Course content:

- dynamical system view
- importance of

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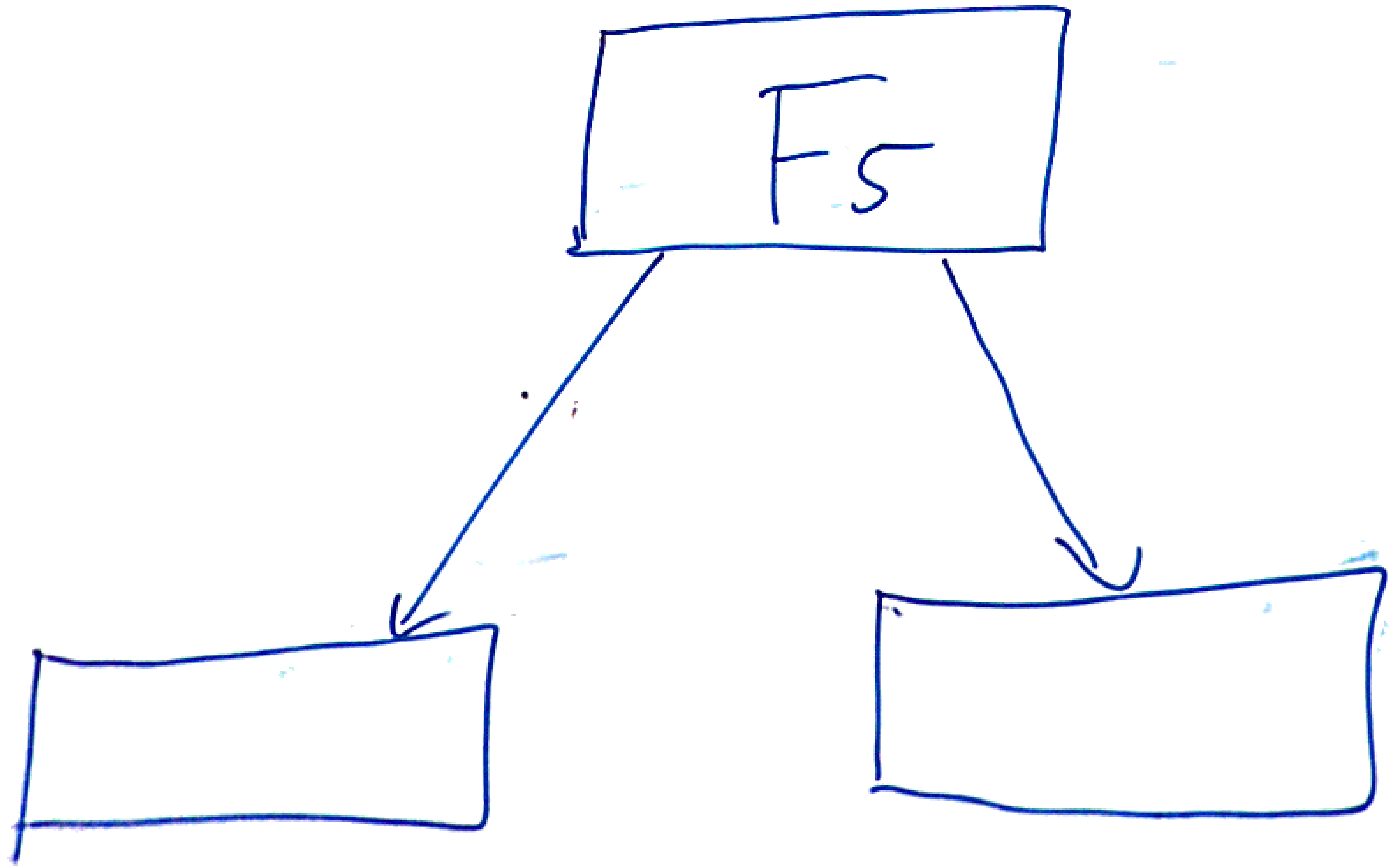
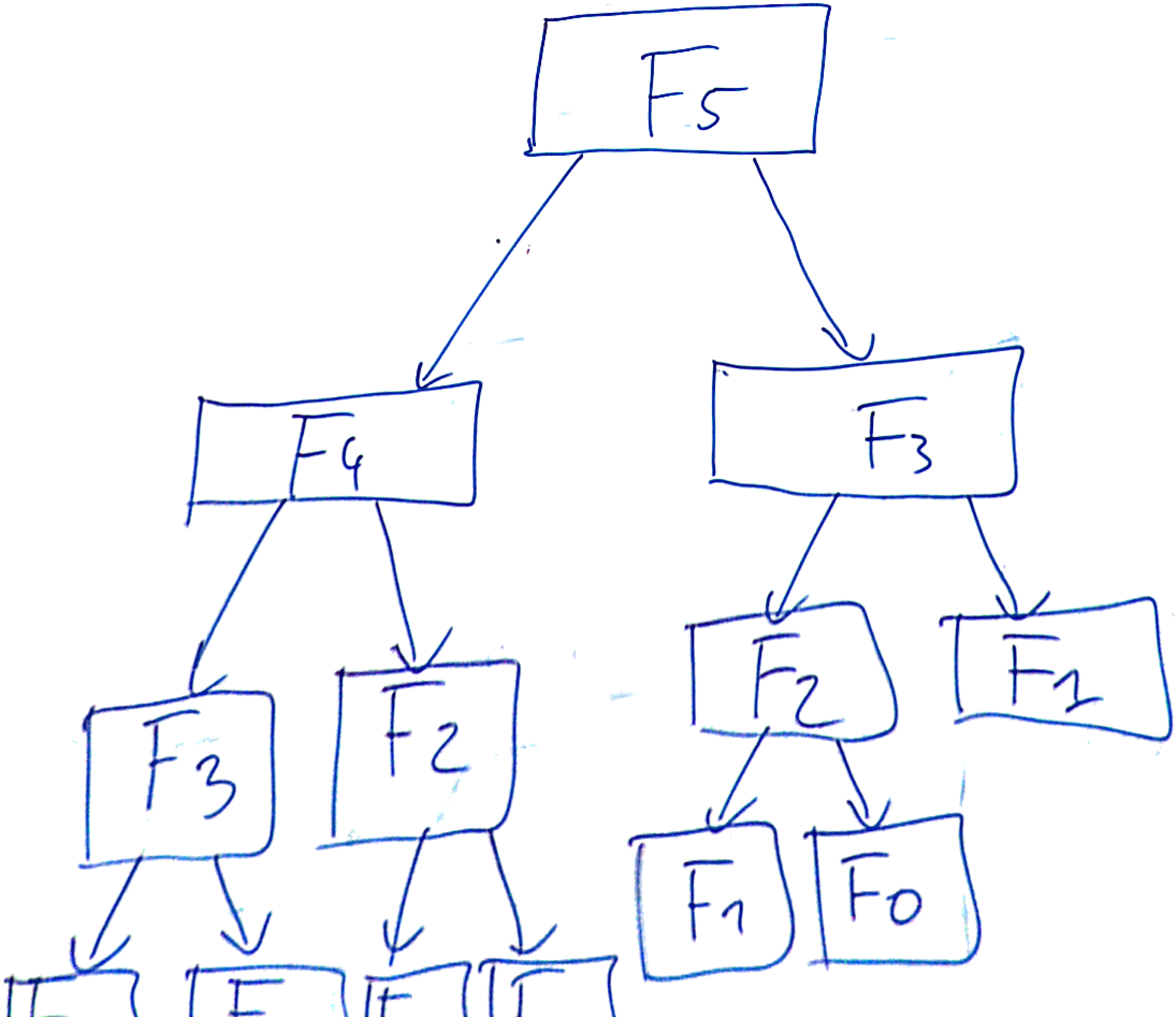
per bound: $T_n \le 2 \cdot T_{n-1} + O(1) = O(2^n)$

