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Data Structures and Algorithms Made Easy in JAVA

Data Structure and Algorithmic Puzzles

CareerMonk Publications

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And
Algorithms
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By
Narasimha Karumanchi

Designed by *Narasimha Karumanchi*

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Mother and *Father*, it is impossible to thank you adequately for everything you have done, from loving me unconditionally to raising me in a stable household, where your persistent efforts and traditional values taught your children to celebrate and embrace life. I could not have asked for better parents or role-models. You showed me that anything is possible with faith, hard work and determination.

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Preface

Dear Reader,

Please hold on! I know many people typically do not read the Preface of a book. But I strongly recommend that you read this particular Preface.

It is not the main objective of this book to present you with the theorems and proofs on *data structures* and *algorithms*. I have followed a pattern of improving the problem solutions with different complexities (for each problem, you will find multiple solutions with different, and reduced, complexities). Basically, it's an enumeration of possible solutions. With this approach, even if you get a new question, it will show you a way to *think* about the possible solutions. You will find this book useful for interview preparation, competitive exams preparation, and campus interview preparations.

As a *job seeker*, if you read the complete book, I am sure you will be able to challenge the interviewers. If you read it as an *instructor*, it will help you to deliver lectures with an approach that is easy to follow, and as a result your students will appreciate the fact that they have opted for Computer Science / Information Technology as their degree.

This book is also useful for *Engineering degree students* and *Masters degree students* during their academic preparations. In all the chapters you will see that there is more emphasis on problems and their analysis rather than on theory. In each chapter, you will first read about the basic required theory, which is then followed by a section on problem sets. In total, there are approximately 700 algorithmic problems, all with solutions.

If you read the book as a *student* preparing for competitive exams for Computer Science / Information Technology, the content covers *all the required topics* in full detail. While writing this book, my main focus was to help students who are preparing for these exams.

In all the chapters you will see more emphasis on problems and analysis rather than on theory. In each chapter, you will first see the basic required theory followed by various problems.

For many problems, *multiple* solutions are provided with different levels of complexity. We start with the *brute force* solution and slowly move toward the *best solution* possible for that problem. For each problem, we endeavor to understand how much time the algorithm takes and how much memory the algorithm uses.

It is recommended that the reader does at least one *complete* reading of this book to gain a full understanding of all the topics that are covered. Then, in subsequent readings you can skip directly to any chapter to refer to a specific topic. Even though many readings have been done for the purpose of correcting errors, there could still be some minor typos in the book. If any are found, they will be updated at *www.CareerMonk.com*. You can monitor this site for any corrections and also for new problems and solutions. Also, please provide your valuable suggestions at: *Info@CareerMonk.com*.

I wish you all the best and I am confident that you will find this book useful.

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CHAPTER

1

INTRODUCTION

The objective of this chapter is to explain the importance of the analysis of algorithms, their notations, relationships and solving as many problems as possible. Let us first focus on understanding the basic elements of algorithms, the importance of algorithm analysis, and then slowly move toward the other topics as mentioned above. After completing this chapter, you should be able to find the complexity of any given algorithm (especially recursive functions).

1.1 Variables

Before getting in to the definition of variables, let us relate them to an old mathematical equation. Many of us would have solved many mathematical equations since childhood. As an example, consider the equation below:

$$x^2 + 2y - 2 = 1$$

We don't have to worry about the use of this equation. The important thing that we need to understand is that the equation has names (x and y), which hold values (data). That means the *names* (x and y) are placeholders for representing data. Similarly, in computer science programming we need something for holding data, and *variables* is the way to do that.

1.2 Data Types

In the above-mentioned equation, the variables x and y can take any values such as integral numbers (10, 20), real numbers (0.23, 5.5), or just 0 and 1. To solve the equation, we need to relate them to the kind of values they can take, and *data type* is the name used in computer science programming for this purpose. A *data type* in a programming language is a set of data with predefined values. Examples of data types are: integer, floating point, unit number, character, string, etc.

Computer memory is all filled with zeros and ones. If we have a problem and we want to code it, it's very difficult to provide the solution in terms of zeros and ones. To help users, programming languages and compilers provide us with data types. For example, *integer* takes 2 bytes (actual value depends on compiler), *float* takes 4 bytes, etc. This says that in memory we are combining 2 bytes (16 bits) and calling it an *integer*. Similarly, combining 4 bytes (32 bits) and calling it a *float*. A data type reduces the coding effort. At the top level, there are two types of data types:

- System-defined data types (also called *Primitive* data types)
- User-defined data types.

System-defined data types (Primitive data types)

Data types that are defined by system are called *primitive* data types. The primitive data types provided by many programming languages are: int, float, char, double, bool, etc. The number of bits allocated for each primitive data type depends on the programming languages, the compiler and the operating system. For the same primitive data type, different languages may use different sizes. Depending on the size of the data types, the total available values (domain) will also change. For example, *"int"* may take 2 bytes or 4 bytes. If it takes 2 bytes (16 bits), then the total possible values are minus 32,768 to plus 32,767 (-2^{15} *to* $2^{15}-1$). If it takes 4 bytes (32 bits), then the possible values are between –2,147,483,648 and +2,147,483,647 (-2^{31} *to* $2^{31}-1$). The same is the case with other data types.

User-defined data types

If the system-defined data types are not enough, then most programming languages allow the users to define their own data types, called *user – defined data types*. Good examples of user defined data types are: structures in *C/C* + + and classes in *Java*. For example, in the snippet below, we are combining many system-defined data types and calling the user defined data type by the name *"newType"*. This gives more flexibility and comfort in dealing with computer memory.

```
public class newType {
   public int data1;
   public int data 2;
   private float data3;
   ...
   private char data;
   //Operations
}
```

1.3 Data Structure

Based on the discussion above, once we have data in variables, we need some mechanism for manipulating that data to solve problems. *Data structure* is a particular way of storing and organizing data in a computer so that it can be used efficiently. A *data structure* is a special format for organizing and storing data. General data structure types include arrays, files, linked lists, stacks, queues, trees, graphs and so on.

Depending on the organization of the elements, data structures are classified into two types:

1) *Linear data structures:* Elements are accessed in a sequential order but it is not compulsory to store all elements sequentially (say, Linked Lists). *Examples:* Linked Lists, Stacks and Queues.
2) *Non – linear data structures:* Elements of this data structure are stored/accessed in a non-linear order. *Examples:* Trees and graphs.

1.4 Abstract Data Types (ADTs)

Before defining abstract data types, let us consider the different view of system-defined data types. We all know that, by default, all primitive data types (int, float, etc.) support basic operations such as addition and subtraction. The system provides the implementations for the primitive data types. For user-defined data types we also need to define operations. The implementation for these operations can be done when we want to actually use them. That means, in general, user defined data types are defined along with their operations.

To simplify the process of solving problems, we combine the data structures with their operations and we call this *Abstract Data Types* (ADTs). An ADT consists of *two parts:*

1. Declaration of data
2. Declaration of operations

Commonly used ADTs include: Linked Lists, Stacks, Queues, Priority Queues, Binary Trees, Dictionaries, Disjoint Sets (Union and Find), Hash Tables, Graphs, and many others. For example, stack uses a LIFO (Last-In-First-Out) mechanism while storing the data in data structures. The last element inserted into the stack is the first element that gets deleted. Common operations are: creating the stack, push an element onto the stack, pop an element from the stack, finding the current top of the stack, finding the number of elements in the stack, etc.

While defining the ADTs do not worry about the implementation details. They come into the picture only when we want to use them. Different kinds of ADTs are suited to different kinds of applications, and some are highly specialized to specific tasks. By the end of this book, we will go through many of them and you will be in a position to relate the data structures to the kind of problems they solve.

1.5 What is an Algorithm?

Let us consider the problem of preparing an *omelette*. To prepare an omelette, we follow the steps given below:

1) Get the frying pan.
2) Get the oil.
    a. Do we have oil?
        i. If yes, put it in the pan.
        ii. If no, do we want to buy oil?
            1. If yes, then go out and buy.
            2. If no, we can terminate.
3) Turn on the stove, etc...

What we are doing is, for a given problem (preparing an omelette), we are providing a step-by-step procedure for solving it. The formal definition of an algorithm can be stated as:

An algorithm is the step-by-step unambiguous instructions to solve a given problem.

In the traditional study of algorithms, there are two main criteria for judging the merits of algorithms: correctness (does the algorithm give solution to the problem in a finite number of steps?) and efficiency (how much resources (in terms of memory and time) does it take to execute the).

Note: We do not have to prove each step of the algorithm.

1.6 Why the Analysis of Algorithms?

To go from city *"A"* to city *"B"*, there can be many ways of accomplishing this: by flight, by bus, by train and also by bicycle. Depending on the availability and convenience, we choose the one that suits us. Similarly, in computer science, multiple algorithms are available for solving the same problem (for example, a sorting problem has many algorithms, like insertion sort, selection sort, quick sort and many more). Algorithm analysis helps us to determine which algorithm is most efficient in terms of time and space consumed.

1.7 Goal of the Analysis of Algorithms

The goal of the *analysis of algorithms* is to compare algorithms (or solutions) mainly in terms of running time but also in terms of other factors (e.g., memory, developer effort, etc.)

1.8 What is Running Time Analysis?

It is the process of determining how processing time increases as the size of the problem (input size) increases. Input size is the number of elements in the input, and depending on the problem type, the input may be of different types. The following are the common types of inputs.

- Size of an array
- Polynomial degree
- Number of elements in a matrix
- Number of bits in the binary representation of the input
- Vertices and edges in a graph.

1.9 How to Compare Algorithms

To compare algorithms, let us define a *few objective measures:*

Execution times? *Not a good measure* as execution times are specific to a particular computer.

Number of statements executed? *Not a good measure*, since the number of statements varies with the programming language as well as the style of the individual programmer.

Ideal solution? Let us assume that we express the running time of a given algorithm as a function of the input size n (i.e., $f(n)$) and compare these different functions corresponding to running times. This kind of comparison is independent of machine time, programming style, etc.

1.10 What is Rate of Growth?

The rate at which the running time increases as a function of input is called *rate of growth*. Let us assume that you go to a shop to buy a car and a bicycle. If your friend sees you there and asks what you are buying, then in general you say *buying a car*. This is because the cost of the car is high compared to the cost of the bicycle (approximating the cost of the bicycle to the cost of the car).

$$Total\ Cost = cost\_of\_car + cost\_of\_bicycle$$
$$Total\ Cost \approx cost\_of\_car\ (approximation)$$

For the above-mentioned example, we can represent the cost of the car and the cost of the bicycle in terms of function, and for a given function ignore the low order terms that are relatively insignificant (for large value of input size, n). As an example, in the case below, n^4, $2n^2$, $100n$ and 500 are the individual costs of some function and approximate to n^4 since n^4 is the highest rate of growth.

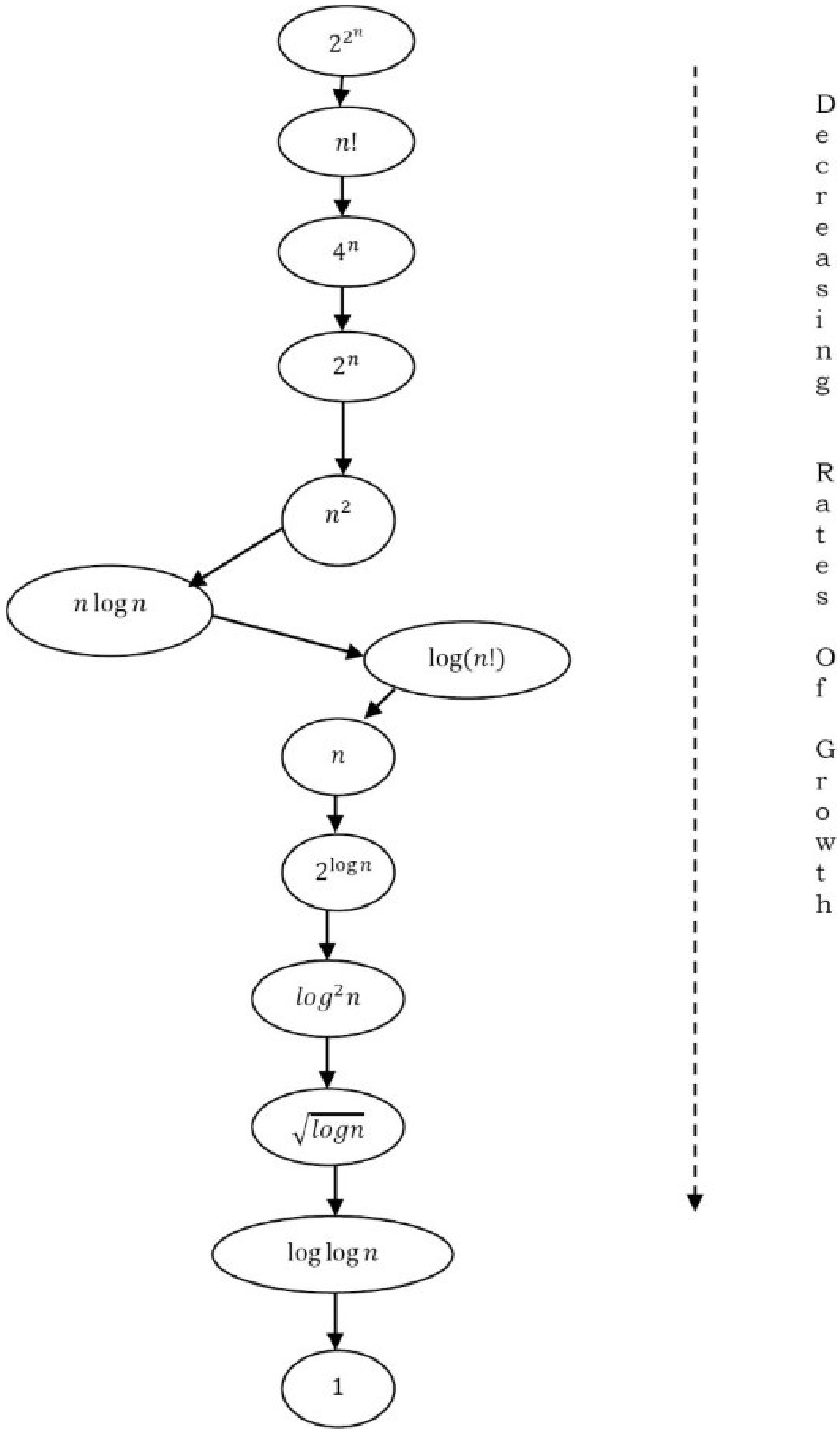
$$n^4 + 2n^2 + 100n + 500 \approx n^4$$

1.11 Commonly used Rates of Growth

Below is the list of growth rates you will come across in the following chapters.

Time Complexity	Name	Example
1	Constant	Adding an element to the front of a linked list
$logn$	Logarithmic	Finding an element in a sorted array
n	Linear	Finding an element in an unsorted array
$nlogn$	Linear Logarithmic	Sorting n items by 'divide-and-conquer'-Mergesort
n^2	Quadratic	Shortest path between two nodes in a graph
n^3	Cubic	Matrix Multiplication
2^n	Exponential	The Towers of Hanoi problem

The diagram below shows the relationship between different rates of growth.



1.12 Types of Analysis

To analyze the given algorithm, we need to know with which inputs the algorithm takes less time (performing well) and with which inputs the algorithm takes a long time. We have already seen that an algorithm can be represented in the form of an expression. That means we represent the algorithm with multiple expressions: one for the case where it takes less time and another for the case where it takes more time.

In general, the first case is called the *best case* and the second case is called the *worst case* for the algorithm. To analyze an algorithm we need some kind of syntax, and that forms the base for asymptotic analysis/notation. There are three types of analysis:

- **Worst case**
  - Defines the input for which the algorithm takes a long time (slowest time to complete).
  - Input is the one for which the algorithm runs the slowest.
- **Best case**
  - Defines the input for which the algorithm takes the least time (fastest time to complete).
  - Input is the one for which the algorithm runs the fastest.
- **Average case**
  - Provides a prediction about the running time of the algorithm.
  - Run the algorithm many times, using many different inputs that come from some distribution that generates these inputs, compute the total running time (by adding the individual times), and divide by the number of trials.
  - Assumes that the input is random.

$$Lower\ Bound <= Average\ Time <= Upper\ Bound$$

For a given algorithm, we can represent the best, worst and average cases in the form of expressions. As an example, let $f(n)$ be the function, which represents the given algorithm.

$$f(n) = n^2 + 500, \text{ for worst case}$$
$$f(n) = n + 100n + 500, \text{ for best case}$$

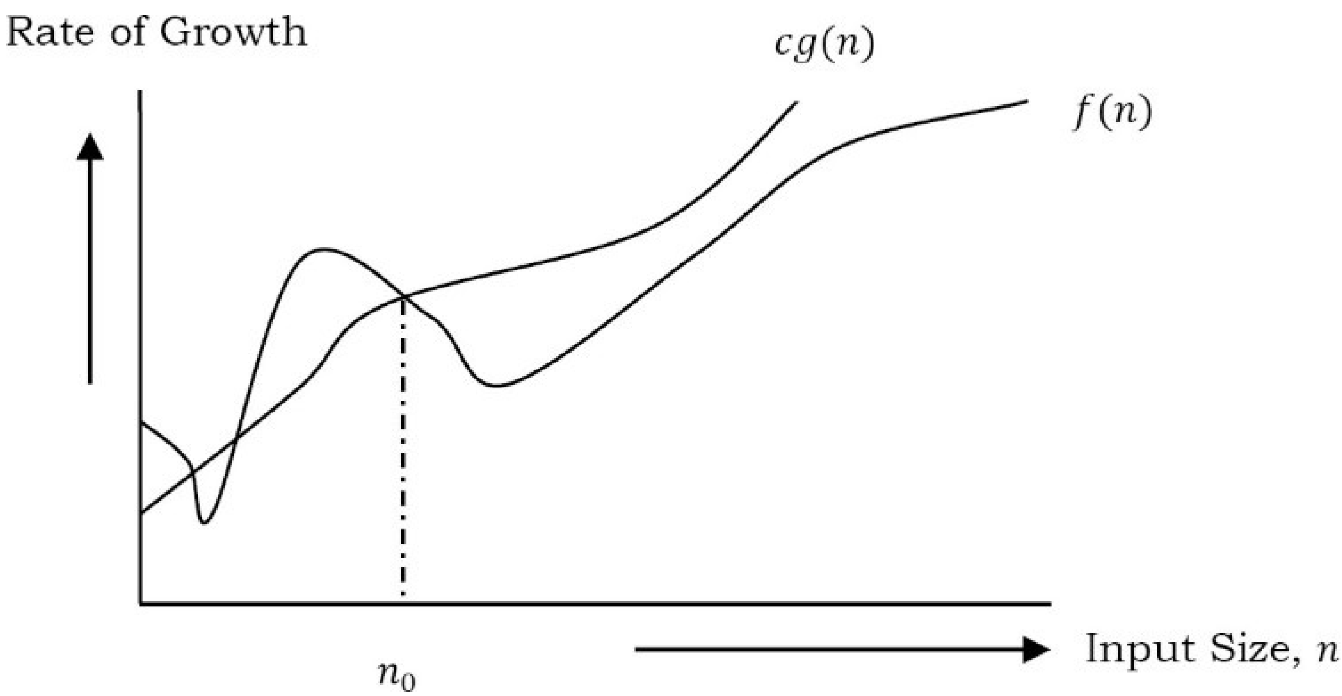
Similarly for the average case. The expression defines the inputs with which the algorithm takes the average running time (or memory).

1.13 Asymptotic Notation

Having the expressions for the best, average and worst cases, for all three cases we need to identify the upper and lower bounds. To represent these upper and lower bounds, we need some kind of syntax, and that is the subject of the following discussion. Let us assume that the given algorithm is represented in the form of function $f(n)$.

1.14 Big-O Notation

This notation gives the *tight* upper bound of the given function. Generally, it is represented *as* $f(n) = O(g(n))$. That means, at larger values of n, the upper bound *of* $f(n)$ is $g(n)$.



For example, if $f(n) = n^4 + 100n^2 + 10n + 50$ is the given algorithm, then n^4 *is* $g(n)$. That means $g(n)$ gives the maximum rate of growth for $f(n)$ at larger values of n.

Let us see the O–notation with a little more detail. O–notation defined as $O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$. $g(n)$ is an asymptotic tight upper bound for $f(n)$. Our objective is to give the smallest rate of growth $g(n)$ which is greater than or equal to the given algorithms' rate of growth $f(n)$.

Generally we discard lower values of n. That means the rate of growth at lower values of n is not important. In the figure, n_0 is the point from which we need to consider the rate of growth for a given algorithm. Below n_0, the rate of growth could be different. n_0 is called threshold for the given function.

Big-O Visualization

$O(g(n))$ is the set of functions with smaller or the same order of growth as $g(n)$. For example; $O(n^2)$ includes $O(1)$, $O(n)$, $O(nlogn)$, etc.

Note: Analyze the algorithms at larger values of n only. What this means is, below n_0 we do not care about the rate of growth.

$O(1)$: $100, 1000, 200, 1, 20, etc.$

$O(n)$: $3n + 100,\ 100n, 2n - 1, 3, etc.$

$O(nlogn)$: $5nlogn, 3n - 100,\ 2n - 1, 100, 100n,\ etc.$

$O(n^2)$: $n^2, 5n - 10, 100,\ n^2 - 2n + 1,\ 5,\ etc.$

Big-O Examples

Example-1 Find upper bound for $f(n) = 3n + 8$

Solution: $3n + 8 \leq 4n$, for all $n \geq 8$

$\therefore 3n + 8 = O(n)$ with c = 4 and $n_0 = 8$

Example-2 Find upper bound for $f(n) = n^2 + 1$

Solution: $n^2 + 1 \leq 2n^2$, for all $n \geq 1$

$\therefore n^2 + 1 = O(n^2)$ with $c = 2$ and $n_0 = 1$

Example-3 Find upper bound for $f(n) = n^4 + 100n^2 + 50$

Solution: $n^4 + 100n^2 + 50 \leq 2n^4$, for all $n \geq 11$

$\therefore n^4 + 100n^2 + 50 = O(n^4)$ with $c = 2$ and $n_0 = 11$

Example-4 Find upper bound for $f(n) = 2n^3 - 2n^2$

Solution: $2n^3 - 2n^2 \leq 2n^3$, for all $n \geq 1$

$\therefore 2n^3 - 2n^2 = O(n^3)$ with $c = 2$ and $n_0 = 1$

Example-5 Find upper bound for $f(n) = n$

Solution: $n \leq n$, for all $n \geq 1$

$\therefore n = O(n)$ with $c = 1$ and $n_0 = 1$

Example-6 Find upper bound for $f(n) = 410$

Solution: $410 \leq 410$, for all $n \geq 1$

$\therefore 410 = O(1)$ with $c = 1$ and $n_0 = 1$

No Uniqueness?

There is no unique set of values for n_0 and c in proving the asymptotic bounds. Let us consider, $100n + 5 = O(n)$. For this function there are multiple n_0 and c values possible.

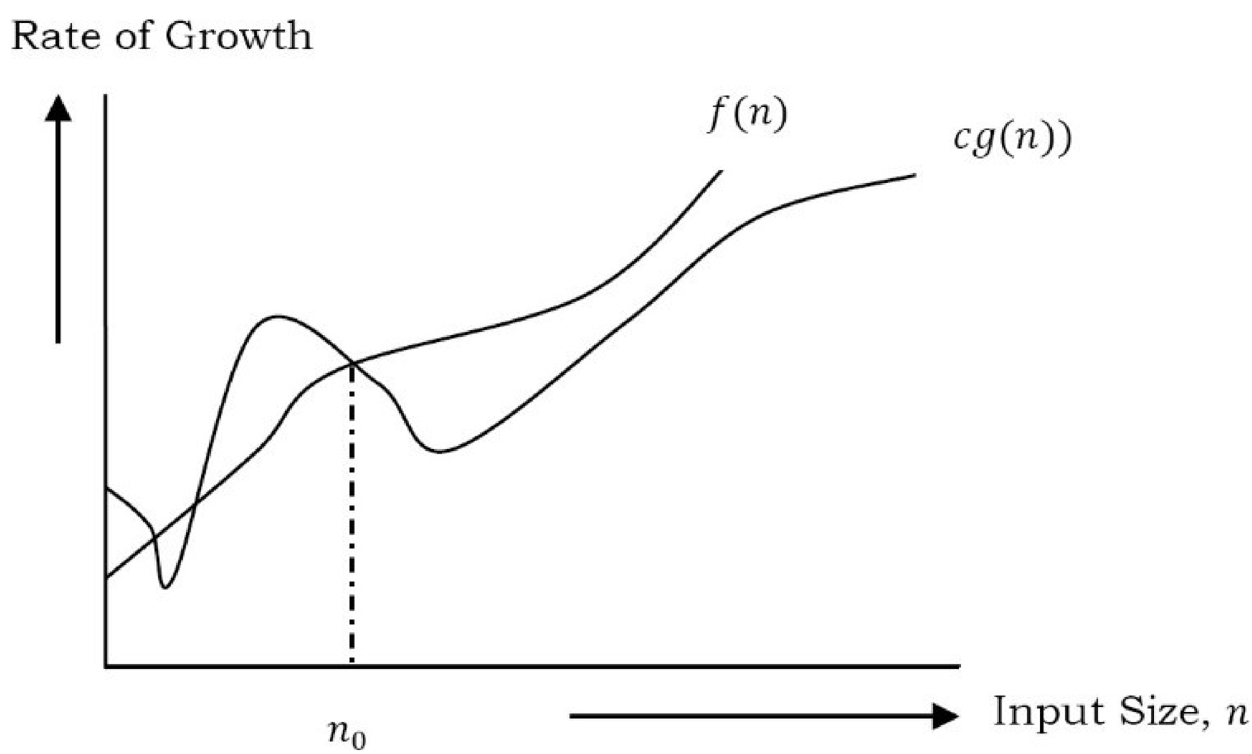
Solution1: $100n + 5 \le 100n + n = 101n \le 101n$, for all $n \ge 5$, $n_0 = 5$ and $c = 101$ is a solution.

Solution2: $100n + 5 \le 100n + 5n = 105n \le 105n$, for all $n \ge 1$, $n_0 = 1$ and $c = 105$ is also a solution.

1.15 Omega-Ω Notation

Similar to the O discussion, this notation gives the tighter lower bound of the given algorithm and we represent it as $f(n) = \Omega(g(n))$. That means, at larger values of n, the tighter lower bound of $f(n)$ is $g(n)$. For example, if $f(n) = 100n^2 + 10n + 50$, $g(n)$ is $\Omega(n^2)$.

The Ω notation can be defined as $\Omega(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$. $g(n)$ is an asymptotic tight lower bound for $f(n)$. Our objective is to give the largest rate of growth $g(n)$ which is less than or equal to the given algorithm's rate of growth $f(n)$.



Ω Examples

Example-1 Find lower bound for $f(n) = 5n^2$.

Solution: $\exists c, n_0$ Such that: $0 \le cn^2 \le 5n^2 \Rightarrow cn^2 \le 5n^2 \Rightarrow c = 5$ and $n_0 = 1$

$\therefore 5n^2 = \Omega(n^2)$ with $c = 5$ and $n_0 = 1$

Example-2 Prove $f(n) = 100n + 5 \neq \Omega(n^2)$.

Solution: $\exists$ c, n_0 Such that: $0 \le cn^2 \le 100n + 5$

$100n + 5 \le 100n + 5n \ (\forall n \ge 1) = 105n$

$cn^2 \le 105n \Rightarrow n(cn - 105) \le 0$

Since n is positive $\Rightarrow cn - 105 \le 0 \Rightarrow n \le 105/c$

$\Rightarrow$ Contradiction: n cannot be smaller than a constant

Example-3 $2n = \Omega(n)$, $n^3 = \Omega(n^3)$, $logn = \Omega(logn)$.

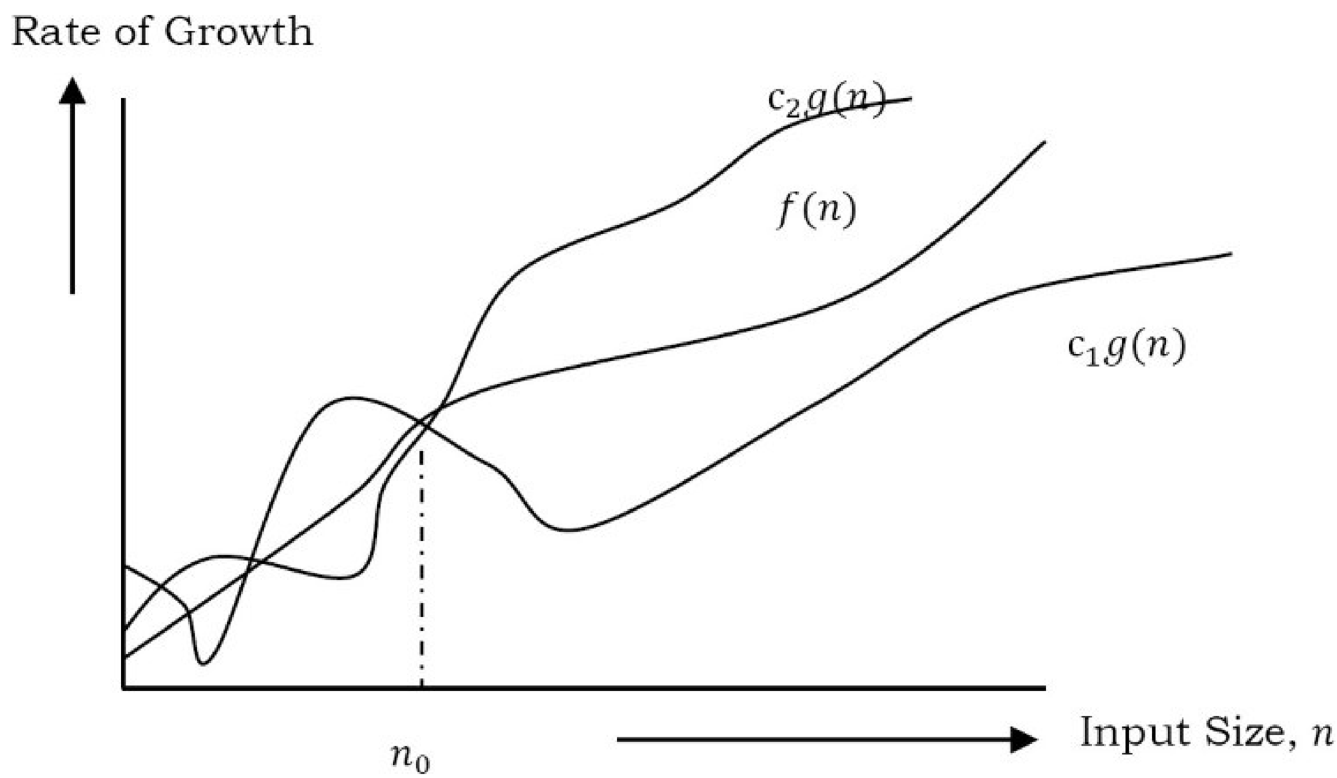
1.16 Theta-Θ Notation

This notation decides whether the upper and lower bounds of a given function (algorithm) are the same. The average running time of an algorithm is always between the lower bound and the upper bound. If the upper bound (O) and lower bound (Ω) give the same result, then the Θ notation will also have the same rate of growth. As an example, let us assume that $f(n) = 10n + n$ is the expression. Then, its tight upper bound $g(n)$ is $O(n)$. The rate of growth in the best case is $g(n) = O(n)$.

In this case, the rates of growth in the best case and worst case are the same. As a result, the average case will also be the same. For a given function (algorithm), if the rates of growth (bounds) for O and Ω are not the same, then the rate of growth for the Θ case may not be the same. In this case, we need to consider all possible time complexities and take the average of those (for example, for a quick sort average case, refer to the *Sorting* chapter).

Now consider the definition *of* Θ notation. *It is defined as* $\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 such that $0 \le c_1g(n) \le f(n) \le c_2g(n)$ for all $n \ge n_0\}$. $g(n)$ is an asymptotic tight bound for $f(n)$. $\Theta(g(n))$ is the set of functions with the same order of growth as $g(n)$.

In this case, the rates of growth in the best case and worst case are the same. As a result, the average case will also be the same. For a given function (algorithm), if the rates of growth (bounds) for O and Ω are not the same, then the rate of growth for the Θ case may not be the same. In this case, we need to consider all possible time complexities and take the average of those (for example, for a quick sort average case, refer to the *Sorting* chapter).



Now consider the definition *of* Θ notation. *It is defined as* $\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0\}$. $g(n)$ is an asymptotic tight bound for $f(n)$. $\Theta(g(n))$ is the set of functions with the same order of growth as $g(n)$.

Θ Examples

Example 1 Find Θ bound for $f(n) = \frac{n^2}{2} - \frac{n}{2}$

Solution: $\frac{n^2}{5} \leq \frac{n^2}{2} - \frac{n}{2} \leq n^2$, for all, $n \geq 2$

$\therefore \frac{n^2}{2} - \frac{n}{2} = \Theta(n^2)$ with $c_1 = 1/5$, $c_2 = 1$ and $n_0 = 2$

Example 2 Prove $n \neq \Theta(n^2)$

Solution: $c_1 n^2 \leq n \leq c_2 n^2 \Rightarrow$ only holds for: $n \leq 1/c_1$

$\therefore n \neq \Theta(n^2)$

Example 3 Prove $6n^3 \neq \Theta(n^2)$

Solution: $c_1 n^2 \leq 6n^3 \leq c_2 n^2 \Rightarrow$ only holds for: $n \leq c_2/6$

$\therefore 6n^3 \neq \Theta(n^2)$

Example 4 Prove $n \neq \Theta(\log n)$

Solution: $c_1 logn \leq n \leq c_2 \, log\, n \Rightarrow c_2 \geq \frac{n}{\log n}, \forall\, n \geq n_0$ – Impossible

1.17 Important Notes

For analysis (best case, worst case and average), we try to give the upper bound (O) and lower bound (Ω) and average running time (Θ). From the above examples, it should also be clear that, for a given function (algorithm), getting the upper bound (O) and lower bound (Ω) and average running time (Θ) may not always be possible. For example, if we are discussing the best case of an algorithm, we try to give the upper bound (O) and lower bound (Ω) and average running time (Θ).

In the remaining chapters, we generally focus on the upper bound (O) because knowing the lower bound (Ω) of an algorithm is of no practical importance, and we use the Θ notation if the upper bound (O) and lower bound (Ω) are the same.

1.18 Why is it called Asymptotic Analysis?

From the discussion above (for all three notations: worst case, best case, and average case), we can easily understand that, in every case for a given function $f(n)$ we are trying to find another function $g(n)$ which approximates $f(n)$ at higher values of n. That means $g(n)$ is also a curve which approximates $f(n)$ at higher values of n.

In mathematics we call such a curve an *asymptotic curve*. In other terms, $g(n)$ is the asymptotic curve for $f(n)$. For this reason, we call algorithm analysis *asymptotic analysis*.

1.19 Guidelines for Asymptotic Analysis

There are some general rules to help us determine the running time of an algorithm.

1) **Loops:** The running time of a loop is, at most, the running time of the statements inside the loop (including tests) multiplied by the number of iterations.

```
// executes n times
for (i=1; i<=n; i++)
    m = m + 2; // constant time, c
```

Total time = a constant $c \times n = c\, n = O(n)$.

2) **Nested loops:** Analyze from the inside out. Total running time is the product of the sizes of

all the loops.

```
//outer loop executed n times
for (i=1; i<=n; i++) {
      // inner loop executed n times
      for (j=1; j<=n; j++)
               k = k+1; //constant time
}
```

Total time $= c \times n \times n = cn^2 = O(n^2)$.

3) Consecutive statements: Add the time complexities of each statement.

```
x = x +1; //constant time
// executed n times
for (i=1; i<=n; i++)
      m = m + 2; //constant time
//outer loop executed n times
for (i=1; i<=n; i++) {
      //inner loop executed n times
      for (j=1; j<=n; j++)
               k = k+1; //constant time
}
```

Total time $= c_0 + c_1n + c_2n^2 = O(n^2)$.

4) If-then-else statements: Worst-case running time: the test, plus *either* the *then* part *or* the *else* part (whichever is the larger).

```
//test: constant
if(length( ) == 0 ) {
      return false; //then part: constant
}
else { // else part: (constant + constant) * n
        for (int n = 0; n < length( ); n++) {
               // another if : constant + constant (no else part)
               if(!list[n].equals(otherList.list[n]))
                        //constant
                        return false;
        }
}
```

Total time = $c_0 + c_1 + (c_2 + c_3) * n = O(n)$.

5) **Logarithmic complexity:** An algorithm is O(*logn*) if it takes a constant time to cut the problem size by a fraction (usually by½). As an example let us consider the following program:

```
for (i=1; i<=n;)
    i = i*2;
```

If we observe carefully, the value of i is doubling every time. Initially $i = 1$, in next step $i = 2$, and in subsequent steps $i = 4$, 8 and so on. Let us assume that the loop is executing some k times. At k^{th} step $2^k = n$, and at $(k + 1)^{th}$ step we come out of the *loop*. Taking logarithm on both sides, gives

$$\begin{aligned} &log(2^k) = logn \\ &klog2 = logn \\ &k = logn \qquad\qquad //\text{if we assume base-2} \end{aligned}$$

Total time = O(*logn*).

Note: Similarly, for the case below, the worst case rate of growth is O(*logn*). The same discussion holds good for the decreasing sequence as well.

```
for (i=n; i>=1;)
    i = i/2;
```

Another example: binary search (finding a word in a dictionary of n pages)

- Look at the center point in the dictionary
- Is the word towards the left or right of center?
- Repeat the process with the left or right part of the dictionary until the word is found.

1.20 Simplyfying properties of asymptotic notations

- Transitivity: $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$. Valid for O and Ω as well.
- Reflexivity: $f(n) = \Theta(f(n))$. Valid for O and Ω.
- Symmetry: $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.
- Transpose symmetry: $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.
- If $f(n)$ is in $O(kg(n))$ for any constant $k > 0$, then $f(n)$ is in $O(g(n))$.
- If $f_1(n)$ is in $O((g_1(n))$ and $f_2(n)$ is in $O(g_2(n))$, then $(f_1 + f_2)(n)$ is in $O(\max(g_1(n), g_1(n)))$.

- If $f_1(n)$ is in $O(g_1(n))$ and $f_2(n)$ is in $O(g_2(n))$ then $f_1(n)\, f_2(n)$ is in $O(g_1(n)\, g_1(n))$.

1.21 Commonly used Logarithms and Summations

Logarithms

$$log\, x^y = y\, log\, x \qquad logn = log_{10}^{n}$$

$$log\, xy = logx + logy \qquad log^k n = (logn)^k$$

$$log\, logn = log(logn) \qquad log\frac{x}{y} = logx - logy$$

$$a^{log_b^x} = x^{log_b^a} \qquad log_b^x = \frac{log_a^x}{log_a^b}$$

Arithmetic series

$$\sum_{K=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometric series

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} (x \neq 1)$$

Harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ \ldots + \ \frac{1}{n} \approx logn$$

Other important formulae

$$\sum_{k=1}^{n} log\, k \approx nlogn$$

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + \cdots + n^p \approx \frac{1}{p+1} n^{p+1}$$

1.22 Master Theorem for Divide and Conquer Recurrences

All divide and conquer algorithms (Also discussed in detail in the *Divide and Conquer* chapter) divide the problem into sub-problems, each of which is part of the original problem, and then perform some additional work to compute the final answer. As an example, a merge sort algorithm [for details, refer to *Sorting* chapter] operates on two sub-problems, each of which is half the size of the original, and then performs O(n) additional work for merging. This gives the running time equation:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

The following theorem can be used to determine the running time of divide and conquer algorithms. For a given program (algorithm), first we try to find the recurrence relation for the problem. If the recurrence is of the below form then we can directly give the answer without fully solving it. If the recurrence is of the form $T(n) = aT(\frac{n}{b}) + \Theta(n^k log^p n)$, where $a \geq 1$, $b > 1$, $k \geq 0$ and p is a real number, then:

1) If $a > b^k$, then $T(n) = \Theta\left(n^{log_b^a}\right)$
2) If $a = b^k$
   a. If $p > -1$, then $T(n) = \Theta\left(n^{log_b^a} log^{p+1} n\right)$
   b. If $p = -1$, then $T(n) = \Theta\left(n^{log_b^a} loglogn\right)$
   c. If $p < -1$, then $T(n) = \Theta\left(n^{log_b^a}\right)$
3) If $a < b^k$
   a. If $p \geq 0$, then $T(n) = \Theta(n^k log^p n)$
   b. If $p < 0$, then $T(n) = O(n^k)$

1.23 Divide and Conquer Master Theorem: Problems & Solutions

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

Problem-1 $T(n) = 3T(n/2) + n^2$

Solution: $T(n) = 3T(n/2) + n^2 \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 3.a)

Problem-2 $T(n) = 4T(n/2) + n^2$

Solution: $T(n) = 4T(n/2) + n^2 \Rightarrow T(n) = \Theta(n^2 logn)$ (Master Theorem Case 2.a)

Problem-3 $T(n) = T(n/2) + n^2$

Solution: $T(n) = T(n/2) + n^2 \Rightarrow \Theta(n^2)$ (Master Theorem Case 3.a)

Problem-4 $T(n) = 2^n T(n/2) + n^n$

Solution: $T(n) = 2^n T(n/2) + n^n \Rightarrow$ Does not apply (a is not constant)

Problem-5 $T(n) = 16T(n/4) + n$

Solution: $T(n) = 16T(n/4) + n \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-6 $T(n) = 2T(n/2) + nlogn$

Solution: $T(n) = 2T(n/2) + nlogn \Rightarrow T(n) = \Theta(nlog^2 n)$ (Master Theorem Case 2.a)

Problem-7 $T(n) = 2T(n/2) + n/logn$

Solution: $T(n) = 2T(n/2) + n/logn \Rightarrow T(n) = \Theta(nloglogn)$ (Master Theorem Case 2. b)

Problem-8 $T(n) = 2T(n/4) + n^{0.51}$

Solution: $T(n) = 2T(n/4) + n^{0.51} \Rightarrow T(n) = \Theta(n^{0.51})$ (Master Theorem Case 3.b)

Problem-9 $T(n) = 0.5T(n/2) + 1/n$

Solution: $T(n) = 0.5T(n/2) + 1/n \Rightarrow$ Does not apply ($a < 1$)

Problem-10 $T(n) = 6T(n/3) + n^2 logn$

Solution: $T(n) = 6T(\mathrm{n}/3) + n^2 logn \Rightarrow T(n) = \Theta(n^2 logn)$ (Master Theorem Case 3.a)

Problem-11 $T(n) = 64T(n/8) - n^2 logn$

Solution: $T(n) = 64T(n/8) - n^2 logn \Rightarrow$ Does not apply (function is not positive)

Problem-12 $T(n) = 7T(n/3) + n^2$

Solution: $T(n) = 7T(n/3) + n^2 \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 3.as)

Problem-13 $T(n) = 4T(n/2) + logn$

Solution: $T(n) = 4T(n/2) + logn \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-14 $T(n) = 16T(n/4) + n!$

Solution: $T(n) = 16T(n/4) + n! \Rightarrow T(n) = \Theta(n!)$ (Master Theorem Case 3.a)

Problem-15 $T(n) = \sqrt{2}T(n/2) + logn$

Solution: $T(n) = \sqrt{2}T(n/2) + logn \Rightarrow T(n) = \Theta(\sqrt{n})$ (Master Theorem Case 1)

Problem-16 $T(n) = 3T(n/2) + n$

Solution: $T(n) = 3T(n/2) + n \Rightarrow T(n) = \Theta(n^{log3})$ (Master Theorem Case 1)

Problem-17 $T(n) = 3T(n/3) + \sqrt{n}$

Solution: $T(n) = 3T(n/3) + \sqrt{n} \Rightarrow T(n) = \Theta(n)$ (Master Theorem Case 1)

Problem-18 $T(n) = 4T(n/2) + cn$

Solution: $T(n) = 4T(n/2) + cn \Rightarrow T(n) = \Theta(n^2)$ (Master Theorem Case 1)

Problem-19 $T(n) = 3T(n/4) + nlogn$

Solution: $T(n) = 3T(n/4) + nlogn \Rightarrow T(n) = \Theta(nlogn)$ (Master Theorem Case 3.a)

Problem-20 $T(n) = 3T(n/3) + n/2$

Solution: $T(n) = 3T(n/3) + n/2 \Rightarrow T(n) = \Theta(nlogn)$ (Master Theorem Case 2.a)

1.24 Master Theorem for Subtract and Conquer Recurrences

Let $T(n)$ be a function defined on positive n, and having the property

$$T(n) = \begin{cases} c, & \text{if } n \leq 1 \\ aT(n-b) + f(n), & \text{if } n > 1 \end{cases}$$

for some constants c, $a > 0$, $b > 0$, $k \geq 0$, and function $f(n)$. If $f(n)$ is in $O(n^k)$, then

$$T(n) = \begin{cases} O(n^k), & \text{if a} < 1 \\ O(n^{k+1}), & \text{if a} = 1 \\ O\left(n^k a^{\frac{n}{b}}\right), & \text{if a} > 1 \end{cases}$$

1.25 Variant of Subtraction and Conquer Master Theorem

The solution to the equation $T(n) = T(\alpha\, n) + T((1 - \alpha)n) + \beta n$, where $0 < \alpha < 1$ and $\beta > 0$ are constants, is $O(nlogn)$.

1.26 Method of Guessing and Confirming

Now, let us discuss a method which can be used to solve any recurrence. The basic idea behind this method is:

guess the answer; and then *prove* it correct by induction.

In other words, it addresses the question: What if the given recurrence doesn’t seem to match with any of these (master theorem) methods? If we guess a solution and then try to verify our guess inductively, usually either the proof will succeed (in which case we are done), or the proof will fail (in which case the failure will help us refine our guess).

As an example, consider the recurrence $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$. This doesn't fit into the form required by the Master Theorems. Carefully observing the recurrence gives us the impression that it is similar to the divide and conquer method (dividing the problem into $\sqrt{n}$ subproblems each with size $\sqrt{n}$). As we can see, the size of the subproblems at the first level of recursion is n. So, let us guess that $T(n) = O(nlogn)$, and then try to prove that our guess is correct.

Let's start by trying to prove an *upper* bound $T(n) \leq cnlogn$:

$$\begin{aligned} T(n) &= \sqrt{n}\, T(\sqrt{n}) + n \\ &\leq \sqrt{n}.\, c\sqrt{n}\, log\sqrt{n} + n \\ &= n.\, c\, log\sqrt{n} + n \\ &= n.c.\tfrac{1}{2}.\, logn + n \\ &\leq cnlogn \end{aligned}$$

The last inequality assumes only that $1 \leq c.\frac{1}{2}.\, logn$. This is correct if n is sufficiently large and for any constant c, no matter how small. From the above proof, we can see that our guess is correct for the upper bound. Now, let us prove the *lower* bound for this recurrence.

$$\begin{aligned} T(n) &= \sqrt{n}\, T(\sqrt{n}) + n \\ &\geq \sqrt{n}.\, k\, \sqrt{n}\, log\sqrt{n} + n \\ &= n.\, k\, log\sqrt{n} + n \\ &= n.k.\tfrac{1}{2}.\, logn + n \\ &\geq knlogn \end{aligned}$$

The last inequality assumes only that $1 \geq k.\frac{1}{2}.\, logn$. This is incorrect if n is sufficiently large and for any constant k. From the above proof, we can see that our guess is incorrect for the lower bound.

From the above discussion, we understood that $\Theta(nlogn)$ is too big. How about $\Theta(n)$? The lower bound is easy to prove directly:

$$T(n) = \sqrt{n}\, T(\sqrt{n}) + n \quad \geq n$$

Now, let us prove the upper bound for this $\Theta(n)$.

$$
\begin{aligned}
T(n) &= \sqrt{n}\,T(\sqrt{n}) + n \\
&\leq \sqrt{n}.c.\sqrt{n} + n \\
&= n.\ c + n \\
&= n\ (c + 1) \\
&\nleq cn
\end{aligned}
$$

From the above induction, we understood that $\Theta(n)$ is too small and $\Theta(nlogn)$ is too big. So, we need something bigger than *n* and smaller than *nlogn*. How about $n\sqrt{logn}$?

Proving the upper bound for $n\sqrt{logn}$:

$$
\begin{aligned}
T(n) &= \sqrt{n}\,T(\sqrt{n}) + n \\
&\leq \sqrt{n}.c.\sqrt{n}\sqrt{log\sqrt{n}} + n \\
&= n.\ c.\frac{1}{\sqrt{2}}\ log\sqrt{n} + n \\
&\leq cnlog\sqrt{n}
\end{aligned}
$$

Proving the lower bound for $n\sqrt{logn}$:

$$
\begin{aligned}
T(n) &= \sqrt{n}\,T(\sqrt{n}) + n \\
&\geq \sqrt{n}.k.\sqrt{n}\sqrt{log\sqrt{n}} + n \\
&= n.\ k.\frac{1}{\sqrt{2}}\ log\sqrt{n} + n \\
&\ngeq knlog\sqrt{n}
\end{aligned}
$$

The last step doesn't work. So, $\Theta(n\sqrt{logn})$ doesn't work. What else is between *n* and *nlogn?* How about *nloglogn?* Proving upper bound for *nloglogn:*

$$
\begin{aligned}
T(n) &= \sqrt{n}\,T(\sqrt{n}) + n \\
&\leq \sqrt{n}.c.\sqrt{n}loglog\sqrt{n} + n \\
&= n.\ c.loglogn\text{-}c.n + n \\
&\leq cnloglogn,\ \text{if } c \geq 1
\end{aligned}
$$

Proving lower bound for *nloglogn*:

$$
\begin{aligned}
T(n) &= \sqrt{n}\,T(\sqrt{n}) + n \\
&\geq \sqrt{n}.k.\sqrt{n}loglog\sqrt{n} + n \\
&= n.\ k.loglogn\text{-}k.n + n \\
&\geq knloglogn,\ \text{if } k \leq 1
\end{aligned}
$$

From the above proofs, we can see that T(n) ≤ $cnloglogn$, if $c \geq 1$ and T(n) ≥ $knloglogn$, if $k \leq 1$. Technically, we're still missing the base cases in both proofs, but we can be fairly confident at this point that T(n) = Θ($nloglogn$).

1.27 Amortized Analysis

Amortized analysis refers to determining the time-averaged running time for a sequence of operations. It is different from average case analysis, because amortized analysis does not make any assumption about the distribution of the data values, whereas average case analysis assumes the data are not "bad" (e.g., some sorting algorithms do well *on average* over all input orderings but very badly on certain input orderings). That is, amortized analysis is a worst-case analysis, but for a sequence of operations rather than for individual operations.

The motivation for amortized analysis is to better understand the running time of certain techniques, where standard worst case analysis provides an overly pessimistic bound. Amortized analysis generally applies to a method that consists of a sequence of operations, where the vast majority of the operations are cheap, but some of the operations are expensive. If we can show that the expensive operations are particularly rare we can *change them* to the cheap operations, and only bound the cheap operations.

The general approach is to assign an artificial cost to each operation in the sequence, such that the total of the artificial costs for the sequence of operations bounds the total of the real costs for the sequence. This artificial cost is called the amortized cost of an operation. To analyze the running time, the amortized cost thus is a correct way of understanding the overall running time – but note that particular operations can still take longer so it is not a way of bounding the running time of any individual operation in the sequence.

When one event in a sequence affects the cost of later events:

- One particular task may be expensive.
- But it may leave data structure in a state that the next few operations becomes easier.

Example: Let us consider an array of elements from which we want to find the k^{th} smallest element. We can solve this problem using sorting. After sorting the given array, we just need to return the k^{th} element from it. The cost of performing the sort (assuming comparison based sorting algorithm) is O($nlogn$). If we perform n such selections then the average cost of each selection is O($nlogn/n$) = O($logn$). This clearly indicates that sorting once is reducing the complexity of subsequent operations.

1.28 Algorithms Analysis: Problems & Solutions

Note: From the following problems, try to understand the cases which have different complexities (O(n),O($logn$),O($loglogn$) etc.).

Problem-21 Find the complexity of the below recurrence:

$$T(n) = \begin{cases} 3T(n-1), if\ n > 0, \\ 1, \qquad otherwise \end{cases}$$

Solution: Let us try solving this function with substitution.

$$T(n) = 3T(n-1)$$

$$T(n) = 3\big(3T(n-2)\big) = 3^2T(n-2)$$

$$T(n) = 3^2(3T(n-3))$$

.

.

$$T(n) = 3^nT(n-n) = 3^nT(0) = 3^n$$

This clearly shows that the complexity of this function is O(3^n).

Note: We can use the *Subtraction and Conquer* master theorem for this problem.

Problem-22 Find the complexity of the below recurrence:

$$T(n) = \begin{cases} 2T(n-1) - 1, if\ n > 0, \\ 1, \qquad otherwise \end{cases}$$

Solution: Let us try solving this function with substitution.

$$T(n) = 2T(n-1) - 1$$

$$T(n) = 2(2T(n-2) - 1) - 1 = 2^2T(n-2) - 2 - 1$$

$$T(n) = 2^2(2T(n-3) - 2 - 1) - 1 = 2^3T(n-4) - 2^2 - 2^1 - 2^0$$

$$T(n) = 2^nT(n-n) - 2^{n-1} - 2^{n-2} - 2^{n-3} \ldots . 2^2 - 2^1 - 2^0$$

$$T(n) = 2^n - 2^{n-1} - 2^{n-2} - 2^{n-3} \ldots . 2^2 - 2^1 - 2^0$$

$$T(n) = 2^n - (2^n - 1)\ [note: 2^{n-1} + 2^{n-2} + \cdots + 2^0 = 2^n]$$

$$T(n) = 1$$

∴ Complexity is O(1). Note that while the recurrence relation looks exponential, the solution to the recurrence relation here gives a different result.

Problem-23 What is the running time of the following function?

```
public void Function(int n) {
        int i=1, s=1;
        while( s <= n) {
                i++;
                s= s+i;
                System.out.println("*");
        }
}
```

Solution: Consider the comments in the below function:

```
public void Function (int n) {
    int i=1, s=1;
    // s is increasing not at rate 1 but i
    while( s <= n) {
        i++;
        s= s+i;
        System.out.println("*");
    }
}
```

We can define the 's' terms according to the relation $s_i = s_{i-1} + i$. The value of 'i' increases by 1 for each iteration. The value contained in 's' at the i^{th} iteration is the sum of the first 'i' positive integers. If k is the total number of iterations taken by the program, then the *while* loop terminates if:

$$1 + 2 + \ldots + k = \frac{k(k+1)}{2} > n \Rightarrow k = O(\sqrt{n}).$$

Problem-24 Find the complexity of the function given below.

```
public void Function(int n) {
        int i, count =0;
        for(i=1; i*i<=n; i++)
                count++;
}
```

Solution:

```
void Function(int n) {
    int i, count =0;
    for(i=1; i*i<=n; i++)
        count++;
}
```

In the above-mentioned function the loop will end, if $i^2 > n \Rightarrow T(n) = O(\sqrt{n})$. The reasoning is smae as that of Problem-23.

Problem-25 What is the complexity of the program given below?

```
public void function(int n) {
        int i, j, k , count =0;
        for(i=n/2; i<=n; i++)
            for(j=1; j + n/2<=n; j++)
                for(k=1; k<=n; k= k * 2)
                    count++;
}
```

Solution: Consider the comments in the following function.

```
public void function(int n) {
    int i, j, k , count =0;
    //Outer loop execute n/2 times
    for(i=n/2; i<=n; i++)
        //Middle loop executes n/2 times
        for(j=1; j + n/2<=n; j++)
            //Inner loop execute logn times
            for(k=1; k<=n; k= k * 2)
                count++;
}
```

The complexity of the above function is $O(n^2 logn)$.

Problem-26 What is the complexity of the program given below?

```
public void function(int n) {
        int i, j, k , count =0;
        for(i=n/2; i<=n; i++)
                for(j=1; j<=n; j= 2 * j)
                        for(k=1; k<=n; k= k * 2)
                                count++;
}
```

Solution: Consider the comments in the following function.

```
public void function(int n) {
    int i, j, k , count =0;
    //Outer loop execute n/2 times
    for(i=n/2; i<=n; i++)
        //Middle loop executes logn times
        for(j=1; j<=n; j= 2 * j)
            //Inner loop execute logn times
            for(k=1; k<=n; k= k*2)
                count++;
}
```

The complexity of the above function is O($nlog^2n$).

Problem-27 Find the complexity of the program given below.

```
public  void function( int n ) {
        if(n == 1) return;
        for(int i = 1 ; i <= n ; i + + ) {
                for(int j= 1 ; j <= n ; j + + ) {
                        System.out.println("*" );
                        break;
                }
        }
}
```

Solution: Consider the comments in the following function.

```
public  void function( int n ) {
      //constant time
      if( n == 1 ) return;
      //Outer loop execute n times
      for(int i = 1 ; i <= n ; i + + ) {
            // Inner loop executes only time due to break statement.
            for(int j= 1 ; j <= n ; j + + )         {
                  System.out.println("*" );
                  break;
            }
      }
}
```

The complexity of the above function is O(n). Even though the inner loop is bounded by n, but due to the break statement it is executing only once.

Problem-28 Write a recursive function for the running time $T(n)$ of the function given below. Prove using the iterative method that $T(n) = \Theta(n^3)$.

```
public  void function( int n ) {
        if( n == 1 ) return;
        for(int i = 1 ; i <= n ; i + + )
                for(int j = 1 ; j <= n ; j + + )
                        System.out.println("*" ) ;
        function( n-3 );
}
```

Solution: Consider the comments in the function below:

```
public void function (int n) {
    //constant time
    if( n == 1 ) return;
    //Outer loop execute n times
    for(int i = 1 ; i <= n ; i + + )
       //Inner loop executes n times
       for(int j = 1 ; j <= n ; j + + )
         //constant time
         System.out.println("*" ) ;
    function( n-3 );
}
```

The recurrence for this code is clearly T(n) = $T(n-3) + cn^2$ for some constant $c > 0$ since each call prints out n^2 asterisks and calls itself recursively on $n - 3$. Using the iterative method we get: $T(n) = T(n-3) + cn^2$. Using the *Subtraction and Conquer* master theorem, we get $T(n) = \Theta(n^3)$.

Problem-29 Determine Θ bounds for the recurrence relation: $T(n) = 2T\left(\frac{n}{2}\right) + nlogn.$

Solution: Using Divide and Conquer master theorem, we get: $O(nlog^2n)$.

Problem-30 Determine Θ bounds for the recurrence: $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n.$

Solution: Substituting in the recurrence equation, we get:

$$T(n) \leq c1 * \frac{n}{2} + c2 * \frac{n}{4} + c3 * \frac{n}{8} + cn \leq k * n \text{ , where } k \text{ is a constant.}$$

Problem-31 Determine Θ bounds for the recurrence relation: $T(n) = T(\lceil n/2 \rceil) + 7$.

Solution: Using Master Theorem we get: $\Theta(logn)$.

Problem-32 Prove that the running time of the code below is $\Omega(logn)$.

```
public void Read(int n) {
        int k = 1;
        while( k < n )
                k = 3k;
}
```

Solution: The *while* loop will terminate once the value of 'k' is greater than or equal to the value of 'n'. In each iteration the value of 'k' is multiplied by 3. If i is the number of iterations, then 'k' has the value of 3^i after i iterations. The loop is terminated upon reaching i iterations when $3^i \geq n \leftrightarrow i \geq \log_3 n$, which shows that $i = \Omega(logn)$.

Problem-33 Solve the following recurrence.

$$T(n) = \begin{cases} 1, & if\ n = 1 \\ T(n-1) + n(n-1), & if\ n \geq 2 \end{cases}$$

Solution: By iteration:

$$T(n) = T(n-2) + (n-1)(n-2) + n(n-1)$$

...

$$T(n) = T(1) + \sum_{i=1}^{n} i(i-1)$$

$$T(n) = T(1) + \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i$$

$$T(n) = 1 + \frac{n((n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$T(n) = \Theta(n^3)$$

Note: We can use the *Subtraction and Conquer* master theorem for this problem.

Problem-34 Consider the following program:

```
Fib[n]
if(n==0) then return 0
else if(n==1) then return 1
else return Fib[n-1]+Fib[n-2]
```

Solution: The recurrence relation for the running time of this program is:

$$T(n) = T(n-1) + T(n-2) + c.$$

Note T(n) has two recurrence calls indicating a binary tree. Each step recursively calls the program for n reduced by 1 and 2, so the depth of the recurrence tree is O(n). The number of leaves at depth n is 2^n since this is a full binary tree, and each leaf takes at least O(1) computations for the constant factor. Running time is clearly exponential in n and it is O(2^n).

Problem-35 Running time of following program?

```
public  void function(n) {
        for(int i = 1 ; i <= n ; i + + )
                for(int j = 1 ; j <= n ; j+ = i )
                        System.out.println("*") ;
}
```

Solution: Consider the comments in the function below:

```
public void function (n) {
    //this loop executes n times
    for(int i = 1 ; i <= n ; i + + )
        //this loop executes j times with j increase by the rate of i
        for(int j = 1 ; j <= n ; j+ = i )
            System.out.println("*") ;
}
```

In the above program, the inner loop executes n/i times for each value of *i*. Its running time is $n \times (\sum_{i=1}^{n} n/i) = O(nlogn)$.

Problem-36 What is the complexity of $\sum_{i=1}^{n} log\ i$?

Solution: Using the logarithmic property, *logxy* = *logx* + *logy*, we can see that this problem is equivalent to

$$\sum_{i=1}^{n} logi = log\,1 + log\,2 + \cdots + log\,n = log(1 \times 2 \times \ldots \times n) = log(n!) \le log(n^n) \le nlogn$$

This shows that the time complexity = O(*nlogn*).

Problem-37 What is the running time of the following recursive function (specified as a function of the input value *n*)? First write the recurrence formula and then find its complexity.

```
public void function(int n) {
        if(n <= 1) return ;
        for (int i=1 ; i <= 3; i++ )
            f(⌈n/3⌉);
}
```

Solution: Consider the comments in the function below:

```
public void function (int n) {
    //constant time
    if(n <= 1) return;
    //this loop executes with recursive loop of n/3 value
    for (int i=1 ; i <= 3; i++ )
        f(⌈n/3⌉);
}
```

We can assume that for asymptotical analysis $k = \lceil k \rceil$ for every integer $k \geq 1$. The recurrence for this code is $T(n) = 3T(\frac{n}{3}) + \Theta(1)$. Using master theorem, we get $T(n) = \Theta(n)$.

Problem-38 What is the running time of the following recursive function (specified as a function of the input value n)? First write a recurrence formula, and show its solution using induction.

```
public void function(int n) {
        if(n <= 1) return;
        for (int i=1 ; i <= 3 ; i++ )
                function (n − 1).
}
```

Solution: Consider the comments in the below function:

```
public void function (int n) {
    //constant time
    if(n <= 1) return;
    //this loop executes 3 times with recursive call of n-1 value
    for (int i=1 ; i <= 3 ; i++ )
        function (n − 1).
}
```

The *if* statement requires constant time [O(1)]. With the *for* loop, we neglect the loop overhead and only count three times that the function is called recursively. This implies a time complexity recurrence:

$$
\begin{aligned}
T(n) &= c, if\ n \leq 1; \\
&= c + 3T(n - 1), if\ n > 1.
\end{aligned}
$$

Using the *Subtraction and Conquer* master theorem, we get $T(n) = \Theta(3^n)$.

Problem-39 Write a recursion formula for the running time $T(n)$ of the function f whose code is given below. What is the running time of *function*, as a function of n?

```
public void function (int n) {
        if(n <= 1)  return;
        for(int i = 1; i < n; i + +)
                System.out.println("*");
        function ( 0.8n ) ;
}
```

Solution: Consider the comments in the below function:

```
public void function (int n) {
    //constant time
    if(n <= 1) return;
    // this loop executes n times with constant time loop
    for(int i = 1; i < n; i + +)
        System.out.println("*");
    //recursive call with 0.8n
    function ( 0.8n ) ;
}
```

The recurrence for this piece of code is $T(n) = T(.8n) + O(n) = \mathrm{T}\left(\frac{4}{5n}\right) + O(n) = \frac{4}{5}T(n) + O(n)$. Applying master theorem, we get T(n) = O(n).

Problem-40 Find the complexity of the recurrence: $T(n) = 2T(\sqrt{n}) + logn$

Solution: The given recurrence is not in the master theorem format. Let us try to convert this to the master theorem format by assuming $n = 2^m$. Applying the logarithm on both sides gives, $logn = mlog2 \Rightarrow m = logn$. Now, the given function becomes:

$$T(n) = T(2^m) = 2T\left(\sqrt{2^m}\right) + m = 2T\left(2^{\frac{m}{2}}\right) + m.$$

To make it simple we assume $S(m) = T(2^m) \Rightarrow S(\frac{m}{2}) = T(2^{\frac{m}{2}}) \Rightarrow S(m) = 2S\left(\frac{m}{2}\right) + m$. Applying the master theorem format would result in $S(m) = O(mlogrn)$. If we substitute $m = logn$ back, $T(n) = S(logn) = O((logn)\ loglogn)$.

Problem-41 Find the complexity of the recurrence: $T(n) = T(\sqrt{n}) + 1$

Solution: Applying the logic of Problem-40 gives $S(m) = S\left(\frac{m}{2}\right) + 1$. Applying the master theorem would result in $S(m) = O(logm)$. Substituting $m = logn$, gives $T(n) = S(logn) = O(loglogn)$.

Problem-42 Find the complexity of the recurrence: $T(n) = 2T(\sqrt{n}) + 1$

Solution: Applying the logic of Problem-40 gives: $S(m) = 2S\left(\frac{m}{2}\right) + 1$. Using the master theorem results $S(m) = O\left(m^{log_2^2}\right) = O(m)$. Substituting $m = logn$ gives $T(n) = O(logn)$.

Problem-43 Find the complexity of the function given below.

```
public int function (int n) {
        if(n <= 2) return 1;
        else
                return (Function (floor(sqrt(n))) + 1);
}
```

Solution: Consider the comments in the below function:

```
public int function (int n) {
    if(n <= 2) return 1;        //constant time
    else
        // executes √n + 1 times
        return (Function (floor(sqrt(n))) + 1);
}
```

For the above function, recurrence function can be given as: $T(n) = T(\sqrt{n}) + 1$. This is same as that of Problem-41.

Problem-44 Analyze the running time of the following recursive pseuedocode as a function of n.

```
public void function(int n) {
        if( n < 2 ) return;
        else    counter = 0;
        for i = 1 to 8 do
                function (n/2);
        for i =1 to n^3 do
                counter = counter + 1;
}
```

Solution: Consider the comments in below pseuedocode and call running time of function(n) as $T(n)$.

```
public void function(int n) {
    if( n < 2 ) return;          //constant time
    else    counter = 0;
    // this loop executes 8 times with n value half in every call
    for i = 1 to 8 do
        function (n/2);
        // this loop executes n^3 times with constant time loop
    for i =1 to n^3 do
        counter = counter + 1;
}
```

$T(n)$ can be defined as follows:

$$T(n) = 1 \; if \; n < 2, \\ = 8T(\frac{n}{2}) + n^3 + 1 \; otherwise.$$

Using the master theorem gives: $T(n) = \Theta(n^{log_2^8} logn) = \Theta(n^3 logn)$.

Problem-45 Find the complexity of the pseudocode given below:

```
temp = 1
repeat
        for i = 1 to n
                temp = temp + 1;
        n = n/2;
until n <= 1
```

Solution: Consider the comments in the pseudocode given below:

```
temp = 1        // constant time
repeat
   // this loops executes n times
   for i = 1 to n
       temp = temp + 1;
   //recursive call with n/2 value
   n = n/2;
until n <= 1
```

The recurrence for this function is $T(n) = T(n/2) + n$. Using master theorem we get: $T(n) = O(n)$.

Problem-46 Running time of the following program?

```
publicvoid function(int n) {
        for(int i = 1 ; i <= n ; i + + )
                for(int j = 1 ; j <= n ; j * = 2 )
                        System.out.println("*");
}
```

Solution: Consider the comments in the function given below:

```
public void function(int n) {
      // this loops executes n times
      for(int i = 1 ; i <= n ; i + + )
            // this loops executes logn times from our logarithms
            //guideline
            for(int j = 1 ; j <= n ; j * = 2 )
                  System.out.println("*");
}
```

Complexity of above program is $O(nlogn)$.

Problem-47 Running time of the following program?

```
public void function(int n) {
        for(int i = 1 ; i <= n/3 ; i + + )
                for(int j = 1 ; j <= n ; j += 4 )
                        System.out.println(" * ");
}
```

Solution: Consider the comments in the function given below:

```
public void function(int n) {
      // this loops executes n/3 times
      for(int i = 1 ; i <= n/3 ; i + + )
            // this loops executes n/4 times
            for(int j = 1 ; j <= n ; j += 4)
                  System.out.println(" * ");
}
```

The time complexity of this program is: $O(n^2)$.

Problem-48 Find the complexity of the below function:

```
public void function(int n) {
    if(n <=  1) return;
    if(n > 1) {
            System.out.println(" * ");
            function( n/2 );
            function( n/2 );
    }
}
```

Solution: Consider the comments in the function given below:

```
void function(int n) {
    if(n <= 1) return;                    //constant time
    if(n > 1) {
        System.out.println(" * ");        //constant time
        //recursion with n/2 value
        function( n/2 );
        //recursion with n/2 value
        function( n/2 );
    }
}
```

The recurrence for this function is: $T(n) = 2T\left(\frac{n}{2}\right) + 1$. Using master theorem, we get $T(n)$ = O(n).

Problem-49 Find the complexity of the below function:

```
public void function(int n) {
        int i=1;
        while (i < n) {
                int j=n;
                while(j > 0)
                        j = j/2;
                i=2*i;
        } // i
}
```

Solution:

```
public void function(int n) {
    int i=1;
    while (i < n) {
        int j=n;
        while(j > 0)
            j = j/2;   //logn code
        i=2*i; //logn times
    } // i
}
```

Time Complexity: $O(logn * logn) = O(log^2n)$.

Problem-50 $\Sigma_{1 \le k \le n} O(n)$, where $O(n)$ stands for order n is:

(a) $O(n)$
(b) $O(n^2)$
(c) $O(n^3)$
(d) $O(3n^2)$
(e) $O(1.5n^2)$

Solution: (b). $\Sigma_{1 \le k \le n} O(n) = O(n) \Sigma_{1 \le k \le n} 1 = O(n^2)$.

Problem-51 Which of the following three claims are correct

I $(n + k)^m = \Theta(n^m)$, where k and m are constants II $2^{n+1} = O(2^n)$ III $2^{2n+1} = O(2^n)$

(a) I and II
(b) I and III
(c) II and III
(d) I, II and III

Solution: (a). (I) $(n + k)^m = n^k + c1*n^{k-1} + ... k^m = \Theta(n^k)$ and (II) $2^{n+1} = 2*2^n = O(2^n)$

Problem-52 Consider the following functions: $f(n) = 2^n$ $g(n) = n!$ $h(n) = n^{logn}$

Which of the following statements about the asymptotic behavior of $f(n)$, $g(n)$, and $h(n)$ is true?

(A) $f(n) = O(g(n))$; $g(n) = O(h(n))$
(B) $f(n) = \Omega(g(n))$; $g(n) = O(h(n))$
(C) $g(n) = O(f(n))$; $h(n) = O(f(n))$
(D) $h(n) = O(f(n))$; $g(n) = \Omega(f(n))$

Solution: (D). According to the rate of growth: $h(n) < f(n) < g(n)$ ($g(n)$ is asymptotically greater than $f(n)$, and $f(n)$ is asymptotically greater than $h(n)$). We can easily see the above order by taking logarithms of the given 3 functions: $lognlogn < n < log(n!)$. Note that, $log(n!) = O(nlogn)$.

Problem-53 Consider the following segment of C-code:

```
int j=1, n;
while (j <=n)
        j = j*2;
```

The number of comparisons made in the execution of the loop for any $n > 0$ is:
(A) ceil(log_2^n)+ 1
(B) n
(C) ceil(log_2^n)
(D) floor(log_2^n) + 1

Solution: (a). Let us assume that the loop executes k times. After k^{th} step the value *of j* is 2^k. Taking logarithms on both sides gives $k = log_2^n$. Since we are doing one more comparison for exiting from the loop, the answer is ceil(log_2^n)+ 1.

Problem-54 Consider the following C code segment. Let T(n) denote the number of times the for loop is executed by the program on input n. Which of the following is true?

```
int IsPrime(int n){
  for(int i=2;i<=sqrt(n);i++)
 if(n%i == 0)
  {printf("Not Prime\n"); return 0;}
 return 1;
}
```

(A) $T(n) = O(\sqrt{n})$ and $T(n) = \Omega(\sqrt{n})$
(B) $T(n) = O(\sqrt{n})$ and $T(n) = \Omega(1)$
(C) $T(n) = O(n)$ and $T(n) = \Omega(\sqrt{n})$
(D) None of the above

Solution: (B). Big O notation describes the tight upper bound and Big Omega notation describes the tight lower bound for an algorithm. The *for* loop in the question is run maximum $\sqrt{n}$ times and minimum 1 time. Therefore, $T(n) = O(\sqrt{n})$ and $T(n) = \Omega(1)$.

Problem-55 In the following C function, let $n \geq m$. How many recursive calls are made by this function?

```
public int gcd(n,m){
        if (n%m ==0) return m;
        n = n%m;
        return gcd(m,n);
}
```

(A) $\Theta(log_2^n)$
(B) $\Omega(n)$

(C) $\Theta(log_2 log_2^n)$
(D) $\Theta(n)$

Solution: No option is correct. Big O notation describes the tight upper bound and Big Omega notation describes the tight lower bound for an algorithm. For m = 2 and for all $n = 2^i$, the running time is O(1) which contradicts every option.

Problem-56 Suppose $T(n) = 2T(n/2) + n$, T(0)=T(1)=1. Which one of the following is FALSE?

(A) $T(n) = O(n^2)$
(B) $T(n) = \Theta(nlogn)$
(C) $T(n) = \Omega(n^2)$
(D) $T(n) = O(nlogn)$

Solution: (C). Big O notation describes the tight upper bound and Big Omega notation describes the tight lower bound for an algorithm. Based on master theorem, we get $T(n) = \Theta(nlogn)$. This indicates that tight lower bound and tight upper bound are the same. That means, $O(nlogn)$ and $\Omega(nlogn)$ are correct for given recurrence. So option (C) is wrong.

Problem-57 Find the complexity of the below function:

```
public void function(int n) {
    for (int i = 0; i<n; i++)
        for(int j=i; j<i*i; j++)
            if (j %i == 0){
                for (int k = 0; k < j; k++)
                    printf(" * ");
            }
}
```

Solution:

```
public void function(int n) {
    for (int i = 0; i<n; i++)                // Executes n times
        for(int j=i; j<i*i; j++)             // Executes n*n times
            if (j %i == 0){
                for (int k = 0; k < j; k++)  // Executes j times = (n*n) times
                    printf(" * ");
            }
}
```

Time Complexity: $O(n^5)$.

Problem-58 To calculate 9^n, give an algorithm and discuss its complexity.

Solution: Start with 1 and multiply by 9 until reaching 9^n.

Time Complexity: There are $n - 1$ multiplications and each takes constant time giving a $\Theta(n)$ algorithm.

Problem-59 For Problem-58, can we improve the time complexity?

Solution: Refer to the *Divide and Conquer* chapter.

Problem-60 Find the complexity of the below function:

```
public void function(int n) {
    int sum = 0;
    for (int i = 0; i<n; i++)
        if (i>j)
    sum = sum +1;
else {
    for (int k = 0; k < n; k++)
        sum = sum -1;
}
}
}
```

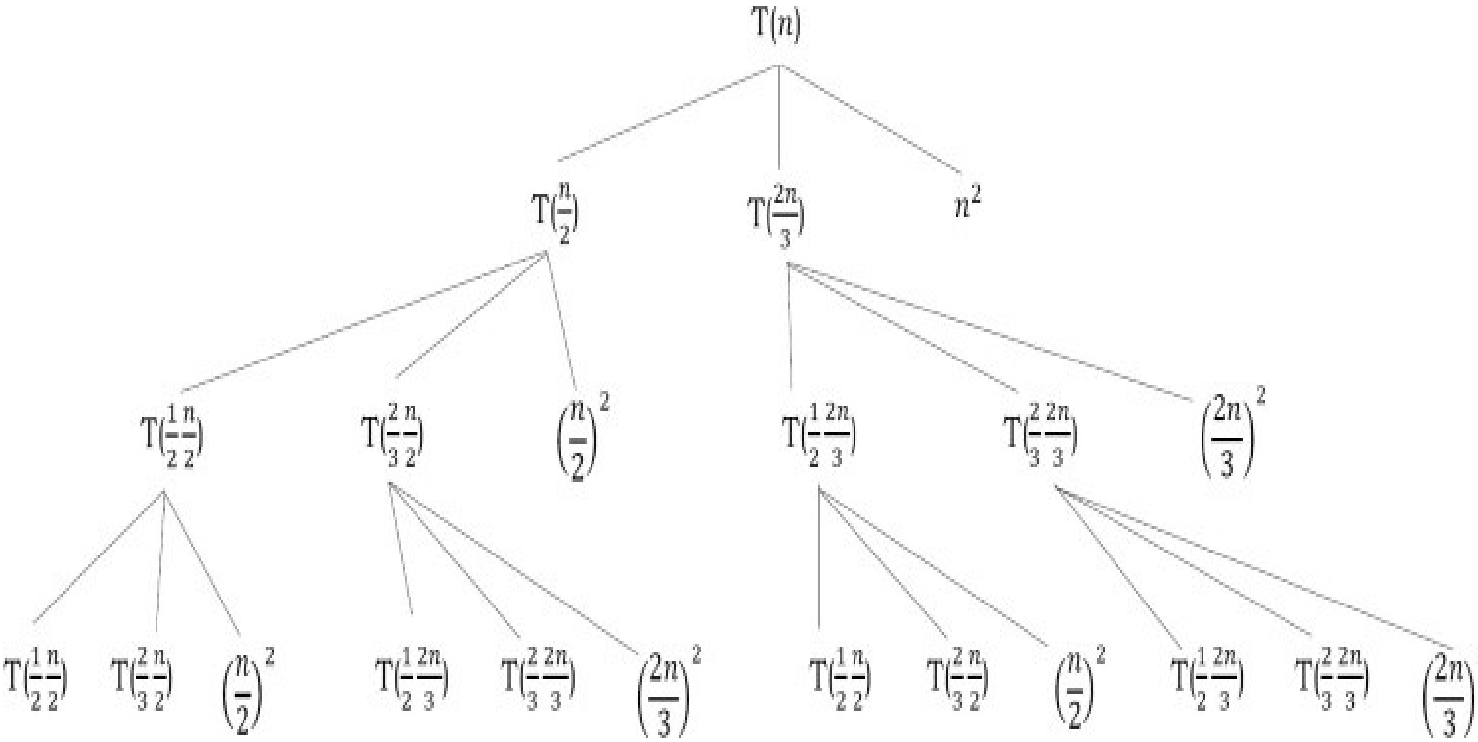
Solution: Consider the worst-case.

```
public void function(int n) {
  int sum = 0;
  for (int i = 0; i<n; i++)              // Executes n times
     if (i>j)
        sum = sum +1;                    // Executes n times
     else {
        for (int k = 0; k < n; k++)      // Executes n times
           sum = sum -1;
     }
  }
}
```

Time Complexity: $O(n^2)$.

Problem-61 Solve the following recurrence relation using the recursion tree method: $T(n)=T(\frac{n}{2}) + T(\frac{2n}{3}) + n^2$.

Solution: How much work do we do in each level of the recursion tree?



In level 0, we take n^2 time. At level 1, the two subproblems take time:

$$\left(\frac{1}{2}n\right)^2 + \left(\frac{2}{3}n\right)^2 = \left(\frac{1}{4} + \frac{4}{9}\right)n^2 = \left(\frac{25}{36}\right)n^2$$

At level 2 the four subproblems are of size $\frac{1}{2}\frac{n}{2}$, $\frac{2}{3}\frac{n}{2}$, $\frac{1}{2}\frac{2n}{3}$, and $\frac{2}{3}\frac{2n}{3}$ respectively. These two subproblems take time:

$$\left(\frac{1}{4}n\right)^2 + \left(\frac{1}{3}n\right)^2 + \left(\frac{1}{3}\right)n^2 + \left(\frac{4}{9}\right)n^2 = \frac{625}{1296}n^2 = \left(\frac{25}{36}\right)^2 n^2$$

Similarly the amount of work at level k is at most $\left(\frac{25}{36}\right)^k n^2$.

Let $\alpha = \frac{25}{36}$, the total runtime is then:

$$
\begin{aligned}
T(n) &\leq \sum_{k=0}^{\infty} \alpha^k n^2 \\
&= \frac{1}{1-\propto} n^2 \\
&= \frac{1}{1-\frac{25}{36}} n^2 \\
&= \frac{1}{\frac{11}{36}} n^2 \\
&= \frac{36}{11} n^2 \\
&= O(n^2)
\end{aligned}
$$

That is, the first level provides a constant fraction of the total runtime.

Problem-62 Find the time complexity of recurrence $T(n) = T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{8}) + n$.

Solution: Let us solve this problem by method of guessing. The total size on each level of the recurrance tree is less than n, so we guess that $f(n) = n$ will dominate. Assume for all $i < n$ that $c_1 n \leq T(i) \leq c_2 n$. Then,

$$
\begin{aligned}
c_1\frac{n}{2} + c_1\frac{n}{4} + c_1\frac{n}{8} + kn \quad &\leq T(n) \leq \quad c_2\frac{n}{2} + c_2\frac{n}{4} + c_2\frac{n}{8} + kn \\
c_1 n(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{k}{c_1}) \quad &\leq T(n) \leq \quad c_2 n(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{k}{c_2}) \\
c_1 n(\frac{7}{8} + \frac{k}{c_1}) \quad &\leq T(n) \leq \quad c_2 n(\frac{7}{8} + \frac{k}{c_2})
\end{aligned}
$$

If $c_1 \geq 8k$ and $c_2 \leq 8k$, then $c_1 n = T(n) = c_2 n$. So, $T(n) = \Theta(n)$. In general, if you have multiple recursive calls, the sum of the arguments to those calls is less than n (in this case $\frac{n}{2} + \frac{n}{4} + \frac{n}{8} < n$), and $f(n)$ is reasonably large, a good guess is $T(n) = \Theta(f(n))$.

Problem-63 Rank the following functions by order of growth: $(n+1)!$, $n!$, 4^n, $n \times 3^n$, $3^n + n^2 + 20n$, $(\frac{3}{2})^n$, $4n^2$, 4^{lgn}, $n^2 + 200$, $20n + 500$, 2^{lgn}, $n^{2/3}$, 1.

Solution:

Function	Rate of Growth	
$(n+1)!$	$O(n!)$	Decreasing rate of growths
$n!$	$O(n!)$	
4^n	$O(4^n)$	
$n \times 3^n$	$O(n3^n)$	
$3^n + n^2 + 20n$	$O(3^n)$	
$(\frac{3}{2})^n$	$O((\frac{3}{2})^n)$	
$4n^2$	$O(n^2)$	
4^{lgn}	$O(n^2)$	
$n^2 + 200$	$O(n^2)$	
$20n + 500$	$O(n)$	
2^{lgn}	$O(n)$	
$n^{2/3}$	$O(n^{2/3})$	
1	$O(1)$	

Problem-64 Can we say $3^{n^{0.75}} = O(3^n)$?

Solution: Yes: because $3^{n^{0.75}} < 3^{n^1}$.

Problem-65 Can we say $2^{3n} = O(2^n)$?

Solution: No: because $2^{3n} = (2^3)^n = 8^n$ not less than 2^n.

CHAPTER 2

RECURSION AND BACKTRACKING

2.1 Introduction

In this chapter, we will look at one of the important topics, *"recursion"*, which will be used in almost every chapter, and also its relative *"backtracking"*.

2.2 What is Recursion?

Any function which calls itself is called *recursive*. A recursive method solves a problem by calling a copy of itself to work on a smaller problem. This is called the recursion step. The recursion step can result in many more such recursive calls. It is important to ensure that the recursion terminates. Each time the function calls itself with a slightly simpler version of the original problem. The sequence of smaller problems must eventually converge on the base case.

2.3 Why Recursion?

Recursion is a useful technique borrowed from mathematics. Recursive code is generally shorter and easier to write than iterative code. Generally, loops are turned into recursive functions when they are compiled or interpreted. Recursion is most useful for tasks that can be defined in terms of similar subtasks. For example, sort, search, and traversal problems often have simple recursive solutions.

2.4 Format of a Recursive Function

A recursive function performs a task in part by calling itself to perform the subtasks. At some point, the function encounters a subtask that it can perform without calling itself. This case, where the function does not recur, is called the *base case*. The former, where the function calls itself to perform a subtask, is referred to as the *cursive case*. We can write all recursive functions using the format:

```
if(test for the base case)
    return some base case value
else if(test for another base case)
    return some other base case value
// the recursive case
else return (some work and then a recursive call)
```

As an example consider the factorial function: $n!$ is the product of all integers between n and 1. The definition of recursive factorial looks like:

$$
\begin{aligned}
n! &= 1, && \text{if } n = 0 \\
n! &= n * (n-1)! && \text{if } n > 0
\end{aligned}
$$

This definition can easily be converted to recursive implementation. Here the problem is determining the value of $n!$, and the subproblem is determining the value of $(n - l)!$. In the recursive case, when n is greater than 1, the function calls itself to determine the value of $(n - l)!$ and multiplies that with n. In the base case, when n is 0 or 1, the function simply returns 1. This looks like the following:

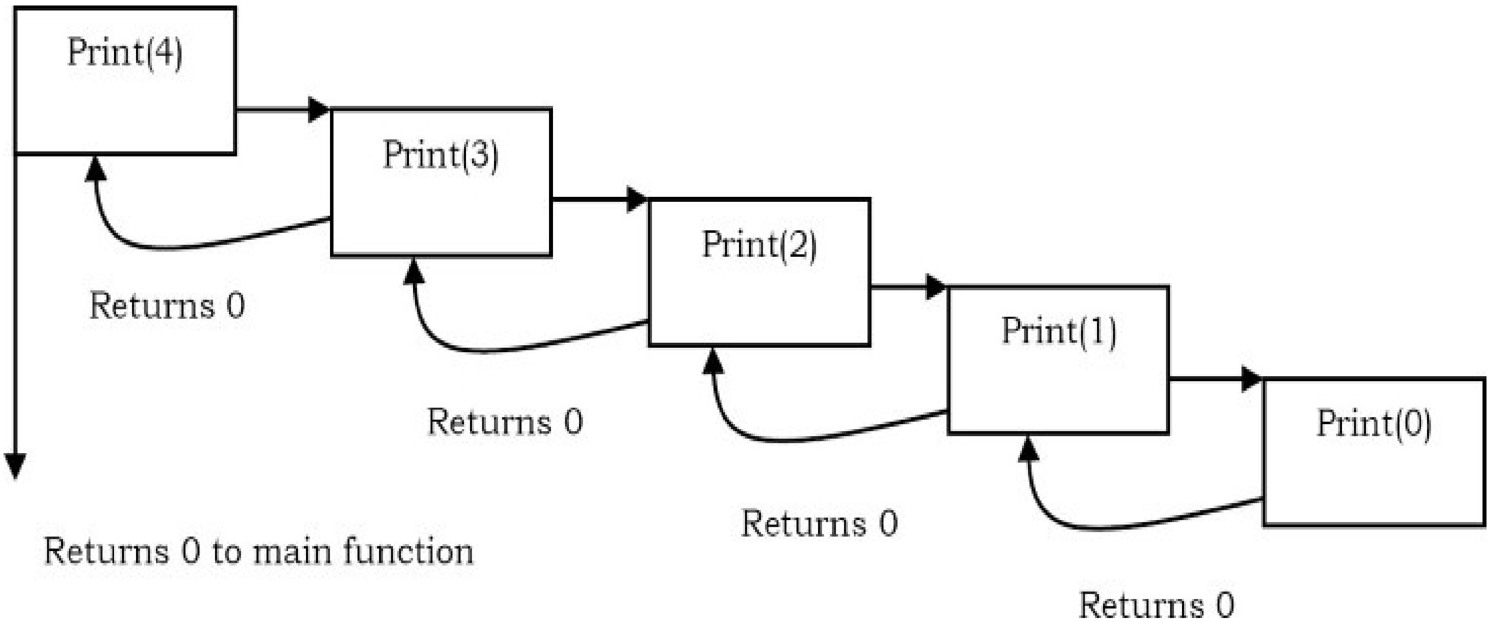
```
// calculates factorial of a positive integer
public int Factorial(int n) {
    // base cases: fact of 0 is 1
    if(n == 0)
            return 1;
    // recursive case: multiply n by (n – 1) factorial
    else
        return n*Factorial(n-1);
}
```

2.5 Recursion and Memory (Visualization)

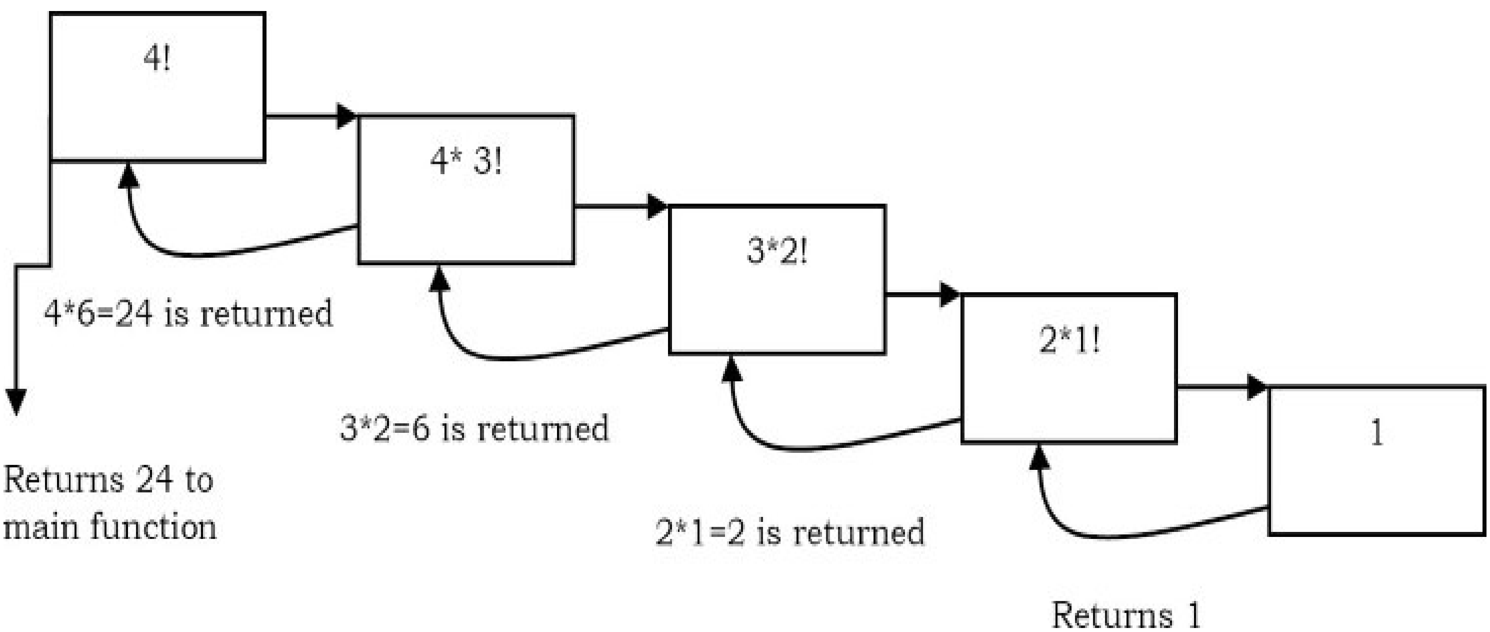
Each recursive call makes a new copy of that method (actually only the variables) in memory. Once a method ends (that is, returns some data), the copy of that returning method is removed from memory. The recursive solutions look simple but visualization and tracing takes time. For better understanding, let us consider the following example.

```
public int Print(int n) {
    if( n = = 0)                    // this is the terminating base case
            return 0;
    else {
            System.out.println(n);
            return Print(n-1);      // recursive call to itself again
    }
}
```

For this example, if we call the print function with n=4, visually our memory assignments may look like:



Now, let us consider our factorial function. The visualization of factorial function with n=4 will look like:



2.6 Recursion versus Iteration

While discussing recursion, the basic question that comes to mind is: which way is better? – iteration or recursion? The answer to this question depends on what we are trying to do. A recursive approach mirrors the problem that we are trying to solve. A recursive approach makes it simpler to solve a problem that may not have the most obvious of answers. But, recursion adds overhead for each recursive call (needs space on the stack frame).

Recursion

- Terminates when a base case is reached.
- Each recursive call requires extra space on the stack frame (memory).
- If we get infinite recursion, the program may run out of memory and result in stack overflow.
- Solutions to some problems are easier to formulate recursively.

Iteration

- Terminates when a condition is proven to be false.
- Each iteration does not require any extra space.
- An infinite loop could loop forever since there is no extra memory being created.
- Iterative solutions to a problem may not always be as obvious as a recursive solution.

2.7 Notes on Recursion

- Recursive algorithms have two types of cases, recursive cases and base cases.
- Every recursive function case must terminate at a base case.
- Generally, iterative solutions are more efficient than recursive solutions [due to the overhead of function calls].
- A recursive algorithm can be implemented without recursive function calls using a stack, but it's usually more trouble than its worth. That means any problem that can be solved recursively can also be solved iteratively.
- For some problems, there are no obvious iterative algorithms.
- Some problems are best suited for recursive solutions while others are not.

2.8 Example Algorithms of Recursion

- Fibonacci Series, Factorial Finding
- Merge Sort, Quick Sort
- Binary Search
- Tree Traversals and many Tree Problems: InOrder, PreOrder PostOrder
- Graph Traversals: DFS [Depth First Search] and BFS [Breadth First Search]
- Dynamic Programming Examples
- Divide and Conquer Algorithms
- Towers of Hanoi
- Backtracking Algorithms [we will discuss in next section]

2.9 Recursion: Problems & Solutions

In this chapter we cover a few problems with recursion and we will discuss the rest in other chapters. By the time you complete reading the entire book, you will encounter many recursion problems.

Problem-1 Discuss Towers of Hanoi puzzle.

Solution: The Towers of Hanoi is a mathematical puzzle. It consists of three rods (or pegs or towers) and a number of disks of different sizes which can slide onto any rod. The puzzle starts with the disks on one rod in ascending order of size, the smallest at the top, thus making a conical shape. The objective of the puzzle is to move the entire stack to another rod, satisfying the following rules:

- Only one disk may be moved at a time.
- Each move consists of taking the upper disk from one of the rods and sliding it onto another rod, on top of the other disks that may already be present on that rod.
- No disk may be placed on top of a smaller disk.

Algorithm

- Move the top $n – 1$ disks from *Source* to *Auxiliary* tower,
- Move the n^{th} disk from *Source* to *Destination* tower,
- Move the $n – 1$ disks from *Auxiliary* tower to *Destination* tower.
- Transferring the top $n – 1$ disks from *Source* to *Auxiliary* tower can again be thought of as a fresh problem and can be solved in the same manner. Once we solve *Towers of Hanoi* with three disks, we can solve it with any number of disks with the above algorithm.

```
public void TowersOfHanoi(int n, char frompeg, char topeg, char auxpeg) {
    /* If only 1 disk, make the move and return */
    if(n==1) {
            System.out.println("Move disk 1 from peg " + frompeg + " to peg " + topeg);
            return;
    }
    /* Move top n-1 disks from A to B, using C as auxiliary */
    TowersOfHanoi(n-1,frompeg,auxpeg,topeg);
    /* Move remaining disks from A to C */
     System.out.println("Move disk from peg" + frompeg + " to peg " + topeg);
    /* Move n-1 disks from B to C using A as auxiliary */
    TowersOfHanoi(n-1,auxpeg,topeg,frompeg);
}
```

Problem-2 Given an array, check whether the array is in sorted order with recursion.

Solution:

```
public int isArrayInSortedOrder(int[] A, int index){
    if(A.length() == 1)
        return 1;
    return (A[index -1] < A[index -2])?0:isArrayInSortedOrder(A, index -1);
}
```

Time Complexity: O(n). Space Complexity: O(n) for recursive stack space.

2.10 What is Backtracking?

Backtracking is an improvement of the brute force approach. It systematically searches for a solution to a problem among all available options. In backtracking, we start with one possible option out of many available options and try to solve the problem if we are able to solve the problem with the selected move then we will print the solution else we will backtrack and select some other option and try to solve it. If none if the options work out we will claim that there is no solution for the problem.

Backtracking is a form of recursion. The usual scenario is that you are faced with a number of options, and you must choose one of these. After you make your choice you will get a new set of options; just what set of options you get depends on what choice you made. This procedure is repeated over and over until you reach a final state. If you made a good sequence of choices, your final state is a goal state; if you didn't, it isn't. Backtracking can be thought of as a selective tree/graph traversal method. The tree is a way of representing some initial starting position (the root node) and a final goal state (one of the leaves). Backtracking allows us to deal with situations in which a raw brute-force approach would explode into an impossible number of options to consider. Backtracking is a sort of refined brute force. At each node, we eliminate choices that are obviously not possible and proceed to recursively check only those that have potential.

What's interesting about backtracking is that we back up only as far as needed to reach a previous decision point with an as-yet-unexplored alternative. In general, that will be at the most recent decision point. Eventually, more and more of these decision points will have been fully explored, and we will have to backtrack further and further. If we backtrack all the way to our initial state and have explored all alternatives from there, we can conclude the particular problem is unsolvable. In such a case, we will have done all the work of the exhaustive recursion and known that there is no viable solution possible.

- Sometimes the best algorithm for a problem is to try all possibilities.
- This is always slow, but there are standard tools that can be used to help.
- Tools: algorithms for generating basic objects, such as binary strings [2^n possibilities for n-bit string], permutations [n!], combinations [$n!/r!\ (n - r)!$], general strings [k –ary strings of length n has k^n possibilities], etc...
- Backtracking speeds the exhaustive search by pruning.

2.11 Example Algorithms of Backtracking

- Binary Strings: generating all binary strings
- Generating *k* – ary Strings
- The Knapsack Problem
- N-Queens Problem
- Generalized Strings
- Hamiltonian Cycles [refer to *Graphs* chapter]
- Graph Coloring Problem

2.12 Backtracking: Problems & Solutions

Problem-3 Generate all the strings of *n* bits. Assume *A*[0..*n* – 1] is an array of size *n*.

Solution:

```
public void Binary(int n) {
    if(n < 1 )
        System.out.println(A);              //Assume array A is a class variable
    else {
            A[n-1] = 0;
            Binary (n - 1);
            A[n-1] = 1;
            Binary(n - 1);
    }
}
```

Let *T*(*n*) be the running time of *binary*(*n*). Assume function *System.out.println* takes time O(1).

$$T(n) = \begin{cases} c, & \text{if } n < 0 \\ 2T(n-1) + d, & \text{otherwise} \end{cases}$$

Using Subtraction and Conquer Master theorem we get: *T*(*n*) = O(2^n). This means the algorithm for generating bit-strings is optimal.

Problem-4 Generate all the strings of length *n* drawn from 0... *k* – 1.

Solution: Let us assume we keep current k-ary string in an array *A*[0.. *n* – 1]. Call function *k-string*(*n*, k):

```
public void k-string(int n, int k) {
    //process all k-ary strings of length m
    if(n < 1 )
        System.out.println(A);              //Assume array A is a class variable
    else {
        for (int j = 0 ; j < k ; j++) {
            A[n-1] = j;
            k-string(n- 1, k);
        }
    }
}
```

Let $T(n)$ be the running time of $k - string(n)$. Then,

$$T(n) = \begin{cases} c, & if\ n < 0 \\ kT(n-1) + d, & otherwise \end{cases}$$

Using Subtraction and Conquer Master theorem we get: $T(n) = O(k^n)$.

