frac{5}{6} n + \frac{1}{3} n^2\right)$$

Ex 7. Solve the recurrence relation

$$a_{n+2} + 3a_{n+1} + 2a_n = 3^n. \text{ for } n \geq 0.$$

given $a_0 = 0$ $a_1 = 1$.

Sol. For the homogeneous part of the given relation, the characteristic eq is

$$K^2 + 3K + 2 = 0 \text{ or } (K+2)(K+1) = 0$$

whose roots are $-2, -1$. Therefore

$$a_n^{(h)} = A \times (-2)^n + B \times (-1)^n. \quad \text{--- (1)}$$

where A and B are arbitrary constants.

Keeping the R.H.S of the given relation in mind, we seek $a_n^{(p)}$ in the form

$$a_n^{(p)} = A_0 \times 3^n \quad \text{--- (2)}$$

putting this for a_n in the given relation, we get

$$A_0 \times 3^{n+2} + 3A_0 \times 3^{n+1} + 2A_0 \times 3^n = 3^n$$

or $A_0 = 1/20$ putting this in (2), we get

$$a_n^{(p)} = \frac{1}{20} \times 3^n \quad \text{--- (3)}$$

Therefore, the general solution of the given relation is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$= A \times (-2)^n + B \times (-1)^n + \frac{1}{20} \times 3^n \quad \text{--- (4)}$$

It is given that $a_0 = 0$ and $a_1 = 1$. Using these in (4) we get

$$0 = A + B + \frac{1}{20}, \qquad 1 = -2A - B + \frac{3}{20}$$

Solving these, we get $A = -4/5$ and $B = 3/4$

putting these into (4), we obtain

$$a_n = \left(-\frac{4}{5}\right) \times (-2)^n + \frac{3}{4}(-1)^n + \frac{1}{20}(3)^n$$

This is the required sol

Ex 8. Solve the recurrence relation

$$a_n + 4a_{n-1} + 4a_{n-2} = 5\times(-2)^n. \quad n\geq 2$$

Sol: For the homogeneous part of the given relation, the characteristic equation is

$$k^2+4k+4=0 \text{ (or) } (k+2)^2=0 \Rightarrow k=-2,-2.$$

$$\therefore a_n^{(h)} = (A+Bn)(-2)^n \quad \text{——— (1)}$$

where A and B are arbitrary constants.

We observe that the R.H.S of the given relation contains $(-2)^n$ as a factor and (-2) is repeated root of the characteristic equation. As such we seek $a_n^{(p)}$ in the form

$$a_n^{(p)} = A_0 n^2 (-2)^n \quad \text{——— (2)}$$

Putting this for a_n in the given relation, we get

$$A_0 n^2(-2)^n + 4A_0(n-1)^2(-2)^{n-1} + 4A_0(n-2)^2(-2)^{n-2} = 5\times(-2)^n$$

Dividing throughout by $(-2)^{n-2}$, this becomes

$$A_0\left[n^2\times(-2)^2 + 4(n-1)^2(-2) + 4(n-2)^2\right] = 5\times(-2)^2$$

$$A_0\left[4n^2 - 8(n^2-2n+1) + 4(n^2-4n+4)\right] = \cancel{5\times(-2)^n}\ 20$$

for which we get $A_0 = \frac{20}{8} = \frac{5}{2}$. Putting this into (2) we get

$$a_n^{(p)} = \frac{5}{2} n^2(-2)^n \quad \text{——— (3)}$$

$\therefore$ Therefore, the general solution of the given relation is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$= \left(A + Bn + \frac{5}{2}n^2\right)(-2)^n$$

Ex 9: For the general solution of the recurrence relation

$$s(K) - 3s(K-1) + 4s(K-2) = 4^K, \quad K \geq 2$$

Sol. In the standard notation, the given relation reads

$$s_n - 3s_{n-1} - 4s_{n-2} = 4^n \quad n \geq 2 \quad \text{--- (1)}$$

For the homogeneous part of the relation, the characteristic equation is $K^2 - 3K - 4 = 0$

$$\Rightarrow K = 4 \text{ and } -1$$

$$\therefore \quad s_n^{(h)} = A \times 4^n + B \times (-1)^n. \quad \text{--- (2)}$$

where A and B are arbitrary constants

We observe that RHS of the relation (1) is 4^n and 4 is a simple root of the characteristic equation.

As such $s_n^{(p)} = A_0 n \times 4^n$ --- (3)

Putting this for s_n in the relation (1) we get

$$A_0 \cdot n \cdot 4^n - 3A_0(n-1) \cdot 4^{n-1} - 4A_0(n-2) \times 4^{n-2} = 4^n$$

Divide throughout by 4^{n-2}, this becomes

$$16 \cdot A_0 \cdot n - 12A_0 \cdot (n-1) - 4A_0(n-2) = 16$$

$\Rightarrow A_0 = 4/5$. Putting this into (3) we get

$$s_n^{(p)} = \frac{4}{5} n \times 4^n \quad \text{--- (4)}$$

$\therefore$ The general sol is $s_n = s_n^{(h)} + s_n^{(p)}$.

$$s_n = \left(A + \frac{4n}{5}\right) \times 4^n + B \times (-1)^n$$

In the given notation, the general solution reads

$$s(K) = \left(A + \frac{4K}{5}\right) \times 4^K + 3 \times (-1)^K$$