$T_n \ge 2 \cdot T_{n-2} + O(1) = O(2^{n/2})$


Fs


F5
F4
F3
F3
F2
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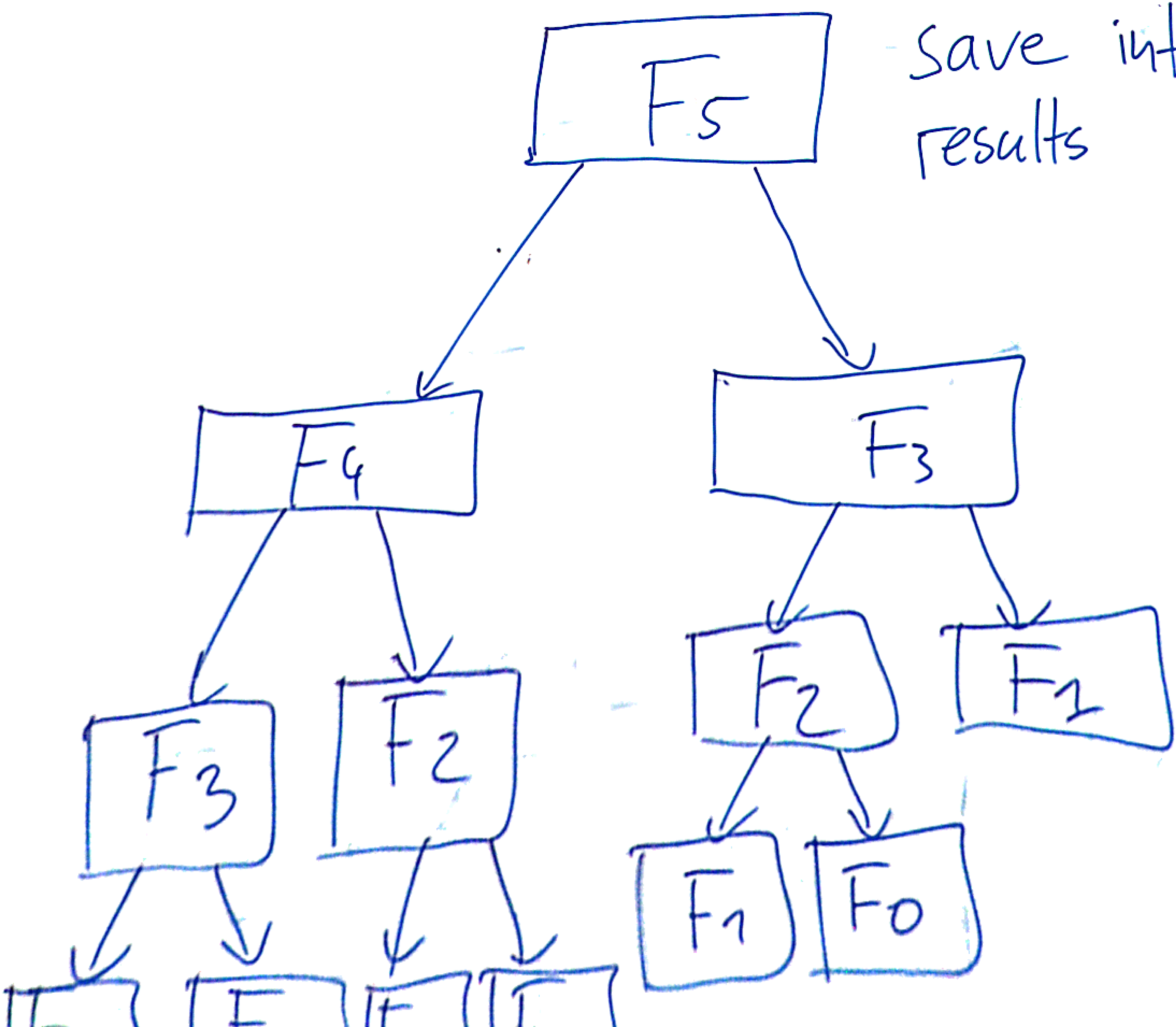
F0



save intermediate results

save intermediate results = "memoization"

$$F_n = F_{n-1} + F_{n-2}$$

Guess: $F_n = r^n$

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$$\Rightarrow \quad 0 = r^2 - r - 1 \; ; \quad r_{1/2} = \frac{1}{2} \pm \sqrt{\frac{5}{4}}$$

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$$F_n = \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n \left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^n$$

The importance of non-linearity

Strogatz p. 355 ff.

Stuart & Humphries, ch. 1

Discrete dynamical systems

$$x_{n+1} = f(x_n), \text{ given } f(\cdot) \text{ and } x_0$$

↳ map ↳ initial condition

Example: $x_{n+1} = \sin(x_n)$

The sequence $\{x_0, x_1, x_2, \dots\}$ is called orbit.

The importance of non-linearity

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Fixed points and linear stability

We call x^* a fixed point if it satisfies

$f(x^*) \quad x^*.$

The importance of non-linearity

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linearize $f(x + \Delta x) = f(x) + \Delta x \frac{df}{dx}$

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$$\cancel{x^*} + \eta_{n+1} = \cancel{x^*} + \eta_n \lambda \Leftrightarrow \eta_{n+1} = \lambda \eta_n$$

$$\eta_n = \lambda^n \eta_0$$

linearize

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The importance of non-linearity

Fixed points and linear stability

If $|\lambda| = |f'(x^*)| < 1 \Rightarrow \eta_n \mapsto 0$ for $n \mapsto \infty$

We say "x^*" is a stable fixed point.

For $|\lambda| > 1$, the fixed point is unstable.

The importance of non-linearity

Example: $x_{n+1} = x_n^2$

Fixed point condition: $x^* = x^{*2}$

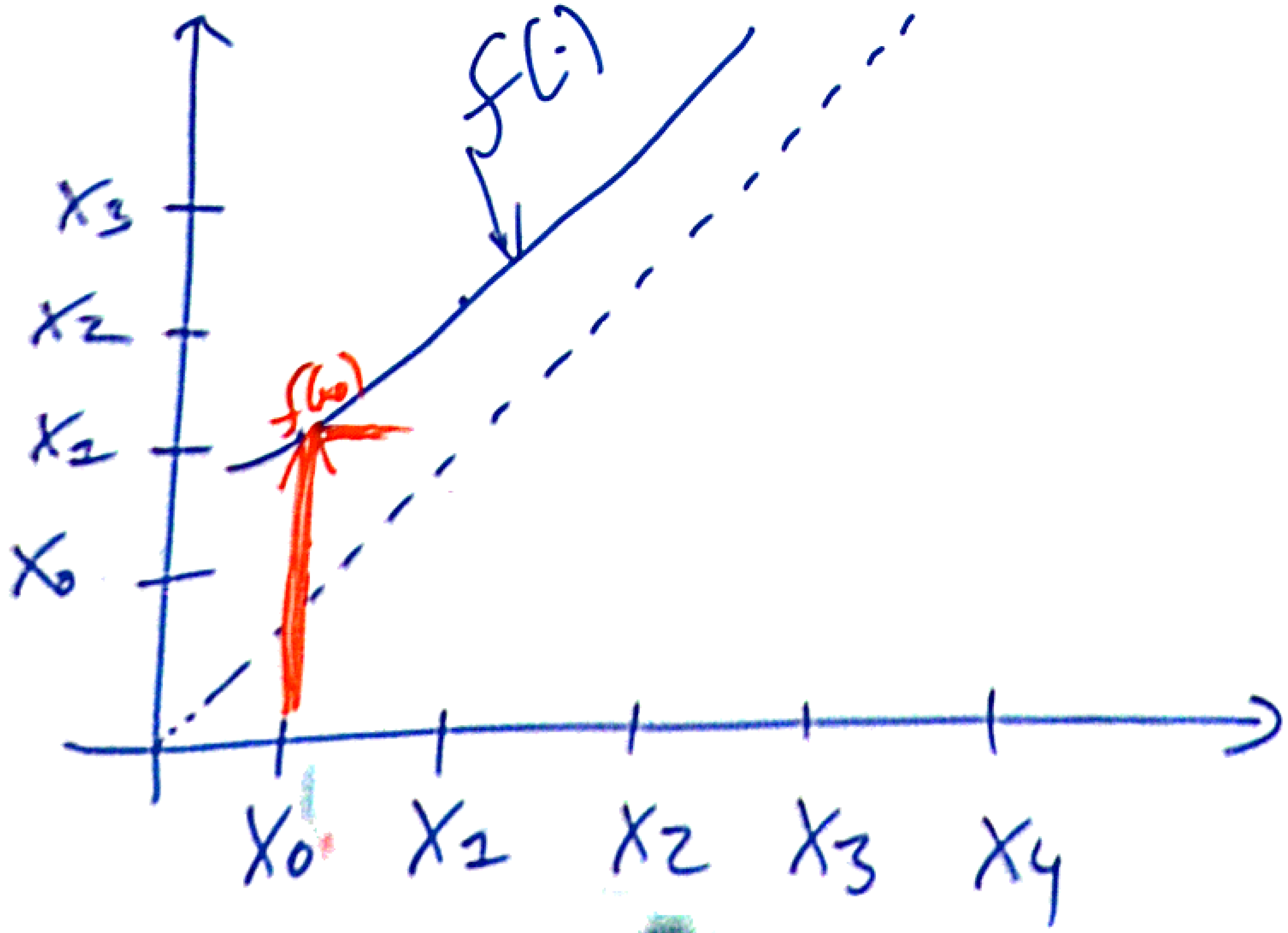
$\Rightarrow$ Two fixed points: 0 and 1.