Note: For more problems, refer to *String Algorithms* chapter.

Problem-5 Solve the recurrence $T(n) = 2T(n-1) + 2^n$.

Solution: At each level of the recurrence tree, the number of problems is double from the previous level, while the amount of work being done in each problem is half from the previous level. Formally, the i^{th} level has 2^i problems, each requiring 2^{n-i} work. Thus the i^{th} level requires exactly 2^n work. The depth of this tree is n, because at the i^{th} level, the originating call will be $T(n-i)$. Thus the total complexity for $T(n)$ is $T(n2^n)$.

CHAPTER

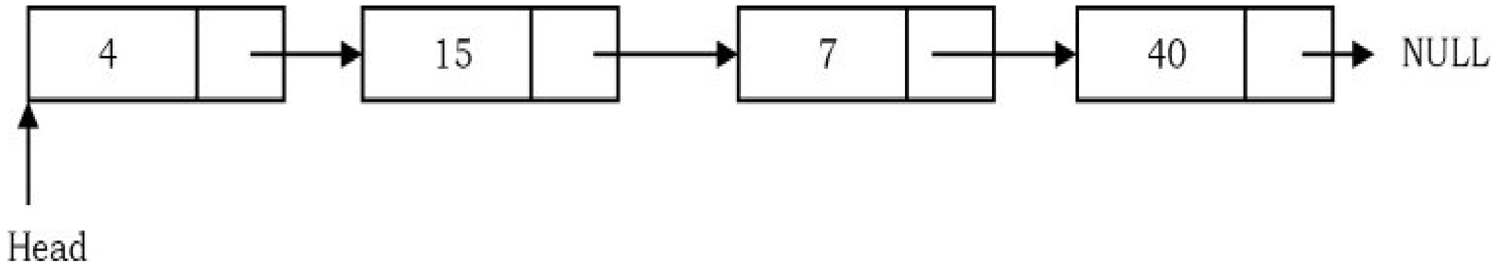
3

LINKED LISTS

3.1 What is a Linked List?

A linked list is a data structure used for storing collections of data. A linked list has the following properties.

- Successive elements are connected by pointers
- The last element points to NULL
- Can grow or shrink in size during execution of a program
- Can be made just as long as required (until systems memory exhausts)
- Does not waste memory space (but takes some extra memory for pointers). It allocates memory as list grows.



3.2 Linked Lists ADT

The following operations make linked lists an ADT:

Main Linked Lists Operations

- Insert: inserts an element into the list
- Delete: removes and returns the specified position element from the list

Auxiliary Linked Lists Operations

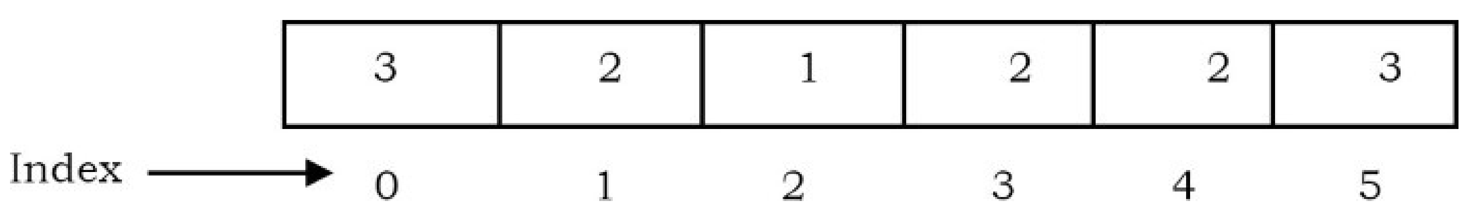
- Delete List: removes all elements of the list (disposees the list)
- Count: returns the number of elements in the list
- Find n^{th} node from the end of the list

3.3 Why Linked Lists?

There are many other data structures that do the same thing as linked lists. Before discussing linked lists it is important to understand the difference between linked lists and arrays. Both linked lists and arrays are used to store collections of data, and since both are used for the same purpose, we need to differentiate their usage. That means in which cases *arrays* are suitable and in which cases *linked lists* are suitable.

3.4 Arrays Overview

One memory block is allocated for the entire array to hold the elements of the array. The array elements can be accessed in constant time by using the index of the particular element as the subscript.



Why Constant Time for Accessing Array Elements?

To access an array element, the address of an element is computed as an offset from the base address of the array and one multiplication is needed to compute what is supposed to be added to the base address to get the memory address of the element. First the size of an element of that data type is calculated and then it is multiplied with the index of the element to get the value to be added to the base address.

This process takes one multiplication and one addition. Since these two operations take constant time, we can say the array access can be performed in constant time.

Advantages of Arrays

- Simple and easy to use
- Faster access to the elements (constant access)

Disadvantages of Arrays

- Preallocates all needed memory up front and wastes memory space for indices in the array that are empty.
- **Fixed size:** The size of the array is static (specify the array size before using it).
- **One block allocation:** To allocate the array itself at the beginning, sometimes it may not be possible to get the memory for the complete array (if the array size is big).
- **Complex position-based insertion:** To insert an element at a given position, we may need to shift the existing elements. This will create a position for us to insert the new element at the desired position. If the position at which we want to add an element is at the beginning, then the shifting operation is more expensive.

Dynamic Arrays

Dynamic array (also called *growable array, resizable array*, *dynamic table*, or *array list*) is a random access, variable-size list data structure that allows elements to be added or removed.

One simple way of implementing dynamic arrays is to initially start with some fixed size array. As soon as that array becomes full, create the new array double the size of the original array. Similarly, reduce the array size to half if the elements in the array are less than half the size.

Note: We will see the implementation for *dynamic arrays* in the *Stacks*, *Queues* and *Hashing* chapters.

Advantages of Linked Lists

Linked lists have both advantages and disadvantages. The advantage of linked lists is that they can be *expanded* in constant time. To create an array, we must allocate memory for a certain number of elements. To add more elements to the array when full, we must create a new array and copy the old array into the new array. This can take a lot of time.

We can prevent this by allocating lots of space initially but then we might allocate more than we need and waste memory. With a linked list, we can start with space for just one allocated element and *add* on new elements easily without the need to do any copying and reallocating.

Issues with Linked Lists (Disadvantages)

There are a number of issues with linked lists. The main disadvantage of linked lists is *access time* to individual elements. Array is random-access, which means it takes O(1) to access any element in the array. Linked lists take O(n) for access to an element in the list in the worst case. Another advantage of arrays in access time is *spacial locality* in memory. Arrays are defined as contiguous blocks of memory, and so any array element will be physically near its neighbors. This greatly benefits from modern CPU caching methods.

Although the dynamic allocation of storage is a great advantage, the *overhead* with storing and retrieving data can make a big difference. Sometimes linked lists are *hard to manipulate*. If the last item is deleted, the last but one must then have its pointer changed to hold a NULL reference. This requires that the list is traversed to find the last but one link, and its pointer set to a NULL reference.

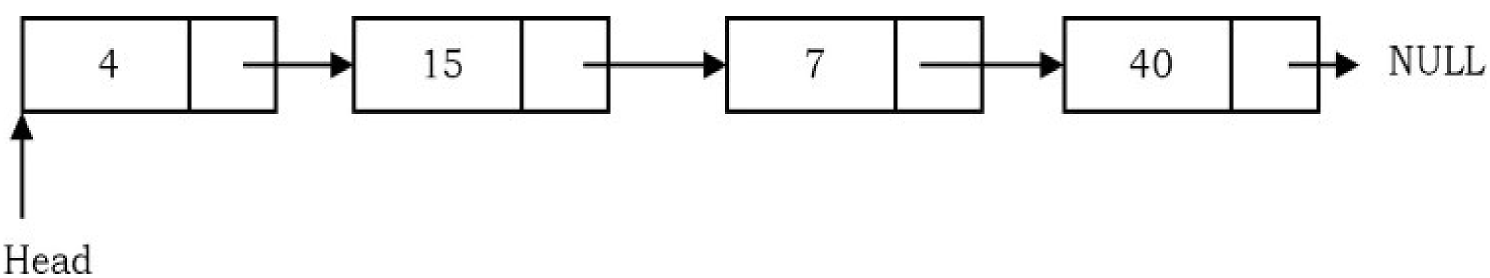
Finally, linked lists waste memory in terms of extra reference points.

3.5 Comparison of Linked Lists with Arrays & Dynamic Arrays

Parameter	Linked list	Array	Dynamic array
Indexing	$O(n)$	$O(1)$	$O(1)$
Insertion/deletion at beginning	$O(1)$	$O(n)$, if array is not full (for shifting the elements)	$O(n)$
Insertion at ending	$O(n)$	$O(1)$, if array is not full	$O(1)$, if array is not full $O(n)$, if array is full
Deletion at ending	$O(n)$	$O(1)$	$O(n)$
Insertion in middle	$O(n)$	$O(n)$, if array is not full (for shifting the elements)	$O(n)$
Deletion in middle	$O(n)$	$O(n)$, if array is not full (for shifting the elements)	$O(n)$
Wasted space	$O(n)$ (for pointers)	0	$O(n)$

3.6 Singly Linked Lists

Generally "linked list" means a singly linked list. This list consists of a number of nodes in which each node has a *next* pointer to the following element. The link of the last node in the list is NULL, which indicates the end of the list.



Following is a type declaration for a linked list:

```java
public class ListNode {
    private int data;
    private ListNode next;
    public ListNode(int data){
        this.data = data;
    }
    public void setData(int data){
        this.data = data;
    }
    public int getData(){
        return data;
    }
    public void setNext(ListNode next){
        this.next = next;
    }
    public ListNode getNext(){
        return this.next;
    }
}
```

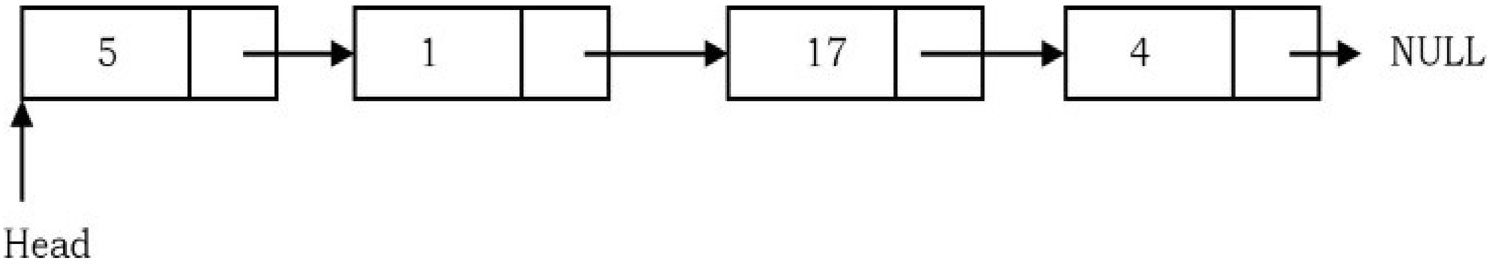
Basic Operations on a List

- Traversing the list
- Inserting an item in the list
- Deleting an item from the list

Traversing the Linked List

Let us assume that the *head* points to the first node of the list. To traverse the list we do the following.

- Follow the pointers.
- Display the contents of the nodes (or count) as they are traversed.
- Stop when the next pointer points to NULL.



The ListLength() function takes a linked list as input and counts the number of nodes in the list. The function given below can be used for printing the list data with extra print function.

```
public int ListLength(ListNode headNode) {
      int length = 0;
      ListNode currentNode = headNode;
      while(currentNode != null){
            length++;
            currentNode = currentNode.getNext();
      }
      return length;
}
```

Time Complexity: O(n), for scanning the list of size n. Space Complexity: O(1), for creating a temporary variable.

Singly Linked List Insertion

Insertion into a singly-linked list has three cases:

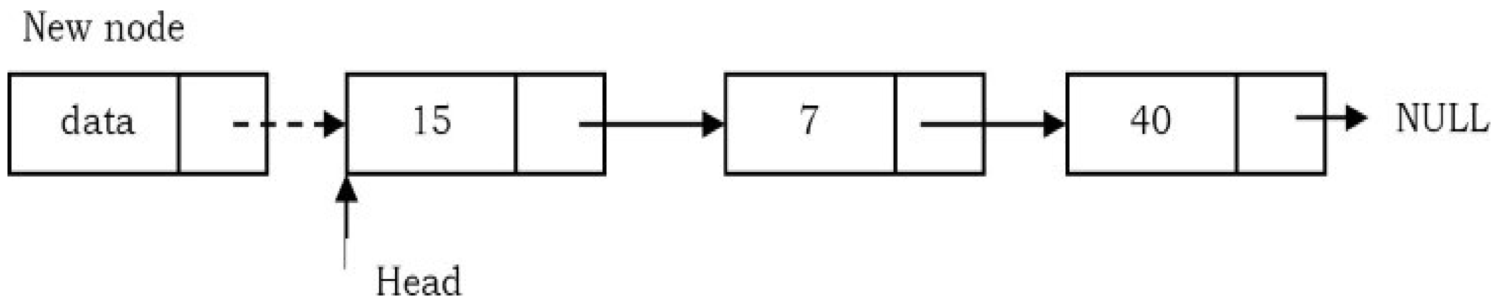
- Inserting a new node before the head (at the beginning)
- Inserting a new node after the tail (at the end of the list)
- Inserting a new node at the middle of the list (random location)

Note: To insert an element in the linked list at some position p, assume that after inserting the element the position of this new node is p.

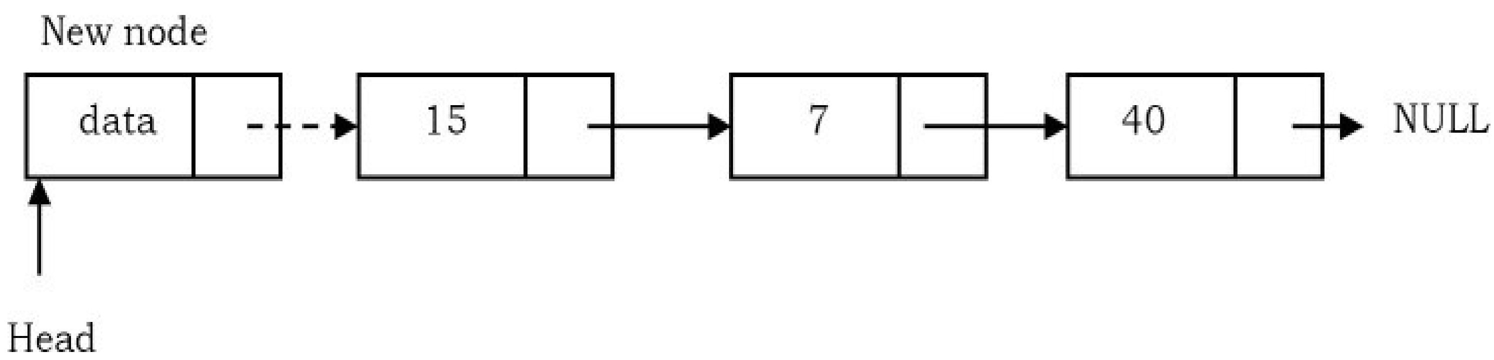
Inserting a Node in Singly Linked List at the Beginning

In this case, a new node is inserted before the current head node. *Only one next pointer* needs to be modified (new node's next pointer) and it can be done in two steps:

- Update the next pointer of new node, to point to the current head.



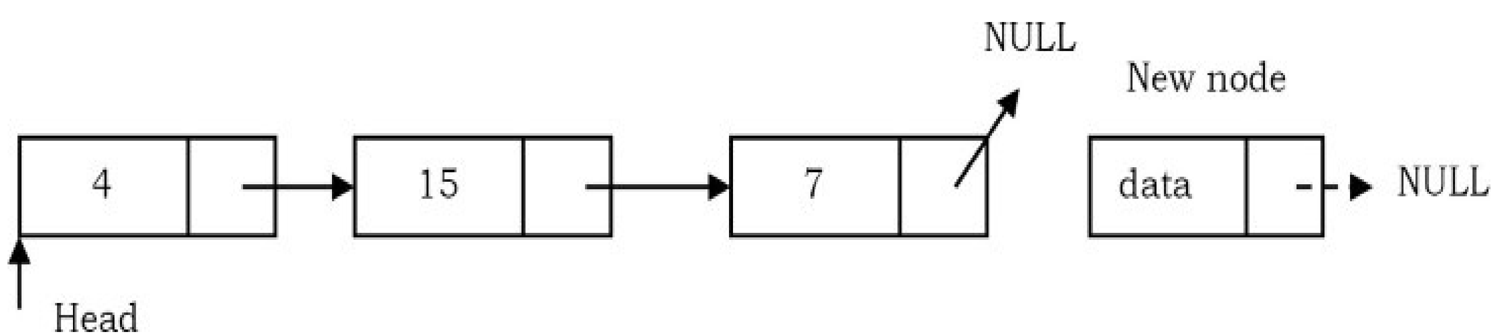
- Update head pointer to point to the new node.



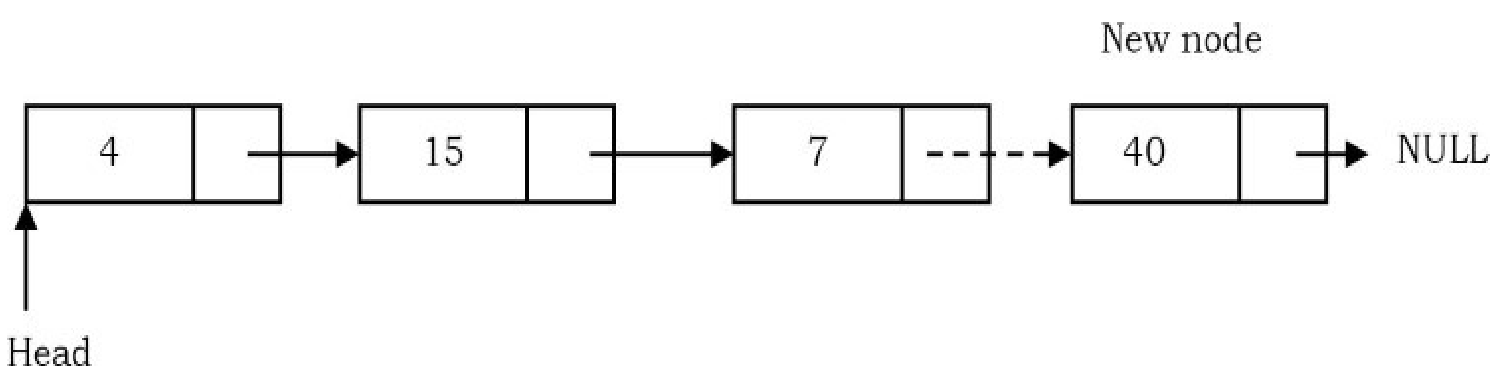
Inserting a Node in Singly Linked List at the Ending

In this case, we need to modify *two next pointers* (last nodes next pointer and new nodes next pointer).

- New nodes next pointer points to NULL.



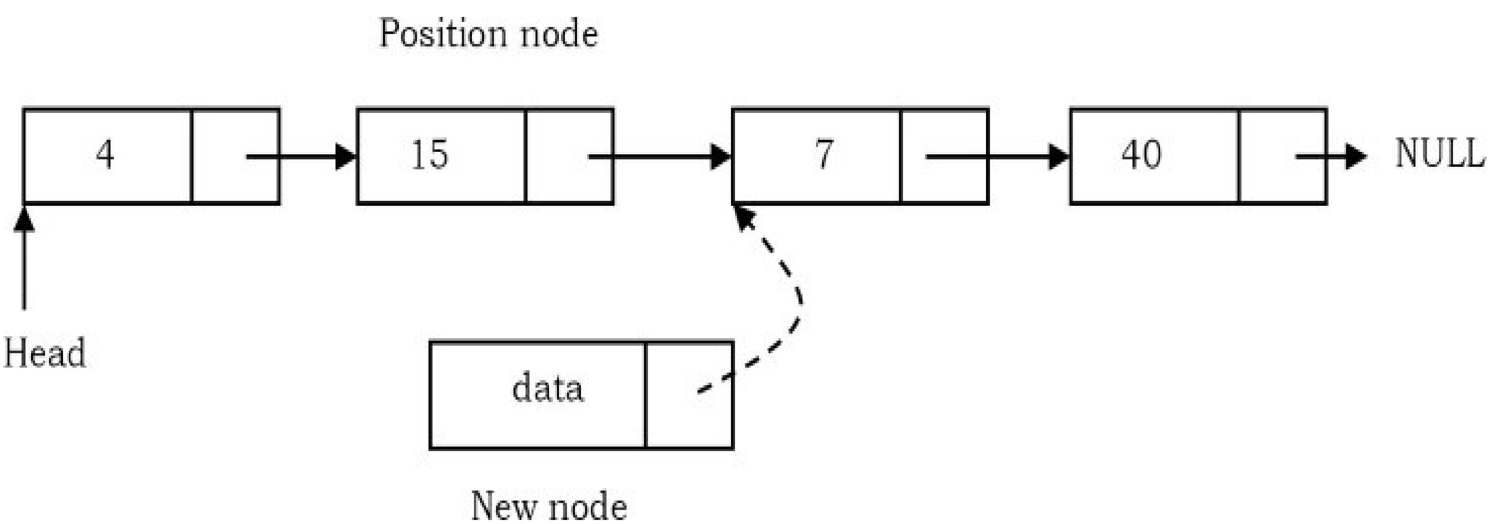
- Last nodes next pointer points to the new node.



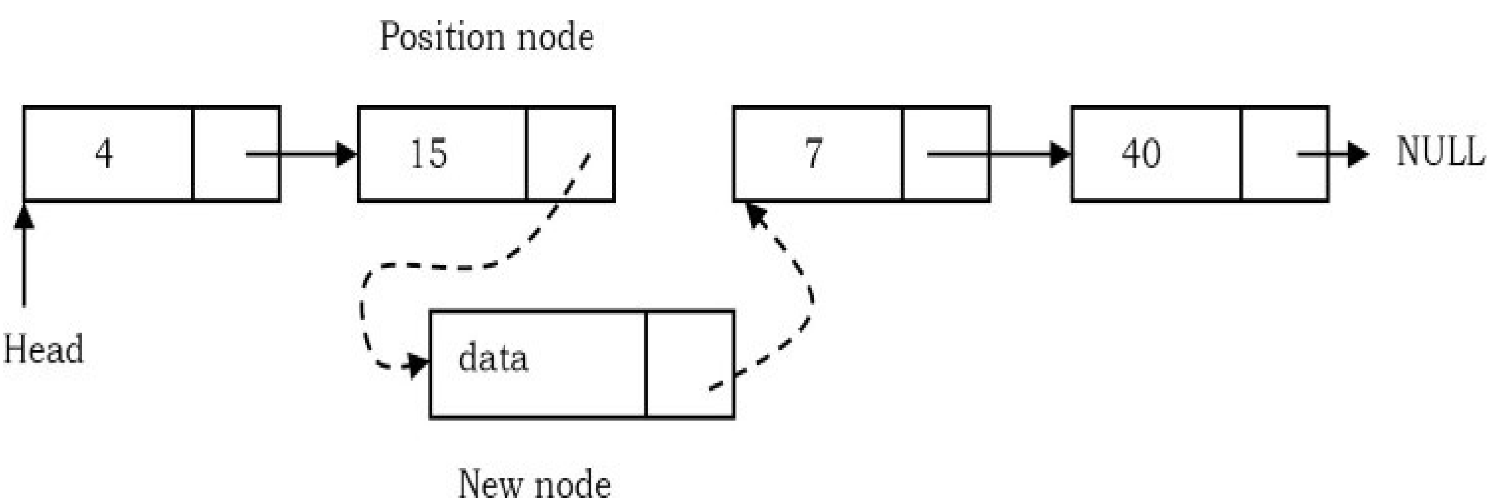
Inserting a Node in Singly Linked List at the Middle

Let us assume that we are given a position where we want to insert the new node. In this case also, we need to modify two next pointers.

- If we want to add an element at position 3 then we stop at position 2. That means we traverse 2 nodes and insert the new node. For simplicity let us assume that the second node is called *position* node. The new node points to the next node of the position where we want to add this node.



- Position node’s next pointer now points to the new node.



Note: We can implement the three variations of the *insert* operation separately.

Time Complexity: O(n), since, in the worst case, we may need to insert the node at the end of the list.
Space Complexity: O(1), for creating one temporary variable.

Singly Linked List Deletion

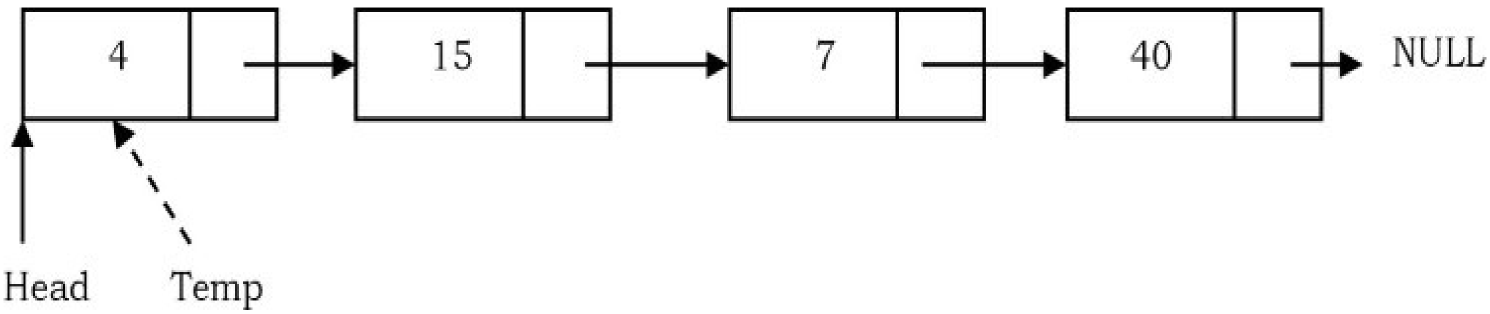
Similar to insertion, here we also have three cases.

- Deleting the first node
- Deleting the last node
- Deleting an intermediate node.

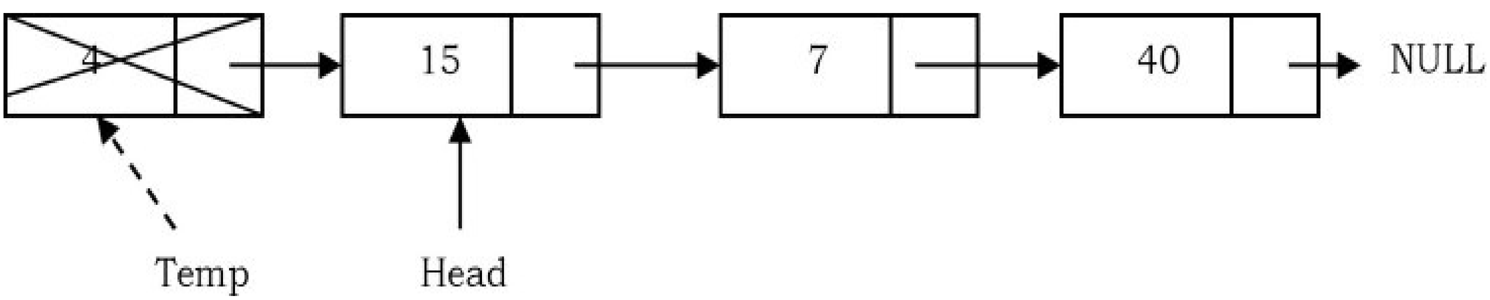
Deleting the First Node in Singly Linked List

First node (current head node) is removed from the list. It can be done in two steps:

- Create a temporary node which will point to the same node as that of head.



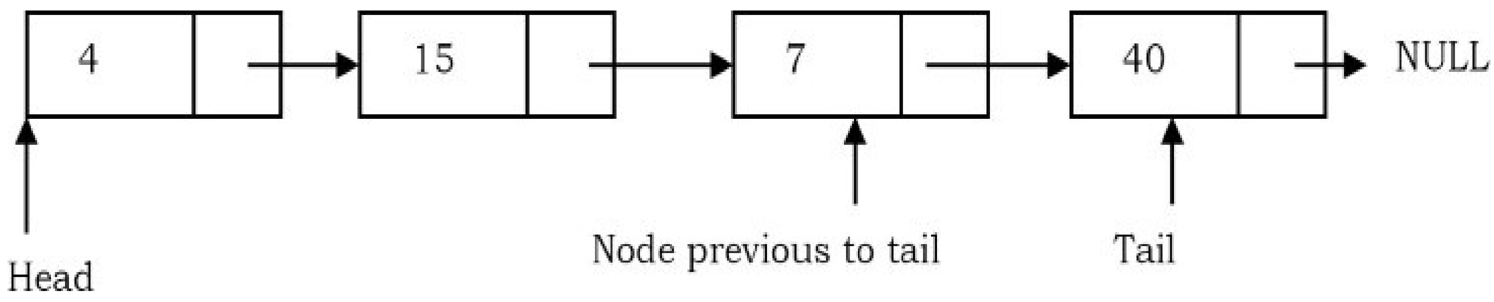
- Now, move the head nodes pointer to the next node and dispose of the temporary node.



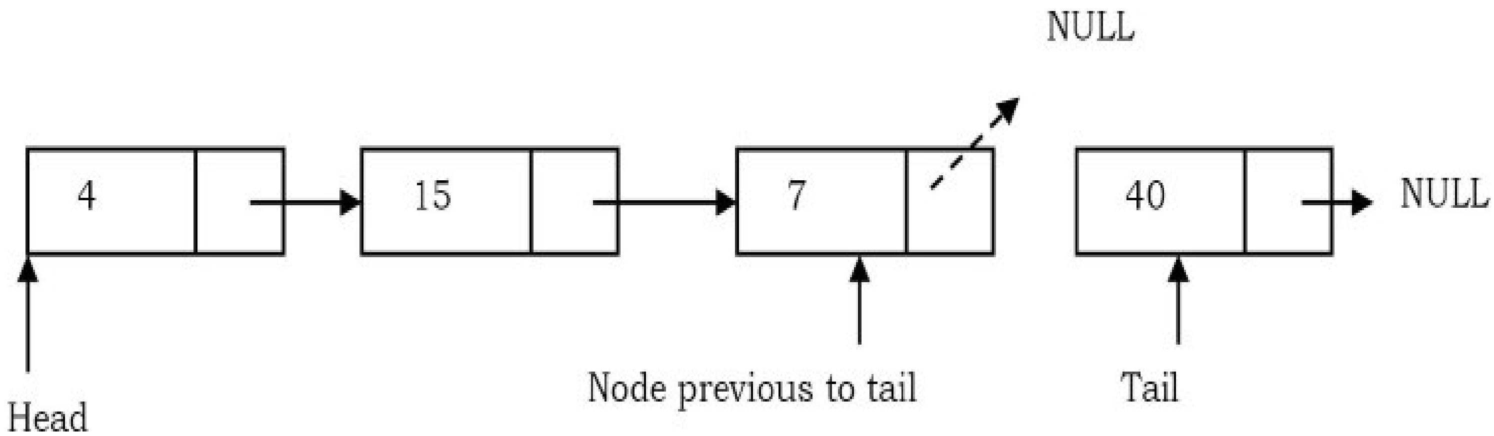
Deleting the Last node in Singly Linked List

In this case, the last node is removed from the list. This operation is a bit trickier than removing the first node, because the algorithm should find a node, which is previous to the tail. It can be done in three steps:

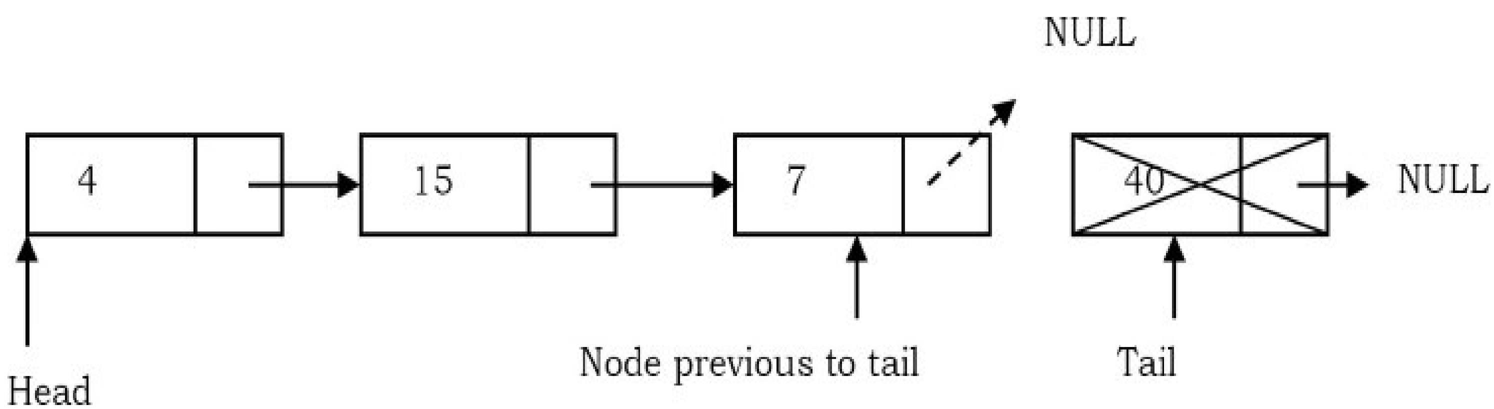
- Traverse the list and while traversing maintain the previous node address also. By the time we reach the end of the list, we will have two pointers, one pointing to the *tail* node and the other pointing to the node *before* the tail node.



- Update previous node's next pointer with NULL.



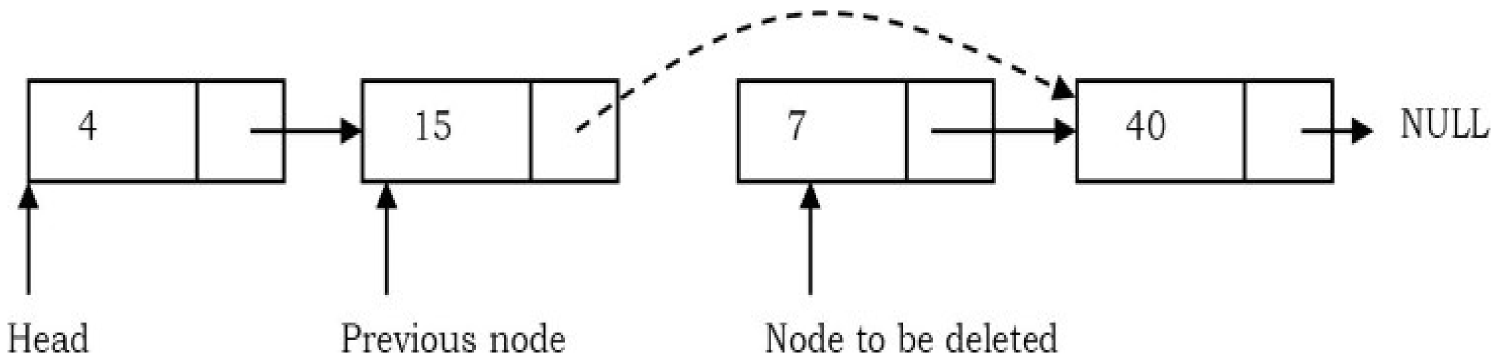
- Dispose of the tail node.



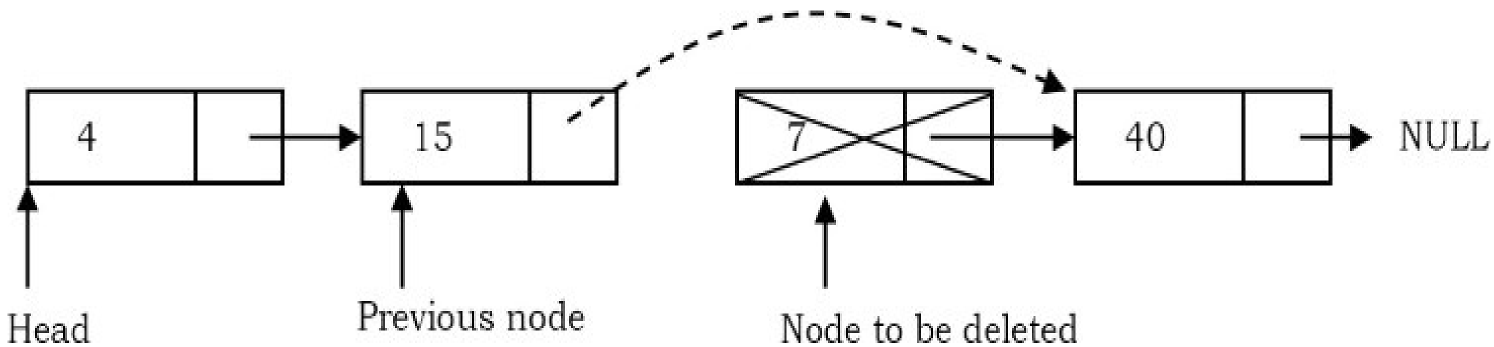
Deleting an Intermediate Node in Singly Linked List

In this case, the node to be removed is *always located between* two nodes. Head and tail links are not updated in this case. Such a removal can be done in two steps:

- Similar to the previous case, maintain the previous node while traversing the list. Once we find the node to be deleted, change the previous node's next pointer to the next pointer of the node to be deleted.



- Dispose of the current node to be deleted.



Time Complexity: O(n). In the worst case, we may need to delete the node at the end of the list. Space Complexity: O(1). Since, we are creating only one temporary variable.

Deleting Singly Linked List

This works by storing the current node in some temporary variable and freeing the current node. After freeing the current node, go to the next node with a temporary variable and repeat this process for all nodes.

Time Complexity: O(n), for scanning the complete list of size n. Space Complexity: O(1), for temporary variable.

Implementation

```
public class LinkedList {
	// This class has a default constructor:
	public LinkedList() {
		length = 0;
	}
	// This is the only field of the class.  It holds the head of the list
	ListNode head;
	// Length of the linked list
	private int length = 0;
	// Return the first node in the list
	public synchronized ListNode getHead() {
		return head;
	}
	// Insert a node at the beginning of the list
	public synchronized void insertAtBegin(ListNode node) {
		node.setNext(head);
		head = node;
		length ++;
	}
	// Insert a node at the end of the list
	public synchronized void insertAtEnd(ListNode node) {
		if (head == null)
			head = node;
		else {
			ListNode p, q;
```

```
                        for(p = head; (q = p.getNext()) != null; p = q);
                                p.setNext(node);
                }
                length ++;
        }
        // Add a new value to the list at a given position.
        // All values at that position to the end move over to make room.
        public void insert(int data, int position) {
                // fix the position
                if (position < 0) {
                        position = 0;
                }
                if (position > length) {
                        position = length;
                }
                // if the list is empty, make it be the only element
                if (head == null) {
                        head = new ListNode(data);
                }
                // if adding at the front of the list...
                else if (position == 0) {
                        ListNode temp = new ListNode(data);
                        temp.next = head;
                        head = temp;
                }
                // else find the correct position and insert
                else {
                        ListNode temp = head;
                        for (int i=1; i<position; i+=1) {
                                temp = temp.getNext();
                        }
                        ListNode newNode = new ListNode(data);
                        newNode.next = temp.next;
                        temp.setNext(newNode);
                }
                // the list is now one value longer
                length += 1;
        }
        // Remove and return the node at the head of the list
        public synchronized ListNode removeFromBegin() {
                ListNode node = head;
                if (node != null) {
                        head = node.getNext();
                        node.setNext(null);
                }
                return node;
        }
        // Remove and return the node at the end of the list
        public synchronized ListNode removeFromEnd() {
                if (head == null)
                        return null;
                ListNode p = head, q = null, next = head.getNext();
                if (next == null) {
                        head = null;
                        return p;
                }
                while((next = p.getNext()) != null) {
                        q = p;
                        p = next;
                }
                q.setNext(null);
                return p;
        }
```

```
// Remove a node matching the specified node from the list.
// Use equals() instead of == to test for a matched node.
public synchronized void removeMatched(ListNode node) {
	if (head == null)
		return;
	if (node.equals(head)) {
		head = head.getNext();
		return;
	}
	ListNode p = head, q = null;
	while((q = p.getNext()) != null) {
		if (node.equals(q)) {
			p.setNext(q.getNext());
			return;
		}
		p = q;
	}
}
// Remove the value at a given position.
// If the position is less than 0, remove the value at position 0.
// If the position is greater than 0, remove the value at the last position.
public void remove(int position) {
	// fix position
	if (position < 0) {
		position = 0;
	}
	if (position >= length) {
		position = length-1;
	}
	// if nothing in the list, do nothing
	if (head == null)
		return;
	// if removing the head element...
	if (position == 0) {
		head = head.getNext();
	}
	// else advance to the correct position and remove
	else {
		ListNode temp = head;
		for (int i=1; i<position; i+=1) {
			temp = temp.getNext();
		}
		temp.setNext(temp.getNext().getNext());
	}
	// reduce the length of the list
	length -= 1;
}
// Return a string representation of this collection, in the form ["str1","str2",...].
public String toString() {
	String result = "[";
	if (head == null) {
		return result+"]";
	}
	result = result + head.getData();
	ListNode temp = head.getNext();;
	while (temp != null) {
		result = result + "," + temp.getData();
		temp = temp.getNext();
	}
	return result + "]";
}
```

```
    // Return the current length of the list.
    public int length() {
        return length;
    }
    // Find the position of the first value that is equal to a given value.
    // The equals method us used to determine equality.
    public int getPosition(int data) {
        // go looking for the data
        ListNode temp = head;
        int pos = 0;
        while (temp != null) {
            if (temp.getData() == data) {
                // return the position if found
                return pos;
            }
            pos += 1;
            temp = temp.getNext();
        }
        // else return some large value
        return Integer.MIN_VALUE;
    }
    // Remove everything from the list.
    public void clearList(){
        head = null;
        length = 0;
    }
}
```

3.7 Doubly Linked Lists

The *advantage* of a doubly linked list (also called *two – way linked list*) is that given a node in the list, we can navigate in both directions. A node in a singly linked list cannot be removed unless we have the pointer to its predecessor. But in a doubly linked list, we can delete a node even if we don't have the previous node's address (since each node has a left pointer pointing to the previous node and can move backward).

The primary *disadvantages* of doubly linked lists are:

- Each node requires an extra pointer, requiring more space.
- The insertion or deletion of a node takes a bit longer (more pointer operations).

Similar to a singly linked list, let us implement the operations of a doubly linked list. If you understand the singly linked list operations, then doubly linked list operations are obvious. Following is a type declaration for a doubly linked list:

```
public class DLLNode {
    private int data;
    private DLLNode prev;
    private DLLNode next;
    public DLLNode(int data) {
        this.data = data;
        prev = null;
        next = null;
    }
    public DLLNode(int data, DLLNode prev, DLLNode next) {
        this.data = data;
        this.prev = prev;
        this.next = next;
    }
    public int getData() {
        return data;
    }
    public void setData(int data) {
        this.data = data;
    }
    public DLLNode getPrev() {
        return prev;
    }
    public DLLNode getNext() {
        return next;
    }
    public void setPrev(DLLNode where) {
        prev = where;
    }
    public void setNext(DLLNode where) {
        next = where;
    }
}
```

Doubly Linked List Insertion

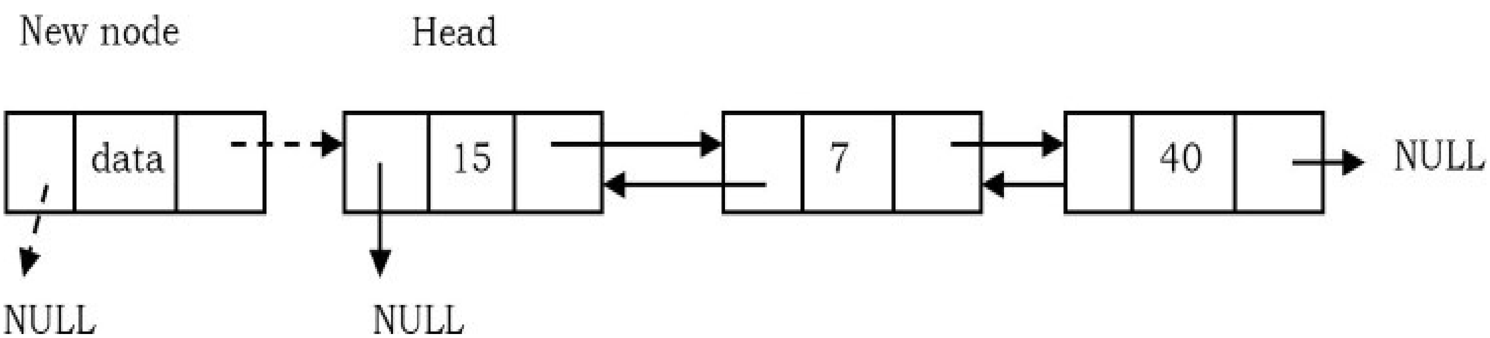
Insertion into a doubly-linked list has three cases (same as a singly linked list).

- Inserting a new node before the head.
- Inserting a new node after the tail (at the end of the list).
- Inserting a new node at the middle of the list.

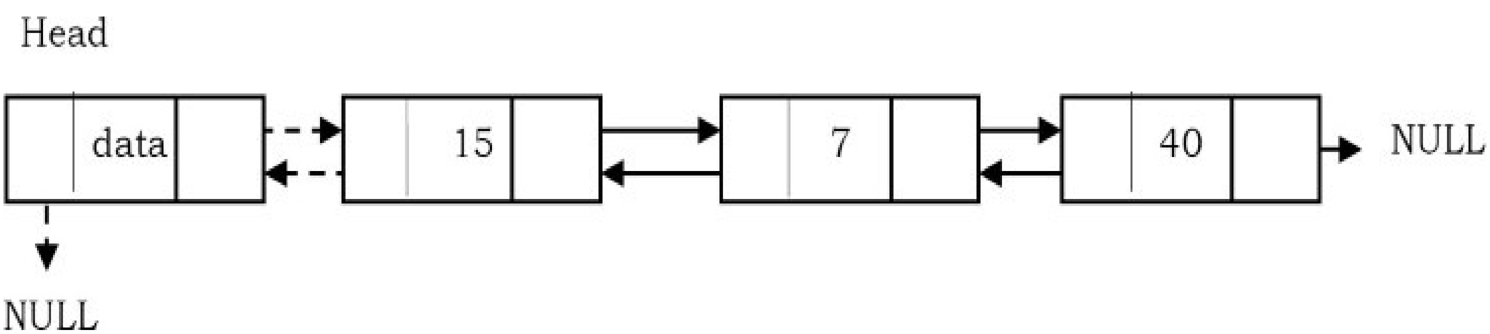
Inserting a Node in Doubly Linked List at the Beginning

In this case, new node is inserted before the head node. Previous and next pointers need to be modified and it can be done in two steps:

- Update the right pointer of the new node to point to the current head node (dotted link in below figure) and also make left pointer of new node as NULL.



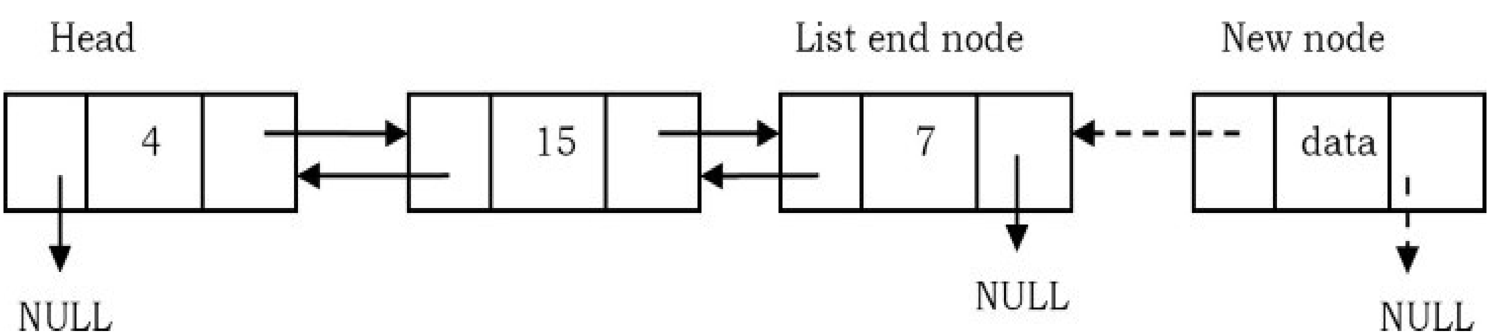
- Update head node's left pointer to point to the new node and make new node as head.



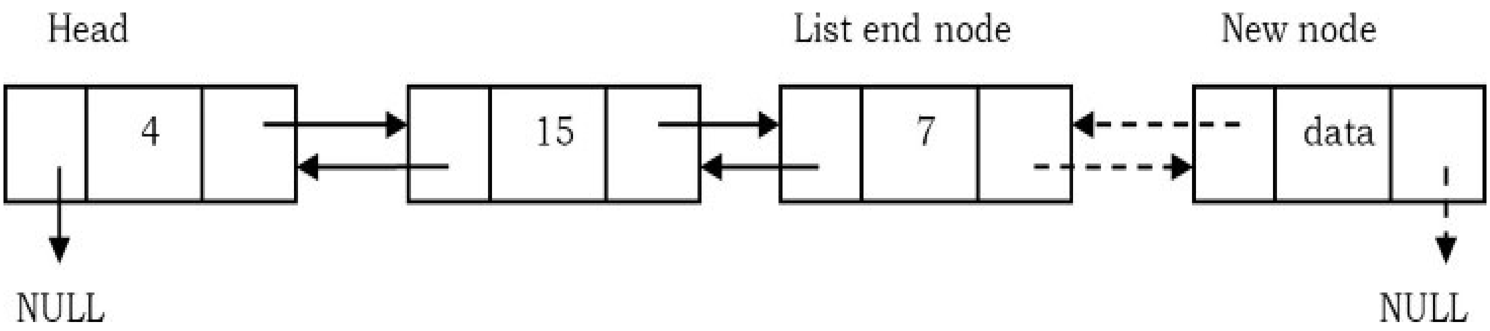
Inserting a Node in Doubly Linked List at the Ending

In this case, traverse the list till the end and insert the new node.

- New node's right pointer points to NULL and left pointer points to the end of the list.



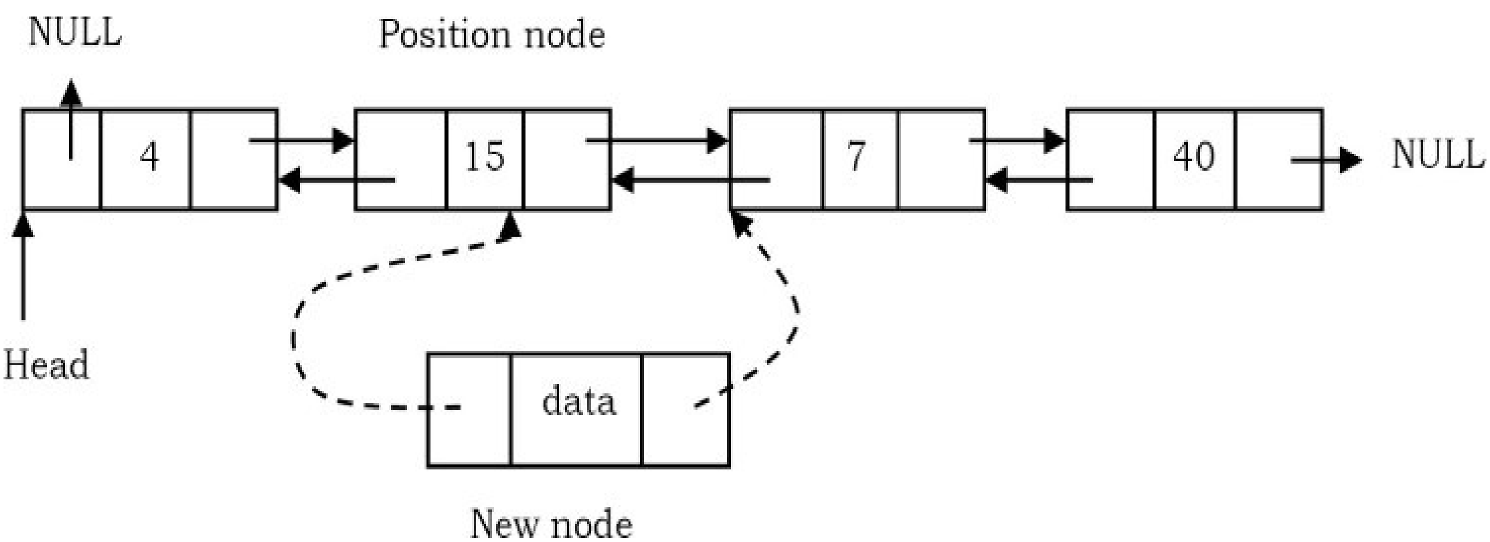
- Update right pointer of last node to point to new node.



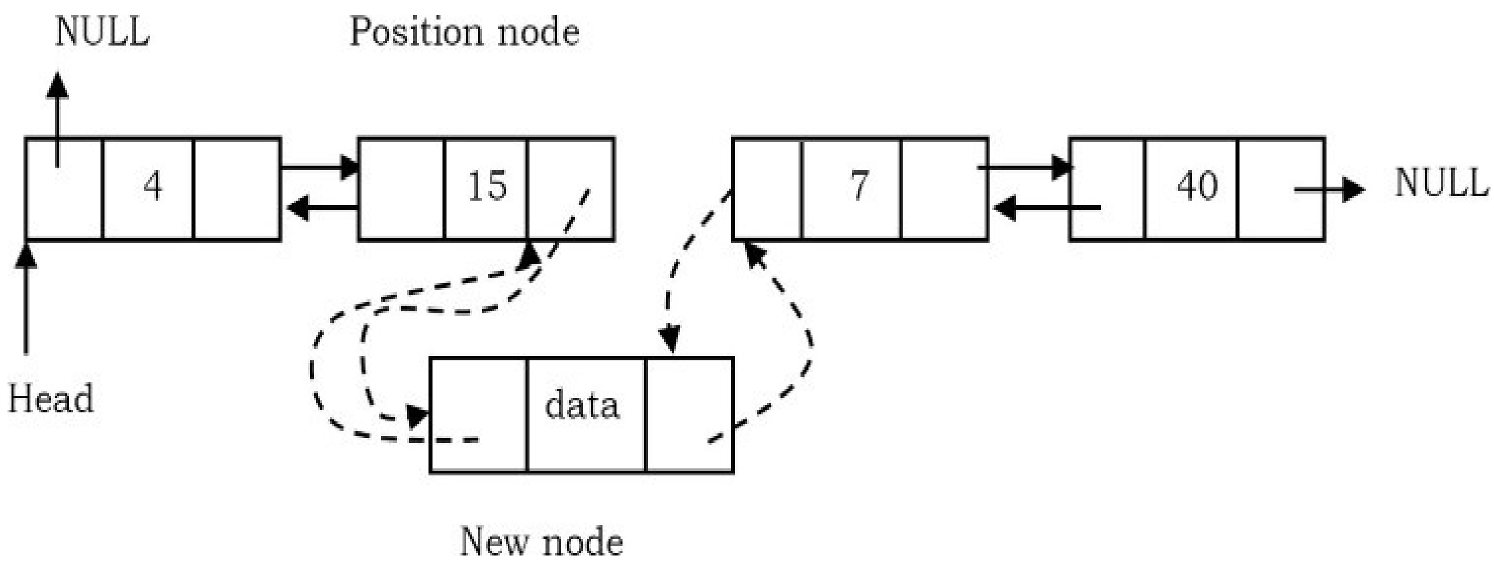
Inserting a Node in Doubly Linked List in the Middle

As discussed in singly linked lists, traverse the list to the position node and insert the new node.

- *New node* right pointer points to the next node of the *position node* where we want to insert the new node. Also, *new node* left pointer points to the *position node*.



- Position node right pointer points to the new node and the *next node* of position node left pointer points to new node.



Time Complexity: O(n). In the worst case, we may need to insert the node at the end of the list.
Space Complexity: O(1), for creating one temporary variable.

Doubly Linked List Deletion

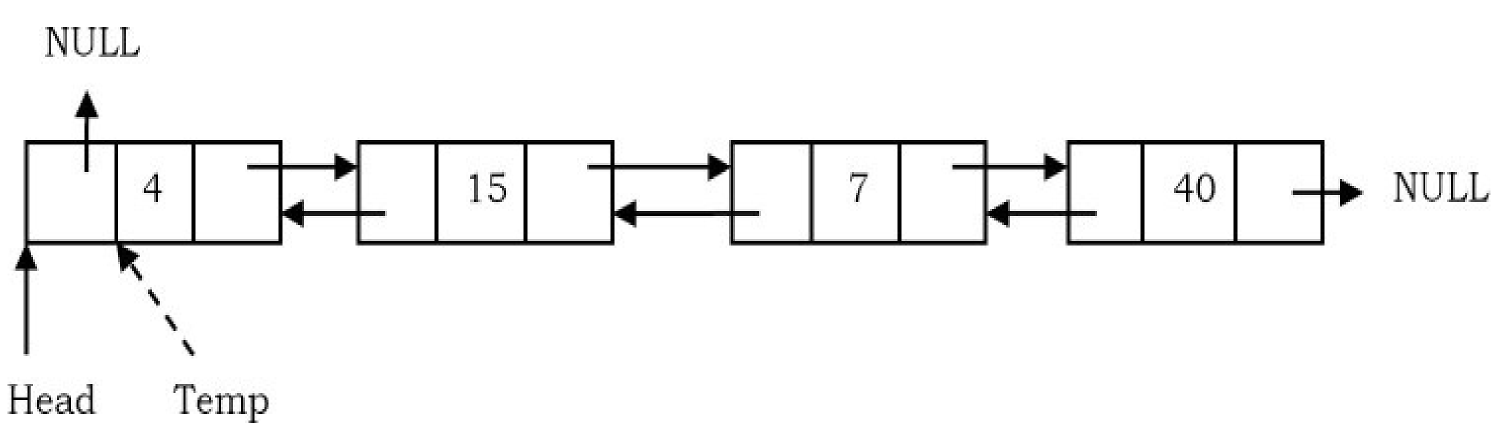
Similar to singly linked list deletion, here we have three cases:

- Deleting the first node
- Deleting the last node
- Deleting an intermediate node

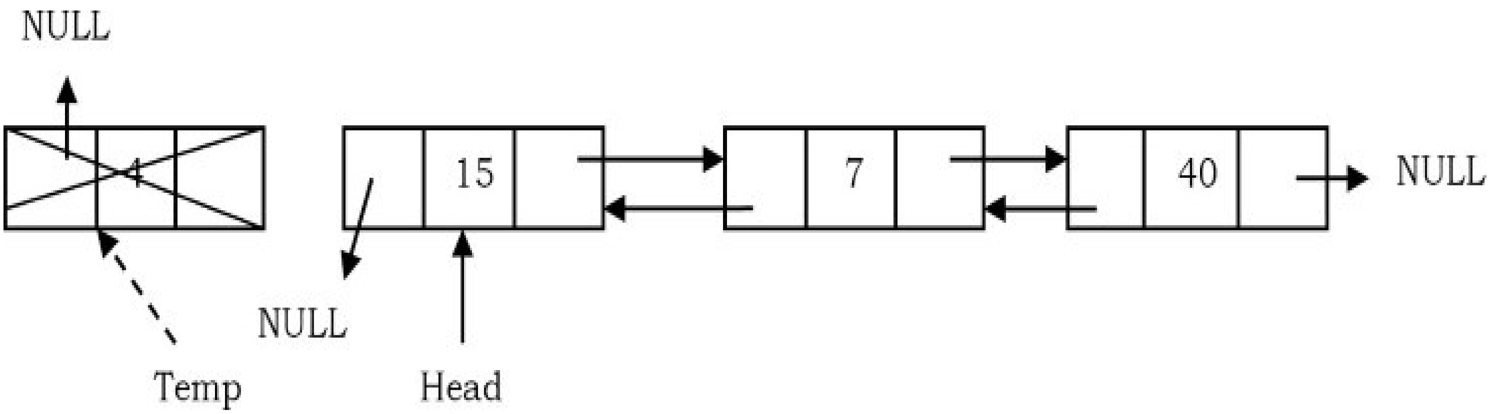
Deleting the First Node in Doubly Linked List

In this case, the first node (current head node) is removed from the list. It can be done in two steps:

- Create a temporary node which will point to the same node as that of head.



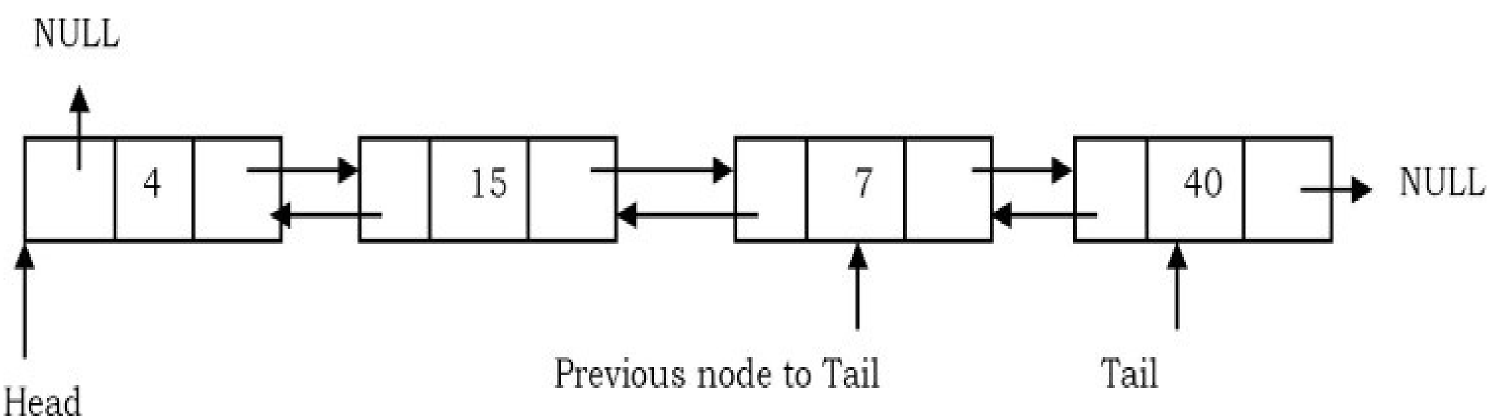
- Now, move the head nodes pointer to the next node and change the heads left pointer to NULL and dispose of the temporary node.



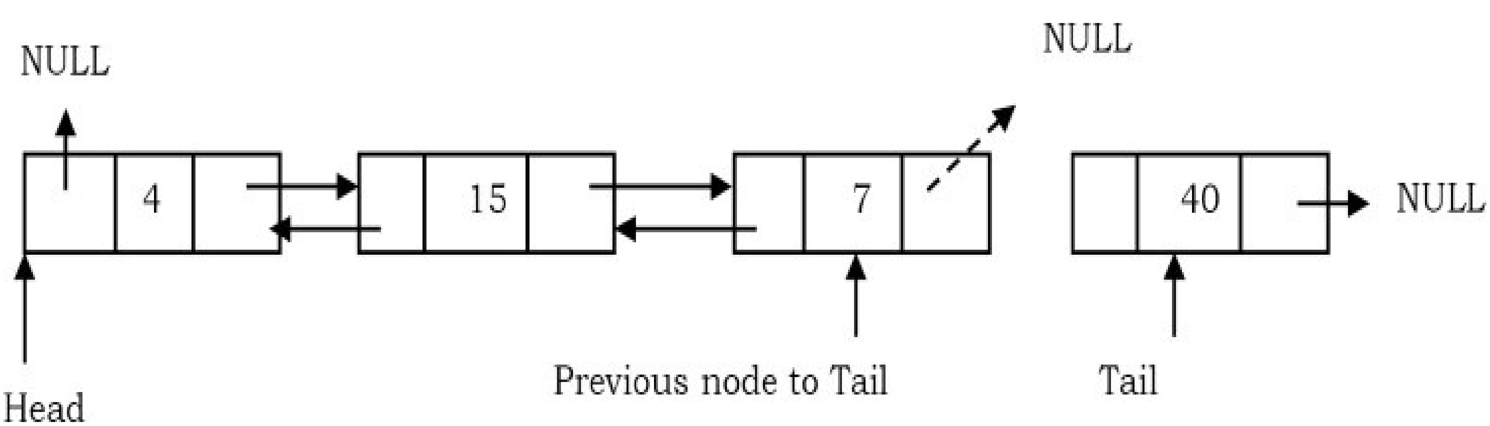
Deleting the Last Node in Doubly Linked List

This operation is a bit trickier than removing the first node, because the algorithm should find a node, which is previous to the tail first. This can be done in three steps:

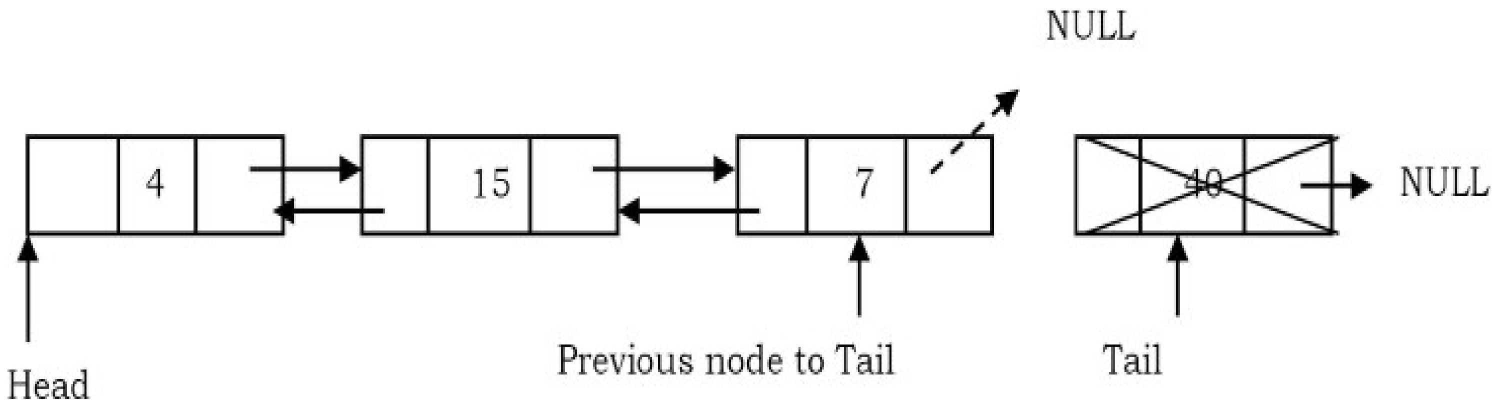
- Traverse the list and while traversing maintain the previous node address also. By the time we reach the end of the list, we will have two pointers, one pointing to the tail and the other pointing to the node before the tail.



- Update the next pointer of previous node to the tail node with NULL.



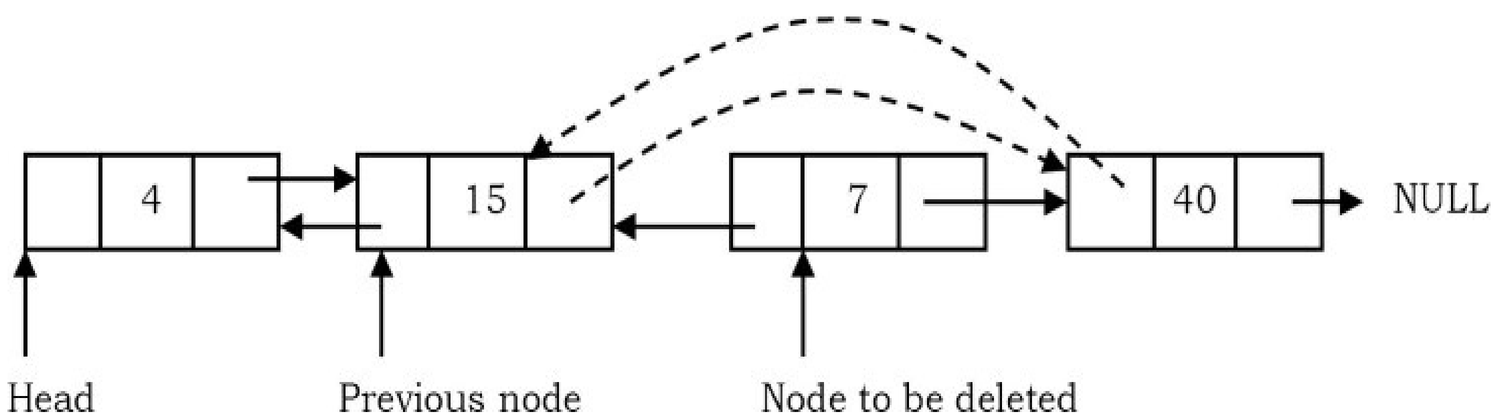
- Dispose of the tail node.



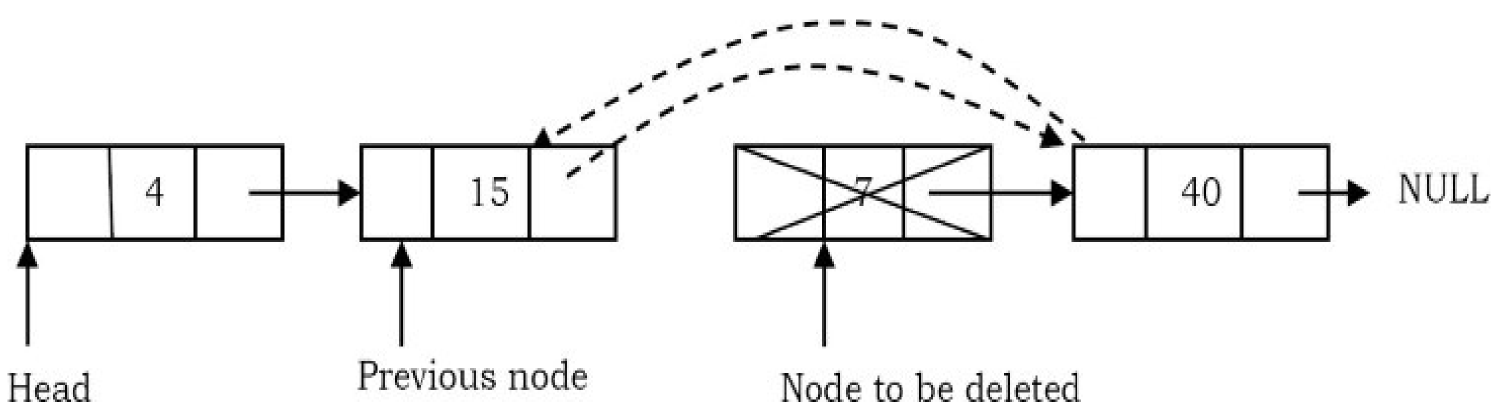
Deleting an Intermediate Node in Doubly Linked List

In this case, the node to be removed is *always located between* two nodes, and the head and tail links are not updated. The removal can be done in two steps:

- Similar to the previous case, maintain the previous node while also traversing the list. Upon locating the node to be deleted, change the previous node's next pointer to the next node of the node to be deleted.



- Dispose of the current node to be deleted.



Time Complexity: O(n), for scanning the complete list of size n.
Space Complexity: O(1), for creating one temporary variable.

Implementation

```
public class DoublyLinkedList{
    // properties
    private DLLNode head;
    private DLLNode tail;
    private int length;
    // Create a new empty list.
    public DoublyLinkedList() {
        head = new DLLNode(Integer.MIN_VALUE,null,null);
        tail = new DLLNode(Integer.MIN_VALUE, head, null);
        head.setNext(tail);
        length = 0;
    }
    // Get the value at a given position.
    public int get(int position) {
        return Integer.MIN_VALUE;
    }
    // Find the position of the head value that is equal to a given value.
    // The equals method us used to determine equality.
    public int getPosition(int data) {
        // go looking for the data
        DLLNode temp = head;
        int pos = 0;
        while (temp != null) {
            if (temp.getData() == data) {
                // return the position if found
                return pos;
            }
            pos += 1;
            temp = temp.getNext();
        }
        // else return some large value
        return Integer.MIN_VALUE;
    }
    // Return the current length of the DLL.
    public int length() {
        return length;
    }
    // Add a new value to the front of the list.
    public void insert(int newValue) {
```

```
		DLLNode newNode = new DLLNode(newValue, null, head.getNext());
		newNode.getNext().setPrev(newNode);
		head = newNode;
		length += 1;
	}
	// Add a new value to the list at a given position.
	// All values at that position to the end move over to make room.
	public void insert(int data, int position) {
		// fix the position
		if (position < 0) {
			position = 0;
		}
		if (position > length) {
			position = length;
		}
		// if the list is empty, make it be the only element
		if (head == null) {
			head = new DLLNode(data);
			tail = head;
		}
		// if adding at the front of the list...
		else if (position == 0) {
			DLLNode temp = new DLLNode(data);
			temp.next = head;
			head = temp;
		}
		// else find the correct position and insert
		else {
			DLLNode temp = head;
			for (int i=1; i<position; i+=1) {
				temp = temp.getNext();
			}
			DLLNode newNode = new DLLNode(data);
			newNode.next = temp.next;
			newNode.prev = temp;
			newNode.next.prev = newNode;
			temp.next = newNode;
		}
		// the list is now one value longer
		length += 1;
	}
	// Add a new value to the rear of the list.
	public void insertTail(int newValue) {
		DLLNode newNode = new DLLNode(newValue,tail.getPrev(),tail);
		newNode.getPrev().setNext(newNode);
		tail.setPrev(newNode);
		length += 1;
	}
	// Remove the value at a given position.
	// If the position is less than 0, remove the value at position 0.
	// If the position is greater than 0, remove the value at the last position.
	public void remove(int position) {
		// fix position
		if (position < 0) {
			position = 0;
		}
		if (position >= length) {
			position = length-1;
		}
		// if nothing in the list, do nothing
		if (head == null)
```

```
				return;
			// if removing the head element...
			if (position == 0) {
				head = head.getNext();
				if (head == null)
					tail = null;
			}
			// else advance to the correct position and remove
			else {
				DLLNode temp = head;
				for (int i=1; i<position; i+=1) {
					temp = temp.getNext();
				}
				temp.getNext().setPrev(temp.getPrev());
				temp.getPrev().setNext(temp.getNext());
			}
			// reduce the length of the list
			length -= 1;
	}
	// Remove a node matching the specified node from the list.
	// Use equals() instead of == to test for a matched node.
	public synchronized void removeMatched(DLLNode node) {
			if (head == null) return;
			if (node.equals(head)) {
				head = head.getNext();
				if (head == null)
				    tail = null;
				return;
			}
			DLLNode p = head;
			while(p != null) {
				if (node.equals(p)) {
					p.prev.next = p.next;
					p.next.prev = p.prev;
					return;
				}
			}
	}
	// Remove the head value from the list. If the list is empty, do nothing.
	public int removeHead() {
			if (length == 0)
			   return Integer.MIN_VALUE;
			DLLNode save = head.getNext();
			head.setNext(save.getNext());
			save.getNext().setPrev(head);
			length -= 1;
			return save.getData();
	}
	// Remove the tail value from the list. If the list is empty, do nothing.
	public int removeTail() {
			if (length == 0)
			    return Integer.MIN_VALUE;
			DLLNode save = tail.getPrev();
			tail.setPrev(save.getPrev());
			save.getPrev().setNext(tail);
			length -= 1;
			return save.getData();
	}
	// Return a string representation of this collection, in the form: ["str1","str2",...].
	public String toString() {
			String result = "[]";
			if (length == 0)
```

```
                return result;
            result = "[" + head.getNext().getData();
            DLLNode temp = head.getNext().getNext();
            while (temp != tail) {
                result += "," + temp.getData();
                temp =  temp.getNext();
            }
            return result + "]";
    }
    // Remove everything from the DLL.
    public void clearList(){
            head = null;
            tail = null;
            length = 0;
    }
}
```

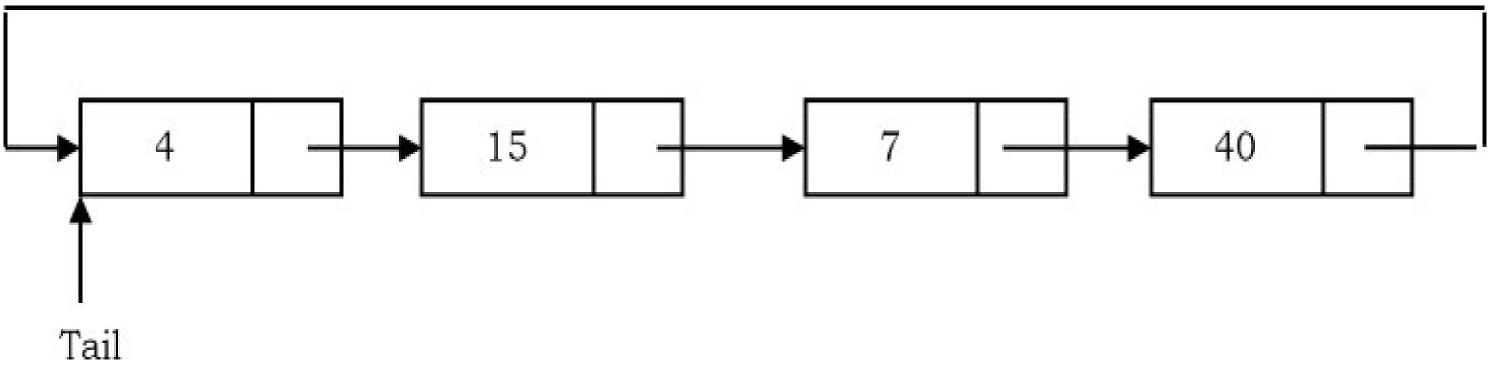
3.8 Circular Linked Lists

In singly linked lists and doubly linked lists, the end of lists are indicated with NULL value. But circular linked lists do not have ends. While traversing the circular linked lists we should be careful; otherwise we will be traversing the list infinitely. In circular linked lists, each node has a successor. Note that unlike singly linked lists, there is no node with NULL pointer in a circularly linked list. In some situations, circular linked lists are useful.

For example, when several processes are using the same computer resource (CPU) for the same amount of time, we have to assure that no process accesses the resource before all other processes do (round robin algorithm). In a circular linked list, we access the elements using the *head* node (similar to *head* node in singly linked list and doubly linked lists). For readability let us assume that the class name of circular linked list is CLLNode.

Counting nodes in a Circular Linked List

The circular list is accessible through the node marked *head*. (also called tail). To count the nodes, the list has to be traversed from the node marked *head*, with the help of a dummy node *current*, and stop the counting when *current* reaches the starting node *head*. If the list is empty, *head* will be NULL, and in that case set *count* = 0. Otherwise, set the current pointer to the first node, and keep on counting till the current pointer reaches the starting node.

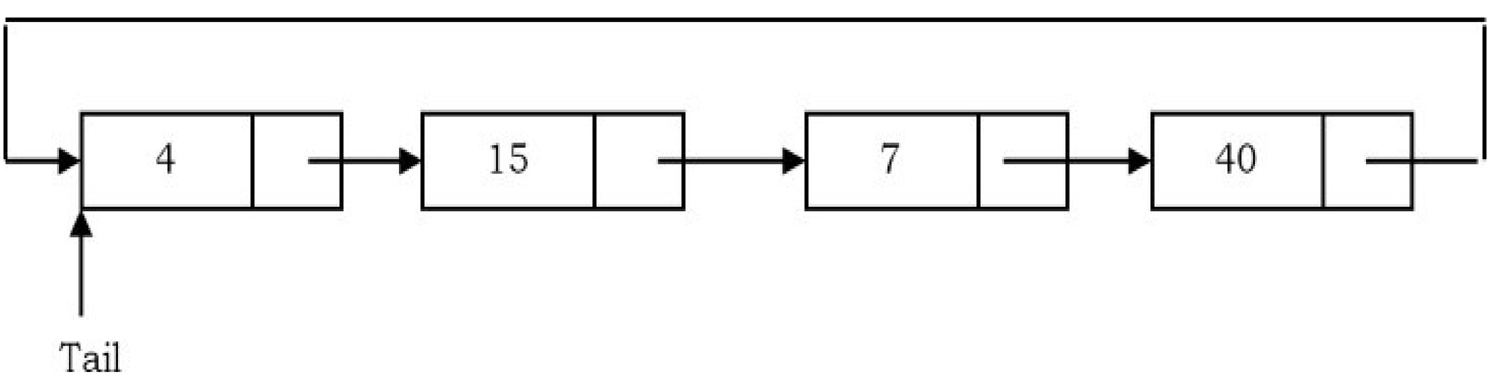



```
public int CircularListLength(CLLNode tail){
	int length = 0;
	CLLNode currentNode = tail.getNext();
	while(currentNode != tail){
		length++;
		currentNode = currentNode.getNext();
	}
	return length;
}
```

Time Complexity: O(n), for scanning the complete list of size n.
Space Complexity: O(1), for temporary variable.

Printing the contents of a circular list

We assume here that the list is being accessed by its *head* node. Since all the nodes are arranged in a circular fashion, the *tail* node of the list will be the node previous to the *head* node. Let us assume we want to print the contents of the nodes starting with the *head* node. Print its contents, move to the next node and continue printing till we reach the *head* node again.



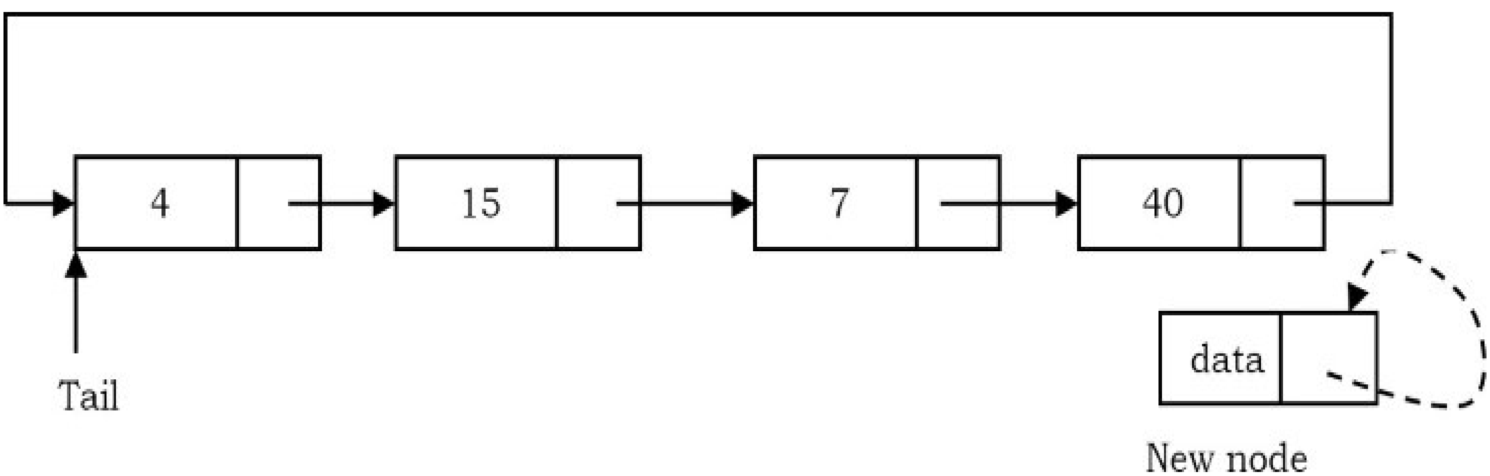
```
public void PrintCircularListData(CLLNode tail){
    CLLNode CLLNode = tail.getNext();
    while(CLLNode != tail){
        System.out.print(CLLNode.getData()+"->");
        CLLNode = CLLNode.getNext();
    }
    System.out.println("(" + CLLNode.getData() + ")headNode");
}
```

Time Complexity: O(*n*), for scanning the complete list of size *n*. Space Complexity: O(1), for temporary variable.

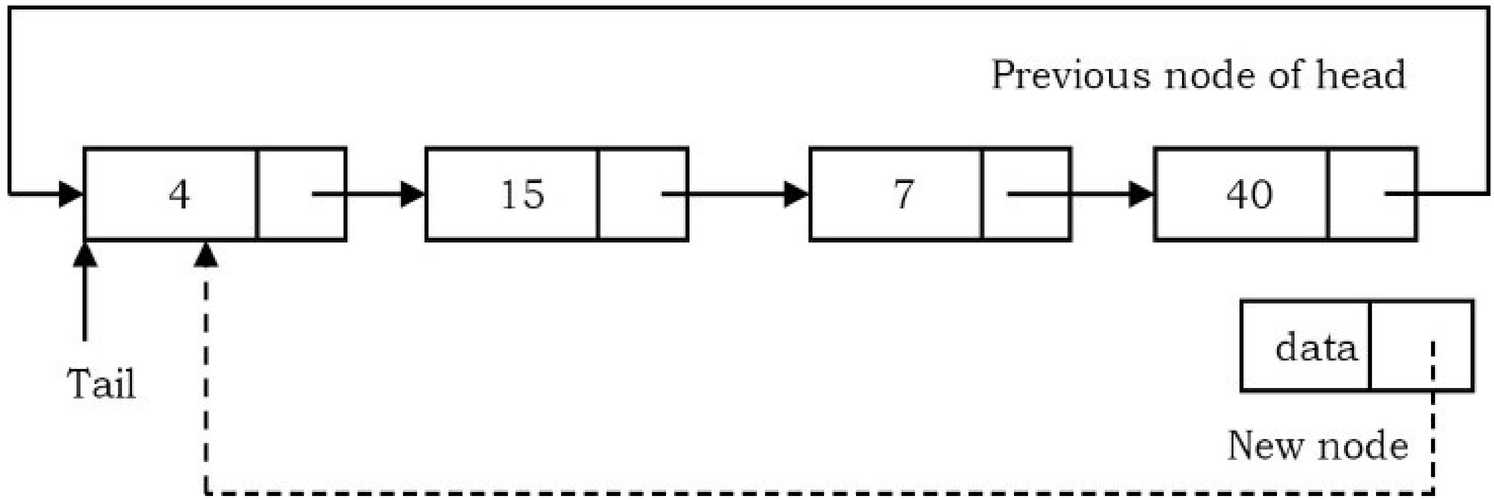
Inserting a Node at the End of a Circular Linked List

Let us add a node containing *data*, at the end of a list (circular list) headed by *head*. The new node will be placed just after the tail node (which is the last node of the list), which means it will have to be inserted in between the tail node and the first node.

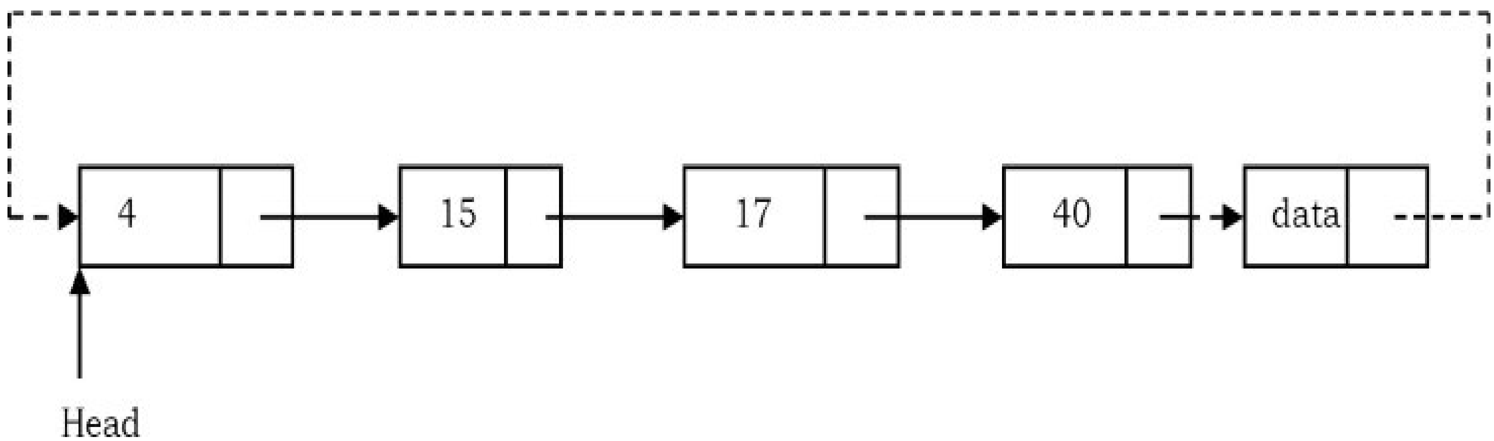
- Create a new node and initially keep its next pointer pointing to itself.



- Update the next pointer of the new node with the head node and also traverse the list to the tail. That means in a circular list we should stop at the node whose next node is head.



- Update the next pointer of the previous node to point to the new node and we get the list as shown below.

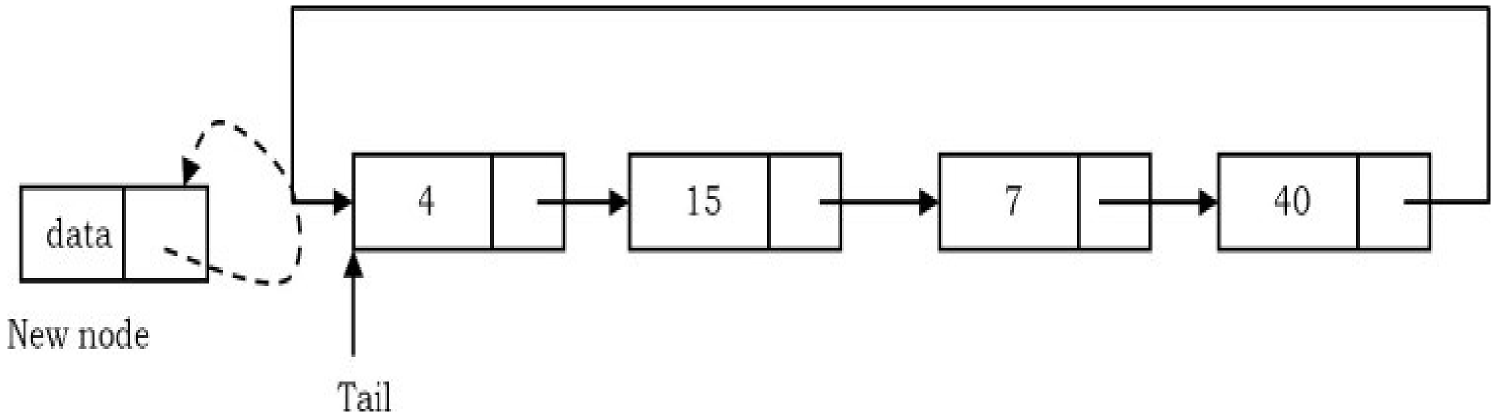


Time Complexity: O(n), for scanning the complete list of size n. Space Complexity: O(1), for temporary variable.

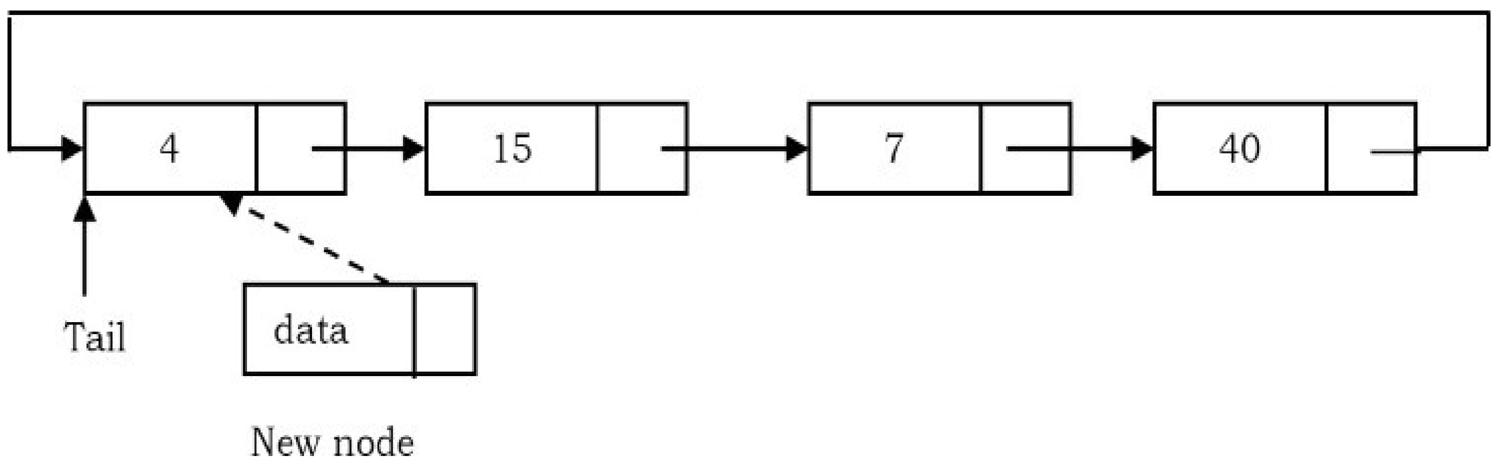
Inserting a Node at the front of a Circular Linked List

The only difference between inserting a node at the beginning and at the end is that, after inserting the new node, we just need to update the pointer. The steps for doing this are given below:

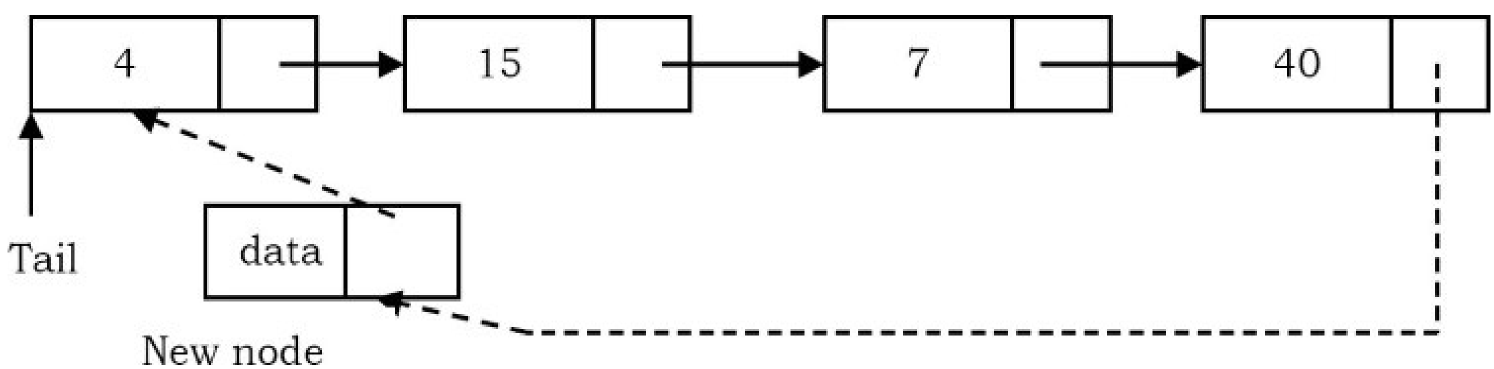
- Create a new node and initially keep its next pointer pointing to itself.



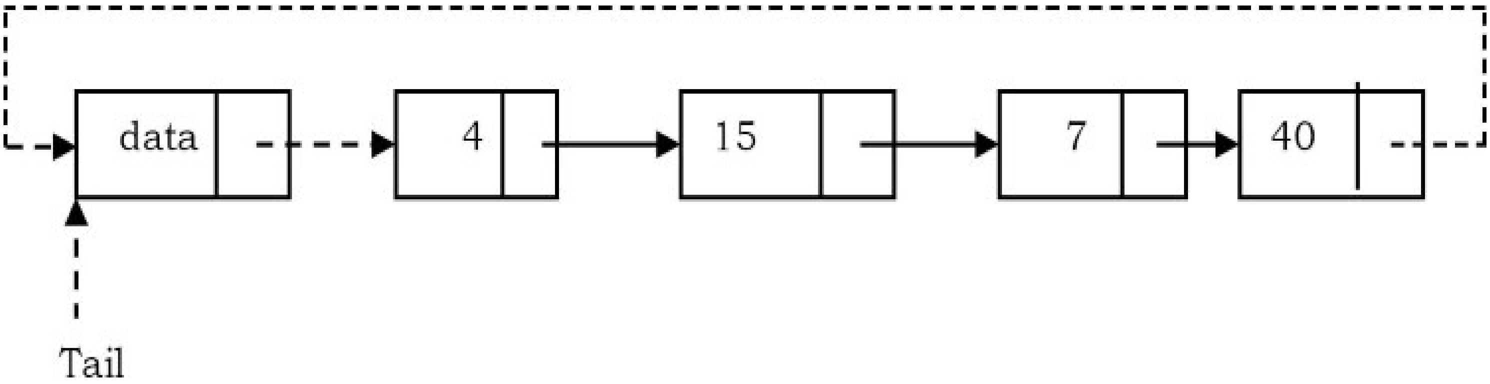
- Update the next pointer of the new node with the head node and also traverse the list until the tail. That means in a circular list we should stop at the node which is its previous node in the list.



- Update the previous head node in the list to point to the new node.



- Make the new node as the head.

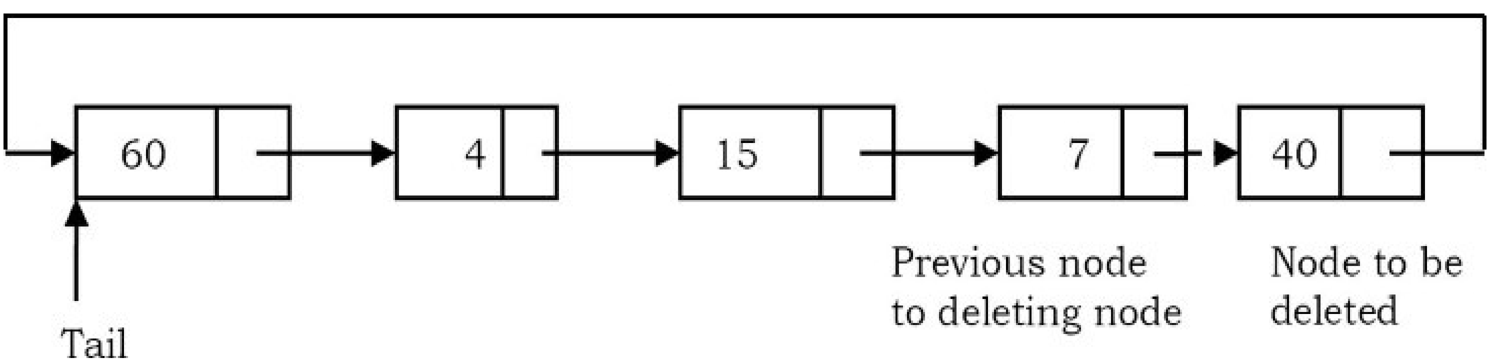


Time Complexity: O(n), for scanning the complete list of size n. Space Complexity: O(1), for temporary variable.

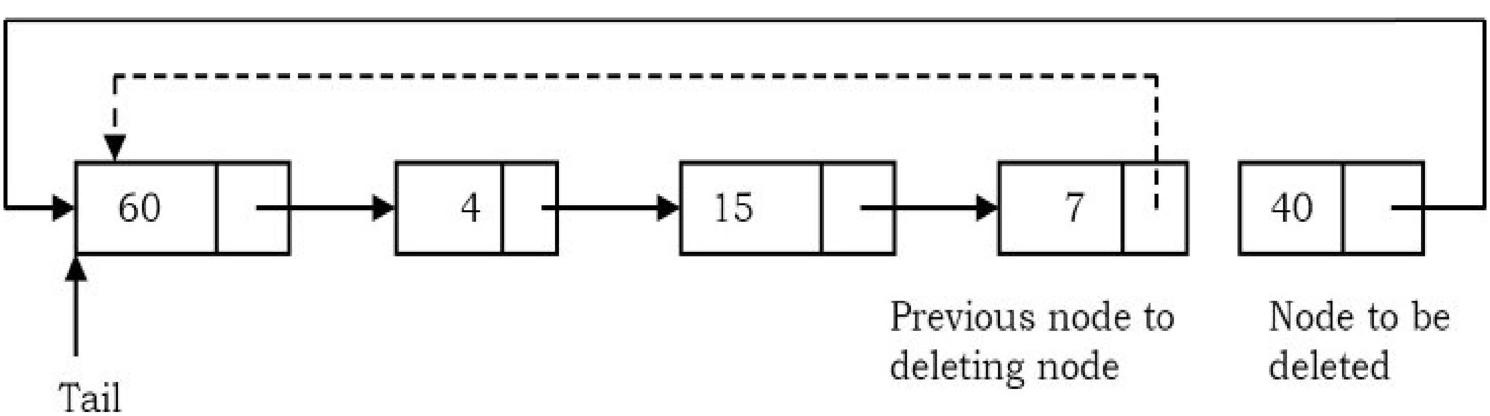
Deleting the last node in a Circular List

The list has to be traversed to reach the last but one node. This has to be named as the tail node, and its next field has to point to the first node. Consider the following list. To delete the last node 40, the list has to be traversed till you reach 7. The next field of 7 has to be changed to point to 60, and this node must be renamed *pTail*.

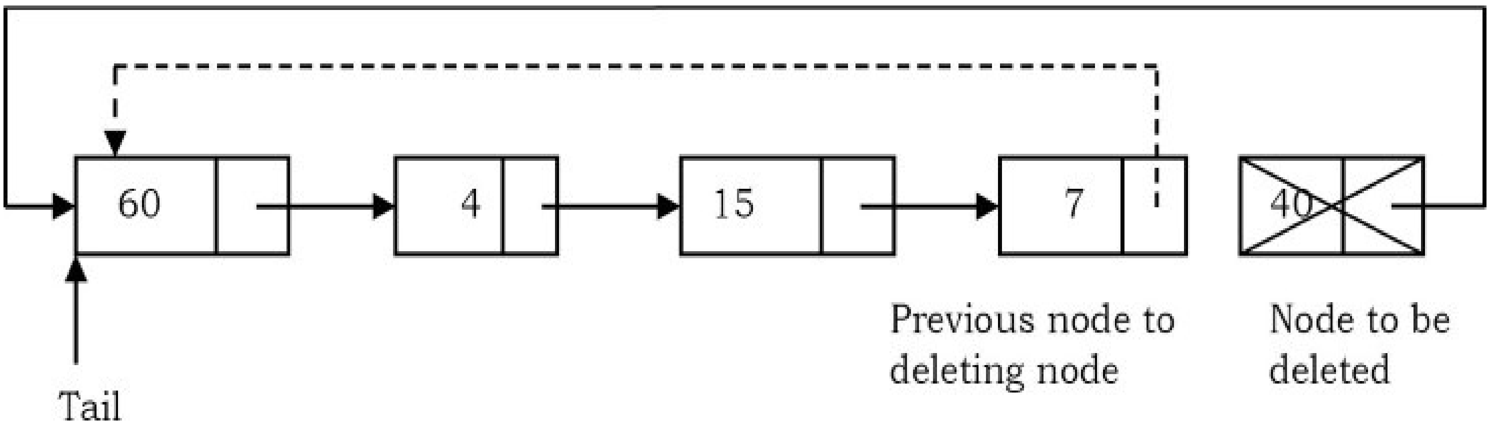
- Traverse the list and find the tail node and its previous node.



- Update the tail node's previous node pointer to point to head.



- Dispose of the tail node.

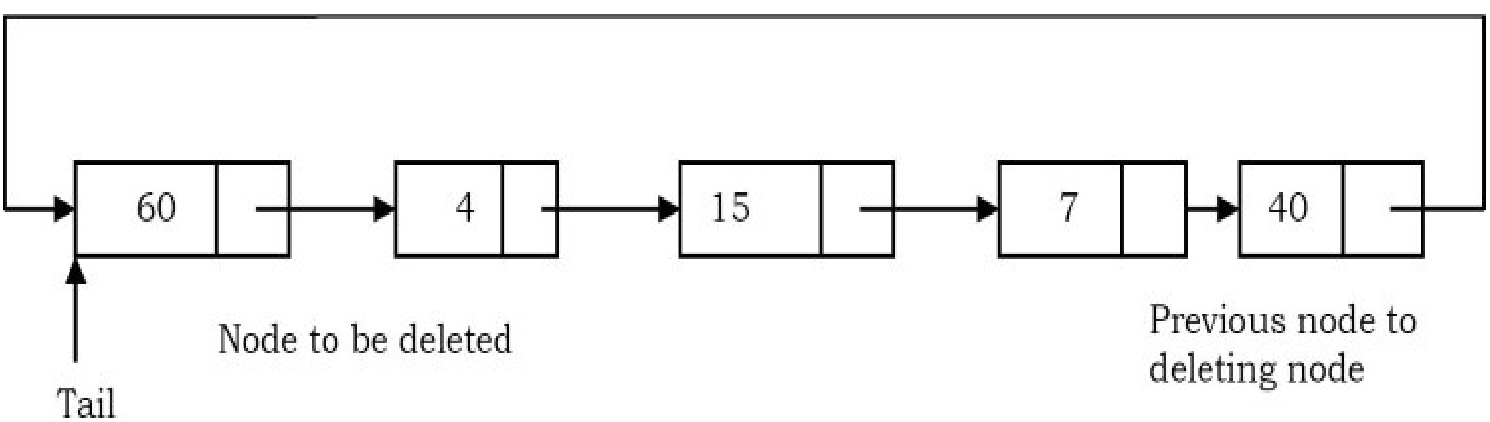


Time Complexity: O(n), for scanning the complete list of size n. Space Complexity: O(1), for a temporary variable.

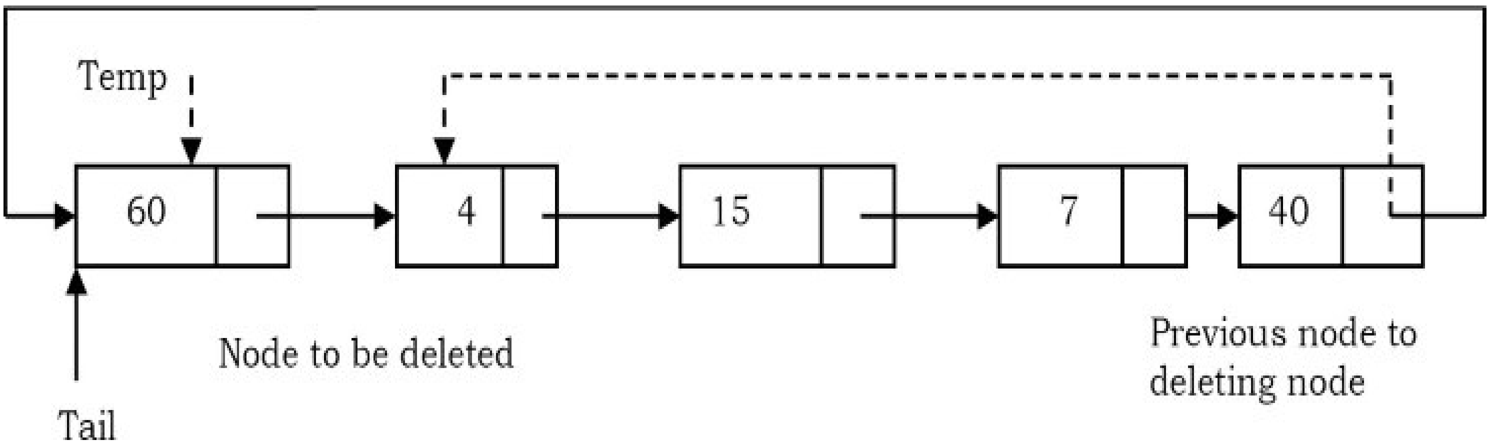
Deleting the First Node in a Circular List

The first node can be deleted by simply replacing the next field of the tail node with the next field of the first node.

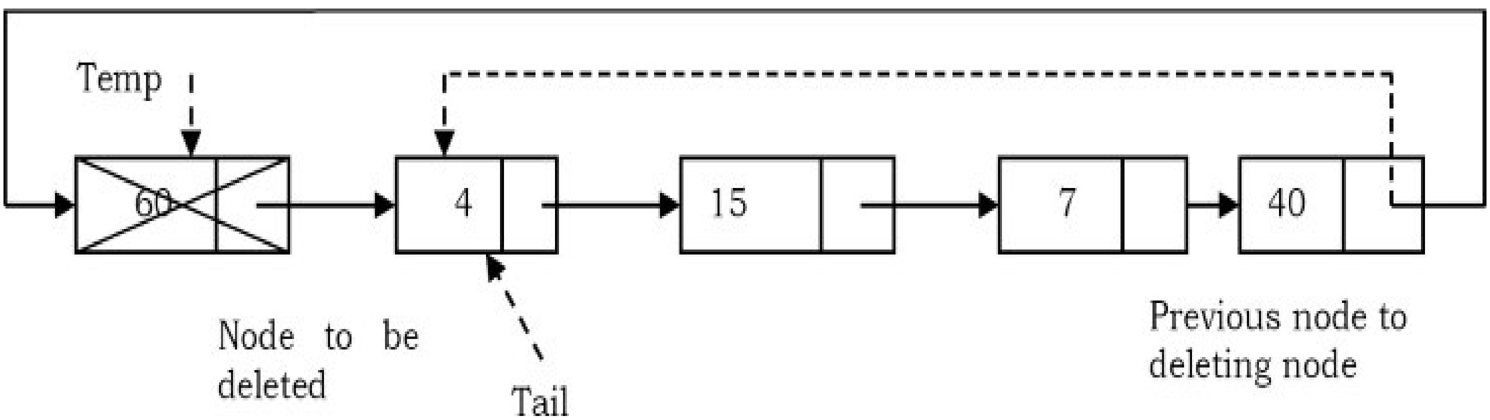
- Find the tail node of the linked list by traversing the list. Tail node is the previous node to the head node which we want to delete.



- Create a temporary node which will point to the head. Also, update the tail nodes next pointer to point to next node of head (as shown below).



- Now, move the head pointer to next node. Create a temporary node which will point to head. Also, update the tail nodes next pointer to point to next node of head (as shown below).



Time Complexity: O(n), for scanning the complete list of size n. Space Complexity: O(1), for a temporary variable.

Applications of Circular List

Circular linked lists are used in managing the computing resources of a computer. We can use circular lists for implementing stacks and queues.

Implementation

```java
public class CircularLinkedList{
	protected CLLNode tail;
	protected int length;
	// Constructs a new circular list
	public CircularLinkedList(){
		tail = null;
		length = 0;
	}
	// Adds data to beginning of list.
	public void add(int data){
		addToHead(data);
	}
	// Adds element to head of list
	public void addToHead(int data){
		CLLNode temp = new CLLNode(data);
		if (tail == null) { // first data added
		   tail = temp;
		   tail.setNext(tail);
		} else { // element exists in list
		   temp.setNext(tail.getNext());
		   tail.setNext(temp);
		}
		length++;
	}
	// Adds element to tail of list
	public void addToTail(int data){
		// new entry:
		addToHead(data);
		tail = tail.getNext();
	}
	// Returns data at head of list
	public int peek(){
		return tail.getNext().getData();
	}
	// Returns data at tail of list
	public int tailPeek(){
		return tail.getData();
	}
	// Returns and removes data from head of list
	public int removeFromHead(){
		CLLNode temp = tail.getNext(); // ie. the head of the list
		if (tail == tail.getNext()) {
			tail = null;
		} else {
			tail.setNext(temp.getNext());
			temp.setNext(null); // helps clean things up; temp is free
		}
		length--;
		return temp.getData();
	}
	// Returns and removes data from tail of list
	public int removeFromTail(){
		if (isEmpty()){
			return Integer.MIN_VALUE;
```

```
        }
        CLLNode finger = tail;
        while (finger.getNext() != tail) {
                finger = finger.getNext();
        }
        // finger now points to second-to-last data
        CLLNode temp = tail;
        if (finger == tail) {
                tail = null;
        } else {
                finger.setNext(tail.getNext());
                tail = finger;
        }
        length--;
        return temp.getData();
}
// Returns true if list contains data, else false
public boolean contains(int data){
        if (tail == null)
            return false;
        CLLNode finger;
        finger = tail.getNext();
        while (finger != tail && (!(finger.getData() == data))){
            finger = finger.getNext();
        }
        return finger.getData() == data;
}
// Removes and returns element equal to data, or null
public int remove(int data){
        if (tail == null) return Integer.MIN_VALUE;
        CLLNode finger = tail.getNext();
        CLLNode previous = tail;
        int compares;
        for (compares = 0; compares < length && (!(finger.getData() == data)); compares++) {
                previous = finger;
                finger = finger.getNext();
        }
        if (finger.getData() == data) {
                // an example of the pigeon-hole principle
                if (tail == tail.getNext()) {
                        tail = null; }
                else {
                        if (finger == tail)
                                tail = previous;
                        previous.setNext(previous.getNext().getNext());
                }
                // finger data free
                finger.setNext(null);        // to keep things disconnected
                length--;                    // fewer elements
                return finger.getData();
        }
        else return Integer.MIN_VALUE;
}
// Return the current length of the CLL.
public int size(){
        return length;
}
// Return the current length of the CLL.
public int length() {
        return length;
}
// Returns true if no elements in list
```

```
        public boolean isEmpty(){
                return tail == null;
        }
        // Remove everything from the CLL.
        public void clear(){
                length = 0;
                tail = null;
        }
        // Return a string representation of this collection, in the form: ["str1","str2",...].
        public String toString(){
                String result = "[";
                if (tail == null) {
                        return result+"]";
                }
                result = result + tail.getData();
                CLLNode temp = tail.getNext();
                while (temp != tail) {
                        result = result + "," + temp.getData();
                        temp = temp.getNext();
                }
                return result + "]";
        }
}
```

3.9 A Memory-efficient Doubly Linked List

In conventional implementation, we need to keep a forward pointer to the next item on the list and a backward pointer to the previous item. That means, elements in doubly linked list implementations consist of data, a pointer to the next node and a pointer to the previous node in the list as shown below.

Conventional Node Definition

```
public class DLLNode {
        private int data;
        private DLLNode next;
        private DLLNode previous;
        .........
}
```

Recently a journal (Sinha) presented an alternative implementation of the doubly linked list ADT, with insertion, traversal and deletion operations. This implementation is based on pointer difference. Each node uses only one pointer field to traverse the list back and forth.

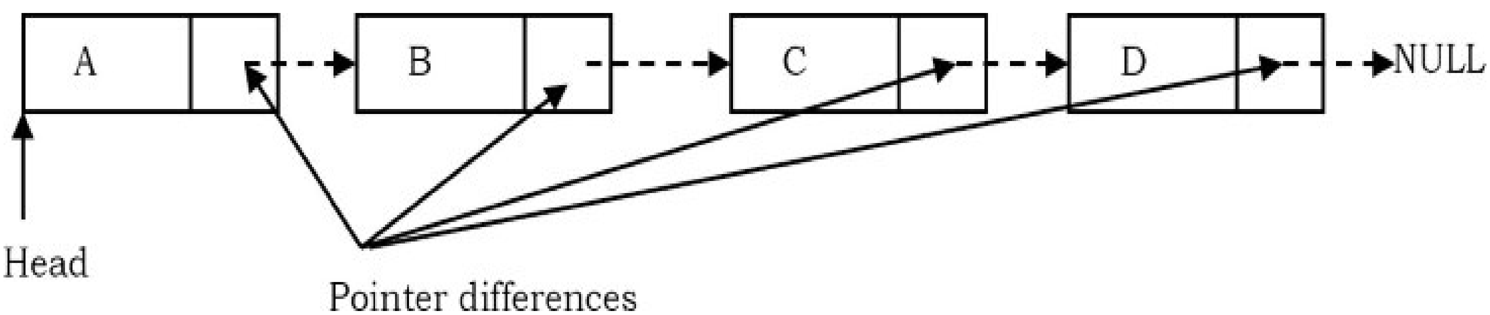
New Node Definition

```
public class ListNode {
    private int data;
    private ListNode ptrdiff;
    .........
}
```

The *ptrdiff* pointer field contains the difference between the pointer to the next node and the pointer to the previous node. The pointer difference is calculated by using exclusive-or (⊕) operation.

$$ptrdiff = pointer\ to\ previous\ node \oplus pointer\ to\ next\ node.$$

The *ptrdiff* of the start node (head node) is the ⊕ of NULL and *next* node (next node to head). Similarly, the *ptrdiff* of end node is the ⊕ of *previous* node (previous to end node) and NULL. As an example, consider the following linked list.



In the example above,

- The next pointer of A is: NULL ⊕ B
- The next pointer of B is: A ⊕ C
- The next pointer of C is: B ⊕ D
- The next pointer of D is: C ⊕ NULL

Why does it work?

To find the answer to this question let us consider the properties of ⊕:

$X \oplus X = 0$
$X \oplus 0 = X$
$X \oplus Y = Y \oplus X$ (symmetric)
$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$ (transitive)

For the example above, let us assume that we are at C node and want to move to B. We know that C's *ptrdiff* is defined as B ⊕ D. If we want to move to B, performing ⊕ on C's *ptrdiff* with D would give B. This is due to the fact that,

$$(B \oplus D) \oplus D = B \text{ (since, } D \oplus D = 0)$$

Similarly, if we want to move to D, then we have to apply ⊕ to C's *ptrdiff* with B to give D.

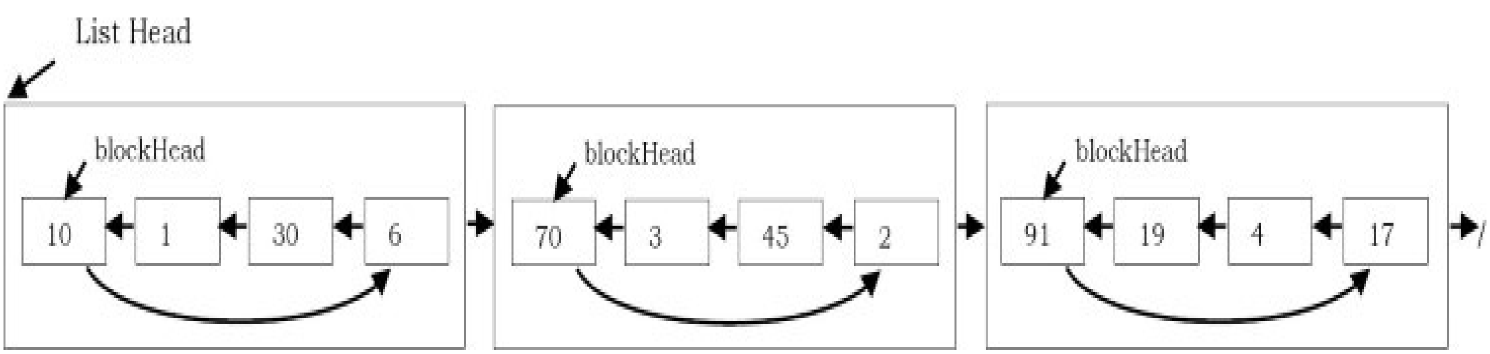
$$(B \oplus D) \oplus B = D \text{ (since, } B \oplus B = 0)$$

From the above discussion we can see that just by using a single pointer, we can move back and forth. A memory-efficient implementation of a doubly linked list is possible with minimal compromising of timing efficiency.

3.10 Unrolled Linked Lists

One of the biggest advantages of linked lists over arrays is that inserting an element at any location takes only O(1) time. However, it takes O(n) to search for an element in a linked list. There is a simple variation of the singly linked list called *unrolled linked lists*.

An unrolled linked list stores multiple elements in each node (let us call it a block for our convenience). In each block, a circular linked list is used to connect all nodes.



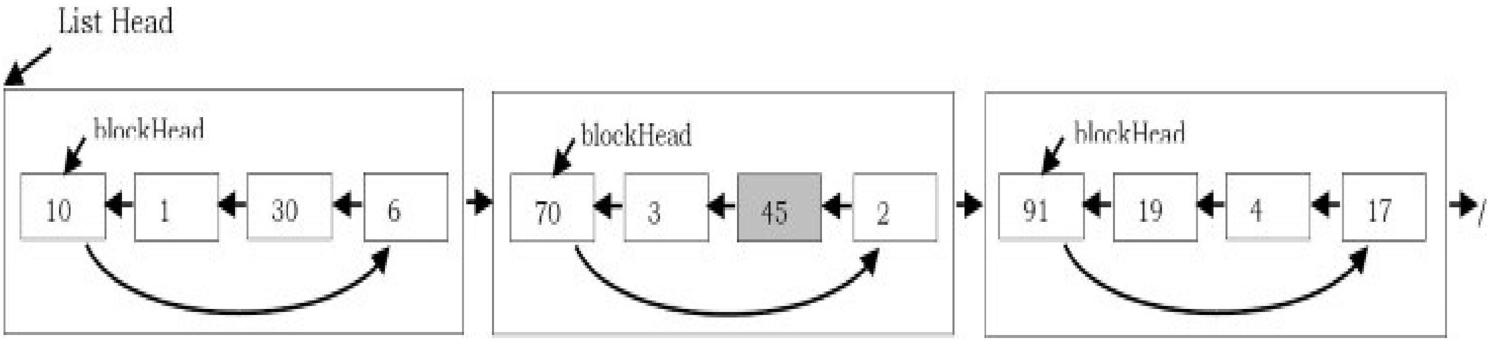
Assume that there will be no more than n elements in the unrolled linked list at any time. To simplify this problem, all blocks, except the last one, should contain exactly $\lceil \sqrt{n} \rceil$ elements. Thus, there will be no more than $\lfloor \sqrt{n} \rfloor$ blocks at any time.

Searching for an element in Unrolled Linked Lists

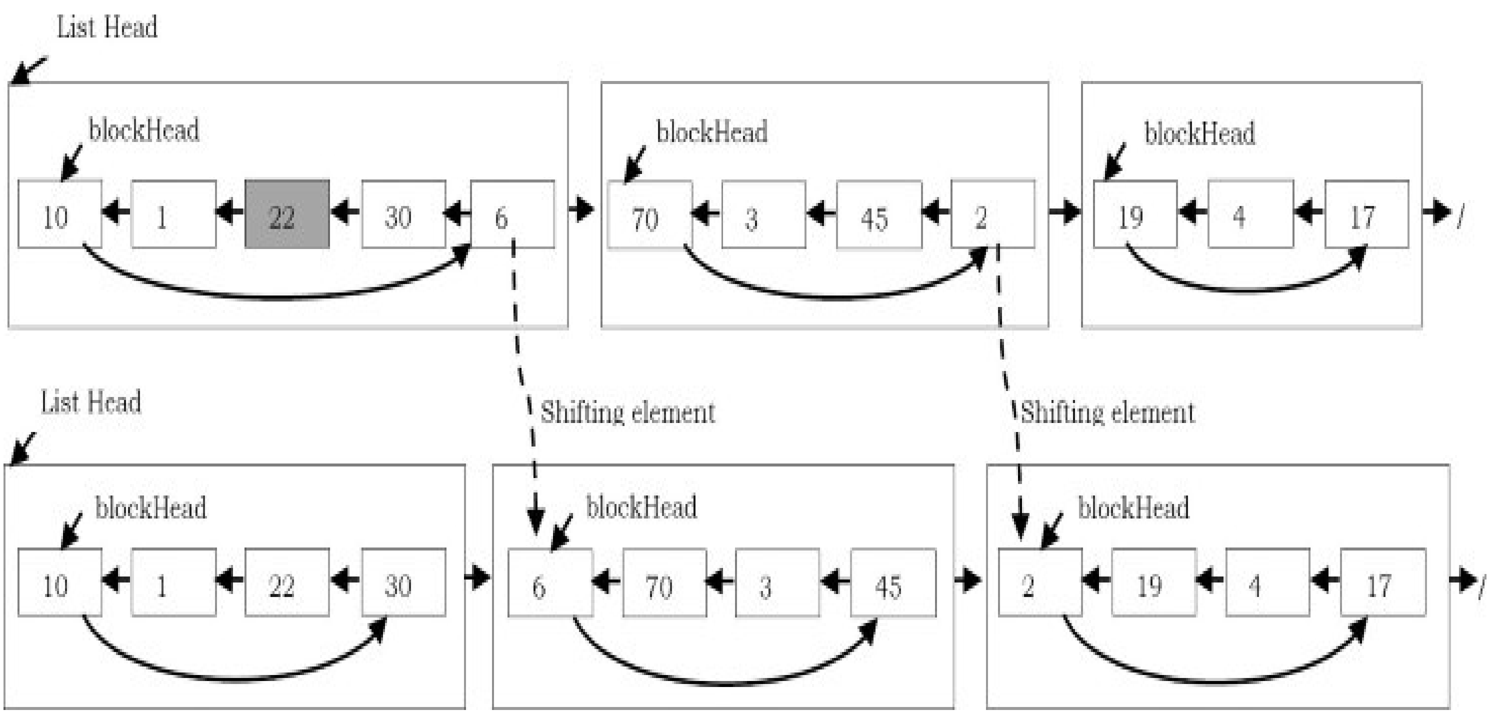
In unrolled linked lists, we can find the k^{th} element in O($\sqrt{n}$):

1. Traverse the *list of blocks* to the one that contains the k^{th} node, i.e., the $\left\lceil \frac{k}{\lceil \sqrt{n} \rceil} \right\rceil^{\text{th}}$ block. It takes O($\sqrt{n}$) since we may find it by going through no more than $\sqrt{n}$ blocks.

2. Find the (k mod $\lceil\sqrt{n}\rceil$)th node in the circular linked list of this block. It also takes O($\sqrt{n}$) since there are no more than $\lceil\sqrt{n}\rceil$ nodes in a single block.



Inserting an element in Unrolled Linked Lists

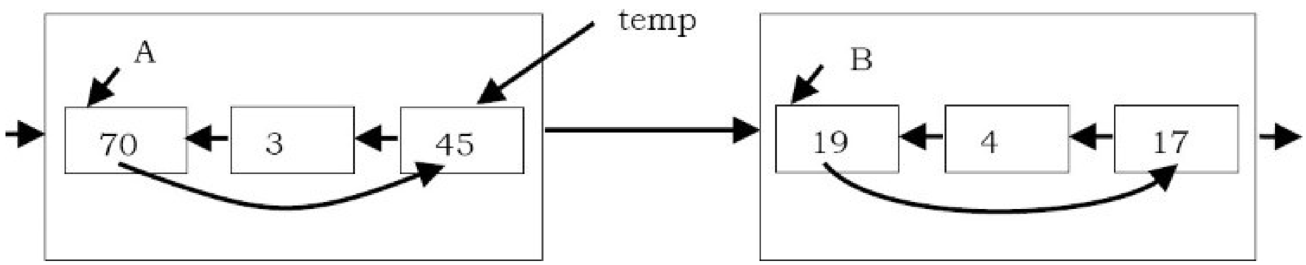


When inserting a node, we have to re-arrange the nodes in the unrolled linked list to maintain the properties previously mentioned, that each block contains $\lceil\sqrt{n}\rceil$ nodes. Suppose that we insert a node x after the i^{th} node, and x should be placed in the j^{th} block. Nodes in the j^{th} block and in the blocks after the j^{th} block have to be shifted toward the tail of the list so that each of them still have $\lceil\sqrt{n}\rceil$ nodes. In addition, a new block needs to be added to the tail if the last block of the list is out of space, i.e., it has more than $\lceil\sqrt{n}\rceil$ nodes.

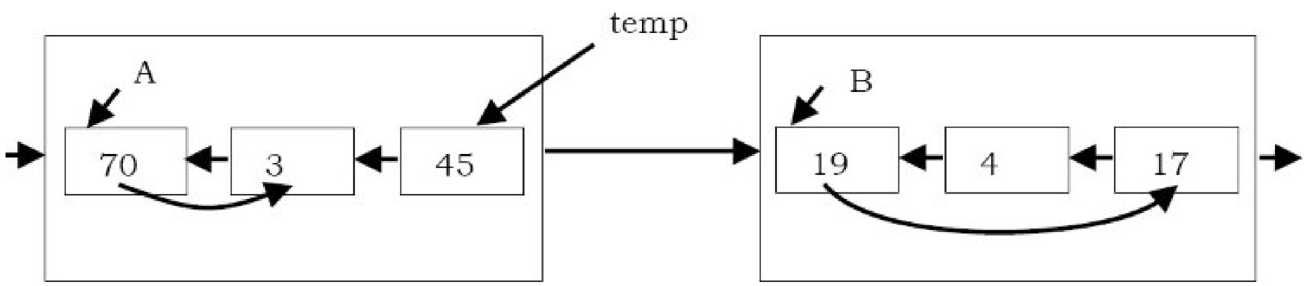
Performing Shift Operation

Note that each *shift* operation, which includes removing a node from the tail of the circular linked list in a block and inserting a node to the head of the circular linked list in the block after, takes only O(1). The total time complexity of an insertion operation for unrolled linked lists is therefore O($\sqrt{n}$); there are at most O($\sqrt{n}$) blocks and therefore at most O($\sqrt{n}$) shift operations.

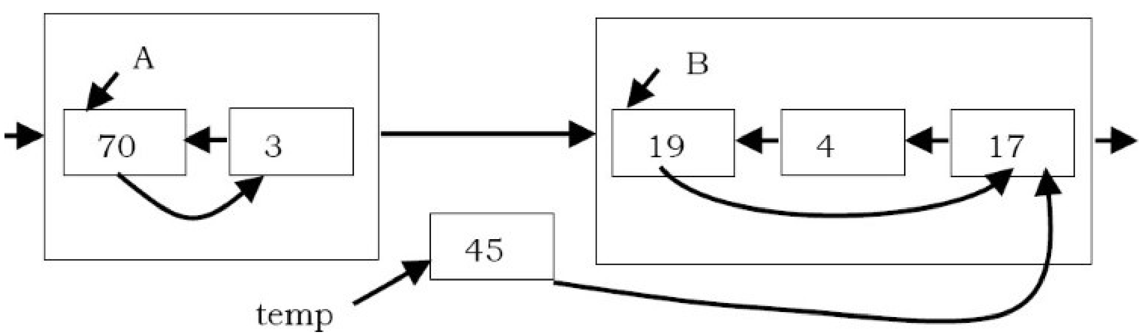
1. A temporary pointer is needed to store the tail of *A*.



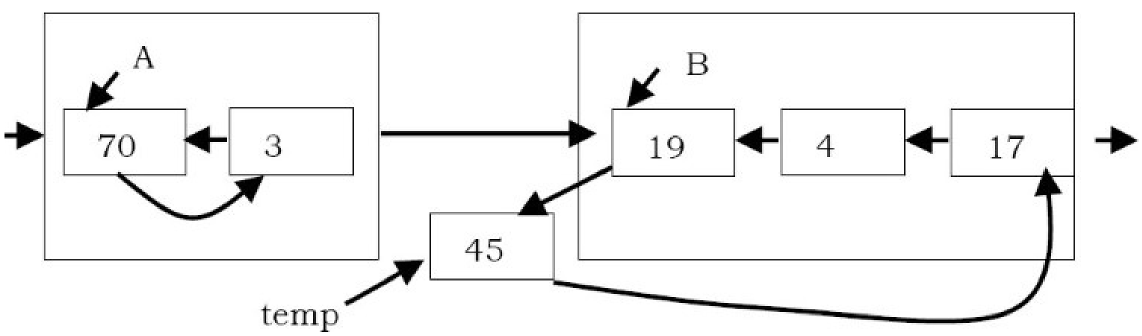
2. In block *A*, move the next pointer of the head node to point to the second-to-last node, so that the tail node of *A* can be removed.



3. Let the next pointer of the node, which will be shifted (the tail node of *A*), point to the tail node of *B*.

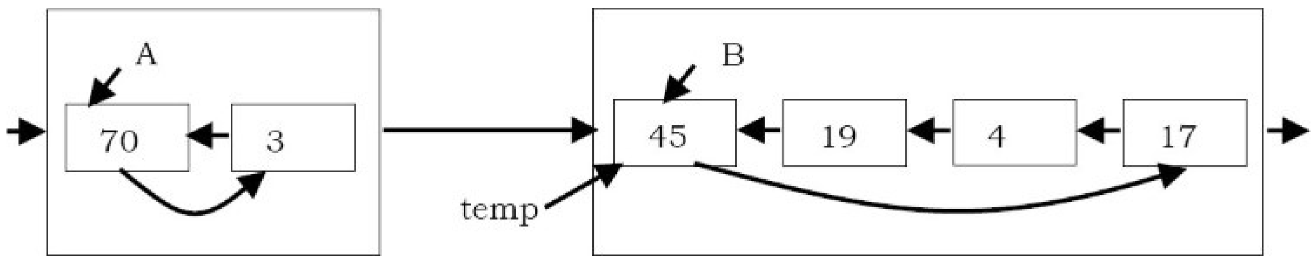


4. Let the next pointer of the head node of *B* point to the node temp points to.

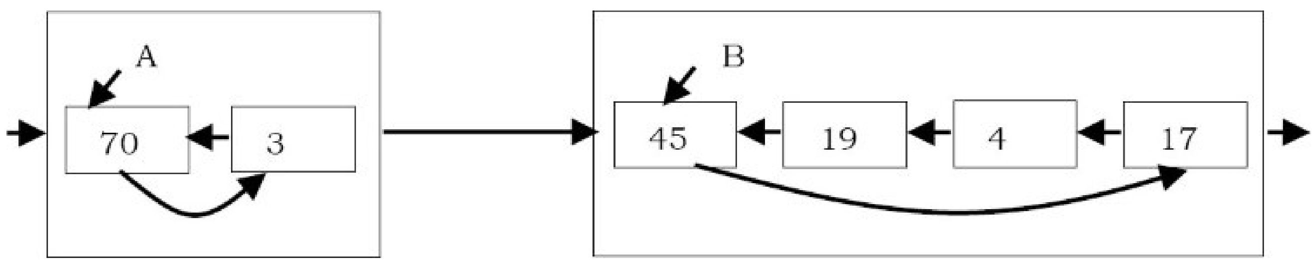


5. Finally, set the head pointer of *B* to point to the node *temp* points to. Now the node temp

points to becomes the new head node of *B*.



6. *temp* pointer can be thrown away. We have completed the shift operation to move the original tail node of *A* to become the new head node of *B*.



Performance

With unrolled linked lists, there are a couple of advantages, one in speed and one in space.

First, if the number of elements in each block is appropriately sized (e.g., at most the size of one cache line), we get noticeably better cache performance from the improved memory locality.

Second, since we have O(n/m) links, where n is the number of elements in the unrolled linked list and m is the number of elements we can store in any block, we can also save an appreciable amount of space, which is particularly noticeable if each element is small.

Comparing Doubly Linked Lists and Unrolled Linked Lists

To compare the overhead for an unrolled list, elements in doubly linked list implementations consist of data, a pointer to the next node, and a pointer to the previous node in the list, as shown below.

Assuming we have 4 byte pointers, each node is going to take 8 bytes. But the allocation overhead for the node could be anywhere between 8 and 16 bytes. Let's go with the best case and assume it will be 8 bytes. So, if we want to store 1K items in this list, we are going to have 16KB of overhead.

Now, let's think about an unrolled linked list node (let us call it *LinkedBlock*). It will look something like this:

Therefore, allocating a single node (12 bytes + 8 bytes of overhead) with an array of 100 elements (400 bytes + 8 bytes of overhead) will now cost 428 bytes, or 4.28 bytes per element. Thinking about our 1K items from above, it would take about 4.2KB of overhead, which is close to 4x better than our original list. Even if the list becomes severely fragmented and the item arrays are only 1/2 full on average, this is still an improvement. Also, note that we can tune the array size to whatever gets us the best overhead for our application.

Implementation

```
public class UnrolledLinkedList<E> extends AbstractList<E> implements List<E>, Serializable {
    //The maximum number of elements that can be stored in a single node.
    private int nodeCapacity;
    //The current size of this list.
    private int size = 0;
    //The first node of this list.
    private ListNode firstNode;
    //The last node of this list.
    private ListNode lastNode;
    //Constructs an empty list with the specified capacity
    public UnrolledLinkedList(int nodeCapacity) throws IllegalArgumentException {
        if (nodeCapacity < 8) {
            throw new IllegalArgumentException("nodeCapacity < 8");
        }
        this.nodeCapacity = nodeCapacity;
        firstNode = new ListNode();
        lastNode = firstNode;
    }
    public UnrolledLinkedList() {
        this(16);
    }
    public int size() {
```

```java
    return size;
}
public boolean isEmpty() {
    return (size == 0);
}
//Returns true if this list contains the specified element.
public boolean contains(Object o) {
    return (indexOf(o) != -1);
}
public Iterator<E> iterator() {
    return new ULLIterator(firstNode, 0, 0);
}
//Appends the specified element to the end of this list.
public boolean add(E e) {
    insertIntoNode(lastNode, lastNode.numElements, e);
    return true;
}
//Removes the first occurrence of the specified element from this list,
public boolean remove(Object o) {
    int index = 0;
    ListNode node = firstNode;
    if (o == null) {
        while (node != null) {
            for (int ptr = 0; ptr < node.numElements; ptr++) {
                if (node.elements[ptr] == null) {
                    removeFromNode(node, ptr);
                    return true;
                }
            }
            index += node.numElements;
            node = node.next;
        }
    } else {
        while (node != null) {
            for (int ptr = 0; ptr < node.numElements; ptr++) {
                if (o.equals(node.elements[ptr])) {
                    removeFromNode(node, ptr);
                    return true;
                }
            }
            index += node.numElements;
            node = node.next;
        }
    }
    return false;
}
//Removes all of the elements from this list.
public void clear() {
    ListNode node = firstNode.next;
    while (node != null) {
        ListNode next = node.next;
        node.next = null;
        node.previous = null;
        node.elements = null;
        node = next;
    }
    lastNode = firstNode;
    for (int ptr = 0; ptr < firstNode.numElements; ptr++) {
        firstNode.elements[ptr] = null;
    }
    firstNode.numElements = 0;
    firstNode.next = null;
    size = 0;
}
```

```
//Returns the element at the specified position in this list.
public E get(int index) throws IndexOutOfBoundsException {
    if (index < 0 || index >= size) {
        throw new IndexOutOfBoundsException();
    }
    ListNode node;
    int p = 0;
    if (size - index > index) {
        node = firstNode;
        while (p <= index - node.numElements) {
            p += node.numElements;
            node = node.next;
        }
    } else {
        node = lastNode;
        p = size;
        while ((p -= node.numElements) > index) {
            node = node.previous;
        }
    }
    return (E) node.elements[index - p];
}
//Replaces the element at the specified position in this list with the specified element.
public E set(int index, E element) {
    if (index < 0 || index >= size) {
        throw new IndexOutOfBoundsException();
    }
    E el = null;
    ListNode node;
    int p = 0;
    if (size - index > index) {
        node = firstNode;
        while (p <= index - node.numElements) {
            p += node.numElements;
            node = node.next;
        }
    } else {
        node = lastNode;
        p = size;
        while ((p -= node.numElements) > index) {
            node = node.previous;
        }
    }
    el = (E) node.elements[index - p];
    node.elements[index - p] = element;
    return el;

}
//Inserts the specified element at the specified position in this list.
//Shifts the element currently at that position (if any) and any
//subsequent elements to the right (adds one to their indices).
public void add(int index, E element) throws IndexOutOfBoundsException {
    if (index < 0 || index > size) {
        throw new IndexOutOfBoundsException();
    }
    ListNode node;
    int p = 0;
    if (size - index > index) {
        node = firstNode;
        while (p <= index - node.numElements) {
            p += node.numElements;
            node = node.next;
        }
    } else {
```

```
            node = lastNode;
            p = size;
            while ((p -= node.numElements) > index) {
                node = node.previous;
            }
        }
        insertIntoNode(node, index - p, element);
    }
    //Removes the element at the specified position in this list.
    //Shifts any subsequent elements to the left (subtracts one from their indices).
    public E remove(int index) throws IndexOutOfBoundsException {
        if (index < 0 || index >= size) {
            throw new IndexOutOfBoundsException();
        }
        E element = null;
        ListNode node;
        int p = 0;
        if (size - index > index) {
            node = firstNode;
            while (p <= index - node.numElements) {
                p += node.numElements;
                node = node.next;
            }
        } else {
            node = lastNode;
            p = size;
            while ((p -= node.numElements) > index) {
                node = node.previous;
            }
        }
        element = (E) node.elements[index - p];
        removeFromNode(node, index - p);
        return element;
    }
    private static final long serialVersionUID = -674052309103045211L;
    private class ListNode {
        ListNode next;
        ListNode previous;
        int numElements = 0;
        Object[] elements;
        ListNode() {
            elements = new Object[nodeCapacity];
        }
    }
    private class ULLIterator implements ListIterator<E> {
        ListNode currentNode;
        int ptr;
        int index;
        private int expectedModCount = modCount;
        ULLIterator(ListNode node, int ptr, int index) {
            this.currentNode = node;
            this.ptr = ptr;
            this.index = index;
        }
        public boolean hasNext() {
            return (index < size - 1)
        }
        public E next() {
            ptr++;
            if (ptr >= currentNode.numElements) {
                if (currentNode.next != null) {
                    currentNode = currentNode.next;
                    ptr = 0;
                } else {
```

```
                throw new NoSuchElementException();
            }
        }
        index++;
        checkForModification();
        return (E) currentNode.elements[ptr];
    }
    public boolean hasPrevious() {
        return (index > 0);
    }
    public E previous() {
        ptr--;
        if (ptr < 0) {
            if (currentNode.previous != null) {
                currentNode = currentNode.previous;
                ptr = currentNode.numElements - 1;
            } else {
                throw new NoSuchElementException();
            }
        }
        index--;
        checkForModification();
        return (E) currentNode.elements[ptr];
    }
    public int nextIndex() {
        return (index + 1);
    }
    public int previousIndex() {
        return (index - 1);
    }
    public void remove() {
        checkForModification();
        removeFromNode(currentNode, ptr);
    }
    public void set(E e) {
        checkForModification();
        currentNode.elements[ptr] = e;
    }
    public void add(E e) {
        checkForModification();
        insertIntoNode(currentNode, ptr + 1, e);
    }
    private void checkForModification() {
        if (modCount != expectedModCount) {
            throw new ConcurrentModificationException();
        }
    }
}
private void insertIntoNode(ListNode node, int ptr, E element) {
    // if the node is full
    if (node.numElements == nodeCapacity) {
        // create a new node
        ListNode newNode = new ListNode();
        // move half of the elements to the new node
        int elementsToMove = nodeCapacity / 2;
        int startIndex = nodeCapacity - elementsToMove;
        for (int i = 0; i < elementsToMove; i++) {
            newNode.elements[i] = node.elements[startIndex + i];
            node.elements[startIndex + i] = null;
        }
        node.numElements -= elementsToMove;
        newNode.numElements = elementsToMove;
        // insert the new node into the list
        newNode.next = node.next;
```

```
            newNode.previous = node;
            if (node.next != null) {
                node.next.previous = newNode;
            }
            node.next = newNode;
            if (node == lastNode) {
                lastNode = newNode;
            }
            // check whether the element should be inserted into
            // the original node or into the new node
            if (ptr > node.numElements) {
                node = newNode;
                ptr -= node.numElements;
            }
        }
        for (int i = node.numElements; i > ptr; i--) {
            node.elements[i] = node.elements[i - 1];
        }
        node.elements[ptr] = element;
        node.numElements++;
        size++;
        modCount++;
    }
    private void removeFromNode(ListNode node, int ptr) {
        node.numElements--;
        for (int i = ptr; i < node.numElements; i++) {
            node.elements[i] = node.elements[i + 1];
        }
        node.elements[node.numElements] = null;
        if (node.next != null && node.next.numElements + node.numElements <= nodeCapacity) {
            mergeWithNextNode(node);
        } else if (node.previous != null && node.previous.numElements + node.numElements <= nodeCapacity) {
            mergeWithNextNode(node.previous);
        }
        size--;
        modCount++;
    }
    //This method does merge the specified node with the next node.
    private void mergeWithNextNode(ListNode node) {
        ListNode next = node.next;
        for (int i = 0; i < next.numElements; i++) {
            node.elements[node.numElements + i] = next.elements[i];
            next.elements[i] = null;
        }
        node.numElements += next.numElements;
        if (next.next != null) {
            next.next.previous = node;
        }
        node.next = next.next.next;
        if (next == lastNode) {
            lastNode = node;
        }
    }
}
```

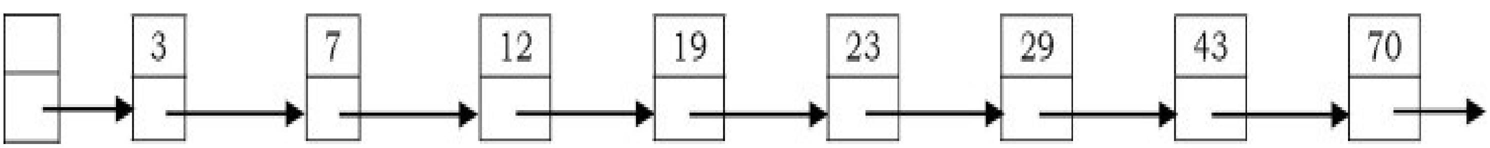
3.11 Skip Lists

Binary trees can be used for representing abstract data types such as dictionaries and ordered lists. They work well when the elements are inserted in a random order. Some sequences of operations, such as inserting the elements in order, produce degenerate data structures that give very poor performance. If it were possible to randomly permute the list of items to be inserted, trees would work well with high probability for any input sequence. In most cases queries must be answered on-line, so randomly permuting the input is impractical. Balanced tree algorithms re-arrange the tree as operations are performed to maintain certain balance conditions and assure good performance.

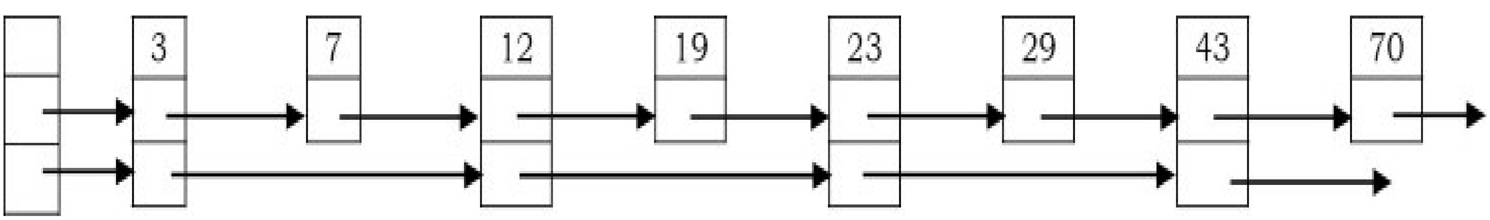
Skip list is a data structure that can be used as an alternative to balanced binary trees (refer to *Trees* chapter). As compared to a binary tree, skip lists allow quick search, insertion and deletion of elements. This is achieved by using probabilistic balancing rather than strictly enforce balancing. It is basically a linked list with additional pointers such that intermediate nodes can be skipped. It uses a random number generator to make some decisions.

In an ordinary sorted linked list, search, insert, and delete are in O(n) because the list must be scanned node-by-node from the head to find the relevant node. If somehow we could scan down the list in bigger steps (skip down, as it were), we would reduce the cost of scanning. This is the fundamental idea behind Skip Lists.

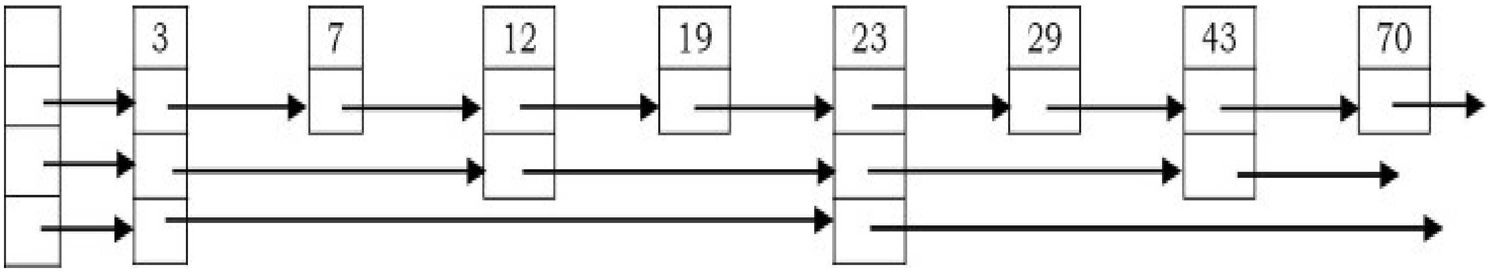
Skip Lists with One Level



Skip Lists with Two Level



Skip Lists with Three Levels



This section gives algorithms to search for, insert and delete elements in a dictionary or symbol table. The Search operation returns the contents of the value associated with the desired key or failure if the key is not present. The Insert operation associates a specified key with a new value (inserting the key if it had not already been present). The Delete operation deletes the specified key. It is easy to support additional operations such as “find the minimum key” or “find the next key”.

Each element is represented by a node, the level of which is chosen randomly when the node is inserted without regard for the number of elements in the data structure. A level i node has i forward pointers, indexed 1 through i. We do not need to store the level of a node in the node. Levels are capped at some appropriate constant *MaxLevel*. The level of a list is the maximum level currently in the list (or 1 if the list is empty). The header of a list has forward pointers at levels one through MaxLevel. The forward pointers of the header at levels higher than the current maximum level of the list point to NULL.

Initialization

An element NIL is allocated and given a key greater than any legal key. All levels of all skip lists are terminated with NIL. A new list is initialized so that the the level of the list is equal to 1 and all forward pointers of the list’s header point to NIL.

Search for an element

We search for an element by traversing forward pointers that do not overshoot the node containing the element being searched for. When no more progress can be made at the current level of forward pointers, the search moves down to the next level. When we can make no more progress at level 1, we must be immediately in front of the node that contains the desired element (if it is in the list).

Insertion and Deletion Algorithms

To insert or delete a node, we simply search and splice. A vector update is maintained so that when the search is complete (and we are ready to perform the splice), update[i] contains a pointer to the rightmost node of level i or higher that is to the left of the location of the insertion/deletion. If an insertion generates a node with a level greater than the previous maximum level of the list, we update the maximum level of the list and initialize the appropriate portions of the update vector. After each deletion, we check if we have deleted the maximum element of the list and if so, decrease the maximum level of the list.

Choosing a Random Level

Initially, we discussed a probability distribution where half of the nodes that have level i pointers also have level i+1 pointers. To get away from magic constants, we say that a fraction p of the nodes with level i pointers also have level i+1 pointers. (for our original discussion, p = 1/2). Levels are generated randomly by an algorithm. Levels are generated without reference to the number of elements in the list

Performance

In a simple linked list that consists of n elements, to perform a search n comparisons are required in the worst case. If a second pointer pointing two nodes ahead is added to every node, the number of comparisons goes down to $n/2 + 1$ in the worst case. Adding one more pointer to every fourth node and making them point to the fourth node ahead reduces the number of comparisons to $\lceil n/2 \rceil + 2$. If this strategy is continued so that every node with i pointers points to $2 * i - 1$ nodes ahead, O($logn$) performance is obtained and the number of pointers has only doubled ($n + n/2 + n/4 + n/8 + n/16 + ... = 2n$).

The find, insert, and remove operations on ordinary binary search trees are efficient, O($logn$), when the input data is random; but less efficient, O(n), when the input data is ordered. Skip List performance for these same operations and for any data set is about as good as that of randomly-built binary search trees - namely O($logn$).

Comparing Skip Lists and Unrolled Linked Lists

In simple terms, Skip Lists are sorted linked lists with two differences:

- The nodes in an ordinary list have one next reference. The nodes in a Skip List have many *next* references (also called *forward* references).
- The number of *forward* references for a given node is determined probabilistically.

We speak of a Skip List node having levels, one level per forward reference. The number of levels in a node is called the *size* of the node. In an ordinary sorted list, insert, remove, and find operations require sequential traversal of the list. This results in O(n) performance per operation. Skip Lists allow intermediate nodes in the list to be skipped during a traversal - resulting in an expected performance of O($logn$) per operation.

Implementation

```
import java.util.Random;
public class SkipList<T extends Comparable<T>, U>{
    private class Node{
        public T key;
        public U value;
        public long level;
        public Node next;
        public Node down;
        public Node(T key, U value, long level, Node next, Node down){
            this.key = key;
            this.value = value;
            this.level = level;
            this.next = next;
            this.down = down;
        }
    }
    private Node head;
    private Random _random;
    private long size;
    private double _p;
    private long level(){
        long level = 0;
        while (level <= size && _random.nextDouble() < _p) {
            level++;
        }
```

```
        return level;
    }
    public SkipList(){
        head = new Node(null, null, 0, null, null);
        _random = new Random();
        size = 0;
        _p = 0.5;
    }
    public void add(T key, U value){
        long level = level();
        if (level > head.level) {
            head = new Node(null, null, level, null, head);
        }
        Node cur = head;
        Node last = null;
        while (cur != null) {
            if (cur.next == null || cur.next.key.compareTo(key) > 0) {
                if (level >= cur.level) {
                    Node n = new Node(key, value, cur.level, cur.next, null);
                    if (last != null) {
                        last.down = n;
                    }
                    cur.next = n;
                    last = n;
                }
                cur = cur.down;
                continue;
            } else if (cur.next.key.equals(key)) {
                cur.next.value = value;
                return;
            }
            cur = cur.next;
        }
        size++;
    }
    public boolean containsKey(T key){
        return get(key) != null;
    }
    public U remove(T key){
        U value = null;
        Node cur = head;
        while (cur != null) {
            if (cur.next == null || cur.next.key.compareTo(key) >= 0) {
                if (cur.next != null && cur.next.key.equals(key)) {
                    value = cur.next.value;
                    cur.next = cur.next.next;
                }
                cur = cur.down;
                continue;
            }
            cur = cur.next;
        }
        size--;
        return value;
    }
    public U get(T key){
        Node cur = head;
        while (cur != null) {
            if (cur.next == null || cur.next.key.compareTo(key) > 0) {
                cur = cur.down;
                continue;
            } else if (cur.next.key.equals(key)) {
```

```
                                return cur.next.value;
                        }
                        cur = cur.next;
                }
                return null;
        }
}
public class SkipListsTest{
        public static void main(String [] args){
                SkipList s = new SkipList();
                s.add(1,100);
                System.out.println(s.get(1));
        }
}
```

3.12 Linked Lists: Problems & Solutions

Problem-1 Implement Stack using Linked List

Solution: Refer to *Stacks* chapter.

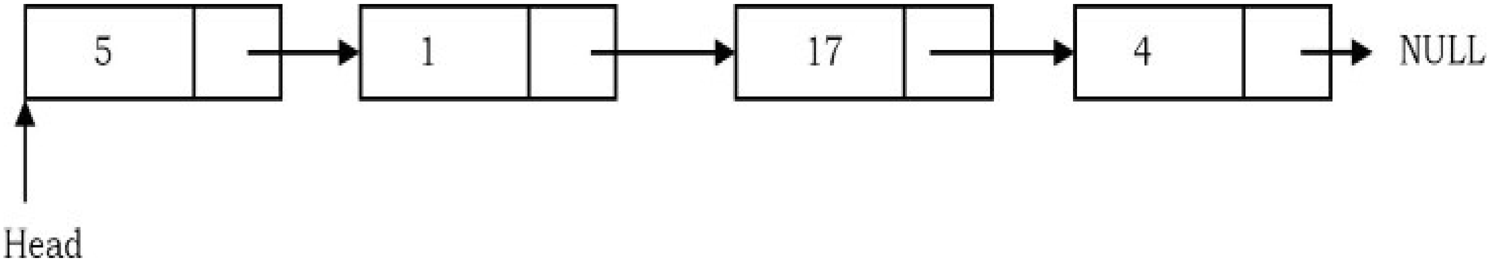
Problem-2 Find n^{th} node from the end of a Linked List.

Solution: Brute-Force Method: Start with the first node and count the number of nodes present after that node. If the number of nodes is < $n - 1$ then return saying "fewer number of nodes in the list". If the number of nodes is > $n - 1$ then go to next node. Continue this until the numbers of nodes after current node are $n - 1$.

Time Complexity: $O(n^2)$, for scanning the remaining list (from current node) for each node.
Space Complexity: O(1).

Problem-3 Can we improve the complexity of Problem-2?

Solution: Yes, using hash table. As an example consider the following list.



In this approach, create a hash table whose entries are < *position of node, node address* >. That means, key is the position of the node in the list and value is the address of that node.

Position in List	Address of Node
1	Address of 5 node
2	Address of 1 node
3	Address of 17 node
4	Address of 4 node

By the time we traverse the complete list (for creating the hash table), we can find the list length. Let us say the list length is M. To find n^{th} from the end of linked list, we can convert this to M - n + 1^{th} from the beginning. Since we already know the length of the list, it is just a matter of returning M - n+ 1^{th} key value from the hash table.

Time Complexity: Time for creating the hash table, Therefore, $T(m) = O(m)$. Space Complexity: Since we need to create a hash table of size m, $O(m)$.

Problem-4 Can we use the Problem-3 approach for solving Problem-2 without creating the hash table?

Solution: Yes. If we observe the Problem-3 solution, what we are actually doing is finding the size of the linked list. That means we are using the hash table to find the size of the linked list. We can find the length of the linked list just by starting at the head node and traversing the list. So, we can find the length of the list without creating the hash table. After finding the length, compute $M - n + 1$ and with one more scan we can get the $M - n + 1^{th}$ node from the beginning. This solution needs two scans: one for finding the length of the list and the other for finding $M - n + 1^{th}$ node from the beginning.

Time Complexity: Time for finding the length + Time for finding the M - n + 1^{th} node from the beginning. Therefore, $T(n = O(n) + O(n) \approx O(n)$.

Space Complexity: O(1). Hence, no need to create the hash table.

Problem-5 Can we solve Problem-2 in one scan?

Solution: Yes. Efficient Approach: Use two pointers *pNthNode* and *pTemp*. Initially, both point to head node of the list. *pNthNode* starts moving only after *pTemp* has made n moves.

From there both move forward until *pTemp* reaches the end of the list. As a result *pNthNode* points to n^{th} node from the end of the linked list.

Note: At any point of time both move one node at a time.

```
public ListNode  NthNodeFromEnd(ListNode head , int NthNode) {
        ListNode pTemp = head, pNthNode = null;
        for(int count =1; count< NthNode;count++) {
            if(pTemp != null)
                pTemp = pTemp.getNext();
        }
        while(pTemp!= null){
            if(pNthNode == null)
                pNthNode = head;
            else
                pNthNode = pNthNode.getNext();
            pTemp = pTemp.getNext();
        }
        if(pNthNode !=  null)
            return pNthNode;
        return null;
}
```

Time Complexity: O(*n*). Space Complexity: O(1).

Problem-6 Can we solve Problem-5 with recursion?

Solution: We can use a global variable to track the post recursive call and when it is same as Nth', return the node.

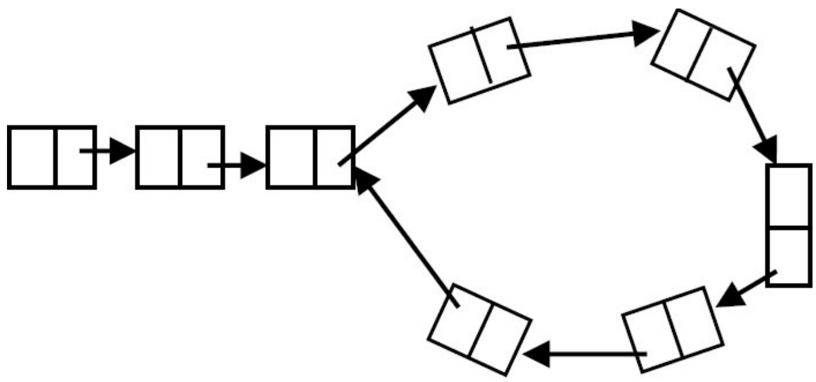
```
public ListNode NthNodeFromEnd(ListNode head, int Nth) {
    if(head != null) {
        NthNodeFromEnd(head.next, Nth);
        counter++;
        if(Nth ==counter) {
            return head;
        }
    }
    return null;
}
```

Time Complexity: O(*n*) for pre recursive calls and O(*n*) for post recursive calls, which is = O(2*n*) = O(*n*).
Space Complexity: O(n) for recursive stack.

Problem-7 Check whether the given linked list is either NULL-terminated or ends in a cycle (cyclic)

Solution: Brute-Force Approach. As an example, consider the following linked list which has a loop in it. The difference between this list and the regular list is that, in this list, there are two nodes whose next pointers are the same. In regular singly linked lists (without a loop) each node's next pointer is unique.

That means the repetition of next pointers indicates the existence of a loop.



One simple and brute force way of solving this is, start with the first node and see whether there is any node whose next pointer is the current node's address. If there is a node with the same address then that indicates that some other node is pointing to the current node and we can say a loop exists.

Continue this process for all the nodes of the linked list.

Does this method work? As per the algorithm, we are checking for the next pointer addresses, but how do we find the end of the linked list (otherwise we will end up in an infinite loop)?

Note: If we start with a node in a loop, this method may work depending on the size of the loop.

Problem-8 Can we use the hashing technique for solving Problem-7?

Solution: Yes. Using Hash Tables we can solve this problem.

Algorithm:

- Traverse the linked list nodes one by one.
- Check if the address of the node is available in the hash table or not.
- If it is already available in the hash table, that indicates that we are visiting the node that was already visited. This is possible only if the given linked list has a loop in it.
- If the address of the node is not available in the hash table, insert that node's address into the hash table.
- Continue this process until we reach the end of the linked list *or* we find the loop.

Time Complexity*:* O(*n*) for scanning the linked list. Note that we are doing a scan of only the input.

Space Complexity: O(n) for hash table.

Problem-9 Can we solve Problem-6 using the sorting technique?

Solution: No . Consider the following algorithm which is based on sorting. Then we see why this algorithm fails.

Algorithm:

- Traverse the linked list nodes one by one and take all the next pointer values into an array.
- Sort the array that has the next node pointers.
- If there is a loop in the linked list, definitely two next node pointers will be pointing to the same node.
- After sorting if there is a loop in the list, the nodes whose next pointers are the same will end up adjacent in the sorted list.
- If any such pair exists in the sorted list then we say the linked list has a loop in it.

Time Complexity: $O(nlogn)$ for sorting the next pointers array.
Space Complexity: O(n) for the next pointers array.

Problem with the above algorithm: The above algorithm works only if we can find the length of the list. But if the list has a loop then we may end up in an infinite loop. Due to this reason the algorithm fails.

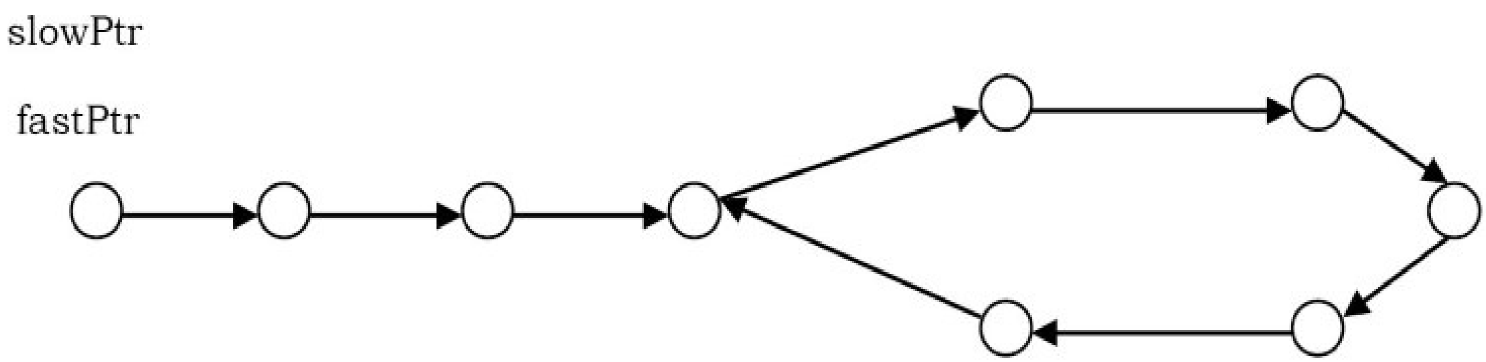
Problem-10 Can we solve the Problem-6 in O(n)?

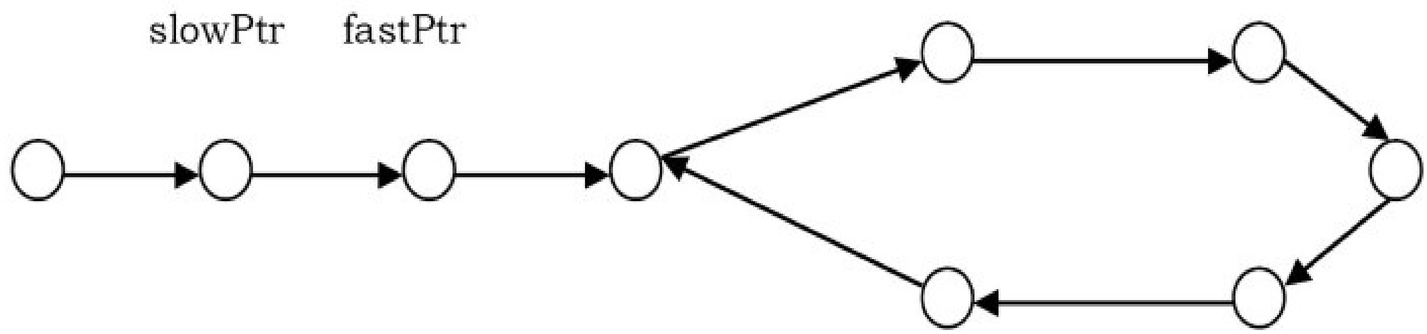
Solution: Yes. Efficient Approach (Memoryless Approach): This problem was solved by *Floyd*. The solution is named the Floyd cycle finding algorithm. It uses *two* pointers moving at different speeds to walk the linked list. Once they enter the loop they are expected to meet, which denotes that there is a loop.

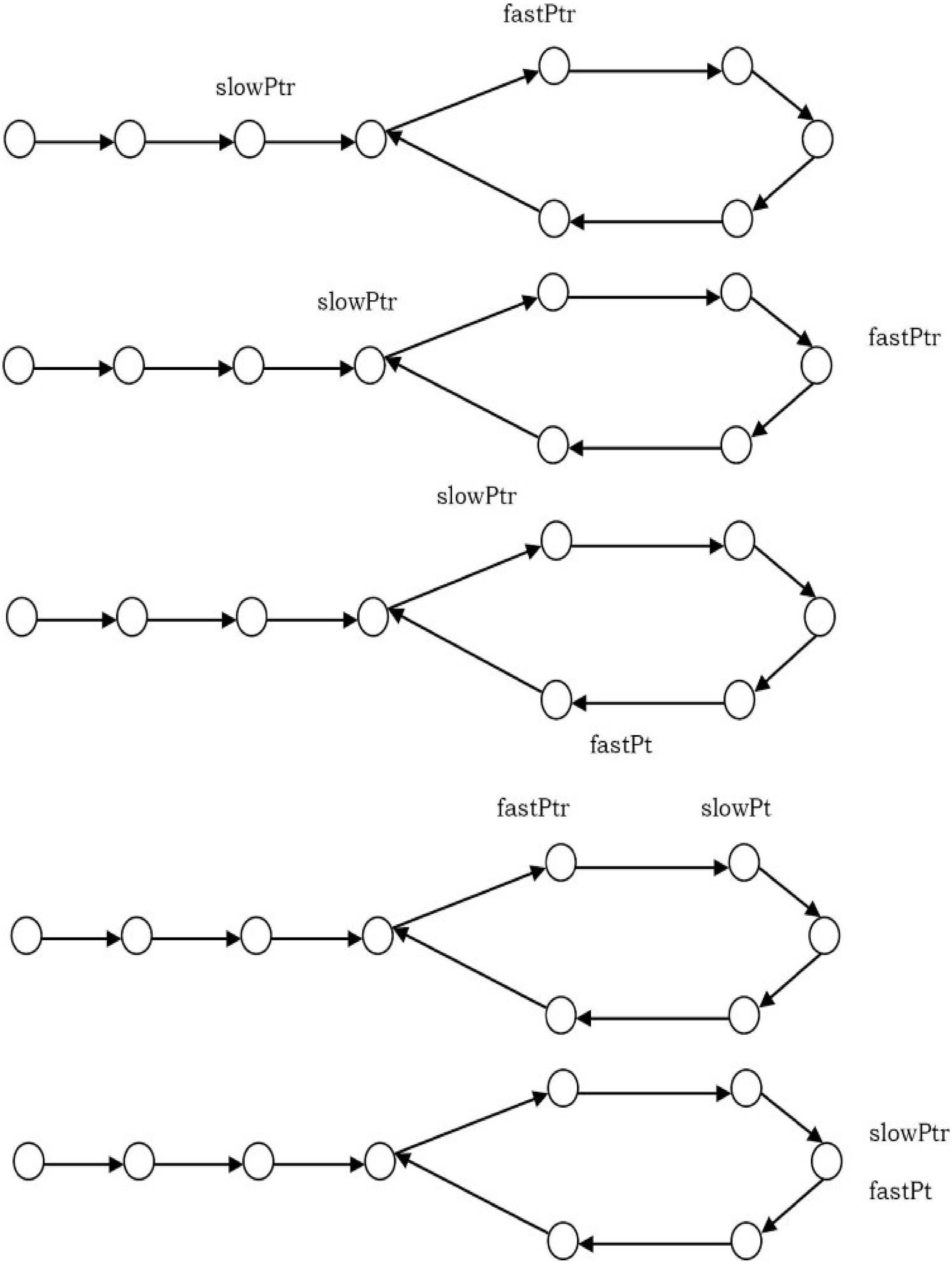
This works because the only way a faster moving pointer would point to the same location as a slower moving pointer is if somehow the entire list or a part of it is circular. Think of a tortoise and a hare running on a track. The faster running hare will catch up with the tortoise if they are running in a loop.

As an example, consider the following example and trace out the Floyd algorithm. From the diagrams below we can see that after the final step they are meeting at some point in the loop which may not be the starting point of the loop.

Note: *slowPtr* (*tortoise*) moves one pointer at a time and *fastPtr* (*hare*) moves two pointers at a time.


slowPtr
fastPtr


slowPtr
fastPtr


fastPtr
slowPtr
slowPtr
fastPtr
slowPtr
fastPt
fastPtr
slowPt
slowPtr
fastPt

```
private static boolean findIfLoopExistsUsingFloyds(ListNode head){
    ListNode fastPtr = head;
    ListNode slowPtr = head;
    while (fastPtr != null && fastPtr.getNext() != null) {
        fastPtr = fastPtr.getNext().getNext();
        slowPtr = slowPtr.getNext();
        if (slowPtr == fastPtr)
            return true;
    }
    return false;
}
```

Time Complexity: O(n) Space Complexity: O(n)

Problem-11 We are given a pointer to the first element of a linked list *L*. There are two possibilities for *L*, it either ends (snake) or its last element points back to one of the earlier elements in the list (snail). Give an algorithm that tests whether a given list L is a snake or a snail.

Solution: It is the same as Problem-6.

Problem-12 Check whether the given linked list is NULL-terminated or not. If there is a cycle find the start node of the loop.

Solution: The solution is an extension to the solution in Problem-10. After finding the loop in the linked list, we initialize the *slowPtr* to the head of the linked list. From that point onwards both *slowPtr* and *fastPtr* move only one node at a time. The point at which they meet is the start of the loop. Generally we use this method for removing the loops. Let x and y be travelers such that y is walking twice as fast as x (i.e. $y = 2x$). Further, let s be the place where x and y first started walking at the same time. Then when x and y meet again, the distance from s to the start of the loop is the exact same distance from the present meeting place of x and y to the start of the loop.

```java
private static ListNode findBeginofLoop(ListNode head){
    ListNode fastPtr = head;
    ListNode slowPtr = head;
    boolean loopExists = false;
    while (fastPtr != null && fastPtr.getNext() != null) {
        fastPtr = fastPtr.getNext().getNext();
        slowPtr = slowPtr.getNext();
        if (slowPtr == fastPtr) {
            loopExists = true;
            break;
        }
    }
    if (loopExists) {
        slowPtr = head;
        while (slowPtr != fastPtr) {
            slowPtr = slowPtr.getNext();
            fastPtr = fastPtr.getNext();
        }
        return fastPtr;
    } else
        return null;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-13 From the previous discussion and problems we understand that the meeting of tortoise and hare concludes the existence of the loop, but how does moving the tortoise to the beginning of the linked list while keeping the hare at the meeting place, followed by moving both one step at a time, make them meet at the starting point of the cycle?

Solution: This problem is at the heart of number theory. In the Floyd cycle finding algorithm, notice that the tortoise and the hare will meet when they are $n \times L$, where L is the loop length. Furthermore, the tortoise is at the midpoint between the hare and the beginning of the sequence because of the way they move. Therefore the tortoise is $n \times L$ away from the beginning of the sequence as well.

If we move both one step at a time, from the position of the tortoise and from the start of the sequence, we know that they will meet as soon as both are in the loop, since they are $n \times L$, a multiple of the loop length, apart. One of them is already in the loop, so we just move the other one in single step until it enters the loop, keeping the other $n \times L$ away from it at all times.

Problem-14 In the Floyd cycle finding algorithm, does it work if we use steps 2 and 3 instead of 1 and 2?

Solution: Yes, but the complexity might be high. Trace out an example.

Problem-15 Check whether the given linked list is NULL-terminated. If there is a cycle, find the length of the loop.

Solution: This solution is also an extension of the basic cycle detection problem. After finding the loop in the linked list, keep the *slowPtr* as it is. The *fastPtr* keeps on moving until it again comes back to *slowPtr*. While moving *fastPtr*, use a counter variable which increments at the rate of 1.

```
private static int findLengthOfTheLoop(ListNode head){
    ListNode fastPtr = head;
    ListNode slowPtr = head;
    boolean loopExists = false;
    while (fastPtr != null && fastPtr.getNext() != null) {
        fastPtr = fastPtr.getNext().getNext();
        slowPtr = slowPtr.getNext();
        if (slowPtr == fastPtr) {
            loopExists = true;
            break;
        }
    }
    int length = 0;
    if (loopExists) {
        do {
            slowPtr = slowPtr.getNext();
            length++;
        } while (slowPtr != fastPtr);
    }
    return length;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-16 Insert a node in a sorted linked list

Solution: Traverse the list and find a position for the element and insert it.

```
public ListNode InsertInSortedList(ListNode head, ListNode newNode) {
    ListNode current = head;
    if(head == null) return newNode;
    // traverse the list until you find item bigger the new node value
    while (current != null && current.getData() <  newNode.getData()){
        temp = current;
        current = current.getNext();
    }
    //  insert the new node before the big item
    newNode.setNext(current);
    temp.setNext(newNode);
    return head;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-17 Reverse a singly linked list.

Solution:

Iterative version:

```
public static ListNode reverseListIterative(ListNode head){
    //initially Current is head
    ListNode current = head;
    //initially previous is null
    ListNode prev = null;
    while (current != null) {
        //Save the next node
        ListNode next = current.getNext();
        //Make current node points to the previous
        current.setNext(prev);
        //update previous
        prev = current;
        //update current
        current = next;
    }
    return prev;
}
```

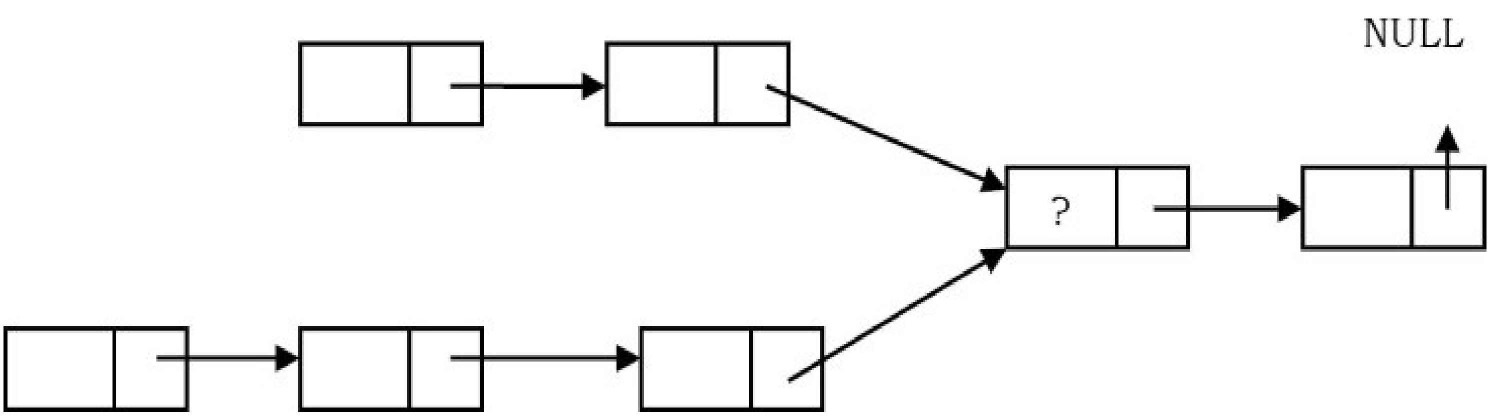
Time Complexity: O(n). Space Complexity: O(1).

Recursive version:

```
public static void reverseLinkedListRecursive(ListNode current, ListNode[] head){
    if (current == null) {
        return;
    }
    ListNode next = current.getNext();
    if (next == null) {
        head[0] = current;
        return;
    }
    reverseLinkedListRecursive(next, head);
    //Make next node points to current node
    next.setNext(current);
    //Remove existing link
    current.setNext(null);
}
```

Time Complexity: O(n). Space Complexity: O(n),for recursive stack.

Problem-18 Suppose there are two singly linked lists both of which intersect at some point and become a single linked list. The head or start pointers of both the lists are known, but the intersecting node is not known. Also, the number of nodes in each of the lists before they intersect is unknown and may be different in each list. *List*1 may have n nodes before it reaches the intersection point, and *List*2 might have m nodes before it reaches the intersection point where m and n may be $m = n, m < n$ or $m > n$. Give an algorithm for finding the merging point.



Solution: Brute-Force Approach: One easy solution is to compare every node pointer in the first list with every other node pointer in the second list by which the matching node pointers will lead us to the intersecting node. But, the time complexity in this case will be O(mn) which will be high.

Time Complexity: O(mn). Space Complexity: O(1).

Problem-19 Can we solve Problem-17 using the sorting technique?

Solution: No. Consider the following algorithm which is based on sorting and see why this algorithm fails.

Algorithm:

- Take first list node pointers and keep them in some array and sort them.
- Take second list node pointers and keep them in some array and sort them.
- After sorting, use two indexes: one for the first sorted array and the other for the second sorted array.
- Start comparing values at the indexes and increment the index according to whichever has the lower value (increment only if the values are not equal).
- At any point, if we are able to find two indexes whose values are the same, then that indicates that those two nodes are pointing to the same node and we return that node.

Time Complexity: Time for sorting lists + Time for scanning (for comparing)
= O($mlogm$) +O($nlogn$) +O($m + n$) We need to consider the one that gives the maximum value.
Space Complexity: O(1).

Any problem with the above algorithm? Yes. In the algorithm, we are storing all the node pointers of both the lists and sorting. But we are forgetting the fact that there can be many repeated elements. This is because after the merging point, all node pointers are the same for both the lists. The algorithm works fine only in one case and it is when both lists have the ending node at their merge point.

Problem-20 Can we solve Problem-18 using hash tables?

Solution: Yes.

Algorithm:

- Select a list which has less number of nodes (If we do not know the lengths beforehand then select one list randomly).
- Now, traverse the other list and for each node pointer of this list check whether the same node pointer exists in the hash table.
- If there is a merge point for the given lists then we will definitely encounter the node pointer in the hash table.

Time Complexity: Time for creating the hash table + Time for scanning the second list = O(m) + O(n) (or O(n) + O(m), depending on which list we select for creating the hash table. But in both cases the time complexity is the same.
Space Complexity: O(n) or O(m).

Problem-21 Can we use stacks for solving Problem-18?

Algorithm:

- Create two stacks: one for the first list and one for the second list.
- Traverse the first list and push all the node addresses onto the first stack.
- Traverse the second list and push all the node addresses onto the second stack.
- Now both stacks contain the node address of the corresponding lists.
- Now compare the top node address of both stacks.
- If they are the same, take the top elements from both the stacks and keep them in some temporary variable (since both node addresses are node, it is enough if we use one temporary variable).
- Continue this process until the top node addresses of the stacks are not the same.
- This point is the one where the lists merge into a single list.
- Return the value of the temporary variable.

Time Complexity: O($m + n$), for scanning both the lists.
Space Complexity: O($m + n$), for creating two stacks for both the lists.

Problem-22 Is there any other way of solving Problem-18?

Solution: Yes. Using "finding the first repeating number" approach in an array (for algorithm refer to *Searching* chapter).

Algorithm:

- Create an array A and keep all the next pointers of both the lists in the array.
- In the array find the first repeating element [Refer to *Searching* chapter for algorithm].
- The first repeating number indicates the merging point of both the lists.

Time Complexity: O($m + n$). Space Complexity: O($m + n$).

Problem-23 Can we still think of finding an alternative solution for Problem-18?

Solution: Yes. By combining sorting and search techniques we can reduce the complexity.

Algorithm:

- Create an array A and keep all the next pointers of the first list in the array.
- Sort these array elements.
- Then, for each of the second list elements, search in the sorted array (let us assume that we are using binary search which gives O($logn$)).
- Since we are scanning the second list one by one, the first repeating element that appears in the array is nothing but the merging point.

Time Complexity: Time for sorting + Time for searching = O(Max($mlogm$, $nlogn$)).
Space Complexity: O(Max(m, n)).

Problem-24 Can we improve the complexity for the Problem-18?

Solution: Yes.

Efficient Approach:

- Find lengths (L1 and L2) of both lists -- O(n) + O(m) = O(max(m, n)).
- Take the difference d of the lengths -- O(1).
- Make d steps in longer list -- O(d).
- Step in both lists in parallel until links to next node match -- O(min(m, n)).
- Total time complexity = O(max(m, n)).
- Space Complexity = O(1).

```java
public static ListNode findIntersectingNode(ListNode list1, ListNode list2) {
    int L1=0, L2=0, diff=0;
    ListNode head1 = list1, head2 = list2;
    while(head1 != null) {
        L1++;
        head1 = head1.getNext();
    }
    while(head2 != null) {
        L2++;
        head2 = head2.getNext();
    }
    if(L1 < L2) {
            head1 = list2;
            head2 = list1;
            diff = L2 - L1;
    } else{
        head1 = list1;
        head2 = list2;
        diff = L1 – L2;
    }
    for(int i = 0; i < diff; i++)
        head1 = head1.getNext();
    while(head1 != null && head2 != null) {
            if(head1 == head2)
                return head1.getData();
            head1= head1.getNext();
            head2= head2.getNext();
    }
    return null;
}
```

Problem-25 How will you find the middle of the linked list?

Solution: Brute-Force Approach: For each of the node count how many nodes are there in the list and see whether it is the middle.

Time Complexity: O(n^2). Space Complexity: O(1).

Problem-26 Can we improve the complexity of Problem-25?

Solution: Yes.
Algorithm:

- Traverse the list and find the length of the list.
- After finding the length, again scan the list and locate $n/2$ node from the beginning.

Time Complexity: Time for finding the length of the list + Time for locating middle node = $O(n) + O(n) \approx O(n)$.
Space Complexity: O(1).

Problem-27 Can we use the hash table for solving Problem-25?

Solution: Yes. The reasoning is the same as that of Problem-3.

Time Complexity: Time for creating the hash table. Therefore, $T(n) = O(n)$. Space Complexity: $O(n)$. Since, we need to create a hash table of size n.

Problem-28 Can we solve Problem-25 just in one scan?

Solution: Efficient Approach: Use two pointers. Move one pointer at twice the speed of the second. When the first pointer reaches the end of the list, the second pointer will be pointing to the middle node.

Note: If the list has an even number of nodes, the middle node will be of $[n/2]$.

```
public static ListNode findMiddle(ListNode head) {
    ListNode ptr1x, ptr2x;
    ptr1x = ptr2x = head;
    int i=0;
    // keep looping until we reach the tail (next will be NULL for the last node)
    while(ptr1x.getNext() != null) {
        if(i == 0) {
            ptr1x = ptr1x.getNext(); //increment only the 1st pointer
            i=1;
        }
        else if( i == 1) {
            ptr1x = ptr1x.getNext(); //increment both pointers
            ptr2x = ptr2x.getNext();
            i = 0;
        }
    }
    return ptr2x;    //now return the ptr2 which points to the middle node
}
```

Time Complexity: $O(n)$. Space Complexity: O(1).

Problem-29 How will you display a linked list from the end?

Solution: Traverse recursively till the end of the linked list. While coming back, start printing the elements.

```
//This Function will print the linked list from end
public static void printListFromEnd(ListNode head) {
    if(head == null)
        return;
    printListFromEnd(head.getNext());
    System.out.println(head.getData());
}
```

Time Complexity: O(n). Space Complexity: O(n)→ for Stack.

Problem-30 Check whether the given Linked List length is even or odd?

Solution: Use a 2x pointer. Take a pointer that moves at 2x [two nodes at a time]. At the end, if the length is even, then the pointer will be NULL; otherwise it will point to the last node.

```
public int IsLinkedListLengthEven(ListNode listHead) {
    while(listHead != null && listHead.getNext() != null)
        listHead = listHead.getNext().getNext();
    if(listHead == null) return 0;
    return 1;
}
```

Time Complexity: O($\lfloor n/2 \rfloor$) ≈O(n). Space Complexity: O(1).

Problem-31 If the head of a linked list is pointing to *kth* element, then how will you get the elements before *kth* element?

Solution: Use Memory Efficient Linked Lists [XOR Linked Lists].

Problem-32 Given two sorted Linked Lists, we need to merge them into the third list in sorted order.

Solution:

```
public ListNode mergeTwoLists(ListNode head1, ListNode head2) {
    if(head1 == null)
        return head2;
    if(head2 == null)
        return head1;
    ListNode head = new ListNode(0);
    if(head1.data <= head2.data){
        head = head1;
        head.next = mergeTwoLists(head1.next, head2);
    }else{
        head = head2;
        head.next = mergeTwoLists(head2.next, head1);
    }
    return head;
}
```

Time Complexity – O(n).

Problem-33 Can we solve Problem-32 without recursion?

Solution:

```
public ListNode mergeTwoLists(ListNode head1, ListNode head2) {
    ListNode head = new ListNode(0);
    ListNode curr = head;
    while(head1 != null && head2 != null){
        if(head1.data <= head2.data){
            curr.next = head1;
            head1 = head1.next;
        }else{
            curr.next = head2;
            head2 = head2.next;
        }
    }
    if(head1 != null)
        curr.next = head1;
    else if(head2 != null)
        curr.next = head2;
    return head.next;
}
```

Time Complexity – O(n).

Problem-34 Reverse the linked list in pairs. If you have a linked list that holds $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow X$, then after the function has been called the linked list would hold $2 \rightarrow 1 \rightarrow 4 \rightarrow 3 \rightarrow X$.

Solution:

```
//Recursive Version
public static ListNode ReversePairRecursive(ListNode head) {
    ListNode temp;
    if(head ==NULL || head.next ==NULL)
            return;  //base case for empty or 1 element list
    else {
            //Reverse first pair
            temp = head.next;
            head.next = temp.next;
            temp.next = head;
            head = temp;
            //Call the method recursively for the rest of the list
            head.next.next = ReversePairRecursive(head.next.next);
            return head;
    }
}
/*Iterative version*/
public static ListNode  ReversePairIterative(ListNode  head) {
    ListNode  temp1 = null;
    ListNode  temp2 = null;
    while (head != null && head.next != null) {
        if (temp1 != null) {
            temp1.next.next = head.next;
        }
        temp1 = head.next;
        head.next = head.next.next;
        temp1.next = head;
        if (temp2 == null)
            temp2 = temp1;
        head = head.next;
    }
    return temp2;
}
```

Time Complexity – O(n). Space Complexity - O(1).

Problem-35 Given a binary tree convert it to doubly linked list.

Solution: Refer *Trees* chapter.

Problem-36 How do we sort the Linked Lists?

Solution: Refer *Sorting* chapter.

Problem-37 If we want to concatenate two linked lists, which of the following gives O(1) complexity?

1) Singly linked lists
2) Doubly linked lists
3) Circular doubly linked lists

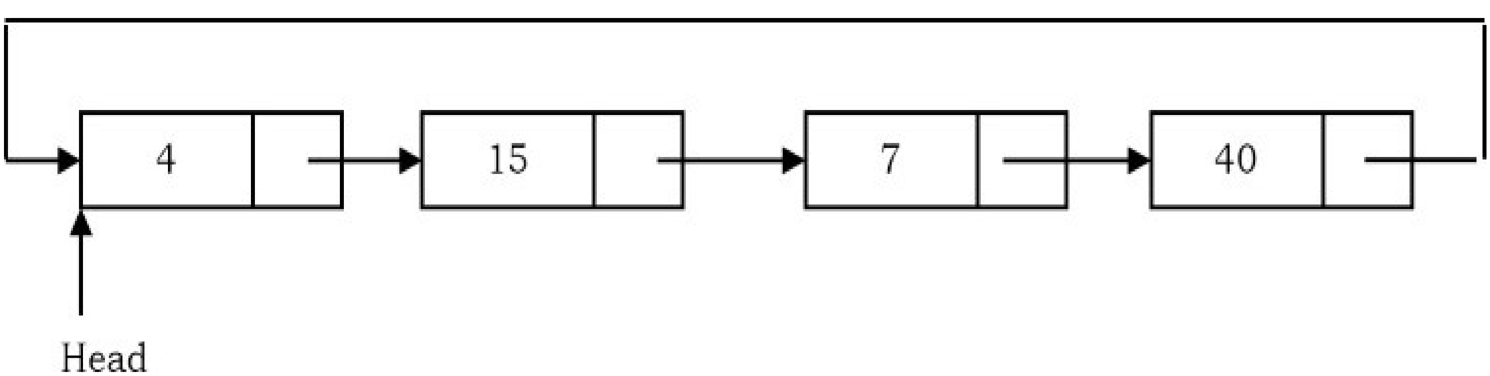
Solution: Circular Doubly Linked Lists. This is because for singly and doubly linked lists, we need to traverse the first list till the end and append the second list. But in the case of circular doubly linked lists we don't have to traverse the lists.

Problem-38 Split a Circular Linked List into two equal parts. If the number of nodes in the list are odd then make first list one node extra than second list.

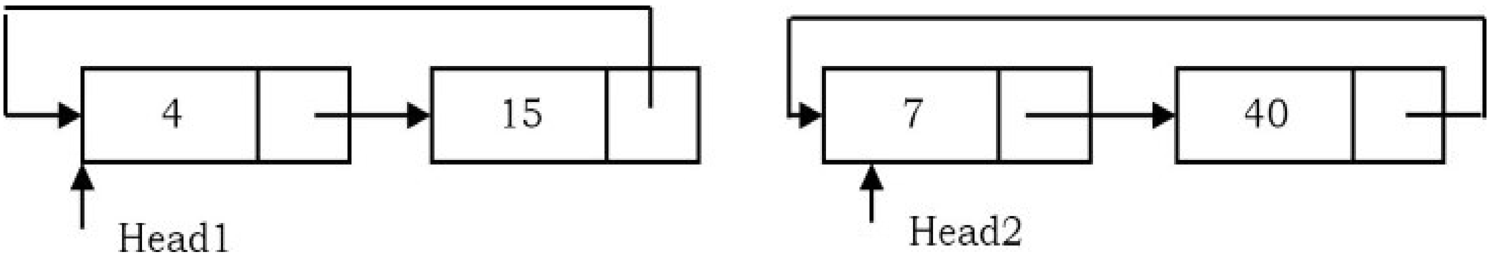
Solution:
Algorithm

- Store the mid and last pointers of the circular linked list using Floyd cycle finding algorithm.
- Make the second half circular.
- Make the first half circular.
- Set head pointers of the two linked lists.

As an example, consider the following circular list.



After the split, the above list will look like:




```
public static void SplitList(ListNode head, ListNode head1, ListNode head2) {
    ListNode slowPtr = head, fastPtr = head;
    if(head == nlu) return;
    /* If there are odd nodes in the circular list then fastPtr.getNext() becomes
       head and for even nodes fastPtr.getNext().getNext() becomes head */
      while(fastPtr.getNext() != head && fastPtr.getNext().getNext() != head)    {
            fastPtr = fastPtr.getNext().getNext();
            slowPtr = slowPtr.getNext();
    }
    /* If there are even elements in list then move fastPtr */
    if(fastPtr.getNext().getNext() == head)
            fastPtr = fastPtr.getNext();
    /* Set the head pointer of first half */
    head1 = head;
    /* Set the head pointer of second half */
    if(head.getNext() != head)
            head2 = slowPtr.getNext();
    /* Make second half circular */
    fastPtr.setNext(slowPtr.getNext());
    /* Make first half circular */
    slowPtr.setNext(head);
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-39 How will you check if the linked list is palindrome or not?

Solution:
Algorithm

1. Get the middle of the linked list.
2. Reverse the second half of the linked list.
3. Compare the first half and second half.
4. Construct the original linked list by reversing the second half again and attaching it back to the first half.

Time Complexity: O(n). Space Complexity: O(1).

Problem-40 Exchange the adjacent elements in a link list.

Solution:

```
public ListNode exchangeAdjacentNodes (ListNode head) {
    ListNode temp = new ListNode(0);
    temp.next = head;
    ListNode prev = temp, curr = head;
    while(curr != null && curr.next != null){
        ListNode tmp = curr.next.next;
        curr.next.next = prev.next;
        prev.next = curr.next;
        curr.next = tmp;
        prev = curr;
        curr = prev.next;
    }
    return temp.next;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-41 For a given K value ($K > 0$) reverse blocks of if nodes in a list.

Example: Input: 1 2 3 4 5 6 7 8 9 10, Output for different K values:
For K = 2: 2 1 4 3 6 5 8 7 10 9,
For K = 3: 3 2 1 6 5 4 9 8 7 10,
For K = 4: 4 3 2 1 8 7 6 5 9 10

Solution:

Algorithm: This is an extension of swapping nodes in a linked list.

1) Check if remaining list has K nodes.
    a. If yes get the pointer of $K + 1^{th}$ node.
    b. Else return.
2) Reverse first K nodes.
3) Set next of last node (after reversal) to $K + 1^{th}$ node.
4) Move to $K + 1^{th}$ node.
5) Go to step 1.
6) $K - 1^{th}$ node of first K nodes becomes the new head if available. Otherwise, we can return the head.

```
public static ListNode reverseKNodesRecursive(ListNode head, int k){
    ListNode current = head;
    ListNode next = null;
    ListNode prev = null;
    int count = k;
    //Reverse K nodes
    while (current != null && count > 0) {
        next = current.getNext();
        current.setNext(prev);
        prev = current;
        current = next;
        count--;
    }
    //Now next points to K+1 th node, returns the pointer to the head node
    if (next != null) {
        head.setNext(reverseKNodesRecursive(next, k));
    }
    //return head node
    return prev;
}
public static ListNode reverseKNodes(ListNode head, int k){
    //Start with head
    ListNode current = head;
    //last node after reverse
    ListNode prevTail = null;
    //first node before reverse
    ListNode prevCurrent = head;
    while (current != null) {
        //loop for reversing K nodes
        int count = k;
        ListNode tail = null;
        while (current != null && count > 0) {
            ListNode next = current.getNext();
            current.setNext(tail);
            tail = current;
            current = next;
            count--;
        }
        //reversed K Nodes
        if (prevTail != null) {
            //Link this set and previous set
            prevTail.setNext(tail);
        } else {
            //We just reversed first set of K nodes, update head point to the Kth Node
            head = tail;
        }
        //save the last node after reverse since we need to connect to the next set.
        prevTail = prevCurrent;
        //Save the current node, which will become the last node after reverse
        prevCurrent = current;
    }
    return head;
}
```

Problem-42 Can we solve Problem-39 with recursion?

Solution:

```
public static ListNode reverseKNodes(ListNode head, int k){
    //Start with head
    ListNode current = head;
    //last node after reverse
    ListNode prevTail = null;
    //first node before reverse
    ListNode prevCurrent = head;
    while (current != null) {
        //loop for reversing K nodes
        int count = k;
        ListNode tail = null;
        while (current != null && count > 0) {
            ListNode next = current.getNext();
            current.setNext(tail);
            tail = current;
            current = next;
            count--;
        }
        //reversed K Nodes
        if (prevTail != null) {
            //Link this set and previous set
            prevTail.setNext(tail);
        } else {
            //We just reversed first set of K nodes, update head point to the Kth Node
            head = tail;
        }
        //save the last node after reverse since we need to connect to the next set.
        prevTail = prevCurrent;
        //Save the current node, which will become the last node after reverse
        prevCurrent = current;
    }
    return head;
}
```

Problem-43 Is it possible to get O(1) access time for Linked Lists?

Solution: Yes. Create a linked list and at the same time keep it in a hash table. For n elements we have to keep all the elements in a hash table which gives a preprocessing time of O(n).To read any element we require only constant time O(1) and to read n elements we require n * 1 unit of time = n units. Hence by using amortized analysis we can say that element access can be performed within O(1) time.

Time Complexity – O(1) [Amortized]. Space Complexity - O(n) for Hash.

Problem-44 **JosephusCircle:** N people have decided to elect a leader by arranging themselves in a circle and eliminating every M^{th} person around the circle, closing ranks as each person drops out. Find which person will be the last one remaining (with rank 1).

Solution: Assume the input is a circular linked list with N nodes and each node has a number (range 1 to N) associated with it. The head node has number 1 as data.

```
public ListNode  GetJosephusPosition(int N, int M) {
    ListNode p, q;
    // Create circular linked list containing all the players:
    p.setData(1);
    q = p;
    for (int i = 2; i <= N; ++i) {
            p = p.getNext();
            p.setData(i);
    }
    p.setNext(q);  // Close the circular linked list by having the last node point to the first.
    // Eliminate every M-th player as long as more than one player remains:
    for (int count = N; count > 1; --count) {
            for (int i = 0; i < M - 1; ++i)
                    p = p.getNext();
            p.setNext(p.getNext().getNext());  // Remove the eiminated player from the list.
    }
    System.out.println("Last player left standing (Josephus Position) is " + p.getData());
}
```

Problem-45 Given a linked list consists of data, a next pointer and also a random pointer which points to a random node of the list. Give an algorithm for cloning the list.

Solution: We can use the hash table to associate newly created nodes with the instances of node in the given list.

Algorithm:

- Scan the original list and for each node X create a new node Y with data of X, then store the pair (X, Y) in hash table using X as a key. Note that during this scan we set Y. *next* and Y. *random* to *NULL* and we will fix them in the next scan

- Now for each node *X* in the original list we have a copy *Y* stored in our hash table. We scan the original list again and set the pointers building the new list.

```
public RandomListNode copyRandomList(RandomListNode head) {
    RandomListNode X = head, Y;
    Map<RandomListNode, RandomListNode> map = new HashMap<RandomListNode, RandomListNode>();
    while(X != null) {
        Y = new RandomListNode(X.label);
        Y.next = null;
        Y.random = null;
        map.put(X, Y);
        X = X.next;
    }
    X = head;
    while(X != null){
        Y = map.get(X);
        Y.next = map.get(X.next);
        Y.random = map.get(X.random);
        X = X.next;
    }
    return map.get(head);
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-46 In a linked list with n nodes, the time taken to insert an element after an element pointed by some pointer is

(A) O(1)
(B) O($logn$)
(C) O(n)
(D) O($n1ogn$)

Solution: A.