$\lambda = f'(x^*) = 2x^* \Rightarrow$ $x_1^* = 0$, $\lambda = 0 \Rightarrow$ it is stable

$x_2^* = 1$, $\lambda = 2 \Rightarrow$ it is unstable

The importance of non-linearity

Cobweb plot:



The importance of non-linearity

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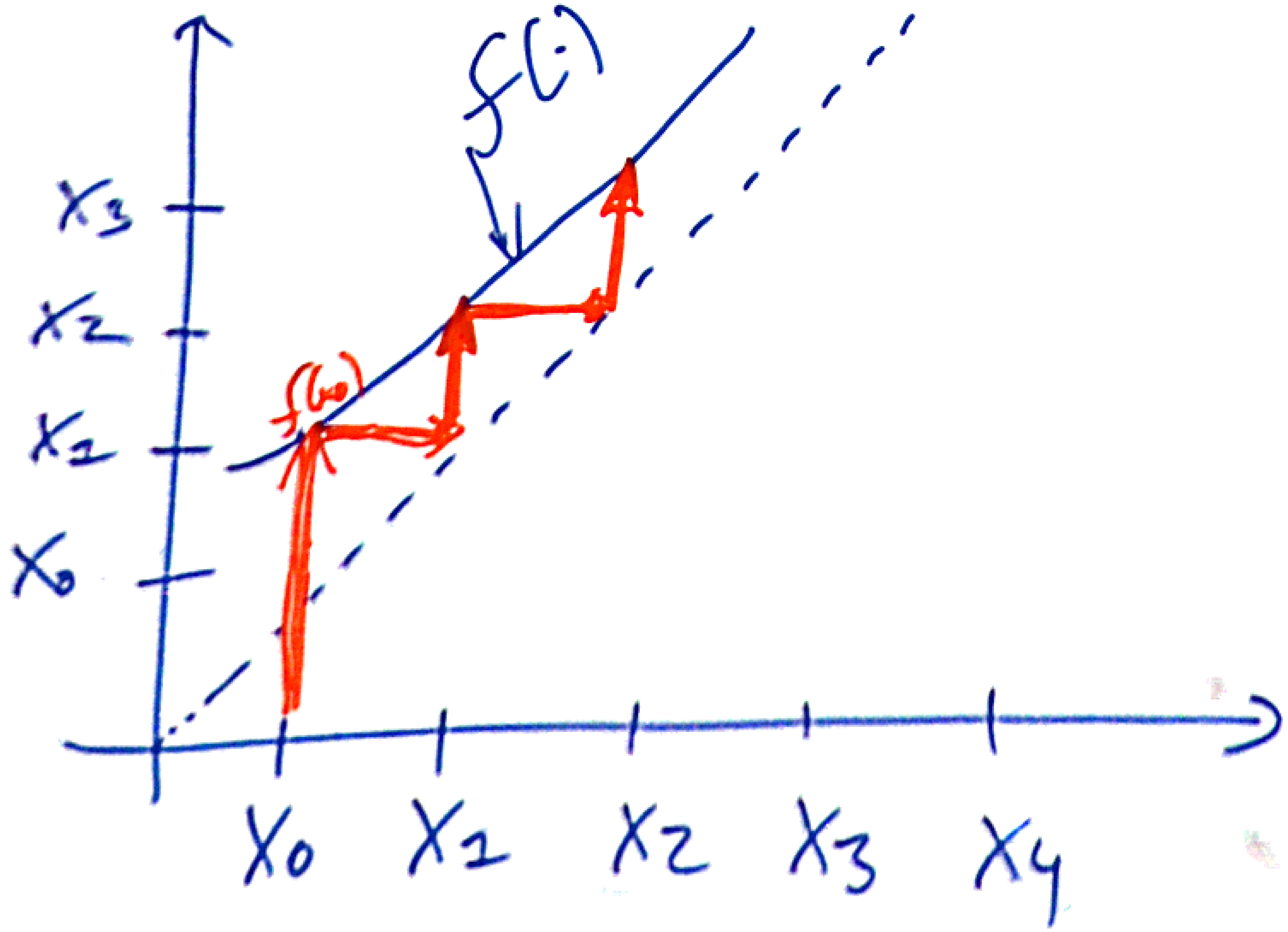

The importance of non-linearity

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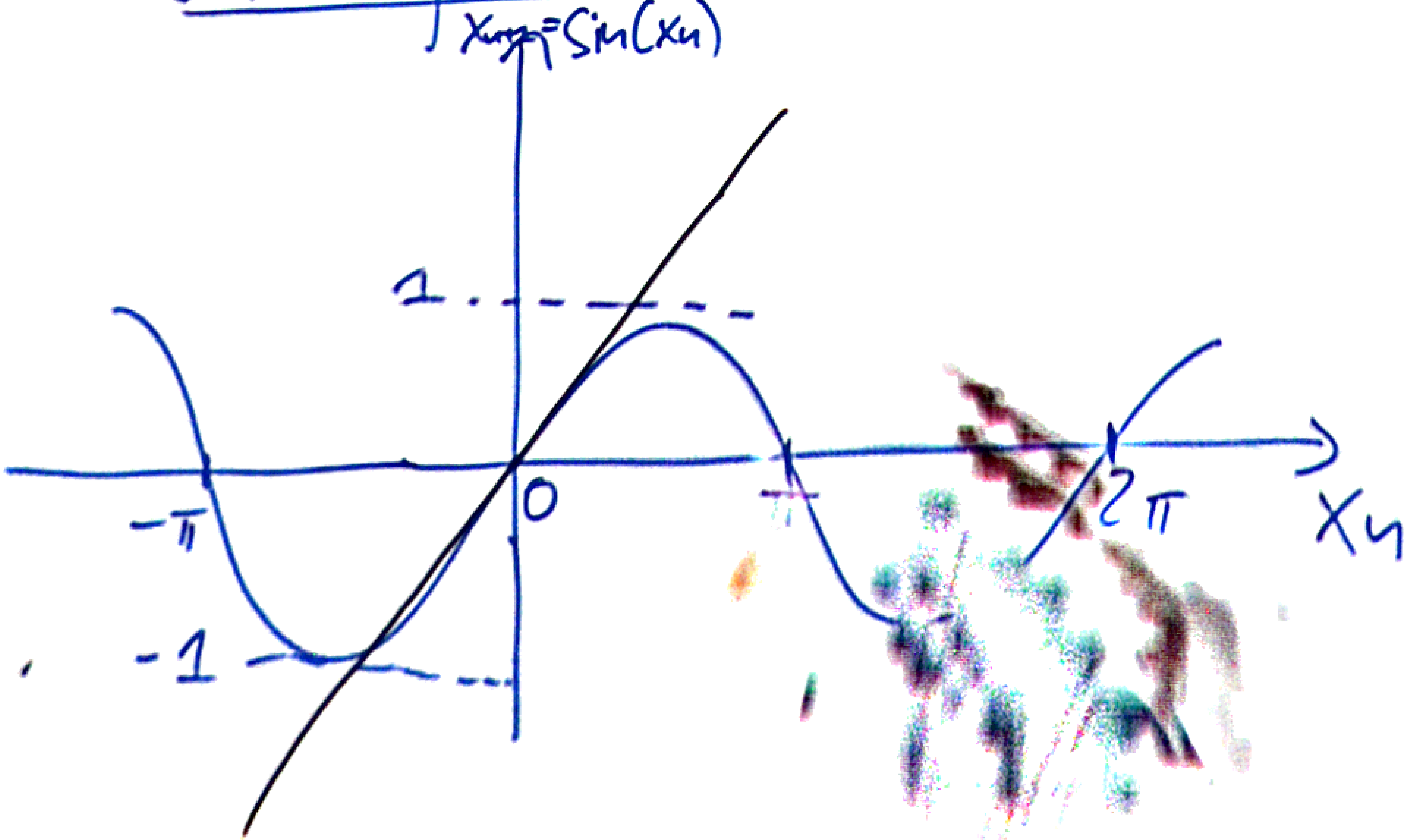