Problem-47 **Find modular node:** Given a singly linked list, write a function to find the last element from the beginning whose $n\%k == 0$, where n is the number of elements in the list and k is an integer constant. For example, if $n = 19$ and $k = 3$ then we should return 18^{th} node.

Solution: For this problem the value of n is not known in advance.

```
public ListNode modularNodes(ListNode head, int k){
    ListNode modularNode;
    int i=0;
    if(k<=0)
        return null;
    for (;head!= null; head = head.getNext()){
        if(i%k == 0){
            modularNode = head;
        }
        i++;
    }
    return modularNode;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-48 **Find modular node from the end:** Given a singly linked list, write a function to find the first element from the end whose n%k == 0, where n is the number of elements in the list and k is an integer constant. For example, if n = 19 and k = 3 then we should return 16^{th} node.

Solution: For this problem the value of n is not known in advance and it is the same as finding the k^{th} element from the end of the the linked list.

```
public ListNode modularNodes(ListNode *head, int k){
    ListNode modularNode=null;
    int i=0;
    if(k<=0) return null;
    for (i=0; i < k; i++){
        if(head!=null)
            head = head.getNext();
        else
            return null;
    }
    while(head!= null)
        modularNode = modularNode.getNext();
        head = head.getNext();
    }
    return modularNode;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-49 **Find fractional node:** Given a singly linked list, write a function to find the $\frac{n}{k}th$ element, where n is the number of elements in the list.

Solution: For this problem the value of n is not known in advance.

```
public ListNode fractionalNodes(ListNode head, int k){
    ListNode fractionalNode;
    int i=0;
    if(k<=0)
        return null;
    for (;head!= null; head = head.getNext()){
        if(i%k == 0){
            if(fractionalNode!=null)
                fractionalNode = head;
            else fractionalNode = fractionalNode.getNext();
        }
        i++;
    }
    return fractionalNode;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-50 Median in an infinite series of integers

Solution: Median is the middle number in a sorted list of numbers (if we have an odd number of elements). If we have an even number of elements, the median is the average of two middle numbers in a sorted list of numbers.

We can solve this problem with linked lists (with both sorted and unsorted linked lists).

First, let us try with an *unsorted* linked list. In an unsorted linked list, we can insert the element either at the head or at the tail. The disadvantage with this approach is that finding the median takes O(n). Also, the insertion operation takes O(1).

Now, let us try with a *sorted* linked list. We can find the median in O(1) time if we keep track of the middle elements. Insertion to a particular location is also O(1) in any linked list. But, finding the right location to insert is not O($logn$) as in a sorted array, it is instead O(n) because we can’t perform binary search in a linked list even if it is sorted.

So, using a sorted linked list isn’t worth the effort as insertion is O(n) and finding median is O(1), the same as the sorted array. In the sorted array the insertion is linear due to shifting, but here it's linear because we can't do a binary search in a linked list.

Note: For an efficient algorithm refer to the *Priority Queues and Heaps* chapter.

Problem-51 Consider the following Java program code, whose runtime F is a function of the input size n.

```
java.util.ArrayList<Integer> list = new java.util.ArrayList<Integer>();
for( int i = 0 ; i < n; i ++)
    list.add (0 , i ) ;
```

Which of the following is correct?

A) $F(n)=\Theta(n)$
B) $F(n)=\Theta(n^2)$
C) $F(n)=\Theta(n^3)$
D) $F(n)=\Theta(n^4)$
F) $F(n)=\Theta(nlogn)$

Solution: B. The operation list.add(0,i) on ArrayList has linear time complexity, with respect to the current size of the data structure. Therefore, overall we have quadratic time complexity.

Problem-52 Consider the following Java program code, whose runtime F is a function of the input size n.

```
java.util.ArrayList<Integer> list = new java.util.ArrayList<Integer>();
for(int i = 0 ; i < n; i ++)
    list.add ( i , i ) ;
for(int j = 0 ; j < n; j++)
    list.remove(n-j -1 );
```

Which of the following is correct?

A) $F(n)=\Theta(n)$
B) $F(n)=\Theta(n^2)$
C) $F(n)=\Theta(n^3)$
D) $F(n)=\Theta(n^4)$
F) $F(n)= \Theta(nlogn)$

Solution: A. Both loop bodies have constant time complexity because they operate the end of the ArrayList.

Problem-53 Consider the following Java program code, whose runtime F is a function of the input size n.

```
java.util.LinkedList<Integer> k = new java.util.LinkedList<Integer>();
for(int i = 0 ; i < n; i ++)
   for(int j = 0 ; j < n; j++)
        k.add(k.size()/2,j);
```

Which of the following is correct?

Solution: D. The LinkedList grows to quadratic size during the execution of the program fragment. Thus the body of the inner loop has quadratic complexity. The inner loop itself is executed n^2 times.

Problem-54 Given a singly linked list L: L_1-> L_2-> L_3...-> L_{n-1}-> L_n, reorder it to: L_1-> L_n-> L_2-> L_{n-1}......

Solution: We divide the list in two parts for instance 1->2->3->4->5 will become 1->2->3 and 4->5, we have to reverse the second list and give a list that alternates both. The split is done with a slow and fast pointer. First solution (using a stack for reversing the list):

```java
public class ReorderLists {
   public void reorderList(ListNode head) {
      if(head == null)
         return;
     ListNode slowPointer = head;
     ListNode fastPointer = head.next;
     while(fastPointer!=null && fastPointer.next !=null){
        fastPointer = fastPointer.next.next;
        slowPointer = slowPointer.next;
     }
      ListNode head2 = slowPointer.next;
      slowPointer.next = null;
      LinkedList<ListNode> queue = new LinkedList<ListNode>();
      while(head2!=null){
         ListNode temp = head2;
         head2 = head2.next;
         temp.next =null;
         queue.push(temp);
      }
      while(!queue.isEmpty()){
         ListNode temp = queue.pop();
         temp.next = head.next;
         head.next = temp;
         head = temp.next;
      }
   }
}
```

Alternative Approach:

```
public class ReorderLists {
  public void reorderList(ListNode head) {
    if(head == null)
      return;
    ListNode slowPointer = head;
    ListNode fastPointer = head.next;
    while(fastPointer!=null && fastPointer.next !=null){
      fastPointer = fastPointer.next.next;
      slowPointer = slowPointer.next;
    }
    ListNode head2 = slowPointer.next;
    slowPointer.next = null;
    head2= reverseList(head2);
    alternate (head, head2);
  }
  private ListNode reverseList(ListNode head){
    if (head == null)
      return null;
    ListNode reversedList = head;
    ListNode pointer = head.next;
    reversedList.next=null;
    while (pointer !=null){
      ListNode temp = pointer;
      pointer = pointer.next;
      temp.next = reversedList;
      reversedList = temp;
    }
    return reversedList;
  }
  private void alternate (ListNode head1, ListNode head2){
    ListNode pointer = head1;
    head1 = head1.next;
    boolean nextList1 = false;
    while(head1 != null || head2 != null){
      if((head2 != null && !nextList1) || (head1==null)){
        pointer.next = head2;
        head2 = head2.next;
        nextList1 = true;
        pointer = pointer.next;
      }
      else {
        pointer.next = head1;
        head1 = head1.next;
        nextList1 = false;
        pointer = pointer.next;
      }
    }
  }
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-55 Which sorting algorithm is easily adaptable to singly linked lists?

Solution: Simple Insertion sort is easily adabtable to singly linked lists. To insert an element, the linked list is traversed until the proper position is found, or until the end of the list is reached. It is inserted into the list by merely adjusting the pointers without shifting any elements, unlike in the array. This reduces the time required for insertion but not the time required for searching for the proper position.

Problem-56 How do you implement insertion sort for linked lists?

Solution:

```java
public static ListNode insertionSortList(ListNode head) {
	if (head == null || head.next == null)
		return head;
	ListNode newHead = new ListNode(head.data);
	ListNode pointer = head.next;
	// loop through each element in the list
	while (pointer != null) {
		// insert this element to the new list
		ListNode innerPointer = newHead;
		ListNode next = pointer.next;
		if (pointer.data <= newHead.data) {
			ListNode oldHead = newHead;
			newHead = pointer;
			newHead.next = oldHead;
		} else {
			while (innerPointer.next != null) {
				if (pointer.data > innerPointer.data && pointer.data <= innerPointer.next.data) {
					ListNode oldNext = innerPointer.next;
					innerPointer.next = pointer;
					pointer.next = oldNext;
				}
				innerPointer = innerPointer.next;
			}
			if (innerPointer.next == null && pointer.data > innerPointer.data) {
				innerPointer.next = pointer;
				pointer.next = null;
			}
		}
		// finally
		pointer = next;
	}
	return newHead;
}
```

Note: For details on insertion sort refer Sorting chapter.

Problem-57 Given a list, rotate the list to the right by k places, where k is non-negative. For example: Given 1_->2->3->4->5->NULL and k = 2, return 4->5->1->2->3->NULL.

Solution:

```
public ListNode rotateRight(ListNode head, int n) {
    if(head == null || head.next == null)
        return head;
    ListNode rotateStart = head, rotateEnd = head;
    while(n-- > 0){
        rotateEnd = rotateEnd.next;
        if(rotateEnd == null){
            rotateEnd = head;
        }
    }
    if(rotateStart == rotateEnd)
        return head;
    while(rotateEnd.next != null){
        rotateStart = rotateStart.next;
        rotateEnd = rotateEnd.next;
    }
    ListNode temp = rotateStart.next;
    rotateEnd.next = head;
    rotateStart.next = null;
    return temp;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-58 You are given two linked lists representing two non-negative numbers. The digits are stored in reverse order and each of their nodes contain a single digit. Add the two numbers and return it as a linked list. For example with input: (3 -> 4 -> 3) + (5 -> 6 -> 4); the output should be 8 -> 0 -> 8.

Solution:

```
public ListNode addTwoNumbers(ListNode list1, ListNode list2) {
    if(list1 == null)
        return list2;
    if(list2 == null)
        return list1;
    ListNode head = new ListNode(0);
    ListNode cur = head;
    int advance = 0;
    while(list1 != null && list2 != null){
        int sum = list1.data + list2.data + advance;
        advance = sum / 10;
        sum = sum % 10;
        cur.next = new ListNode(sum);
        cur = cur.next;
        list1 = list1.next;
        list2 = list2.next;
    }
    if(list1 != null){
        if(advance != 0)
            cur.next = addTwoNumbers(list1, new ListNode(advance));
        else
            cur.next = list1;
    }else if(list2 != null){
        if(advance != 0)
            cur.next = addTwoNumbers(list2, new ListNode(advance));
        else
            cur.next = list2;
    }else if(advance != 0){
        cur.next = new ListNode(advance);
    }
    return head.next;
}
```

Problem-59 Given a linked list and a value K, partition it such that all nodes less than K come before nodes greater than or equal to K. You should preserve the original relative order of the nodes in each of the two partitions. For example, given 1->4->3->2->5->2 and K = 3, return 1->2->2->4->3->5.

Solution:

```
public ListNode partition(ListNode head, int K) {
    ListNode root = new ListNode(0);
    ListNode pivot = new ListNode(K);
    ListNode rootNext = root, pivotNext = pivot;
    ListNode currentNode = head;
    while(currentNode != null){
        ListNode next = currentNode.next;
        if(currentNode.data >= K){
            pivotNext.next = currentNode;
            pivotNext = currentNode;
            pivotNext.next = null;
        }else{
            rootNext.next = currentNode;
            rootNext = currentNode;
        }
        currentNode = next;
    }
    rootNext.next = pivot.next;
    return root.next;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-60 Merge k sorted linked lists and return it as one sorted list.

Solution: Refer Priority Queues chapter.

Problem-66 Given a unordered linked list, how do you remove duplicates in it?

Solution:

```java
public static void removeDuplicates2(ListNode head) {
        ListNode curr = head;
        if(curr == null || curr.getNext() == null) {
                return; // 0 or 1 nodes in the list so no duplicates
        }
        ListNode curr2;
        ListNode prev;
        while(curr != null) {
                curr2 = curr.getNext();
                prev = curr;
                while(curr2 != null) {
                        // check and see if curr and curr2 values are the same, if they are then delete curr2
                        if(curr.getData()==curr2.getData()) {
                                prev.setNext(curr2.getNext());
                        }
                        prev = curr2;
                        curr2 = curr2.getNext();
                }
                curr = curr.getNext();
        }
}
```

Time Complexity: O(n^2). Space Complexity: O(1).

Problem-67 Can reduce the time complexity of Problem-61?

Solution: We can simply use hash table and check whether an element already exist.

```
// using a temporary buffer O(n)
public static void removeDuplicates(ListNode head) {
        Map<Integer, Boolean> mapper = new HashMap<Integer, Boolean>();
        ListNode curr = head;
        ListNode next;
        while (curr.getNext() != null) {
                next = curr.getNext();
                if(mapper.get(next.getData()) != null) {
                        // already seen it, so delete
                        curr.setNext(next.getNext());
                } else {
                        mapper.put(next.getData(), true);
                        curr = curr.getNext();
                }
        }
}
```

Time Complexity: O(n). Space Complexity: O(n) for hash table.

Problem-68 Given a linked list with even and odd numbers, create an algorithm for making changes to the list in such a way that all even numbers appear at the beginning.

Solution: To solve this problem, we can use the splitting logic. While traversing the list, split the linked list into two: one contains all even nodes and the other contains all odd nodes. Now, to get the final list, we can simply append the odd node linked list after the even node linked list.

To split the linked list, traverse the original linked list and move all odd nodes to a separate linked list of all odd nodes. At the end of the loop, the original list will have all the even nodes and the odd node list will have all the odd nodes. To keep the ordering of all nodes the same, we must insert all the odd nodes at the end of the odd node list.

Time Complexity: O(n). Space Complexity: O(1).

Problem-69 Given two sorted linked lists, given an algorithm for the printing common elements of them.

Solution: The solution is based on merge sort logic. Assume the given two linked lists are: list1 and list2. Since the elements are in sorted order, we run a loop till we reach the end of either of the list. We compare the values of list1 and list2. If the values are equal, we add it to the common list. We move list1 / list2/ both nodes ahead to the next pointer if the values pointed by list1 was less / more / equal to the value pointed by list2.

Time complexity O(m + n), where m is the length of list1 and n is the length of list2. Space Complexity: O(1).

```
public static ListNode commonElement(ListNode list1, ListNode list2) {
	ListNode temp =  new ListNode(0);
	ListNode head = temp;
	while (list1 != null && list2 != null) {
		if (list1.data == list2.data) {
			head.next = new ListNode(list1.data); // Copy common element.
			list1 = list1.next;
			list2 = list2.next;
			head = head.next;
		} else if (list1.data > list2.data) {
			list2 = list2.next;
		} else { // list1.data < list2.data
			list1 = list1.next;
		}
	}
	return temp.next;
}
```

Stacks

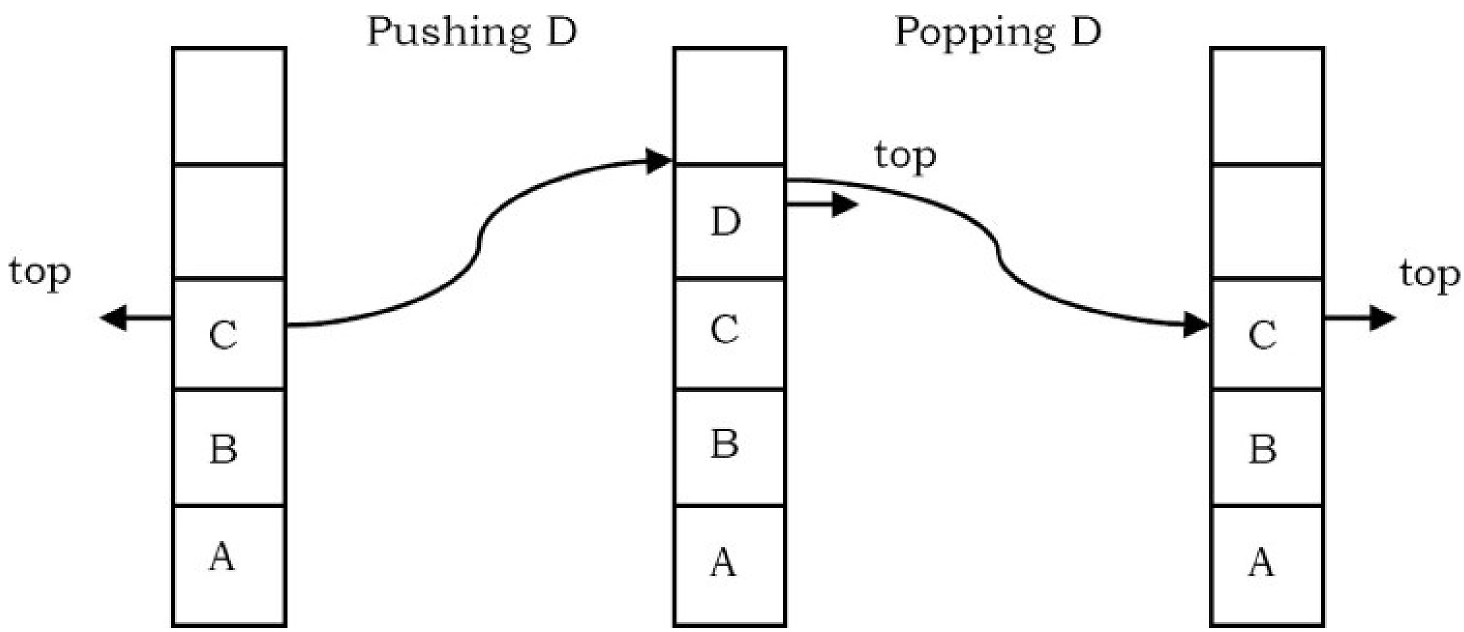
Chapter 4

4.1 What is a Stack?

A stack is a simple data structure used for storing data (similar to Linked Lists). In a stack, the order in which the data arrives is important. A pile of plates in a cafeteria is a good example of a stack. The plates are added to the stack as they are cleaned and they are placed on the top. When a plate, is required it is taken from the top of the stack. The first plate placed on the stack is the last one to be used.

Definition: A *stack* is an ordered list in which insertion and deletion are done at one end, called *top*. The last element inserted is the first one to be deleted. Hence, it is called the Last in First out (LIFO) or First in Last out (FILO) list.

Special names are given to the two changes that can be made to a stack. When an element is inserted in a stack, the concept is called *push*, and when an element is removed from the stack, the concept is called *pop*. Trying to pop out an empty stack is called *underflow* and trying to push an element in a full stack is called *overflow*. Generally, we treat them as exceptions. As an example, consider the snapshots of the stack.



4.2 How Stacks are used

Consider a working day in the office. Let us assume a developer is working on a long-term project. The manager then gives the developer a new task which is more important. The developer puts the long-term project aside and begins work on the new task. The phone rings, and this is the highest priority as it must be answered immediately. The developer pushes the present task into the pending tray and answers the phone.

When the call is complete the task that was abandoned to answer the phone is retrieved from the pending tray and work progresses. To take another call, it may have to be handled in the same manner, but eventually the new task will be finished, and the developer can draw the long-term project from the pending tray and continue with that.

4.3 Stack ADT

The following operations make a stack an ADT. For simplicity, assume the data is an integer type.

Main stack operations

- void push(int data): Inserts *data* onto stack.
- int pop(): Removes and returns the last inserted element from the stack.

Auxiliary stack operations

- int Top(): Returns the last inserted element without removing it.
- int Size(): Returns the number of elements stored in the stack.
- int IsEmptyStack(): Indicates whether any elements are stored in the stack or not.
- int IsFullStack(): Indicates whether the stack is full or not.

4.4 Exceptions

Attempting the execution of an operation may sometimes cause an error condition, called an exception. Exceptions are said to be "thrown" by an operation that cannot be executed. In the Stack ADT, operations pop and top cannot be performed if the stack is empty. Attempting the execution of pop (top) on an empty stack throws an exception. Trying to push an element in a full stack throws an exception.

4.5 Applications

Following are some of the applications in which stacks play an important role.

Direct applications

- Balancing of symbols
- Infix-to-postfix conversion
- Evaluation of postfix expression
- Implementing function calls (including recursion)
- Finding of spans (finding spans in stock markets, refer to *Problems* section)
- Page-visited history in a Web browser [Back Buttons]
- Undo sequence in a text editor
- Matching Tags in HTML and XML

Indirect applications

- Auxiliary data structure for other algorithms (Example: Tree traversal algorithms)
- Component of other data structures (Example: Simulating queues, refer *Queues* chapter)

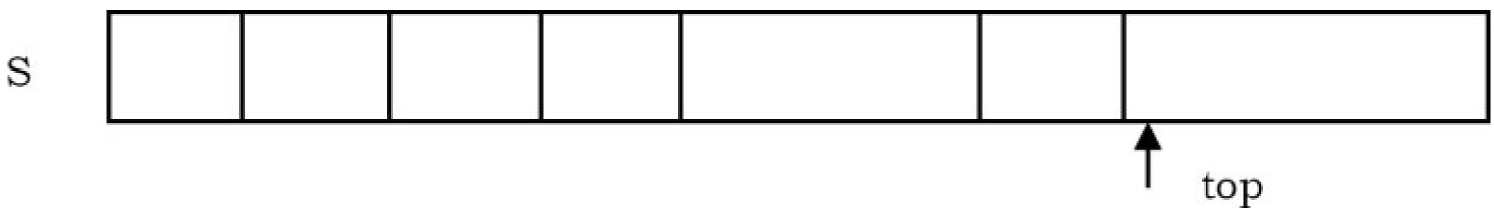
4.6 Implementation

There are many ways of implementing stack ADT; below are the commonly used methods.

- Simple array based implementation
- Dynamic array based implementation
- Linked lists implementation

Simple Array Implementation

This implementation of stack ADT uses an array. In the array, we add elements from left to right and use a variable to keep track of the index of the top element.



The array storing the stack elements may become full. A push operation will then throw a *full stack exception*. Similarly, if we try deleting an element from an empty stack it will throw *stack empty exception*.

```java
public class FixedSizeArrayStack{
	// Length of the array used to implement the stack.
	protected int capacity;
	// Default array capacity.
	public static final int CAPACITY = 10;
	// Array used to implement the stack.
	protected int[] stackRep;
	// Index of the top element of the stack in the array.
	protected int top = -1;
	// Initializes the stack to use an array of default length.
	public FixedSizeArrayStack() {
		this(CAPACITY); // default capacity
	}
	// Initializes the stack to use an array of given length.
	public FixedSizeArrayStack(int cap) {
		capacity = cap;
		stackRep = new int[capacity]; // compiler may give warning, but this is ok
	}
	// Returns the number of elements in the stack. This method runs in O(1) time.
	public int size() {
		return (top + 1);
	}
	// Testes whether the stack is empty. This method runs in O(1) time.
	public boolean isEmpty() {
		return (top < 0);
	}
	// Inserts an element at the top of the stack. This method runs in O(1) time.
	public void push(int data) throws Exception {
		if (size() == capacity)
			throw new Exception("Stack is full.");
		stackRep[++top] = data;
	}
	// Inspects the element at the top of the stack. This method runs in O(1) time.
	public int top() throws Exception {
		if (isEmpty())
			throw new Exception("Stack is empty.");
		return stackRep[top];
	}
	// Removes the top element from the stack. This method runs in O(1) time.
	public int pop() throws Exception {
		int data;
		if (isEmpty())
		   throw new Exception("Stack is empty.");
		data = stackRep[top];
		stackRep[top--] = Integer.MIN_VALUE;
		return data;
	}
	// Returns a string representation of the stack as a list of elements, with
	// the top element at the end: [ ... , prev, top ]. This method runs in O(n)
	// time, where n is the size of the stack.
	public String toString() {
		String s;
		s = "[";
		if (size() > 0)
			s += stackRep[0];
		if (size() > 1)
			for (int i = 1; i <= size() - 1; i++) {
				s += ", " + stackRep[i];
			}
		return s + "]";
	}
}
```

Performance & Limitations

Performance: Let *n* be the number of elements in the stack. The complexities of stack operations with this representation can be given as:

Space Complexity (for *n* push operations)	O(*n*)
Time Complexity of push()	O(1)
Time Complexity of pop()	O(1)
Time Complexity of size()	O(1)
Time Complexity of isEmpty()	O(1)
Time Complexity of isFullStack()	O(1)
Time Complexity of deleteStack()	O(1)

Limitations: The maximum size of the stack must first be defined and it cannot be changed. Trying to push a new element into a full stack causes an implementation-specific exception.

Dynamic Array Implementation

First, let's consider how we implemented a simple array based stack. We took one index variable *top* which points to the index of the most recently inserted element in the stack. To insert (or push) an element, we increment *top* index and then place the new element at that index.

Similarly, to delete (or pop) an element we take the element at *top* index and then decrement the *top* index. We represent an empty queue with *top* value equal to –1. The issue that still needs to be resolved is what we do when all the slots in the fixed size array stack are occupied?

First try: What if we increment the size of the array by 1 every time the stack is full?

- Push(); increase size of S[] by 1
- Pop(): decrease size of S[] by 1

Problems with this approach?

This way of incrementing the array size is too expensive. Let us see the reason for this. For example, at $n = 1$, to push an element create a new array of size 2 and copy all the old array elements to the new array, and at the end add the new element. At $n = 2$, to push an element create a new array of size 3 and copy all the old array elements to the new array, and at the end add the new element.

Similarly, at $n = n - 1$, if we want to push an element create a new array of size n and copy all the old array elements to the new array and at the end add the new element. After n push operations the total time $T(n)$ (number of copy operations) is proportional to $1 + 2 + ... + n \approx O(n^2)$.

Alternative Approach: Repeated Doubling

Let us improve the complexity by using the array *doubling* technique. If the array is full, create a new array of twice the size, and copy the items. With this approach, pushing n items takes time proportional to n (not n^2).

For simplicity, let us assume that initially we started with $n = 1$ and moved up to $n = 32$. That means, we do the doubling at 1, 2, 4, 8, 16. The other way of analyzing the same approach is: at $n = 1$, if we want to add (push) an element, double the current size of the array and copy all the elements of the old array to the new array.

At $n = 1$, we do 1 copy operation, at $n = 2$, we do 2 copy operations, and at $n = 4$, we do 4 copy operations and so on. By the time we reach $n = 32$, the total number of copy operations is $1+2+4+8+16 = 31$ which is approximately equal to $2n$ value (32). If we observe carefully, we are doing the doubling operation *logn* times.

Now, let us generalize the discussion. For n push operations we double the array size *logn* times. That means, we will have *logn* terms in the expression below. The total time $T(n)$ of a series of n push operations is proportional to

$$\begin{aligned}1 + 2 + 4 + 8 \ldots + \frac{n}{4} + \frac{n}{2} + n &= n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} \ldots + 4 + 2 + 1 \\ &= n\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots + \frac{4}{n} + \frac{2}{n} + \frac{1}{n}\right) \\ &= n(2) \approx 2n = O(n)\end{aligned}$$

$T(n)$ is $O(n)$ and the amortized time of a push operation is $O(1)$.

```
public class DynamicArrayStack{
    // Length of the array used to implement the stack.
    protected int capacity;
```

```
// Default array capacity.
public static final int CAPACITY = 16;     // power of 2
public static int MINCAPACITY=1<<15;   // power of 2
// Array used to implement the stack.
protected int[] stackRep;
// Index of the top element of the stack in the array.
protected int top = -1;
// Initializes the stack to use an array of default length.
public DynamicArrayStack() {
        this(CAPACITY);                        // default capacity
}
// Initializes the stack to use an array of given length.
public DynamicArrayStack(int cap) {
        capacity = cap;
        stackRep = new int[capacity];     // compiler may give warning, but this is ok
}
// Returns the number of elements in the stack. This method runs in O(1) time.
public int size() {
        return (top + 1);
}
// Testes whether the stack is empty. This method runs in O(1) time.
public boolean isEmpty() {
        return (top < 0);
}
// Inserts an element at the top of the stack. This method runs in O(1) time.
public void push(int data) throws Exception {
        if (size() == capacity)
            expand();
        stackRep[++top] = data;
}
private void expand() {
        int length = size();
        int[] newstack=new int[length<<1];
        System.arraycopy(stackRep,0,newstack,0,length);
        stackRep=newstack;
        this.capacity = this.capacity<<1;
}
// dynamic array operation: shrinks to 1/2 if more than than 3/4 empty
private void shrink() {
        int length = top + 1;
        if(length<=MINCAPACITY || top<<2 >= length)
            return;
        length=length + (top<<1); // still means shrink to at 1/2 or less of the heap
        if(top<MINCAPACITY) length = MINCAPACITY;
        int[] newstack=new int[length];
        System.arraycopy(stackRep,0,newstack,0,top+1);
        stackRep=newstack;
        this.capacity = length;
}
// Inspects the element at the top of the stack. This method runs in O(1) time.
public int top() throws Exception {
        if (isEmpty())
            throw new Exception("Stack is empty.");
        return stackRep[top];
}
// Removes the top element from the stack. This method runs in O(1) time.
public int pop() throws Exception {
        int data;
        if (isEmpty())
                throw new Exception("Stack is empty.");
        data = stackRep[top];
        stackRep[top--] = Integer.MIN_VALUE; // dereference S[top] for garbage collection.
        shrink();
        return data;
```

```
	}
	// Returns a string representation of the stack as a list of elements, with
	// the top element at the end: [ ... , prev, top ]. This method runs in O(n)
	// time, where n is the size of the stack.
	public String toString() {
		String s;
		s = "[";
		if (size() > 0)
			s += stackRep[0];
		if (size() > 1)
			for (int i = 1; i <= size() - 1; i++) {
				s += ", " + stackRep[i];
			}
		return s + "]";
	}
}
```

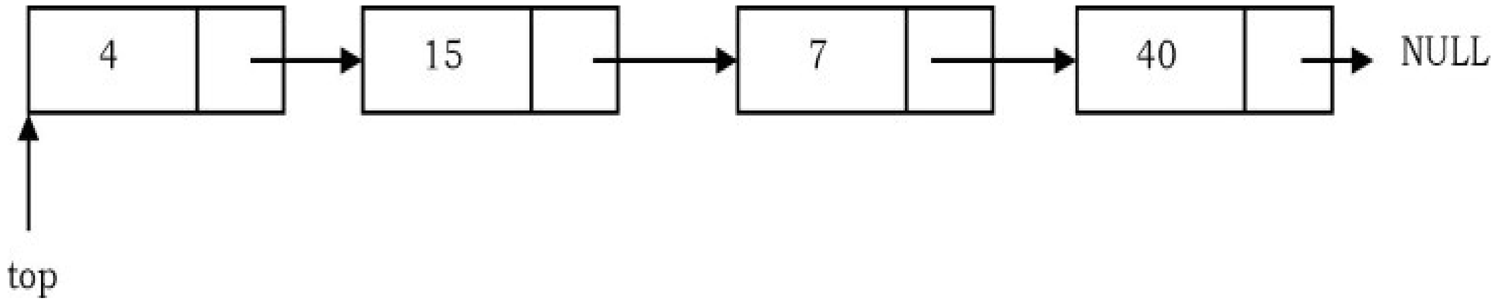
Performance: Let *n* be the number of elements in the stack. The complexities for operations with this representation can be given as:

Space Complexity (for *n* push operations)	O(*n*)
Time Complexity of create Stack: DynArrayStack ()	O(1)
Time Complexity of push()	O(1) (Average)
Time Complexity of pop()	O(1)
Time Complexity of top()	O(1)
Time Complexity of isEmpty()	O(1)
Time Complexity of isStackFull ()	O(1)
Time Complexity of deleteStack()	O(1)

Note: Too many doublings may cause memory overflow exception.

Linked List Implementation

The other way of implementing stacks is by using Linked lists. Push operation is implemented by inserting element at the beginning of the list. Pop operation is implemented by deleting the node from the beginning (the header/top node).


4
15
7
40
NULL
top

```
public class LinkedStack<T>{
    private int length;                    // indicates the size of the linked list
    private ListNode top;
    //  Constructor: Creates an empty stack.
    public LinkedStack()  {
            length = 0;
            top = null;
    }
    //  Adds the specified data to the top of this stack.
    public void push (int data)  {
            ListNode temp = new ListNode (data);
            temp.setNext(top);
            top = temp;
            length++;
    }
    //  Removes the data at the top of this stack and returns a
    //  reference to it. Throws an EmptyStackException if the stack
    //  is empty.
    public int pop() throws EmptyStackException{
            if (isEmpty())
                    throw new EmptyStackException();
            int result = top.getdata();
            top = top.getNext();
            length--;
            return result;
    }
    // Returns a reference to the data at the top of this stack.
    // The data is not removed from the stack.  Throws an
    // EmptyStackException if the stack is empty.
    public int peek() throws EmptyStackException{
            if (isEmpty())
                    throw new EmptyStackException();
            return top.getdata();
    }
    // Returns true if this stack is empty and false otherwise.
    public boolean isEmpty(){
            return (length == 0);
    }
    // Returns the number of elements in the stack.
    public int size(){
            return length;
    }
    // Returns a string representation of this stack.
    public String toString(){
            String result = "";
            ListNode current = top;
            while (current != null){
                    result = result + current.toString() + "\n";
                    current = current.getNext();
            }
            return result;
    }
}
```

Performance

Let n be the number of elements in the stack. The complexities for operations with this representation can be given as:

Space Complexity (for n push operations)	$O(n)$
Time Complexity of create Stack: DynArrayStack()	$O(1)$
Time Complexity of push()	$O(1)$ (Average)
Time Complexity of pop()	$O(1)$
Time Complexity of top()	$O(1)$
Time Complexity of isEmpty()	$O(1)$
Time Complexity of deleteStack()	$O(n)$

4.7 Comparison of Implementations

Comparing Incremental Strategy and Doubling Strategy

We compare the incremental strategy and doubling strategy by analyzing the total time $T(n)$ needed to perform a series of n push operations. We start with an empty stack represented by an array of size 1. We call *amortized* time of a push operation is the average time taken by a push over the series of operations, that is, $T(n)/n$.

Incremental Strategy: The amortized time (average time per operation) of a push operation is $O(n)$ $[O(n^2)/n]$.

Doubling Strategy: In this method, the amortized time of a push operation is $O(1)$ $[O(n)/n]$.

Note: For reasoning, refer to the *Implementation* section.

Comparing Array Implementation & Linked List Implementation

Array Implementation

- Operations take constant time.
- Expensive doubling operation every once in a while.
- Any sequence of n operations (starting from empty stack) - “amortized” bound takes

time proportional to *n*.

Linked List Implementation

- Grows and shrinks gracefully.
- Every operation takes constant time O(1).
- Every operation uses extra space and time to deal with references.

4.8 Stacks: Problems & Solutions

Problem-1 Discuss how stacks can be used for checking balancing of symbols.

Solution: Stacks can be used to check whether the given expression has balanced symbols. This algorithm is very useful in compilers. Each time the parser reads one character at a time. If the character is an opening delimiter such as (, {, or [- then it is written to the stack. When a closing delimiter is encountered like), }, or]- the stack is popped. The opening and closing delimiters are then compared. If they match, the parsing of the string continues. If they do not match, the parser indicates that there is an error on the line. A linear-time O(*n*) algorithm based on stack can be given as:

Algorithm

a) Create a stack.
b) while (end of input is not reached)
   1) If the character read is not a symbol to be balanced, ignore it.
   2) If the character is an opening symbol like (, [, {, push it onto the stack
   3) If it is a closing symbol like),],}, then if the stack is empty report an error. Otherwise pop the stack.
   4) If the symbol popped is not the corresponding opening symbol, report an error.
c) At end of input, if the stack is not empty report an error

Example	Valid?	Description
(A+B)+(C-D)	Yes	The expression has a balanced symbol
((A+B)+(C-D)	No	One closing brace is missing
((A+B)+[C-D])	Yes	Opening and immediate closing braces correspond
((A+B)+[C-D]}	No	The last closing brace does not correspond with the first opening parenthesis

For tracing the algorithm let us assume that the input is: () (() [()])

Input Symbol, A[i]	Operation	Stack	Output
(	Push (	(	
)	Pop (Test if (and A[i] match? YES		
(	Push (	(	
(	Push (	((	
)	Pop (Test if(and A[i] match? YES	(	
[	Push [	([	
(	Push (	([(	
)	Pop (Test if(and A[i] match? YES	([	
]	Pop [Test if [and A[i] match? YES	(	
)	Pop (Test if(and A[i] match? YES		
	Test if stack is Empty? YES		TRUE

Time Complexity: O(n). Since we are scanning the input only once. Space Complexity: O(n) [for stack].

```
public boolean isValidSymbolPattern(String s) {
    Stack<Character> stk = new Stack<Character>();
    if(s == null || s.length() == 0)
        return true;
    for(int i = 0; i < s.length(); i++){
        if(s.charAt(i) == ')'){
            if(!stk.isEmpty() && stk.peek() == '(')
                stk.pop();
            else
                return false;
        }else if(s.charAt(i) == ']'){
            if(!stk.isEmpty() && stk.peek() == '[')
                stk.pop();
            else
                return false;
        }else if(s.charAt(i) == '}'){
            if(!stk.isEmpty() && stk.peek() == '{')
                stk.pop();
            else
                return false;
        }else{
            stk.push(s.charAt(i));
        }
    }
    if(stk.isEmpty())
        return true;
    else
        return false;
}
```

Problem-2 Discuss infix to postfix conversion algorithm using stack.

Solution: Before discussing the algorithm, first let us see the definitions of infix, prefix and postfix expressions.

Infix: An infix expression is a single letter, or an operator, proceeded by one infix string and followed by another Infix string.

A
A+B
(A+B)+ (C-D)

Prefix: A prefix expression is a single letter, or an operator, followed by two prefix strings. Every prefix string longer than a single variable contains an operator, first operand and second operand.

A
+AB
++AB-CD

Postfix: A postfix expression (also called Reverse Polish Notation) is a single letter or an operator, preceded by two postfix strings. Every postfix string longer than a single variable contains first and second operands followed by an operator.

A
AB+
AB+CD-+

Prefix and postfix notions are methods of writing mathematical expressions without parenthesis. Time to evaluate a postfix and prefix expression is O(*n*), were *n* is the number of elements in the array.

Infix	Prefix	Postfix
A+B	+AB	AB+
A+B-C	-+ABC	AB+C-
(A+B)*C-D	-*+ABCD	AB+C*D-

Now, let us focus on the algorithm. In infix expressions, the operator precedence is implicit unless we use parentheses. Therefore, for the infix to postfix conversion algorithm we have to define the operator precedence (or priority) inside the algorithm. The table shows the precedence and their associativity (order of evaluation) among operators.

Token	Operator	Precedence	Associativity
( ) [ ] → .	function call array element struct or union member	17	left-to-right
-- ++	increment, decrement	16	left-to-right
-- ++ ! - - + & * sizeof	decrement, increment logical not one's complement unary minus or plus address or indirection size (in bytes)	15	right-to-left
(type)	type cast	14	right-to-left
* / %	multiplicative	13	Left-to-right
+ -	binary add or subtract	12	left-to-right
<< >>	shift	11	left-to-right
> >= < <=	relational	10	left-to-right
== !=	equality	9	left-to-right
&	bitwise and	8	left-to-right
^	bitwise exclusive or	7	left-to-right
|	bitwise or	6	left-to-right
&&	logical and	5	left-to-right
||	logical or	4	left-to-right
?:	conditional	3	right-to-left
= += -= /= *= %= <<= >>= &= ^=	assignment	2	right-to-left
,	comma	1	left-to-right

Important Properties

- Let us consider the infix expression 2 + 3 * 4 and its postfix equivalent 2 3 4 *+. Notice that between infix and postfix the order of the numbers (or operands) is unchanged. It is 2 3 4 in both cases. But the order of the operators * and + is affected in the two expressions.
- Only one stack is enough to convert an infix expression to postfix expression. The stack that we use in the algorithm will be used to change the order of operators from infix to postfix. The stack we use will only contain operators and the open parentheses symbol '('. Postfix expressions do not contain parentheses. We shall not output the parentheses in the postfix output.

Algorithm

a) Create a stack
b) for each character t in the input stream

```
if(t is an operand)  output t
else if(t is a right parenthesis)
        Pop and output tokens until a left parenthesis is popped (but not output)
else // t is an operator or left parenthesis
        pop and output tokens until one of lower priority than t is encountered or a left  parenthesis
        is encountered or the stack is empty
        Push t
```

c) pop and output tokens until the stack is empty

For better understanding let us trace out an example: A * B- (C + D) + E

Input Character	Operation on Stack	Stack	Postfix Expression
A		Empty	A
*	Push	*	A
B		*	AB
-	Check and Push	-	AB*
(	Push	-(	AB*
C		-(	AB*C
+	Check and Push	-(+	AB*C
D			AB*CD
)	Pop and append to postfix till '('	-	AB*CD+
+	Check and Push	+	AB*CD+-
E		+	AB*CD+-E
End of input	Pop till empty		AB*CD+-E+

Problem-3 For a given array with *n* symbols how many stack permutations are possible?

Solution: The number of stack permutations with *n* symbols is represented by *Catalan number* and we will discuss this in *Dynamic Programming* chapter.

Problem-4 Discuss postfix evaluation using stacks?

Solution:

Algorithm:

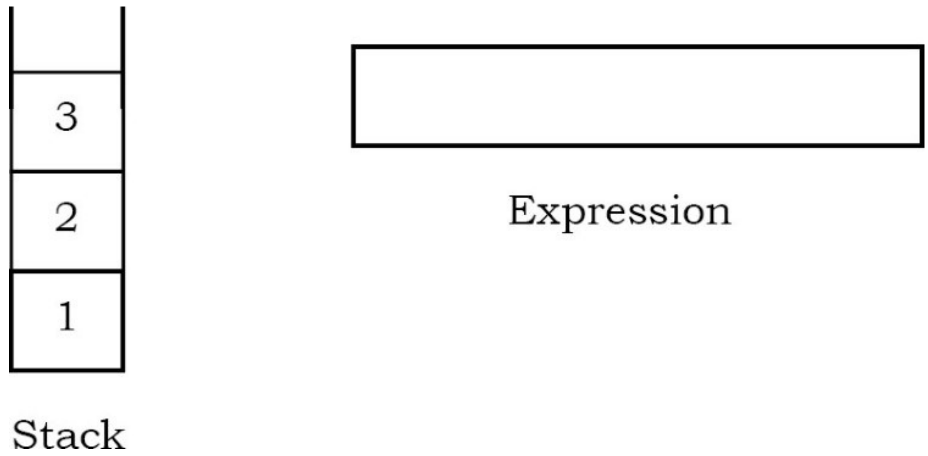
1. Scan the Postfix string from left to right.
2. Initialize an empty stack.
3. Repeat steps 4 and 5 till all the characters are scanned.
4. If the scanned character is an operand, push it onto the stack.
5. If the scanned character is an operator, and if the operator is a unary operator, then

pop an element from the stack. If the operator is a binary operator, then pop two elements from the stack. After popping the elements, apply the operator to those popped elements. Let the result of this operation be retVal onto the stack.

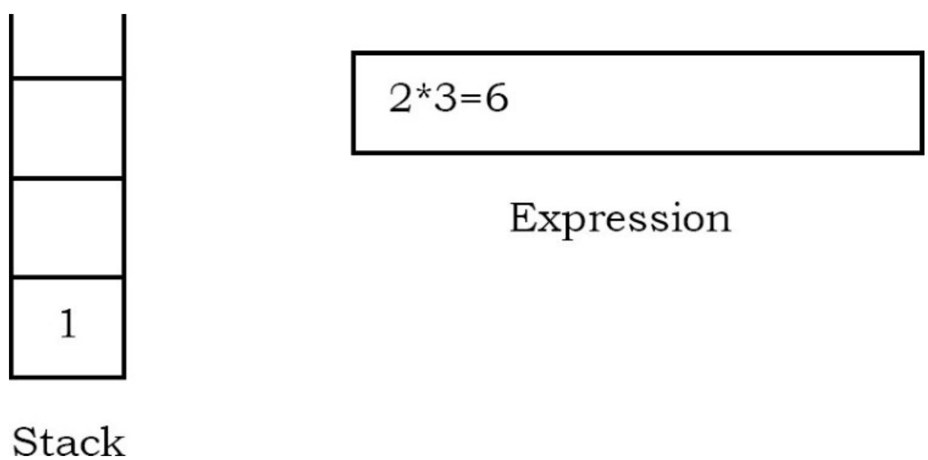
6 After all characters are scanned, we will have only one element in the stack.

7 Return top of the stack as result.

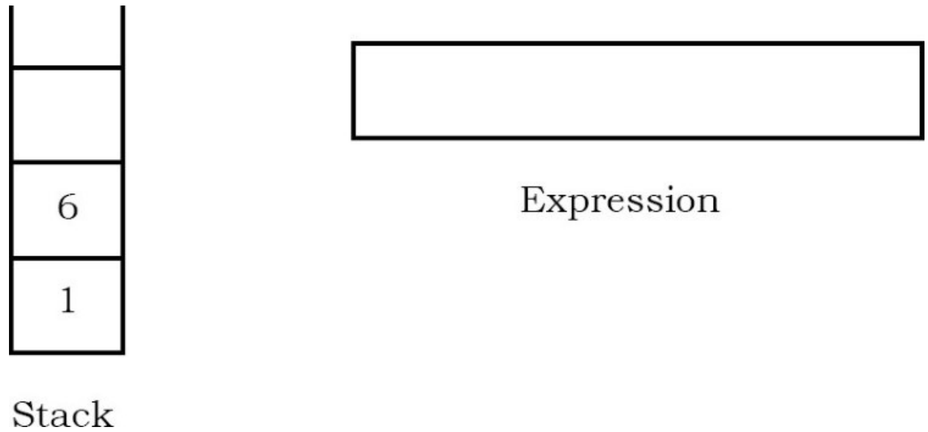
Example: Let us see how the above algorithm works using an example. Assume that the postfix string is 123*+5-. Initially the stack is empty. Now, the first three characters scanned are 1, 2 and 3, which are operands. They will be pushed into the stack in that order.



The next character scanned is “*”, which is an operator. Thus, we pop the top two elements from the stack and perform the “*” operation with the two operands. The second operand will be the first element that is popped.

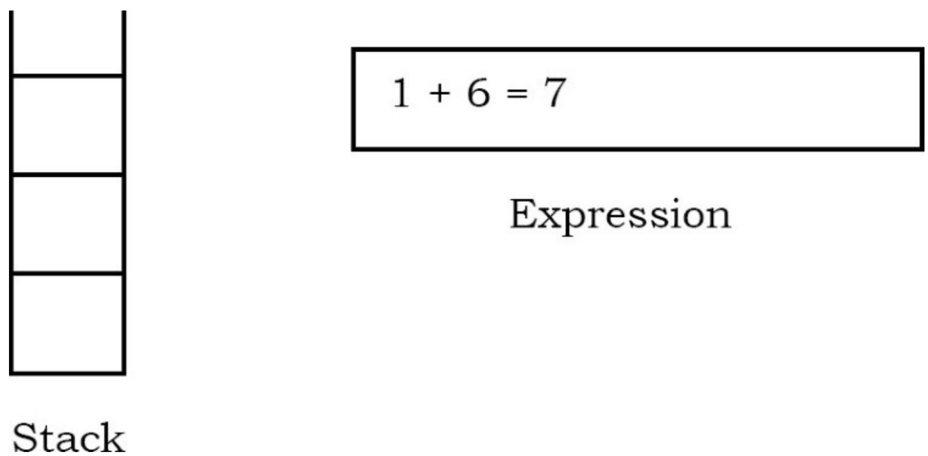


The value of the expression (2*3) that has been evaluated (6) is pushed into the stack.

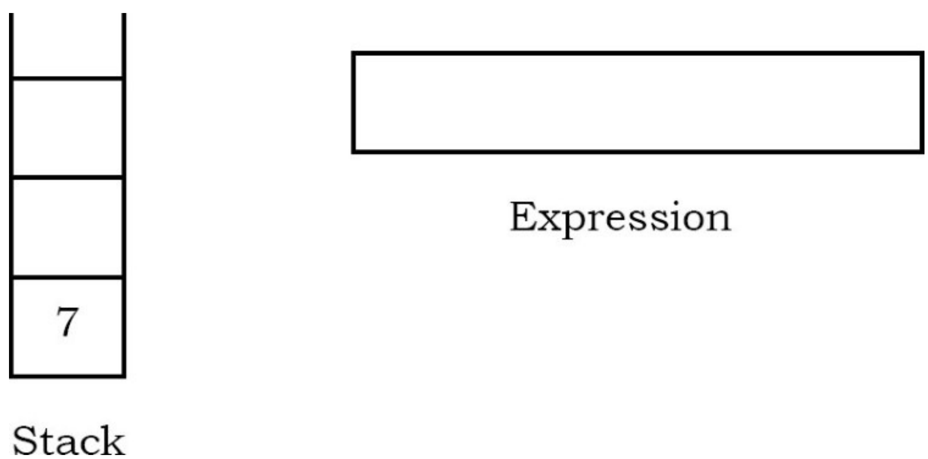


The next character scanned is “+”, which is an operator. Thus, we pop the top two elements from the stack and perform the “+” operation with the two operands.

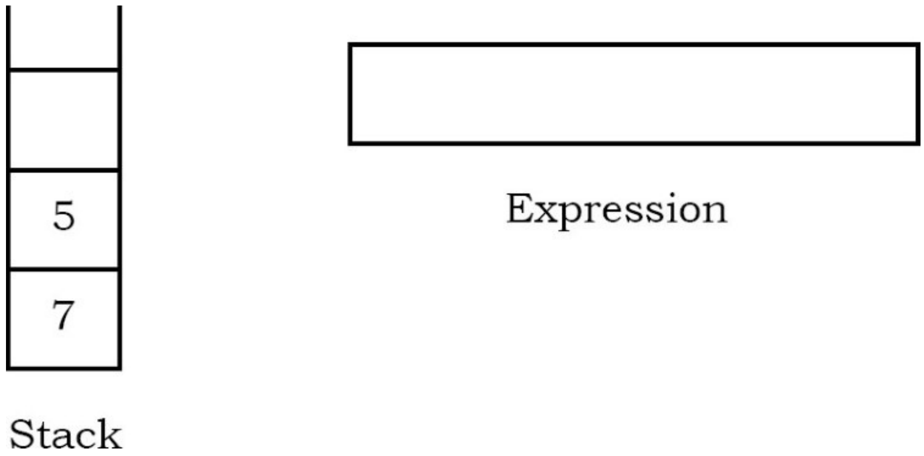
The second operand will be the first element that is popped.



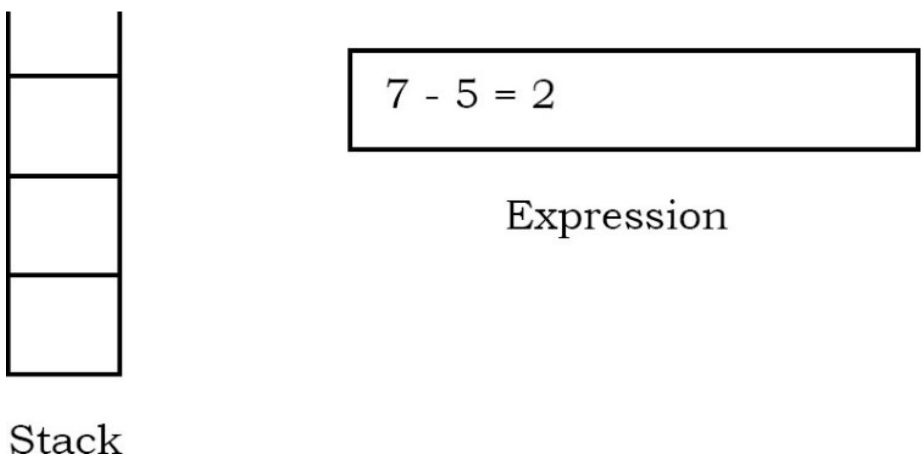
The value of the expression (1+6) that has been evaluated (7) is pushed into the stack.



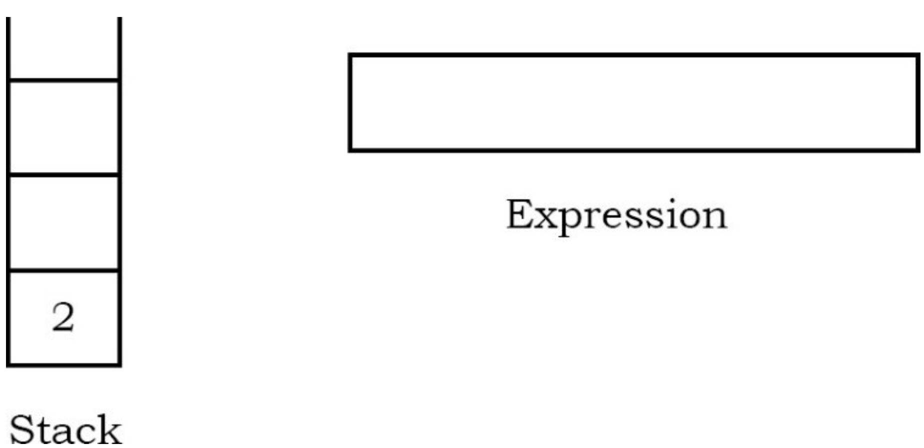
The next character scanned is “5”, which is added to the stack.



The next character scanned is “-”, which is an operator. Thus, we pop the top two elements from the stack and perform the “-” operation with the two operands. The second operand will be the first element that is popped.



The value of the expression(7-5) that has been evaluated(23) is pushed into the stack.



Now, since all the characters are scanned, the remaining element in the stack (there will be only one element in the stack) will be returned. End result:

- Postfix String : 123*+5-
- Result : 2

```java
public int expressionEvaluation(String[] tokens) {
    Stack<Integer> s = new Stack<Integer>();
    for(String token : tokens){
        if(token.equals("+")){
            int op1 = s.pop();
            int op2 = s.pop();
            int res = op1+op2;
            s.push(res);
        }else if(token.equals("-")){
            int op1 = s.pop();
            int op2 = s.pop();
            int res = op2-op1;
            s.push(res);
        }else if(token.equals("*")){
            int op1 = s.pop();
            int op2 = s.pop();
            int res = op1 * op2;
            s.push(res);
        }else if(token.equals("/")){
            int op1 = s.pop();
            int op2 = s.pop();
            int res = op2 / op1;
            s.push(res);
        }else{
            s.push(Integer.parseInt(token));
        }
    }
    return s.pop();
}
```

Problem-5 Can we evaluate the infix expression with stacks in one pass?

Solution: Using 2 stacks we can evaluate an infix expression in 1 pass without converting to postfix.

Algorithm

1) Create an empty operator stack
2) Create an empty operand stack
3) For each token in the input string
    a. Get the next token in the infix string
    b. If next token is an operand, place it on the operand stack
    c. If next token is an operator

i. Evaluate the operator (next op)

4) While operator stack is not empty, pop operator and operands (left and right), evaluate left operator right and push result onto operand stack

5) Pop result from operator stack

Problem-6 How to design a stack such that getMinimum( ) should be O(1)?

Solution: Take an auxiliary stack that maintains the minimum of all values in the stack. Also, assume that each element of the stack is less than its below elements. For simplicity let us call the auxiliary stack *min stack*.

When we *pop* the main stack, *pop* the min stack too. When we push the main stack, push either the new element or the current minimum, whichever is lower. At any point, if we want to get the minimum, then we just need to return the top element from the min stack. Let us take an example and trace it out. Initially let us assume that we have pushed 2, 6, 4, 1 and 5. Based on the above-mentioned algorithm the *min stack* will look like:

Main stack	Min stack
5 →top	1 →top
1	1
4	2
6	2
2	2

After popping twice we get:

Main stack	Min stack
4 -→ top	2→top
6	2
2	2

Based on the discussion above, now let us code the push, pop and GetMinimum() operations.

```java
public class AdvancedStack implements Stack{
    private Stack elementStack = new LLStack();
    private Stack minStack = new LLStack();
    public void push(int data){
            elementStack.push(data);
            if(minStack.isEmpty() || (Integer)minStack.top() >= (Integer)data){
                        minStack.push(data);
            }else{
                  minStack.push(minStack.top());
            }
    }
    public int pop(){
            if(elementStack.isEmpty()) return null;
            minStack.pop();
            return elementStack.pop();
    }
    public int getMinimum(){
            return minStack.top();
    }
    public int top(){
            return elementStack.top();
    }
    public boolean isEmpty(){
            return elementStack.isEmpty();
    }
}
```

Time complexity: O(1). Space complexity: O(n) [for Min stack]. This algorithm has much better space usage if we rarely get a "new minimum or equal".

Problem-7 For Problem-6 is it possible to improve the space complexity?

Solution: Yes. The main problem of the previous approach is, for each push operation we are pushing the element on to *min stack* also (either the new element or existing minimum element). That means, we are pushing the duplicate minimum elements on to the stack.

Now, let us change the algorithm to improve the space complexity. We still have the min stack, but we only pop from it when the value we pop from the main stack is equal to the one on the min stack. We only *push* to the min stack when the value being pushed onto the main stack is less than *or equal* to the current min value. In this modified algorithm also, if we want to get the minimum then we just need to return the top element from the min stack. For example, taking the original version and pushing 1 again, we'd get:

Main stack	Min stack
1 → top	
5	
1	
4	1 → top
6	1
2	2

Popping from the above pops from both stacks because 1 == 1, leaving:

Main stack	Min stack
5 → top	
1	
4	
6	1 → top
2	2

Popping again *only* pops from the main stack, because 5 > 1:

Main stack	Min stack
1 → top	
4	
6	1 → top
2	2

Popping again pops both stacks because 1 == 1:

Main stack	Min stack
4 → top	
6	
2	2 → top

Note: The difference is only in push & pop operations.

```
public class AdvancedStack  implements Stack{
      private Stack elementStack = new LLStack();
      private Stack minStack = new LLStack();
      public void Push(int data){
            elementStack.push(data);
            if(minStack.isEmpty() || (Integer)minStack.top() >= (Integer)data){
                        minStack.push(data);
            }
      }
      public int Pop(){
            if(elementStack.isEmpty())
                  return null;
            Integer minTop = (Integer) minStack.top();
            Integer elementTop = (Integer) elementStack.top();
            if(minTop.intValue() == elementTop.intValue())
                  minStack.pop();
            return elementStack.pop();
      }
      public int GetMinimum(){
            return minStack.top();
      }
      public int Top(){
            return elementStack.top();
      }
      public boolean isEmpty(){
            return elementStack.isEmpty();
      }
}
```

Time complexity: O(1). Space complexity: O(n) [for Min stack]. But this algorithm has much better space usage if we rarely get a “new minimum or equal”.

Problem-8 Given an array of characters formed with a’s and b’s. The string is marked with special character X which represents the middle of the list (for example: ababa...ababXbabab…..baaa). Check whether the string is palindrome.

Solution: This is one of the simplest algorithms. What we do is, start two indexes, one at the beginning of the string and the other at the end of the string. Each time compare whether the values at both the indexes are the same or not. If the values are not the same then we say that the given string is not a palindrome.

If the values are the same then increment the left index and decrement the right index. Continue this process until both the indexes meet at the middle (at X) or if the string is not palindrome.

```
public int isPalindrome(String inputStr) {
    int i=0, j = inputStr.length;
    while(i < j && A[i] == A[j]) {
        i++;
        j--;
    }
    if(i < j ) {
        System.out.println("Not a Palindrome");
        return 0;
    }
    else {
        System.out.println("Palindrome");
        return 1;
    }
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-9 For Problem-8, if the input is in singly linked list then how do we check whether the list elements form a palindrome (That means, moving backward is not possible).

Solution: Refer to *Linked Lists* chapter.

Problem-10 Can we solve Problem-8 using stacks?

Solution: Yes.

Algorithm

- Traverse the list till we encounter X as input element.
- During the traversal push all the elements (until X) on to the stack.
- For the second half of the list, compare each element's content with top of the stack. If they are the same then pop the stack and go to the next element in the input list.
- If they are not the same then the given string is not a palindrome.
- Continue this process until the stack is empty or the string is not a palindrome.

```java
public boolean isPalindrome(String inputStr){
	char inputChar[] = inputStr.toCharArray();
	Stack s = new LLStack();
	int i=0;
	while(inputChar[i] != 'X'){
		s.push(inputChar[i]);
		i++;
	}
	i++;
	while(i<inputChar.length){
		if(s.isEmpty()) return false;
		if(inputChar[i] != ((Character)s.pop()).charValue()) return false;
		i++;
	}
	return true;
}
```

Time Complexity: O(n). Space Complexity: O(n/2) ≈ O(n).

Problem-11 Given a stack, how to reverse the contents of the stack using only stack operations (push and pop)?

Solution:

Algorithm

- First pop all the elements of the stack till it becomes empty.
- For each upward step in recursion, insert the element at the bottom of the stack.

```java
public class StackReversal {
    public static void reverseStack(Stack stack){
        if(stack.isEmpty()) return;
        int temp = stack.pop();
        reverseStack(stack);
        insertAtBottom(stack, temp);
    }
    private static void insertAtBottom(Stack stack , int data){
        if(stack.isEmpty()){
            stack.push(data);
            return;
        }
        int temp = stack.pop();
        insertAtBottom(stack, data);
        stack.push(temp);
    }
}
```

Time Complexity: O(n^2). Space Complexity: O(n), for recursive stack.

Problem-12 Show how to implement one queue efficiently using two stacks. Analyze the running time of the queue operations.

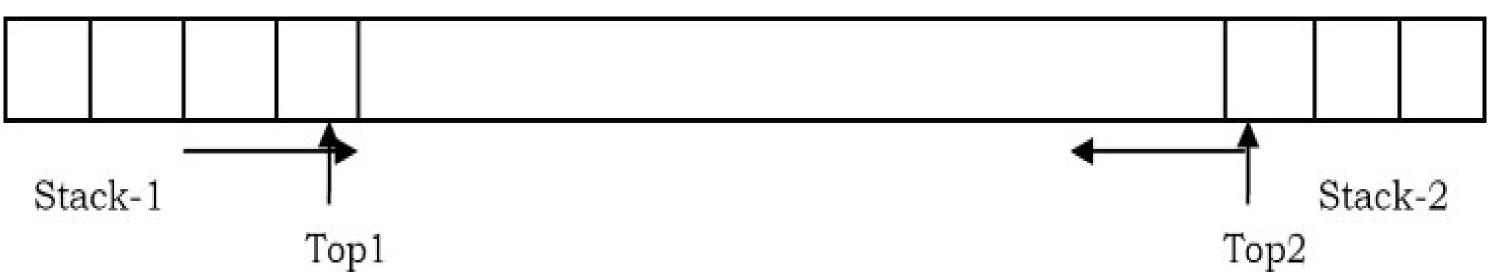
Solution: Refer *Queues* chapter.

Problem-13 Show how to implement one stack efficiently using two queues. Analyze the running time of the stack operations.

Solution: Refer *Queues* chapter.

Problem-14 How do we implement *two* stacks using only one array? Our stack routines should not indicate an exception unless every slot in the array is used?

Solution:



Algorithm:

- Start two indexes one at the left end and the other at the right end.
- The left index simulates the first stack and the right index simulates the second stack.

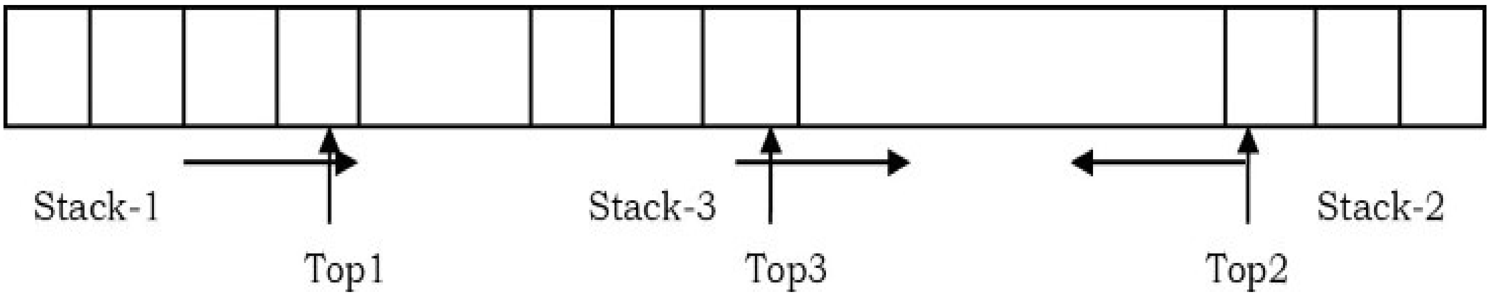
- If we want to push an element into the first stack then put the element at the left index.
- Similarly, if we want to push an element into the second stack then put the element at the right index.
- The first stack grows towards the right, and the second stack grows towards the left.

Time Complexity of push and pop for both stacks is O(1). Space Complexity is O(1).

```
public class ArrayWithTwoStacks{
    private int[] dataArray;
    private int size, topOne,  topTwo;
    public ArrayWithTwoStacks(int size){
            if(size<2) throw new IllegalStateException("size < 2 is no persmissible");
            dataArray = new int[size];
            this.size = size;
            topOne = -1;
            topTwo = size;
    }
    public void push(int stackId, int data){
            if(topTwo == topOne+1) throw new StackOverflowException("Array is full");
            if(stackId == 1){
                    dataArray[++topOne] = data;
            }else if(stackId == 2){
                    dataArray[--topTwo] = data;
            }else return;
    }
    public int pop(int stackId){
            if(stackId == 1){
                    if(topOne == -1) throw new EmptyStackException("First Stack is Empty");
                    int toPop = dataArray[topOne];
                    dataArray[topOne--] = null;
                    return toPop;
            }else if(stackId == 2){
                    if(topTwo == this.size) throw new EmptyStackException("Second Stack is Empty");
                    int toPop = dataArray[topTwo];
                    dataArray[topTwo++] = null;
                    return toPop;
            }else return null;
    }
    public int top(int stackId){
            if(stackId == 1){
                    if(topOne == -1) throw new EmptyStackException("First Stack is Empty");
                            return dataArray[topOne];
                    }else if(stackId == 2){
                            if(topTwo == this.size)
                                    throw new EmptyStackException("Second Stack is Empty");
                            return dataArray[topTwo];
                    }else return null;
    }
    public boolean isEmpty(int stackId){
            if(stackId == 1){
                    return topOne == -1;
            }else if(stackId == 2){
                    return topTwo == this.size;
            }else return true;
    }
}
```

Problem-15 3 stacks in one array: How to implement 3 stacks in one array?

Solution: For this problem, there could be other ways of solving it. Given below is one possibility and it works as long as there is an empty space in the array.



To implement 3 stacks we keep the following information.

- The index of the first stack (Top 1): this indicates the size of the first stack.
- The index of the second stack (Top2): this indicates the size of the second stack.
- Starting index of the third stack (base address of third stack).
- op index of the third stack.

Now, let us define the push and pop operations for this implementation.

Pushing:

- For pushing on to the first stack, we need to see if adding a new element causes it to bump into the third stack. If so, try to shift the third stack upwards. Insert the new element at (start1 + Top1).
- For pushing to the second stack, we need to see if adding a new element causes it to bump into the third stack. If so, try to shift the third stack downward. Insert the new element at (start2 – Top2).
- When pushing to the third stack, see if it bumps into the second stack. If so, try to shift the third stack downward and try pushing again. Insert the new element at (start3 + Top3).

Time Complexity: O(*n*). Since, we may need to adjust the third stack. Space Complexity: O(1).

Popping: For popping, we don’t need to shift, just decrement the size of the appropriate stack. Time Complexity: O(1). Space Complexity: O(1).

One Possible Implementation

```
public class ArrayWithThreeStacks {
    private int[] dataArray;
    private int size, topOne, topTwo, baseThree, topThree;
    public ArrayWithThreeStacks(int size){
        if(size<3) throw new IllegalStateException("Size < 3 is no persmissible");
        dataArray = new int[size];
        this.size = size;
        topOne = -1;
        topTwo = size;
        baseThree = size/2;
        topThree = baseThree;
    }
    public void push(int stackId, int data){
        if(stackId == 1){
            if(topOne+1 == baseThree){
                if(stack3IsRightShiftable()){
                    shiftStack3ToRight();
                    dataArray[++topOne] = data;
                }else throw new StackOverflowException("Stack1 has reached max limit");
            }else dataArray[++topOne] = data;
        }else if(stackId == 2){
            if(topTwo-1 == topThree){
                if(stack3IsLeftShiftable()){
                    shiftStack3ToLeft();
                    dataArray[--topTwo] = data;
                }else throw new StackOverflowException("Stack2 has reached max limit");
            }else dataArray[--topTwo] = data;
        }else if(stackId == 3){
            if(topTwo-1 == topThree){
                if(stack3IsLeftShiftable()){
                    shiftStack3ToLeft();
                    dataArray[++topThree] = data;
                }else throw new StackOverflowException("Stack3 has reached max limit");
            }else dataArray[++topThree] = data;
        }else return;
    }
    public int pop(int stackId){
        if(stackId == 1){
            if(topOne == -1) throw new EmptyStackException("First Stack is Empty");
            int toPop = dataArray[topOne];
            dataArray[topOne--] = null;
            return toPop;
        }else if(stackId == 2){
```

```
                    if(topTwo == this.size) throw new EmptyStackException("Second Stack is Empty");
                    int toPop = dataArray[topTwo];
                    dataArray[topTwo++] = null;
                    return toPop;
            }else if(stackId == 3){
                    if(topThree == this.size && dataArray[topThree] == null)
                            throw new EmptyStackException("Third Stack is Empty");
                    int toPop = dataArray[topThree];
                    if(topThree > baseThree) dataArray[topThree--] = null;
                    if(topThree == baseThree) dataArray[topThree] = null;
                    return toPop;
            }else return null;
    }
    public int top(int stackId){
            if(stackId == 1){
                    if(topOne == -1) throw new EmptyStackException("First Stack is Empty");
                            return dataArray[topOne];
                    }else if(stackId == 2){
                            if(topTwo == this.size)
                                            throw new EmptyStackException("Second Stack is Empty");
                            return dataArray[topTwo];
                    }else if(stackId == 3){
                            if(topThree == baseThree && dataArray[baseThree] == null)
                                    throw new EmptyStackException("Third Stack is Empty");
                            return dataArray[topThree];
                    }else return null;
    }
    public boolean isEmpty(int stackId){
            if(stackId == 1){
                            return topOne == -1;
            }else if(stackId == 2){
                    return topTwo == this.size;
            }else if(stackId == 3){
                    return (topThree == baseThree) && (dataArray[baseThree] == null);
            }else return true;
    }
    private void shiftStack3ToLeft() {
            for(int i=baseThree-1; i<=topThree-1;i++){
                    dataArray[i] = dataArray[i+1];
            }
            dataArray[topThree--] = null;
            baseThree--;
    }
    private boolean stack3IsLeftShiftable() {
            if(topOne+1 < baseThree){
                    return true;
            }
            return false;
    }
    private void shiftStack3ToRight() {
            for(int i=topThree+1; i>=baseThree+1;i--){
                    dataArray[i] = dataArray[i-1];
            }
            dataArray[baseThree++] = null;
            topThree++;
    }
    private boolean stack3IsRightShiftable() {
            if(topThree+1 < topTwo){
                    return true;
            }
            return false;
    }
}
```

Problem-16 For Problem-15, is there any other way of implementing the middle stack?

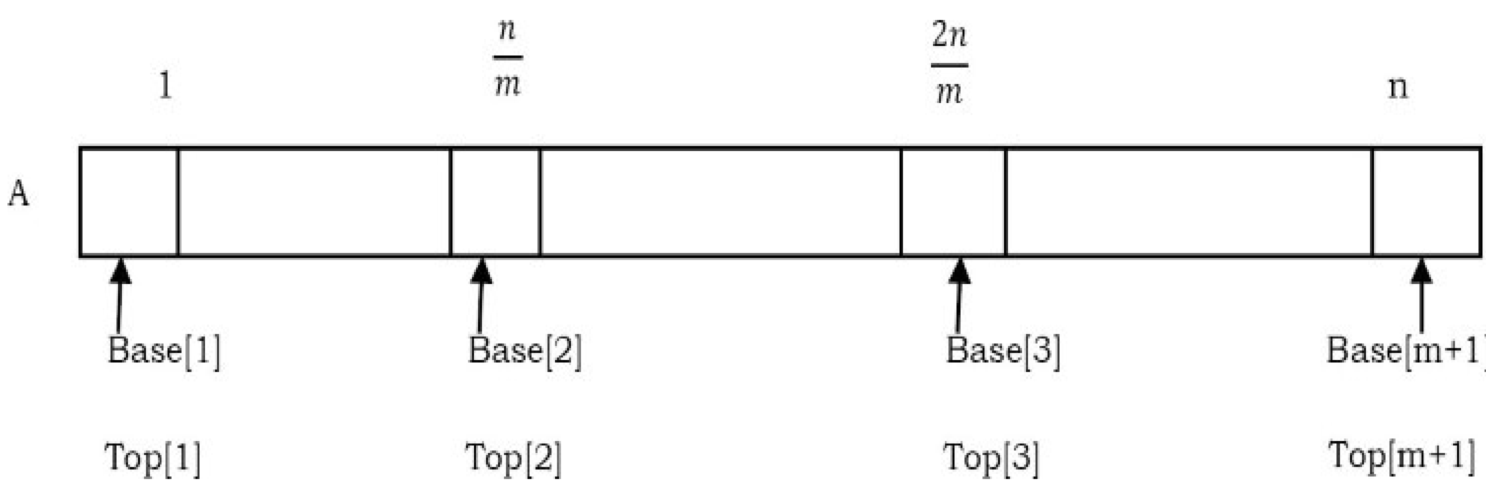
Solution: Yes. When either the left stack (which grows to the right) or the right stack (which grows to the left) bumps into the middle stack, we need to shift the entire middle stack to make room. The same happens if a push on the middle stack causes it to bump into the right stack.

To solve the above-mentioned problem (number of shifts) what we can do is: alternating pushes can be added at alternating sides of the middle list (For example, even elements are pushed to the left, odd elements are pushed to the right). This would keep the middle stack balanced in the center of the array but it would still need to be shifted when it bumps into the left or right stack, whether by growing on its own or by the growth of a neighboring stack.

We can optimize the initial locations of the three stacks if they grow/shrink at different rates and if they have different average sizes. For example, suppose one stack doesn't change much. If we put it at the left, then the middle stack will eventually get pushed against it and leave a gap between the middle and right stacks, which grow toward each other. If they collide, then it's likely we've run out of space in the array. There is no change in the time complexity but the average number of shifts will get reduced.

Problem-17 Multiple (m) stacks in one array: Similar to Problem-15, what if we want to implement m stacks in one array?

Solution: Let us assume that array indexes are from 1 to n. Similar to the discussion in Problem-15, to implement m stacks in one array, we divide the array into m parts (as shown below). The size of each part is $\frac{n}{m}$.



From the above representation we can see that, first stack is starting at index 1 (starting index is stored in Base[l]), second stack is starting at index $\frac{n}{m}$ (starting index is stored in Base[2]), third stack is starting at index $\frac{2n}{m}$ (starting index is stored in Base[3]), and so on. Similar to *Base* array, let us assume that *Top* array stores the top indexes for each of the stack. Consider the following terminology for the discussion.

- Top[i], for $1 \le i \le m$ will point to the topmost element of the stack i.
- If Base[i] == Top[i], then we can say the stack i is empty.
- If Top[i] == Base[i+1], then we can say the stack i is full. Initially Base[i] = Top[i] $= \frac{n}{m}(i-1)$, for $1 \le i \le m$.
- The i^{th} stack grows from Base[i]+1 to Base[i+1].

Pushing on to i^{th} stack:

1) For pushing on to the i^{th} stack, we check whether the top of i^{th} stack is pointing to Base[i+1] (this case defines that i^{th} stack is full). That means, we need to see if adding a new element causes it to bump into the $i + 1^{th}$ stack. If so, try to shift the stacks from $i + 1^{th}$ stack to m^{th} stack toward the right. Insert the new element at (Base[i] + Top[i]).
2) If right shifting is not possible then try shifting the stacks from 1 to $i - 1^{th}$ stack toward the left.
3) If both of them are not possible then we can say that all stacks are full.

```
public void Push(int StackID, int data){
    if(Top[i] == Base[i+1])
            Print ith Stack is full and does the necessary action (shifting);
    Top[i] = Top[i]+1;
    A[Top[i]] = data;
}
```

Time Complexity: O(n). Since we may need to adjust the stacks. Space Complexity: O(1).

Popping from i^{th} stack: For popping, we don't need to shift, just decrement the size of the appropriate stack. The only case we need to check is stack empty case.

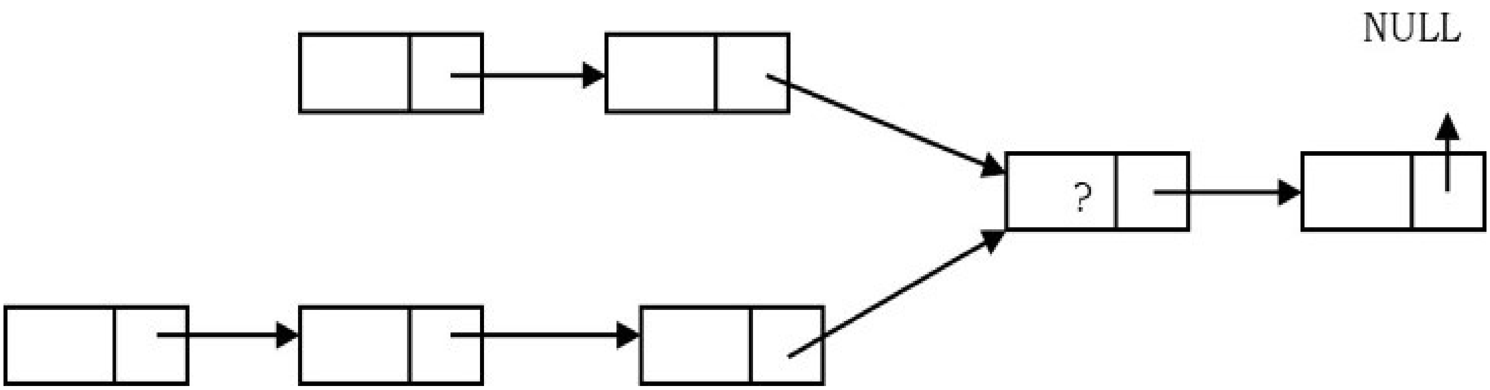
```
public int Pop(int StackID) {
    if(Top[i] == Base[i])
            Print ith Stack is empty;
    return A[Top[i]--];
}
```

Time Complexity: O(1). Space Complexity: O(1).

Problem-18 Consider an empty stack of integers. Let the numbers 1, 2, 3, 4, 5, 6 be pushed on to this stack in the order they appear from left to right. Let 5 indicate a push and X indicate a pop operation. Can they be permuted in to the order 325641(output) and order 154623? (If a permutation is possible give the order string of operations.

Solution: SSSXXSSXSXXX outputs 325641. 154623 cannot be output as 2 is pushed much before 3 so can appear only after 3 is output.

Problem-19 Suppose there are two singly linked lists which intersect at some point and become a single linked list. The head or start pointers of both the lists are known, but the intersecting node is not known. Also, the number of nodes in each of the lists before they intersect are unknown and both lists may have a different number. *List*1 may have *n* nodes before it reaches the intersection point and *List*2 may have *m* nodes before it reaches the intersection point where *m* and *n* may be $m = n$, $m < n$ or $m > n$. Can we find the merging point using stacks?



Solution: Yes. For algorithm refer to *Linked Lists* chapter.

Problem-20 Earlier in this chapter, we discussed that for dynamic array implementation of stacks, the 'repeated doubling' approach is used. For the same problem, what is the complexity if we create a new array whose size is $n + K$ instead of doubling?

Solution: Let us assume that the initial stack size is 0. For simplicity let us assume that $K = 10$. For inserting the element we create a new array whose size is $0 + 10 = 10$. Similarly, after 10 elements we again create a new array whose size is $10 + 10 = 20$ and this process continues at values: 30, 40 ... That means, for a given *n* value, we are creating the new arrays at: $\frac{n}{10}, \frac{n}{20}, \frac{n}{30}, \frac{n}{40}$ ... The total number of copy operations is:

$$= \frac{n}{10} + \frac{n}{20} + \frac{n}{30} + \cdots 1 = \frac{n}{10}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}\right) = \frac{n}{10} logn \approx O(nlogn)$$

If we are performing *n* push operations, the cost per operation is O(*logn*).

Problem-21 Given a string containing *n S*'s and *n X*'s where 5 indicates a push operation and Vindicates a pop operation, and with the stack initially empty, formulate a rule to check whether a given string 5 of operations is admissible or not?

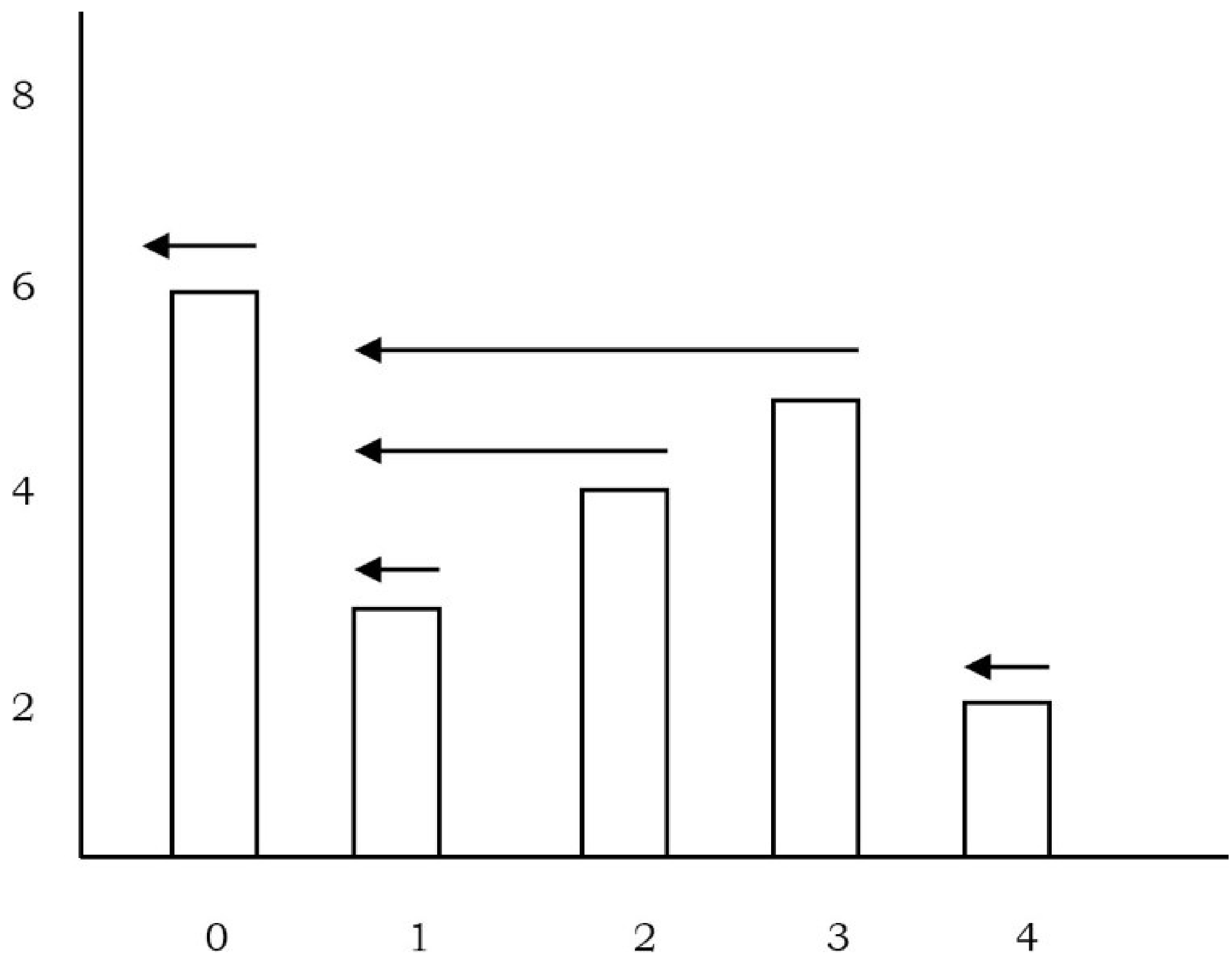
Solution: Given a string of length 2*n*, we wish to check whether the given string of operations is permissible or not with respect to its functioning on a stack. The only restricted operation is pop whose prior requirement is that the stack should not be empty. So while traversing the string from left to right, prior to any pop the stack shouldn't be empty, which means the number of *S*'s is always greater than or equal to that of *X*'s. Hence the condition is at any stage of processing of the string, the number of push operations (5) should be greater than the number of pop operations (*X*).

Problem-22 **Finding of Spans:** Given an array A the span $S[i]$ of $A[i]$ is the maximum number of consecutive elements $A[j]$ immediately preceding $A[i]$ and such that $A[j] \leq A[j + 1]$?
Another way of asking: Given an array A of integers, find the maximum of $j - i$ subjected to the constraint of $A[i] < A[j]$.

Solution: This is a very common problem in stock markets to find the peaks. Spans are used in financial analysis (E.g., stock at 52-week high). The span of a stock price on a certain day, i, is the maximum number of consecutive days (up to the current day) the price of the stock has been less than or equal to its price on i.

As an example, let us consider the table and the corresponding spans diagram. In the figure the arrows indicate the length of the spans.

Day: Index i	Input Array A[i]	S[i]: Span of A[i]
0	6	1
1	3	1
2	4	2
3	5	3
4	2	1



Now, let us concentrate on the algorithm for finding the spans. One simple way is, each day, check how many contiguous days have a stock price that is less than the current price.

```
public int[] FindingSpans(int[] inputArray){
    int[] spans = new int[inputArray.length];
    for(int i=0;i<inputArray.length;i++){
        int span = 1;
        int j=i-1;
        while(j>=0 && inputArray[j]<=inputArray[j+1]){
            span++;
            j--;
        }
        spans[i] = span;
    }
    return spans;
}
```

Time Complexity: O(n^2). Space Complexity: O(1).

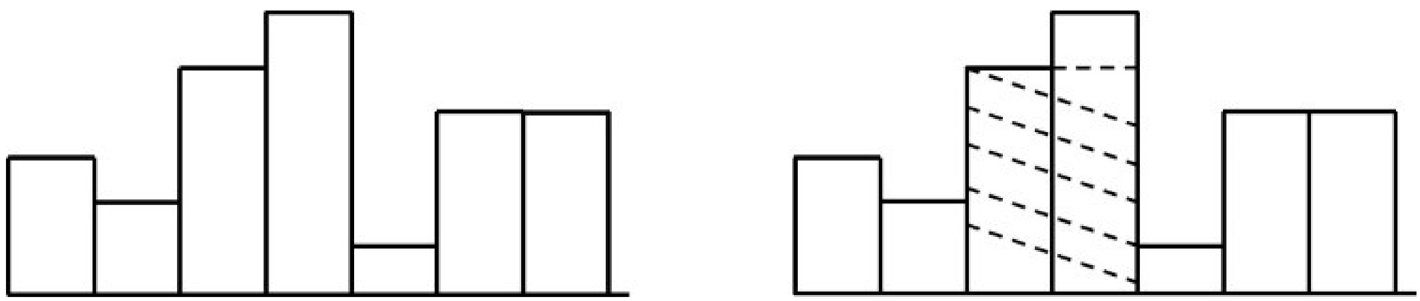
Problem-23 Can we improve the complexity of Problem-22?

Solution: From the example above, we can see that span $S[i]$ on day i can be easily calculated if we know the closest day preceding i, such that the price is greater on that day than the price on day i. Let us call such a day as P. If such a day exists then the span is now defined as $S[i] = i – P$.

```
public int[] FindingSpans(int[] inputArray){
    int[] spans = new int[inputArray.length];
    Stack stack = new LLStack();
    int p = 0;
    for(int i=0;i<inputArray.length;i++){
        while (!stack.isEmpty() && inputArray[i] > inputArray[(Integer) stack.top()])
            stack.pop();
        if( stack.isEmpty())
            p = -1;
        else    p = (Integer) stack.top();
        spans[i] = i - p;
        stack.push(i);
    }
    return spans;
}
```

Time Complexity: Each index of the array is pushed into the stack exactly once and also popped from the stack at most once. The statements in the while loop are executed at most n times. Even though the algorithm has nested loops, the complexity is $O(n)$ as the inner loop is executing only n times during the course of the algorithm (trace out an example and see how many times the inner loop becomes successful). Space Complexity: $O(n)$ [for stack].

Problem-24 **Largest rectangle under histogram:** A histogram is a polygon composed of a sequence of rectangles aligned at a common base line. For simplicity, assume that the rectangles have equal widths but may have different heights. For example, the figure on the left shows a histogram that consists of rectangles with the heights 3, 2, 5, 6, 1, 4, 4, measured in units where 1 is the width of the rectangles. Here our problem is: given an array with heights of rectangles (assuming width is 1), we need to find the largest rectangle possible. For the given example, the largest rectangle is the shared part.



Solution: A straightforward answer is to go to each bar in the histogram and find the maximum possible area in the histogram for it. Finally, find the maximum of these values. This will require $O(n^2)$.

Problem-25 For **Error! Reference source not found.,** can we improve the time complexity?

Solution: Linear search using a stack of incomplete subproblems: There are many ways of solving this problem. *Judge* has given a nice algorithm for this problem which is based on stack. We process the elements in left-to-right order and maintain a stack of information about started but yet unfinished sub histograms. If the stack is empty, open a new subproblem by pushing the element onto the stack. Otherwise compare it to the element on top of the stack. If the new one is greater we again push it. If the new one is equal we skip it. In all these cases, we continue with the next new element.

If the new one is less, we finish the topmost subproblem by updating the maximum area with respect to the element at the top of the stack. Then, we discard the element at the top, and repeat the procedure keeping the current new element. This way, all subproblems are finished when the stack becomes empty, or its top element is less than or equal to the new element, leading to the actions described above. If all elements have been processed, and the stack is not yet empty, we finish the remaining subproblems by updating the maximum area with respect to the elements at the top.

```java
public class MaxRectangleAreaInHistrogram {
  public int MaxRectangleArea(int[] A) {
    Stack<Integer> s = new Stack<Integer>();
    if(A == null || A.length == 0)
      return 0;
    // Initialize max area
    int maxArea = 0;
    int i = 0;
    // run through all bars of given histogram
    while(i < A.length) {
      // If current bar is higher than the bar of the stack peek, push it to stack.
      if(s.empty() || A[s.peek()] <= A[i])
        s.push(i++);
      else {
        // if current bar is lower than the stack peek,
        // calculate area of rectangle with stack top as the smallest bar.
        // 'i' is 'right index' for the top and element before top in stack is 'left index'
        int top = s.pop();
        // calculate the area with A[top] stack as smallest bar and update maxArea if needed
        maxArea = Math.max(maxArea, A[top] * (s.empty() ? i : i - s.peek() - 1));
      }
    }
    // Now pop the remaining bars from stack and calculate area with every popped bar as the smallest bar.
    while(!s.isEmpty()) {
      int top = s.pop();
      maxArea = Math.max(maxArea, A[top] * (s.empty() ? i : i - s.peek() - 1));
    }
    return maxArea;
  }
}
```

At the first impression, this solution seems to be having O(n^2) complexity. But if we look carefully, every element is pushed and popped at most once, and in every step of the function at least one element is pushed or popped. Since the amount of work for the decisions and the update is constant, the complexity of the algorithm is O(n) by amortized analysis.
Space Complexity: O(n) [for stack].

Problem-26 Give an algorithm for sorting a stack in ascending order. We should not make any assumptions about how the stack is implemented.

Solution:

```
public static Stack<Integer> sort(Stack<Integer> stk) {
    Stack<Integer> rstk = new Stack<Integer>();
    while(!stk.isEmpty()){
        int tmp = stk.pop();
        while(!rstk.isEmpty() && rstk.peek() > tmp){
            stk.push(rstk.pop());
        }
        rstk.push(tmp);
    }
    return rstk;
}
```

Time Complexity: O(n^2). Space Complexity: O(n),for stack.

Problem-27 Given a stack of integers, how do you check whether each successive pair of numbers in the stack is consecutive or not. The pairs can be increasing or decreasing, and if the stack has an odd number of elements, the element at the top is left out of a pair. For example, if the stack of elements are [4, 5, -2, -3, 11, 10, 5, 6, 20], then the output should be true because each of the pairs (4, 5), (-2, -3), (11, 10), and (5, 6) consists of consecutive numbers.

Solution: Refer to *Queues* chapter.

Problem-28 Recursively remove all adjacent duplicates: Given an array of numbers, recursively remove adjacent duplicate numbers. The output array should not have any adjacent duplicates.

Input: 1,5,6, 8,8,8,0,1,1,0,6,5 *Output*: 1	*Input*: 1,9,6, 8,8,8,0,1,1,0,6,5} *Output*: 1, 9, 5

Solution: This solution runs with the concept of in-place stack. When element on stack doesn’t match to the current number, we add it to stack. When it matches to stack top, we skip numbers until the element match the top of stack and remove the element from stack.

```java
public class RemoveAdjacentDuplicates {
    public int removeAdjacentDuplicates(int []A){
        int stkptr=-1;
        int i=0;
        while (i<A.length){
            if (stkptr == -1 || A[stkptr]!=A[i]){
                stkptr++;
                A[stkptr]=A[i];
                i++;
            }else {
                while(i < A.length&& A[stkptr]==A[i])
                    i++;
                stkptr--;
            }
        }
        return stkptr;
    }
}
public class TestRemoveAdjacentDuplicates {
    public static void main(String[] args) {
        RemoveAdjacentDuplicates obj = new RemoveAdjacentDuplicates();
        int[] A = {1,5,6, 8,8,8,0,1,1,0,6,5};
        int index = obj.removeAdjacentDuplicates(A);
        for (int i = 0; i <= index; i++) {
            System.out.print(" " + A[i]);
        }
    }
}
```

Time Complexity: O(n). Space Complexity: O(1) as the stack simulation is done inplace.

Problem-28 If the stack gets too high, it might overbalance. There-fore, in real life, we would likely start a new stack when the previous stack exceeds some threshold. Implement a data structure that mimics this and composed of several stacks, and should create a new stack once the previous one exceeds capacity. push() and pop() of this class should behave identically to a regular stack.

Solution: Follow the comments in code below.

```
class StackForStackSets {
    private int top = -1;
    private int[] arr;
    // Maximum size of stack
    private int capacity;
    StackForStackSets(int capacity){
            this.capacity = capacity;
            arr = new int[capacity];
    }
    public void push(int v){
            arr[++top] = v;
    }
    public int pop(){
            return arr[top--];
    }
    // if the stack is at capacity
    public Boolean isAtCapacity(){
            return capacity == top + 1;
    }
    //return the size of the stack
    public int size(){
            return top+1;
    }
    public String toString(){
            String s = "";
            int index = top;
            while(index >= 0){
                s += "[" + arr[index--] + "]"+ " --> ";
            }
            return s;
    }
}
public class StackSets{
    // Number of elements for each stack
    private int threshold;
    private ArrayList<StackForStackSets> listOfStacks = new  ArrayList<StackForStackSets>();
    StackSets(int threshold) {
            this.threshold = threshold;
    }
    //get the last stack
    public StackForStackSets getLastStack(){
            int size = listOfStacks.size();
            if(size <= 0)
                        return null;
            else return listOfStacks.get(size - 1);
    }
```

```
    //get the nth stack
    public StackForStackSets getNthStack(int n){
            System.out.println(n);
            int size = listOfStacks.size();
            if(size <= 0)
                    return null;
            else return listOfStacks.get(n - 1);
    }
    //push value
    public void push(int value){
            StackForStackSets lastStack = this.getLastStack();
            if(lastStack == null){
               lastStack = new StackForStackSets(threshold);
               lastStack.push(value);
               listOfStacks.add(lastStack);
            }else {
               if( !lastStack.isAtCapacity())
                    lastStack.push(value);
               else {
                    StackForStackSets newLastStack = new StackForStackSets(threshold);
                    newLastStack.push(value);
                    listOfStacks.add(newLastStack);
               }
            }
    }
    // the pop
    public int pop(){
            StackForStackSets lastStack = this.getLastStack();
            int v = lastStack.pop();
            if(lastStack.size() == 0) listOfStacks.remove(listOfStacks.size() -  1);
            return v;
    }
    //pop from the nth stack
    public int pop(int nth){
            StackForStackSets nthStack = this.getNthStack(nth);
             int v = nthStack.pop();
             if(nthStack.size() == 0) listOfStacks.remove(listOfStacks.size() -  1);
             return v;
    }
    public String toString(){
            String s = "";
            for(int i = 0; i < listOfStacks.size(); i++){
                    StackForStackSets stack = listOfStacks.get(i);
               s = "Stack "+ i + ": " + stack.toString() +  s;
            }
            return s;
    }
    public static void main(String[] args){
            StackSets stacks = new StackSets(3);
            stacks.push(10); stacks.push(9); stacks.push(8);
            stacks.push(7); stacks.push(6); stacks.push(5);
            stacks.push(4); stacks.push(3); stacks.push(2);
            System.out.println("Popping " + stacks.pop(2));
            System.out.println("Popping from stack 1" + stacks.pop(1));
            System.out.println("Popping " + stacks.pop(3));
            System.out.println("Popping " + stacks.pop(2));
            System.out.println("the stack is: " + stacks);
    }
}
```

CHAPTER

5

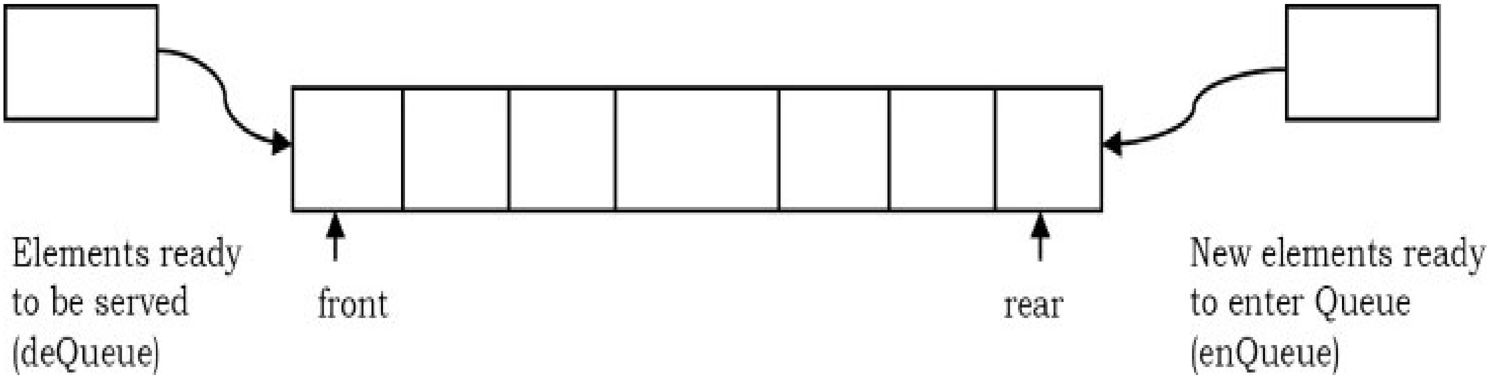
QUEUES

5.1 What is a Queue?

A queue is a data structure used for storing data (similar to Linked Lists and Stacks). In queue, the order in which data arrives is important. In general, a queue is a line of people or things waiting to be served in sequential order starting at the beginning of the line or sequence.

Definition: A *queue* is an ordered list in which insertions are done at one end (*rear*) and deletions are done at other end (*front*). The first element to be inserted is the first one to be deleted. Hence, it is called First in First out (FIFO) or Last in Last out (LILO) list.

Similar to *Stacks*, special names are given to the two changes that can be made to a queue. When an element is inserted in a queue, the concept is called *EnQueue*, and when an element is removed from the queue, the concept is called *DeQueue*. *DeQueueing* an empty queue is called *underflow* and *EnQueuing* an element in a full queue is called *overflow*. Generally, we treat them as exceptions. As an example, consider the snapshot of the queue.



5.2 How are Queues Used?

The concept of a queue can be explained by observing a line at a reservation counter. When we enter the line we stand at the end of the line and the person who is at the front of the line is the one who will be served next. He will exit the queue and be served.

As this happens, the next person will come at the head of the line, will exit the queue and will be served. As each person at the head of the line keeps exiting the queue, we move towards the head of the line. Finally we will reach the head of the line and we will exit the queue and be served. This behavior is very useful in cases where there is a need to maintain the order of arrival.

5.3 Queue ADT

The following operations make a queue an ADT. Insertions and deletions in the queue must follow the FIFO scheme. For simplicity we assume the elements are integers.

Main Queue Operations

- enQueue(int data): Inserts an element at the end of the queue
- int deQueue(): Removes and returns the element at the front of the queue

Auxiliary Queue Operations

- int Front(): Returns the element at the front without removing it
- int QueueSize(): Returns the number of elements stored in the queue
- int IsEmptyQueue(): Indicates whether no elements are stored in the queue or not

5.4 Exceptions

Similar to other ADTs, executing *DeQueue* on an empty queue throws an *“Empty Queue Exception”* and executing *EnQueue* on a full queue throws a *“Full Queue Exception”*.

5.5 Applications

Following are the some of the applications that use queues.

Direct Applications

- Operating systems schedule jobs (with equal priority) in the order of arrival (e.g., a print queue).
- Simulation of real-world queues such as lines at a ticket counter, or any other first-come first-served scenario requires a queue.
- Multiprogramming.
- Asynchronous data transfer (file IO, pipes, sockets).
- Waiting times of customers at call center.
- Determining number of cashiers to have at a supermarket.

Indirect Applications

- Auxiliary data structure for algorithms
- Component of other data structures

5.6 Implementation

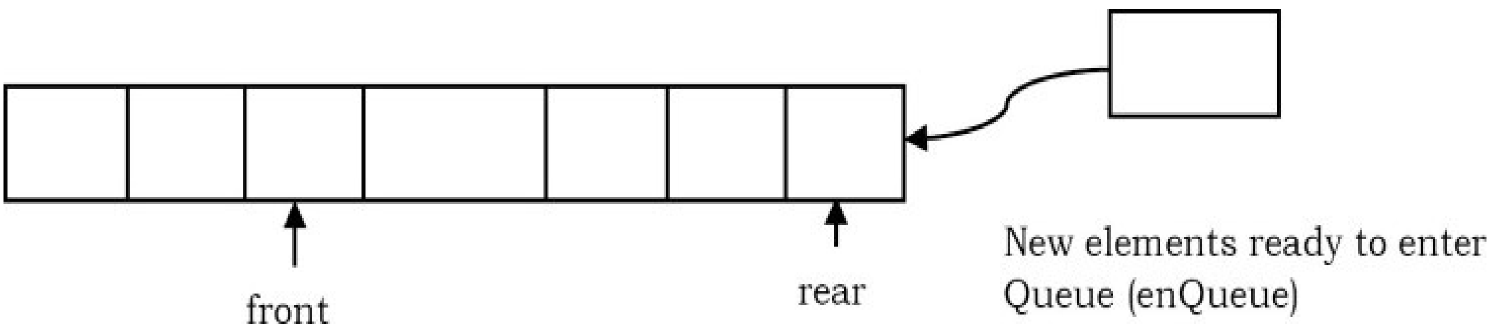
There are many ways (similar to Stacks) of implementing queue operations and some of the commonly used methods are listed below.

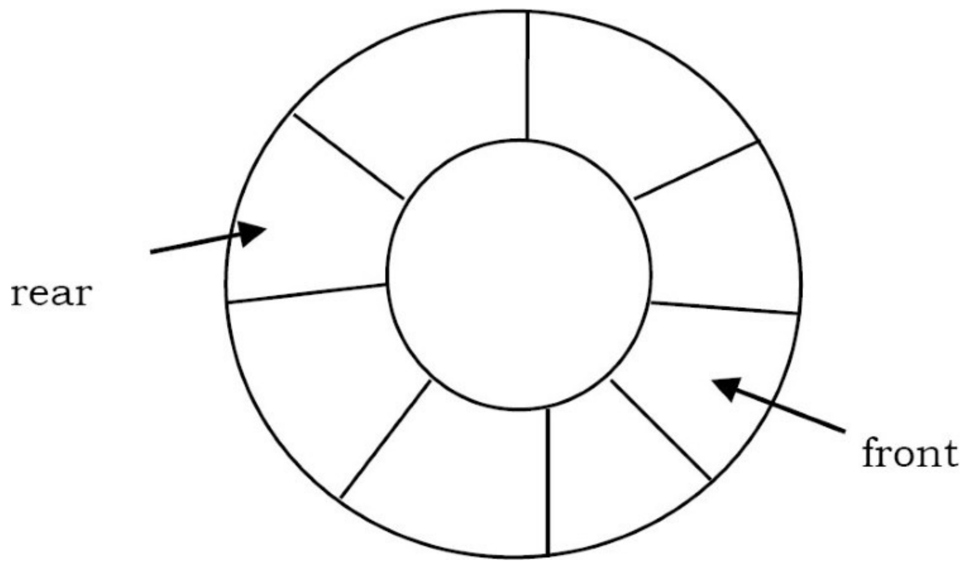
- Simple circular array based implementation
- Dynamic circular array based implementation
- Linked list implementation

Why Circular Arrays?

First, let us see whether we can use simple arrays for implementing queues as we have done for stacks. We know that, in queues, the insertions are performed at one end and deletions are performed at the other end. After performing some insertions and deletions the process becomes easy to understand.

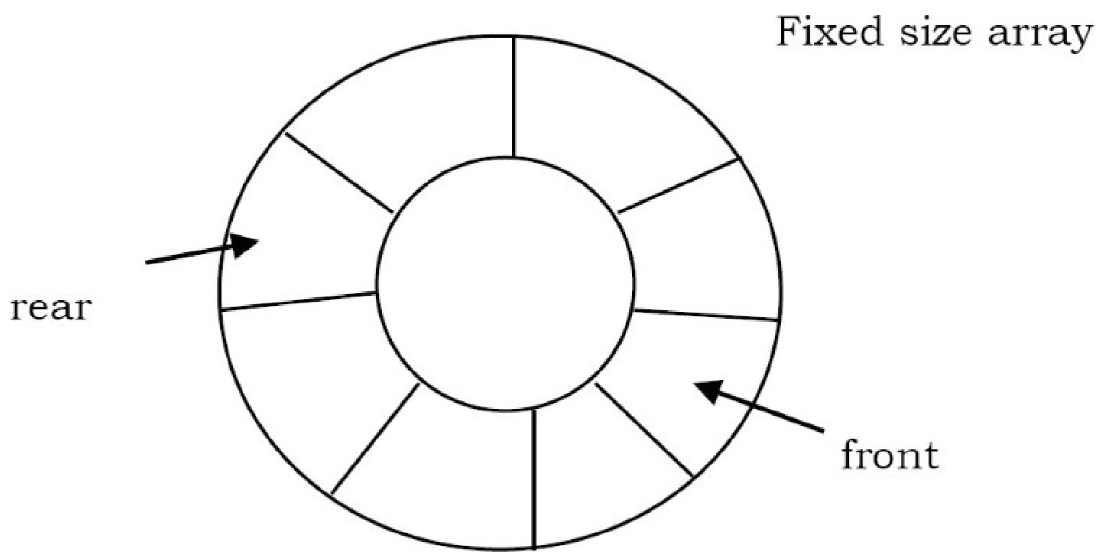
In the example shown below, it can be seen clearly that the initial slots of the array are getting wasted. So, simple array implementation for queue is not efficient. To solve this problem we assume the arrays as circular arrays. That means, we treat the last element and the first array elements as contiguous. With this representation, if there are any free slots at the beginning, the rear pointer can easily go to its next free slot.





Note: The simple circular array and dynamic circular array implementations are very similar to stack array implementations. Refer to *Stacks* chapter for analysis of these implementations.

Simple Circular Array Implementation



This simple implementation of Queue ADT uses an array. In the array, we add elements circularly and use two variables to keep track of the start element and end element. Generally, *front* is used to indicate the start element and *rear* is used to indicate the end element in the queue.

The array storing the queue elements may become full. An *EnQueue* operation will then throw a *full queue exception*. Similarly, if we try deleting an element from an empty queue it will throw *empty queue exception*.

Note: Initially, both front and rear points to -1 which indicates that the queue is empty.

```
public class FixedSizeArrayQueue{
    // Array used to implement the queue.
    private int[] queueRep;
    private int size, front, rear;

    // Length of the array used to implement the queue.
    private static final int CAPACITY = 16;    //Default Queue size

    // Initializes the queue to use an array of default length.
    public FixedSizeArrayQueue (){
        queueRep = new int [CAPACITY];
        size  = 0; front = 0; rear  = 0;
    }

    // Initializes the queue to use an array of given length.
    public FixedSizeArrayQueue (int cap){
        queueRep = new int [ cap];
        size  = 0; front = 0; rear  = 0;
    }

    // Inserts an element at the rear of the queue. This method runs in O(1) time.
    public void enQueue (int data)throws NullPointerException, IllegalStateException{
            if (size == CAPACITY)
                    throw new IllegalStateException ("Queue is full: Overflow");
            else {
                    size++;
                    queueRep[rear] = data;
                    rear = (rear+1) % CAPACITY;
            }
```

```
    }
    // Removes the front element from the queue. This method runs in O(1) time.
    public int deQueue () throws IllegalStateException{
            // Effects:   If queue is empty, throw IllegalStateException,
            // else remove and return oldest element of this
            if (size == 0)
                    throw new IllegalStateException ("Queue is empty: Underflow");
            else {
                    size--;
                    int data = queueRep [ (front % CAPACITY) ];
                    queueRep [front] = Integer.MIN_VALUE;
                    front = (front+1) % CAPACITY;
                    return data;
            }
    }
    // Checks whether the queue is empty. This method runs in O(1) time.
    public boolean isEmpty(){
        return (size == 0);
    }
    // Checks whether the queue is full. This method runs in O(1) time.
    public boolean isFull(){
        return (size == CAPACITY);
    }
    // Returns the number of elements in the queue. This method runs in O(1) time.
    public int size() {
        return size;
    }
    // Returns a string representation of the queue as a list of elements, with
    // the front element at the end: [ ... , prev, rear ]. This method runs in O(n)
    // time, where n is the size of the queue.
    public String toString(){
            String result = "[";
            for (int i = 0; i < size; i++){
                    result += Integer.toString(queueRep[ (front + i) % CAPACITY ]);
                    if (i < size -1) {
                         result += ", ";
                    }
            }
            result += "]";
            return result;
    }
}
```

Performance and Limitations

Performance: Let *n* be the number of elements in the queue:

Space Complexity (for *n* enQueue operations)	O(*n*)
Time Complexity of enQueueQ	O(1)
Time Complexity of deQueueQ	O(1)
Time Complexity of isEmpty()	O(1)
Time Complexity of isFull ()	O(1)
Time Complexity of getQueueSize ()	O(1)

Limitations

The maximum size of the queue must be defined as prior and cannot be changed. Trying to *EnQueue* a new element into a full queue causes an implementation-specific exception.

Dynamic Circular Array Implementation

```
public class DynamicArrayQueue{
    // Array used to implement the queue.
```

```java
private int[] queueRep;
private int size, front, rear;
// Length of the array used to implement the queue.
private static int CAPACITY = 16;                    //Default Queue size
public static int MINCAPACITY=1<<15;                 // power of 2
// Initializes the queue to use an array of default length.
public DynamicArrayQueue (){
        queueRep = new int [CAPACITY];
        size = 0; front = 0; rear = 0;
}
// Initializes the queue to use an array of given length.
public DynamicArrayQueue (int cap){
        queueRep = new int [ cap];
        size = 0; front = 0; rear = 0;
}
// Inserts an element at the rear of the queue. This method runs in O(1) time.
public void enQueue (int data)throws NullPointerException, IllegalStateException{
        if (size == CAPACITY)
                expand();
        size++;
        queueRep[rear] = data;
        rear = (rear+1) % CAPACITY;
}
// Removes the front element from the queue. This method runs in O(1) time.
public int deQueue () throws IllegalStateException{
        // Effects:   If queue is empty, throw IllegalStateException,
        // else remove and return oldest element of this
        if (size == 0)
                throw new IllegalStateException ("Queue is empty: Underflow");
        else {
                size--;
                int data = queueRep [ (front % CAPACITY) ];
                queueRep [front] = Integer.MIN_VALUE;
                front = (front+1) % CAPACITY;
                return data;
        }
}
// Checks whether the queue is empty. This method runs in O(1) time.
public boolean isEmpty(){
        return (size == 0);
}
// Checks whether the queue is full. This method runs in O(1) time.
public boolean isFull(){
        return (size == CAPACITY);
}
// Returns the number of elements in the queue. This method runs in O(1) time.
public int size() {
        return size;
}
// Increases the queue size by double
private void expand(){
        int length = size();
        int[] newQueue = new int[length<<1];  // or 2* length
        //copy items
        for(int i = front; i <= rear; i ++)
                newQueue[i-front] = queueRep[i%CAPACITY];
        queueRep = newQueue;
        front = 0;
        rear = size-1;
        CAPACITY *= 2;
```

```
    }
    // dynamic array operation: shrinks to 1/2 if more than than 3/4 empty
    private void shrink() {
        int length = size;
        if(length <= MINCAPACITY || length <<2 >= length)
            return;
        if(length<MINCAPACITY) length = MINCAPACITY;
        int[] newQueue=new int[length];
        System.arraycopy(queueRep,0,newQueue,0,length+1);
        queueRep=newQueue;
    }
    // Returns a string representation of the queue as a list of elements, with
    // the front element at the end: [ ... , prev, rear ]. This method runs in O(n)
    // time, where n is the size of the queue.
    public String toString(){
        String result = "[";
        for (int i = 0; i < size; i++){
            result += Integer.toString(queueRep[ (front + i) % CAPACITY ]);
            if (i < size -1) {
                result += ", ";
            }
        }
        result += "]";
        return result;
    }
}
```

Performance

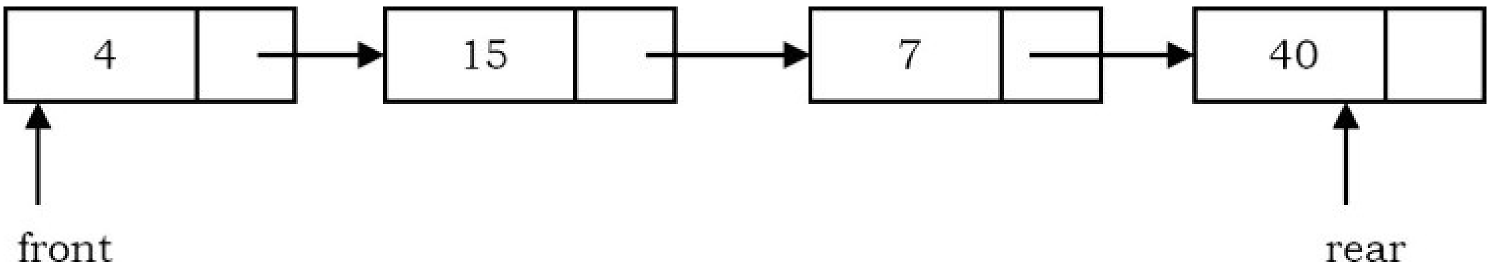
Let *n* be the number of elements in the queue.

Space Complexity (for *n* enQueue operations)	O(*n*)
Time Complexity of enQueue()	O(1) (Average)
Time Complexity of deQueue()	O(1)
Time Complexity of getQueueSize()	O(1)
Time Complexity of isEmpty()	O(1)

Time Complexity of isFull()	O(1)

Linked List Implementation

Another way of implementing queues is by using Linked lists. *EnQueue* operation is implemented by inserting an element at the end of the list. *DeQueue* operation is implemented by deleting an element from the beginning of the list.



```
public class LinkedQueue{
    private int length;
    private ListNode front, rear;
    // Creates an empty queue.
    public LinkedQueue(){
            length = 0;
            front = rear = null;
    }
    // Adds the specified data to the rear of the queue.
    public void enqueue (int data){
            ListNode node = new ListNode(data);
            if (isEmpty())
                    front = node;
            else
                    rear.setNext (node);
            rear = node;
            length++;
    }
    // Removes the data at the front of the queue and returns a
    // reference to it. Throws an Exception if the
    // queue is empty.
    public int dequeue() throws Exception{
            if (isEmpty())
                    throw new Exception ("queue");
            int result = front.getData();
            front = front.getNext();
            length--;
            if (isEmpty())
                    rear = null;
            return result;
    }
    // Returns a reference to the data at the front of the queue.
    // The data is not removed from the queue.  Throws an
    // Exception if the queue is empty.
    public int first() throws Exception{
            if (isEmpty())
                    throw new Exception();

            return front.getData();
    }
    // Returns true if this queue is empty and false otherwise.
    public boolean isEmpty(){
            return (length == 0);
    }
    // Returns the number of elements in this queue.
    public int size(){
            return length;
    }
    // Returns a string representation of this queue.
    public String toString(){
            String result = "";
            ListNode current = front;
            while (current != null){
                    result = result + current.toString() + "\n";
                    current = current.getNext();
            }
            return result;
    }
}
```

Performance

Let *n* be the number of elements in the queue, then

Space Complexity (for *n* enQueue operations)	O(*n*)
Time Complexity of enQueue()	O(1) (Average)
Time Complexity of deQueue()	O(1)
Time Complexity of isEmpty()	O(1)
Time Complexity of deleteQueue()	O(1)

Comparison of Implementations

Note: Comparison is very similar to stack implementations and *Stacks* chapter.

5.7 Queues: Problems & Solutions

Problem-1 Give an algorithm for reversing a queue *Q*. To access the queue, you are only allowed to use the methods of queue ADT.

Solution:

```
public class QueueReversal {
    public static Queue reverseQueue(Queue queue){
        Stack stack = new LLStack();
        while(!queue.isEmpty()){
            stack.push(queue.deQueue());
        }
        while(!stack.isEmpty()){
            queue.enQueue(stack.pop());
        }
        return queue;
    }
}
```

Time Complexity: O(*n*).

Problem-2 How can you implement a queue using two stacks?

Solution: Let S1 and S2 be the two stacks to be used in the implementation of queue. All we have to do is to define the enQueue and deQueue operations for the queue.

EnQueue Algorithm:

- Just push on to stack S1

Time Complexity: O(1).

DeQueue Algorithm:

- If stack S2 is not empty then pop from S2 and return that element.
- If stack is empty, then transfer all elements from S1 to S2 and pop the top element from S2 and return that popped element [we can optimize the code a little by transferring only $n - 1$ elements from S1 to S2 and pop the n^{th} element from S1 and return that popped element].
- If stack S1 is also empty then throw error.

Time Complexity: From the algorithm, if the stack S2 is not empty then the complexity is O(1). If the stack S2 is empty, then we need to transfer the elements from SI to S2. But if we carefully observe, the number of transferred elements and the number of popped elements from S2 are equal. Due to this the average complexity of pop operation in this case is O(1).The amortized complexity of pop operation is O(1).

```
public class QueuewithTwoStacks<T> {
    private Stack<T> S1 = new Stack<T>();
    private Stack<T> S2 = new Stack<T>();
    public void enqueue(T data){
        S1.push(data);
    }
    public T dequeue(){
        if(S2.empty())
            while(!S1.isEmpty()){
                S2.push(S1.pop());
            }
        return S2.pop();
    }
}
```

Problem-3 Show how you can efficiently implement one stack using two queues. Analyze the running time of the stack operations.

Solution: Yes, it is possible to implement the Stack ADT using 2 implementations of the Queue ADT. One of the queues will be used to store the elements and the other to hold them temporarily during the *pop* and *top* methods. The *push* method would *enqueue* the given element onto the storage queue. The *top* method would transfer all but the last element from the storage queue onto the temporary queue, save the front element of the storage queue to be returned, transfer the last element to the temporary queue, then transfer all elements back to the storage queue. The *pop* method would do the same as top, except instead of transferring the last element onto the temporary queue after saving it for return, that last element would be discarded. Let Q1 and Q2 be the two queues to be used in the implementation of stack. All we have to do is to define the push and pop operations for the stack.

In the algorithms below, we make sure that one queue is always empty.

Push Operation Algorithm: Insert the element in whichever queue is not empty.

- Check whether queue Q1 is empty or not. If Q1 is empty then Enqueue the element into Q2.
- Otherwise enQueue the element into Q1.

Time Complexity: O(1).

Pop Operation Algorithm: Transfer n – 1 elements to the other queue and delete last from queue for performing pop operation.

- If queue Q1 is not empty then transfer n – 1 elements from Q1 to Q2 and then, deQueue the last element of Q1 and return it.
- If queue Q2 is not empty then transfer n – 1 elements from Q2 to Q1 and then, deQueue the last element of Q2 and return it.

Time Complexity: Running time of pop operation is O(n) as each time pop is called, we are transferring all the elements from one queue to the other.

```
public class StackwithTwoQueues<T> {
    private Queue<T> Q1 = new LinkedList<T>();
    private Queue<T> Q2 = new LinkedList<T>();
    public void push(T data){
        if(Q1.isEmpty())
            Q2.offer(data);
        else    Q1.offer(data);
    }
    public T pop(){
        int i=0, size;
        if(Q2.isEmpty()) {
            size = Q1.size();
            while(i < size-1) {
                Q2.offer(Q1.poll());
                i++;
            }
            return Q1.poll();
        }
        else {
            size = Q2.size();
            while(i < size-1) {
                Q1.offer(Q2.poll());
                i++;
            }
            return Q2.poll();
        }
    }
}
```

Problem-4 **Maximum sum in sliding window:** Given array A[] with sliding window of size *w* which is moving from the very left of the array to the very right. Assume that we can only see the *w* numbers in the window. Each time the sliding window moves rightwards by one position. For example: The array is [1 3 -1 -3 5 3 6 7], and *w* is 3.

Window position	Max
[1 3 -1] -3 5 3 6 7	3
1 [3 -1 -3] 5 3 6 7	3
1 3[-l -3 5] 3 6 7	5
1 3 -1[-3 5 3] 6 7	5

1 3 -1 -3 [5 3 6] 7	6
1 3 -1 -3 5 [3 6 7]	7

Input: A long array A[], and a window width w. **Output:** An array B[], B[i] is the maximum value from A[i] to A[i+w-1]. **Requirement:** Find a good optimal way to get B[i]

Solution: This problem can be solved with doubly ended queue (which supports insertion and deletion at both ends). Refer *Priority Queues* chapter for algorithms.

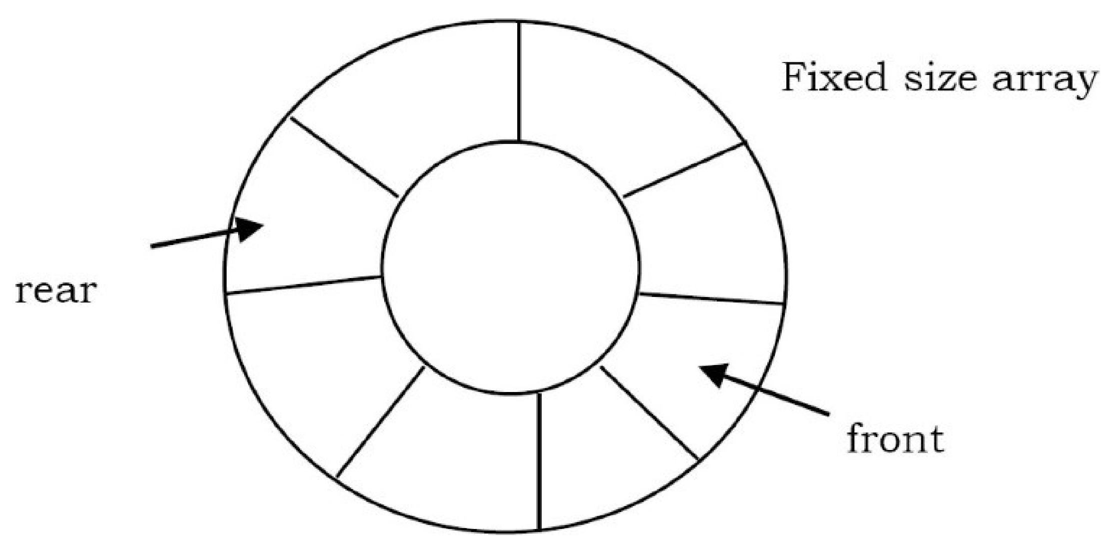
Problem-5 Given a queue Q containing n elements, transfer these items on to a stack S (initially empty) so that front element of Q appears at the top of the stack and the order of all other items is preserved. Using enqueue and dequeue operations for the queue, and push and pop operations for the stack, outline an efficient O(n) algorithm to accomplish the above task, using only a constant amount of additional storage.

Solution: Assume the elements of queue Q are $a_1, a_2 \ldots a_n$. Dequeuing all elements and pushing them onto the stack will result in a stack with a_n at the top and a_1 at the bottom. This is done in O(n) time as dequeue and each push require constant time per operation. The queue is now empty. By popping all elements and pushing them on the queue we will get a_1 at the top of the stack. This is done again in O(n) time.

As in big-oh arithmetic we can ignore constant factors. The process is carried out in O(n) time. The amount of additional storage needed here has to be big enough to temporarily hold one item.

Problem-6 A queue is set up in a circular array A[0..n - 1] with front and rear defined as usual. Assume that $n - 1$ locations in the array are available for storing the elements (with the other element being used to detect full/empty condition). Give a formula for the number of elements in the queue in terms of *rear, front*, and n.

Solution: Consider the following figure to get a clear idea of the queue.



- Rear of the queue is somewhere clockwise from the front.
- To enqueue an element, we move *rear* one position clockwise and write the element in that position.
- To dequeue, we simply move *front* one position clockwise.
- Queue migrates in a clockwise direction as we enqueue and dequeue.
- Emptiness and fullness to be checked carefully.
- Analyze the possible situations (make some drawings to see where *front* and *rear* are when the queue is empty, and partially and totally filled). We will get this:

$$Number\ Of\ Elements = \begin{cases} rear - front + 1 & \text{if rear } = \text{ front} \\ rear - front + n & \text{otherwise} \end{cases}$$

Problem-7 What is the most appropriate data structure to print elements of queue in reverse order?

Solution: Stack.

Problem-8 Given a stack of integers, how do you check whether each successive pair of numbers in the stack is consecutive or not. The pairs can be increasing or decreasing, and if the stack has an odd number of elements, the element at the top is left out of a pair. For example, if the stack of elements are [4, 5, -2, -3, 11, 10, 5, 6, 20], then the output should be true because each of the pairs (4, 5), (-2, -3), (11, 10), and (5, 6) consists of consecutive numbers.

Solution:

```java
public static boolean checkStackPairwiseOrder(Stack<Integer> s) {
    Queue<Integer> q = new LinkedList<Integer>();
    boolean pairwiseOrdered = true;
    while (!s.isEmpty())
        q.add(s.pop());
    while (!q.isEmpty())
        s.push(q.remove());
    while (!s.isEmpty()) {
        int n = s.pop();
        q.add(n);
        if (!s.isEmpty()) {
            int m = s.pop();
            q.add(m);
            if (Math.abs(n - m) != 1) {
                pairwiseOrdered = false;
            }
        }
    }
    while (!q.isEmpty())
        s.push(q.remove());
    return pairwiseOrdered;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-9 Given a queue of integers, rearrange the elements by interleaving the first half of the list with the second half of the list. For example, suppose a queue stores the following sequence of values: [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Consider the two halves of this list: first half: [11, 12, 13, 14, 15] second half: [16, 17, 18, 19, 20]. These are combined in an alternating fashion to form a sequence of interleave pairs: the first values from each half (11 and 16), then the second values from each half (12 and 17), then the third values from each half (13 and 18), and so on. In each pair, the value from the first half appears before the value from the second half. Thus, after the call, the queue stores the following values: [11, 16, 12, 17, 13, 18, 14, 19, 15, 20].

Solution:

```
public void interLeavingQueue(Queue <Integer> q) {
    if (q.size() % 2 != 0)
        throw new IllegalArgumentException();
    Stack<Integer> s = new ArrayStack<Integer>();
    int halfSize = q.size() / 2;
    for (int i = 0; i &lt; halfSize; i++)
        s.push(q.dequeue());
    while (!s.isEmpty())
        q.enqueue(s.pop());
         for (int i = 0; i &lt; halfSize; i++)
        q.enqueue(q.dequeue());
    for (int i = 0; i &lt; halfSize; i++)
        s.push(q.dequeue());
    while (!s.isEmpty()) {
        q.enqueue(s.pop());
        q.enqueue(q.dequeue());
    }
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-10 Given an integer k and a queue of integers, how do you reverse the order of the first k elements of the queue, leaving the other elements in the same relative order? For example, if k=4 and queue has the elements [10, 20, 30, 40, 50, 60, 70, 80, 90]; the output should be [40, 30, 20, 10, 50, 60, 70, 80, 90].

Solution:

```java
public static void reverseQueueFirstKElements(int k, Queue<Integer> q) {
    if (q == null || k > q.size()) {
        throw new IllegalArgumentException();
    }
    else if (k > 0) {
        Stack<Integer> s = new Stack<Integer>();
        for (int i = 0; i < k; i++) {
            s.push(q.remove());
        }
        while (!s.isEmpty()) {
            q.add(s.pop());
        }
        for (int i = 0; i < q.size() - k; i++) { // wrap around rest of elements
            q.add(q.remove());
        }
    }
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*).

CHAPTER 6

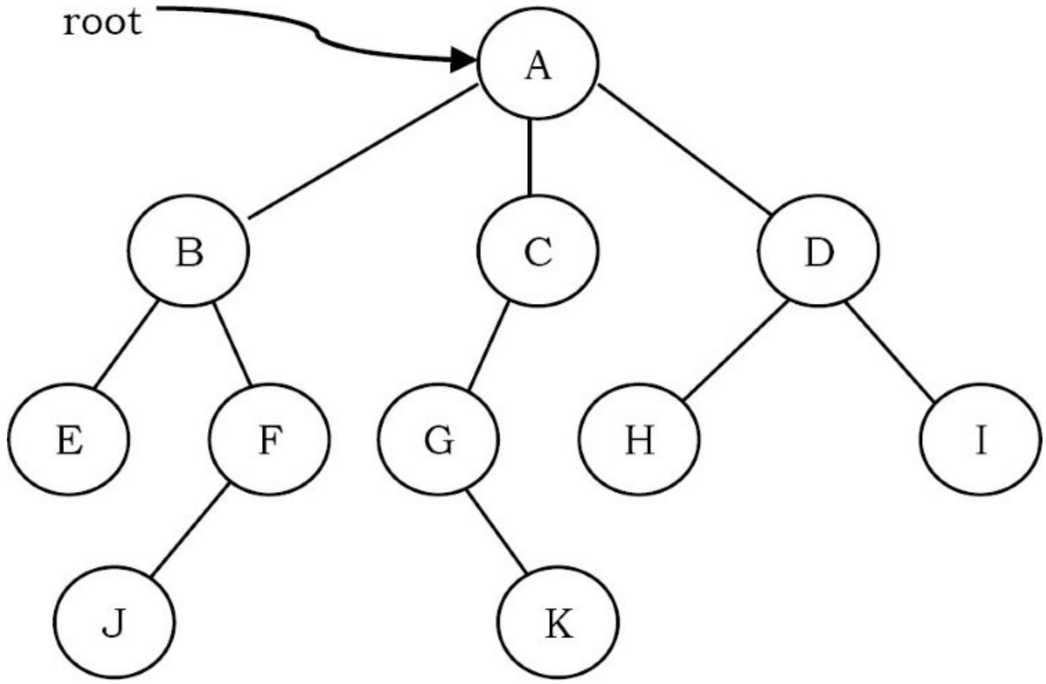
TREES

6.1 What is a Tree?

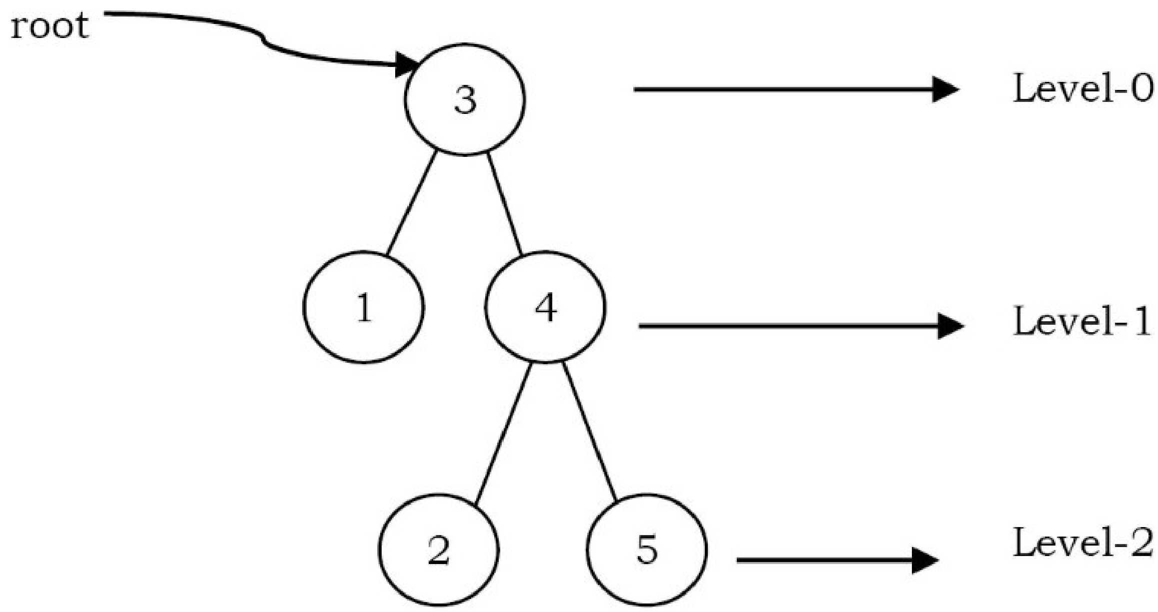
A *tree* is a data structure similar to a linked list but instead of each node pointing simply to the next node in a linear fashion, each node points to a number of nodes. Tree is an example of non-linear data structures. A *tree* structure is a way of representing the hierarchical nature of a structure in a graphical form.

In trees ADT (Abstract Data Type), the order of the elements is not important. If we need ordering information, linear data structures like linked lists, stacks, queues, etc. can be used.

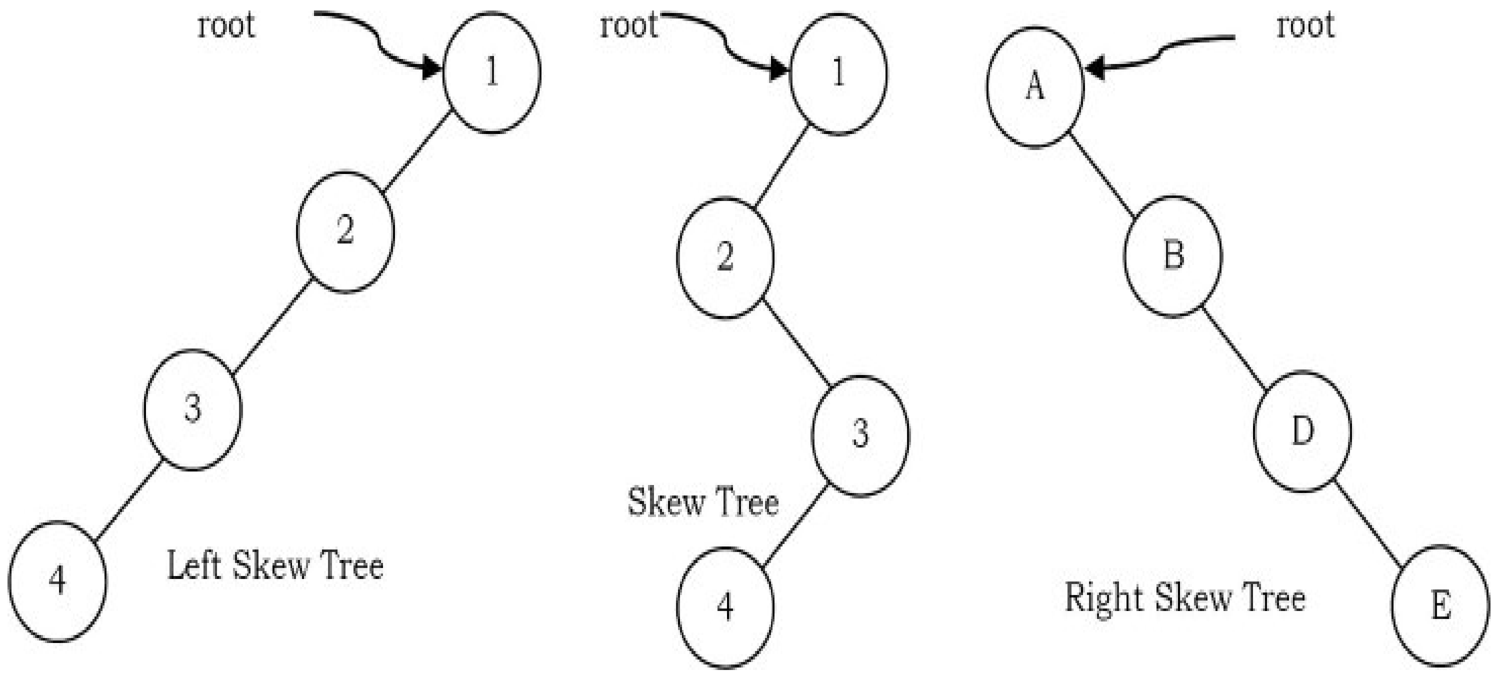
6.2 Glossary



- The *root* of a tree is the node with no parents. There can be at most one root node in a tree (node *A* in the above example).
- An *edge* refers to the link from parent to child (all links in the figure).
- A node with no children is called *leaf* node (*E, J, K, H* and *I*).
- Children of same parent are called *siblings* (*B, C, D* are siblings of *A*, and *E, F* are the siblings of *B*).
- A node p is an *ancestor* of node *q* if there exists a path from *root* to *q* and p appears on the path. The node *q* is called a *descendant* of p. For example, *A, C* and *G* are the ancestors of *K*.
- The set of all nodes at a given depth is called the *level* of the tree (*B, C* and *D* are the same level). The root node is at level zero.
- The *depth* of a node is the length of the path from the root to the node (depth of *G* is *2, A – C – G*).
- The *height* of a node is the length of the path from that node to the deepest node. The height of a tree is the length of the path from the root to the deepest node in the tree. A (rooted) tree with only one node (the root) has a height of zero. In the previous example, the height of *B* is 2 (*B – F – J*).
- *Height of the tree* is the maximum height among all the nodes in the tree and *depth of the tree* is the maximum depth among all the nodes in the tree. For a given tree, depth and height returns the same value. But for individual nodes we may get different results.



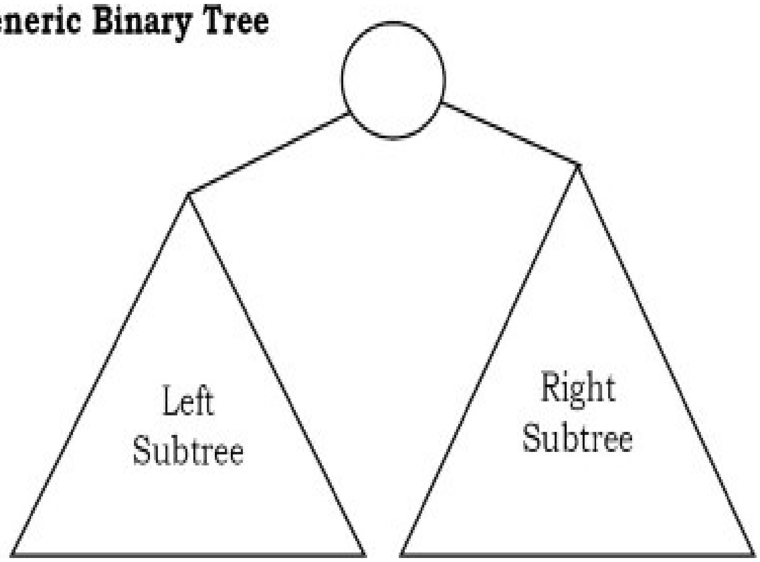
- The *depth* of a node is the length of the path from the root to the node (depth of *G* is 2, *A* – *C* – *G*).
- The *height* of a node is the length of the path from that node to the deepest node. The height of a tree is the length of the path from the root to the deepest node in the tree. A (rooted) tree with only one node (the root) has a height of zero. In the previous example, the height of *B* is 2 (*B* – *F* – *J*).
- *Height of the tree* is the maximum height among all the nodes in the tree and *depth of the tree* is the maximum depth among all the nodes in the tree. For a given tree, depth and height returns the same value. But for individual nodes we may get different results.
- The size of a node is the number of descendants it has including itself (the size of the subtree *C* is 3).
- If every node in a tree has only one child (except leaf nodes) then we call such trees *skew trees*. If every node has only left child then we call them *left skew trees*. Similarly, if every node has only right child then we call them *right skew trees*.



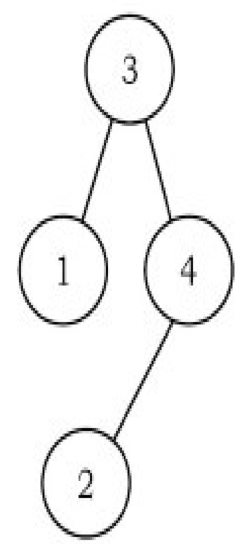
6.3 Binary Trees

A tree is called *binary tree* if each node has zero child, one child or two children. Empty tree is also a valid binary tree. We can visualize a binary tree as consisting of a root and two disjoint binary trees, called the left and right subtrees of the root.

Generic Binary Tree

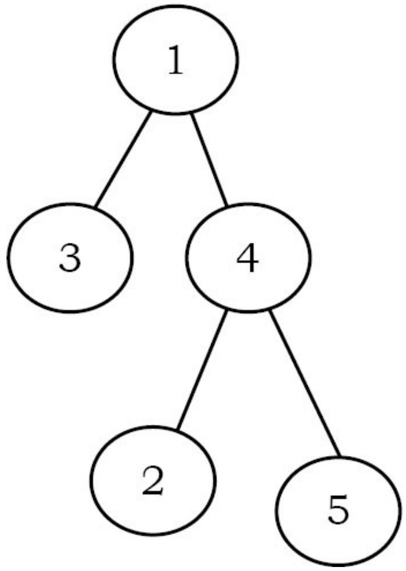


Example

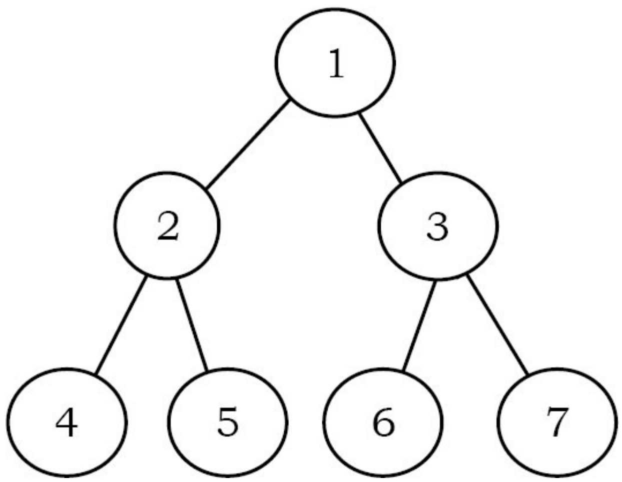


Types of Binary Trees

Strict Binary Tree: A binary tree is called *strict binary tree* if each node has exactly two children or no children.

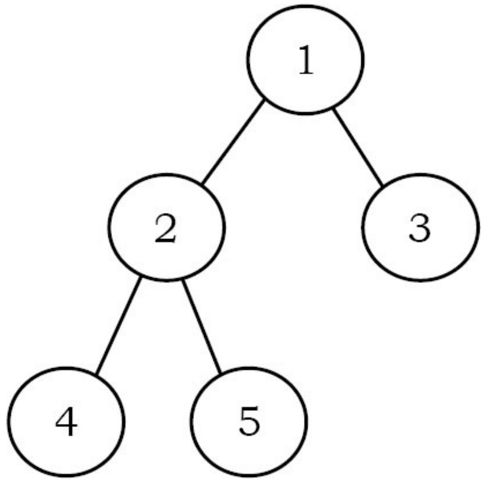


Full Binary Tree: A binary tree is called *full binary tree* if each node has exactly two children and all leaf nodes are at the same level.



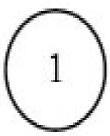
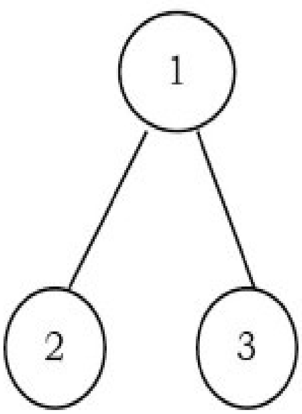
Complete Binary Tree: Before defining the *complete binary tree*, let us assume that the height of the binary tree is h. In complete binary trees, if we give numbering for the nodes by starting at the root (let us say the root node has 1) then we get a complete sequence from 1 to the number of nodes in the tree.

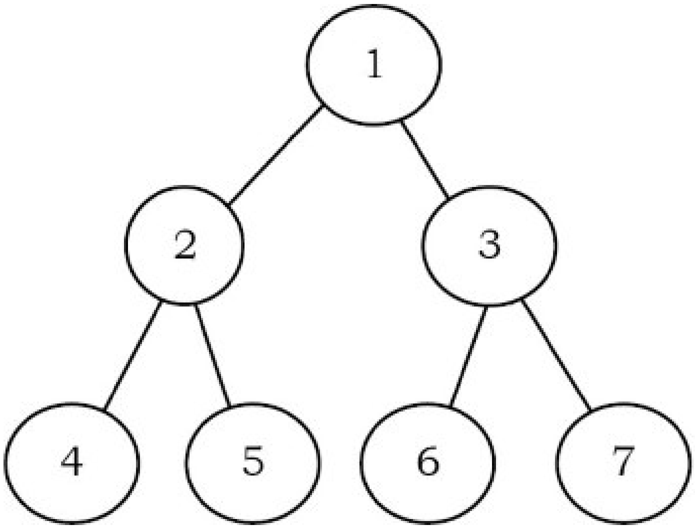
While traversing we should give numbering for NULL pointers also. A binary tree is called *complete binary tree* if all leaf nodes are at height h or $h - 1$ and also without any missing number in the sequence.



Properties of Binary Trees

For the following properties, let us assume that the height of the tree is *h*. Also, assume that root node is at height zero.

	Height	**Number of nodes at level h**
	$h = 0$	$2^0 = 1$
	$h = 1$	$2^1 = 2$



$h = 2$ $\qquad$ $2^2 = 4$

From the below diagram we can infer the following properties:

- The number of nodes n in a full binary tree is $2^{h+1} - 1$. Since, there are h levels we need to add all nodes at each level $[2^0 + 2^1 + 2^2 + \ldots + 2^h = 2^{h+1} - 1]$.
- The number of nodes n in a complete binary tree is between 2^h (minimum) and $2^{h+1} - 1$ (maximum). For more information on this, refer to *Priority Queues* chapter.
- The number of leaf nodes in a full binary tree is 2^h.
- The number of NULL links (wasted pointers) in a complete binary tree of n nodes is $n + 1$.

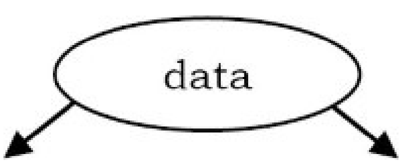
Structure of Binary Trees

Now let us define structure of the binary tree. For simplicity, assume that the data of the nodes are integers. One way to represent a node (which contains data) is to have two links which point to left and right children along with data fields as shown below:

Or

```
public class BinaryTreeNode {
    public int data;
    public BinaryTreeNode left, right;
    public BinaryTreeNode(int data){
        this.data = data;
        left = null;
        right = null;
    }
    public int getData() {
        return data;
    }
    public void setData(int data) {
        this.data = data;
    }
    public BinaryTreeNode getLeft() {
        return left;
    }
    public void setLeft(BinaryTreeNode left) {
        this.left = left;
    }
    public BinaryTreeNode getRight() {
        return right;
    }
    public void setRight(BinaryTreeNode right) {
        this.right = right;
    }
}
```

Note: In trees, the default flow is from parent to children and it is not mandatory to show directed branches. For our discussion, we assume both the representations shown below are the same.



Operations on Binary Trees

Basic Operations

- Inserting an element into a tree
- Deleting an element from a tree
- Searching for an element
- Traversing the tree

Auxiliary Operations

- Finding the size of the tree
- Finding the height of the tree
- Finding the level which has maximum sum
- Finding the least common ancestor (LCA) for a given pair of nodes, and many more.

Applications of Binary Trees

Following are the some of the applications where *binary trees* play an important role:

- Expression trees are used in compilers.
- Huffman coding trees that are used in data compression algorithms.
- Binary Search Tree (BST), which supports search, insertion and deletion on a collection of items in O(*logn*) (average).
- Priority Queue (PQ), which supports search and deletion of minimum (or maximum) on a collection of items in logarithmic time (in worst case).

6.4 Binary Tree Traversals

In order to process trees, we need a mechanism for traversing them, and that forms the subject of this section. The process of visiting all nodes of a tree is called *tree traversal*. Each node is processed only once but it may be visited more than once. As we have already seen in linear data structures (like linked lists, stacks, queues, etc.), the elements are visited in sequential order. But, in tree structures there are many different ways.

Tree traversal is like searching the tree, except that in traversal the goal is to move through the tree in a particular order. In addition, all nodes are processed in the *traversal but searching* stops when the required node is found.

Traversal Possibilities

Starting at the root of a binary tree, there are three main steps that can be performed and the order in which they are performed defines the traversal type. These steps are: performing an action on the current node (referred to as "visiting" the node and denoted with "*D*"), traversing to the left child node (denoted with "*L*"), and traversing to the right child node (denoted with "*R*"). This process can be easily described through recursion. Based on the above definition there are 6 possibilities:

1. *LDR:* Process left subtree, process the current node data and then process right subtree
2. *LRD:* Process left subtree, process right subtree and then process the current node data
3. *DLR:* Process the current node data, process left subtree and then process right subtree
4. *DRL:* Process the current node data, process right subtree and then process left subtree
5. *RDL:* Process right subtree, process the current node data and then process left subtree
6. *RLD:* Process right subtree, process left subtree and then process the current node data

Classifying the Traversals

The sequence in which these entities (nodes) are processed defines a particular traversal method. The classification is based on the order in which current node is processed. That means, if we are classifying based on current node (*D*) and if *D* comes in the middle then it does not matter whether *L* is on left side of *D* or *R* is on left side of *D*.

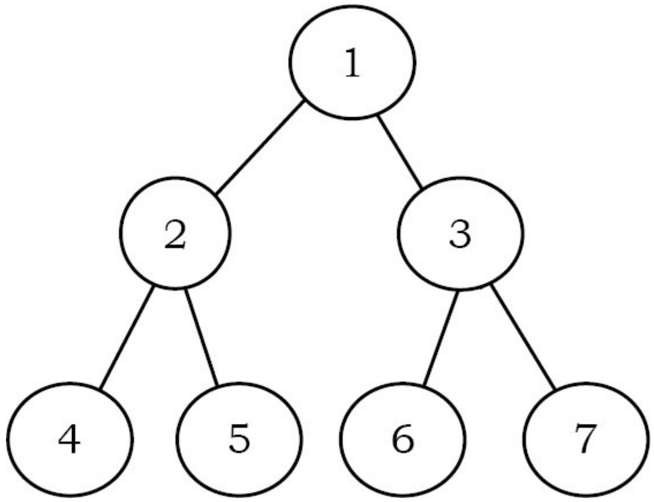
Similarly, it does not matter whether *L* is on right side of *D* or *R* is on right side of *D*. Due to this, the total 6 possibilities are reduced to 3 and these are:

- Preorder (*DLR*) Traversal
- Inorder (*LDR*) Traversal
- Postorder (*LRD*) Traversal

There is another traversal method which does not depend on the above orders and it is:

- Level Order Traversal: This method is inspired from Breadth First Traversal (BFS of Graph algorithms).

Let us use the diagram below for the remaining discussion.



PreOrder Traversal

In preorder traversal, each node is processed before (pre) either of its subtrees. This is the simplest traversal to understand. However, even though each node is processed before the subtrees, it still requires that some information must be maintained while moving down the tree. In the example above, 1 is processed first, then the left subtree, and this is followed by the right subtree.

Therefore, processing must return to the right subtree after finishing the processing of the left subtree. To move to the right subtree after processing the left subtree, we must maintain the root information. The obvious ADT for such information is a stack. Because of its LIFO structure, it is possible to get the information about the right subtrees back in the reverse order.

Preorder traversal is defined as follows:

- Visit the root.
- Traverse the left subtree in Preorder.
- Traverse the right subtree in Preorder.

The nodes of tree would be visited in the order: 1 2 4 5 3 6 7

```
public void PreOrder(BinaryTreeNode root){
    if(root != null) {
        System.out.println(root.data);
        PreOrder(root.left);
        PreOrder(root.right);
    }
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*).

Iterative Preorder Traversal

In the recursive version, a stack is required as we need to remember the current node so that after completing the left subtree we can go to the right subtree. To simulate the same, first we process the current node and before going to the left subtree, we store the current node on stack. After completing the left subtree processing, *pop* the element and go to its right subtree. Continue this process until stack is nonempty.

```
public ArrayList<Integer> preorderTraversal(BinaryTreeNode root) {
    ArrayList<Integer> res = new ArrayList<Integer>();
    if(root == null)
        return res;
    Stack<BinaryTreeNode> s = new Stack<BinaryTreeNode>();
    s.push(root);
    while(!s.isEmpty()){
        BinaryTreeNode tmp = s.pop();
        res.add(tmp.data);
        // pay more attention to this sequence.
        if(tmp.right != null)
            s.push(tmp.right);
        if(tmp.left != null)
            s.push(tmp.left);
    }
    return res;
}
```

Time Complexity: O(n). Space Complexity: O(n).

InOrder Traversal

In Inorder Traversal the root is visited between the subtrees. Inorder traversal is defined as follows:

- Traverse the left subtree in Inorder.
- Visit the root.
- Traverse the right subtree in Inorder.

The nodes of tree would be visited in the order: 4 2 5 16 3 7

```
public void InOrder(BinaryTreeNode root){
    if(root != null) {
        InOrder(root.left);
        System.out.println(root.data);
        InOrder(root.right);
    }
}
```

Time Complexity: O(n). Space Complexity: O(n).

Non-Recursive Inorder Traversal

The Non-recursive version of Inorder traversal is similar to Preorder. The only change is, instead of processing the node before going to left subtree, process it after popping (which is indicated after completion of left subtree processing).

```
public ArrayList<Integer> inorderTraversal(BinaryTreeNode root) {
    ArrayList<Integer> res = new ArrayList<Integer>();
    Stack<BinaryTreeNode> s = new Stack<BinaryTreeNode>();
    BinaryTreeNode currentNode = root;
    boolean done = false;
    while(!done){
        if(currentNode != null){
            s.push(currentNode);
            currentNode = currentNode.left;
        }else{
            if(s.isEmpty())
                done = true;
            else{
                    currentNode = s.pop();
                res.add(currentNode.data);
                currentNode = currentNode.right;
            }
        }
    }
    return res;
}
```

Time Complexity: O(n). Space Complexity: O(n).

PostOrder Traversal

In postorder traversal, the root is visited after both subtrees. Postorder traversal is defined as follows:

- Traverse the left subtree in Postorder.
- Traverse the right subtree in Postorder.
- Visit the root.

The nodes of the tree would be visited in the order: 4 5 2 6 7 3 1

```
public void PostOrder(BinaryTreeNode root){
    if(root != null) {
        PostOrder(root.left);
        PostOrder(root.right);
        System.out.println(root.data);
    }
}
```

Time Complexity: O(n). Space Complexity: O(n).

Non-Recursive Postorder Traversal

In preorder and inorder traversals, after popping the stack element we do not need to visit the same vertex again. But in postorder traversal, each node is visited twice. That means, after processing the left subtree we will visit the current node and after processing the right subtree we will visit the same current node. But we should be processing the node during the second visit. Here the problem is how to differentiate whether we are returning from the left subtree or the right subtree.

We use a *previous* variable to keep track of the earlier traversed node. Let's assume *current* is the current node that is on top of the stack. When *previous* is *current's* parent, we are traversing down the tree. In this case, we try to traverse to *current's* left child if available (i.e., push left child to the stack). If it is not available, we look at *current's* right child. If both left and right child do not exist (ie, *current* is a leaf node), we print *current's* value and pop it off the stack.

If prev is *current's* left child, we are traversing up the tree from the left. We look at *current's* right child. If it is available, then traverse down the right child (i.e., push right child to the stack); otherwise print *current's* value and pop it off the stack. If *previous* is *current's* right child, we are traversing up the tree from the right. In this case, we print *current's* value and pop it off the stack.

```
public ArrayList<Integer> postorderTraversal(BinaryTreeNode root) {
        ArrayList<Integer> res = new ArrayList<Integer>();
        if(root == null)
            return res;
        Stack<BinaryTreeNode> s = new Stack<BinaryTreeNode>();
        s.push(root);
        BinaryTreeNode prev = null;
        while(!s.isEmpty()){
                BinaryTreeNode curr = s.peek();
            if(prev == null || prev.left == curr || prev.right == curr){
                // traverse from top to bottom, and if curr has left child or right child,
                // push into the stack; otherwise, pop out.
                if(curr.left != null)
                    s.push(curr.left);
                else if(curr.right != null)
                    s.push(curr.right);
            }else if(curr.left == prev){
                if(curr.right != null)
                    s.push(curr.right);
            }else{
                res.add(curr.data);
                s.pop();
            }
            prev = curr;
        }
        return res;
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*).

Level Order Traversal

Level order traversal is defined as follows:

- Visit the root.
- While traversing level 1, keep all the elements at level 1+1 in queue.
- Go to the next level and visit all the nodes at that level.
- Repeat this until all levels are completed.

The nodes of the tree are visited in the order: 1 2 3 4 5 6 7

```java
public ArrayList<ArrayList<Integer>> levelOrder(BinaryTreeNode root) {
    ArrayList<ArrayList<Integer>> res = new ArrayList<ArrayList<Integer>>();
        if (root == null)
                return res;
        // Initialization
        Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
        q.offer(root);
        q.offer(null);
        ArrayList<Integer> curr = new ArrayList<Integer>();
        while (!q.isEmpty()) {
                BinaryTreeNode tmp = q.poll();
                if (tmp != null) {
                        curr.add(tmp.data);
                        if (tmp.left != null)
                                q.offer(tmp.left);
                        if (tmp.right != null)
                                q.offer(tmp.right);
                } else {
                        ArrayList<Integer> c_curr = new ArrayList<Integer>(curr);
                        res.add(c_curr);
                        curr.clear(); // Java will clear the reference, so have to new an new ArrayList.
                        // completion of a level;
                        if (!q.isEmpty())
                                q.offer(null);
                }
        }
        return res;
}
```

Time Complexity: O(n). Space Complexity: O(n). Since, in the worst case, all the nodes on the entire last level could be in the queue simultaneously.

Binary Trees: Problems & Solutions

Problem-1 Give an algorithm for finding maximum element in binary tree.

Solution: One simple way of solving this problem is: find the maximum element in left subtree, find the maximum element in right sub tree, compare them with root data and select the one which is giving the maximum value. This approach can be easily implemented with recursion.

```
public int maxInBinaryTree(BinaryTreeNode root){
    int maxValue = Integer.MIN_VALUE;
    if (root != null){
        int leftMax = maxInBinaryTree(root.left);
        int rightMax = maxInBinaryTree(root.right);
        if (leftMax > rightMax)
            maxValue = leftMax;
        else
            maxValue = rightMax;
        if (root.data > maxValue)
            maxValue = root.data;
        }
    return maxValue;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-2 Give an algorithm for finding maximum element in binary tree without recursion.

Solution: Using level order traversal: just observe the element's data while deleting.

```
public int maxInBinaryTreeLevelOrder(BinaryTreeNode root){
    if(root == null)
      return Integer.MIN_VALUE;
  int max = Integer.MIN_VALUE;
  Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
  q.offer(root);
  while(!q.isEmpty()){
    BinaryTreeNode tmp = q.poll();
    if (tmp.getData() > max){
        max = tmp.getData();
    }
    if(tmp != null){
      if(tmp.getLeft() != null)
        q.offer(tmp.getLeft());
      if(tmp.right != null)
        q.offer(tmp.right);
    }
  }
  return max;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-3 Give an algorithm for searching an element in binary tree.

Solution: Given a binary tree, return true if a node with data is found in the tree. Recurse down the tree, choose the left or right branch by comparing data with each nodes data.

```
// Tests whether the root argument contains within itself the data argument.
public static boolean findInBT(BinaryTreeNode root, int  data) {
        if (root == null)
            return false;
        if (root.getData() == data)
            return true;
        return findInBT(root.getLeft(), data)  || findInBT(root.getRight(), data);
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-4 Give an algorithm for searching an element in binary tree without recursion.

Solution: We can use level order traversal for solving this problem. The only change required in level order traversal is, instead of printing the data, we just need to check whether the root data is equal to the element we want to search.

```java
// Tests whether the root argument contains within itself the data argument.
public boolean findInBT(BinaryTreeNode root, int data){
        if(root == null)
            return false;
        Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
        q.offer(root);
        while(!q.isEmpty()){
                BinaryTreeNode tmp = q.poll();
                if (tmp.getData() == data){
                        return true;
                }
            if(tmp != null){
                if(tmp.getLeft() != null)
                    q.offer(tmp.getLeft());
                if(tmp.right != null)
                    q.offer(tmp.right);
            }
        }
        return false;
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*).

Problem-5 Give an algorithm for inserting an element into binary tree.

Solution: Since the given tree is a binary tree, we can insert the element wherever we want. To insert an element, we can use the level order traversal and insert the element wherever we find the node whose left or right child is NULL.

```
public BinaryTreeNode insertInBinaryTreeLevelOrder(BinaryTreeNode root, int data){
    if(root == null)
        return null;
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    while(!q.isEmpty()){
        BinaryTreeNode tmp = q.poll();
        if(tmp != null){
            if(tmp.getLeft() != null)
                q.offer(tmp.getLeft());
            else{
                tmp.left = new BinaryTreeNode(data);
                return root;
            }
        if(tmp.right != null)
            q.offer(tmp.right);
        else{
            tmp.right = new BinaryTreeNode(data);
            return root;
        }
      }
    }
    return root;
}
```

Recursive Approach

```
public void insert(BinaryTreeNode root, int data) {
	if (root == null) {
		root = new BinaryTreeNode(data);
	} else {
		insertHelper(root, data);
	}
}
private void insertHelper(BinaryTreeNode root, int data) {
	if (root.data >= data) {    // It is not compulsory to put this check.
		if (root.left == null) {
			root.left = new BinaryTreeNode(data);
		} else {
			insertHelper(root.right, data);
		}
	} else {
		if (root.right == null) {
			root.right = new BinaryTreeNode(data);
		} else {
			insertHelper(root.right, data);
		}
	}
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-6 Give an algorithm for finding the size of binary tree.

Solution: Calculate the size of left and right subtrees recursively, add 1 (current node) and return to its parent.

```
// Returns the total number of nodes in this binary tree (include the root in the count).
public int size(BinaryTreeNode root) {
	int leftCount = root.left == null ? 0 : size(root.left);
	int rightCount = root.right == null ? 0 : size(root.right);
	return 1 + leftCount + rightCount;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-7 Can we solve Problem-6 without recursion?

Solution: Yes, using level order traversal.

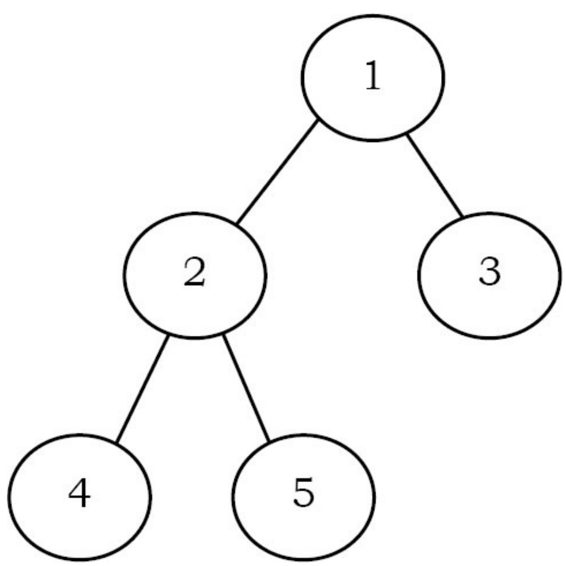
```
// Returns the total number of nodes in this binary tree (include the root in the count).
public int size(BinaryTreeNode root) {
        int count = 0;
        if(root == null)
           return 0;
        Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
        q.offer(root);
        while(!q.isEmpty()){
                BinaryTreeNode tmp = q.poll();
                count++;
           if(tmp.getLeft() != null)
                q.offer(tmp.getLeft());
           if(tmp.right != null)
                q.offer(tmp.right);
        }
        return count;
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*).

Problem-8 Give an algorithm for deleting the tree.

Solution: To delete a tree, we must traverse all the nodes of the tree and delete them one by one. So which traversal should we use: Inorder, Preorder, Postorder or Level order Traversal?

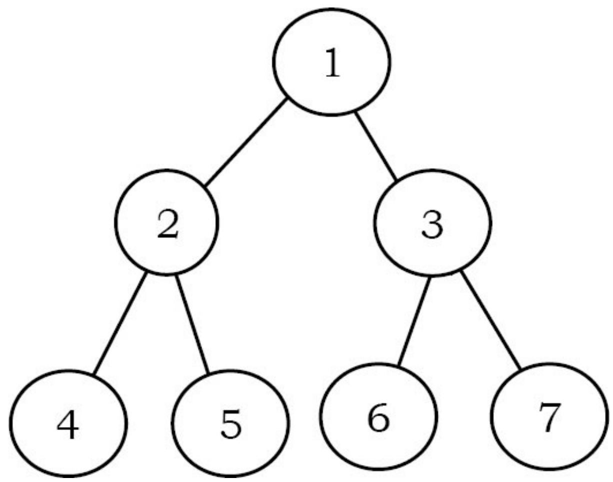
Before deleting the parent node we should delete its children nodes first. We can use postorder traversal as it does the work without storing anything. We can delete tree with other traversals also with extra space complexity. For the following, tree nodes are deleted in order - 4,5,2,3,1.



```
public void deleteBinaryTree(BinaryTreeNode root) {
        root = null;                    //In Java, it will be taken care by garbage collector
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*).

Problem-9 Give an algorithm for printing the level order data in reverse order. For example, the output for the below tree should be: 4 5 6 7 2 3 1



Solution:

```
public static void levelOrderTraversalInReverse(BinaryTreeNode root) {
    if(root == null)
        return;
    Stack<BinaryTreeNode> s = new Stack<BinaryTreeNode>();
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    while(!q.isEmpty()){
        BinaryTreeNode tmp = q.poll();
        if(tmp.getLeft() != null)
            q.offer(tmp.getLeft());
        if(tmp.right != null)
            q.offer(tmp.right);
        s.push(tmp);
    }
    while(!s.isEmpty())
            System.out.println(s.pop().getData()+ " ");
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*).

Problem-10 Give an algorithm for finding the height (or depth) of the binary tree.

Solution: Recursively calculate height of left and right subtrees of a node and assign height to the node as max of the heights of two children plus 1. This is similar to *PreOrder* tree traversal (and *DFS* of Graph algorithms).

```
// Returns the depth of this binary tree. The depth of a binary tree is the
// length of the longest path from this node to a leaf. The depth of a
// binary tree with no descendants (that is, just a leaf) is zero.
public int maxDepthRecursive(BinaryTreeNode root) {
   if(root == null)
         return 0;
   // Compute the depth of each subtree
   int leftDepth = maxDepthRecursive(root.left);
   int rightDepth = maxDepthRecursive(root.right);
   return (leftDepth > rightDepth) ? leftDepth + 1 : rightDepth + 1;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-11 Can we solve Problem-10 with stack?

Solution: We can use stack to simulate recursion.

```
// Returns the depth of this binary tree. The depth of a binary tree is the
// length of the longest path from this node to a leaf. The depth of a
// binary tree with no descendants (that is, just a leaf) is zero.
public int maxDepthIterative(BinaryTreeNode root){
    if(root == null)
            return 0;
    Stack<BinaryTreeNode> s = new Stack<BinaryTreeNode>();
    s.push(root);
    int maxDepth = 0;
    BinaryTreeNode prev = null;
    while(!s.isEmpty()){
        BinaryTreeNode curr = s.peek();
        if(prev == null || prev.left == curr || prev.right == curr){
            if(curr.left != null) s.push(curr.left);
            else if(curr.right != null) s.push(curr.right);
        }else if(curr.left == prev){
            if(curr.right != null) s.push(curr.right);
        }else
            s.pop();
        prev = curr;
        if(s.size() > maxDepth)
            maxDepth = s.size();
    }
    return maxDepth;
}
```

Problem-12 Can we solve Problem-10 without recursion?

Solution: Yes. Using level order traversal. This is similar to *BFS* of Graph algorithms. End of level is identified with NULL.

```java
// Returns the depth of this binary tree. The depth of a binary tree is the
// length of the longest path from this node to a leaf. The depth of a
// binary tree with no descendants (that is, just a leaf) is zero.
public int minDepth(BinaryTreeNode root) {
    if(root == null) return 0;
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    q.offer(null);
    int count = 1;
    while(!q.isEmpty()){
        BinaryTreeNode currentNode = q.poll();
        if(currentNode != null){
            if(currentNode.left == null && currentNode.right == null){
                return count;
            }
            if(currentNode.left != null){
                q.offer(currentNode.left);
            }
            if(currentNode.right != null){
                q.offer(currentNode.right);
            }
        }else{
            if(!q.isEmpty()){
                count++;
                q.offer(null);
            }
        }
    }
    return count;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-13 Give an algorithm for finding the minimum depth of the binary tree.

Solution:

```
public int minDepth(BinaryTreeNode root) {
    if(root == null) return 0;
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    q.offer(null);
    int count = 1;
    while(!q.isEmpty()){
        BinaryTreeNode currentNode = q.poll();
        if(currentNode != null){
            if(currentNode.left == null && currentNode.right == null){
                return count;
            }
            if(currentNode.left != null){
                q.offer(currentNode.left);
            }
            if(currentNode.right != null){
                q.offer(currentNode.right);
            }
        }else{
            if(!q.isEmpty()){
                count++;
                q.offer(null);
            }
        }
    }
    return count;
}
```

Problem-14 Give an algorithm for finding the deepest node of the binary tree.

Solution: The last node processed from queue in level order is the deepest node in binary tree.

```java
public BinaryTreeNode deepestNodeinBinaryTree(BinaryTreeNode root) {
    BinaryTreeNode tmp = null;
    if(root == null)
          return null;
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    while(!q.isEmpty()){
        tmp = q.poll();
       if(tmp.getLeft() != null)
          q.offer(tmp.getLeft());
       if(tmp.right != null)
          q.offer(tmp.right);
    }
    return tmp;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-15 Give an algorithm for deleting an element (assuming data is given) from binary tree.

Solution: The deletion of a node in binary tree can be implemented as

- Starting at root, find the node which we want to delete.
- Find the deepest node in the tree.
- Replace the deepest node's data with node to be deleted.
- Then delete the deepest node.

Problem-16 Give an algorithm for finding the number of leaves in the binary tree without using recursion.

Solution: The set of nodes whose both left and right children are NULL are called leaf nodes.

```
public int numberOfLeavesInBTusingLevelOrder(BinaryTreeNode root) {
    int count = 0;
    if(root == null)
        return 0;
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    while(!q.isEmpty()){
        BinaryTreeNode tmp = q.poll();
        if(tmp.getLeft()== null && tmp.getRight()== null)
            count++;
        if(tmp.getLeft() != null)
            q.offer(tmp.getLeft());
        if(tmp.right != null)
            q.offer(tmp.right);
    }
    return count;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-17 Give an algorithm for finding the number of full nodes in the binary tree without using recursion.

Solution: The set of all nodes with both left and right children are called full nodes.

```
public int numberOfFullNodesInBTusingLevelOrder(BinaryTreeNode root) {
    int count = 0;
    if(root == null)
        return 0;
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    while(!q.isEmpty()){
        BinaryTreeNode tmp = q.poll();
        if(tmp.getLeft() != null && tmp.getRight() != null)
        count++;
        if(tmp.getLeft() != null)
            q.offer(tmp.getLeft());
        if(tmp.right != null)
            q.offer(tmp.right);
    }
    return count;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-18 Give an algorithm for finding the number of half nodes (nodes with only one child) in the binary tree without using recursion.

Solution: The set of all nodes with either left or right child (but not both) are called half nodes.

```
public int numberOfHalfNodesInBTusingLevelOrder(BinaryTreeNode root) {
    int count = 0;
    if(root == null)
        return 0;
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    while(!q.isEmpty()){
        BinaryTreeNode tmp = q.poll();
        if((tmp.getLeft() == null && tmp.getRight() != null) ||
            (tmp.getLeft() != null && tmp.getRight() != null))
            count++;
        if(tmp.getLeft() != null)
            q.offer(tmp.getLeft());
        if(tmp.right != null)
            q.offer(tmp.right);
    }
    return count;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-19 Given two binary trees, return true if they are structurally identical.

Solution:
Algorithm:

- If both trees are NULL then return true.
- If both trees are not NULL, then recursively check left and right subtree structures.

```
//Return true if they are structurally identical.
public boolean checkStructurullySameTrees(BinaryTreeNode  root1, BinaryTreeNode  root2) {
    if(root1 == null && root2 == null)
        return true;
    if(root1 == null || root2 == null)
        return false;
    return (checkStructurullySameTrees(root1.getLeft(), root2.right) &&
        checkStructurullySameTrees(root1.right, root2.getLeft()));
}
```

Time Complexity: O(n). Space Complexity: O(n), for recursive stack.

Problem-20 Give an algorithm for finding the diameter of the binary tree. The diameter of a tree (sometimes called the *width*) is the number of nodes on the longest path between two leaves in the tree.

Solution: To find the diameter of a tree, first calculate the diameter of left subtree and right subtrees recursively. Among these two values, we need to send maximum value along with current level (+1).

```
public int diameterOfTree(BinaryTreeNode root){
    int left, right;
    if(root == null)
        return 0;
    left = diameterOfTree(root.getLeft());
    right = diameterOfTree(root.getRight());
    if(left + right > diameter)
        diameter = left + right;
    return Math.max(left, right)+1;
}
// Alternative Coding
public int diameter(BinaryTreeNode root){
    if(root==null) return 0;
    //the path goes through the root
    int len1 = height(root.left) + height(root.right) +3;
    //the path does not pass the root
    int len2 = Math.max(diameter(root.left), diameter(root.right));
    return Math.max(len1, len2);
}
public int height(BinaryTreeNode root) {
    if(root == null)
        return 0;
    /* compute the depth of each subtree */
    int leftDepth = height(root.getLeft());
    int rightDepth = height(root.right);
    return (leftDepth > rightDepth) ? leftDepth + 1 : rightDepth + 1;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-21 Give an algorithm for finding the width of the binary tree. The diameter of a tree is the maximum number of nodes at any level (or depth) in the tree.

Solution:

```
//The width of a binary tree is the maximum number of elements on one level of the tree.
public int width(BinaryTreeNode root){
	int max = 0;
	int height = maxDepthRecursive(root);
	for(int k = 0; k <= height; k++){
		int tmp = width(root, k);
		if(tmp > max) max = tmp;
	}
	return max;
}
// Returns the number of node on a given level
public int width(BinaryTreeNode root, int depth){
	if(root==null)
		return 0;
	else
		if(depth == 0)
			return 1;
		else
			return width(root.left, depth-1) + width(root.right, depth-1);
}
```

Problem-22 Give an algorithm for finding the level that has the maximum sum in the binary tree.

Solution: The logic is very much similar to finding the number of levels. The only change is, we need to keep track of the sums as well.

```java
public int findLevelwithMaxSum(BinaryTreeNode root) {
        int maxSum = 0, currentSum = 0;
        if (root == null)
                return 0;
        // Initialization
        Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
        q.offer(root);
        q.offer(null);
        while (!q.isEmpty()) {
                BinaryTreeNode tmp = q.poll();
                if (tmp != null) {
                        if (tmp.getLeft() != null)
                                q.offer(tmp.getLeft());
                        if (tmp.right != null)
                                q.offer(tmp.right);
                } else {
                        if (currentSum > maxSum){
                                maxSum =  currentSum;
                        }
                        currentSum = 0;
                        // completion of a level;
                        if (!q.isEmpty())
                                q.offer(null);
                }
        }
        return maxSum;
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*).

Problem-23 Given a binary tree, print out all its root-to-leaf paths.

Solution: Refer to comments in functions.

```java
public void printPaths(BinaryTreeNode root) {
	int[] path = new int[256];
	printPaths(root, path, 0);
}
private void printPaths(BinaryTreeNode root, int[] path, int pathLen) {
	if (root == null)
		return;
	// append this node to the path array
	path[pathLen] = root.getData();
	pathLen++;
	// it's a leaf, so print the path that led to here
	if (root.getLeft() == null && root.getRight() == null) {
		printArray(path, pathLen);
	}
	else {	// otherwise try both subtrees
		printPaths(root.getLeft(), path, pathLen);
		printPaths(root.getRight(), path, pathLen);
	}
}
private void printArray(int[] ints, int len) {
	for (int i=0; i<len; i++) {
		System.out.print(ints[i] + " ");
	}
	System.out.println();
}
```

Time Complexity: O(n). Space Complexity: O(n), for recursive stack.

Problem-24 Give an algorithm for checking the existence of path with given sum. That means, given a sum, check whether there exists a path from root to any of the nodes.

Solution: For this problem, the strategy is: subtract the node value from the sum before calling its children recursively, and check to see if the sum is 0 when we run out of tree.

```
public boolean hasPathSum(TreeNode root, int sum) {
    if(root == null)
        return false;
    if(root.left == null && root.right == null && root.data == sum)
        return true;
    else
        return hasPathSum(root.left, sum - root.data) || hasPathSum(root.right, sum - root.data);
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-25 Give an algorithm for finding the sum of all elements in binary tree.

Solution: Recursively, call left subtree sum, right subtree sum and add their values to current nodes data.

```
public int addBT(BinaryTreeNode  root) {
        if(root == null) return 0;
        else return(root.getData() + addBT(root.getLeft()) +  addBT(root.getRight()));
}
```

Time Complexity: O(n). Space Complexity: O(n).

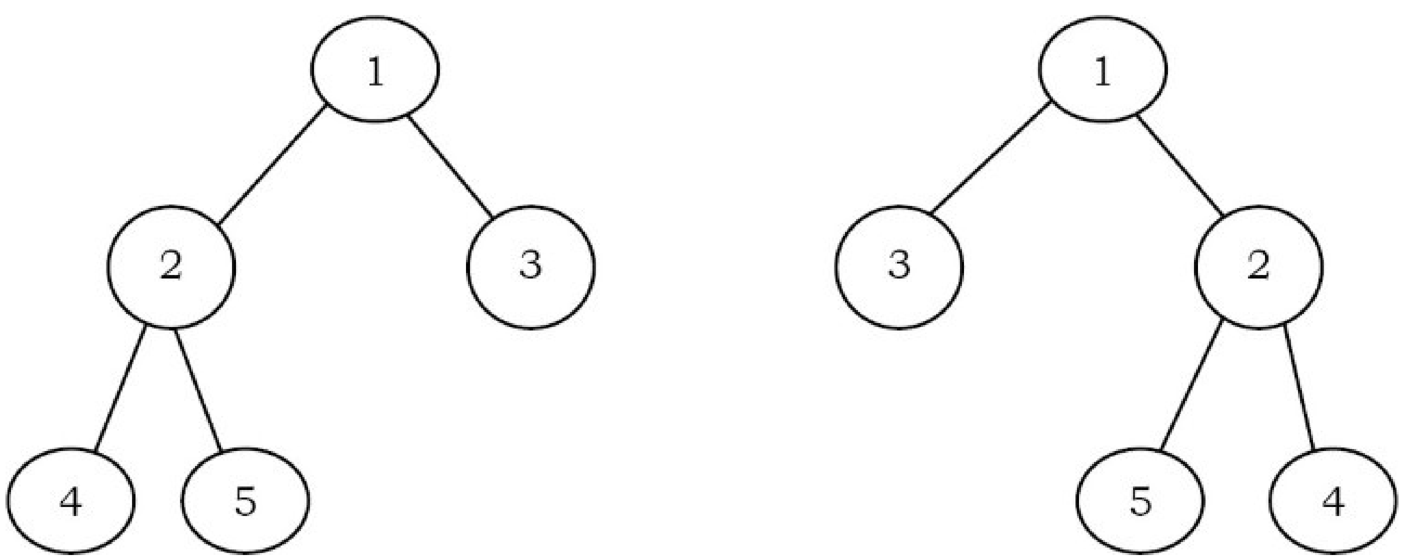
Problem-26 Can we solve Problem-25 without recursion?

Solution: We can use level order traversal with simple change. Every time after deleting an element from queue, add the node's data value to *sum* variable.

```
public int addBT(BinaryTreeNode  root) {
    int sum = 0;
    if(root == null)
        return 0;
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    while(!q.isEmpty()){
        BinaryTreeNode tmp = q.poll();
        sum += tmp.data;
        if(tmp.getLeft() != null)
            q.offer(tmp.getLeft());
        if(tmp.right != null)
            q.offer(tmp.right);
    }
    return sum;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-27 Give an algorithm for converting a tree to its mirror. Mirror of a tree is another tree with left and right children of all non-leaf nodes interchanged. The trees below are mirrors to each other.



Solution:

```
public BinaryTreeNode mirrorOfBinaryTree(BinaryTreeNode root) {
    BinaryTreeNode  temp;
    if(root != null) {
        mirrorOfBinaryTree (root.left);
        mirrorOfBinaryTree (root.right);
        /* swap the pointers in this node */
        temp  = root.left;
        root.left = root.right;
        root.right = temp;
    }
    return root;
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-28 Given two trees, give an algorithm for checking whether they are mirrors of each other.

Solution:

```
public boolean areMirrors(BinaryTreeNode  root1, BinaryTreeNode  root2) {
    if(root1 == null && root2 == null)
        return true;
    if(root1 == null || root2 == null)
        return false;
    if(root1.data != root2.data)
        return false;
    else return (areMirrors(root1.left, root2.right) && areMirrors(root1.right, root2.left));
}
```

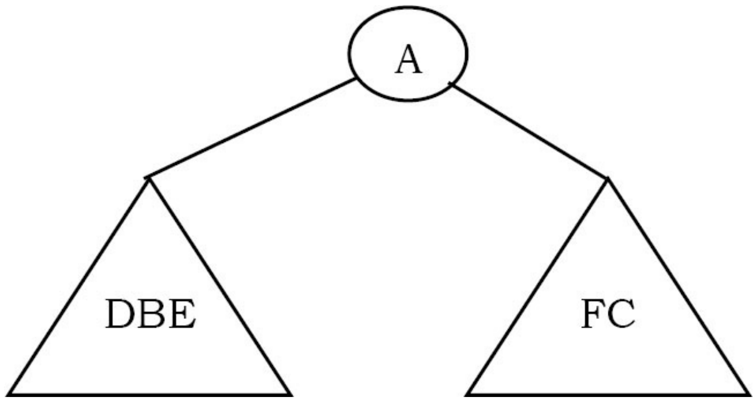
Time Complexity: O(n). Space Complexity: O(n).

Problem-29 Give an algorithm for constructing binary tree from given Inorder and Preorder traversals.

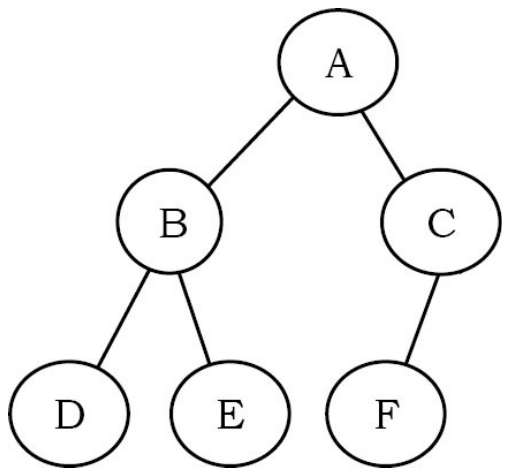
Solution: Let us consider the traversals below:

Inorder sequence: D B E A F C
Preorder sequence: A B D E C F

In a Preorder sequence, leftmost element denotes the root of the tree. So we know '*A*' is the root for given sequences. By searching '*A*' in Inorder sequence we can find out all elements on the left side of '*A*', which come under the left subtree, and elements on the right side of '*A*', which come under the right subtree. So we get the structure as seen below.



We recursively follow the above steps and get the following tree.



Algorithm: BuildTree()

1. Select an element from Preorder. Increment a Preorder index variable (preIndex in below code) to pick next element in next recursive call.
2. Create a new tree node (newNode) with the data as selected element.
3. Find the selected element's index in Inorder. Let the index be inIndex.
4. Call BuildBinaryTree for elements before inIndex and make the built tree as left subtree of newNode.
5. Call BuildBinaryTree for elements after inIndex and make the built tree as right subtree of newNode.
6. return newNode.

```
public BinaryTreeNode buildBinaryTree(int[] preorder, int[] inorder) {
    if(preorder.length == 0 || inorder.length != preorder.length)
        return null;
    return buildBT(preorder, 0, preorder.length - 1, inorder, 0, inorder.length - 1);
}
public BinaryTreeNode buildBT(int[] preOrder, int preStart, int preEnd, int[] inOrder, int inStart, int inEnd){
    if(preStart > preEnd || inStart > inEnd)
        return null;
    int data = preOrder[preStart];
    BinaryTreeNode cur = new BinaryTreeNode(data);
    int offset = inStart;
    for(; offset < inEnd; offset++){
        if(inOrder[offset] == data)
            break;
    }
    cur.left = buildBT(preOrder, preStart + 1, preStart + offset - inStart, inOrder, inStart, offset - 1);
    cur.right = buildBT(preOrder, preStart + offset - inStart + 1, preEnd, inOrder, offset + 1, inEnd);
    return cur;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-30 Give an algorithm for constructing binary tree from given Inorder and Postorder traversals.

Solution:

```java
public BinaryTreeNode buildBinaryTree(int[] inorder, int[] postorder) {
    if(postorder.length == 0 || postorder.length != inorder.length)
        return null;
    return buildBT(postorder, 0, postorder.length - 1, inorder, 0, inorder.length - 1);
}
public BinaryTreeNode buildBT(int[] postOrder, int postStart, int postEnd, int[] inOrder,
                                                    int inStart, int inEnd){
    if(postStart > postEnd || inStart > inEnd)
        return null;
    int val = postOrder[postEnd];
    int offset = inStart;
    BinaryTreeNode cur = new BinaryTreeNode(val);
    // search for the inorder index
    for(; offset < inEnd; offset++){
        if(inOrder[offset] == val)
            break;
    }
    cur.left = buildBT(postOrder, postStart, postStart + offset - inStart - 1, inOrder, inStart, offset - 1);
    cur.right = buildBT(postOrder, postStart + offset - inStart, postEnd - 1, inOrder, offset + 1, inEnd);
    return cur;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-31 If we are given two traversal sequences, can we construct the binary tree uniquely?

Solution: It depends on what traversals are given. If one of the traversal methods is *Inorder* then the tree can be constructed uniquely, otherwise not.

Therefore, the following combinations can uniquely identify a tree:

- Inorder and Preorder
- Inorder and Postorder
- Inorder and Level-order

The following combinations do not uniquely identify a tree.

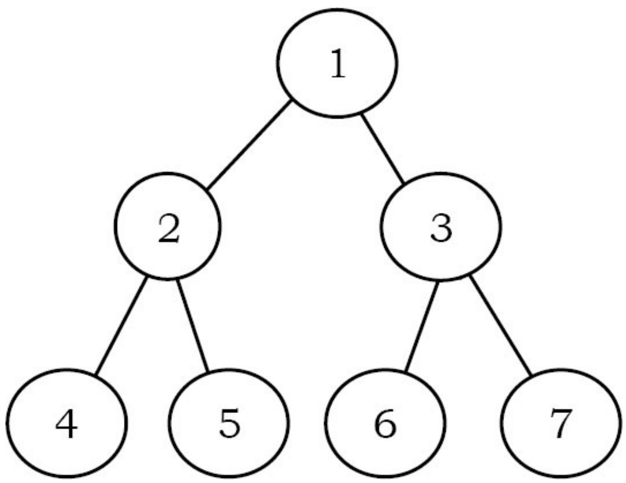
- Postorder and Preorder
- Preorder and Level-order
- Postorder and Level-order

For example, Preorder, Level-order and Postorder traversals are the same for the above trees:

Preorder Traversal = AB
Postorder Traversal = BA
Level-order Traversal = AB

So, even if three of them (PreOrder, Level-Order and PostOrder) are given, the tree cannot be constructed uniquely.

Problem-32 Give an algorithm for printing all the ancestors of a node in a Binary tree. For the tree below, for 7 the ancestors are 1 3 7.



Solution: Apart from the Depth First Search of this tree, we can use the following recursive way to print the ancestors.

```
public static boolean printAllAncestors(BinaryTreeNode root, BinaryTreeNode node){
	if(root == null)
		return false;
	if(root.getLeft() == node || root.getRight() == node ||
		printAllAncestors(root.getLeft(), node) || printAllAncestors(root.getRight(), node)) {
		System.out.println(root.getData());
		return true;
	}
	return false;
}
```

Time Complexity: O(n). Space Complexity: O(n) for recursion.

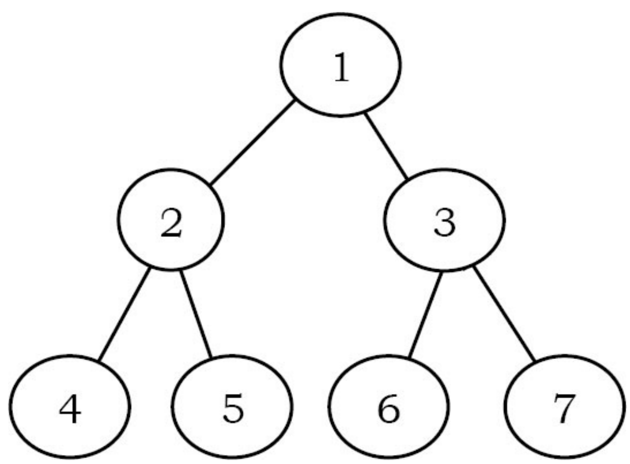
Problem-33 Give an algorithm for finding LCA (Least Common Ancestor) of two nodes in a Binary Tree.

Solution:

```
public BinaryTreeNode LCA(BinaryTreeNode root, BinaryTreeNode a, BinaryTreeNode b) {
    BinaryTreeNode left, right;
    if (root == null)
        return root;
    if (root == a || root == b)
        return root;
    left = LCA(root.left, a, b);
    right = LCA(root.right, a, b);
    if (left != null && right != null)
        return root;          // nodes are each on a separate branch
    else
        return (left != null ? left : right);
    // either one node is on one branch,
    // or none was found in any of the branches
}
```

Time Complexity: O(n). Space Complexity: O(n) for recursion.

Problem-34 **Zigzag Tree Traversal:** Give an algorithm to traverse a binary tree in Zigzag order. For example, the output for the tree below should be: 1 3 2 4 5 6 7



Solution: This problem can be solved easily using two stacks. Assume the two stacks are: *currentLevel* and *nextLevel*. We would also need a variable to keep track of the current level order (whether it is left to right or right to left).

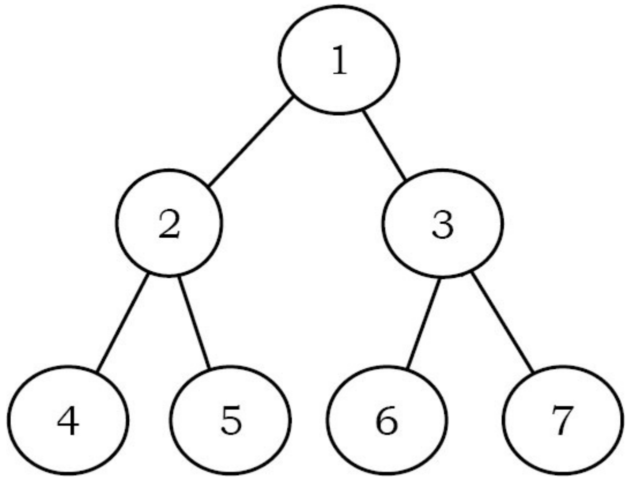
We pop from *currentLevel* stack and print the node's value. Whenever the current level order is from left to right, push the node's left child, then its right child, to stack *nextLevel*. Since a stack is a Last In First Out (*LIFO*) structure, the next time that nodes are popped off nextLevel, it will be in the reverse order.

On the other hand, when the current level order is from right to left, we would push the node's right child first, then its left child. Finally, don't forget to swap those two stacks at the end of each level (*i. e.,* when *currentLevel* is empty).

```java
public ArrayList<ArrayList<Integer>> zigzagLevelOrder(BinaryTreeNode root) {
    ArrayList<ArrayList<Integer>> res = new ArrayList<ArrayList<Integer>>();
    if(root == null)
        return res;
    Queue<BinaryTreeNode> q = new LinkedList<BinaryTreeNode>();
    q.offer(root);
    q.offer(null);
    boolean leftToRight = true;
    ArrayList<Integer> curr = new ArrayList<Integer>();
    while(!q.isEmpty()){
        BinaryTreeNode tmp = q.poll();
        if(tmp != null){
            curr.add(tmp.data);
            if(tmp.left != null)
                q.offer(tmp.left);
            if(tmp.right != null)
                q.offer(tmp.right);
        }else{
            if(leftToRight){
                ArrayList<Integer> c_curr = new ArrayList<Integer>(curr);
                res.add(c_curr);
                curr.clear();
            }else{
                Stack<Integer> s = new Stack<Integer>();
                s.addAll(curr);
                ArrayList<Integer> c_curr = new ArrayList<Integer>();
                while(!s.isEmpty()){
                    c_curr.add(s.pop());
                }
                res.add(c_curr);
                curr.clear();
            }
            if(!q.isEmpty()){
                q.offer(null);
                leftToRight = !leftToRight;
            }
        }
    }
    return res;
}
```

Time Complexity: O(n). Space Complexity: Space for two stacks = O(n) + O(n) = O(n).

Problem-35 Give an algorithm for finding the vertical sum of a binary tree. For example,



The tree has 5 vertical lines

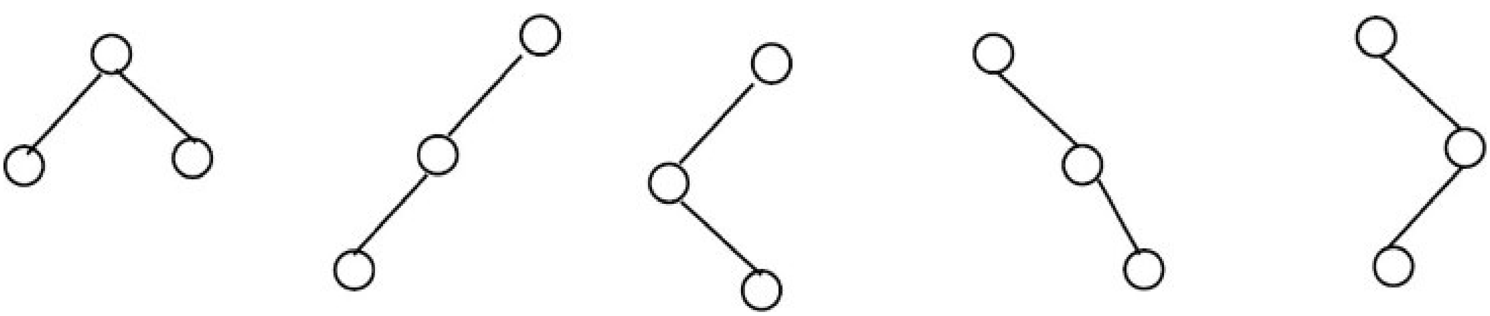
Vertical-1: nodes-4 => vertical sum is 4
Vertical-2: nodes-2 => vertical sum is 2
Vertical-3: nodes-1,5,6 => vertical sum is 1 + 5 + 6 = 12
Vertical-4: nodes-3 => vertical sum is 3
Vertical-5: nodes-7 => vertical sum is 7
We need to output: 4 2 12 3 7

Solution: We can do an inorder traversal and hash the column. We call VerticalSumInBinaryTree(root, 0) which means the root is at column 0. While doing the traversal, hash the column and increase its value by *root.getData*().

```java
public static void vSum(HashMap <Integer, Integer> hash,BinaryTreeNode root, int c){
	if(root.left!=null)
		vSum(hash, root.left, c-1);
	if(root.right!=null)
		vSum(hash,root.right, c+1);
	int data=0;
	if(hash.containsKey(c))
		data=hash.get(c);
	hash.put(c, root.data+data);
}
public static void verticalSum(BinaryTreeNode root){
	HashMap <Integer, Integer> hash = new HashMap<Integer, Integer>();
	vSum(hash, root, 0);
	System.out.println();
	for(int k:hash.keySet()){
		System.out.println("key(pos): "+k+ " sum: "+ hash.get(k)+" ");
	}
}
```

Problem-36 How many different binary trees are possible with *n* nodes?

Solution: For example, consider a tree with 3 nodes ($n = 3$). It will have the maximum combination of 5 different (i.e., $2^3 - 3 = 5$) trees.



In general, if there are *n* nodes, there exist $2^n - n$ different trees.

Programatically we can count by using the code beow.

```java
public int numOfBSTs(int n) {
    int[] count = new int[n + 1];
    count[0] = 1;
    count[1] = 1;
    for(int i = 2; i <= n; i++){
        for(int j = 0; j < i; j++){
            count[i] += count[j] * count[i - j - 1];
        }
    }
    return count[n];
}
```

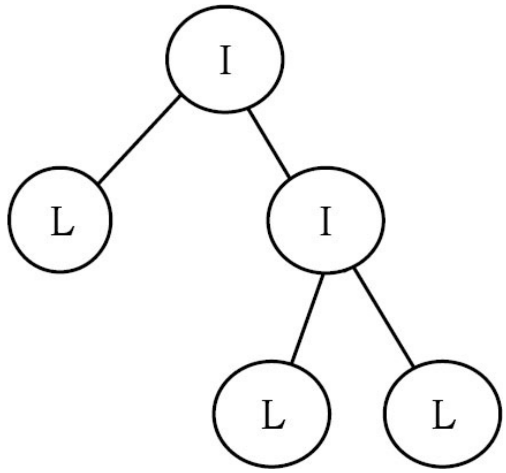
Time Complexity: O(n^2). Space Complexity: O(n).

Problem-37 For Problem-36, how do we generate all different BSTs with n nodes?