The importance of non-linearity

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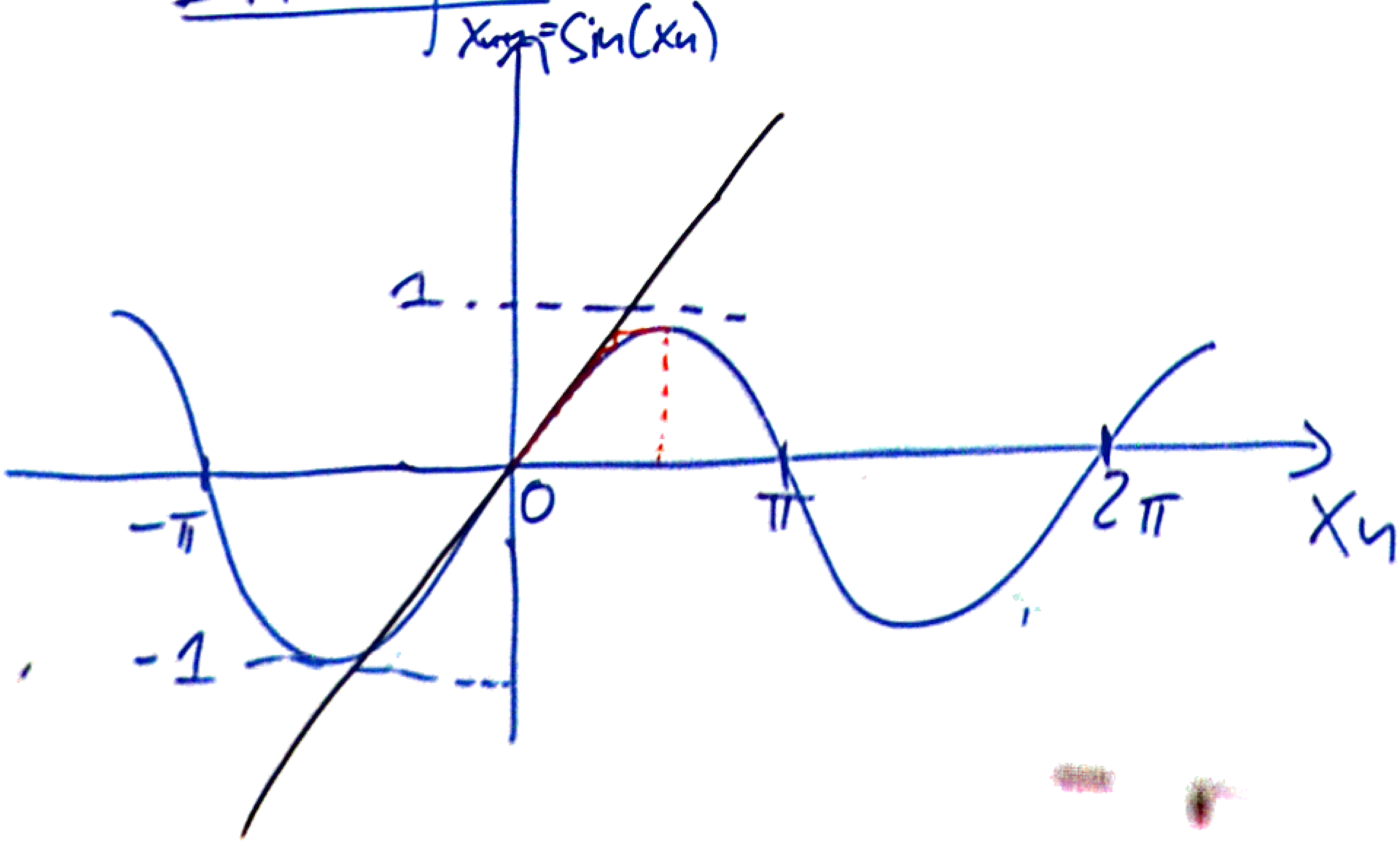

The importance of non-linearity

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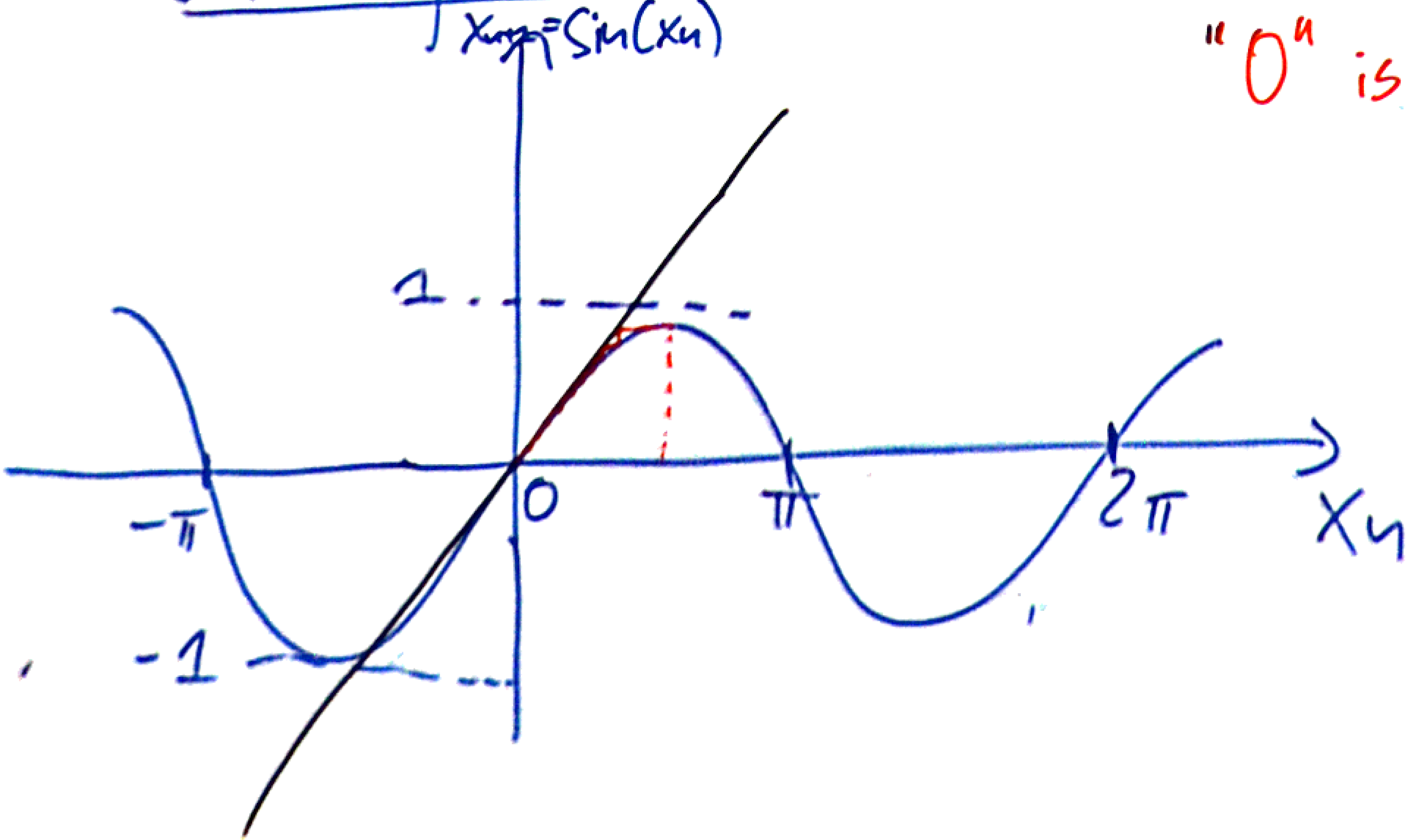

"0" is globally stable.

The importance of non-linearity

The logistic map

Simple model of population growth:

$$x_{n+1} = r x_n (1 - x_n)$$

The importance of non-linearity

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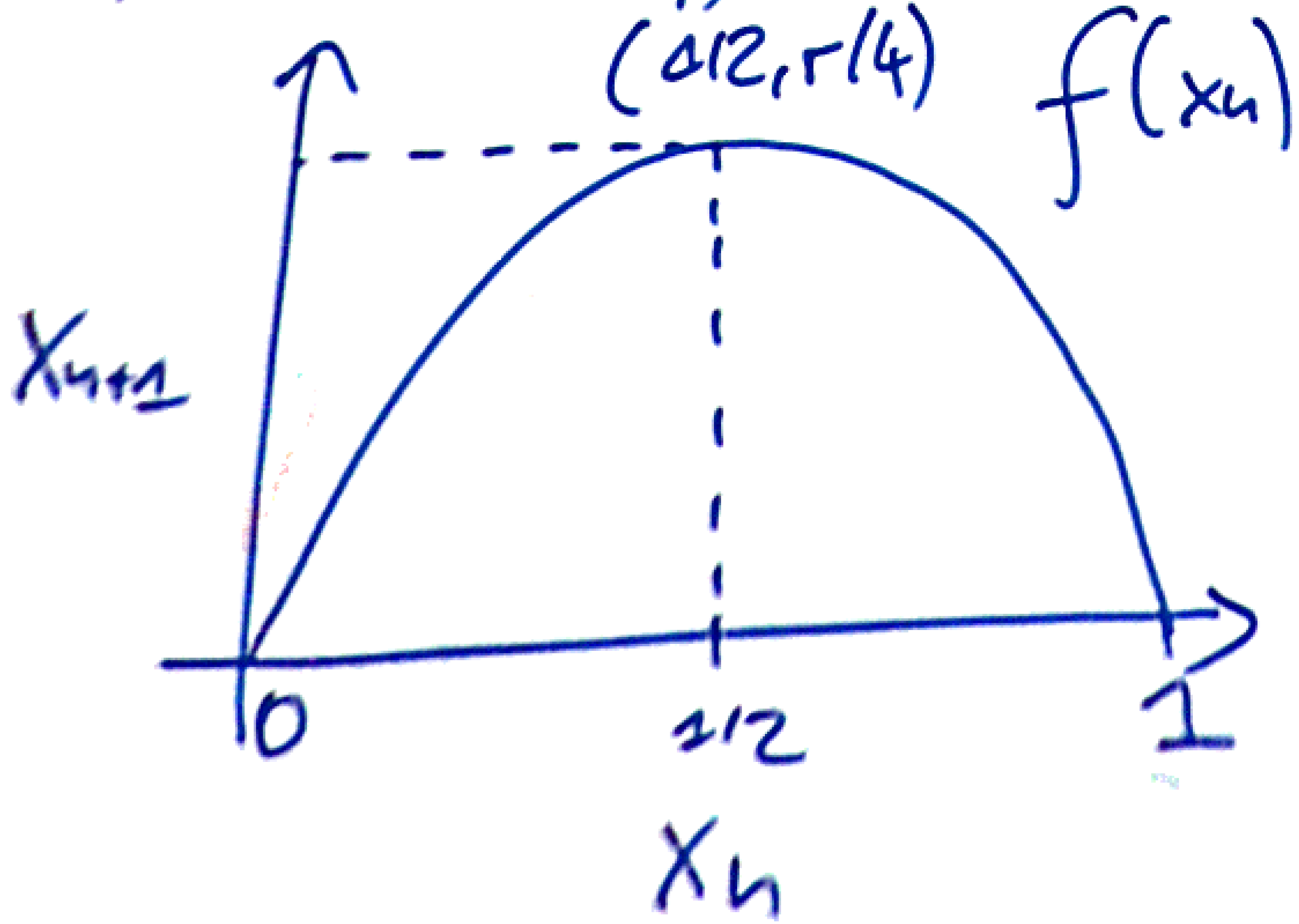

$$f(x_n) = r x_n (1 - x_n)$$

$$0 \leq r \leq 4$$

The importance of non-linearity

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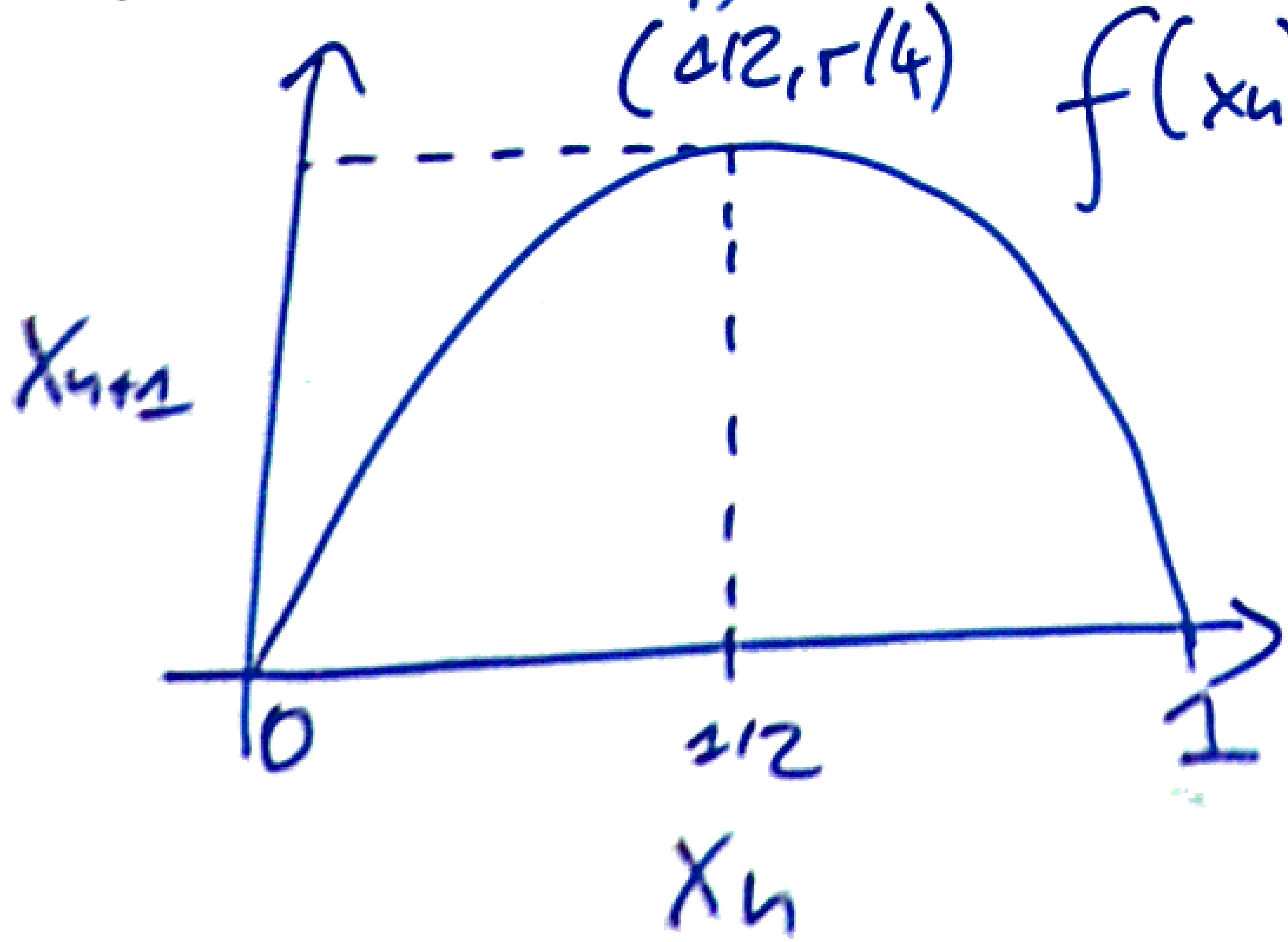

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Period doubling:

$r < 1$: population goes extinct ($x_n \to 0$, $n \to \infty$)

$1 < r < 3$: population reaches a non-zero steady

for larger r:

oscillations and chaos

The importance of non-linearity

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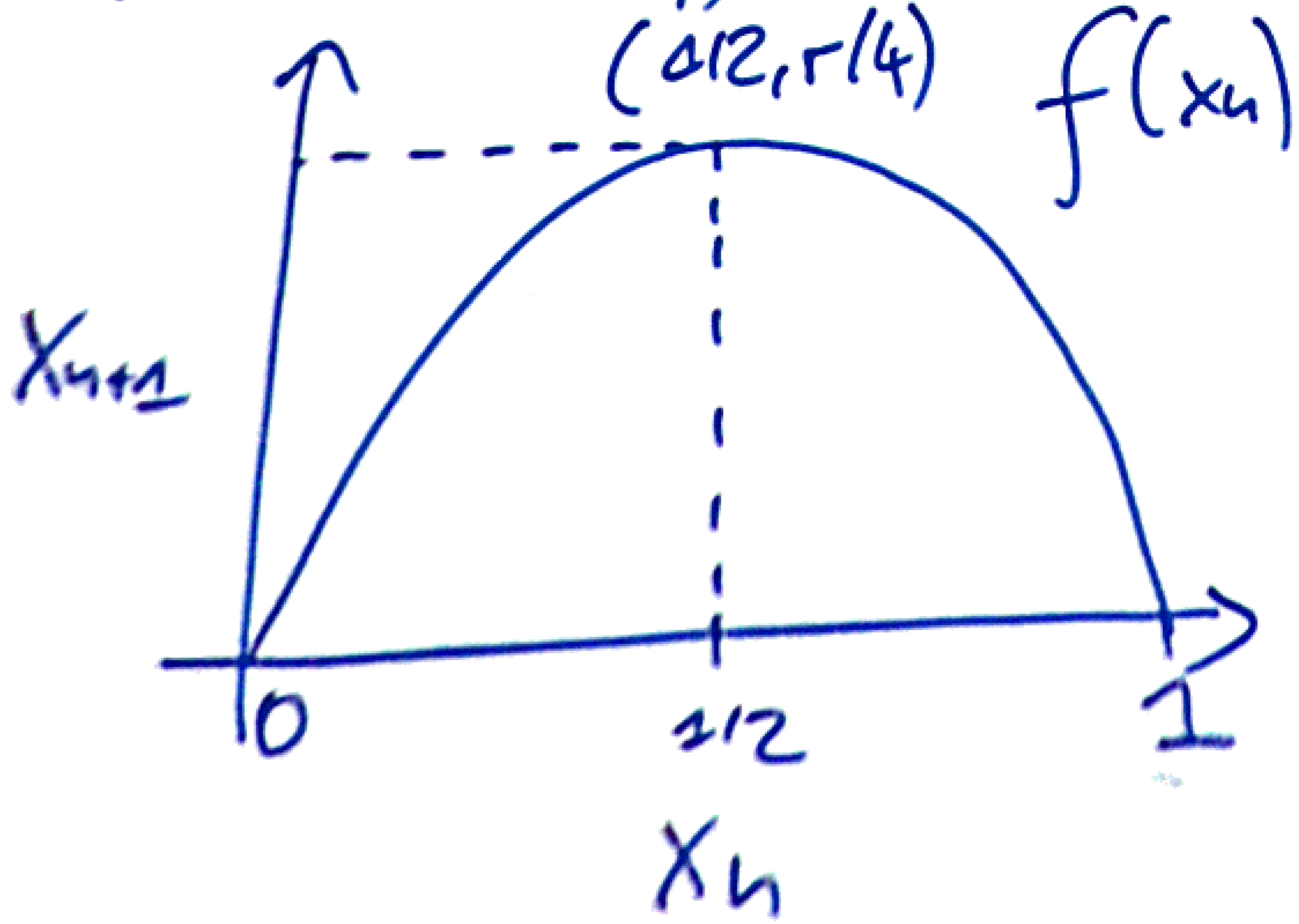

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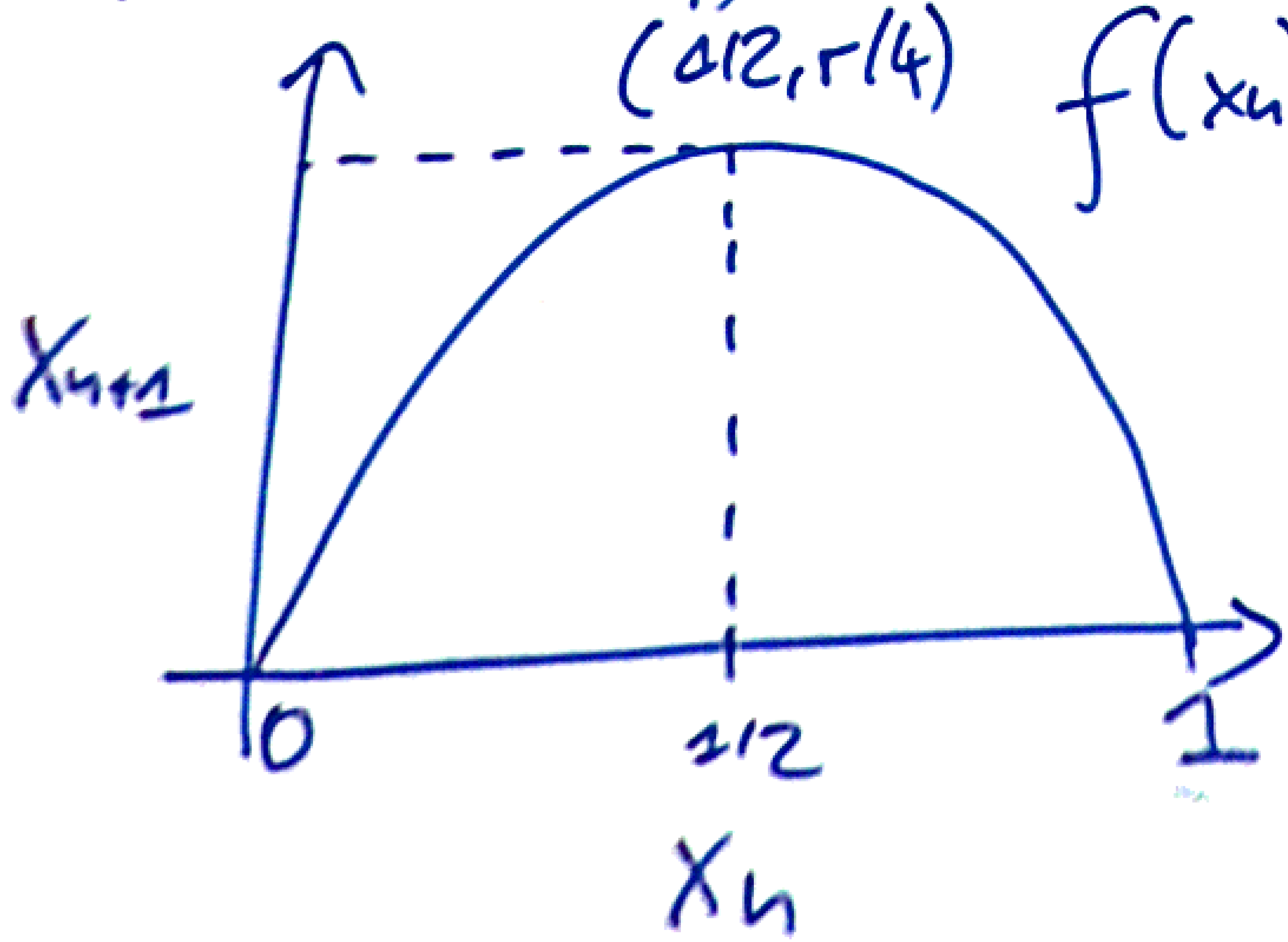

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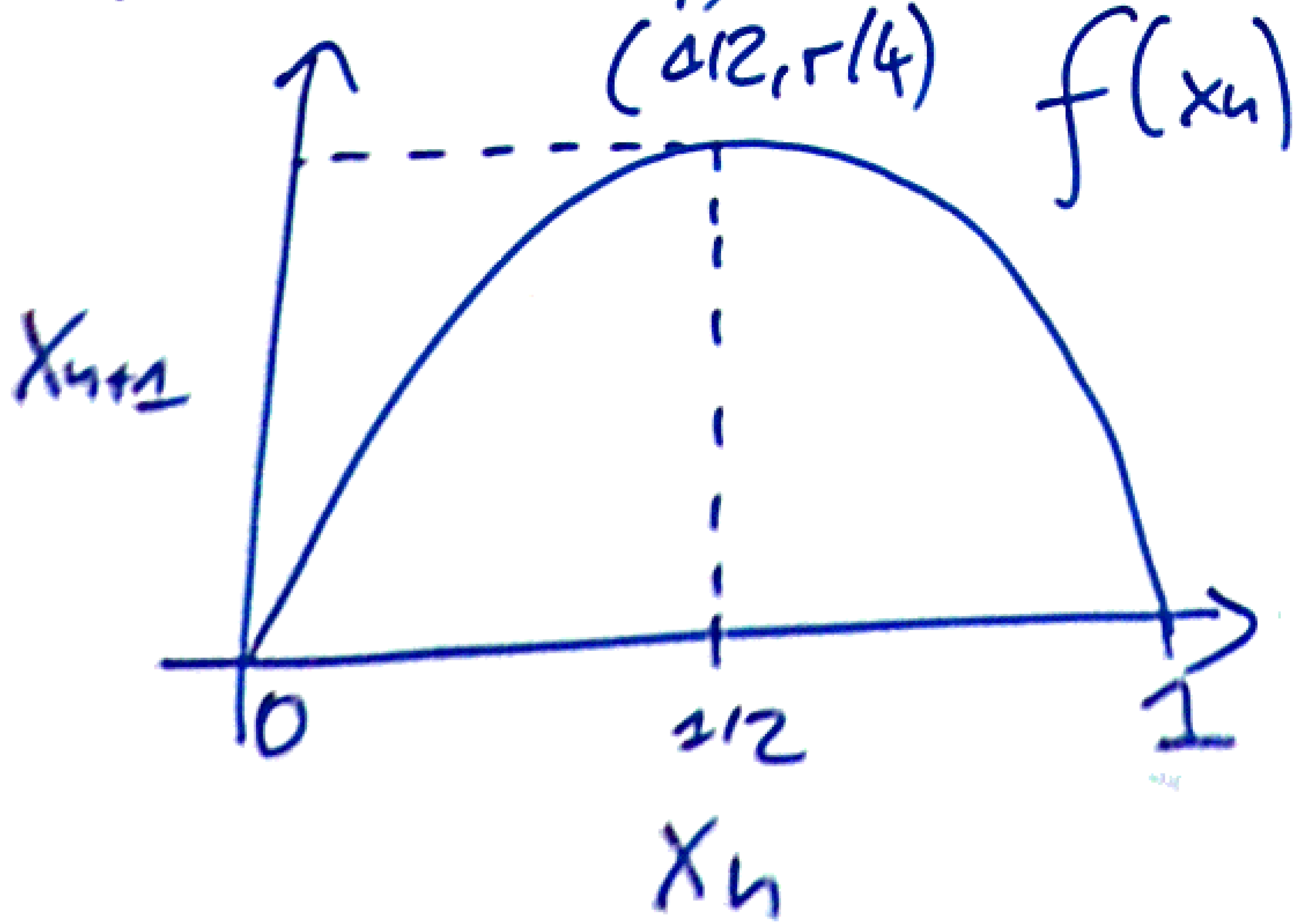

$f(x_n) = r x_n (1 - x_n)$; $x_n \in [0,1]$

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$f'(x_1^* = 0) = r \Rightarrow x_1^*$ stable for $r < 1$

$f'(x_2^* = 1 - 1/r) = 2 - r \Rightarrow$ stable for r>

The importance of non-linearity

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The importance of non-linearity

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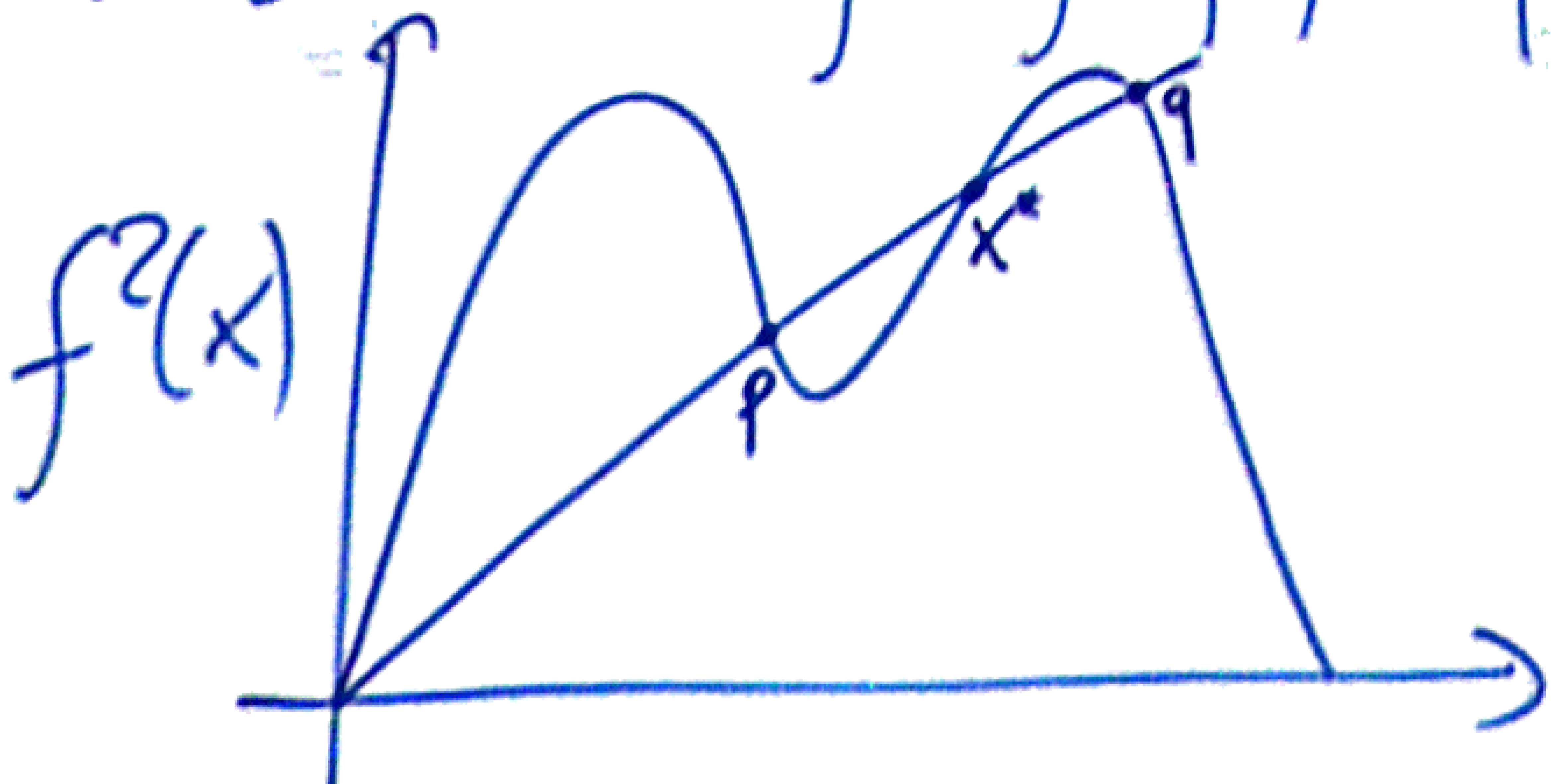

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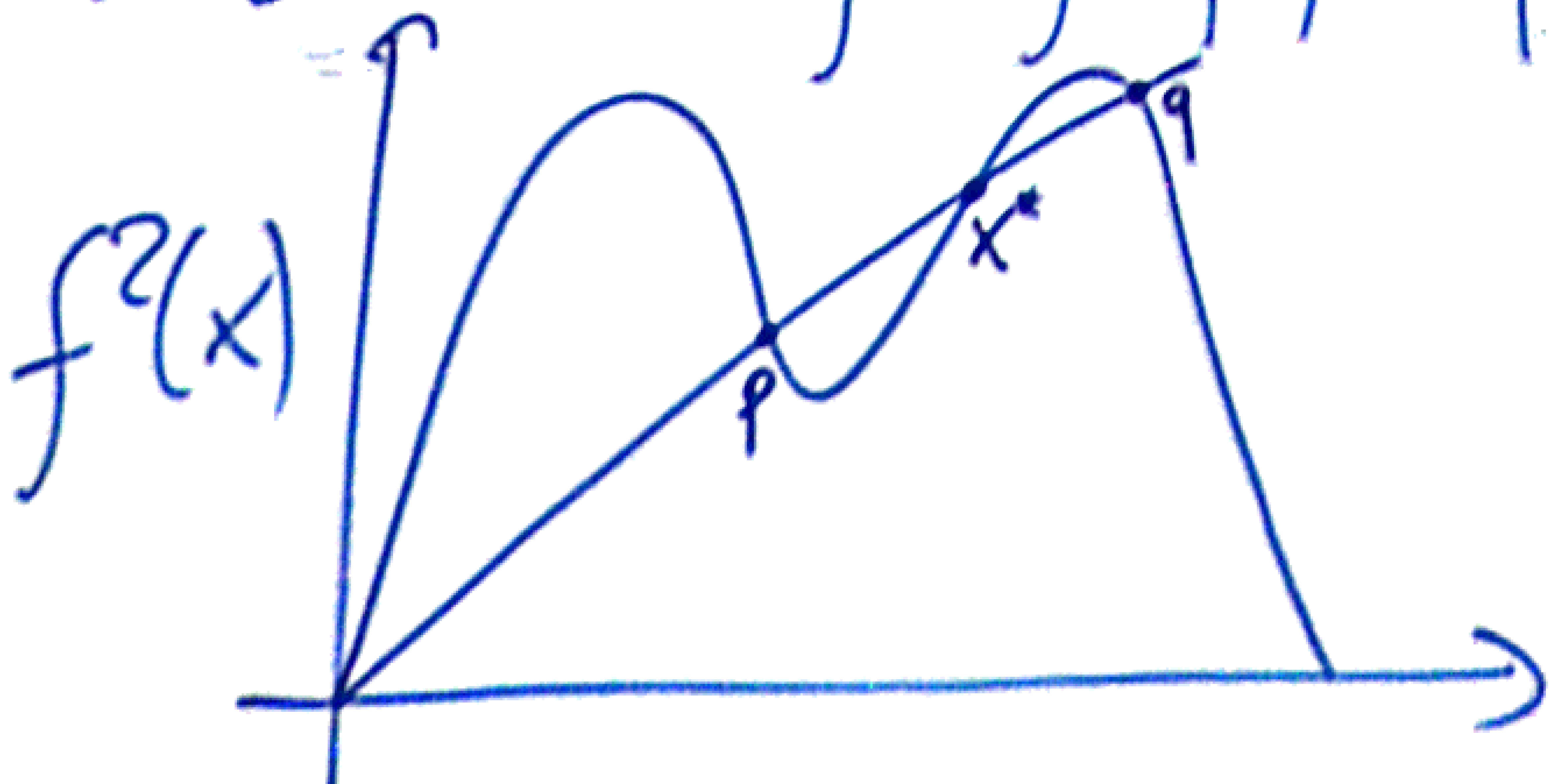