Solution:

```java
public ArrayList<BinarySearchTreeNode> generateTrees(int n) {
    if(n == 0)
        return generateTrees(1, 0);
    return generateTrees(1, n);
}
public ArrayList<BinarySearchTreeNode> generateTrees(int start, int end) {
    ArrayList<BinarySearchTreeNode> subTrees = new ArrayList<BinarySearchTreeNode>();
    if(start > end){
        subTrees.add(null);
        return subTrees;
    }
    for(int i = start; i <= end; i++){
        for(BinarySearchTreeNode left : generateTrees(start, i - 1)){
            for(BinarySearchTreeNode right : generateTrees(i + 1, end)){
                    BinarySearchTreeNode aTree = new BinarySearchTreeNode(i);
                aTree.left = left;
                aTree.right = right;
                subTrees.add(aTree);
            }
        }
    }
    return subTrees;
}
```

Problem-38 Given a tree with a special property where leaves are represented with 'L' and internal node with 'I'. Also, assume that each node has either 0 or 2 children. Given preorder traversal of this tree, construct the tree. **Example:** Given preorder string => ILILL



Solution: First, we should see how preorder traversal is arranged. Pre-order traversal means first put root node, then pre-order traversal of left subtree and then pre-order traversal of right subtree. In a normal scenario, it's not possible to detect where left subtree ends and right subtree starts using only pre-order traversal. Since every node has either 2 children or no child, we can surely say that if a node exists then its sibling also exists. So every time when we are computing a subtree, we need to compute its sibling subtree as well.

Secondly, whenever we get 'L' in the input string, that is a leaf and we can stop for a particular subtree at that point. After this 'L' node (left child of its parent 'L'), its sibling starts. If 'L' node is right child of its parent, then we need to go up in the hierarchy to find the next subtree to compute.

Keeping the above invariant in mind, we can easily determine when a subtree ends and the next one starts. It means that we can give any start node to our method and it can easily complete the subtree it generates going outside of its nodes. We just need to take care of passing the correct start nodes to different sub-trees.

```java
public BinaryTreeNode BuildTreeFromPreOrder(char[] A, int i) {
        if(A == null)                           //Boundary Condition
                return null;
        if(A.length == i)                       //Boundary Condition
                return null;
        BinaryTreeNode newNode = new BinaryTreeNode();
        newNode.setData(A[i]);
        newNode.setLeft(null);
        newNode.setRight(null);
        if(A[i] == 'L')                                 //On reaching leaf node, return
                return newNode;
        i = i + 1;                                      //Populate left sub tree
        newNode.setLeft(BuildTreeFromPreOrder(A, i));
        i = i + 1;                                      //Populate right sub tree
        newNode.setRight(BuildTreeFromPreOrder(A, i));
        return newNode;
}
```

Time Complexity: O(n).

Problem-39 Given a binary tree with three pointers (left, right and nextSibling), give an algorithm for filling the *nextSibling* pointers assuming they are NULL initially.

Solution: We can use simple queue.

```
public class SiblingBinaryTreeNode {
        public int data;
        public SiblingBinaryTreeNode left;
        public SiblingBinaryTreeNode right;
        public SiblingBinaryTreeNode nextSibling;
        public SiblingBinaryTreeNode(int data){
                this.data = data;
                left = null;
                right = null;
                nextSibling = null;
        }
        public SiblingBinaryTreeNode(int data, SiblingBinaryTreeNode left,
                                     SiblingBinaryTreeNode right,
                                     SiblingBinaryTreeNode nextSibling){
                this.data = data;
                this.left = left;
                this.right = right;
                this.nextSibling = nextSibling;
        }
```

```
	public int getData() {
		return data;
	}
	public void setData(int data) {
		this.data = data;
	}
	public SiblingBinaryTreeNode getLeft() {
		return left;
	}
	public void setLeft(SiblingBinaryTreeNode left) {
		this.left = left;
	}
	public SiblingBinaryTreeNode getRight() {
		return right;
	}
	public void setRight(SiblingBinaryTreeNode right) {
		this.right = right;
	}
	public SiblingBinaryTreeNode getNextSibling() {
		return nextSibling;
	}
	public void setNextSibling(SiblingBinaryTreeNode nextSibling) {
		this.nextSibling = nextSibling;
	}
	// Sets the data in this BinaryTreeNode node.
	public void setValue(int data) {
		this.data = data;
	}
	// Tests whether this node is a leaf node.
	public boolean isLeaf() {
		return left == null && right == null;
	}
}
public static void fillNextSiblings(SiblingBinaryTreeNode root) {
	SiblingBinaryTreeNode tmp = null;
	if (root == null)
		return;
	// Initialization
	Queue<SiblingBinaryTreeNode> q = new LinkedList<SiblingBinaryTreeNode>();
	q.offer(root);
	q.offer(null);
	while (!q.isEmpty()) {
		tmp = q.poll();
		if (tmp != null) {
			tmp.setNextSibling(q.peek());
			if (tmp.getLeft() != null)
				q.offer(tmp.getLeft());
			if (tmp.right != null)
				q.offer(tmp.right);
		} else {
			// completion of a level;
			if (!q.isEmpty())
				q.offer(null);
		}
	}
}
```

Time Complexity: O(n). Space Complexity: O(n).

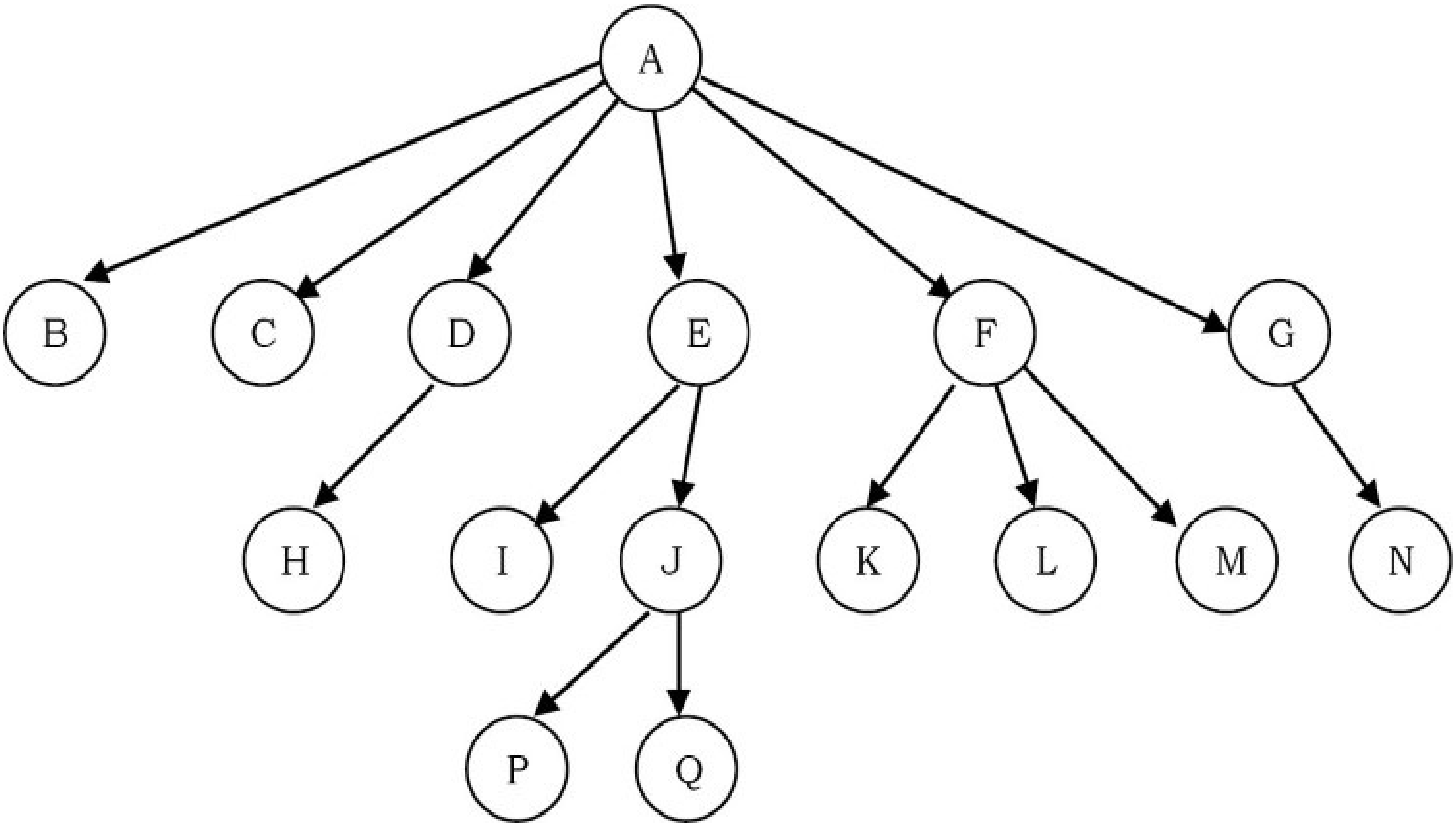
Problem-40 Is there any other way of solving Problem-39?

Solution: The trick is to re-use the populated *nextSibling* pointers. As mentioned earlier, we just need one more step for it to work. Before we pass the *left* and *right* to the recursion function itself, we connect the right child's *nextSibling* to the current node's nextSibling left child. In order for this to work, the current node *nextSibling* pointer must be populated, which is true in this case.

```
public static void fillNextSiblings(SiblingBinaryTreeNode root) {
	if (root == null)
		return;
	if (root.getLeft() != null)
		root.getLeft().setNextSibling(root.getRight());
	if (root.getRight() != null)
		if(root.getNextSibling() != null)
			root.getRight().setNextSibling(root.getNextSibling().getLeft());
		else	root.getRight().setNextSibling(null);
	fillNextSiblings(root.getLeft());
	fillNextSiblings(root.getRight());
}
```

Time Complexity: O(n).

6.5 Generic Trees (N-ary Trees)



In the previous section we discussed binary trees where each node can have a maximum of two children and these are represented easily with two pointers. But suppose if we have a tree with many children at every node and also if we do not know how many children a node can have, how do we represent them?

For example, consider the tree shown above.

How do we represent the tree?

In the above tree, there are nodes with 6 children, with 3 children, with 2 children, with 1 child, and with zero children (leaves). To present this tree we have to consider the worst case (6 children) and allocate that many child pointers for each node. Based on this, the node representation can be given as:

```
public class TreeNode {
        public int data;
        public TreeNode firstChild;
        public TreeNode secondChild;
        public TreeNode thirdChild;
        public TreeNode fourthChild;
        public TreeNode fifthChild;
        public TreeNode sixthChild;
        .....
}
```

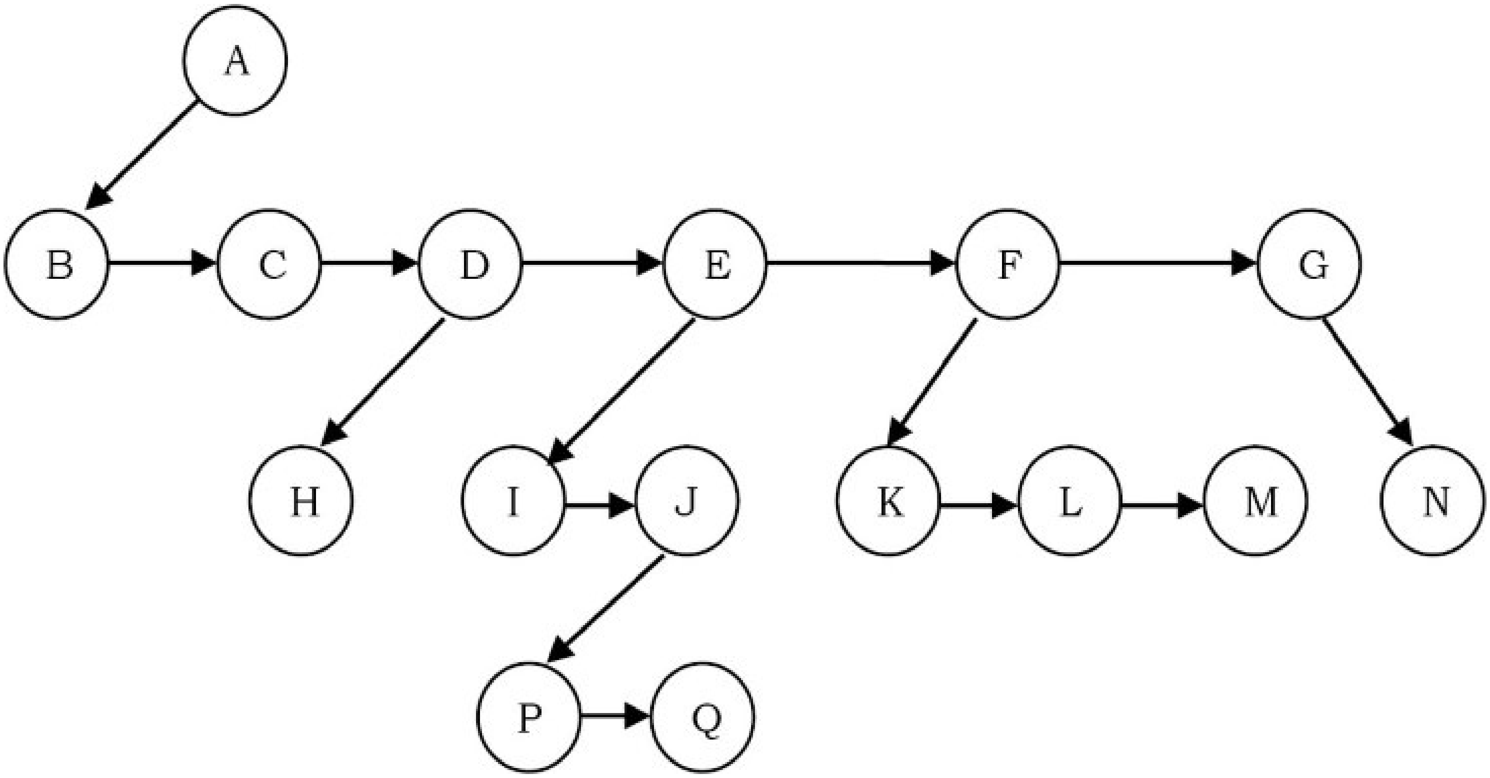
Since we are not using all the pointers in all the cases, there is a lot of memory wastage. Another problem is that we do not know the number of children for each node in advance. In order to solve this problem we need a representation that minimizes the wastage and also accepts nodes with any number of children.

Representation of Generic Trees

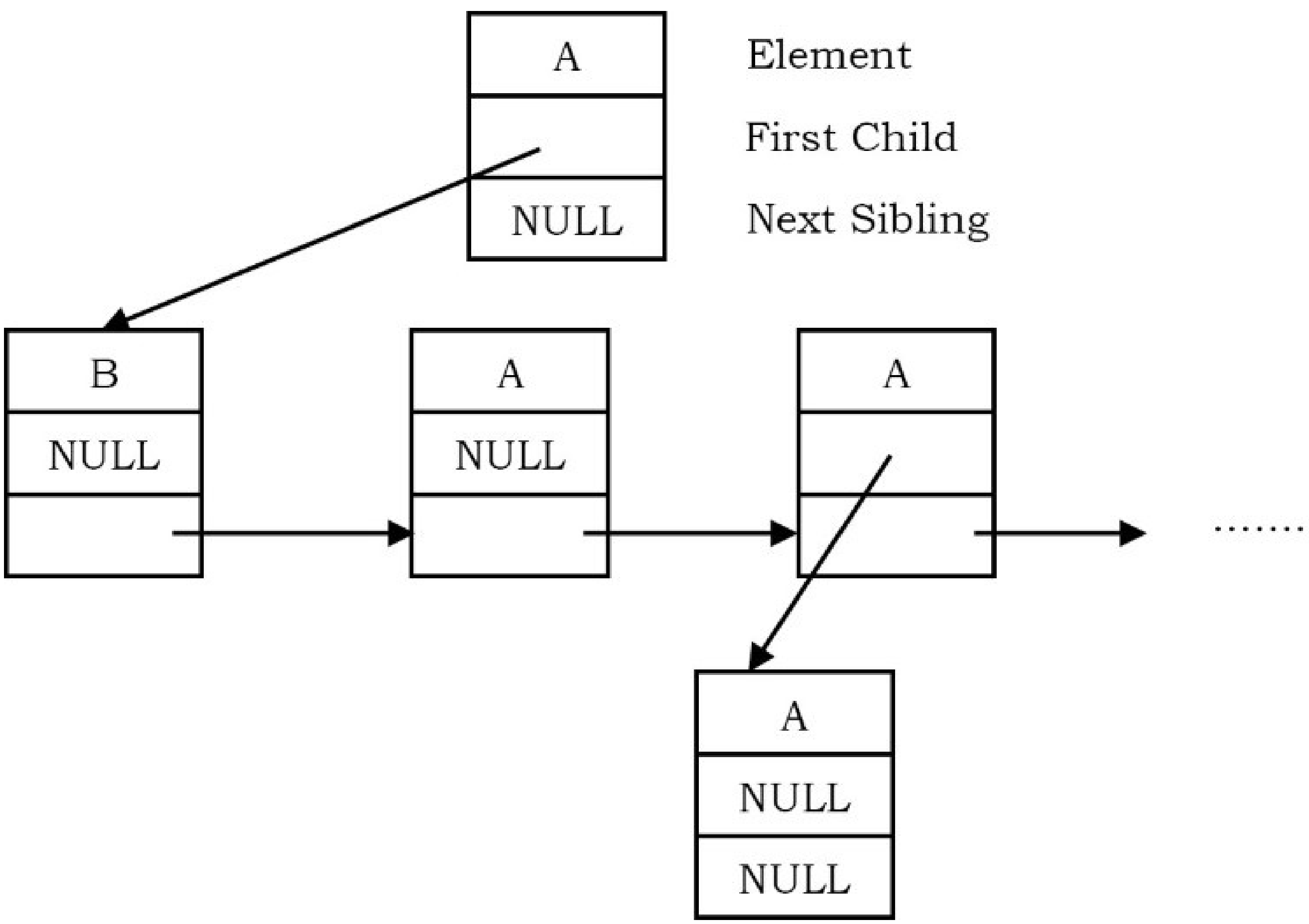
Since our objective is to reach all nodes of the tree, a possible solution to this is as follows:

- At each node link children of same parent (siblings) from left to right.
- Remove the links from parent to all children except the first child.

What these above statements say is if we have a link between children then we do not need extra links from parent to all children. This is because we can traverse all the elements by starting at the first child of the parent. So if we have a link between parent and first child and also links between all children of same parent then it solves our problem.



This representation is sometimes called first child/next sibling representation. First child/next sibling representation of the generic tree is shown above. The actual representation for this tree is:



Based on this discussion, the tree node declaration for general tree can be given as:

```
public class TreeNode {
        public int data;
        public TreeNode firstChild;
        public TreeNode nextSibling;
        public int getData() {
                return data;
        }
        public void setData(int data) {
                this.data = data;
        }
        public BinaryTreeNode getFirstChild() {
                return firstChild;
        }
        public void setFirstChild(BinaryTreeNode firstChild) {
                this.firstChild = firstChild;
        }
        public BinaryTreeNode getNextSibling () {
                return nextSibling;
        }
        public void setNextSibling(BinaryTreeNode nextSibling) {
                this.nextSibling = nextSibling;
        }
}
```

Note: Since we are able to convert any generic tree to binary representation, in practice we use binary trees. We can treat all generic trees with a first child/next sibling representation as binary trees.

Generic Trees: Problems & Solutions

Problem-41 Given a tree, give an algorithm for finding the sum of all the elements of the tree.

Solution: The solution is similar to what we have done for simple binary trees. That means, traverse the complete list and keep on adding the values. We can either use level order traversal or simple recursion.

```
public int FindSum(BinaryTreeNode root) {
        if(root == null)
            return 0;
        return root.getData() + FindSum(root.getFirstChild()) + FindSum(root.getNextSibling());
}
```

Time Complexity: O(n). Space Complexity: O(1) (if we do not consider stack space), otherwise O(n).

Note: All problems which we have discussed for binary trees are applicable for generic trees also. Instead of left and right pointers we just need to use firstChild and nextSibling.

Problem-42 For a 4-ary tree (each node can contain maximum of 4 children), what is the maximum possible height with 100 nodes? Assume height of a single node is 0.

Solution: In 4-ary tree each node can contain 0 to 4 children, and to get maximum height, we need to keep only one child for each parent. With 100 nodes, the maximum possible height we can get is 99. If we have a restriction that at least one node has 4 children, then we keep one node with 4 children and the remaining nodes with 1 child. In this case, the maximum possible height is 96. Similarly, with n nodes the maximum possible height is $n - 4$.

Problem-43 For a 4-ary tree (each node can contain maximum of 4 children), what is the minimum possible height with n nodes?

Solution: Similar to the above discussion, if we want to get minimum height, then we need to fill all nodes with maximum children (in this case 4). Now let's see the following table, which indicates the maximum number of nodes for a given height.

Height, h	Maximum Nodes at height, $h = 4^h$	Total Nodes height $h = \frac{4^{h+1}-1}{3}$
0	1	1
1	4	1+4
2	4×4	$1 + 4 \times 4$
3	$4 \times 4 \times 4$	$1 + 4 \times 4 + 4 \times 4 \times 4$

For a given height h the maximum possible nodes are: $\frac{4^{h+1}-1}{3}$. To get minimum height, take logarithm on both sides:

$$n = \frac{4^{h+1}-1}{3} \Rightarrow 4^{h+1} = 3n + 1 \Rightarrow (h+1)log4 = \log(3n+1)$$
$$\Rightarrow h + 1 = \log_4(3n+1) \Rightarrow h = \log_4(3n+1) - 1$$

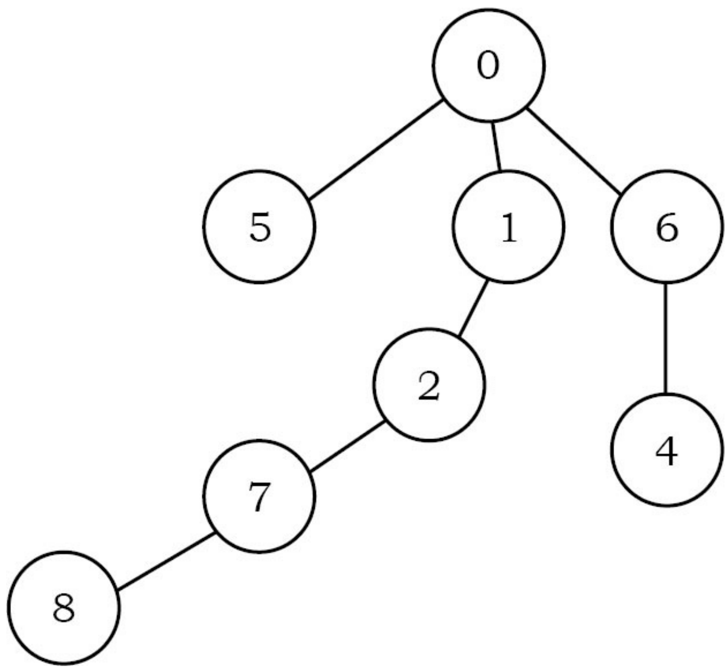
Problem-44 Given a parent array P, where $P[i]$ indicates the parent of i^{th} node in the tree (assume parent of root node is indicated with –1). Give an algorithm for finding the height or depth of the tree.

Solution: From the problem definition, the given array represents the parent array. That means, we need to consider the tree for that array and find the depth of the tree.

For example: if the P is

-1	0	1	6	6	0	0	2	7
0	1	2	3	4	5	6	7	8

Its corresponding tree is:



The depth of this given tree is 4. If we carefully observe, we just need to start at every node and keep going to its parent until we reach –1 and also keep track of the maximum depth among all nodes.

```
public int FindDepthInGenericTree(int[] P) {
    int maxDepth =-1, currentDepth =-1, j;
    for(int i = 0; i < P.length; i++)    {
        currentDepth = 0; j = i;
        while(P[j] != -1) {
            currentDepth++; j = P[j];
        }
        if(currentDepth > maxDepth)
            maxDepth = currentDepth;
    }
    return maxDepth;
}
```

Time Complexity: O(n^2). For skew trees we will be calculating the same values again and again. Space Complexity: O(1).

Note: We can optimize the code by storing the previous calculated nodes' depth in some hash table or other array. This reduces the time complexity but uses extra space.

Problem-45 Given a node in the generic tree, find the number of siblings for that node.

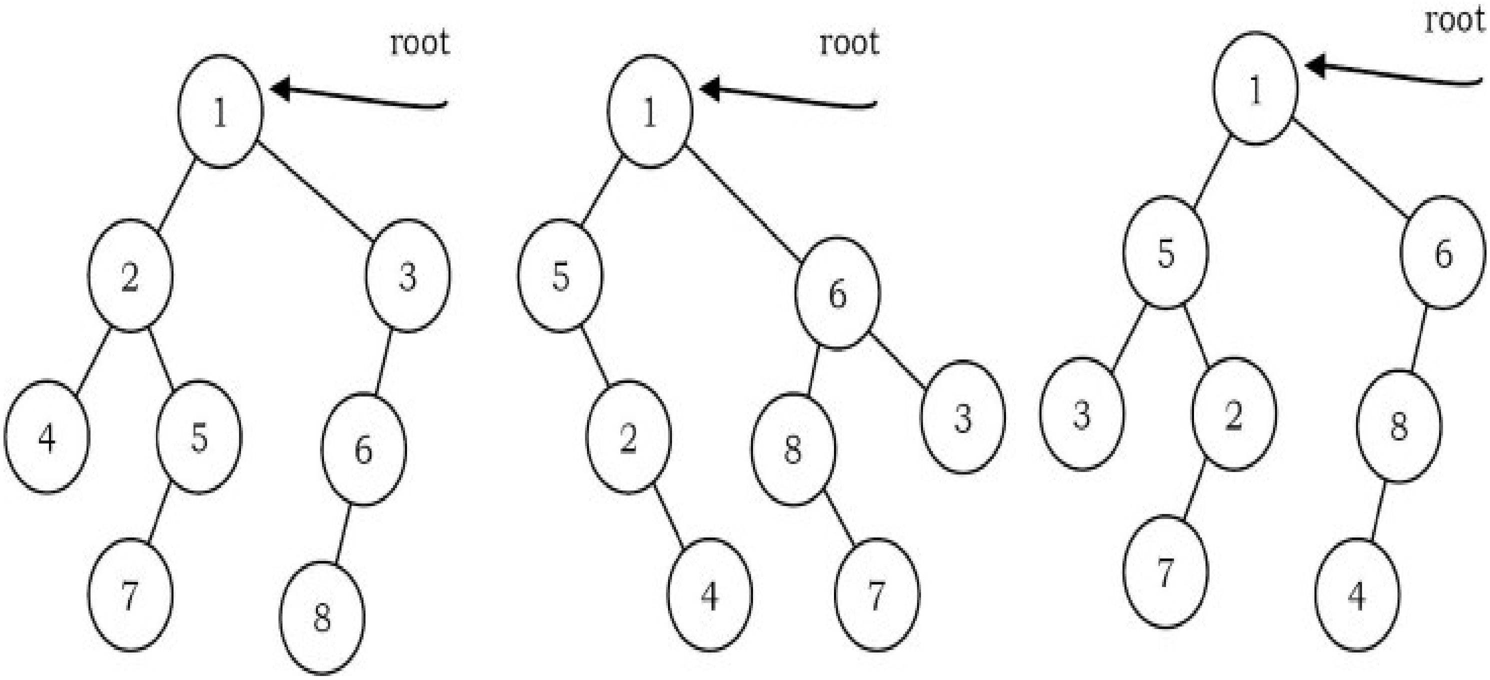
Solution: For a given node in the tree, we just need to traverse all its next siblings.

```
public int SiblingsCount(TreeNode current) {
    int count = 0;
    while(current)  {
        count++;
        current = current.getNextSibling();
    }
    reutrn count;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-46 Given two trees how do we check whether the trees are isomorphic to each other or not?

Solution: Two binary trees *root*1 and *root*2 are isomorphic if they have the same structure. The values of the nodes does not affect whether two trees are isomorphic or not. In the diagram below, the tree in the middle is not isomorphic to the other trees, but the tree on the right is isomorphic to the tree on the left.

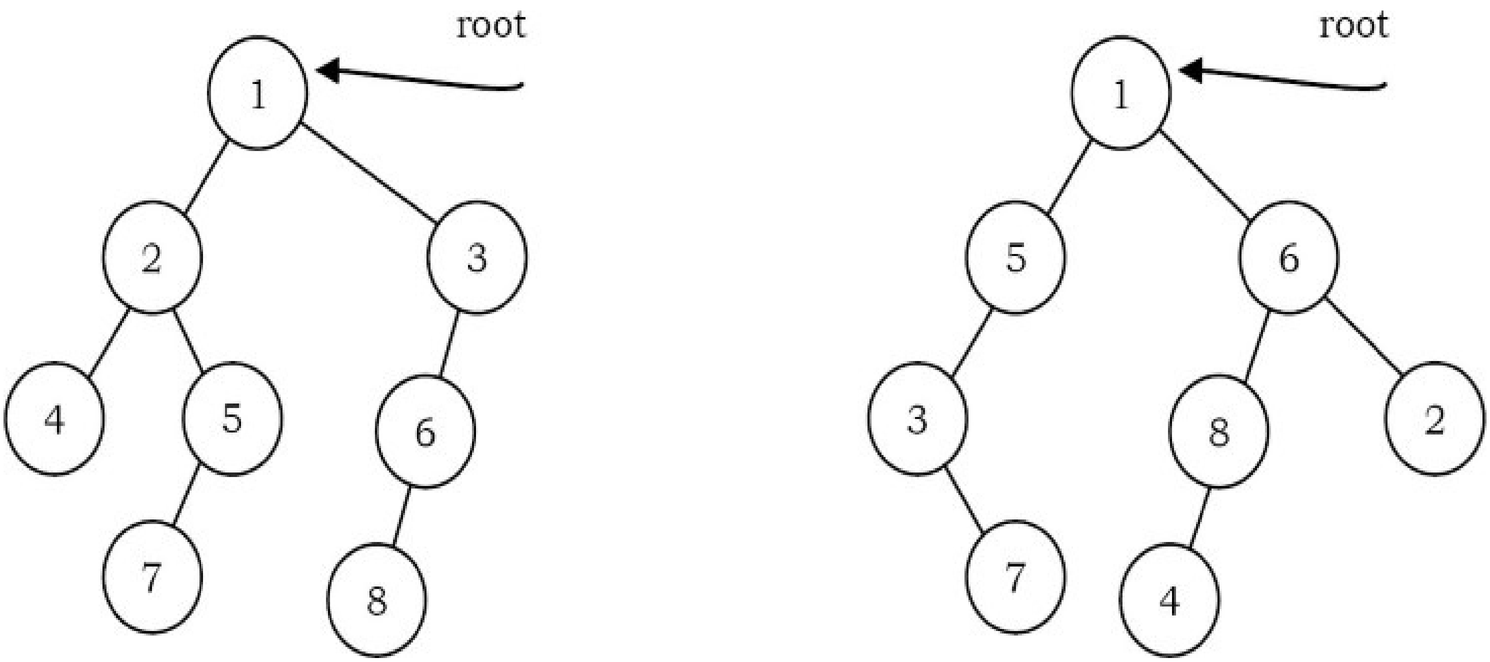



```
public int IsIsomorphic(TreeNode  root1, TreeNode root2) {
    if(root1 == null && root2 == null)
            return 1;
    if((root1 == null && root2 != null) || (root1 != null && root2 == null))
            return 0;
    return (IsIsomorphic(root1.getLeft(), root2.getLeft())  && IsIsomorphic(root1.getRight(), root2.getRight()));
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-47 Given two trees how do we check whether they are quasi-isomorphic to each other or not?

Solution: Two trees *root*1 and *root*2 are quasi-isomorphic if *root*1 can be transformed into *root*2 by swapping the left and right children of some of the nodes of *root*1. Data in the nodes are not important in determining quasi-isomorphism; only the shape is important. The trees below are quasi-isomorphic because if the children of the nodes on the left are swapped, the tree on the right is obtained.




```
public int QuasiIsomorphic(TreeNode  root1, TreeNode  root2) {
    if(root1 == null && root2 == null)
        return true;
    if((root1 == null && root2 != null) || (root1 != null && root2 == null))
        return false;
    return (QuasiIsomorphic(root1.getLeft(), root2.getLeft()) &&
        QuasiIsomorphic(root1.getRight(), root2.getRight()) ||
        QuasiIsomorphic(root1.getRight(), root2.getLeft()) &&
        QuasiIsomorphic(root1.getLeft(), root2.getRight()));
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*).

Problem-48 Given a node in the generic tree, give an algorithm for counting the number of children for that node.

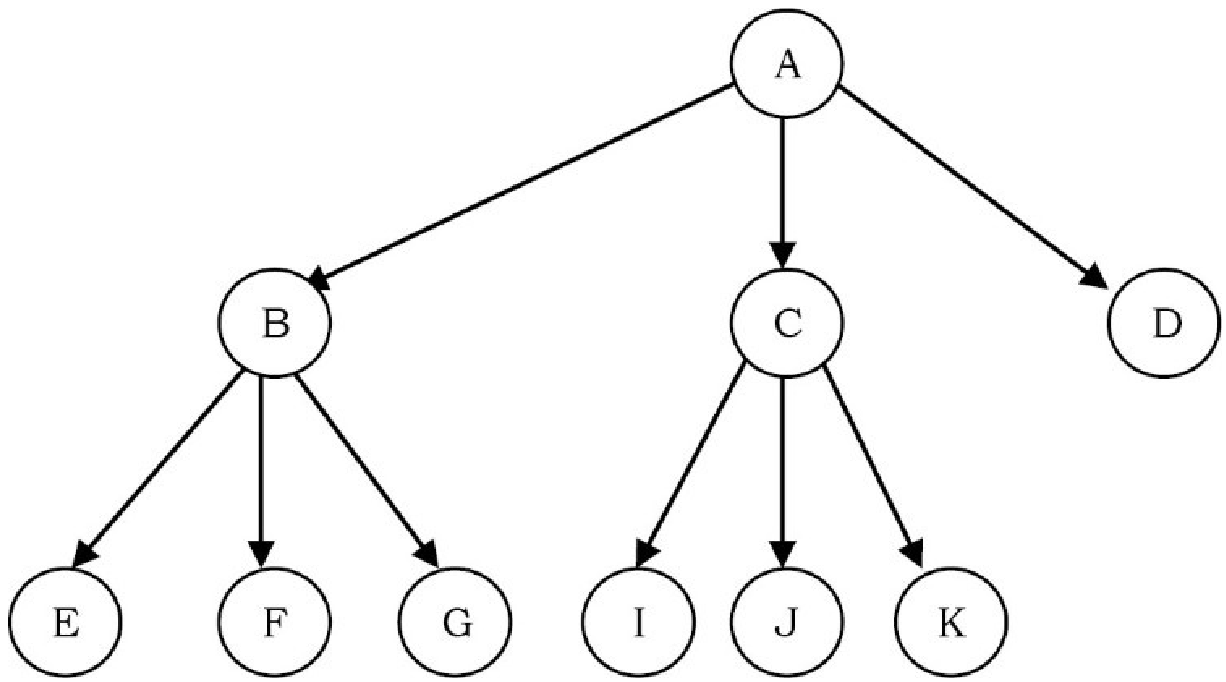
Solution: For a given node in the tree, we just need to point to its first child and keep traversing all its nextsiblings.

```
public int ChildCount(TreeNode current) {
    int count = 0;
    current = current.getFirstChild();
    while(current != null)    {
            count++;
            current = current.getNextSibling();
    }
    reutrn count;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-49 A full k –ary tree is a tree where each node has either 0 or k children. Given an array which contains the preorder traversal of full k –ary tree, give an algorithm for constructing the full k –ary tree.

Solution: In k –ary tree, for a node at i^{th} position its children will be at $k * i + 1$ to $k * i + k$. For example, the example below is for full 3-ary tree.



As we have seen, in preorder traversal first left subtree is processed then followed by root node and right subtree. Because of this, to construct a full k-ary, we just need to keep on creating the nodes without bothering about the previous constructed nodes. We can use this trick to build the tree recursively by using one global index. The declaration for k-ary tree can be given as:

```
public class K-aryTreeNode {
    public int data;
    public K-aryTreeNode[] child;
    public K-aryTreeNode(int k){
            child = new K-aryTreeNode[k];
    }
    public void setData(int dataInput){
            data = dataInput;
    }
    public int getChild(){
            return data;
    }
    public void setChild(int i, K-aryTreeNode childNode){
            child[i]= childNode;
    }
    public K-aryTreeNode getChild(int i){
            return child[i];
    }
    //...
}
int Ind = 0;
public K-aryTreeNode BuildK-aryTree(int A[], int n, int k) {
    if(n <= 0)
        return null;
    K-aryTreeNode newNode = new K-aryTreeNode(k);
    if(newNode == null) {
            System.out.println("Memory Error");
            return;
    }
    newNode.setData(A[Ind]);
     for(int i = 0; i<k; i++) {
          if(k * Ind + i <n) {
              Ind++;
              newNode.setChild(BuildK-aryTree(A, n, k,Ind ));
          }
          else  newNode.setChild(null);
     }
     return newNode;
}
```

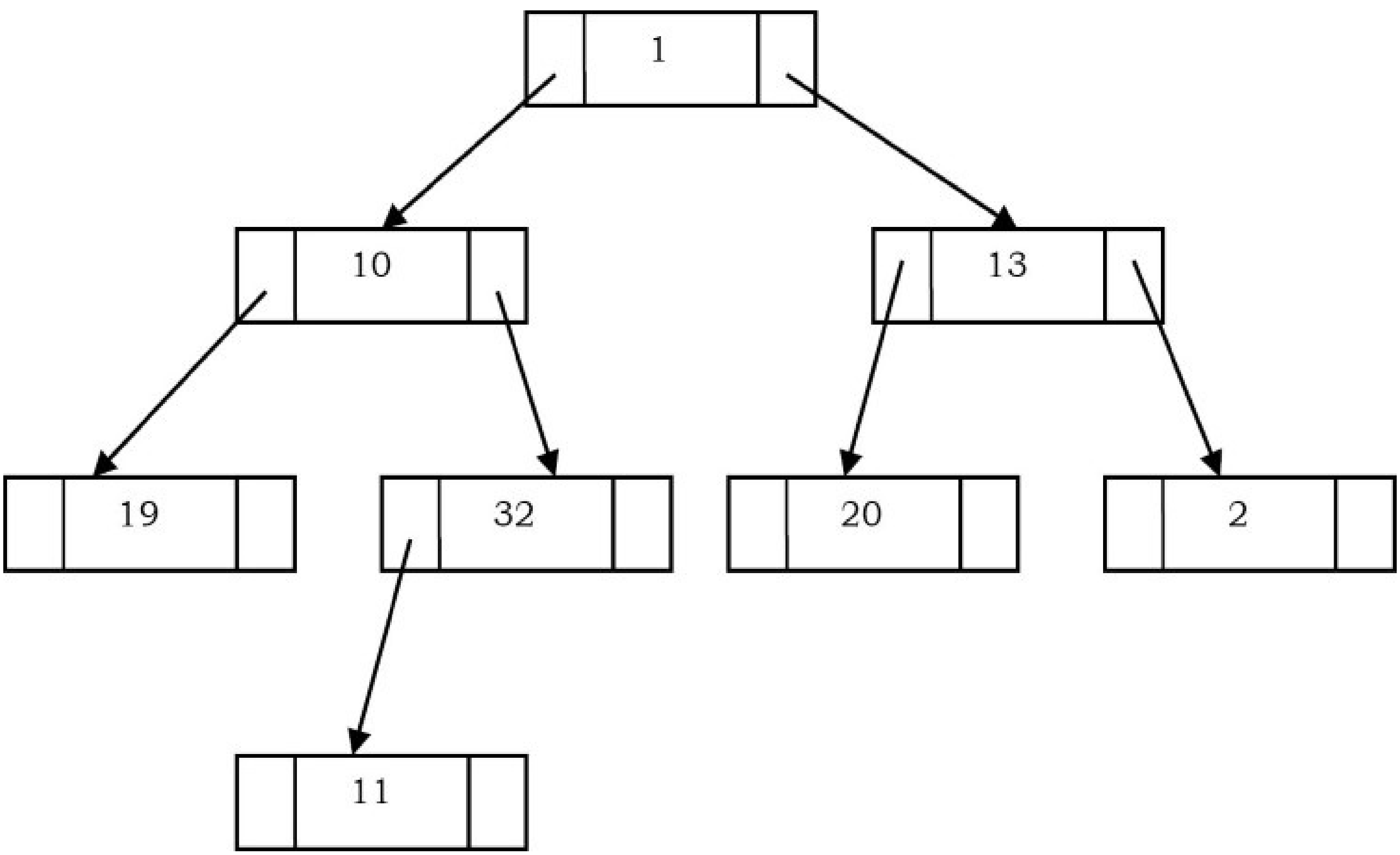
Time Complexity: O(n), where n is the size of the pre-order array. This is because we are moving sequentially and not visiting the already constructed nodes.

6.6 Threaded (Stack or Queue less) Binary Tree Traversals

In earlier sections we have seen that, *preorder, inorder, and postorder* binary tree traversals used stacks and *level order* traversal used queues as an auxiliary data structure. . In this section we will discuss new traversal algorithms which do not need both stacks and queues. Such traversal algorithms are called *threaded binary tree traversals* or *stack/queue less traversals*.

Issues with Regular Binary Tree Traversals

- The storage space required for the stack and queue is large.
- The majority of pointers in any binary tree are NULL. For example, a binary tree with n nodes has $n + 1$ NULL pointers and these were wasted.
- It is difficult to find successor node (preorder, inorder and postorder successors) for a given node.



Motivation for Threaded Binary Trees

To solve these problems, one idea is to store some useful information in NULL pointers. If we observe the previous traversals carefully, stack/queue is required because we have to record the current position in order to move to the right subtree after processing the left subtree. If we store the useful information in NULL pointers, then we don’t have to store such information in stack/ queue.

The binary trees which store such information in NULL pointers are called *threaded binary trees*. From the above discussion, let us assume that we want to store some useful information in NULL pointers. The next question is what to store?

The common convention is to put predecessor/successor information. That means, if we are dealing with preorder traversals, then for a given node, NULL left pointer will contain preorder predecessor information and NULL right pointer will contain preorder successor information. These special pointers are called *threads*.

Classifying Threaded Binary Trees

The classification is based on whether we are storing useful information in both NULL pointers or only in one of them.

- If we store predecessor information in NULL left pointers only then we call such binary trees as *left threaded binary trees*.
- If we store successor information in NULL right pointers only then we call such binary trees as *right threaded binary trees*.
- If we store predecessor information in NULL left pointers and store successor information in NULL right pointers, then we call such binary trees as *fully threaded binary trees* or simply *threaded binary trees*.

Note: For the remaining discussion we consider only (*fully*) *threaded binary trees*.

Types of Threaded Binary Trees

Based on above discussion we get three representations for threaded binary trees.

- *Preorder Threaded Binary Trees:* NULL left pointer will contain PreOrder predecessor information and NULL right pointer will contain PreOrder successor information.
- *Inorder Threaded Binary Trees:* NULL left pointer will contain InOrder predecessor information and NULL right pointer will contain InOrder successor information.
- *Postorder Threaded Binary Trees:* NULL left pointer will contain PostOrder predecessor information and NULL right pointer will contain PostOrder successor information.

Note: As the representations are similar, for the remaining discussion, we will use InOrder threaded binary trees.

Threaded Binary Tree structure

Any program examining the tree must be able to differentiate between a regular *left/right* pointer and a *thread*. To do this, we use two additional fields in each node, giving us, for threaded trees, nodes of the following form:

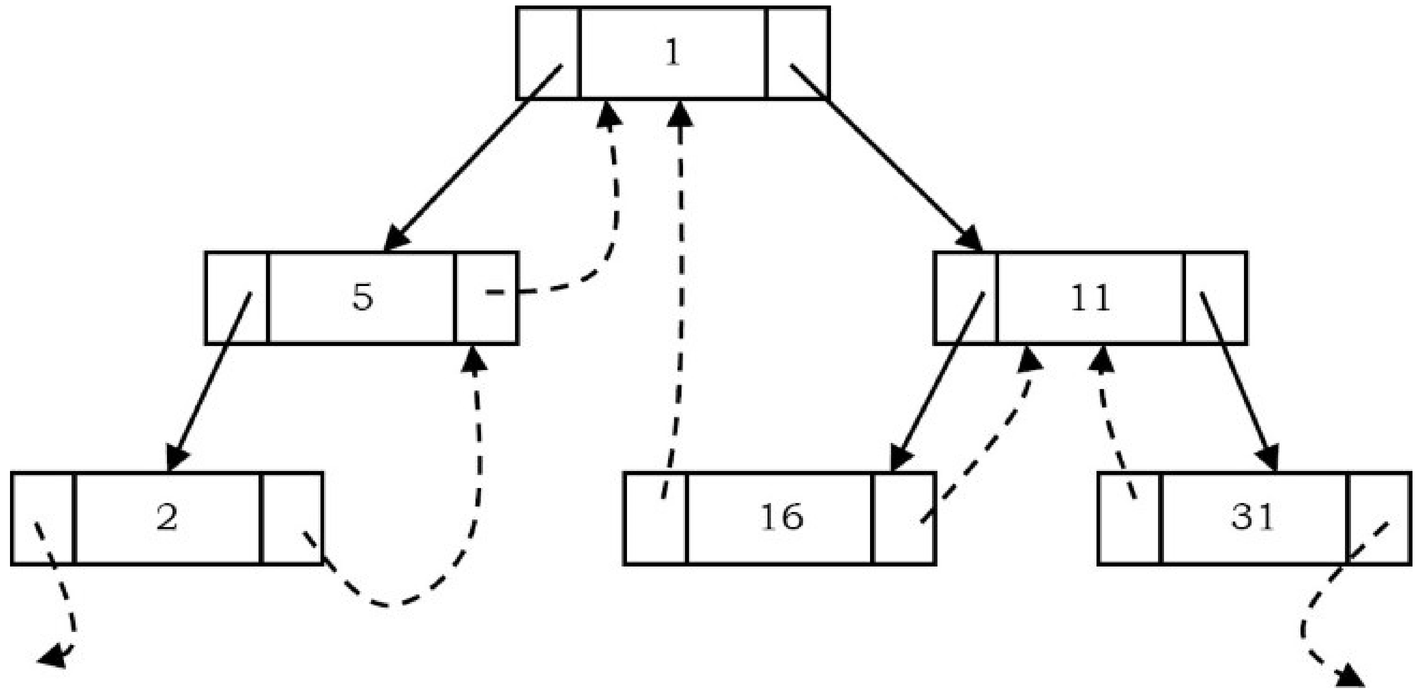

```
public class ThreadedBinaryTreeNode {
    public ThreadedBinaryTreeNode left;
    public int LTag;
    public int data;
    public int RTag;
    public ThreadedBinaryTreeNode right;
    .....
}
```

Difference between Binary Tree and Threaded Binary Tree Structures

	Regular Binary Trees	Threaded Binary Trees
if LTag == 0	NULL	left points to the in-order predecessor
if LTag == 1	left points to the left child	left points to left child
if RTag == 0	NULL	right points to the in-order successor
if RTag == 1	right points to the right child	right points to the right child

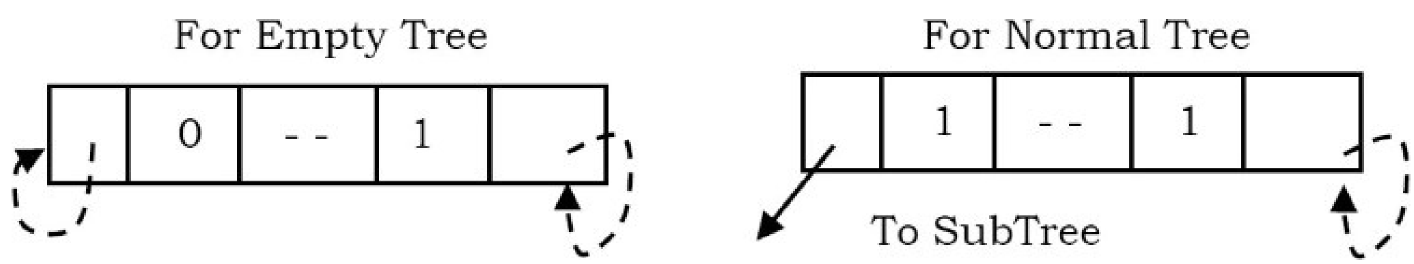
Note: Similarly, we can define for preorder/postorder differences as well.

As an example, let us try representing a tree in inorder threaded binary tree form. The tree below shows what an inorder threaded binary tree will look like. The dotted arrows indicate the threads. If we observe, the left pointer of left most node (2) and right pointer of right most node (31) are hanging.

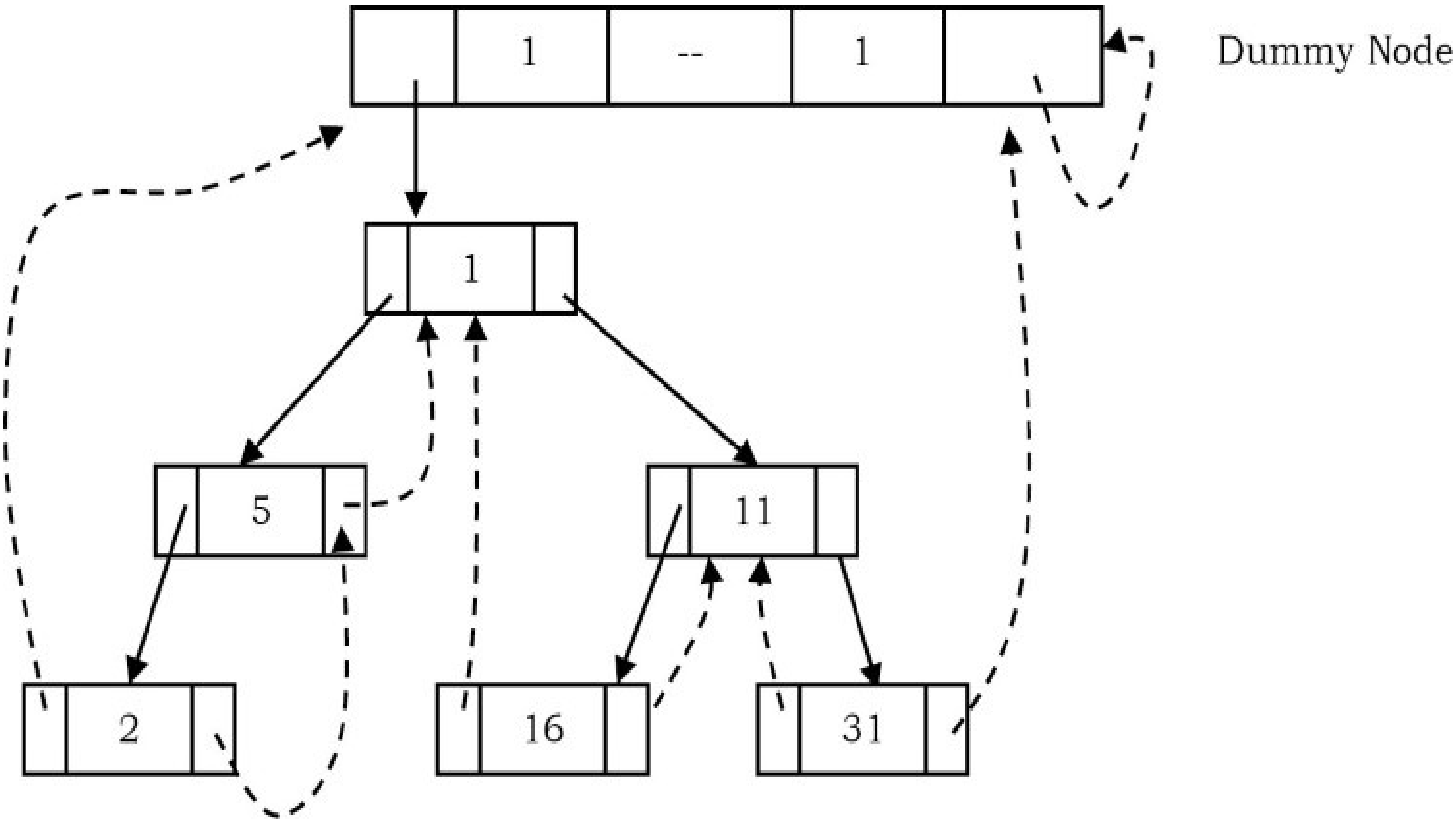


What should leftmost and rightmost pointers point to?

In the representation of a threaded binary tree, it is convenient to use a special node *Dummy* which is always present even for an empty tree. Note that right tag of Dummy node is 1 and its right child points to itself.



With this convention the above tree can be represented as:



Finding Inorder Successor in Inorder Threaded Binary Tree

To find inorder successor of a given node without using a stack, assume that the node for which we want to find the inorder successor is *P*.

Strategy: If *P* has a no right subtree, then return the right child of P. If *P* has right subtree, then return the left of the nearest node whose left subtree contains *P*.

```
public ThreadedBinaryTreeNode InorderSuccessor(ThreadedBinaryTreeNode P) {
    ThreadedBinaryTreeNode Position;
    if(P→RTag == 0)
        return P.getRight();
    else {
        Position = P.getRight();
        while(Position.getLTag() == 1)
            Position = Position.getLeft();
        return Position;
    }
}
```

Time Complexity: O(n). Space Complexity: O(1).

Inorder Traversal in Inorder Threaded Binary Tree

We can start with *dummy* node and call InorderSuccessor() to visit each node until we reach *dummy* node.

```
public void InorderTraversal(ThreadedBinaryTreeNode root) {
    ThreadedBinaryTreeNode P = InorderSuccessor(root);
    while(P != root) {
            P = InorderSuccessor(P);
            System.out.println(P.getData());
    }
}
```

Alternative way of coding:

```
public void InorderTraversal(ThreadedBinaryTreeNode root) {
    ThreadedBinaryTreeNode P = root;
    while(1) {
            P = InorderSuccessor(P);
            if(P == root)
                return;
            System.out.println(P.getData());
    }
}
```

Time Complexity: O(n). Space Complexity: O(1).

Finding PreOrder Successor in InOrder Threaded Binary Tree

Strategy: If P has a left subtree, then return the left child of P. If P has no left subtree, then return the right child of the nearest node whose right subtree contains *P*.

```
public ThreadedBinaryTreeNode* PreorderSuccessor(ThreadedBinaryTreeNode P) {
    ThreadedBinaryTreeNode Position;
    if(P.getLTag() == 1)  return P.getLeft();
    else {
            Position = P;
            while(Position.getRTag() == 0)
                    Position = Position.getRight();
            return Position.getRight();
    }
}
```

Time Complexity: O(n). Space Complexity: O(1).

PreOrder Traversal of InOrder Threaded Binary Tree

As in inorder traversal, start with *dummy* node and call PreorderSuccessor() to visit each node until we get *dummy* node again.

```
public void PreorderTraversal(ThreadedBinaryTreeNode root) {
    ThreadedBinaryTreeNode P;
    P = PreorderSuccessor(root);
    while(P != root) {
            P = PreorderSuccessor(P);
            System.out.println(P.getData());
    }
}
```

Alternative way of coding:

```
public void PreorderTraversal(ThreadedBinaryTreeNode root) {
    ThreadedBinaryTreeNode P = root;
    while(1) {
            P = PreorderSuccessor(P);
            if(P == root)
                return;
            System.out.println(P.getData());
    }
}
```

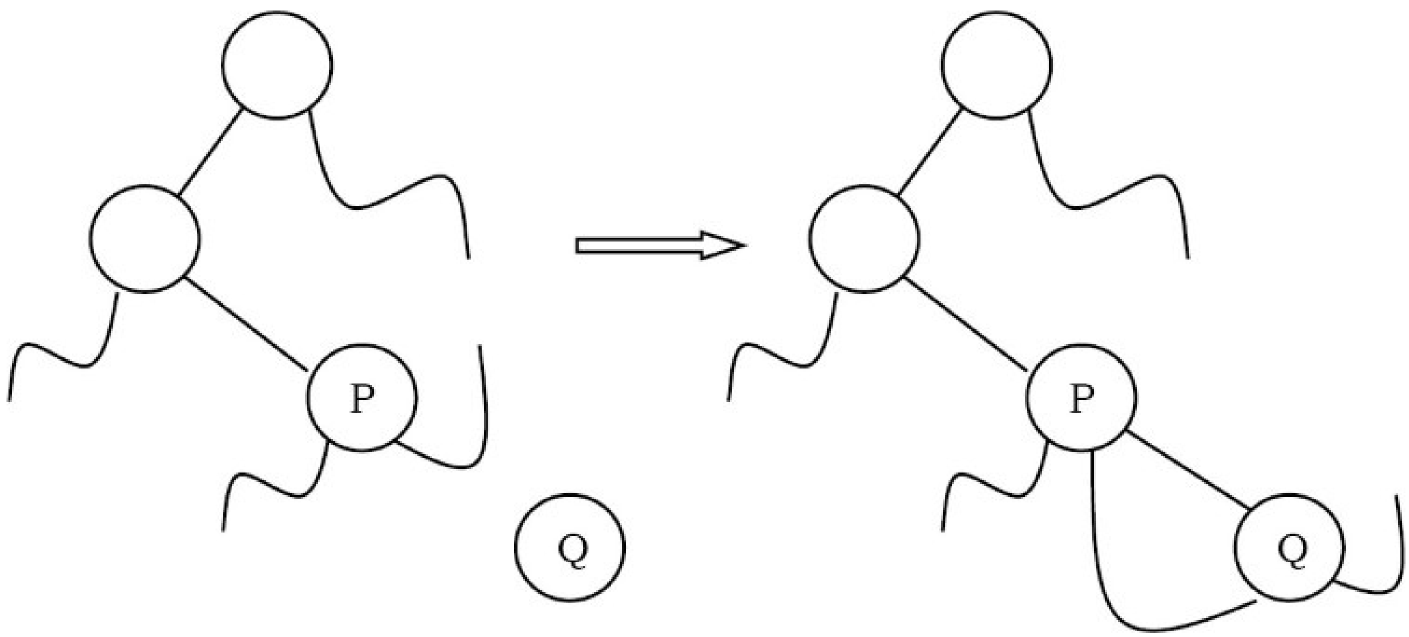
Time Complexity: O(n). Space Complexity: O(1).

Note: From the above discussion, it should be clear that inorder and preorder successor finding is easy with threaded binary trees. But finding postorder successor is very difficult if we do not use stack.

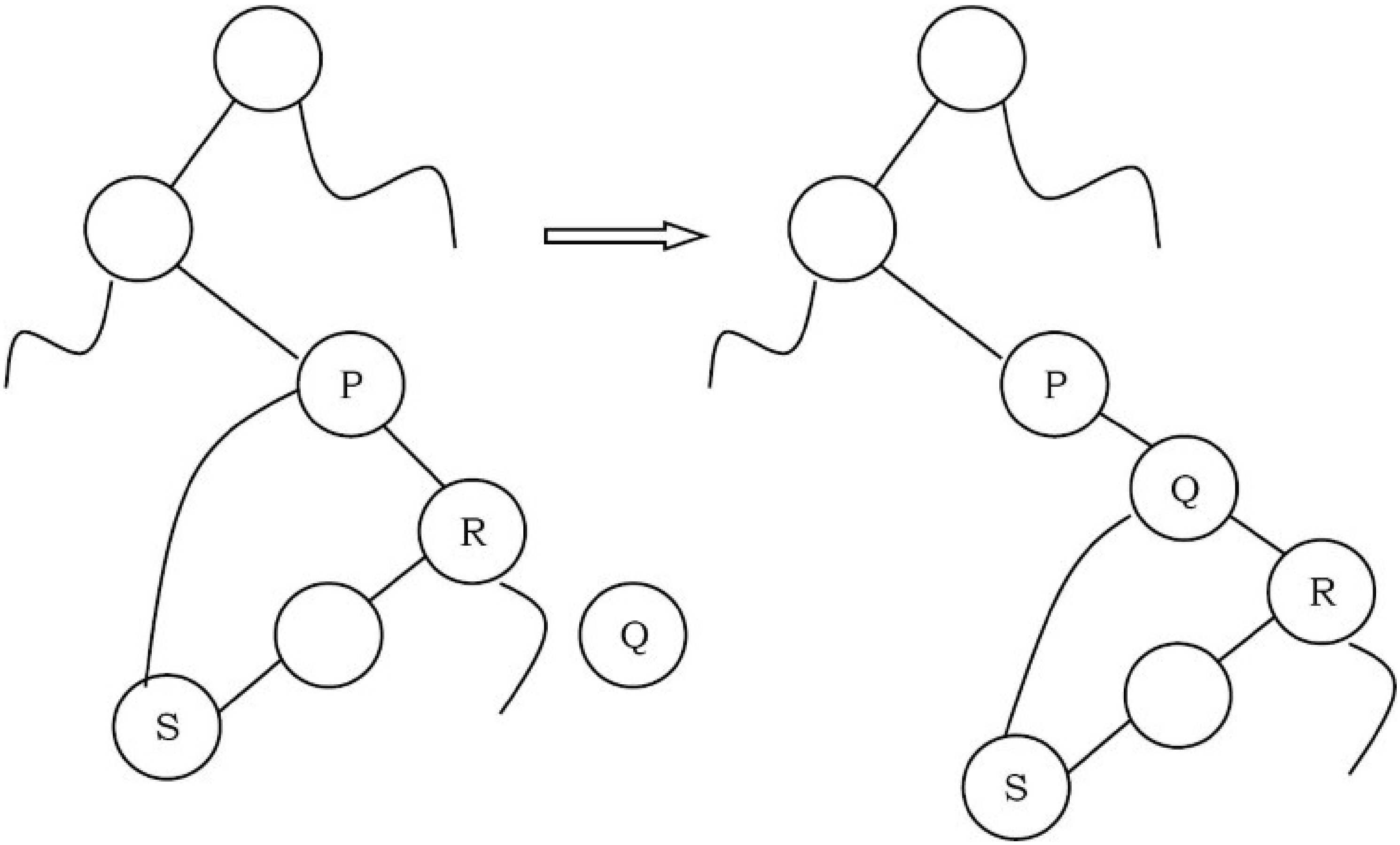
Insertion of Nodes in InOrder Threaded Binary Trees

For simplicity, let us assume that there are two nodes *P* and *Q* and we want to attach *Q* to right of P. For this we will have two cases.

- Node P does not have right child: In this case we just need to attach *Q* to P and change its left and right pointers.



- Node *P* has right child (say, *R*)*:* In this case we need to traverse *R's* left subtree and find the left most node and then update the left and right pointer of that node (as shown below).




```
public void InsertRightInInorderTBT(ThreadedBinaryTreeNode P,  ThreadedBinaryTreeNode Q) {
    ThreadedBinaryTreeNode Temp;
    Q.setRight(P.getRight());
    Q.setRTag(P.getRTag());
    Q.setLeft(P);
    Q.setLTag(0);
    P.setRight(Q);
    P.setRTag( 1);
    if(Q.getRTag() == 1) {                              //Case-2
            Temp = Q.getRight();
            while(Temp.getLTag())
                    Temp = Temp.getLeft();
            Temp.setLeft(Q);
    }
}
```

Time Complexity: O(n). Space Complexity: O(1).

Threaded Binary Trees: Problems & Solutions

Problem-50 For a given binary tree (not threaded) how do we find the preorder successor?

Solution: For solving this problem, we need to use an auxiliary stack 5. On the first call, the parameter node is a pointer to the head of the tree, and thereafter its value is NULL. Since we are simply asking for the successor of the node we got the last time we called the function. It is necessary that the contents of the stack *S* and the pointer *P* to the last node "visited" are preserved from one call of the function to the next; they are defined as static variables.

```
// pre-order successor for an unthreaded binary tree
public BinaryTreeNode PreorderSuccssor(BinaryTreeNode node) {
    static BinaryTreeNode P;
    LLStack S = new LLStack();
    if(node != null)
        P = node;
    if(P.getLeft() != null) {
        S.push(P);
        P = P.getLeft();
    }
    else {
        while(P.getRight() == null)
            P = S.pop();
        P = P.getRight();
    }
    return P;
}
```

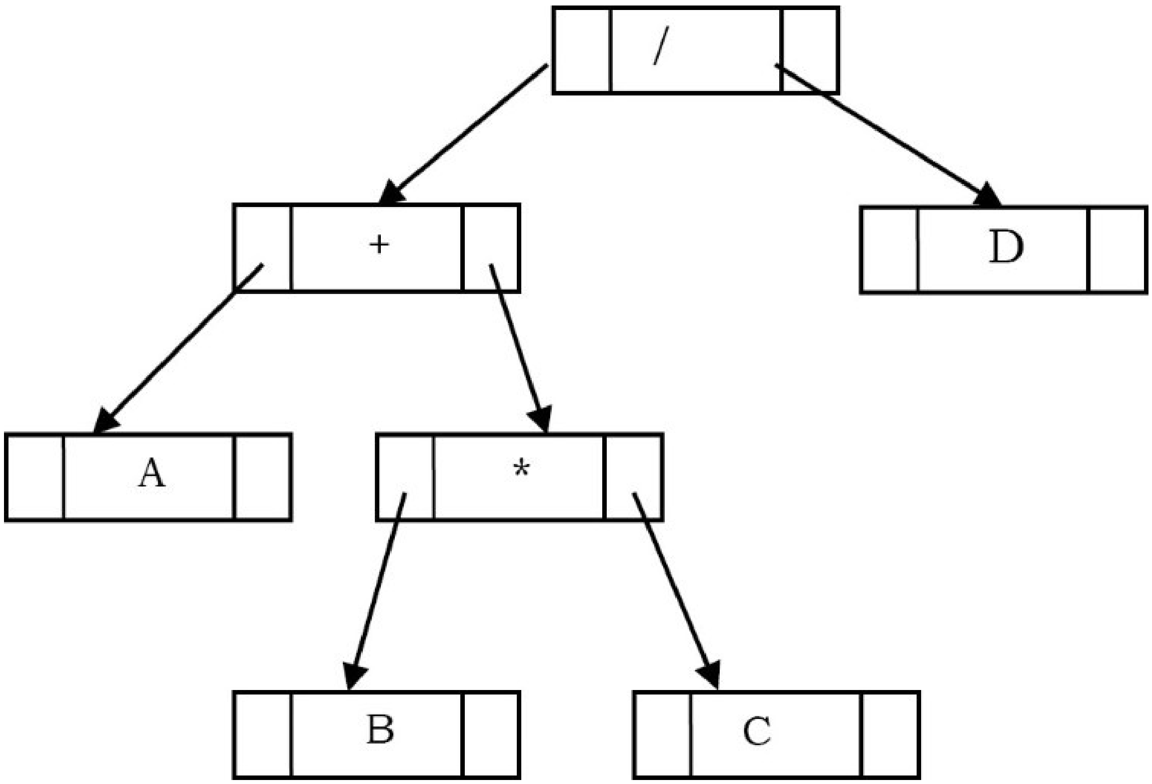
Problem-51 For a given binary tree (not threaded) how do we find the inorder successor?

Solution: Similar to the above discussion, we can find the inorder successor of a node as:

```
// In-order successor for an unthreaded binary tree
public BinaryTreeNode InorderSuccssor(BinaryTreeNode node) {
    static BinaryTreeNode P;
    LLStack S = new LLStack();
    if(node != null)
        P = node;
    if(P.getRight() == null)
        P = S.pop();
    else {
        P = P.getRight();
        while(P.getLeft() != null)
            S.push(P);
        P = P.getLeft();
    }
    return P;
}
```

6.7 Expression Trees

A tree representing an expression is called an *expression tree*. In expression trees, leaf nodes are operands and non-leaf nodes are operators. That means, an expression tree is a binary tree where internal nodes are operators and leaves are operands. An expression tree consists of binary expression. But for a u-nary operator, one subtree will be empty. The figure below shows a simple expression tree for (A + B * C) / D.

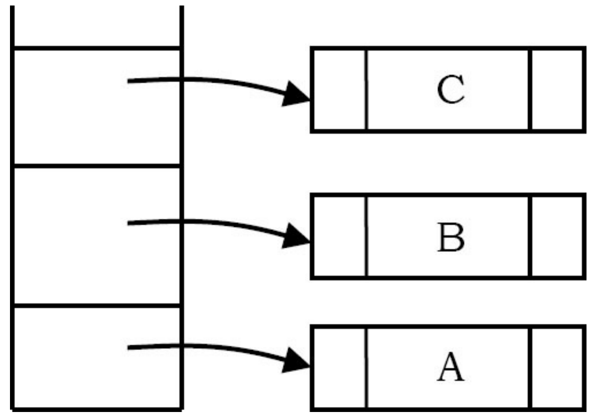


Algorithm for Building Expression Tree from Postfix Expression

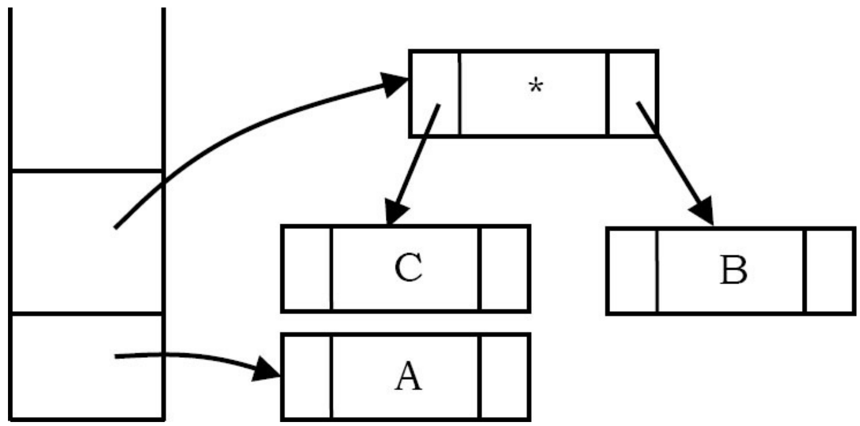
```
public BinaryTreeNode BuildExprTree(char postfixExpr[], int size) {
    LLStack S = new LLStack();
    for(int i = 0; i< size; i++) {
            if(postfixExpr[i] is an operand) {
                    BinaryTreeNode newNode = new BinaryTreeNode();
                    if(newNode == null) {
                            System.out.println("Memory Error");
                            return null;
                    }
                    newNode.setData(postfixExpr[i]);
                    newNode.setLeft(null);
                    newNode.setRight(null);
                    S.push(newNode);
            }
            else {
                    BinaryTreeNode T2 = S.pop(), T1 = S.pop();
                    BinaryTreeNode newNode = new BinaryTreeNode();
                    if(newNode == null) {
                            System.out.println("Memory Error");
                            return null;
                    }
                    newNode.setData(postfixExpr[i]);
                    //Make T2 as right child and T1 as left child for new node
                    newNode.setLeft(T1); newNode.setRight(T2);
                    S.push(newNode);
            }
    }
    return S;
}
```

Example

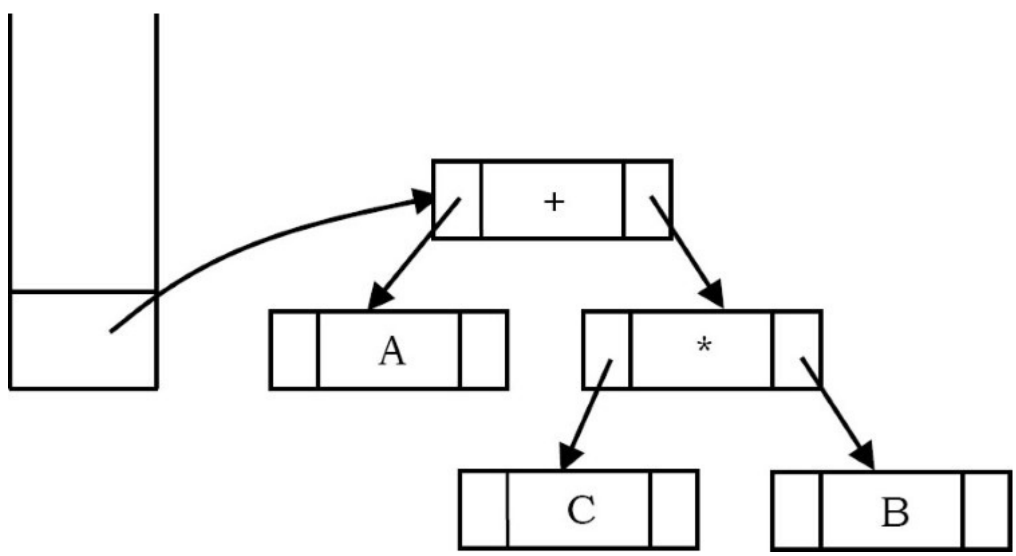
Example: Assume that one symbol is read at a time. If the symbol is an operand, we create a tree node and push a pointer to it onto a stack. If the symbol is an operator, pop pointers to two trees T_1 and T_2 from the stack (T_1 is popped first) and form a new tree whose root is the operator and whose left and right children point to T_2 and T_1 respectively. A pointer to this new tree is then pushed onto the stack. As an example, assume the input is A B C * + D /. The first three symbols are operands, so create tree nodes and push pointers to them onto a stack as shown below.



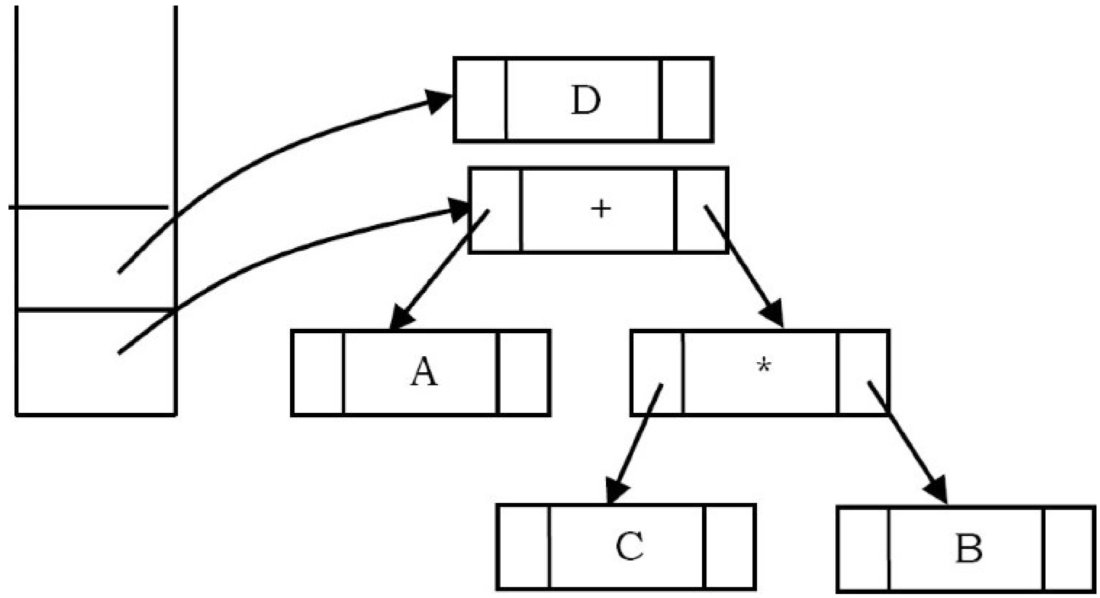
Next, an operator '*' is read, so two pointers to trees are popped, a new tree is formed and a pointer to it is pushed onto the stack.



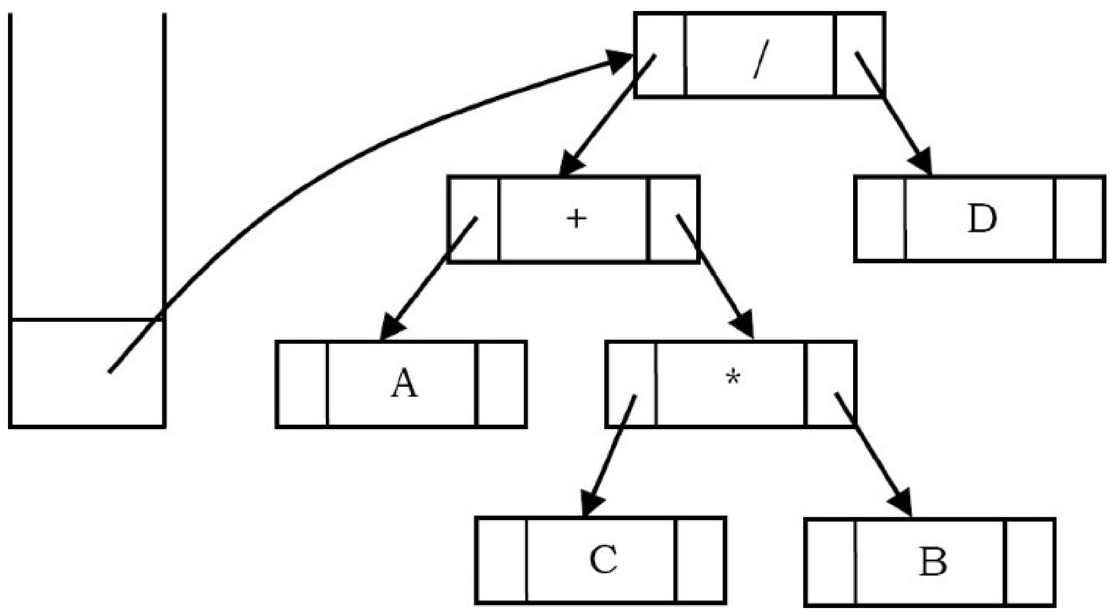
Next, an operator '+' is read, so two pointers to trees are popped, a new tree is formed and a pointer to it is pushed onto the stack.



Next, an operand 'D' is read, a one-node tree is created and a pointer to the corresponding tree is pushed onto the stack.



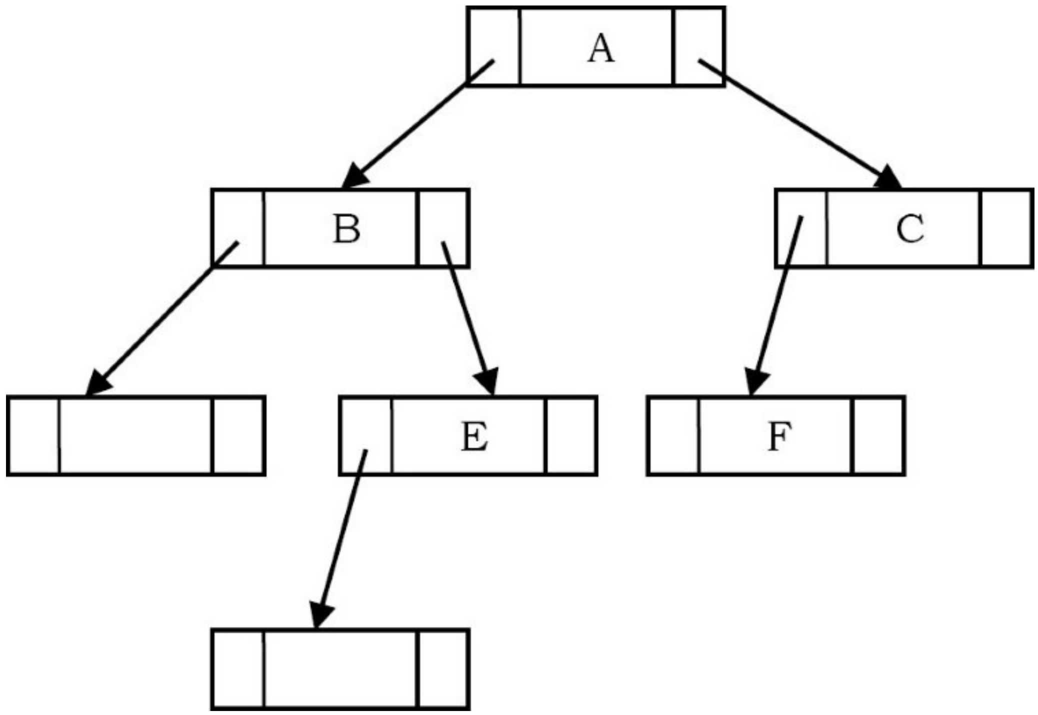
Finally, the last symbol ('/') is read, two trees are merged and a pointer to the final tree is left on the stack.



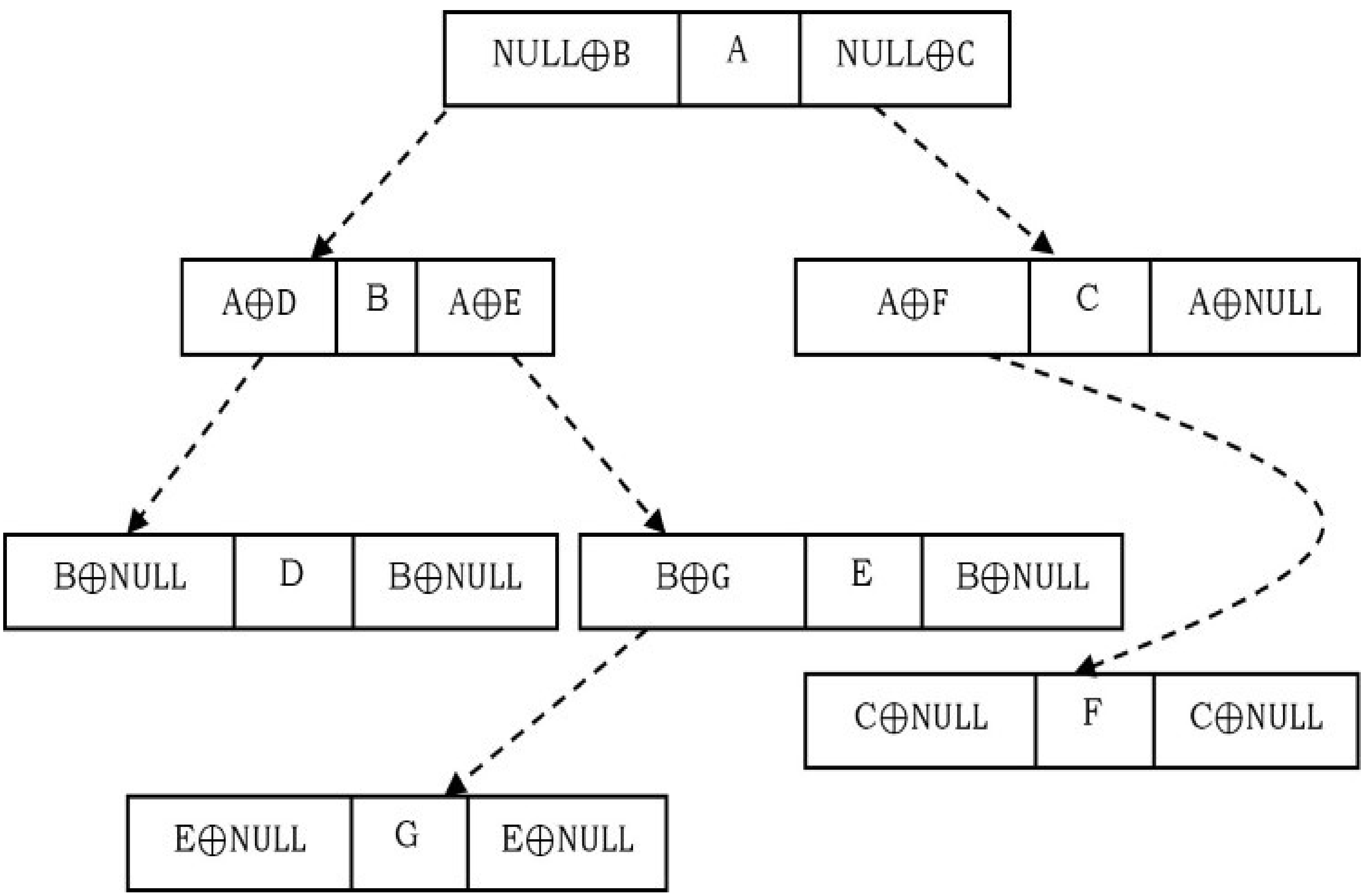
6.8 XOR Trees

This concept is similar to *memory efficient doubly linked lists* of *Linked Lists* chapter. Also, like threaded binary trees this representation does not need stacks or queues for traversing the trees. This representation is used for traversing back (to parent) and forth (to children) using ⊕ operation. To represent the same in XOR trees, for each node below are the rules used for representation:

- Each nodes left will have the ⊕ of its parent and its left children.
- Each nodes right will have the ⊕ of its parent and its right children.
- The root nodes parent is NULL and also leaf nodes children are NULL nodes.



Based on the above rules and discussion, the tree can be represented as:



The major objective of this presentation is the ability to move to parent as well to children. Now, let us see how to use this representation for traversing the tree. For example, if we are at node B and want to move to its parent node A, then we just need to perform ⊕ on its left content with its left child address (we can use right child also for going to parent node).

Similarly, if we want to move to its child (say, left child D) then we have to perform ⊕ on its left content with its parent node address. One important point that we need to understand about this representation is: When we are at node B, how do we know the address of its children D? Since the traversal starts at node root node, we can apply ⊕ on root's left content with NULL. As a result we get its left child, B. When we are at B, we can apply ⊕ on its left content with A address.

6.9 Binary Search Trees (BSTs)

Why Binary Search Trees?

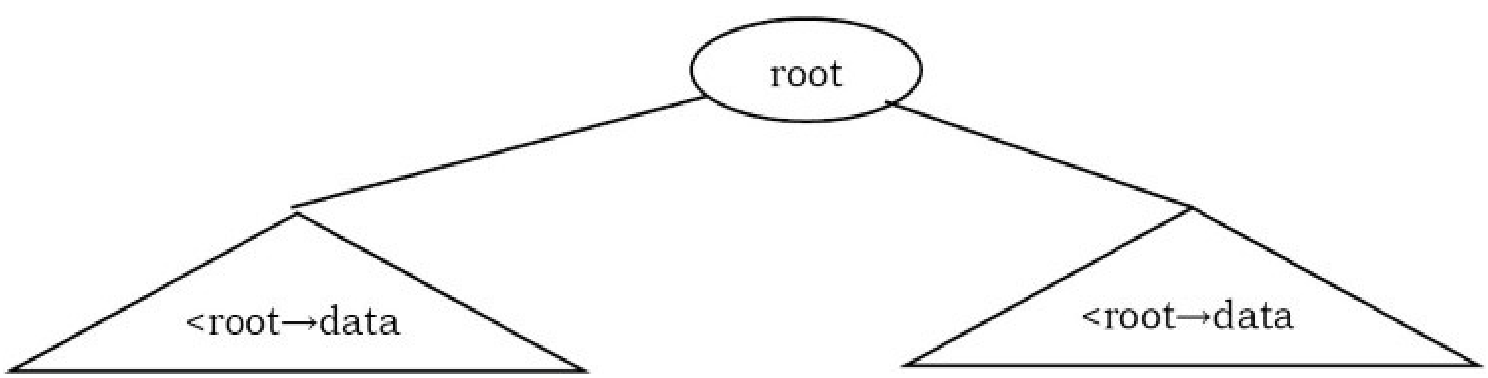
In previous sections we have discussed different tree representations and in all of them we did not impose any restriction on the nodes data. As a result, to search for an element we need to check both in left subtree and in right subtree. Due to this, the worst case complexity of search operation is O(n).

In this section, we will discuss another variant of binary trees: Binary Search Trees (BSTs). As the name suggests, the main use of this representation is for *searching*. In this representation we impose restriction on the kind of data a node can contain. As a result, it reduces the worst case average search operation to O($logn$).

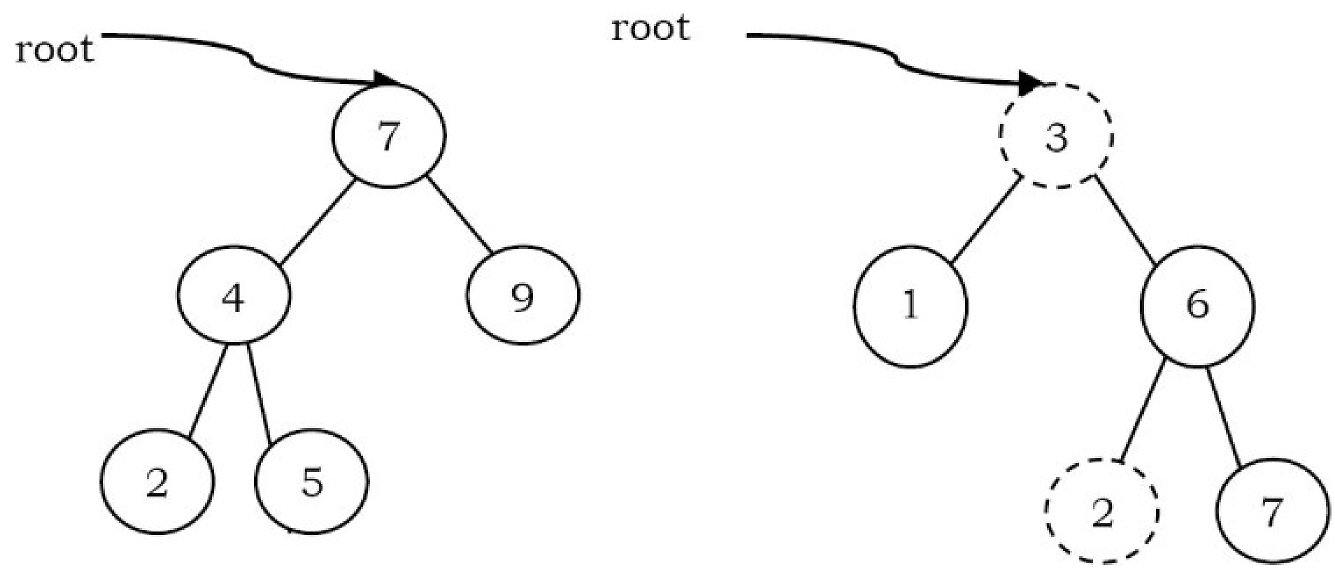
Binary Search Tree Property

In binary search trees, all the left subtree elements should be less than root data and all the right subtree elements should be greater than root data. This is called binary search tree property. Note that, this property should be satisfied at every node in the tree.

- The left subtree of a node contains only nodes with keys less than the nodes key.
- The right subtree of a node contains only nodes with keys greater than the nodes key.
- Both the left and right subtrees must also be binary search trees.



Example: The left tree is a binary search tree and the right tree is not a binary search tree (at node 6 it's not satisfying the binary search tree property).



Binary Search Tree Declaration

There is no difference between regular binary tree declaration and binary search tree declaration. The difference is only in data but not in structure. But for our convenience we change the structure name as:

```
public class BinarySearchTreeNode {
    private int data;
    private BinarySearchTreeNode left;
    private BinarySearchTreeNode right;
    public int getData() {
        return data;
    }
    public void setData(int data) {
        this.data = data;
    }
    public BinarySearchTreeNode getLeft() {
        return left;
    }
    public void setLeft(BinarySearchTreeNode left) {
        this.left = left;
    }
    public BinarySearchTreeNode getRight() {
        return right;
    }
    public void setRight(BinarySearchTreeNode right) {
        this.right = right;
    }
}
```

Operations on Binary Search Trees

Main operations

Following are the main operations that are supported by binary search trees:

- Find/ Find Minimum / Find Maximum in binary search trees
- Inserting an element in binary search trees
- Deleting an element from binary search trees

Auxiliary operations

- Checking whether the given tree is a binary search tree or not
- Finding k^{th}-smallest element in tree
- Sorting the elements of binary search tree and many more

Notes on Binary Search Trees

- Since root data is always between left subtree data and right subtree data, performing inorder traversal on binary search tree produces a sorted list.

- While solving problems on binary search trees, first we process left subtree, then root data, and finally we process right subtree. This means, depending on the problem, only the intermediate step (processing root data) changes and we do not touch the first and third steps.
- If we are searching for an element and if the left subtree root data is less than the element we want to search, then skip it. The same is the case with the right subtree.. Because of this, binary search trees take less time for searching an element than regular binary trees. In other words, the binary search trees consider either left or right subtrees for searching an element but not both.

- The basic operations that can be performed on binary search tree (BST) are insertion of element, deletion of element, and searching for an element. While performing these operations on BST the height of the tree gets changed each time. Hence there exists variations in time complexities of best case, average case, and worst case.
- The basic operations on a binary search tree take time proportional to the height of the tree. For a complete binary tree with node n, such operations runs in O(lgn) worst-case time. If the tree is a linear chain of *n* nodes (skew-tree), however, the same operations takes O(*n*) worst-case time.

Finding an Element in Binary Search Trees

Find operation is straightforward in a BST. Start with the root and keep moving left or right using the BST property. If the data we are searching is same as nodes data then we return current node. If the data we are searching is less than nodes data then search left subtree of current node; otherwise search right subtree of current node. If the data is not present, we end up in a *null* link.

```
public BinarySearchTreeNode Find(BinarySearchTreeNode root, int data) {
    if( root == null)
            return null;
    if( data < root.getData() )
            return Find(root.getLeft(), data);
    else if( data > root.getData() )
            return( Find(root.getRight(), data );
    return root;
}
```

Time Complexity: O(*n*), in worst case (when the given binary search tree is a skew tree).
Space Complexity: O(*n*), for recursive stack.

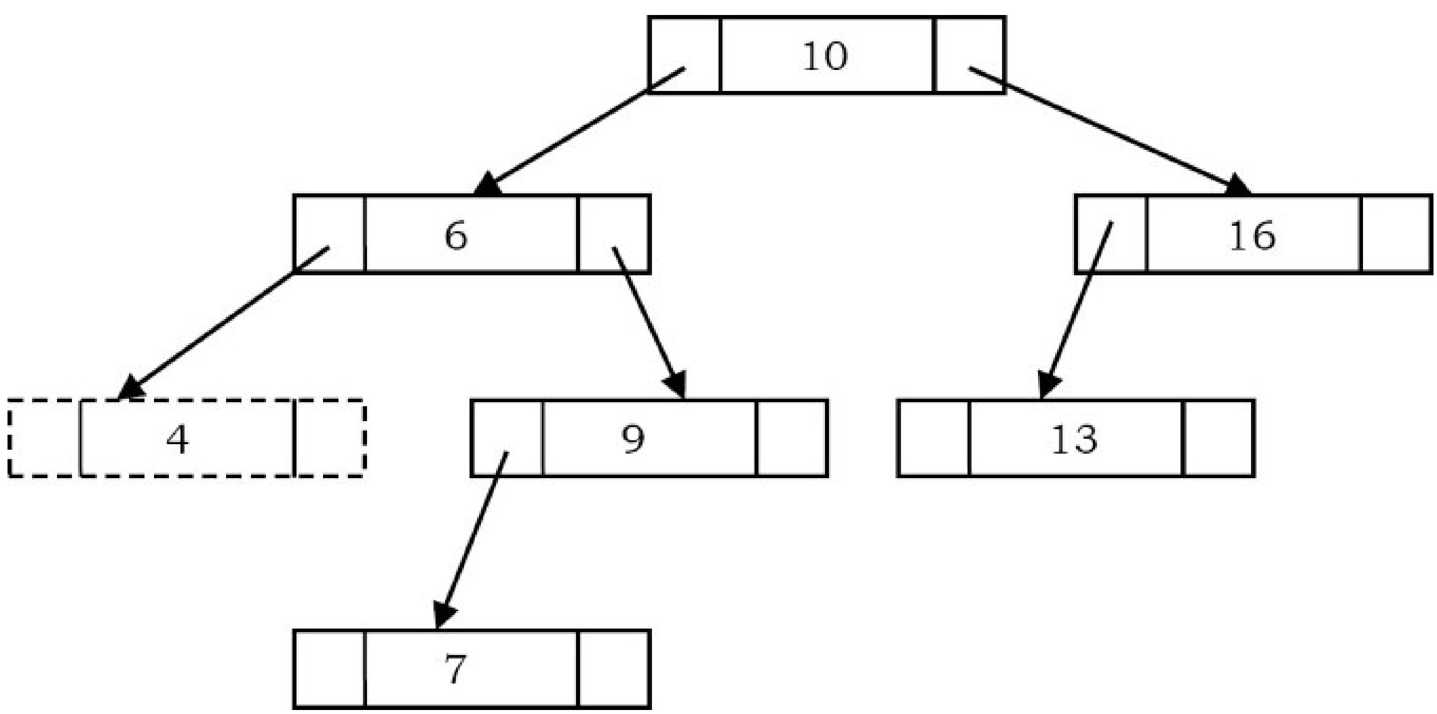
Non recursive version of the above algorithm can be given as:

```
public BinarySearchTreeNode Find(BinarySearchTreeNode root, int data ) {
    if( root == null)
        return null;
    while(root != null) {
            if(data == root.getData())
                    return root;
            else if(data > root.getData())
                    root = root.getRight();
            else    root = root.getLeft();
    }
    return null;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Finding Minimum Element in Binary Search Trees

In BSTs, the minimum element is the left-most node, which does not have left child. In the BST below, the minimum element is **4** .



```
public BinarySearchTreeNode FindMin(BinarySearchTreeNode root) {
    if(root == null)
        return null;
    else
        if( root.getLeft() == null )
            return root;
        else return FindMin(root.getLeft() );
}
```

Time Complexity: O(n), in worst case (when BST is a *left skew* tree).
Space Complexity: O(n), for recursive stack.

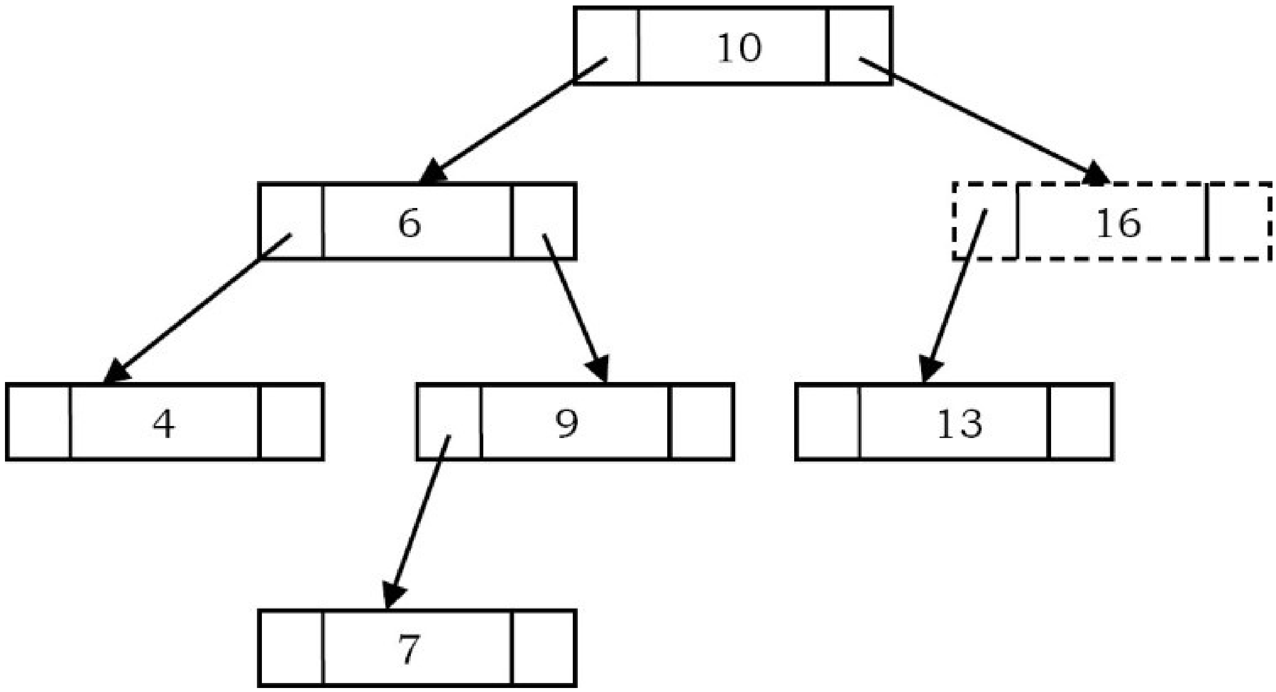
Non recursive version of the above algorithm can be given as:

```
public BinarySearchTreeNode FindMin(BinarySearchTreeNode  root ) {
    if( root == null)
        return null;
    while( root.getLeft() != null) root = root.getLeft();
    return root;
}
```

Time Complexity: O(n). Space Complexity: O(1).

Finding Maximum Element in Binary Search Trees

In BSTs, the maximum element is the right-most node, which does not have right child. In the BST below, the maximum element is **16** .