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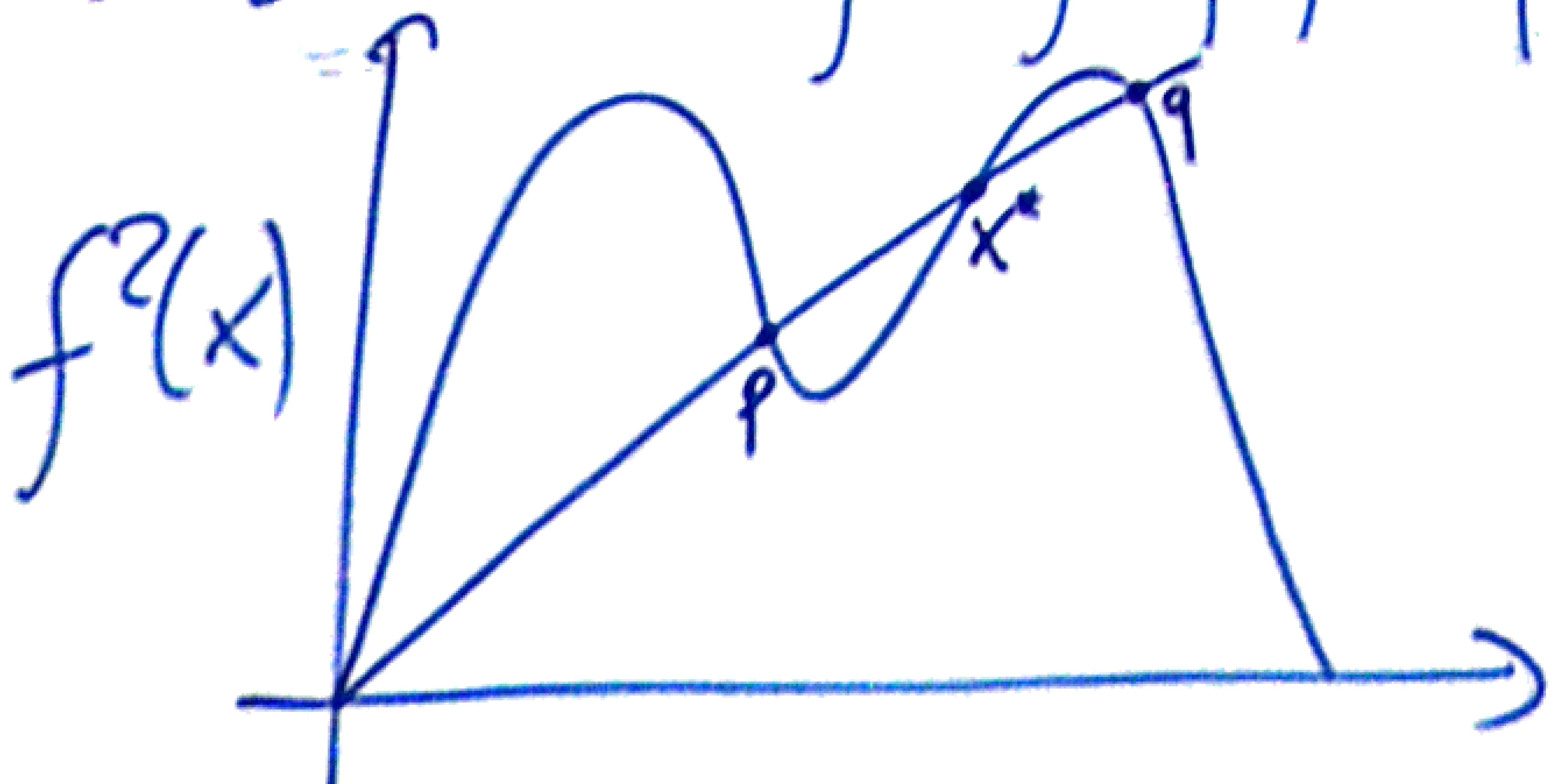

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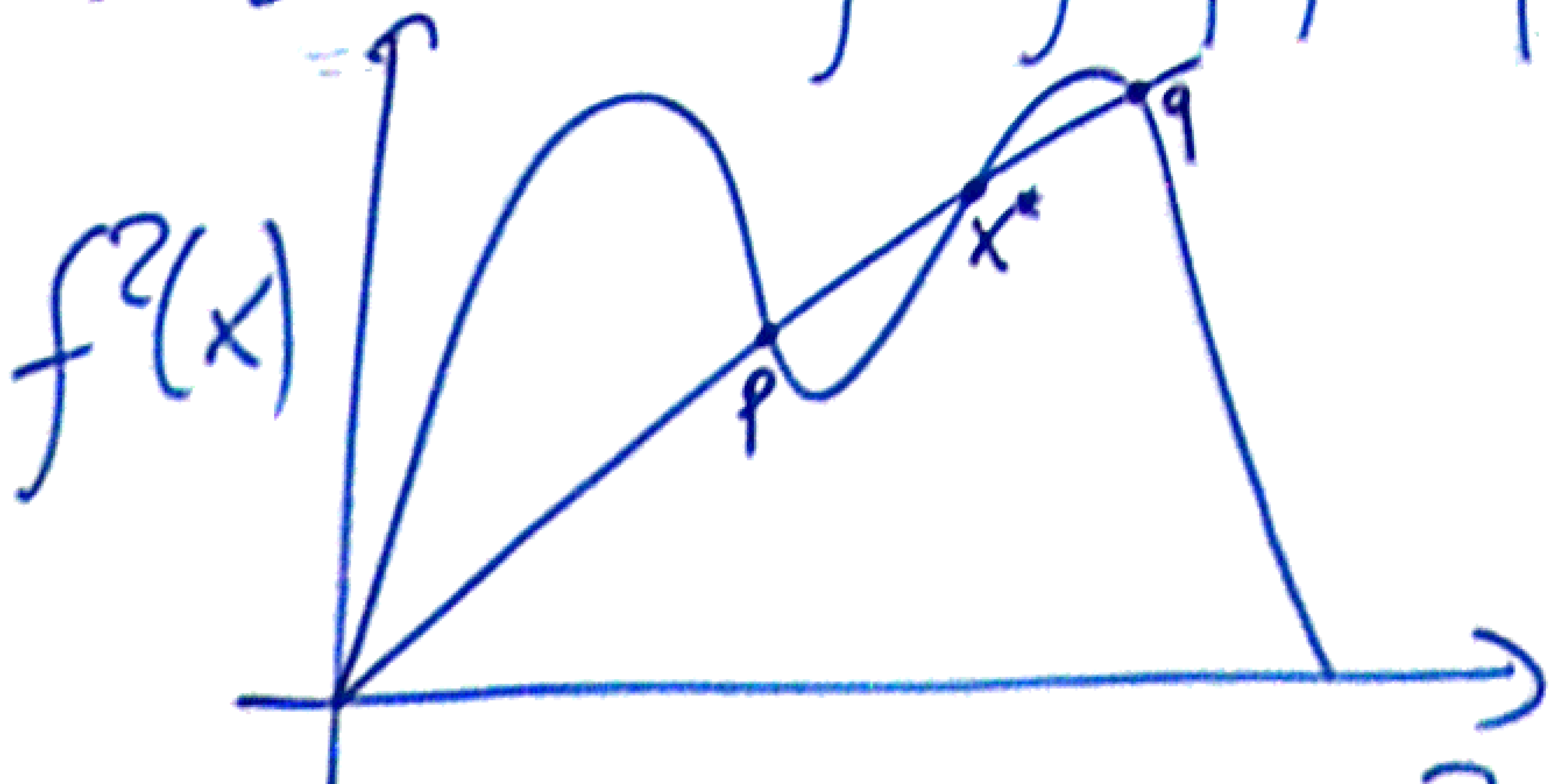

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The importance of non-linearity

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Period doubling:

$r_1 = 3$ period 2

$r_2 = 3.449\ldots$ period 4

$r_3 = 3.544\ldots$ — " — 8

$\vdots$

$r_\infty = 3.569\ldots$ — " — ∞

The importance of non-linearity

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