```
public BinarySearchTreeNode FindMax(BinarySearchTreeNode root) {
    if(root == null)
       return null;
    else
       if( root.getRight() == null)
          return root;
        else  return FindMax( root.getRight());
}
```

Time Complexity: O(n), in worst case (when BST is a *right skew* tree).
Space Complexity: O(n), for recursive stack.

Non recursive version of the above algorithm can be given as:

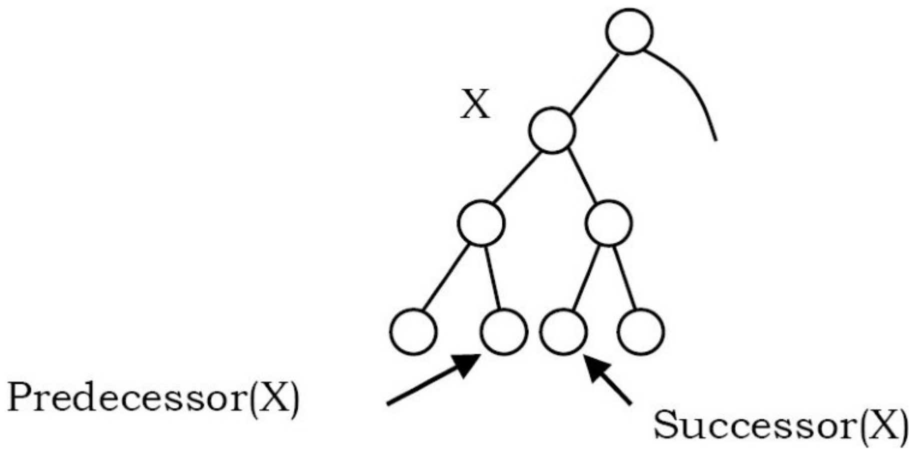
```
public BinarySearchTreeNode FindMax(BinarySearchTreeNode  root ) {
    if( root == null)
          return null;
    while( root.getRight() != null)
        root = root.getRight();
    return root;
}
```

Time Complexity: O(n). Space Complexity: O(1).

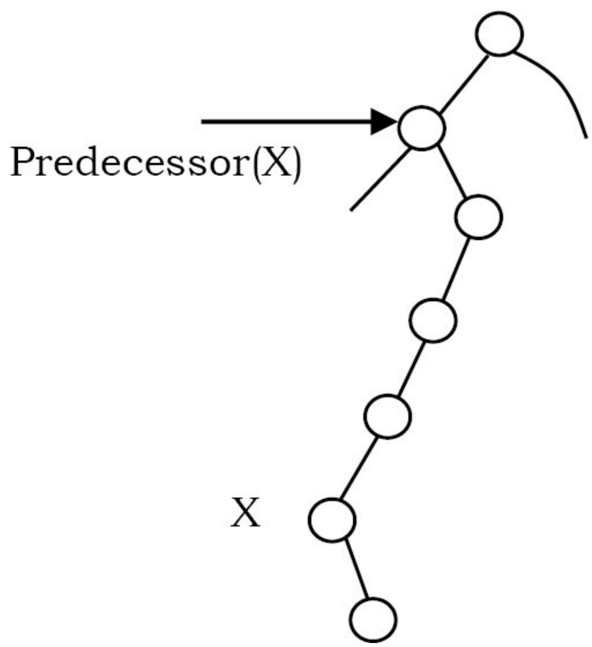
Where is Inorder Predecessor and Successor?

Where is the inorder predecessor and successor of node *X* in a binary search tree assuming all keys are distinct?

If *X* has two children then its inorder predecessor is the maximum value in its left subtree and its inorder successor the minimum value in its right subtree.



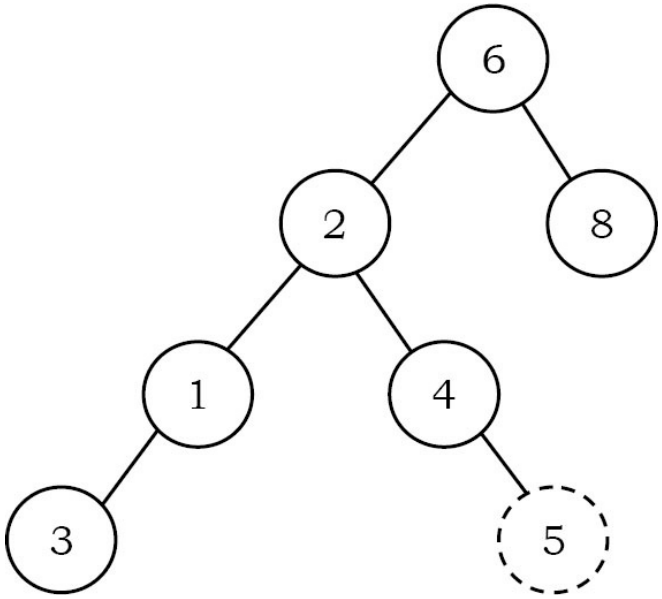
If it does not have a left child, then a nodes inorder predecessor is its first left ancestor.



Inserting an Element from Binary Search Tree

To insert *data* into binary search tree, first we need to find the location for that element. We can find the location of insertion by following the same mechanism as that of *find* operation. While finding the location, if the *data* is already there then we can simply neglect and come out. Otherwise, insert *data* at the last location on the path traversed.

As an example let us consider the following tree. The dotted node indicates the element (5) to be inserted. To insert 5, traverse the tree using *find* function. At node with key 4, we need to go right, but there is no subtree, so 5 is not in the tree, and this is the correct location for insertion.




```
public BinarySearchTreeNode Insert(BinarySearchTreeNode root, int data) {
    if( root == null) {
            root = new BinarySearchTreeNode();
            if( root == null) {
                System.out.println("Memory Error"); return;
            }
            else {
                root.setData(data);
                root.setLeft(null); root.setRight(null);
            }
    }
    else {
            if( data < root.getData() )
                root.setLeft(Insert(root.getLeft(), data));
            else if( data > root.getData() )
                root.setRight(Insert(root.getRight(), data));
    }
    return root;
}
```

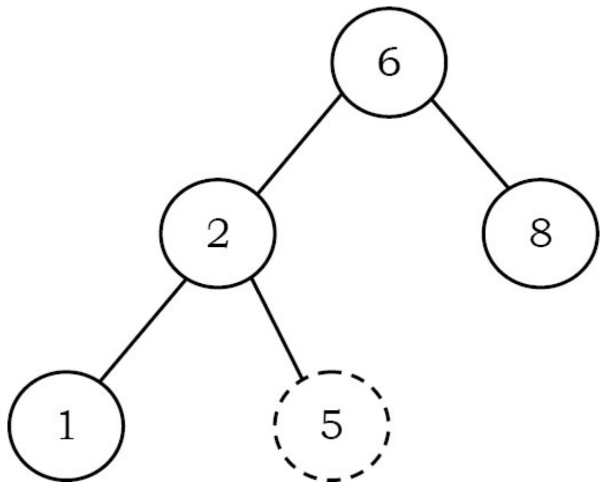
Note: In the above code, after inserting an element in subtrees, the tree is returned to its parent. As a result, the complete tree will get updated.

Time Complexity: O(*n*).
Space Complexity: O(*n*), for recursive stack. For iterative version, space complexity is O(1).

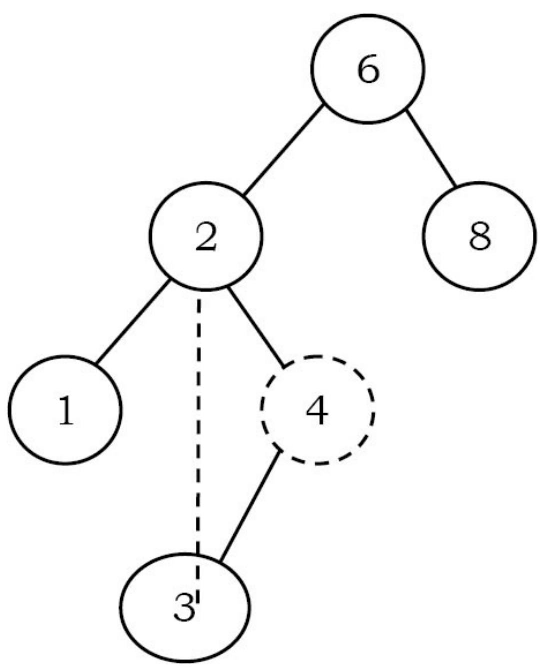
Deleting an Element from Binary Search Tree

The delete operation is more complicated than other operations. This is because the element to be deleted may not be the leaf node. In this operation also, first we need to find the location of the element which we want to delete. Once we have found the node to be deleted, consider the following cases:

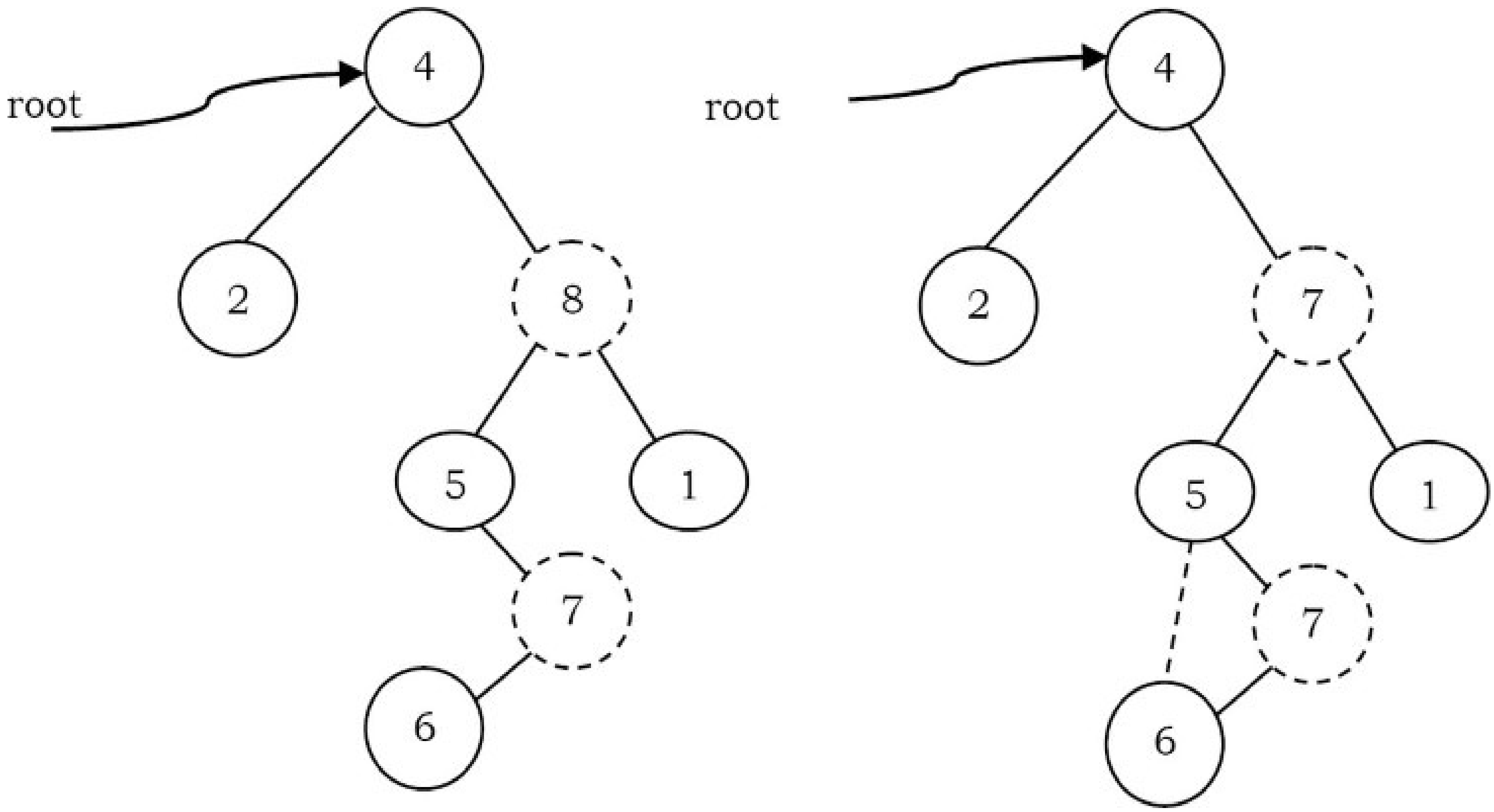
- If the element to be deleted is a leaf node: return NULL to its parent. That means make the corresponding child pointer NULL. In the tree below to delete 5, set NULL to its parent node 2.



- If the element to be deleted has one child: In this case we just need to send the current node's child to its parent. In the tree below, to delete 4, 4 left subtree is set to its parent node 2.



- If the element to be deleted has both children: The general strategy is to replace the key of this node with the largest element of the left subtree and recursively delete that node (which is now empty). The largest node in the left subtree cannot have a right child, so the second *delete* is an easy one. As an example, let us consider the following tree. In the tree below, to delete 8, it is the right child of the root. The key value is 8. It is replaced with the largest key in its left subtree (7), and then that node is deleted as before (second case).



Note: We can replace with minimum element in right subtree also.

```
public BinarySearchTreeNode Delete(BinarySearchTreeNode root, int data) {
    BinarySearchTreeNode temp;
    if( root == null)
        System.out.println("Element not there in tree");
    else if(data < root.data)
        root.left = Delete(root.getLeft(),data);
    else if(data > root→ data)
        root.right = Delete(root.getRight(), data);
    else {      //Found element
            if( root.getLeft() != null && root.getRight() !=null ) {
                    /* Replace with largest in left subtree */
                    temp = FindMax( root.getLeft() );
                    root.setData(temp.data);
                    root.left = Delete(root.getLeft(), root.getData());
            }
            else {      /* One child */
                    temp = root;
                    if( root.getLeft() == null )
                            root = root.getRight();
                    if( root.getRight() == null)
                            root = root.getLeft();
            }
    }
    return root;
}
```

Time Complexity: O(n).
Space Complexity: O(n) for recursive stack. For iterative version, space complexity is O(1).

Binary Search Trees: Problems & Solutions

Note: For ordering related problems with binary search trees and balanced binary search trees, Inorder traversal has advantages over others as it gives the sorted order.

Problem-52 Give an algorithm for finding the shortest path between two nodes in a BST.

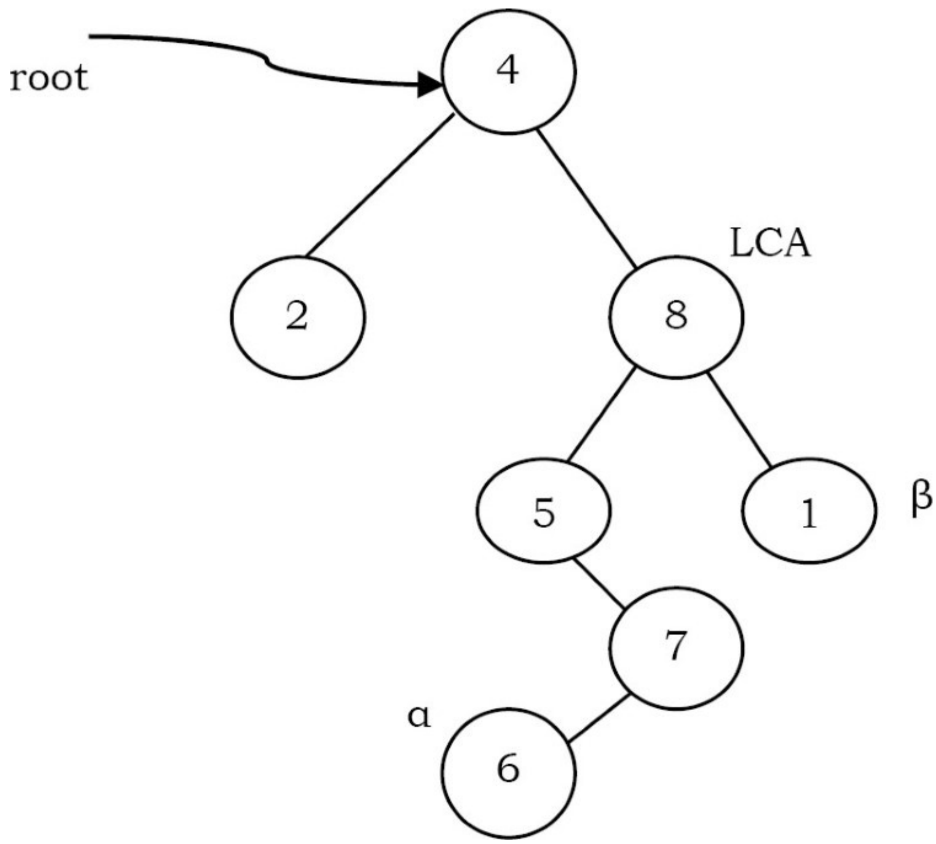
Solution: It's nothing but finding the LCA of two nodes in BST.

Problem-53 Give an algorithm for counting the number of BSTs possible with n nodes.

Solution: This is a DP problem. Refer to *Dynamic Programming* chapter for the algorithm.

Problem-54 Given pointers to two nodes in a binary search tree, find the lowest common ancestor (*LCA*). Assume that both values already exist in the tree.

Solution: The main idea of the solution is: while traversing BST from root to bottom, the first node we encounter with value between α and β, i.e., $\alpha < node \rightarrow data < \beta$, is the Least Common Ancestor(LCA) of α and β (where $\alpha < \beta$). So just traverse the BST in pre-order, and if we find a node with value in between α and β, then that node is the LCA. If its value is greater than both α and β, then the LCA lies on the left side of the node, and if its value is smaller than both a and β, then the LCA lies on the right side.




```
public BinarySearchTreeNode LCA(BinarySearchTreeNode root,
                                        BinarySearchTreeNode a, BinarySearchTreeNode b) {
    if (root == null)
            return root;
    if (root == a || root == b)
            return root;
    if (Math.max(a.data, b.data) < root.data)         // a.data < root.data && b.data < root.data
            return LCA(root.left, a, b);
    else if (Math.min(a.data, b.data) > root.data)    // a.data > root.data && b.data > root.data
            return LCA(root.right, a, b);
    else
            return root;
}
```

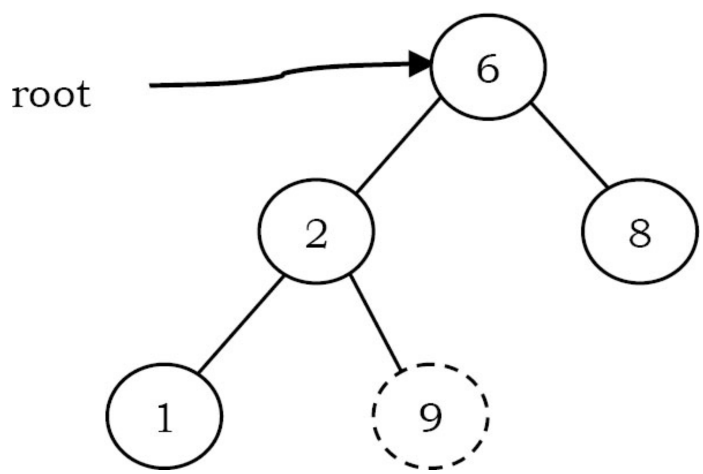
Time Complexity: O(n). Space Complexity: O(n), for skew trees.

Problem-55 Give an algorithm to check whether the given binary tree is a BST or not.

Solution: Consider the following simple program. For each node, check if the node on its left is smaller and check if the node on its right is greater.

```
public boolean IsBST(BinaryTreeNode root) {
    if(root == null) return true;
    /* false if left is > than root */
    if(root.getLeft() != null && root.getLeft().getData() > root.getData())
            return false;
    /* false if right is < than root */
    if(root.getRight() != null && root.getRight().getData() < root.getData())
            return false;
    /* false if, recursively, the left or right is not a BST */
    if(!IsBST(root.getLeft()) || !IsBST(root.getRight()))
            return false;
    /* passing all that, it's a BST */
    return true;
}
```

This approach is wrong as this will return true for binary tree below. Checking only at current node is not enough.



Problem-56 Can we think of getting the correct algorithm?

Solution: For each node, check if max value in left subtree is smaller than the current node data and min value in right subtree greater than the node data.

```
/* Returns true if a binary tree is a binary search tree */
public boolean IsBST(BinaryTreeNode root) {
    if(root == null)
          return true;
    /* false if the max of the left is > than us */
    if(root.getLeft() != null && FindMax(root.getLeft()) > root.getData())
            return false;
    /* false if the min of the right is <= than us */
    if(root.getRight() != null && FindMin(root.getRight()) < root.getData())
            return false;
    /* false if, recursively, the left or right is not a BST */
    if(!IsBST(root.getLeft()) || !IsBST(root.getRight()))
            return false;
    /* passing all that, it's a BST */
    return true;
}
```

It is assumed that we have helper functions *FindMin*() and *FindMax*() that return the min or max integer value from a non-empty tree.
Time Complexity: O(n^2). Space Complexity: O(n).

Problem-57 Can we improve the complexity of Problem-56?

Solution: Yes. A better solution is to look at each node only once. The trick is to write a utility helper function IsBSTUtil(BinaryTreeNode* root, int min, int max) that traverses down the tree keeping track of the narrowing min and max allowed values as it goes, looking at each node only once. The initial values for min and max should be INT_MIN and INT_MAX — they narrow from there.

```
Initial call: isBST(root, Integer.MIN_VALUE, Integer.MAX_VALUE);
public boolean isBST(BinarySearchTreeNode root, int min, int max) {
     if (root == null)
          return true;
     return (root.data > min && root.data < max &&
               isBST(root.left, min, root.data) && isBST(root.right, root.data, max));
}
```

Time Complexity: O(n). Space Complexity: O(n), for stack space.

Problem-58 Can we further improve the complexity of Problem-56?

Solution: Yes, by using inorder traversal. The idea behind this solution is that inorder traversal of BST produces sorted lists. While traversing the BST in inorder, at each node check the condition that its key value should be greater than the key value of its previous visited node. Also, we need to initialize the prev with possible minimum integer value (say, Integer. MIN_VALUE).

```
public class CheckValidBSTRecursiveSingleVariable {
    private int prev = Integer.MIN_VALUE;
    public boolean checkValidBST(BinarySearchTreeNode root) {
        return isBST(root);
    }
    public boolean isBST(BinarySearchTreeNode root) {
        if (root == null)
            return true;
        if(!isBST(root.left))
            return false;
        if(root.data < prev)
            return false;
        prev = root.data;
        return isBST(root.right);
    }
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*), for stack space.

Problem-59 Give an algorithm for converting BST to circular DLL with space complexity O(1).

Solution: Convert left and right subtrees to DLLs and maintain end of those lists. Then, adjust the pointers.

```
public BinarySearchTreeNode BST2DLL(BinarySearchTreeNode root, BinarySearchTreeNode Ltail) {
    BinarySearchTreeNode left, ltail, right, rtail;
    if(root == null) {
        ltail = null;
        return null;
    }
    left = BST2DLL(root.getLeft(), ltail);
    right = BST2DLL(root.getRight(), rtail);
    root.setLeft(ltail);
    root.setRight(right);
    if(right == null)
        ltail = root;
    else {
        right.setLeft(root);
        ltail = rtail;
    }
    if(left == null)
        return root;
    else {
        ltail.setRight(root);
        return left;
    }
}
```

Time Complexity: O(n).

Problem-60 Given a sorted doubly linked list, give an algorithm for converting it into balanced binary search tree.

Solution: Find the list length and construct the tree bottom-up.

```
public BinarySearchTreeNode sortedListToBST(ListNode head) {
  int len = 0;
  ListNode currentNode = head;
  while(currentNode != null){
    len++;
    currentNode = currentNode.next;
  }
  return construct(head, 0, len - 1);
}
public BinarySearchTreeNode construct(ListNode head, int start, int end){
   if(start > end)
     return null;
   int mid = start + (end - start) / 2;
   // build left first, since it is the bottom up approach.
   BinarySearchTreeNode left = construct(head, start, mid - 1);
   BinarySearchTreeNode root = new BinarySearchTreeNode(head.data);
   root.left = left;
   if(head.next != null){
     head.data = head.next.data;
     head.next = head.next.next;
   }
   root.right = construct(head, mid + 1, end);
   return root;
}
```

Time Complexity: $2T(n/2) + O(n)$ [for finding the middle node] = $O(nlogn)$.

Problem-61 Given a sorted array, give an algorithm for converting the array to BST.

Solution: If we have to choose an array element to be the root of a balanced BST, which element should we pick? The root of a balanced BST should be the middle element from the sorted array. We would pick the middle element from the sorted array in each iteration. We then create a node in the tree initialized with this element. After the element is chosen, what is left? Could you identify the sub-problems within the problem?

There are two arrays left — the one on its left and the one on its right. These two arrays are the sub-problems of the original problem, since both of them are sorted. Furthermore, they are subtrees of the current node's left and right child.

The code below creates a balanced BST from the sorted array in O(n) time (n is the number of elements in the array). Compare how similar the code is to a binary search algorithm. Both are using the divide and conquer methodology.

```java
public BinaryTreeNode  BuildBST(int A[], int left, int right) {
    BinaryTreeNode  newNode;
    if(left > right)
        return null;
    newNode = new BinaryTreeNode();
    if(newNode == null) {
        System.out.println("Memory Error"); return;
    }
    if(left == right) {
            newNode.setData(A[left]);
            newNode.setLeft(null);
            newNode.setRight(null);
    }
    else {
            int mid = left + (right-left)/ 2;
            newNode.setData(A[mid]);
            newNode.setLeft(BuildBST(A, left, mid - 1));
            newNode.setRight(BuildBST(A, mid + 1, right));
    }
    return newNode;
}
```

Time Complexity: O(*n*). Space Complexity: O(*n*), for stack space.

Problem-62 Given a singly linked list where elements are sorted in ascending order, convert it to a height balanced BST.

Solution: A naive way is to apply the Problem-60 solution directly. In each recursive call, we would have to traverse half of the list's length to find the middle element. The run time complexity is clearly O(*nlogn*), where *n* is the total number of elements in the list. This is because each level of recursive call requires a total of *n*/2 traversal steps in the list, and there are a total of *logn* number of levels (ie, the height of the balanced tree).

Problem-63 For Problem-62, can we improve the complexity?

Solution: Hint: How about inserting nodes following the list order? If we can achieve this, we no longer need to find the middle element as we are able to traverse the list while inserting nodes to the tree.

Best Solution: As usual, the best solution requires us to think from another perspective. In other words, we no longer create nodes in the tree using the top-down approach. Create nodes bottom-up, and assign them to their parents. The bottom-up approach enables us to access the list in its order while creating nodes [42].

Isn’t the bottom-up approach precise? Any time we are stuck with the top-down approach, we can give bottom-up a try. Although the bottom-up approach is not the most natural way we think, it is helpful in some cases. However, we should prefer top-down instead of bottom-up in general, since the latter is more difficult to verify.

Below is the code for converting a singly linked list to a balanced BST. Please note that the algorithm requires the list length to be passed in as the function parameters. The list length can be found in O(n) time by traversing the entire list once. The recursive calls traverse the list and create tree nodes by the list order, which also takes O(n) time. Therefore, the overall run time complexity is still O(n).

```
public BinaryTreeNode SortedListToBST(ListNode list, int start, int end) {
	if(start > end)
		return null;
	// same as (start+end)/2, avoids overflow
	int mid = start + (end - start) / 2;
	BinaryTreeNode leftChild = SortedListToBST(list, start, mid-1);
	BinaryTreeNode  parent = new BinaryTreeNode();
	if(parent == null)
		System.out.println(“Memory Error”); return;
	parent.setData(list.getData());
	parent.setLeft(leftChild);
	list = list.getNext();
	parent.setRight(SortedListToBST(list, mid+1, end));
	return parent;
}
public BinaryTreeNode  SortedListToBST(ListNode head, int n) {
	return SortedListToBST(head, 0, n-1);
}
```

Problem-64 Give an algorithm for finding the k^{th} smallest element in BST.

Solution: The idea behind this solution is that, inorder traversal of BST produces sorted lists. While traversing the BST in inorder, keep track of the number of elements visited.

```
public BinarySearchTreeNode kthSmallestInBST(BinarySearchTreeNode root, int k, int count) {
    if(root == null)
        return null;
    BinarySearchTreeNode left = kthSmallestInBST(root.getLeft(), k, count);
    if( left != null)
        return left;
    if(++count == k)
        return root;
    return kthSmallestInBST(root.getRight(), k, count);
}
```

Time Complexity: O(n). Space Complexity: O(1).

Problem-65 **Floor and ceiling:** If a given key is less than the key at the root of a BST then the floor of the key (the largest key in the BST less than or equal to the key) must be in the left subtree. If the key is greater than the key at the root, then the floor of the key could be in the right subtree, but only if there is a key smaller than or equal to the key in the right subtree; if not (or if the key is equal to the the key at the root) then the key at the root is the floor of the key. Finding the ceiling is similar, with interchanging right and left. For example, if the sorted with input array is {1, 2, 8, 10, 10, 12, 19}, then
For x = 0: floor doesn't exist in array, ceil = 1,
For x = 1: floor = 1, ceil = 1
For x = 5: floor =2, ceil =8,
For x = 20: floor = 19, ceil doesn't exist in array

Solution: The idea behind this solution is that, inorder traversal of BST produces sorted lists. While traversing the BST in inorder, keep track of the values being visited. If the roots data is greater than the given value then return the previous value which we have maintained during traversal. If the roots data is equal to the given data then return root data.

```
public BinaryTreeNode FloorInBST(BinaryTreeNode root, int data) {
    BinaryTreeNode prev=null;
    return FloorInBSTUtil(root, prev, data);
}
public BinaryTreeNode FloorInBSTUtil(BinaryTreeNode root, BinaryTreeNode prev, int data) {
    if(root == null) return null;
    if(!FloorInBSTUtil(root.getLeft(), prev, data))
        return 0;
    if(root.getData() == data) return root;
    if(root.getData() > data) return prev;
      prev = root;
    return FloorInBSTUtil(root.getRight(), prev, data);
}
```

Time Complexity: O(n). Space Complexity: O(n), for stack space.

For ceiling, we just need to call the right subtree first, followed by left subtree.

```
public BinaryTreeNode CeilingInBST(BinaryTreeNode root, int data) {
        BinaryTreeNode prev=null;
        return CeilingInBSTUtil(root, prev, data);
}
public BinaryTreeNode CeilingInBSTUtil(BinaryTreeNode root, BinaryTreeNode prev, int data) {
        if(root == null) return null;
        if(!CeilingInBSTUtil(root.getRight(), prev, data)) return 0;
        if(root.getData() == data)
                return root;
        if(root.getData() < data)
                return prev;
        prev = root;
        return CeilingInBSTUtil(root.getLeft(), prev, data);
}
```

Time Complexity: O(n). Space Complexity: O(n), for stack space.

Problem-66 Give an algorithm for finding the union and intersection of BSTs. Assume parent pointers are available (say threaded binary trees). Also, assume the lengths of two BSTs are m and n respectively.

Solution: If parent pointers are available then the problem is same as merging of two sorted lists. This is because if we call inorder successor each time we get the next highest element. It’s just a matter of which InorderSuccessor to call.

Time Complexity: O($m + n$). Space complexity: O(1).

Problem-67 For Problem-66, what if parent pointers are not available?

Solution: If parent pointers are not available, the BSTs can be converted to linked lists and then merged.

1. Convert both the BSTs into sorted doubly linked lists in O($n + m$) time. This produces 2 sorted lists.
2. Merge the two double linked lists into one and also maintain the count of total elements in O($n + m$) time.
3. Convert the sorted doubly linked list into height balanced tree in O($n + m$) time.

Problem-68 For Problem-66, is there any alternative way of solving the problem?

Solution: **Yes,** by using inorder traversal.

- Perform inorder traversal on one of the BSTs.
- While performing the traversal store them in table (hash table).
- After completion of the traversal of first *BST*, start traversal of second *BST* and compare them with hash table contents.

Time Complexity: O($m + n$). Space Complexity: O($Max(m, n)$).

Problem-69 Given a *BST* and two numbers *K*1 and *K*2, give an algorithm for printing all the elements *of BST* in the range *K*1 and *K*2.

Solution:

```
public void RangePrinter(BinarySearchTreeNode root, int  K1, int K2) {
   if(root == null)
         return;
   if(root.getData() >= K1)
          RangePrinter(root.getLeft(), K1, K2);
     if(root.getData() >= K1 && root.getData() <= K2)
          System.out.println( root.getData());
     if(root.getData() <= K2)
          RangePrinter(root.getRight(), K1, K2);
}
```

Time Complexity: O(n). Space Complexity: O(n), for stack space.

Problem-70 For Problem-69, is there any alternative way of solving the problem?

Solution: We can use level order traversal: while adding the elements to queue check for the range.

```
public void RangeSeachLevelOrder(BinarySearchTreeNode root, int K1, int K2) {
    BinarySearchTreeNode temp;
    LLQueue Q = new LLQueue();
    if(root == null)
        return null;
    Q.enQueue( root);
    while(!Q.isEmpty()) {
            temp=Q.deQueue();
            if(temp.getData() >= K1 && temp.getData() <= K2)
                    System.out.println(temp.getData());
            if(temp.getLeft() && temp.getData() >= K1)
                    Q.enQueue( temp.getLeft());
            if(temp.getRight() && temp.getData() <= K2)
                    Q.enQueue( temp.getRight());
    }
    Q.deleteQueue();
    return null;
}
```

Time Complexity: O(n). Space Complexity: O(n), for queue.

Problem-71 For Problem-69, can we still think of an alternative way to solve the problem?

Solution: First locate $K1$ with normal binary search and after that use InOrder successor until we encounter $K2$. For algorithm, refer to *Problems* section of threaded binary trees.

Problem-72 Given root of a Binary Search tree, trim the tree, so that all elements returned in the new tree are between the inputs A and B.

Solution: It's just another way of asking Problem-69.

Problem-73 Given two BSTs, check whether the elements of them are the same or not. For example: two BSTs with data 10 5 20 15 30 and 10 20 15 30 5 should return true and the dataset with 10 5 20 15 30 and 10 15 30 20 5 should return false. **Note:** BSTs data can be in any order.

Solution: One simple way is performing an inorder traversal on first tree and storing its data in hash table. As a second step, perform inorder traversal on second tree and check whether that data is already there in hash table or not (if it exists in hash table then mark it with -1 or some unique value).

During the traversal of second tree if we find any mismatch return false. After traversal of second tree check whether it has all -Is in the hash table or not (this ensures extra data available in second tree).

Time Complexity: O($max(m, n)$), where m and n are the number of elements in first and second

BST. Space Complexity: O($max(m, n)$). This depends on the size of the first tree.

Problem-74 For Problem-73, can we reduce the time complexity?

Solution: Instead of performing the traversals one after the other, we can perform *in – order* traversal of both the trees in parallel. Since the *in – order* traversal gives the sorted list, we can check whether both the trees are generating the same sequence or not.

Time Complexity: O($max(m, n)$). Space Complexity: O(1). This depends on the size of the first tree.

Problem-75 Given a BST of size n, in which each node r has an additional field $r \rightarrow size$, the number of the keys in the sub-tree rooted at r (including the root node r). Give an O(h) algorithm *GreaterthanConstant*(r, k) to find the number of keys that are strictly greater than k (h is the height of the binary search tree).

Solution:

```
int GreaterthanConstant (struct BinarySearchTreeNode r, int k){
   keysCount = 0;
   while (r != null ){
      if (k < r.data){
         keysCount = keysCount + r.right.size + 1;
         r = r.left;
      }
      else if (k > r.data)
         r = r.right;
      else{ // k = r.key
         keysCount = keysCount + r.right.size;
         break;
      }
   }
   return keysCount;
}
```

The suggested algorithm works well if the key is a unique value for each node. Otherwise when reaching $k=r.data$, we should start a process of moving to the right until reaching a node y with a key that is bigger then k, and then we should return $keysCount + y.size$. Time Complexity: O(h) where h=O(n) in the worst case and O($logn$) in the average case.

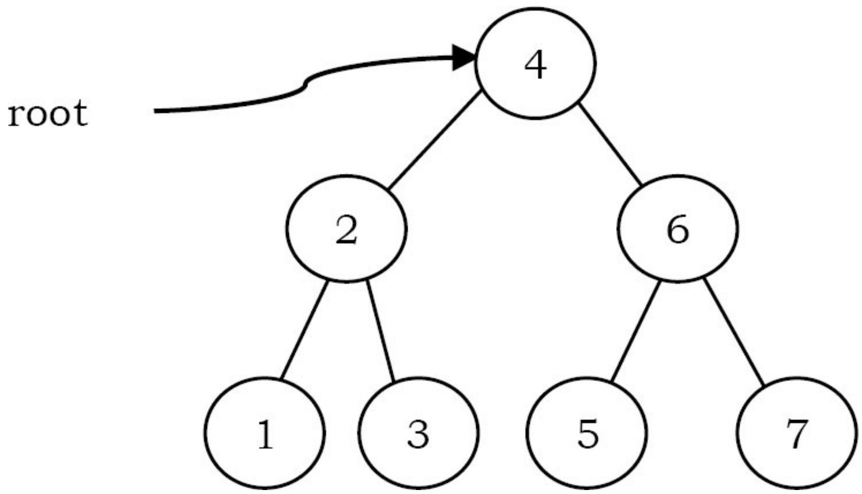
6.10 Balanced Binary Search Trees

In earlier sections we have seen different trees whose worst case complexity is O(n), where n is the number of nodes in the tree. This happens when the trees are skew trees. In this section we will try to reduce this worst case complexity to $O(logn)$ by imposing restrictions on the heights.

In general, the height balanced trees are represented with $HB(k)$, where k is the difference between left subtree height and right subtree height. Sometimes k is called balance factor.

Full Balanced Binary Search Trees

In $HB(k)$, if $k = 0$ (if balance factor is zero), then we call such binary search trees as *full* balanced binary search trees. That means, in $HB(0)$ binary search tree, the difference between left subtree height and right subtree height should be at most zero. This ensures that the tree is a full binary tree. For example,



Note: For constructing $HB(0)$ tree refer to *Problems* section.

6.11 AVL (Adelson-Velskii and Landis) Trees

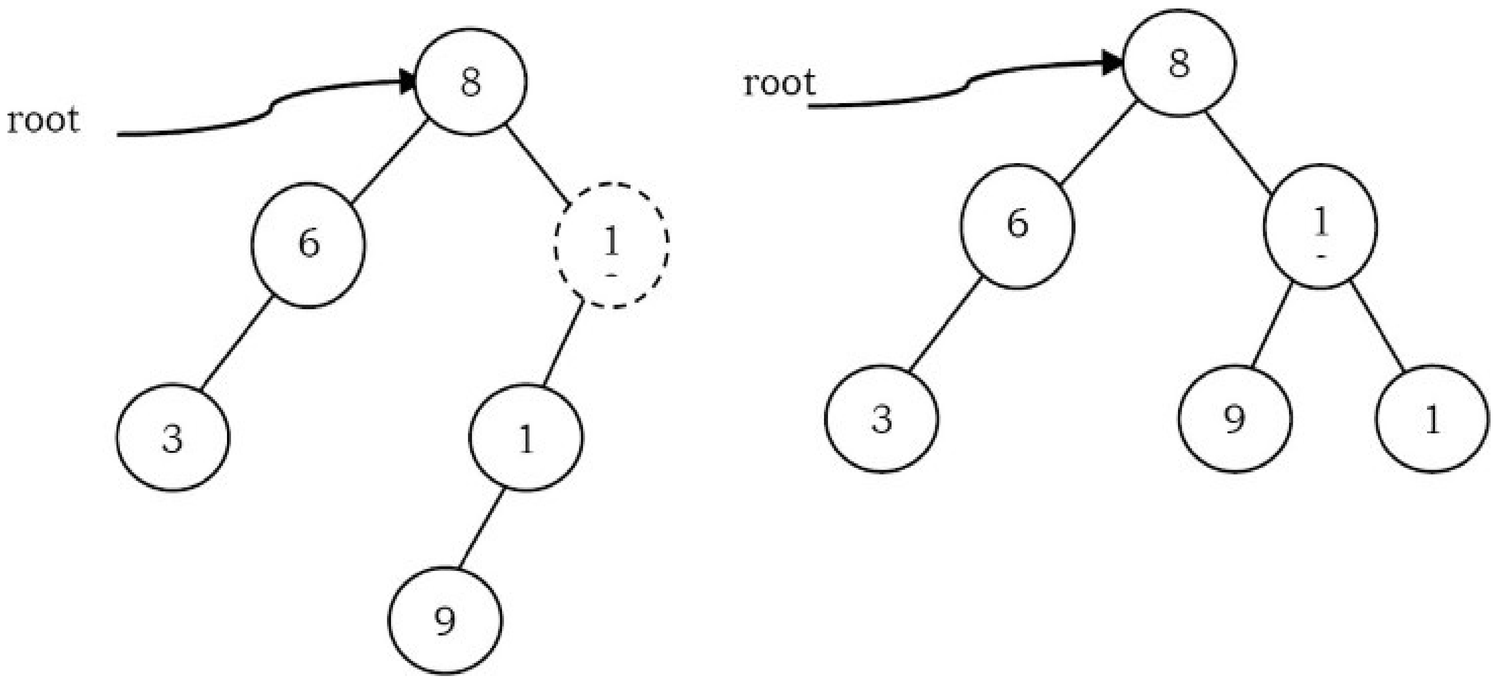
In $HB(k)$, if $k = 1$ (if balance factor is one), such a binary search tree is called an *AVL tree*. That means an AVL tree is a binary search tree with a *balance* condition: the difference between left subtree height and right subtree height is at most 1.

Properties of AVL Trees

A binary tree is said to be an AVL tree, if:

- It is a binary search tree, and
- For any node X, the height of left subtree of X and height of right subtree of X differ by at most 1.

As an example, among the above binary search trees, the left one is not an AVL tree, whereas the right binary search tree is an AVL tree.



Minimum/Maximum Number of Nodes in AVL Tree

For simplicity let us assume that the height of an AVL tree is h and $N(h)$ indicates the number of nodes in AVL tree with height h. To get the minimum number of nodes with height h, we should fill the tree with the minimum number of nodes possible. That means if we fill the left subtree with height $h – 1$ then we should fill the right subtree with height $h – 2$. As a result, the minimum number of nodes with height h is:

$$N(h) = N(h-1) + N(h-2) + 1$$

In the above equation:

- $N(h – 1)$ indicates the minimum number of nodes with height $h – 1$.
- $N(h – 2)$ indicates the minimum number of nodes with height $h – 2$.
- In the above expression, “1” indicates the current node.

We can give $N(h – 1)$ either for left subtree or right subtree. Solving the above recurrence gives:

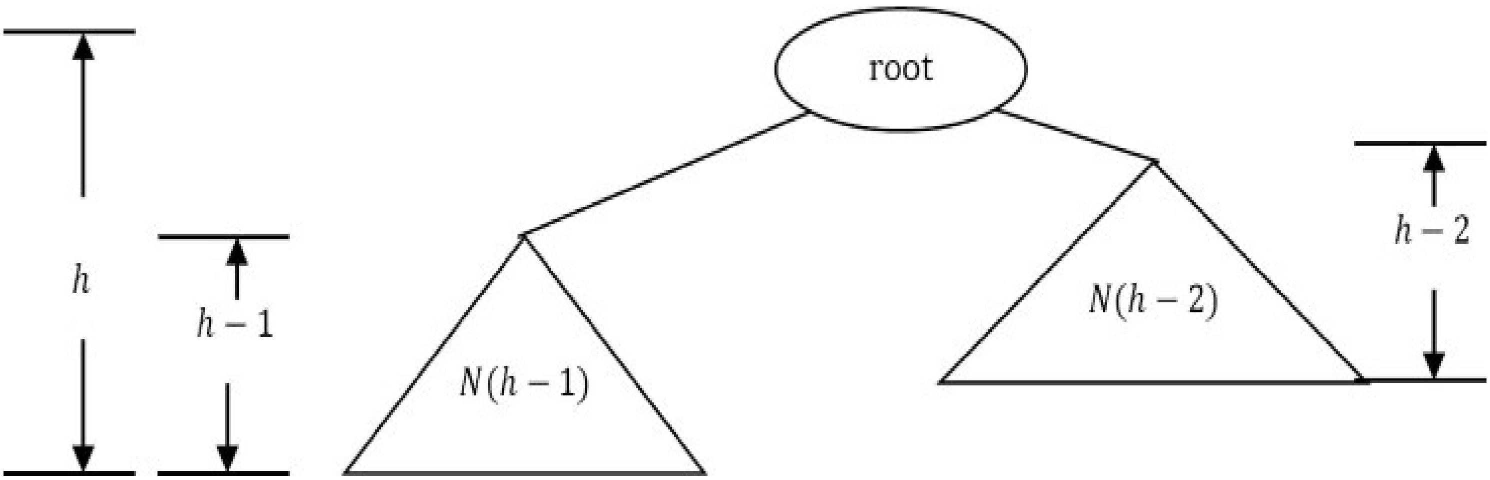
$$N(h) = O(1.618^h) \Longrightarrow h = 1.44 logn \approx O(logn)$$

Where n is the number of nodes in AVL tree. Also, the above derivation says that the maximum height in AVL trees is O($logn$).

Similarly, to get maximum number of nodes, we need to fill both left and right subtrees with height $h – 1$. As a result, we get $N(h) = N(h – 1) + N(h – 1) + 1 = 2N(h – 1) + 1$. The above expression defines the case of full binary tree. Solving the recurrence we get:

$$N(h) = O(2^h) \Rightarrow h = logn \approx O(logn)$$

∴ In both the cases, AVL tree property is ensuring that the height of an AVL tree with n nodes is O($logn$).



AVL Tree Declaration

Since AVL tree is a BST, the declaration of AVL is similar to that of BST. But just to simplify the operations, we also include the height as part of the declaration.

```
public class AVLTreeNode {
    private int data, height;
    private AVLTreeNode left, right;
    public int getData() {  return data; }
    public void setData(int data) {
        this.data = data;
    }
    public int getHeight() {
        return height;
    }
    public void setHeight(int height) {
        this.height = height;
    }
    public AVLTreeNode getLeft() {
        return left;
    }
    public void setLeft(AVLTreeNode left) {
        this.left = left;
    }
    public AVLTreeNode getRight() {
        return right;
    }
    public void setRight(AVLTreeNode right) {
        this.right = right;
    }
}
```

Finding the Height of an AVL tree

```
public int Height(AVLTreeNode root ) {
    if( root == null)  return -1;
    else  return root.getHeight();
}
```

Time Complexity: O(1).

Rotations

When the tree structure changes (e.g., with insertion or deletion), we need to modify the tree to restore the AVL tree property. This can be done using single rotations or double rotations. Since an insertion/deletion involves adding/deleting a single node, this can only increase/decrease the height of a subtree by 1.

So, if the AVL tree property is violated at a node *X*, it means that the heights of left(*X*) and right(*X*) differ by exactly 2. This is because, if we balance the AVL tree every time, then at any point, the difference in heights of left(*X*) and right(*X*) differ by exactly 2. Rotations is the technique used for restoring the AVL tree property. This means, we need to apply the rotations for the node *X*.

Observation: One important observation is that, after an insertion, only nodes that are on the path from the insertion point to the root might have their balances altered, because only those nodes have their subtrees altered. To restore the AVL tree property, we start at the insertion point and keep going to the root of the tree.

While moving to the root, we need to consider the first node that is not satisfying the AVL property. From that node onwards, every node on the path to the root will have the issue.

Also, if we fix the issue for that first node, then all other nodes on the path to the root will automatically satisfy the AVL tree property. That means we always need to care for the first node that is not satisfying the AVL property on the path from the insertion point to the root and fix it.

Types of Violations

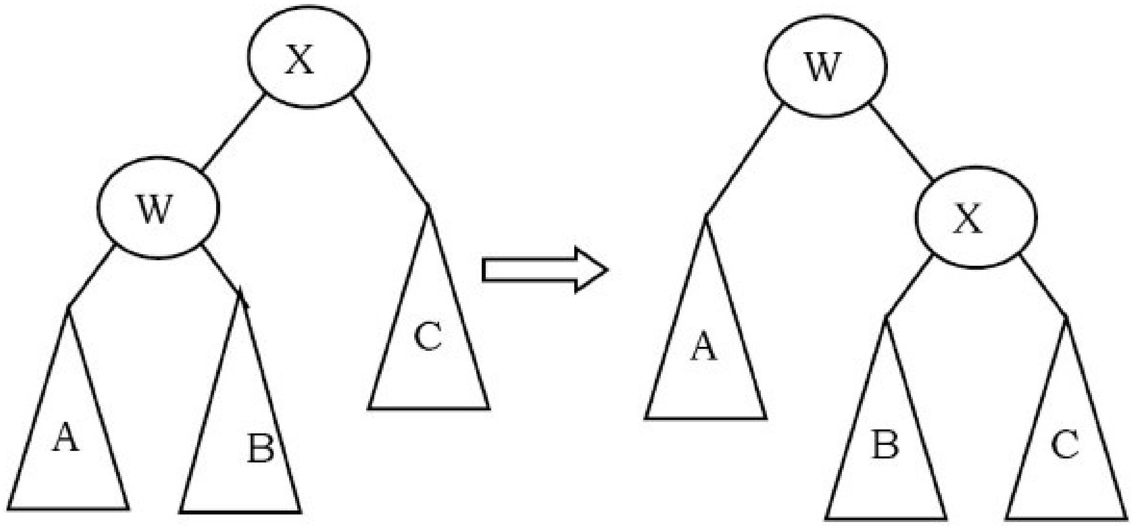
Let us assume the node that must be rebalanced is *X*. Since any node has at most two children, and a height imbalance requires that *X's* two subtree heights differ by two, we can easily observe that a violation might occur in four cases:

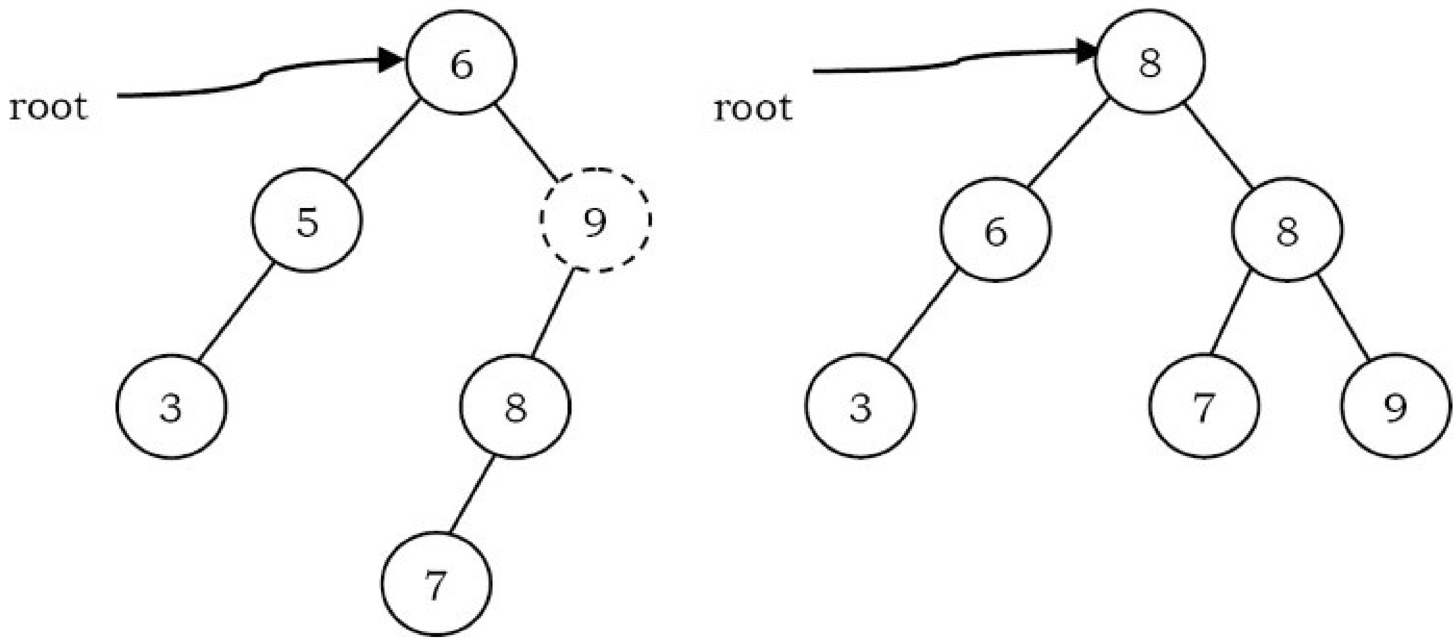
1. An insertion into the left subtree of the left child of *X*.
2. An insertion into the right subtree of the left child of *X*.
3. An insertion into the left subtree of the right child of *X*.
4. An insertion into the right subtree of the right child of *X*.

Cases 1 and 4 are symmetric and easily solved with single rotations. Similarly, cases 2 and 3 are also symmetric and can be solved with double rotations (needs two single rotations).

Single Rotations

Left Left Rotation (LL Rotation) [Case-1]: In the case below, node *X* is not satisfying the AVL tree property. As discussed earlier, the rotation does not have to be done at the root of a tree. In general, we start at the node inserted and travel up the tree, updating the balance information at every node on the path. For example, in below figure, after the insertion of 7 in the original AVL tree on the left, node 9 becomes unbalanced. So, we do a single left-left rotation at 9. As a result we get the tree on the right.

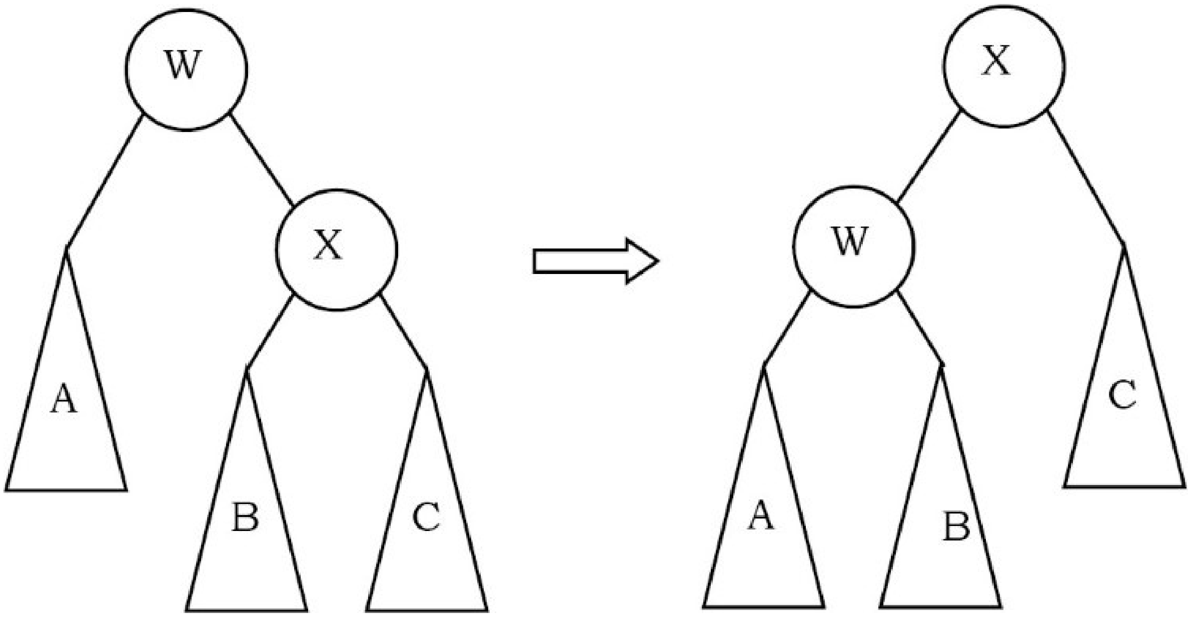




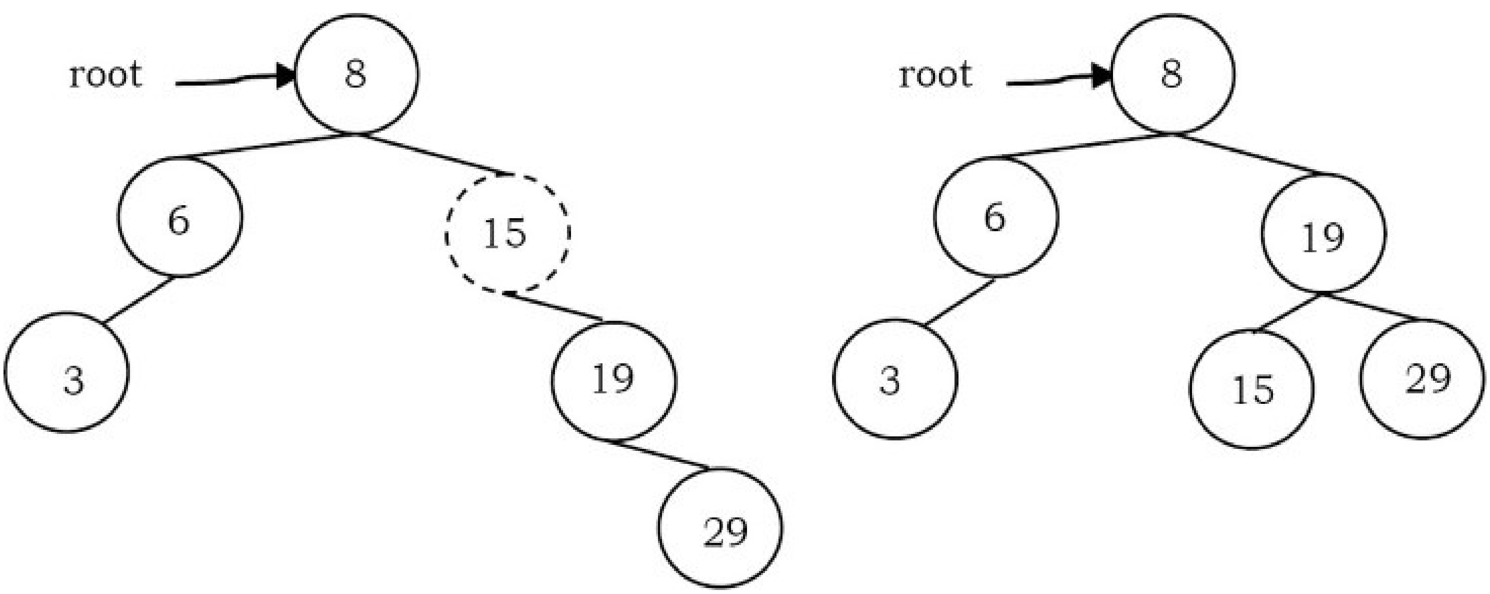

```
public AVLTreeNode SingleRotateLeft(AVLTreeNode X ) {
    AVLTreeNode W = X.getLeft();
    X.setLeft(W.getRight());
    W.setRight(X);
    X.setHeight(Math.max( Height(X→left), Height(X.getRight()) ) + 1);
    W.setHeight(Math.max( Height(W→left), X→height ) + 1);
    return W;        /* New root */
}
```

Time Complexity: O(1). Space Complexity: O(1).

Right Right Rotation (RR Rotation) [Case-4]: In this case, node *X* is not satisfying the AVL tree property.



For example, in the above figure, after the insertion of 29 in the original AVL tree on the left, node 15 becomes unbalanced. So, we do a single right-right rotation at 15. As a result we get the tree on the right.

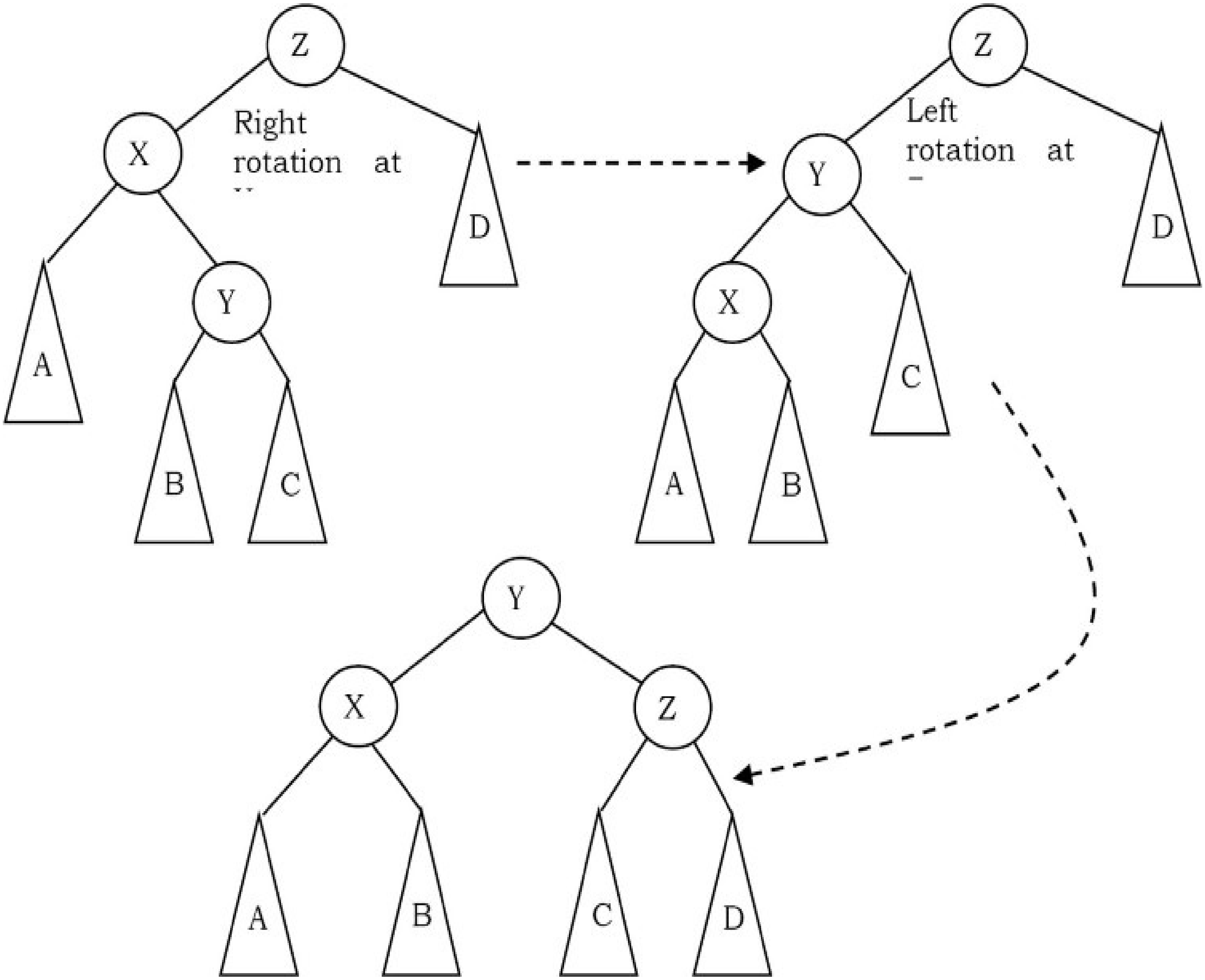


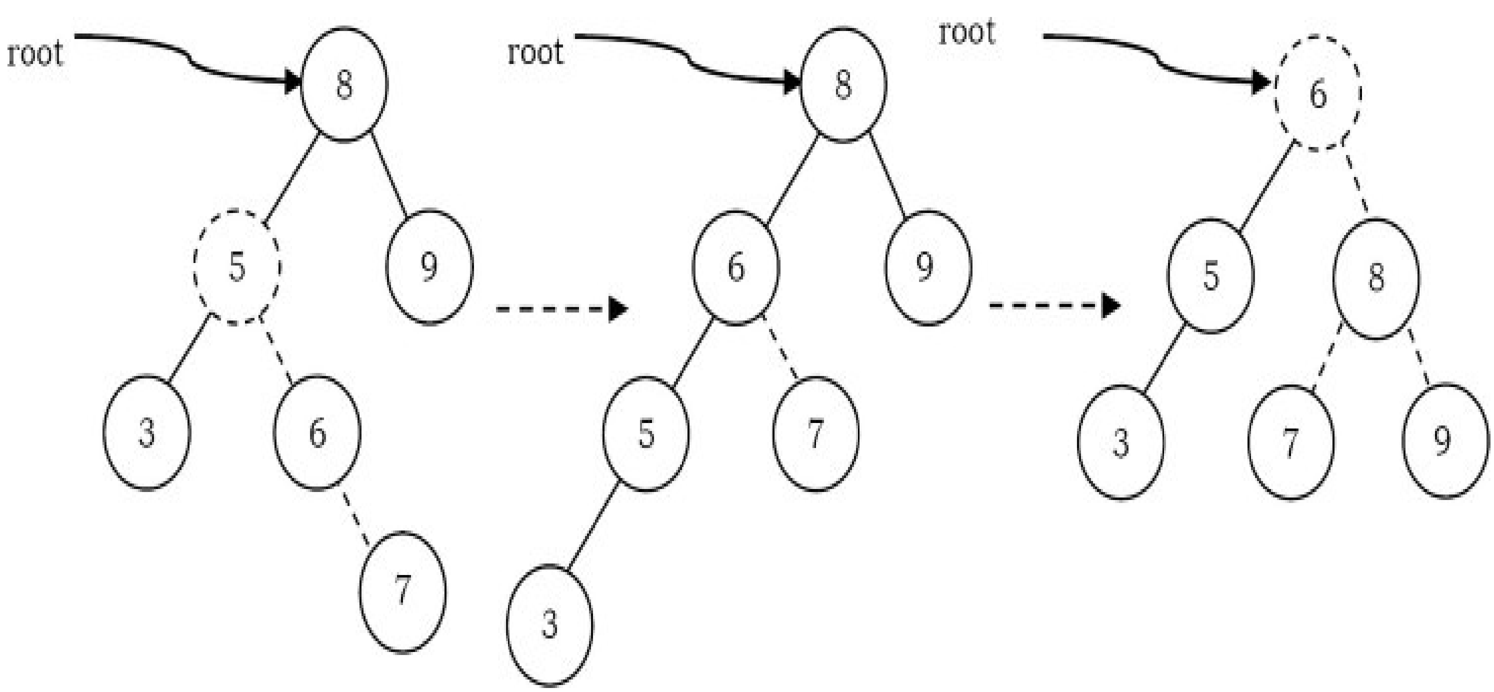
```
public AVLTreeNode SingleRotateRight(AVLTreeNode W ) {
    AVLTreeNode X  = W.getRight();
    W.setRight(X.getLeft());
     X.setLeft(W);
    W.setHeight(Math.max( Height(W.getRight()), Height(W.getLeft()) ) + 1);
    X.setHeight(Math.max( Height(X.getRight()), W→height) + 1);
    return X;
}
```

Time Complexity: O(1). Space Complexity: O(1).

Double Rotations

Left Right Rotation (LR Rotation) [Case-2]: For case-2 and case-3 single rotation does not fix the problem. We need to perform two rotations.

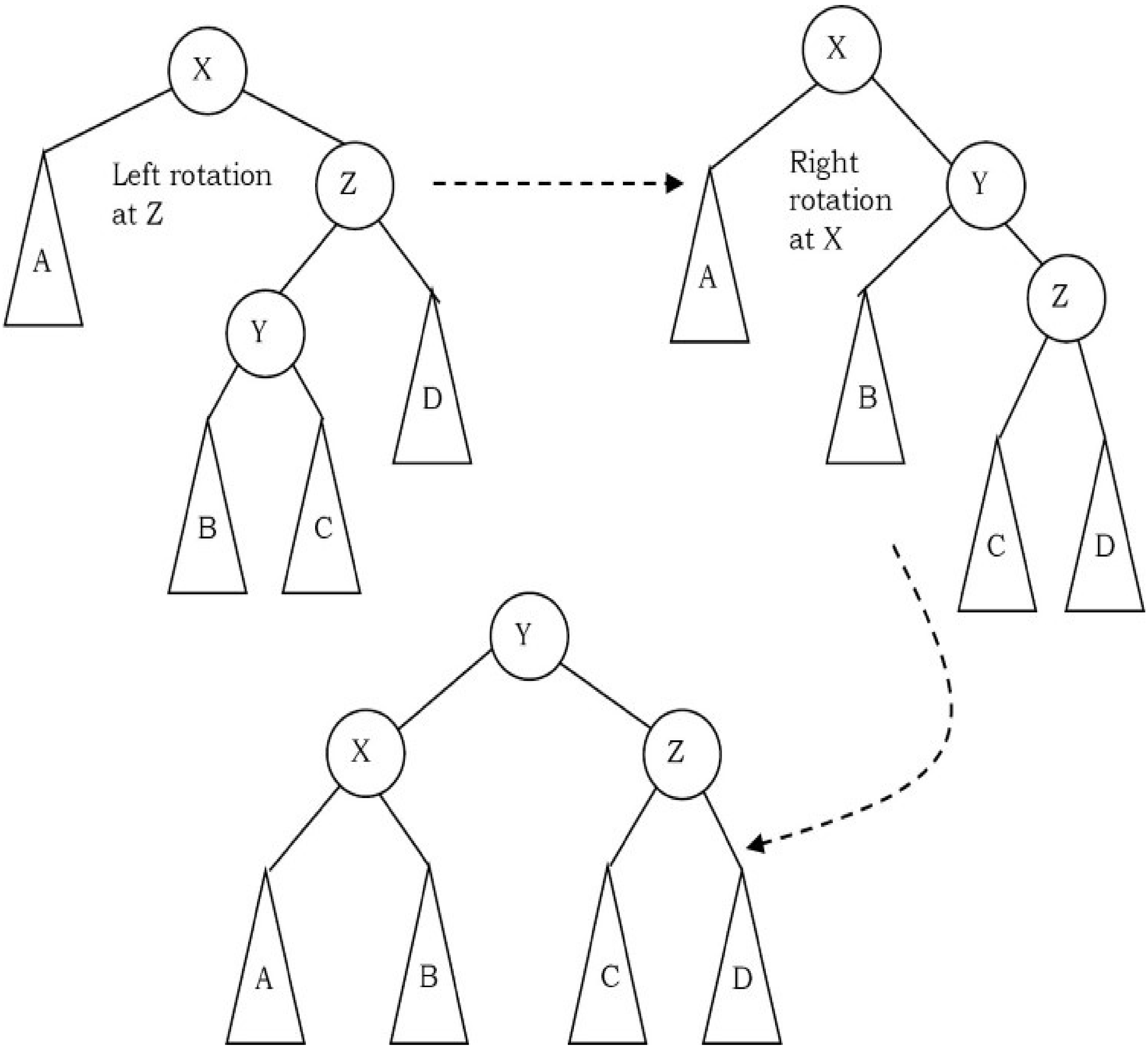

Z
X
Right
rotation at
D
Y
A
B
C
Z
Left
rotation at
Y
D
X
C
A
B
Y
X
Z
A
B
C
D


root
8
5
9
3
6
7
root
8
6
9
5
7
3
root
6
5
8
3
7
9

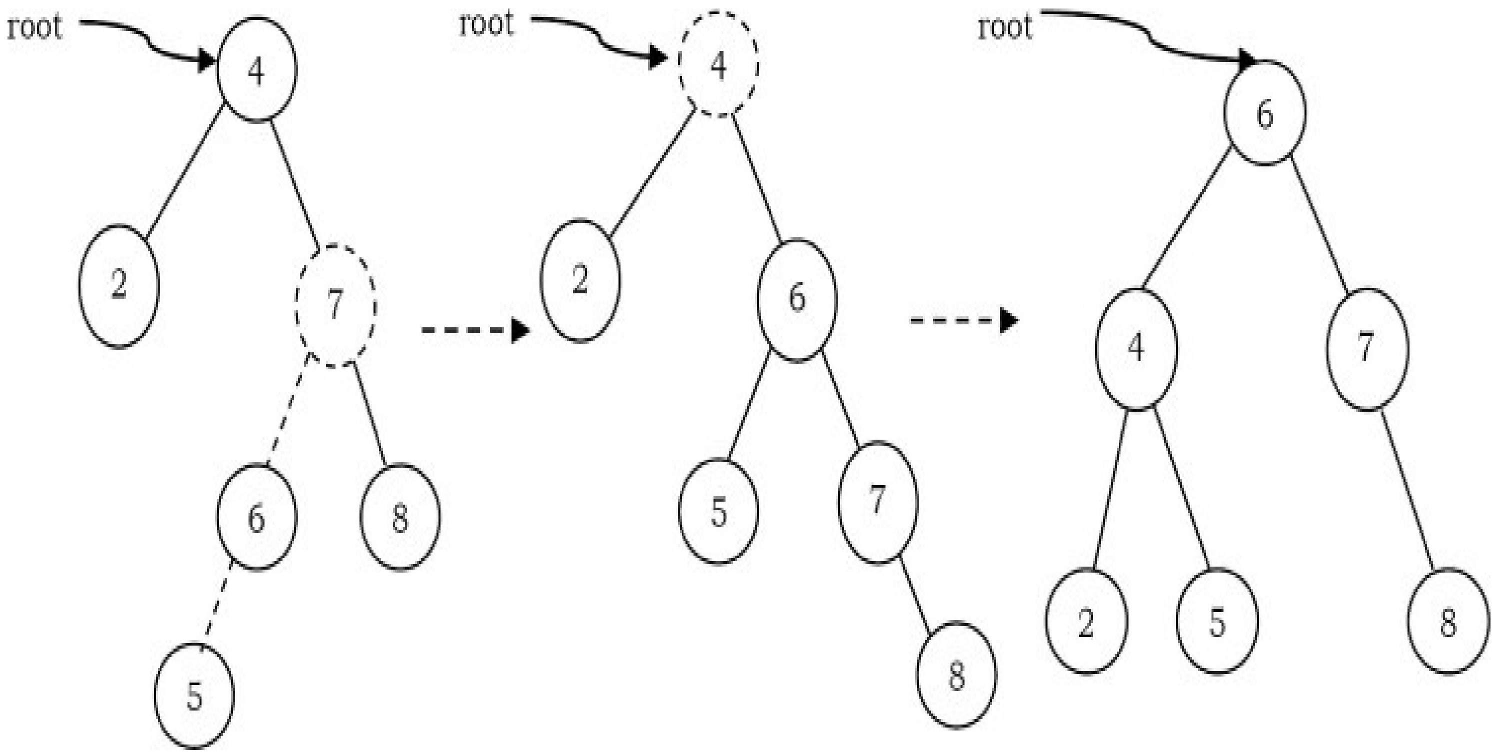
As an example, let us consider the following tree: The insertion of 7 is creating the case-2 scenario and the right side tree is the one after the double rotation.

```
public AVLTreeNode DoubleRotatewithLeft( AVLTreeNode Z ) {
    Z.setLeft( SingleRotateRight(Z.getLeft()) );
    return SingleRotateLeft(Z);
}
```

Right Left Rotation (RL Rotation) [Case-3]: Similar to case-2, we need to perform two rotations to fix this scenario.



As an example, let us consider the following tree: The insertion of 6 is creating the case-3 scenario and the right side tree is the one after the double rotation.



Insertion into an AVL tree

Insertion into an AVL tree is similar to a BST insertion. After inserting the element, we just need to check whether there is any height imbalance. If there is an imbalance, call the appropriate rotation functions.

```
public AVLTreeNode Insert( AVLTreeNode root, AVLTreeNode parent, int data) {
    if( root == null) {
        root = new AVLTreeNode();
        root.setData(data);
        root.setHeight(0);
        root.setLeft(null);
        root.setRight(null);
    }
    else if( data < root.getData() ) {
        root.setLeft(Insert( root.getLeft(), root, data ));
        if( ( Height( root.getLeft() ) - Height( root.getRight() ) ) == 2 ) {
            if( data < root.getLeft().getData() )
                root = SingleRotateLeft( root );
            else
                root = DoubleRotateLeft( root );
        }
    }
    else if( data > root.getData() ) {
        root.setRight(Insert( root.getRight(), root, data ) );
        if( ( Height( root.getRight() ) - Height( root.getLeft() ) ) == 2 ) {
            if( data < root.getRight().getData() )
                root = SingleRotateRight( root );
            else
                root = DoubleRotateRight( root );
        }
    }
    /* Else data is in the tree already. We'll do nothing */
    root.setHeight(Math.max( Height(root.getLeft()), Height(root.getRight()) ) + 1);
    return root;
}
```

Time Complexity: O(*logn*). Space Complexity: O(*logn*).

AVL Trees: Problems & Solutions

Problem-76 Given a height *h*, give an algorithm for generating the *HB*(0).

Solution: As we have discussed, *HB*(0) is nothing but generating full binary tree. In full binary tree the number of nodes with height *h* is: $2^{h+1} - 1$ (let us assume that the height of a tree with one node is 0). As a result the nodes can be numbered as: 1 to $2^{h+1} - 1$.

```
public BinarySearchTreeNode BuildHB0(int h) {
    BinarySearchTreeNode temp;
    if(h == 0)
     return null;
    temp = newBinarySearchTreeNode();
    temp.setLeft(BuildHB0 (h-1));
    temp.getData(count++); //assume count is a global variable
    temp.setRight(BuildHB0 (h-1));
    return temp;
}
```

Time Complexity: O(n). Space Complexity: O($logn$), where $logn$ indicates the maximum stack size which is equal to height of the tree.

Problem-77 Is there any alternative way of solving Problem-76?

Solution: Yes, we can solve it following Mergesort logic. That means, instead of working with height, we can take the range. With this approach we do not need any global counter to be maintained.

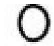
```
public BinarySearchTreeNode BuildHB0(int l, int r) {
    BinarySearchTreeNode temp;
    int mid = l + (r-l)/2;
    if( l > r)
       return null;
    temp = (BinarySearchTreeNode ) malloc (sizeof(BinarySearchTreeNode));
    temp.setData(mid);
    temp.setLeft(BuildHB0(l, mid-1));
    temp.setRight(BuildHB0(mid+1, r));
    return temp;
}
```

The initial call to *BuildHB*0 function could be: *BuildHB*0(1, 1 ≪ h). 1 ≪ h does the shift operation for calculating the $2^{h+1} - 1$. Time Complexity: O(n). Space Complexity: O($logn$). Where $logn$ indicates maximum stack size which is equal to the height of the tree.

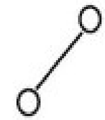
Problem-78 Construct minimal AVL trees of height 0,1,2,3,4, and 5. What is the number of nodes in a minimal AVL tree of height 6?

Solution Let $N(h)$ be the number of nodes in a minimal AVL tree with height h.

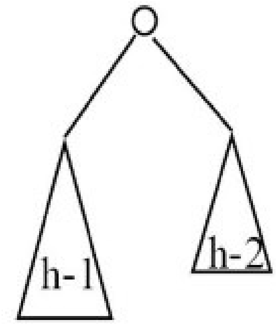
$$N(0) = 1$$



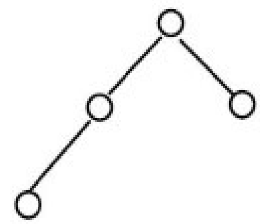
$$N(1) = 2$$



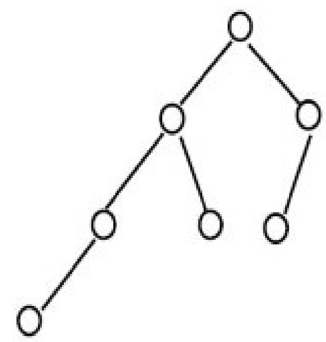
$$N(h) = 1 + N(h-1) + N(h-2)$$



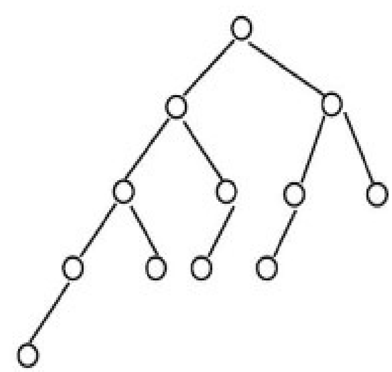
$$N(2) = 1 + N(1) + N(0)$$
$$= 1 + 2 + 1 = 4$$



$$N(3) = 1 + N(2) + N(1)$$
$$= 1 + 4 + 2 = 7$$



$$N(4) = 1 + N(3) + N(2)$$
$$= 1 + 7 + 4 = 12$$



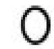
$$N(5) = 1 + N(4) + N(3)$$
$$= 1 + 12 + 7 = 20$$

...

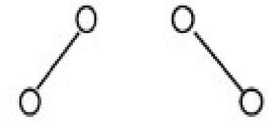
Problem-79 For Problem-76, how many different shapes can there be of a minimal AVL tree of height h?

Solution: Let $NS(h)$ be the number of different shapes of a minimal AVL tree of height h.

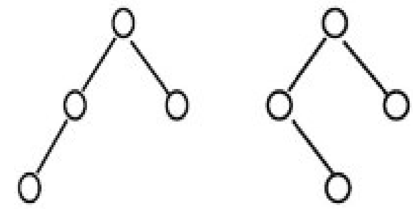
$NS(0) = 1$



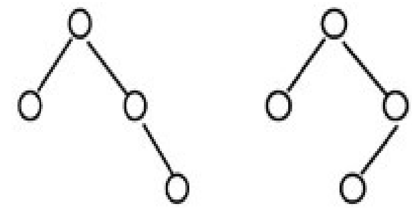
$NS(1) = 2$



$$NS(2) = 2 * NS(1) * NS(0)$$
$$= 2 * 2 * 1 = 4$$

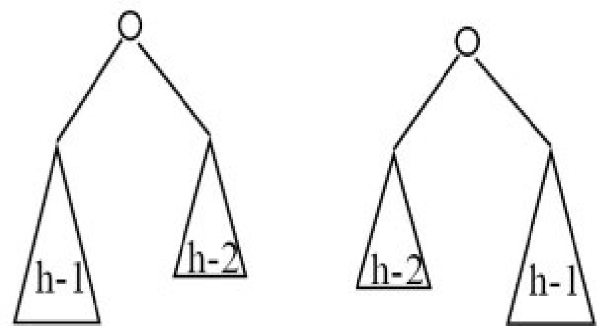


$$NS(3) = 2 * NS(2) * NS(1)$$
$$= 2 * 4 * 1 = 8$$



...

$$NS(h) = 2 * NS(h-1) * NS(h-2)$$



Problem-80 Given a binary search tree, check whether it is an AVL tree or not?

Solution: Let us assume that *IsAVL* is the function which checks whether the given binary search tree is an AVL tree or not. *IsAVL* returns –1 if the tree is not an AVL tree. During the checks each node sends its height to its parent.

```
public boolean isAVL(BinarySearchTreeNoderoot) {
    if(root == null)
        return true;
    return isAVL(root.left) && isAVL(root.right) && Math.abs(getHeight(root.left) - getHeight(root.right)) <= 1;
}
public int getHeight(BinarySearchTreeNoderoot){
    int leftHeight, rightHeight;
    if(root == null)
        return 0;
    else{
        leftHeight = getHeight(root.left);
        rightHeight = getHeight(root.right);
        if(leftHeight > rightHeight)
            return leftHeight + 1;
        else
            return rightHeight + 1;
    }
}
```

Time Complexity: O(n). Space Complexity: O(n).

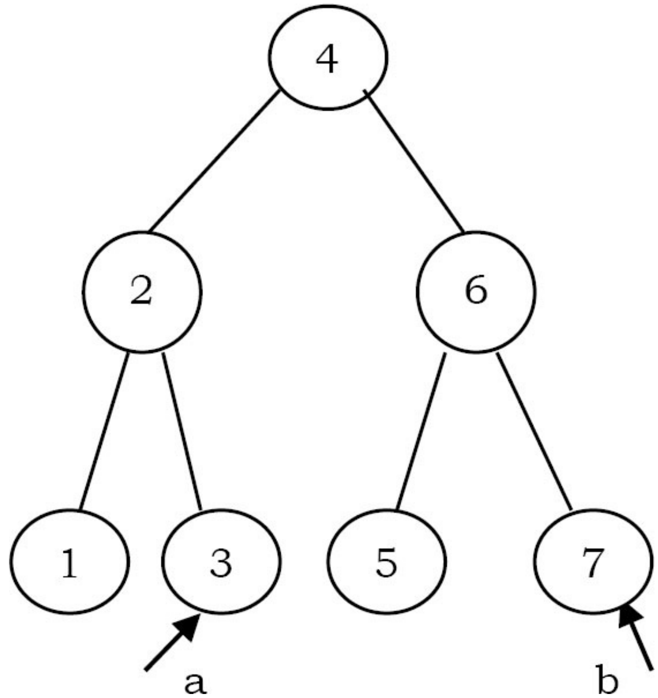
Problem-81 Given a height h, give an algorithm for generating an AVL tree with minimum number of nodes.

Solution: To get minimum number of nodes, fill one level with $h - 1$ and the other with $h - 2$.

```
public AVLTreeNode GenerateAVLTree(int h) {
    AVLTreeNode temp;
    if(h == 0)
            return null;
    temp = new AVLTreeNode();
    temp.setLeft(GenerateAVLTree(h-1));
    temp.getData(count++); //assume count is a global variable
    temp.setRight(GenerateAVLTree(h-2));
    temp.setHeight(temp.getLeft().getHeight()+1); // or temp→height = h;
    return temp;
}
```

Problem-82 Given an AVL tree with n integer items and two integers a and b, where a and b can be any integers with $a <= b$. Implement an algorithm to count the number of nodes in the range [a, b].

Solution:



The idea is to make use of the recursive property of binary search trees. There are three cases to consider: whether the current node is in the range [*a, b*], on the left side of the range [*a, b*], or on the right side of the range [*a, b*]. Only subtrees that possibly contain the nodes will be processed under each of the three cases.

```
public int RangeCount(AVLNode root, int a, int b) {
    if(root == null)
        return 0;
    else if(root.getData() > b)
        return RangeCount(root.getLeft(), a, b);
    else if(root.getData() < a)
        return RangeCount(root.getRight(), a, b);
    else if(root.getData() >= a && root.getData() <= b)
        return RangeCount(root.getLeft(), a, b) + RangeCount(root.getRight(), a, b) + 1;
}
```

The complexity is similar to *in – order* traversal of the tree but skipping left or right sub-trees when they do not contain any answers. So in the worst case, if the range covers all the nodes in the tree, we need to traverse all the *n* nodes to get the answer. The worst time complexity is therefore O(*n*).

If the range is small, which only covers a few elements in a small subtree at the bottom of the tree, the time complexity will be O(*h*) = O(*logn*), where *h* is the height of the tree. This is because only a single path is traversed to reach the small subtree at the bottom and many higher level subtrees have been pruned along the way.

Note: Refer to similar problem in BST.

Problem-83 Median in an infinite series of integers

Solution: Median is the middle number in a sorted list of numbers (if we have odd number of elements). If we have even number of elements, median is the average of two middle numbers in a sorted list of numbers.

For solving this problem we can use a binary search tree with additional information at each node, and the number of children on the left and right subtrees. We also keep the number of total nodes in the tree. Using this additional information we can find the median in O(*logn*) time, taking the appropriate branch in the tree based on the number of children on the left and right of the current node. But, the insertion complexity is O(*n*) because a standard binary search tree can degenerate into a linked list if we happen to receive the numbers in sorted order. So, let's use a balanced binary search tree to avoid worst case behavior of standard binary search trees. For this problem, the balance factor is the number of nodes in the left subtree minus the number of nodes in the right subtree. And only the nodes with a balance factor of +1 or 0 are considered to be balanced. So, the number of nodes on the left subtree is either equal to or 1 more than the number of nodes on the right subtree, but not less.

If we ensure this balance factor on every node in the tree, then the root of the tree is the median, if the number of elements is odd. In the number of elements is even, the median is the average of the root and its inorder successor, which is the leftmost descendent of its right subtree.

So, the complexity of insertion maintaining a balanced condition is O(*logn*) and finding a median operation is O(1) assuming we calculate the inorder successor of the root at every insertion if the number of nodes is even.

Insertion and balancing is very similar to AVL trees. Instead of updating the heights, we update the number of nodes information. Balanced binary search trees seem to be the most optimal solution, insertion is O(*logn*) and find median is O(1).

Note: For an efficient algorithm refer to the *Priority Queues and Heaps* chapter.

Problem-84 Given a binary tree, how do you remove all the half nodes (which have only one child)? Note that we should not touch leaves.

Solution: By using post-order traversal we can solve this problem efficiently. We first process the left children, then the right children, and finally the node itself. So we form the new tree bottom up, starting from the leaves towards the root. By the time we process the current node, both its left and right subtrees have already been processed.

```
public BinaryTreeNode removeHalfNodes(BinaryTreeNode root){
    if (root == null)
        return null;
    root.left=removeHalfNodes(root.getLeft());
    root.right=removeHalfNodes(root.getRight());
    if (root.getLeft() == null && root.getRight() == null)
        return root;
    if (root.getLeft() == null)
        return root.getRight();
    if (root.getRight() == null)
        return root.getLeft();
    return root;
}
```

Time Complexity: O(n).

Problem-85 Given a binary tree, how do you remove its leaves?

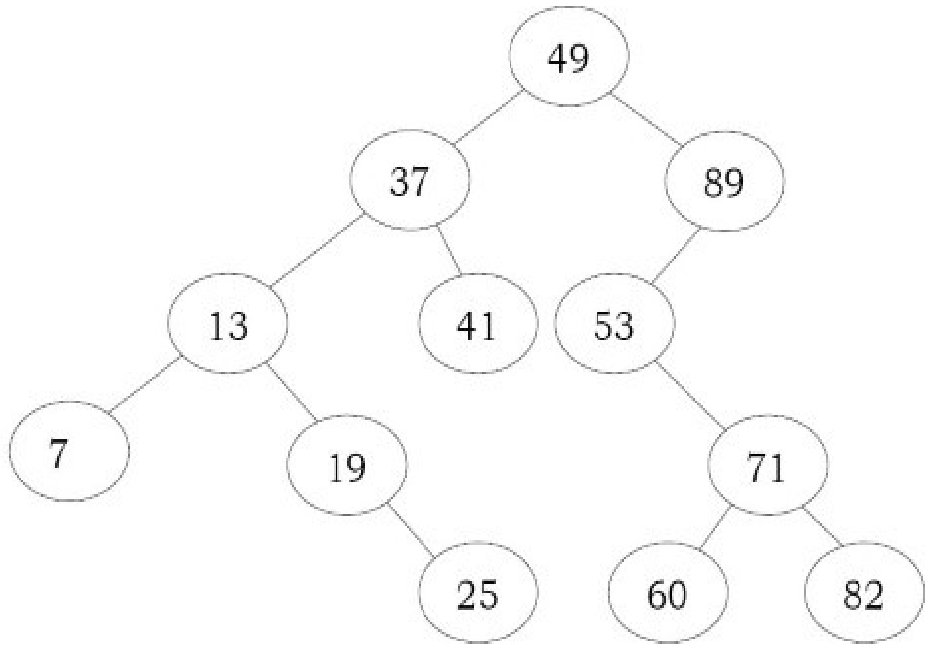
Solution: By using post-order traversal we can solve this problem (other traversals would also work).

```
public BinaryTreeNode removeLeaves(BinaryTreeNode root) {
    if (root != null) {
        if (root.getLeft() == null && root.getRight() == null) {
            root = null;
        } else {
            root.left = removeLeaves(root.getLeft());
            root.right = removeLeaves(root.getRight());
        }
    }
    return root;
}
```

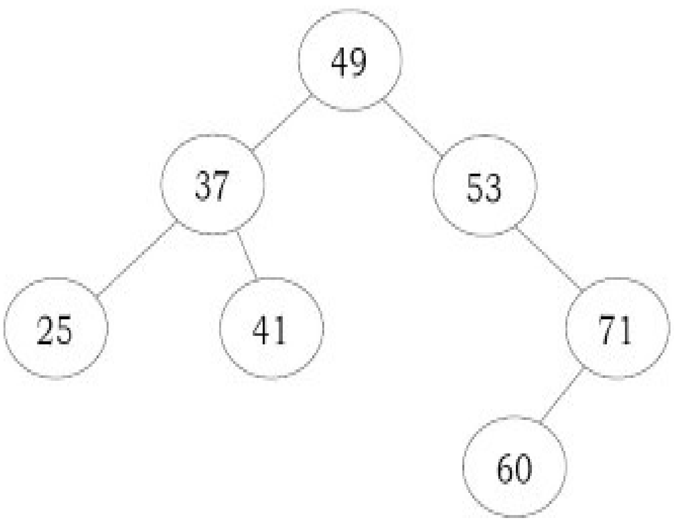
Time Complexity: O(n).

Problem-86 Given a BST and two integers (minimum and maximum integers) as parameters, how do you remove (prune) elements elements that are not within that range?

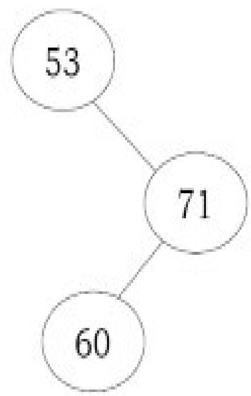
Sample Tree



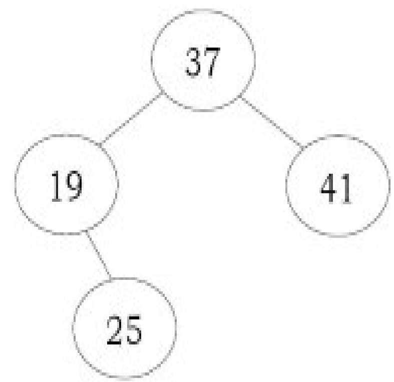
PruneBST(24,71);



PruneBST(53,79);



PruneBST(17,41);



Solution: Observation: Since we need to check each and every element in the tree, and the subtree changes should be reflected in the parent, we can think about using post order traversal. So we process the nodes starting from the leaves towards the root. As a result, while processing the node itself, both its left and right subtrees are valid pruned BSTs. At each node we will return a pointer based on its value, which will then be assigned to its parent's left or right child pointer, depending on whether the current node is the left or right child of the parent. If the current node's value is between *A* and *B* (*A* <= *node's data* <= *B*) then no action needs to be taken, so we return the reference to the node itself.

If the current node's value is less than *A*, then we return the reference to its right subtree and discard the left subtree. Because if a node's value is less than *A*, then its left children are definitely less than A since this is a binary search tree. But its right children may or may not be less than A; we can't be sure, so we return the reference to it. Since we're performing bottom-up post-order traversal, its right subtree is already a trimmed valid binary search tree (possibly

NULL), and its left subtree is definitely NULL because those nodes were surely less than A and they were eliminated during the post-order traversal.

A similar situation occurs when the node's value is greater than *B*, so we now return the reference to its left subtree. Because if a node's value is greater than *B*, then its right children are definitely greater than *B*. But its left children may or may not be greater than *B;* So we discard the right subtree and return the reference to the already valid left subtree.

```
public BinarySearchTreeNode PruneBST(BinarySearchTreeNode root, int  A, int B){
    if(root == null)
        return null;
    root.left= PruneBST(root->getLeft(),A,B);
    root.right= PruneBST(root->getRight(),A,B);
    if(A<=root.getData() && root.getData()<=B)
        return root;
    if(root.getData()<A)
        return root.getRight();
    if(root.getData()>B)
        return root.getLeft();
}
```

Time Complexity: O(*n*) in worst case and in average case it is O(*logn*).

Note: If the given BST is an AVL tree then O(*n*) is the average time complexity.

Problem-87 Given a binary tree, how do you connect all the adjacent nodes at the same level? Assume that given binary tree has next pointer along with left and right pointers.

Solution: One simple approach is to use level-order traversal and keep updating the next pointers. While traversing, we will link the nodes on the next level. If the node has left and right node, we will link left to right. If node has next node, then link rightmost child of current node to leftmost child of next node.

```
public void linkLevelNodes(BinaryTreeNode root){
    Queue Q = CreateQueue();
    BinaryTreeNode prev;     // Pointer to the revious node of the current level
    BinaryTreeNode temp;
    int currentLevelNodeCount;
    int nextLevelNodeCount;
    if(root == null)
        return;
    EnQueue(Q, root);
    currentLevelNodeCount = 1;
    nextLevelNodeCount = 0;
    prev = NULL;
    while (!IsEmptyQueue(Q)) {
        temp = DeQueue(Q);
        if (temp.left != null){
            EnQueue(Q, temp.left);
            nextLevelNodeCount++;
        }
        if (temp.right != null){
            EnQueue(Q, temp.right);
            nextLevelNodeCount++;
        }
        // Link the previous node of the current level to this node
        if (prev)
            prev.next = temp;
        //Set the previous node to the current
        prev = temp;
        currentLevelNodeCount--;
        if (currentLevelNodeCount == 0) {   // if this is the last node of the current level
            currentLevelNodeCount = nextLevelNodeCount;
            nextLevelNodeCount = 0;
            prev = NULL;
        }
    }
}
```

Time Complexity: O(n). Space Complexity: O(n).

Problem-88 Can we improve space complexity for Problem-87?

Solution: We can process the tree level by level, but without a queue. The logical part is that when we process the nodes of the next level, we make sure that the current level has already been linked.

```
public void linkLevelNodes(BinaryTreeNode root) {
    if(root==null)
        return;
    BinaryTreeNode rightMostNode = null;
    BinaryTreeNode nextHead = null;
    BinaryTreeNode temp = root;
    //connect next level of current root node level
    while(temp!=null){
        if(temp.left!=null)
            if(rightMostNode==null){
                rightMostNode=temp.left;
                nextHead=temp.left;
            }
            else{
                rightMostNode.next = temp.left;
                rightMostNode = rightMostNode.next;
            }
        if(temp.right!=null)
            if(rightMostNode==null){
                rightMostNode=temp.right;
                nextHead=temp.right;
            }
            else{
                rightMostNode.next = temp.right;
                rightMostNode = rightMostNode.next;
            }
        temp=temp.next;
    }
    connect(nextHead);
}
```

Time Complexity: O(n). Space Complexity: O(*depth of tree*) for stack space.

Problem-89 Assume that a set S of n numbers are stored in some form of balanced binary search tree; i.e. the depth of the tree is O($logn$). In addition to the key value and the pointers to children, assume that every node contains the number of nodes in its subtree. Specify a reason(s) why a balanced binary tree can be a better option than a complete binary tree for storing the set S.

Solution: Implementation of a balanced binary tree requires less RAM space as we do not need to keep complete tree in RAM (since they use pointers).

Problem-90 For the Problem-89, specify a reason (s) why a complete binary tree can be a better option than a balanced binary tree for storing the set S.

Solution: A complete binary tree is more space efficient as we do not need any extra flags. A balanced binary tree usually takes more space since we need to store some flags. For example, in a Red-Black tree we need to store a bit for the color. Also, a complete binary tree can be stored in a RAM as an array without using pointers.

Problem-90 Let T be a proper binary tree with root r. Consider the following algorithm.

```
Algorithm TreeTraversal(r):
    if r is a leaf then return 1
    else {
            a = TreeTraversal(left child of r)
            b = TreeTraversal(right child of r)
            return a + b
    }
```

What does the algorithm do?
A. It always returns the value 1.
B. It computes the number of nodes in the tree.
C. It computes the depth of the nodes.
D. It computes the height of the tree.
E. It computes the number of leaves in the tree.

Solution: E.

6.12 Other Variations on Trees

In this section, let us enumerate the other possible representations of trees. In the earlier sections, we have looked at AVL trees, which is a binary search tree (BST) with balancing property. Now, let us look at a few more balanced binary search trees: Red-black Trees and Splay Trees.

6.12.1 Red-Black Trees

In Red-black trees each node is associated with an extra attribute: the color, which is either red or black. To get logarithmic complexity we impose the following restrictions.

Definition: A Red-black tree is a binary search tree that satisfies the following properties:

- Root Property: the root is black
- External Property: every leaf is black
- Internal Property: the children of a red node are black
- Depth Property: all the leaves have the same black

Similar to AVL trees, if the Red-black tree becomes imbalanced, then we perform rotations to reinforce the balancing property. With Red-black trees, we can perform the following operations in O(*logn*) in worst case, where *n* is the number of nodes in the trees.

- Insertion
- Deletion
- Find predecessor
- Find successor
- Find minimum
- Find maximum

6.12.2 Splay Trees

Splay-trees are BSTs with a self-adjusting property. Another interesting property of splay-trees is: starting with an empty tree, any sequence of *K* operations with maximum of *n* nodes takes O(*Klogn*) time complexity in worst case.

Splay trees are easier to program and also ensure faster access to recently accessed items. Similar to *AVL* and Red-Black trees, at any point that the splay tree becomes imbalanced, we can perform rotations to reinforce the balancing property.

Splay-trees cannot guarantee the O(*logn*) complexity in worst case. But it gives amortized O(*logn*) complexity. Even though individual operations can be expensive, any sequence of operations gets the complexity of logarithmic behavior. One operation may take more time (a single operation may take O(*n*) time) but the subsequent operations may not take worst case complexity and on the average *per operation* complexity is O(*logn*).

6.14.3 B-Trees

B-Tree is like other self-balancing trees such as AVL and Red-black tree such that it maintains its balance of nodes while opertions are performed against it. B-Tree has the following properties:

- Minimum degree "*t*" where, except root node, all other nodes must have no less than $t - 1$ keys
- Each node with *n* keys has $n + 1$ children
- Keys in each node are lined up where $k_1 < k_2 < .. k_n$
- Each node cannot have more than 2t-l keys, thus 2t children
- Root node at least must contain one key. There is no root node if the tree is empty.

- Tree grows in depth only when root node is split.

Unlike a binary-tree, each node of a b-tree may have a variable number of keys and children. The keys are stored in non-decreasing order. Each key has an associated child that is the root of a subtree containing all nodes with keys less than or equal to the key but greater than the preceeding key. A node also has an additional rightmost child that is the root for a subtree containing all keys greater than any keys in the node.

A b-tree has a minumum number of allowable children for each node known as the *minimization factor*. If t is this *minimization factor*, every node must have at least $t - 1$ keys. Under certain circumstances, the root node is allowed to violate this property by having fewer than $t - 1$ keys. Every node may have at most $2t - 1$ keys or, equivalently, 2t children.

Since each node tends to have a large branching factor (a large number of children), it is typically neccessary to traverse relatively few nodes before locating the desired key. If access to each node requires a disk access, then a B-tree will minimize the number of disk accesses required. The minimzation factor is usually chosen so that the total size of each node corresponds to a multiple of the block size of the underlying storage device. This choice simplifies and optimizes disk access. Consequently, a B-tree is an ideal data structure for situations where all data cannot reside in primary storage and accesses to secondary storage are comparatively expensive (or time consuming).

To *search* the tree, it is similar to binary tree except that the key is compared multiple times in a given node because the node contains more than 1 key. If the key is found in the node, the search terminates. Otherwise, it moves down where at child pointed by ci where key $k < k_i$.

Key *insertions* of a B-tree happens from the bottom fasion. This means that it walk down the tree from root to the target child node first. If the child is not full, the key is simply inserted. If it is full, the child node is split in the middle, the median key moves up to the parent, then the new key is inserted. When inserting and walking down the tree, if the root node is found to be full, it's split first and we have a new root node. Then the normal insertion operation is performed.

Key *deletion* is more complicated as it needs to maintain the number of keys in each node to meet the constraint. If a key is found in leaf node and deleting it still keeps the number of keys in the nodes not too low, it's simply done right away. If it's done to the inner node, the predecessor of the key in the corresonding child node is moved to replace the key in the inner node. If moving the predecessor will cause the child node to violate the node count constraint, the sibling child nodes are combined and the key in the inner node is deleted.

6.12.4 Augmented Trees

In earlier sections, we have seen various problems like finding the K^{th} – smallest – element in the tree and other similar ones. Of all the problems the worst complexity is O(n), where n is the number of nodes in the tree. To perform such operations in O($logn$), augmented trees are useful. In these trees, extra information is added to each node and that extra data depends on the problem we are trying to solve.

For example, to find the k^{th} element in a binary search tree, let us see how augmented trees solve the problem. Let us assume that we are using Red-Black trees as balanced BST (or any balanced BST) and augmenting the size information in the nodes data. For a given node X in Red-Black tree with a field $size(X)$ equal to the number of nodes in the subtree and can be calculated as:

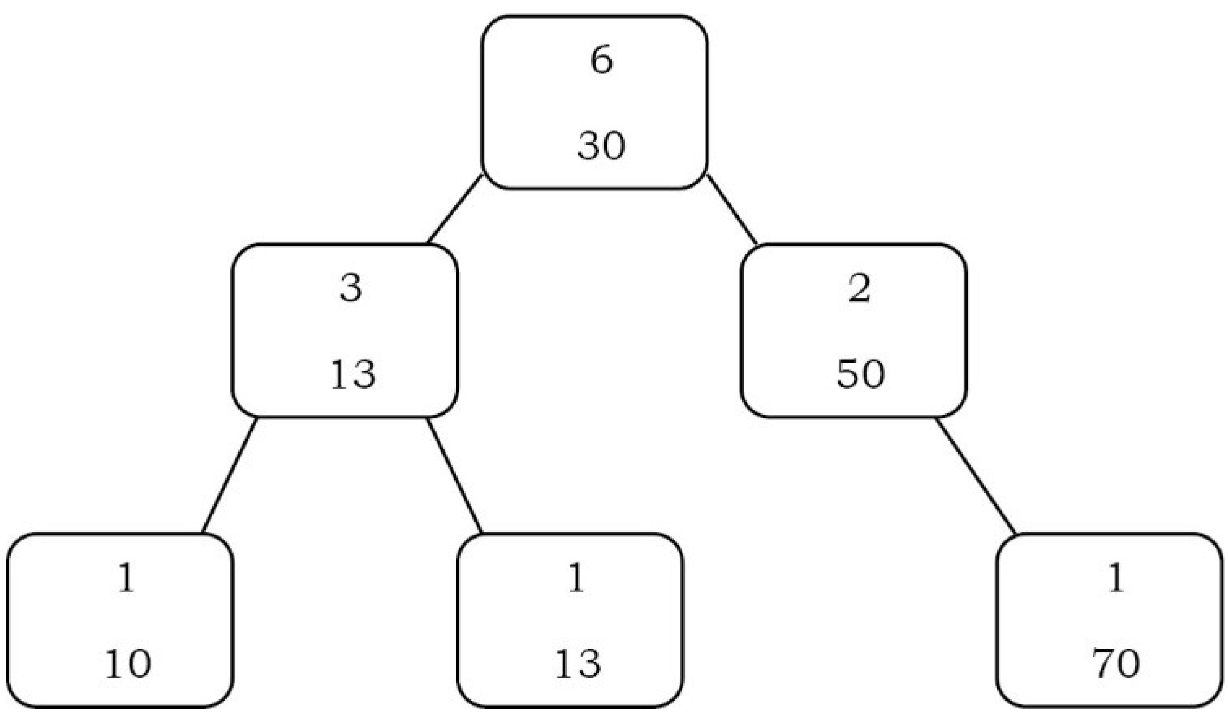
$$size(X) = size(X \rightarrow left) + size(X \rightarrow right)) + 1$$

Kth – smallest – operation can be defined as:

```
public BinarySearcTreeNode KthSmallest (BinarySearcTreeNode X, int K) {
    int r = size(X.getLeft()) + 1;
    if(K == r)
        return X;
    if(K < r)
        return KthSmallest (X.getLeft(), K);
    if(K > r)
        return KthSmallest (X.getRight(), K-r);
}
```

Time Complexity: O($logn$). Space Complexity: O($logn$).

Example: With the extra size information, the augmented tree will look like:



6.12.5 Scapegoat Trees

Scapegoat tree is a self-balancing binary search tree, discovered by Arne Andersson. It provides worst-case O(*logn*) search time, and O(*logn*) amortized (average) insertion and deletion time.

AVL tree rebalance whenever the heights of two sibling subtrees differ by more than one, scapegoat tree rebalance whenever the size of a child exceeds a certain ratio of its parent's, a ratio known as a. After inserting the element, we traverse back up the tree. If we find an imbalance where a child's size exceeds the parent's size times alpha, we must rebuild the subtree at the parent, the *scapegoat*.

There might be more than possible scapegoat, but we only have to pick one. The most optimal scapegoat is actually determined by height balance. When removing, we see if the total size of the tree is less than alpha of the largest size since the last rebuilding of the tree. If so, we rebuild the entire tree. The alpha for a scapegoat tree can be any number between 0.5 and 1.0. The value 0.5 will force perfect balance, while 1.0 will cause rebalancing to never occur, effectively turning it into a BST.

6.12.6 Interval Trees

We often face questions that involve queries made in an array based on range. For example, for a given array of integers, what is the maximum number in the range α to β, where α and β are of course within array limits. To iterate over those entries with intervals containing a particular value, we can use a simple array. But if we need more efficient access, we need a more sophisticated data structure.

An array-based storage scheme and a brute-force search through the entire array is acceptable only if a single search is to be performed, or if the number of elements is small. For example, if you know all the array values of interest in advance, you need to make only one pass through the array. However, if you can interactively specify different search operations at different times, the brute-force search becomes impractical because every element in the array must be examined during each search operation.

If you sort the array in ascending order of the array values, you can terminate the sequential search when you reach the object whose low value is greater than the element we are searching. Unfortunately, this technique becomes increasingly ineffective as the low value increases, because fewer search operations are eliminated. That means, what if we have to answer a large number of queries like this? - is brute force still a good option?

Another example is when we need to return a sum in a given range. We can brute force this too, but the problem for a large number of queries still remains. So, what can we do? With a bit of thinking we can come up with an approach like maintaining a separate array of *n* elements, where *n* is the size of the original array, where each index stores the sum of all elements from 0 to that index. So essentially we have with a bit of preprocessing brought down the query time from a worst case O(n) to O(1). Now this is great as far as static arrays are concerned, but, what if we are required to perform updates on the array too?

The first approach gives us an O(n) query time, but an O(1) update time. The second approach, on the other hand, gives us O(1) query time, but an O(n) update time. So, which one do we choose?

Interval trees are also binary search trees and they store interval information in the node structure. That means, we maintain a set of n intervals $[i_1, i_2]$ such that one of the intervals containing a query point Q (if any) can be found efficiently. Interval trees are used for performing range queries efficiently.

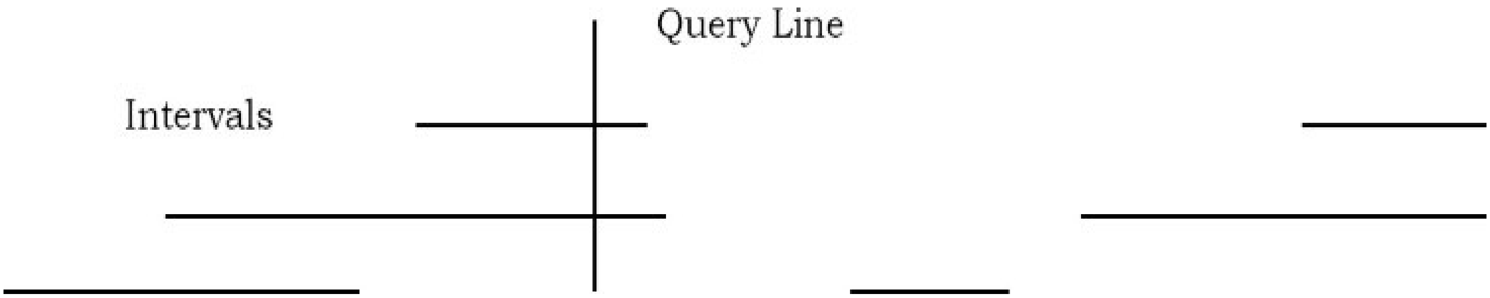
A segment tree is a heap-like data structure that can be used for making update/query operations upon array intervals in logarithmical time. We define the segment tree for the interval $[i, j]$ in the following recursive manner:

- The root (first node in the array) node will hold the information for the interval $[i, j]$
- If $i < j$ the left and right children will hold the information for the intervals $[i, \frac{i+j}{2}]$ and $[\frac{i+j}{2}+1, j]$

Segment trees (also called *segtrees* and *interval trees*) is a cool data structure, primarily used for range queries. It is a height balanced binary tree with a static structure. The nodes of a segment tree correspond to various intervals, and can be augmented with appropriate information pertaining to those intervals. It is somewhat less powerful than a balanced binary tree because of its static structure, but due to the recursive nature of operations on the segtree, it is incredibly easy to think about and code.

We can use segment trees to solve range minimum/maximum query problems. The time complexity is T($nlogn$) where O(n) is the time required to build the tree and each query takes O($logn$) time.

Example: Given a set of intervals: 5= {[2-5], [6-7], [6-10], [8-9], [12-15], [15-23], [25-30]}. A query with Q = 9 returns [6, 10] or [8, 9] (assume these are the intervals which contain 9 among all the intervals). A query with Q = 23 returns [15, 23].



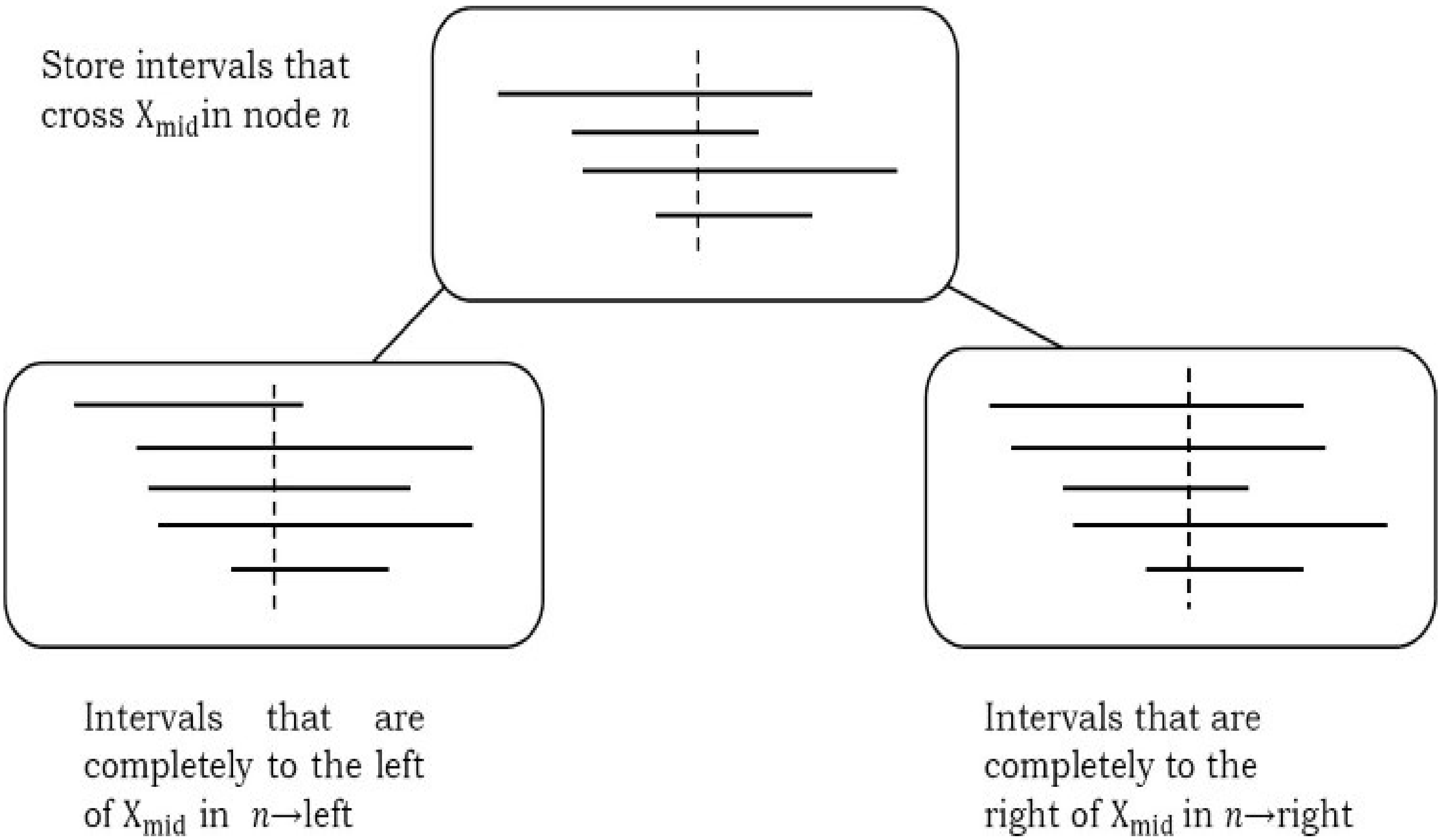
Construction of Interval Trees

Let us assume that we are given a set S of n intervals (also called segments). These n intervals will have $2n$ endpoints. Now, let us see how to construct the interval tree.

Algorithm

Recursively build the tre\e on interval set S as follows:

- Sort the $2n$ endpoints
- Let X_{mid} be the median point



Time Complexity for building interval trees: O($nlogn$). Since we are choosing the median, Interval Trees will be approximately balanced. This ensures that we split the set of end points in half each time. The depth of the tree is O($logn$). To simplify the search process, generally X_{mid} is stored with each node.

CHAPTER

7

PRIORITY QUEUES AND HEAPS

7.1 What is a Priority Queue?

In some situations we may need to find the minimum/maximum element among a collection of elements. We can do this with the help of Priority Queue ADT. A priority queue ADT is a data structure that supports the operations *Insert* and *DeleteMin* (which returns and removes the minimum element) or *DeleteMax* (which returns and removes the maximum element).

These operations are equivalent to *EnQueue* and *DeQueue* operations of a queue. The difference is that, in priority queues, the order in which the elements enter the queue may not be the same in which they were processed. An example application of a priority queue is job scheduling, which is prioritized instead of serving in first come first serve.

A priority queue is called an *ascending – priority* queue, if the item with the smallest key has the highest priority (that means, delete the smallest element always). Similarly, a priority queue is said to be a *descending —priority* queue if the item with the largest key has the highest priority (delete the maximum element always). Since these two types are symmetric we will be concentrating on one of them: ascending-priority queue.

7.2 Priority Queue ADT

The following operations make priority queues an ADT.

Main Priority Queues Operations

A priority queue is a container of elements, each having an associated key.

- Insert (key, data): Inserts data with *key* to the priority queue. Elements are ordered based on key.
- DeleteMin/DeleteMax: Remove and return the element with the smallest/largest key.
- GetMinimum/GetMaximum: Return the element with the smallest/largest key without deleting it.

Auxiliary Priority Queues Operations

- k^{th} – Smallest/k^{th} – Largest: Returns the k^{th} – Smallest/k^{th} –Largest key in priority queue.
- Size: Returns number of elements in priority queue.
- Heap Sort: Sorts the elements in the priority queue based on priority (key).

7.3 Priority Queue Applications

Priority queues have many applications – a few of them are listed below:

- Data compression: Huffman Coding algorithm
- Shortest path algorithms: Dijkstra’s algorithm
- Minimum spanning tree algorithms: Prim’s algorithm
- Event-driven simulation: customers in a line

- Selection problem: Finding k^{th}- smallest element

7.4 Priority Queue Implementations

Before discussing the actual implementation, let us enumerate the possible options.

Unordered Array Implementation

Elements are inserted into the array without bothering about the order. Deletions (DeleteMax) are performed by searching the key and then deleting.

Insertions complexity: O(1). DeleteMin complexity: O(n).

Unordered List Implementation

It is very similar to array implementation, but instead of using arrays, linked lists are used.

Insertions complexity: O(1). DeleteMin complexity: O(n).

Ordered Array Implementation

Elements are inserted into the array in sorted order based on key field. Deletions are performed at only one end.

Insertions complexity: O(n). DeleteMin complexity: O(1).

Ordered List Implementation

Elements are inserted into the list in sorted order based on key field. Deletions are performed at only one end, hence preserving the status of the priority queue. All other functionalities associated with a linked list ADT are performed without modification.

Insertions complexity: O(n). DeleteMin complexity: O(1).

Binary Search Trees Implementation

Both insertions and deletions take O($logn$) on average if insertions are random (refer to *Trees* chapter).

Balanced Binary Search Trees Implementation

Both insertions and deletion take O(*logn*) in the worst case (refer to *Trees* chapter).

Binary Heap Implementation

In subsequent sections we will discuss this in full detail. For now, assume that binary heap implementation gives O(*logn*) complexity for search, insertions and deletions and O(1) for finding the maximum or minimum element.

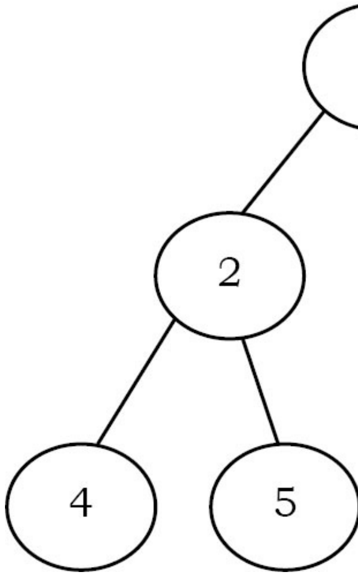
Comparing Implementations

Implementation	Insertion	Deletion (DeleteMax)	Find Min
Unordered array	1	n	n
Unordered list	1	n	n
Ordered array	n	1	1
Ordered list	n	1	1
Binary Search Trees	$logn$ (average)	$logn$ (average)	$logn$ (average)
Balanced Binary Search Trees	$logn$	$logn$	$logn$
Binary Heaps	$logn$	$logn$	1

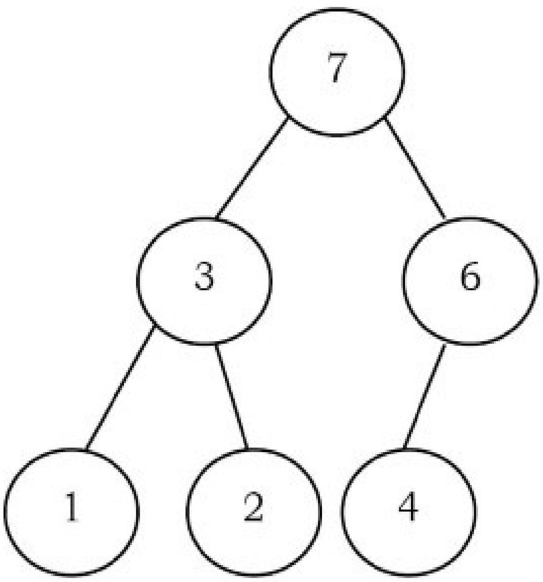
7.5 Heaps and Binary Heaps

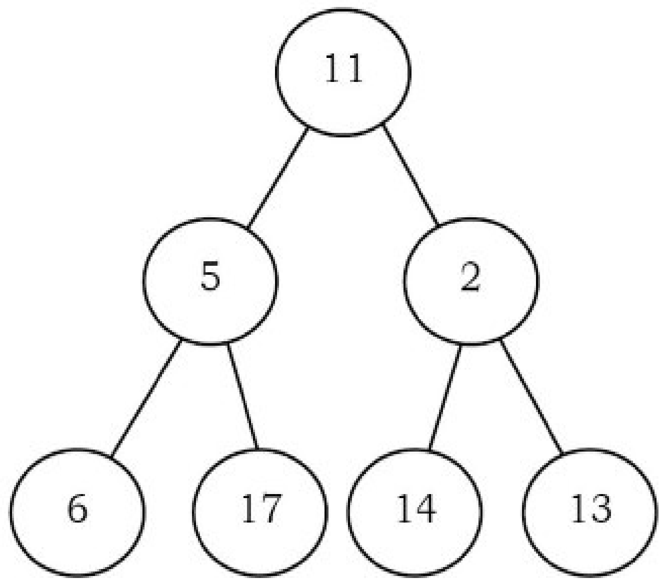
What is a Heap?

A heap is a tree with some special properties. The basic requirement of a heap is that the value of a node must be > (or <) than the values of its children. This is called *heap property*. A heap also has the additional property that all leaves should be at h or $h – 1$ levels (where h is the height of the tree) for some $h > 0$ (*complete binary trees*). That means heap should form a *complete binary tree* (as shown below).



In the examples below, the left tree is a heap (each element is greater than its children) and the right tree is not a heap (since 11 is greater than 2).

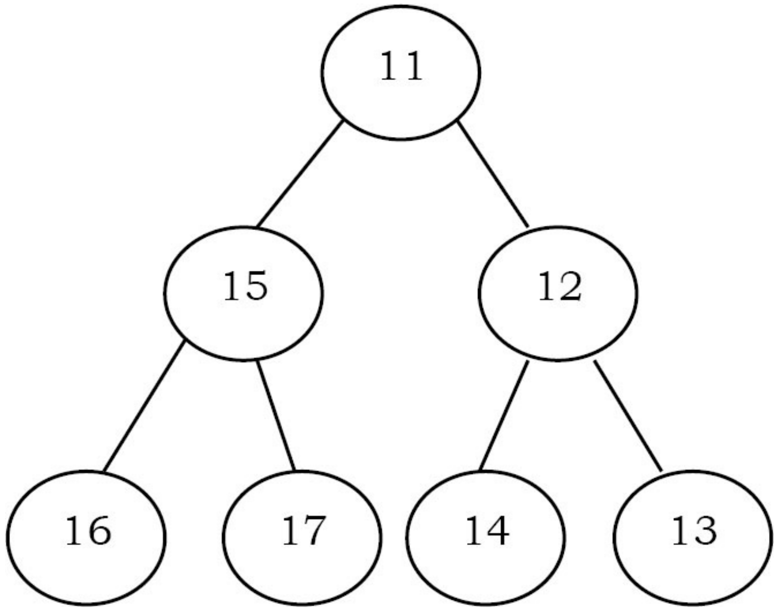




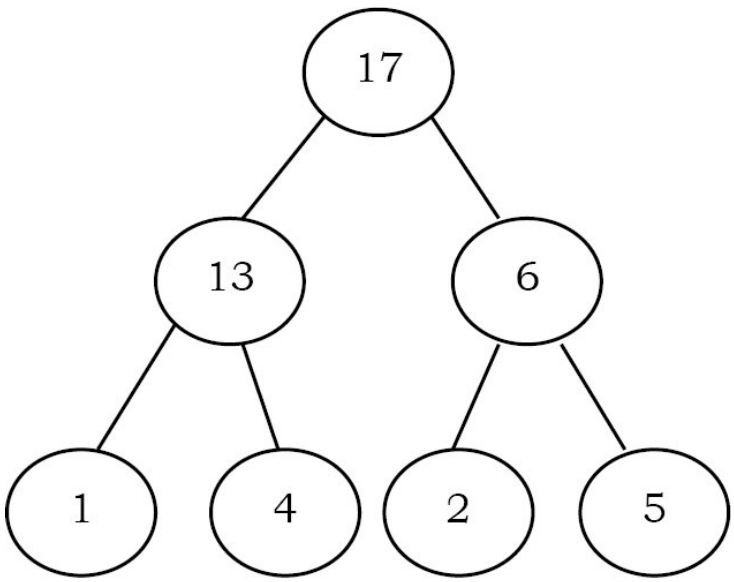
Types of Heaps?

Based on the property of a heap we can classify heaps into two types:

- **Min heap:** The value of a node must be less than or equal to the values of its children



- **Max heap:** The value of a node must be greater than or equal to the values of its children



7.6 Binary Heaps

In binary heap each node may have up to two children. In practice, binary heaps are enough and we concentrate on binary min heaps and binary max heaps for the remaining discussion.

Representing Heaps: Before looking at heap operations, let us see how heaps can be represented. One possibility is using arrays. Since heaps are forming complete binary trees, there will not be any wastage of locations. For the discussion below let us assume that elements are stored in arrays, which starts at index 0. The previous max heap can be represented as:

17	13	6	1	4	2	5
0	1	2	3	4	5	6

Note: For the remaining discussion let us assume that we are doing manipulations in max heap.

Declaration of Heap

```
public class  Heap {
      public int[] array;
      public int count;                                    // Number of elements in Heap
      public int capacity;                                 // Size of the heap
      public int heap_type;                                // Min Heap or Max Heap
      public Heap(int capacity, int heap_type)            { //Refer Below sections }
      public Parent(int capacity, int heap_type)          { //Refer Below sections }
      public int LeftChild(int i)                          {//Refer Below sections }
      public int RightChild(int i)                         {//Refer Below sections }
      public int GetMaximum(int i)                         {//Refer Below sections }
      .........
}
```

Note: Assume all the below functions are part of class.

Creating Heap

```
public Heap(int capacity, int heap_type) {
      this.heap_type = heap_type;
      this.count = 0;
      this.capacity = capacity;
      this.array = new int[capacity];
}
```

Time Complexity: O(1).

Parent of a Node

For a node at *ith* location, its parent is at $\frac{i-1}{2}$ location. In the previous example, the element 6 is at second location and its parent is at 0*th* location.

```
public int Parent (int i) {
    if(i <= 0 || i >= this.count)
            return -1;
    return i-1/2;
}
```

Time Complexity: O(1).

Children of a Node

Similar to the above discussion, for a node at *ith* location, its children are at 2 * *i* + 1 and 2 * *i* + 2 locations. For example, in the above tree the element 6 is at second location and its children 2 and 5 are at5(2 * *i* + l = 2 * 2 + 1) and 6(2 * *i* + 2 = 2 * 2 + 2) locations.

```
public int LeftChild(int i) {
    int left = 2 * i + 1;
    if(left >= this.count)
        return -1;
    return left;
}
Time Complexity: O(1).
```

```
public int RightChild(int i) {
    int right = 2 * i + 2;
    if(right >= this.count)
        return -1;
    return right;
}
Time Complexity: O(1).
```

Getting the Maximum Element

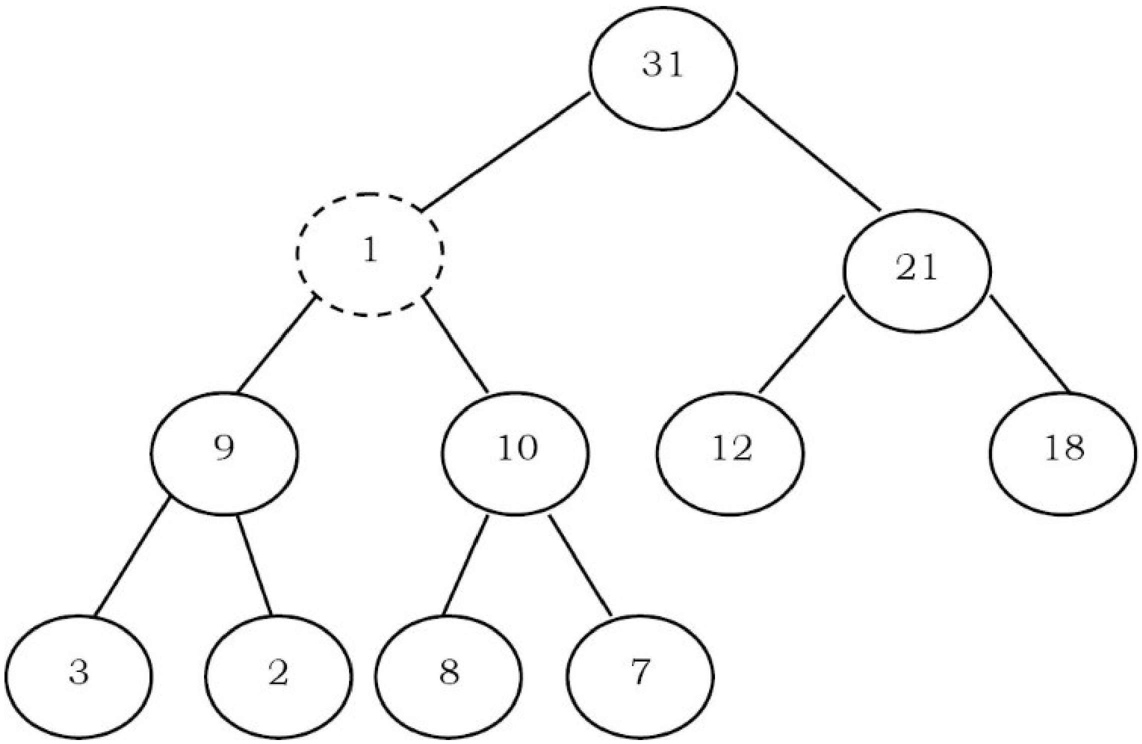
Since the maximum element in max heap is always at root, it will be stored at this.array[0].

```
public int GetMaximum() {
    if(this.count == 0)
        return -1;
    return this.array[0];
}
```

Time Complexity: O(1).

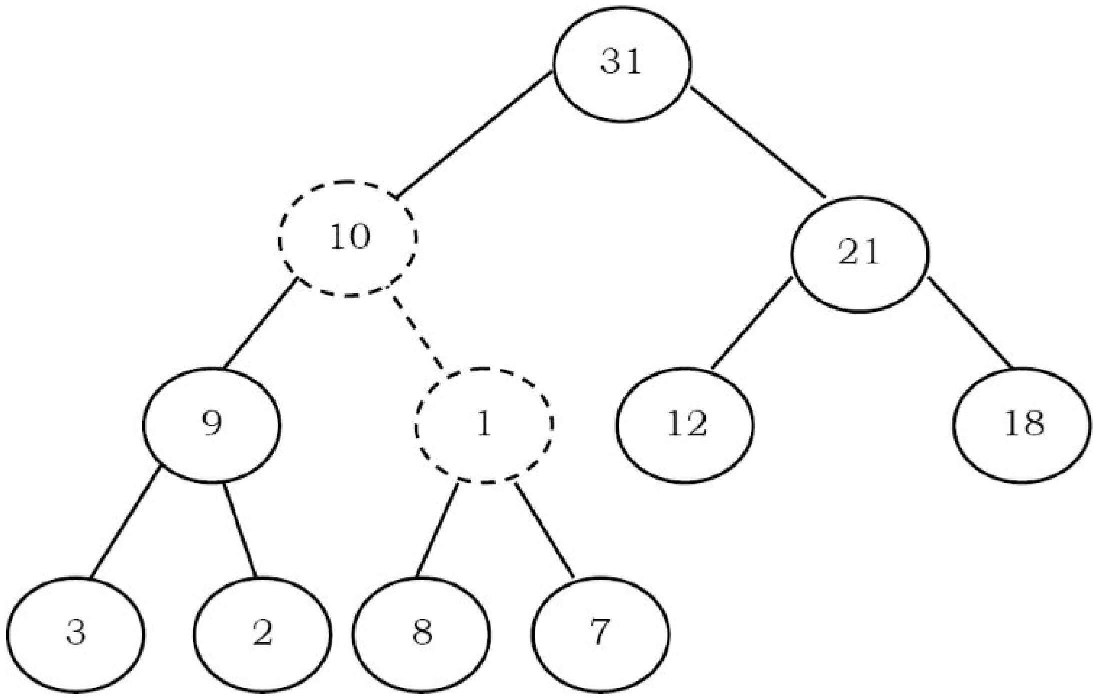
Heapifying an Element

After inserting an element into heap, it may not satisfy the heap property. In that case we need to adjust the locations of the heap to make it heap again. This process is called *heapifying*. In max-heap, to heapify an element, we have to find the maximum of its children and swap it with the current element and continue this process until the heap property is satisfied at every node.

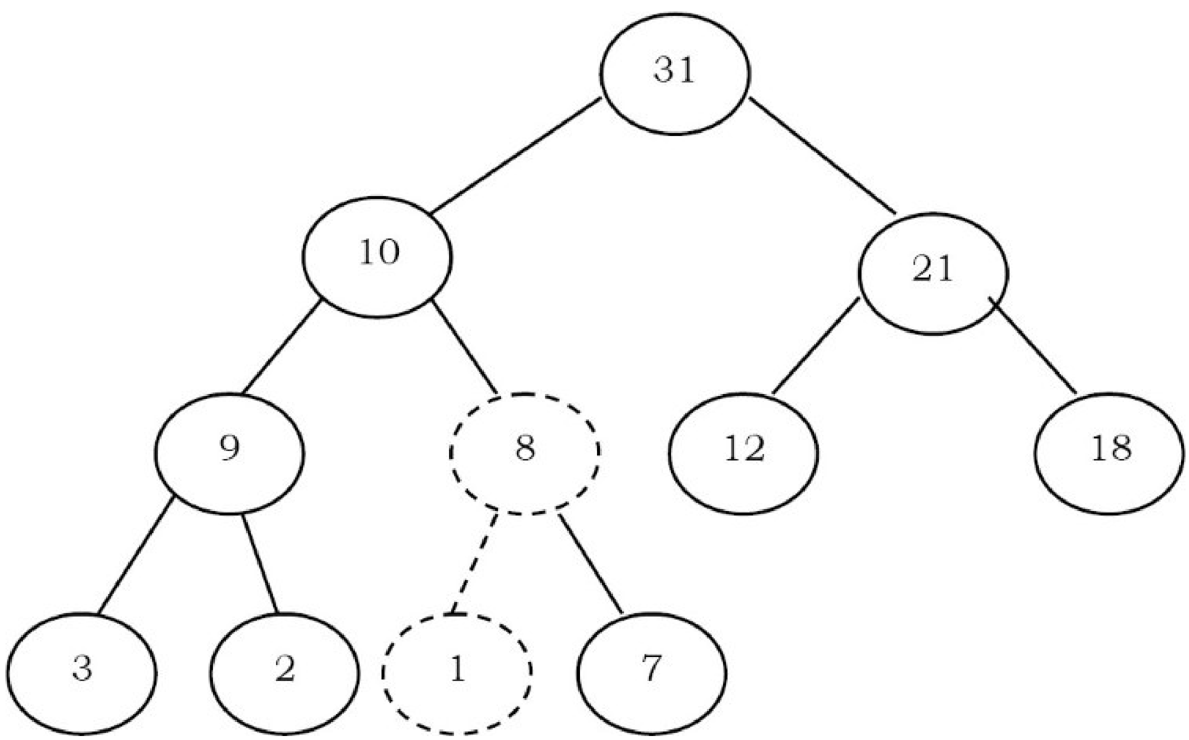


Observation: One important property of heap is that, if an element is not satisfying the heap property, then all the elements from that element to the root will have the same problem. In the example below, element 1 is not satisfying the heap property and its parent 31 is also having the issue. Similarly, if we heapify an element, then all the elements from that element to the root will also satisfy the heap property automatically. Let us go through an example. In the above heap, the element 1 is not satisfying the heap property. Let us try heapifying this element.

To heapify 1, find the maximum of its children and swap with that.



We need to continue this process until the element satisfies the heap properties. Now, swap 1 with 8.



Now the tree is satisfying the heap property. In the above heapifying process, since we are moving from top to bottom, this process is sometimes called *percolate down*. Similarly, if we start heapifying from any other node to root, we can that process *percolate up* as move from bottom to top.

```
//Heapifying the element at location i.
public void PercolateDown(int i) {
      int l, r, max, temp;
      l = LeftChild(i);
      r = RightChild(i);
      if(l != -1 && this.array[l] > this.array[i])
           max = l;
      else  max = i;
      if(r != -1&& this.array[r] > this.array[max])
           max = r;
      if(max != i) {  //Swap this.array[i] and  this.array[max];
           temp = this.array[i]; this.array[i] = this.array[max];
           this.array[max] = temp;
      }
      PercolateDown(max);
}
```

Time Complexity: O(*logn*). Heap is a complete binary tree and in the worst case we start at the root and come down to the leaf. This is equal to the height of the complete binary tree. Space Complexity: O(1).

Deleting an Element

To delete an element from heap, we just need to delete the element from the root. This is the only operation (maximum element) supported by standard heap. After deleting the root element, copy the last element of the heap (tree) and delete that last element.

After replacing the last element, the tree may not satisfy the heap property. To make it heap again, call the *PercolateDown* function.

- Copy the first element into some variable
- Copy the last element into first element location
- *PercolateDown* the first element

```
int DeleteMax() {
      if(this.count == 0)
              return -1;
      int data = this.array[0];
      this.array[0] = this.array[this.count-1];
      this.count--; //reducing the heap size
      PercolateDown(0);
      return data;
}
```

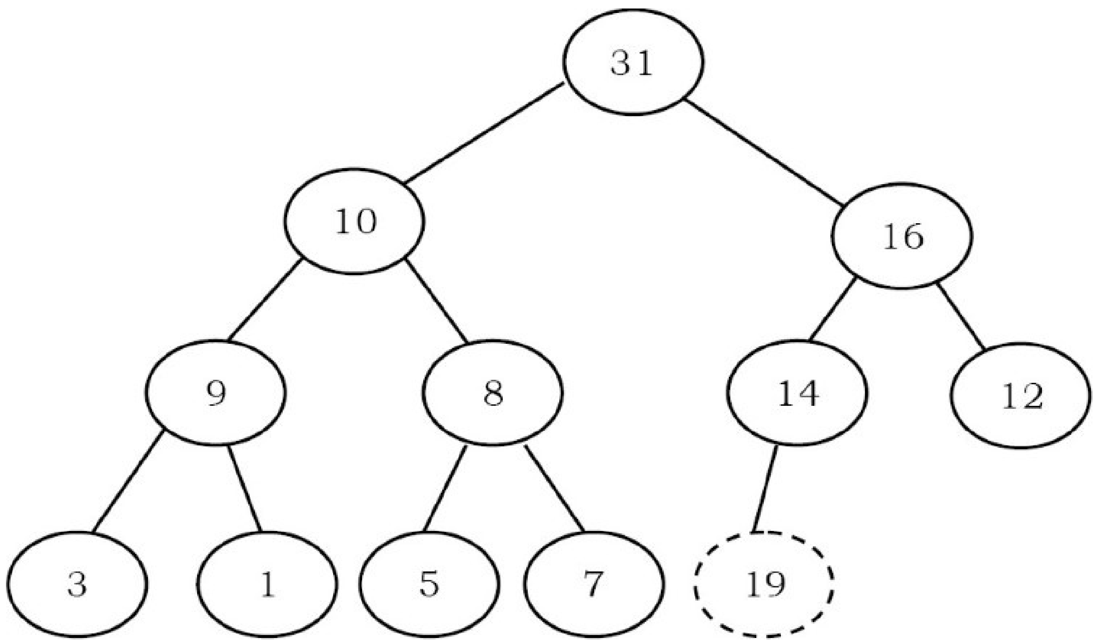
Note: Deleting an element uses *PercolateDown*, and inserting an element uses *PercolateUp*.
Time Complexity: same as Heapify function and it is O(*logn*).

Inserting an Element

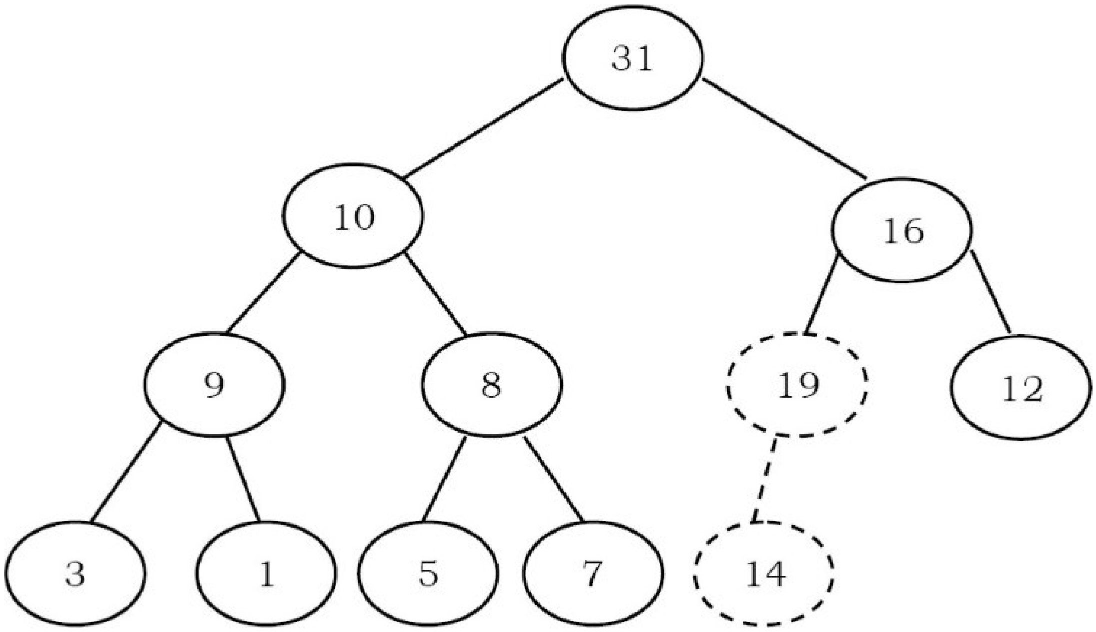
Insertion of an element is similar to the heapify and deletion process.

- Increase the heap size
- Keep the new element at the end of the heap (tree)
- Heapify the element from bottom to top (root)

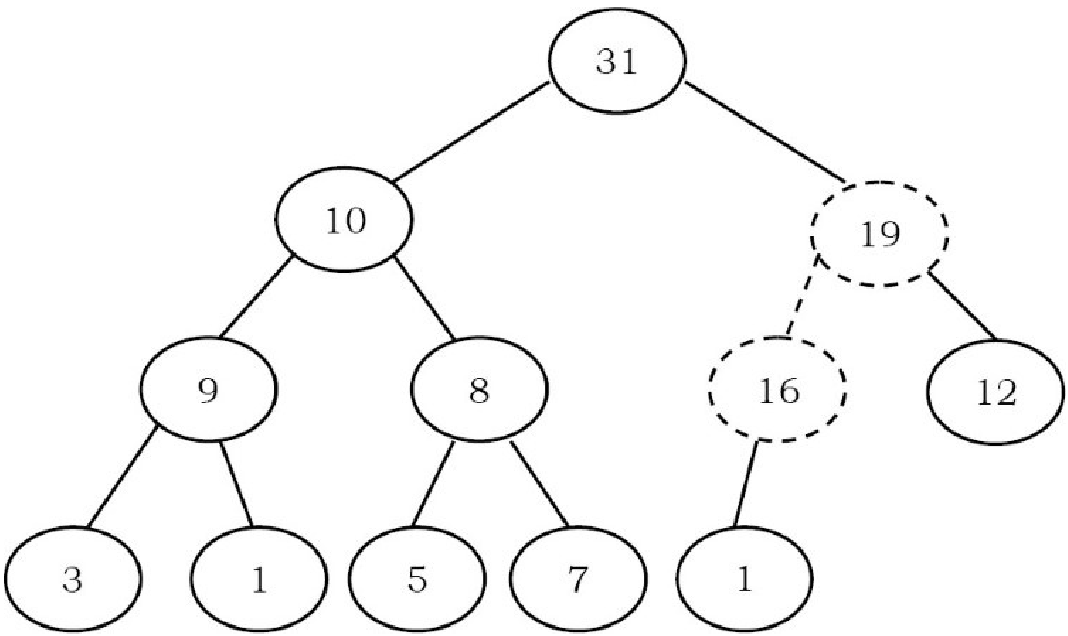
Before going through code, let us look at an example. We have inserted the element 19 at the end of the heap and this is not satisfying the heap property.



In order to heapify this element (19), we need to compare it with its parent and adjust them. Swapping 19 and 14 gives:



Again, swap 19 andl6:



Now the tree is satisfying the heap property. Since we are following the bottom-up approach we sometimes call this process *percolate up*.

```
public int Insert(int data) {
    int i;
    if(this.count == this.capacity)
        ResizeHeap();
    this.count++;          //increasing the heap size to hold this new item
    i = this.count-1;
    while(i >=0 && data > this.array[(i-1)/2]) {
        this.array[i] = this.array[(i-1)/2];
        i = i-1/2;
    }
    this.array[i] = data;
}
public void ResizeHeap() {
    int[] array_old = new int[this.capacity];
    System.arraycopy(this.array, 0, array_old, this.count-1);
    this.array = new int[this.capacity * 2];
    if(this.array == null) {
        System.out.println("Memory Error");
        return;
    }
    for (int i = 0; i < this.capacity; i ++)
        this.array[i] = array_old[i];
    this.capacity *= 2;
    array_old = null;
}
```

Time Complexity: O(*logn*). The explanation is the same as that of the *Heapify* function.

Destroying Heap

```
public void DestroyHeap() {
    this.count = 0;
    this.array = null;
}
```

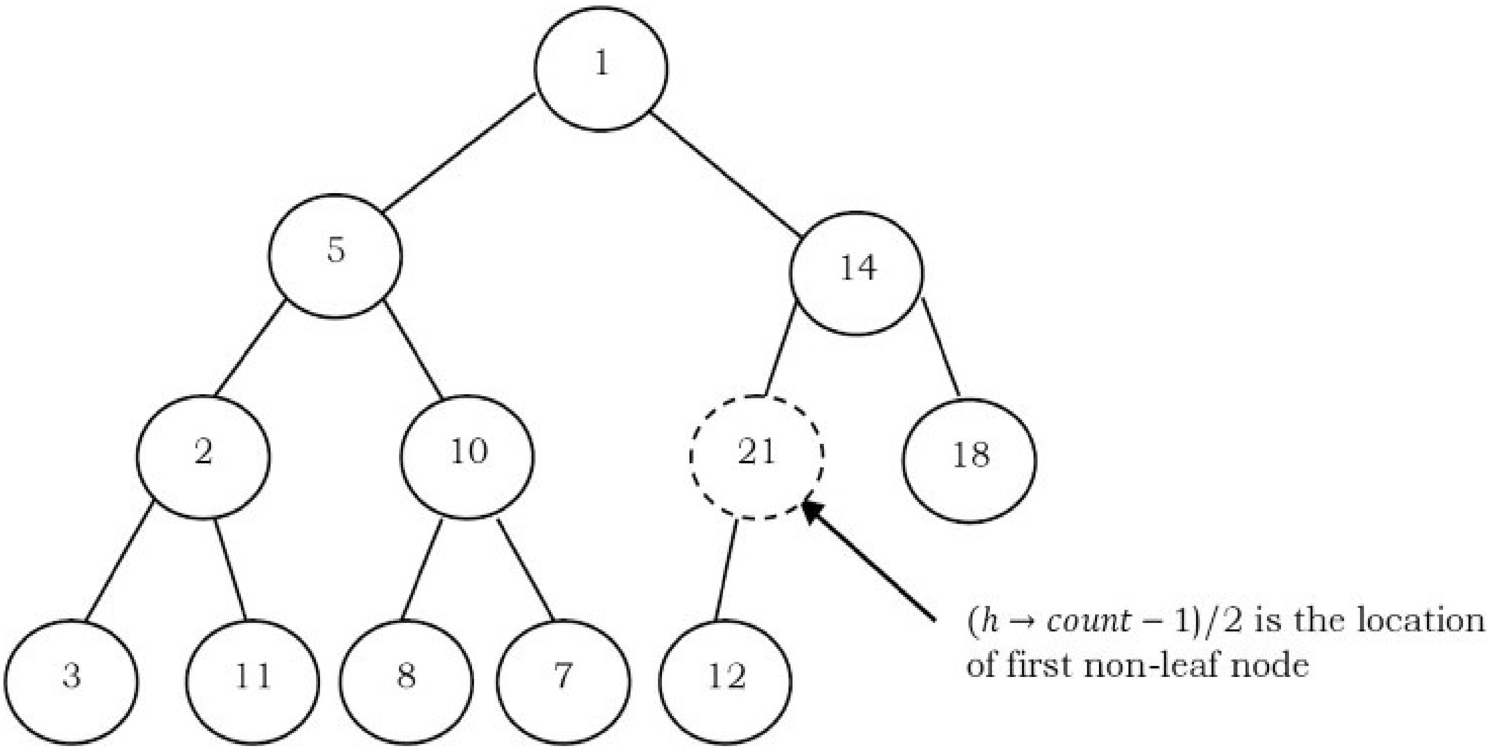
Heapifying the Array

One simple approach for building the heap is, take *n* input items and place them into an empty heap. This can be done with *n* successive inserts and takes O(*nlogn*) in the worst case. This is due to the fact that each insert operation takes O(*logn*).

To finish our discussion of binary heaps, we will look at a method to build an entire heap from a list of keys. The first method you might think of may be like the following. Given a list of keys, you could easily build a heap by inserting each key one at a time. Since you are starting with a list of one item, the list is sorted and you could use binary search to find the right position to insert the next key at a cost of approximately O(*logn*) operations. However, remember that inserting an item in the middle of the list may require O(*n*) operations to shift the rest of the list over to make room for the new key. Therefore, to insert *n* keys into the heap would require a total of O(*nlogn*) operations. However, if we start with an entire list then we can build the whole heap in O(*n*) operations.

Observation: Leaf nodes always satisfy the heap property and do not need to care for them. The leaf elements are always at the end and to heapify the given array it should be enough if we heapify the non-leaf nodes. Now let us concentrate on finding the first non-leaf node. The last element of the heap is at location $h \rightarrow count - 1$, and to find the first non-leaf node it is enough to find the parent of the last element.

```
public void BuildHeap(Heap h, int[] A, int n) {
    if(h == null) return;
    while (n > this.capacity)
        h.ResizeHeap();
    for (int i = 0; i < n; i ++)
        h.array[i] = A[i];
    this.count = n;
    for (int i = (n-1)/2; i >=0; i --)
        h.PercolateDown(i);
}
```



Time Complexity: The linear time bound of building heap can be shown by computing the sum of the heights of all the nodes. For a complete binary tree of height h containing $n = 2^{h+1} - 1$ nodes, the sum of the heights of the nodes is $n - h - 1 = n - logn - 1$ (for proof refer to *Problems Section*). That means, building the heap operation can be done in linear time (O(n)) by applying a *PercolateDown* function to the nodes in reverse level order.

Heapsort

One main application of heap ADT is sorting (heap sort). The heap sort algorithm inserts all elements (from an unsorted array) into a heap, then removes them from the root of a heap until the heap is empty. Note that heap sort can be done in place with the array to be sorted. Instead of deleting an element, exchange the first element (maximum) with the last element and reduce the heap size (array size). Then, we heapify the first element. Continue this process until the number of remaining elements is one.

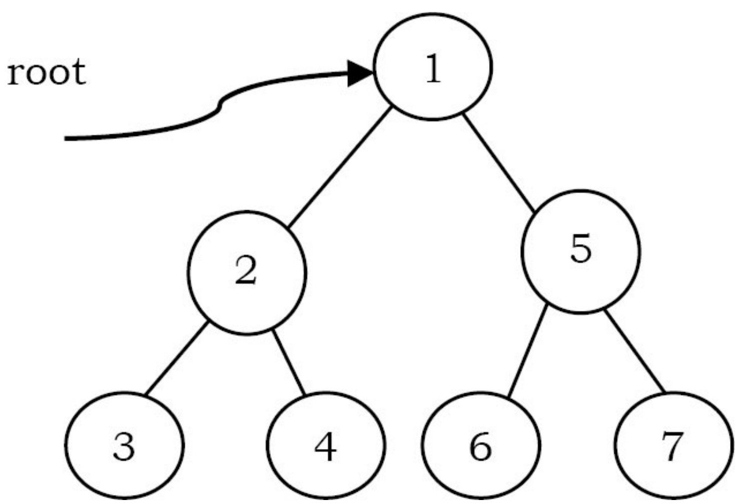
```
public void Heapsort(int[] A, in n) {
    Heap h = new Heap(n, 0);
    int old_size, i, temp;
    BuildHeap(h, A, n);
    old_size = h.count;
    for(i = n-1; i > 0; i--) { //h.array[0] is the largest element
        temp = h.array[0]; h.array[0] = h.array[h.count-1];
        h.count--;
        h.PercolateDown(0);
    }
    h.count = old_size;
}
```

Time complexity: As we remove the elements from the heap, the values become sorted (since maximum elements are always *root* only). Since the time complexity of both the insertion algorithm and deletion algorithm is O(*logn*) (where *n* is the number of items in the heap), the time complexity of the heap sort algorithm is O(*nlogn*).

7.7 Priority Queues [Heaps]: Problems 85 Solutions

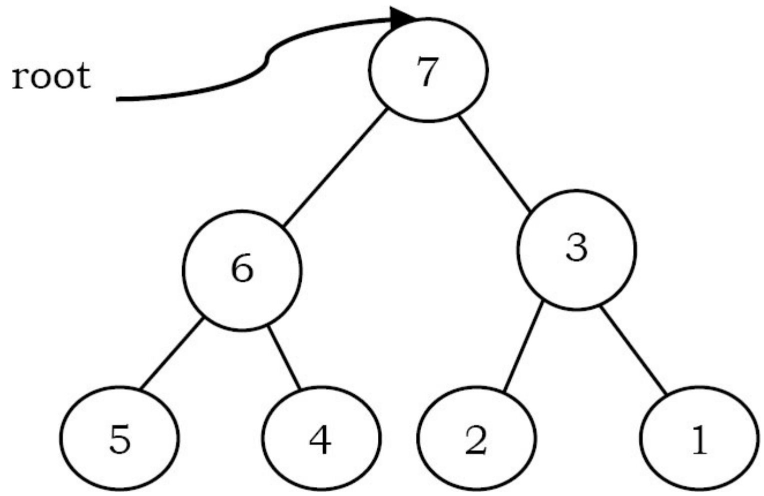
Problem-1 Is there a min-heap with seven distinct elements so that the preorder traversal of it gives the elements in sorted order?

Solution: Yes. For the tree below, preorder traversal produces ascending order.



Problem-2 Is there a max-heap with seven distinct elements so that the preorder traversal of it gives the elements in sorted order?

Solution: Yes. For the tree below, preorder traversal produces descending order.

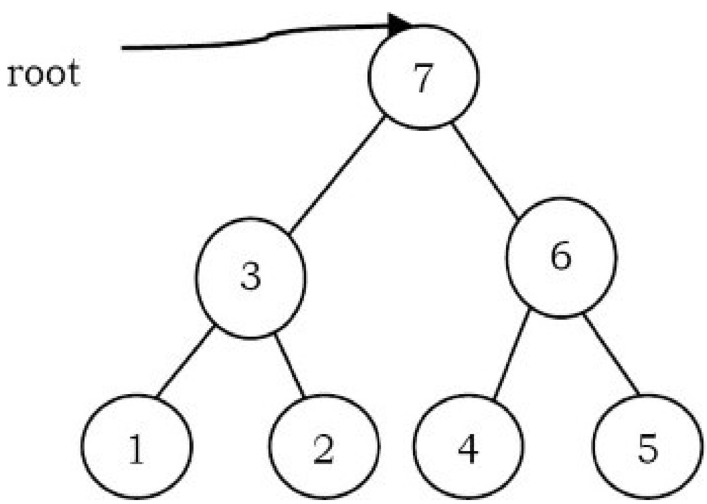


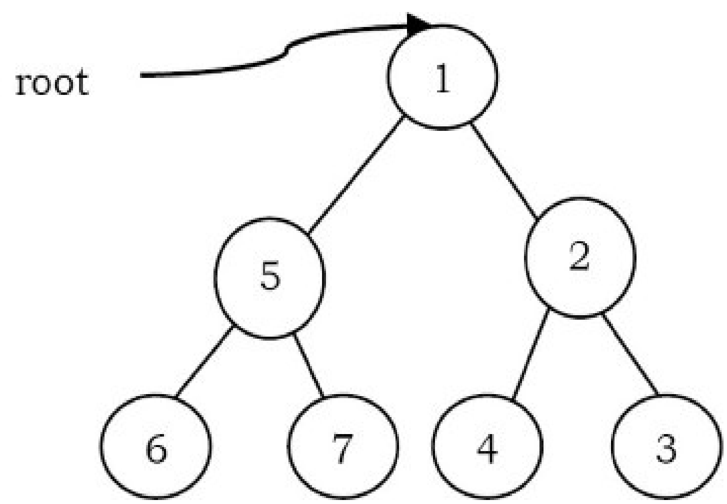
Problem-3 Is there a min-heap/max-heap with seven distinct elements so that the inorder traversal of it gives the elements in sorted order?

Solution: No. Since a heap must be either a min-heap or a max-heap, the root will hold the smallest element or the largest. An inorder traversal will visit the root of the tree as its second step, which is not the appropriate place if the tree's root contains the smallest or largest element.

Problem-4 Is there a min-heap/max-heap with seven distinct elements so that, the postorder traversal of it gives the elements in sorted order?

Solution: Yes, if the tree is a max-heap and we want descending order (below left), or if the tree is a min-heap and we want ascending order (below right).





Problem-5 What are the minimum and maximum number of elements in a heap of height h?

Solution: Since heap is a complete binary tree (all levels contain full nodes except possibly the lowest level), it has at most $2^{h+1} - 1$ elements (if it is complete). This is because, to get maximum nodes, we need to fill all the h levels completely and the maximum number of nodes is nothing but the sum of all nodes at all h levels. To get minimum nodes, we should fill the $h - 1$ levels fully and the last level with only one element. As a result, the minimum number of nodes is nothing but the sum of all nodes from $h - 1$ levels plus 1 (for the last level) and we get $2^h - 1 + 1 = 2^h$ elements (if the lowest level has just 1 element and all the other levels are complete).

Problem-6 Show that the height of a heap with n elements is $logn$?

Solution: A heap is a complete binary tree. All the levels, except the lowest, are completely full. A heap has at least 2^h elements and at most elements $2^h \leq n \leq 2^{h+1} - 1$. This implies, $h \leq logn \leq h + 1$. Since h is an integer, $h = logn$.

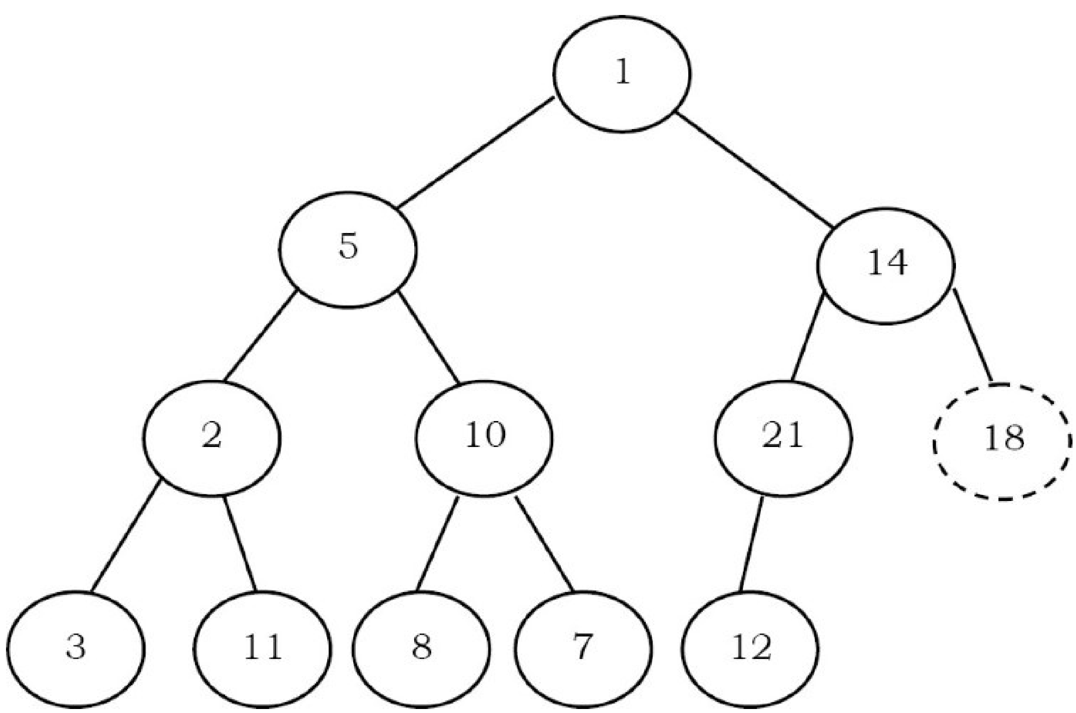
Problem-7 Given a min-heap, give an algorithm for finding the maximum element.

Solution: For a given min heap, the maximum element will always be at leaf only. Now, the next question is how to find the leaf nodes in the tree. If we carefully observe, the next node of the last element's parent is the first leaf node. Since the last element is always at the $h \rightarrow count - 1^{th}$ location, the next node of its parent (parent at location $\frac{h \rightarrow count - 1}{2}$) can be calculated as:

$$\frac{h \rightarrow count - 1}{2} + 1 \approx \frac{h \rightarrow count + 1}{2}$$

Now, the only step remaining is scanning the leaf nodes and finding the maximum among them.

```
public int FindMaxInMinHeap(Heap h) {
      int Max = -1;
      for(int i = (h.count+1)/2; i < h.count; i++)
            if(h→array[i] > Max)
                  Max = h.array[i];
}
```



Time Complexity: $O(\frac{n}{2}) \approx O(n)$.

Problem-8 Give an algorithm for deleting an arbitrary element from min heap.

Solution: To delete an element, first we need to search for an element. Let us assume that we are using level order traversal for finding the element. After finding the element we need to follow the DeleteMin process.

Time Complexity = Time for finding the element + Time for deleting an element
= $O(n) + O(logn) \approx O(n)$. //Time for searching is dominated.

Problem-9 Give an algorithm for deleting the i^{th} indexed element in a given min-heap.

Solution:

```
public int Delete(Heap h, int i) {
    int key;
    if(n < i) {
      System.out.println("Wrong position");
      return;
    }
    key = h.array[i];
    h.array[i]= h.array[h.count-1];
    h.count--;
    h.PercolateDown(i);
    return key;
}
```

Time Complexity = $O(logn)$.

Problem-10 Prove that, for a complete binary tree of height h the sum of the height of all nodes is $O(n - h)$.

Solution: A complete binary tree has 2^i nodes on level (.Also, a node on level i has depth i and height $h - i$. Let us assume that S denotes the sum of the heights of all these nodes and S can be calculated as:

$$S = \sum_{i=0}^{h} 2^i(h-i)$$

$$S = h + 2(h-1) + 4(h-2) + \cdots + 2^{h-1}(1)$$

Multiplying with 2 on both sides gives: $2S = 2h + 4(h-1) + 8(h-2) + \cdots + 2^h(1)$

Now, subtract S from $2S$: $2S - S = -h + 2 + 4 + \cdots + 2^h \Rightarrow S = (2^{h+1} - 1) - (h - 1)$

But, we already know that the total number of nodes n in a complete binary tree with height h is $n = 2^{h+1} - 1$. This gives us: $h = \log(n + 1)$.

Finally, replacing $2^{h+1} - 1$ with n, gives: $S = n - (h - 1) = O(n - logn) = O(n - h)$.

Problem-11 Give an algorithm to find all elements less than some value of k in a binary heap.

Solution: Start from the root of the heap. If the value of the root is smaller than k then print its value and call recursively once for its left child and once for its right child. If the value of a node is greater or equal than k then the function stops without printing that value.

The complexity of this algorithm is $O(n)$, where n is the total number of nodes in the heap. This bound takes place in the worst case, where the value of every node in the heap will be smaller than k, so the function has to call each node of the heap.

Problem-12 Give an algorithm for merging two binary max-heaps. Let us assume that the size of the first heap is $m + n$ and the size of the second heap is n.

Solution: One simple way of solving this problem is:

- Assume that the elements of the first array (with size $m + n$) are at the beginning. That means, first m cells are filled and remaining n cells are empty.
- Without changing the first heap, just append the second heap and heapify the array.
- Since the total number of elements in the new array is $m + n$, each heapify operation takes $O(log(m + n))$.

The complexity of this algorithm is : $O((m + n)log(m + n))$.

Problem-13 Can we improve the complexity of Problem-12?

Solution: Instead of heapifying all the elements of the $m + n$ array, we can use the technique of "building heap with an array of elements (heapifying array)". We can start with non-leaf nodes and heapify them. The algorithm can be given as:

- Assume that the elements of the first array (with size $m + n$) are at the beginning. That means, the first m cells are filled and the remaining n cells are empty.
- Without changing the first heap, just append the second heap.
- Now, find the first non-leaf node and start heapifying from that element.

In the theory section, we have already seen that building a heap with n elements takes $O(n)$ complexity. The complexity of merging with this technique is: $O(m + n)$.

Problem-14 Is there an efficient algorithm for merging 2 max-heaps (stored as an array)? Assume both arrays have n elements.

Solution: The alternative solution for this problem depends on what type of heap it is. If it's a standard heap where every node has up to two children and which gets filled up so that the leaves are on a maximum of two different rows, we cannot get better than $O(n)$ for the merge.

There is an $O(logm \times logn)$ algorithm for merging two binary heaps with sizes m and n. For $m = n$, this algorithm takes $O(log^2 n)$ time complexity. We will be skipping it due to its difficulty and scope.

For better merging performance, we can use another variant of binary heap like a *Fibonacci-Heap* which can merge in O(1) on average (amortized).

Problem-15 Give an algorithm for finding the k^{th} smallest element in min-heap.

Solution: One simple solution to this problem is: perform deletion *k* times from min-heap.

```
public int FindKthLargestEle(Heap h, int k) {
      //Just delete first k-1 elements and return the k-th element.
    for(int i=0;i<k-1;i++)
          h.DeleteMin();
    return h.DeleteMin();
}
```

Time Complexity: O($klogn$). Since we are performing deletion operation k times and each deletion takes O($logn$).

Problem-16 For Problem-15, can we improve the time complexity?

Solution: Assume that the original min-heap is called *HOrig* and the auxiliary min-heap is named *HAux*. Initially, the element at the top *of HOrig*, the minimum one, is inserted into *HAux*. Here we don't do the operation of DeleteMin with *HOrig*.

```
Heap HOrig, HAux;
public int FindKthLargestEle( int k ) {
      int heapElement;//Assuming heap data is of integers
      int count=1;
      HAux.Insert(HOrig.DeleteMin());
      while( true ) {
            //return the minimum element and delete it from the HA heap
            heapElement = HAux.DeleteMin();
            if(++count == k ) {
                  return heapElement;
            }
            else {     //insert the left and right children in HO into the HA
                  HAux.Insert(heapElement.LeftChild());
                 HAux.Insert(heapElement.RightChild());
            }
      }
}
```

Every while-loop iteration gives the k^{th} smallest element and we need *k* loops to get the k^{th}

smallest elements. Because the size of the auxiliary heap is always less than k, every while-loop iteration the size of the auxiliary heap increases by one, and the original heap *HOrig* has no operation during the finding, the running time is O($klogk$).

Note: The above algorithm is useful if the k value is too small compared to n. If the k value is approximately equal to n, then we can simply sort the array (let's say, using *couting* sort or any other linear sorting algorithm) and return k^{th} smallest element from the sorted array. This gives O(n) solution.

Problem-17 Find k max elements from max heap.

Solution: One simple solution to this problem is: build max-heap and perform deletion k times.

$$T(n) = \text{DeleteMin from heap } k \text{ times} = \Theta(klogn).$$

Problem-18 For Problem-17, is there any alternative solution?

Solution: We can use the Problem-16 solution. At the end, the auxiliary heap contains the k-largest elements. Without deleting the elements we should keep on adding elements to *HAux*.

Problem-19 How do we implement stack using heap?

Solution: To implement a stack using a priority queue PQ (using min heap), let us assume that we are using one extra integer variable c. Also, assume that c is initialized equal to any known value (e.g., 0). The implementation of the stack ADT is given below. Here c is used as the priority while inserting/deleting the elements from PQ.

```
public void Push(int element) {
	PQ.Insert(c, element);
	c--;
}
public  int Pop() {
	return PQ.DeleteMin();
}
public int Top() {
	return PQ.Min();
}
public int Size() {
	return PQ.Size();
}
public int isEmpty() {
	return PQ.isEmpty();
}
```

Note: We could also increment c back when popping.

Observation: We could use the negative of the current system time instead of *c* (to avoid overflow). The implementation based on this can be given as:

```
public void Push(int element) {
      PQ.insert(-gettime(),element);
}
```

Problem-20 How do we implement Queue using heap?

Solution: To implement a queue using a priority queue PQ (using min heap), as similar to stacks simulation, let us assume that we are using one extra integer variable, *c*. Also, assume that *c* is initialized equal to any known value (e.g., 0). The implementation of the queue ADT is given below. Here the *c* is used as the priority while inserting/deleting the elements from PQ.

```
public void Push(int element) {
      PQ.Insert(c, element);
      c++;
}
public int Pop() {
      return PQ.DeleteMin();
}
public int Top() {
      return PQ.Min();
}
public int Size() {
      return PQ.Size();
}
public int isEmpty() {
      return PQ.isEmpty();
}
```

Note: We could also decrement *c* when popping.

Observation: We could use just the negative of the current system time instead of *c* (to avoid overflow). The implementation based on this can be given as:

```
public void Push(int element) {
      PQ.insert(gettime(),element);
}
```

Note: The only change is that we need to take a positive *c* value instead of negative.

Problem-21 Given a big file containing billions of numbers, how can you find the 10 maximum numbers from that file?

Solution: Always remember that when you need to find max *n* elements, the best data structure to use is priority queues.

One solution for this problem is to divide the data in sets of 1000 elements (let's say 1000) and make a heap of them, and then take 10 elements from each heap one by one. Finally heap sort all the sets of 10 elements and take the top 10 among those. But the problem in this approach is where to store 10 elements from each heap. That may require a large amount of memory as we have billions of numbers.

Reusing the top 10 elements (from the earlier heap) in subsequent elements can solve this problem. That means take the first block of 1000 elements and subsequent blocks of 990 elements each. Initially, Heapsort the first set of 1000 numbers, take max 10 elements, and mix them with 990 elements of the 2^{nd} set. Again, Heapsort these 1000 numbers (10 from the first set and 990 from the 2^{nd} set), take 10 max elements, and mix them with 990 elements of the 3^{rd} set. Repeat till the last set of 990 (or less) elements and take max 10 elements from the final heap. These 10 elements will be your answer.

Time Complexity: $O(n) = n/1000 \times$(complexity of Heapsort 1000 elements) Since complexity of heap sorting 1000 elements will be a constant so the $O(n) = n$ i.e. linear complexity.

Problem-22 **Merge *k* sorted lists with total of *n* elements:** We are given *k* sorted lists with total *n* inputs in all the lists. Give an algorithm to merge them into one single sorted list.

Solution: Since there are *k* equal size lists with a total of *n* elements, the size of each list is $\frac{n}{k}$. One simple way of solving this problem is:

- Take the first list and merge it with the second list. Since the size of each list is $\frac{n}{k}$, this step produces a sorted list with size $\frac{2n}{k}$. This is similar to merge sort logic. The time complexity of this step is: $\frac{2n}{k}$. This is because we need to scan all the elements of both the lists.
- Then, merge the second list output with the third list. As a result, this step produces a sorted list with size $\frac{3n}{k}$. The time complexity of this step is: $\frac{3n}{k}$. This is because we need to scan all the elements of both lists (one with size $\frac{2n}{k}$ and the other with size $\frac{n}{k}$).
- Continue this process until all the lists are merged to one list.

Total time complexity: $= \frac{2n}{k} + \frac{3n}{k} + \frac{4n}{k} + \cdots . \frac{kn}{k} = \sum_{i=2}^{n} \frac{in}{k} = \frac{n}{k} \sum_{i=2}^{n} i \approx \frac{n(k^2)}{k} \approx O(nk)$.
Space Complexity: O(1).

Problem-23 For the Problem-22, can we improve the time complexity?

Solution:

1 Divide the lists into pairs and merge them. That means, first take two lists at a time

and merge them so that the total elements parsed for all lists is O(n). This operation gives k/2 lists.

2 Repeat step-1 until the number of lists becomes one.

Time complexity: Step-1 executes $logk$ times and each operation parses all n elements in all the lists for making k/2 lists. For example, if we have 8 lists, then the first pass would make 4 lists by parsing all n elements. The second pass would make 2 lists by again parsing n elements and the third pass would give 1 list by again parsing n elements. As a result the total time complexity is O($nlogn$).
Space Complexity: O(n).

Problem-24 For Problem-23, can we improve the space complexity?

Solution: Let us use heaps for reducing the space complexity.

1. Build the max-heap with all the first elements from each list in O(k).
2. In each step, extract the maximum element of the heap and add it at the end of the output.
3. Add the next element from the list of the one extracted. That means we need to select the next element of the list which contains the extracted element of the previous step.
4. Repeat step-2 and step-3 until all the elements are completed from all the lists.

Time Complexity = O($nlogk$). At a time we have k elements max-heap and for all n elements we have to read just the heap in $logk$ time, so total time = O($nlogk$).
Space Complexity: O(k) [for Max-heap].

Problem-25 Given 2 arrays A and B each with n elements. Give an algorithm for finding largest n pairs ($A[i]$, $B[j]$).

Solution:

Algorithm:

- Heapify A and B. This step takes O(2n) ≈O(n).
- Then keep on deleting the elements from both the heaps. Each step takes O(2$logn$) ≈O($logn$).

Total Time complexity: O($nlogn$).

Problem-26 **Min-Max heap:** Give an algorithm that supports min and max in O(1) time, insert, delete min, and delete max in O($logn$) time. That means, design a data structure which supports the following operations:

Operation	Complexity
Init	O(n)
Insert	O($logn$)

FindMin	O(1)
FindMax	O(1)
Delete Min	O(*logn*)
Delete Max	O(*logn*)

Solution: This problem can be solved using two heaps. Let us say two heaps are: Minimum-Heap H_{min} and Maximum-Heap H_{max}. Also, assume that elements in both the arrays have mutual pointers. That means, an element in H_{min} will have a pointer to the same element in H_{max} and an element in H_{max} will have a pointer to the same element in H_{min}.

Init	Build H_{min} in O(*n*) and H_{max} in O(*n*)
Insert(x)	Insert x to H_{min} in O(*logn*). Insert x to H_{max} in O(*logn*). Update the pointers in O(1)
FindMin()	Return root(H_{min}) in O(1)
FindMax	Return root(H_{max}) in O(1)
DeleteMin	Delete the minimum from H_{min} in O(*logn*). Delete the same element from H_{max} by using the mutual pointer in O(*logn*)
DeleteMax	Delete the maximum from H_{max} in O(*logn*). Delete the same element from H_{min} by using the mutual pointer in O(*logn*)

Problem-27 Dynamic median finding. Design a heap data structure that supports finding the median.

Solution: In a set of *n* elements, median is the middle element, such that the number of elements lesser than the median is equal to the number of elements larger than the median. If *n* is odd, we can find the median by sorting the set and taking the middle element. If *n* is even, the median is usually defined as the average of the two middle elements. This algorithm works even when some of the elements in the list are equal. For example, the median of the multiset {1, 1, 2, 3, 5} is 2, and the median of the multiset {1, 1, 2, 3, 5, 8} is 2.5.

"Median heaps" are the variant of heaps that give access to the median element. A median heap can be implemented using two heaps, each containing half the elements. One is a max-heap, containing the smallest elements; the other is a min-heap, containing the largest elements. The size of the max-heap may be equal to the size of the min-heap, if the total number of elements is even. In this case, the median is the average of the maximum element of the max-heap and the minimum

element of the min-heap. If there is an odd number of elements, the max-heap will contain one more element than the min-heap. The median in this case is simply the maximum element of the max-heap.

Problem-28 **Maximum sum in sliding window:** Given array A[] with sliding window of size w which is moving from the very left of the array to the very right. Assume that we can only see the *w* numbers in the window. Each time the sliding window moves rightwards by one position. For example: The array is [1 3 -1 -3 5 3 6 7], and *w* is 3.

Window position	Max
[1 3 -1] -3 5 3 6 7	3
1 [3 -1 -3] 5 3 6 7	3
1 3 [-1 -3 5] 3 6 7	5
1 3 -1 [-3 5 3] 6 7	5
1 3 -1 -3 [5 3 6] 7	6
1 3 -1 -3 5 [3 6 7]	7

Input: A long array A[], and a window width *w*. **Output:** An array B[], B[i] is the maximum value of from A[i] to A[i+w-1]
Requirement: Find a good optimal way to get B[i]

Solution: Brute force solution is, every time the window is moved we can search for a total of *w* elements in the window.

Time complexity: O(*nw*).

Problem-29 For Problem-28, can we reduce the complexity?

Solution: Yes, we can use heap data structure. This reduces the time complexity to O(*nlogw*). Insert operation takes O(*logw*) time, where w is the size of the heap. However, getting the maximum value is cheap; it merely takes constant time as the maximum value is always kept in the root (head) of the heap. As the window slides to the right, some elements in the heap might not be valid anymore (range is outside of the current window). How should we remove them? We would need to be somewhat careful here. Since we only remove elements that are out of the window's range, we would need to keep track of the elements' indices too.

Problem-30 For Problem-28, can we further reduce the complexity?

Solution: Yes, The double-ended queue is the perfect data structure for this problem. It supports insertion/deletion from the front and back. The trick is to find a way such that the largest element in the window would always appear in the front of the queue. How would you maintain this requirement as you push and pop elements in and out of the queue?

Besides, you will notice that there are some redundant elements in the queue that we shouldn't even consider. For example, if the current queue has the elements: [10 5 3], and a new element in the window has the element 11. Now, we could have emptied the queue without considering elements 10, 5, and 3, and insert only element 11 into the queue.

Typically, most people try to maintain the queue size the same as the window's size. Try to break away from this thought and think out of the box. Removing redundant elements and storing only elements that need to be considered in the queue is the key to achieving the efficient O(n) solution below. This is because each element in the list is being inserted and removed at most once. Therefore, the total number of insert + delete operations is 2n.

```
public void MaxSlidingWindow(int[] A, int w, int[] B) {
    DoubleEndQueue Q = new DoubleEndQueue();
    for (int i = 0; i < w; i++) {
        while (!Q.isEmpty() && A[i] >= A[Q.QBack()])
            Q.PopBack();
        Q.PushBack(i);
    }
    for (int i = w; i < A.length; i++) {
        B[i-w] = A[Q.QFront()];
        while (!Q.isEmpty() && A[i] >= A[Q.QBack()])
            Q.PopBack();
        while (!Q.isEmpty() && Q.QFront() <= i-w)
            Q.PopFront();
        Q.PushBack(i);
    }
    B[n-w] = A[Q.QFront()];
}
```

Problem-31 A priority queue is a list of items in which each item has associated with it a priority. Items are withdrawn from a priority queue in order of their priorities starting with the highest priority item first. If the maximum priority item is required, then a heap is constructed such than priority of every node is greater than the priority of its children.

Design such a heap where the item with the middle priority is withdrawn first. If there are n items in the heap, then the number of items with the priority smaller than the middle priority is $\frac{n}{2}$ if n is odd, else $\frac{n}{2} \mp 1$.

Explain how the withdraw and insert operations work, calculate their complexity, and how the data structure is constructed.

Solution: We can use one min heap and one max heap such that root of the min heap is larger than the root of the max heap. The size of the min heap should be equal or one less than the size of the max heap. So the middle element is always the root of the max heap.

For the insert operation, if the new item is less than the root of max heap, then insert it into the max heap; else insert it into the min heap. After the withdraw or insert operation, if the size of heaps are not as specified above than transfer the root element of the max heap to min heap or vice-versa. With this implementation, insert and withdraw operation will be in O(*logn*) time.

Problem-32 Given two heaps, how do you merge (union) them?

Solution: Binary heap supports various operations quickly: Find-min, insert, decrease-key. If we have two min-heaps, H1 and H2, there is no efficient way to combine them into a single min-heap.

For solving this problem efficiently, we can use mergeable heaps. Mergeable heaps support efficient union operation. It is a data structure that supports the following operations:

- Create-Heap(): creates an empty heap
- Insert(H,X,K) : insert an item x with key K into a heap H
- Find-Min(H) : return item with min key
- Delete-Min(H) : return and remove
- Union(H1, H2) : merge heaps H1 and H2

Examples of mergeable heaps are:

- Binomial Heaps
- Fibonacci Heaps

Both heaps also support:

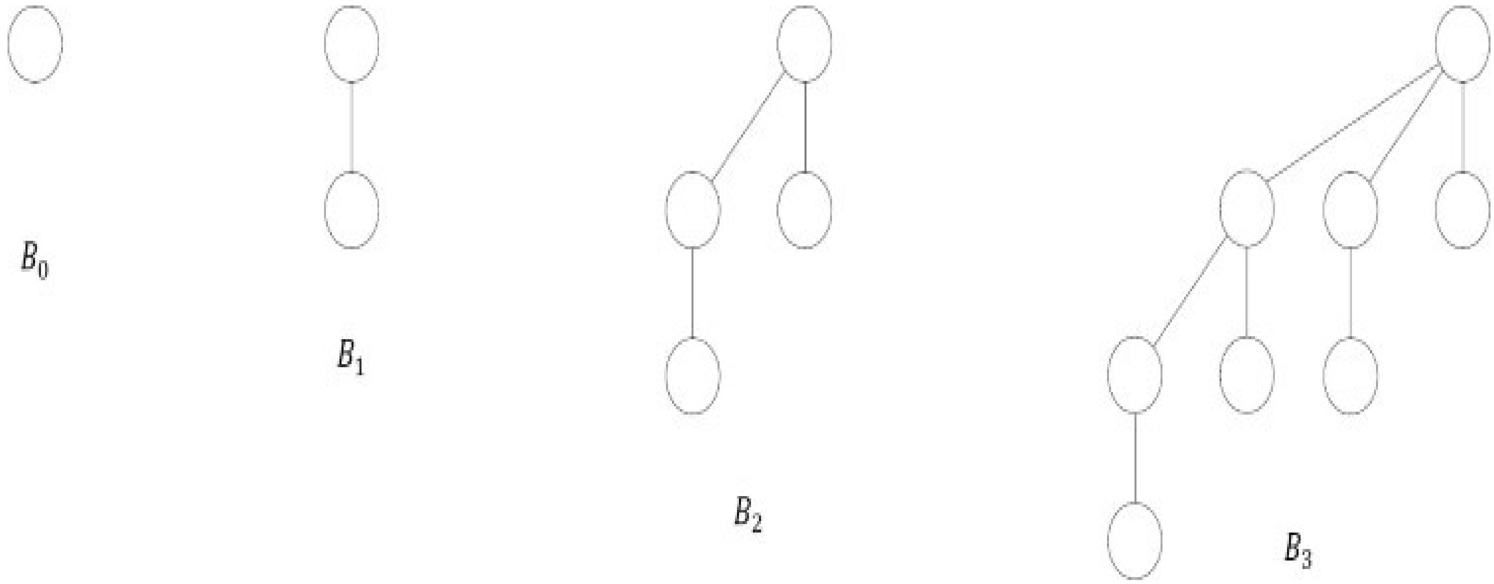
- Decrease-Key(H,X,K): assign item Y with a smaller key K
- Delete(H,X) : remove item X

Binomial Heaps: Unlike binary heap which consists of a single tree, a *binomial* heap consists of a small set of component trees and no need to rebuild everything when union is performed. Each component tree is in a special format, called a *binomial tree*.

A binomial tree of order k, denoted by B_k is defined recursively as follows:

- B_0 is a tree with a single node
- For $k \geq 1$, B_k is formed by joining two B_{k-i}, such that the root of one tree becomes the leftmost child of the root of the other.

Example:



Fibonacci Heaps: Fibonacci heap is another example of mergeable heap. It has no good worst-case guarantee for any operation (except Insert/Create-Heap). Fibonacci Heaps have excellent amortized cost to perform each operation. Like *binomial* heap, *fibonacci* heap consists of a set of min-heap ordered component trees. However, unlike binomial heap, it has

- No limit on number of trees (up to O(n)), and
- No limit on height of a tree (up to O(n))

Also, *Find-Min, Delete-Min, Union, Decrease-Key, Delete* all have worst-case O(n) running time. However, in the amortized sense, each operation performs very quickly.

Operation	Binary Heap	Binomial Heap	Fibonacci Heap
Create-Heap	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
Find-Min	$\Theta(1)$	$\Theta(logn)$	$\Theta(1)$
Delete-Min	$\Theta(logn)$	$\Theta(logn)$	$\Theta(logn)$
Insert	$\Theta(logn)$	$\Theta(logn)$	$\Theta(1)$
Delete	$\Theta(logn)$	$\Theta(logn)$	$\Theta(logn)$
Decrease-Key	$\Theta(logn)$	$\Theta(logn)$	$\Theta(1)$
Union	$\Theta(n)$	$\Theta(logn)$	$\Theta(1)$

Problem-33 Median in an infinite series of integers

Solution: Median is the middle number in a sorted list of numbers (if we have odd number of elements). If we have even number of elements, median is the average of two middle numbers in a sorted list of numbers.

We can solve this problem efficiently by using 2 heaps: One MaxHeap and one MinHeap.

1. MaxHeap contains the smallest half of the received integers
2. MinHeap contains the largest half of the received integers

The integers in MaxHeap are always less than or equal to the integers in MinHeap. Also, the number of elements in MaxHeap is either equal to or 1 more than the number of elements in the MinHeap. In the stream if we get $2n$ elements (at any point of time), MaxHeap and MinHeap will both contain equal number of elements (in this case, n elements in each heap). Otherwise, if we have received $2n + 1$ elements, MaxHeap will contain $n + 1$ and MinHeap n.

Let us find the Median: If we have $2n + 1$ elements (odd), the Median of received elements will be the largest element in the MaxHeap (nothing but the root of MaxHeap). Otherwise, the Median of received elements will be the average of largest element in the MaxHeap (nothing but the root of MaxHeap) and smallest element in the MinHeap (nothing but the root of MinHeap). This can be calculated in O(1).

Inserting an element into heap can be done in O($logn$). Note that, any heap containing $n + 1$ elements might need one delete operation (and insertion to other heap) as well.

Example:

Insert 1: Insert to MaxHeap.
MaxHeap: {1}, MinHeap:{}
Insert 9: Insert to MinHeap. Since 9 is greater than 1 and MinHeap maintains the maximum elements.
MaxHeap: {1}, MinHeap:{9}
Insert 2: Insert MinHeap. Since 2 is less than all elements of MinHeap.
MaxHeap: {1,2}, MinHeap:{9}
Insert 0: Since MaxHeap already has more than half; we have to drop the max element from MaxHeap and insert it to MinHeap. So, we have to remove 2 and insert into MinHeap. With that it becomes:
MaxHeap: {1}, MinHeap:{2,9}
Now, insert 0 to MaxHeap.

Total Time Complexity: O($logn$).

Problem-34 Merge k sorted linked lists and return it as one sorted list.

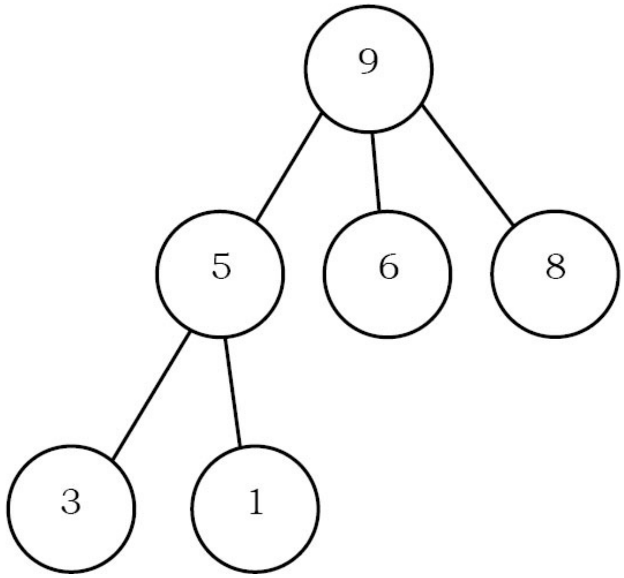
Solution:

```java
public ListNode mergeKLists(ArrayList<ListNode> lists) {
    if (lists == null || lists.isEmpty())
        return null;
    PriorityQueue<ListNode> heap = new PriorityQueue<ListNode>(lists.size(), new Comparator<ListNode>(){
        public int compare(ListNode o1, ListNode o2) {
            return o1.data > o2.data ? 1 : (o1.data < o2.data ? -1 : 0);
        }
    });
    for (ListNode node : lists) {
        if (node != null) {
            heap.add(node);
        }
    }
    ListNode head = null, cur = null;
    while (!heap.isEmpty()) {
        if (head == null) {
            head = heap.poll();
            cur = head;
        } else {
            cur.next = heap.poll();
            cur = cur.next;
        }
        if (cur.next != null) {
            heap.add(cur.next);
        }
    }
    return head;
}
```

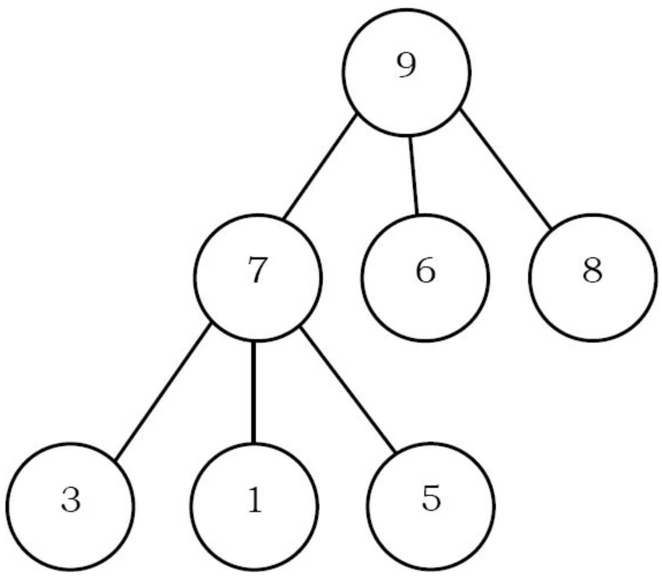
Problem-35 Suppose the elements 7, 2, 10 and 4 are inserted, in that order, into the valid 3-ary max heap found in the above question, Which one of the following is the sequence of items in the array representing the resultant heap?

(A) 10, 7, 9, 8, 3, 1, 5, 2, 6, 4
(B) 10, 9, 8, 7, 6, 5, 4, 3, 2, 1
(C) 10, 9, 4, 5, 7, 6, 8, 2, 1, 3
(D) 10, 8, 6, 9, 7, 2, 3, 4, 1, 5

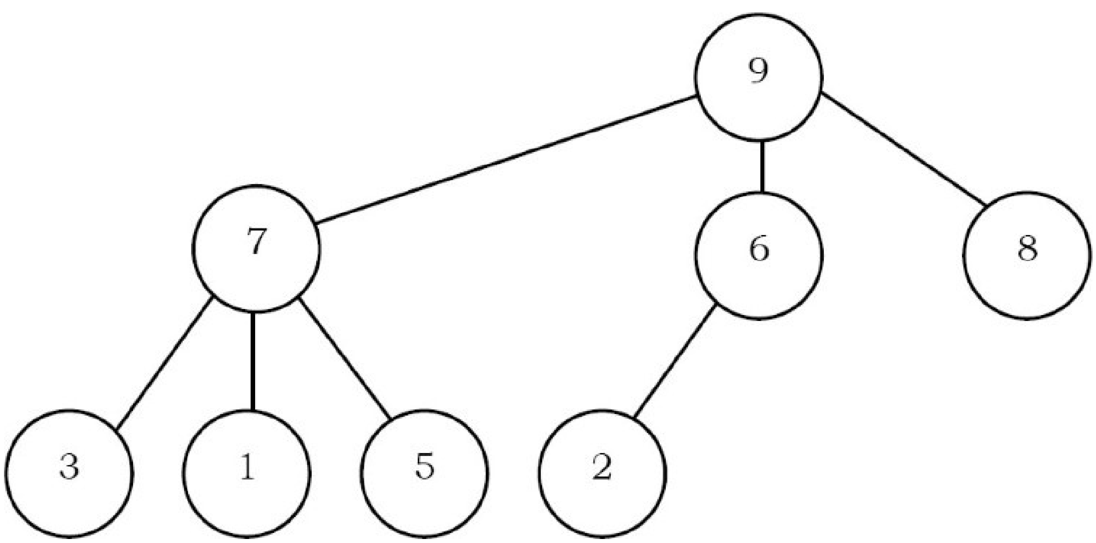
Solution: The 3-ary max heap with elements 9, 5, 6, 8, 3, 1 is:



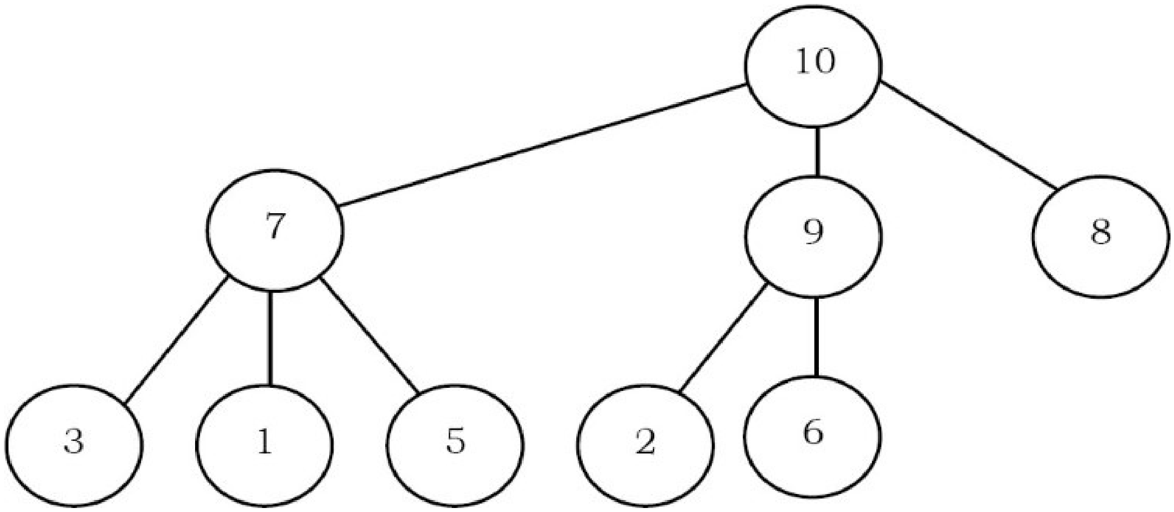
After Insertion of 7:



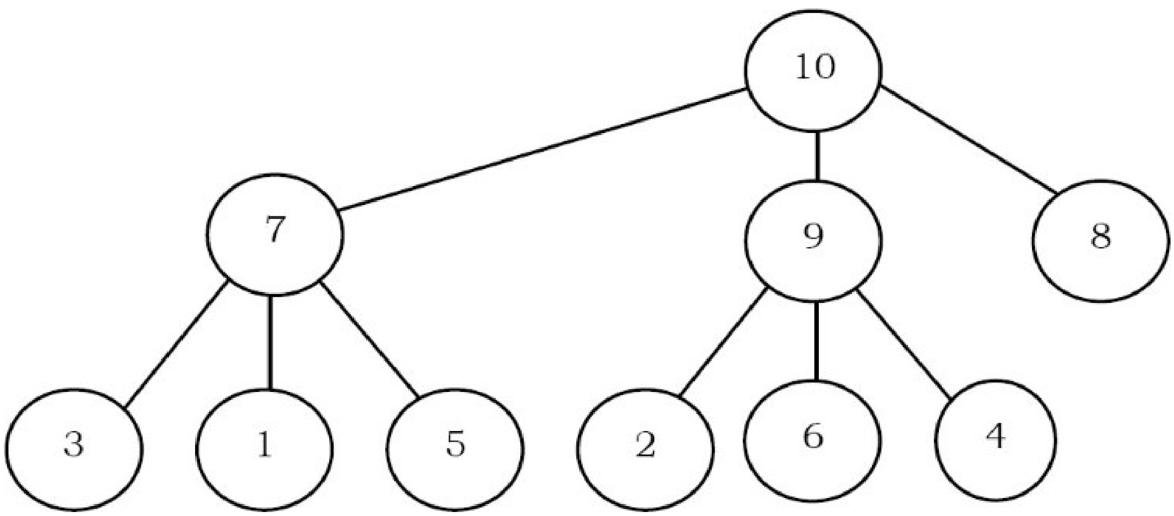
After Insertion of 2:



After Insertion of 10:

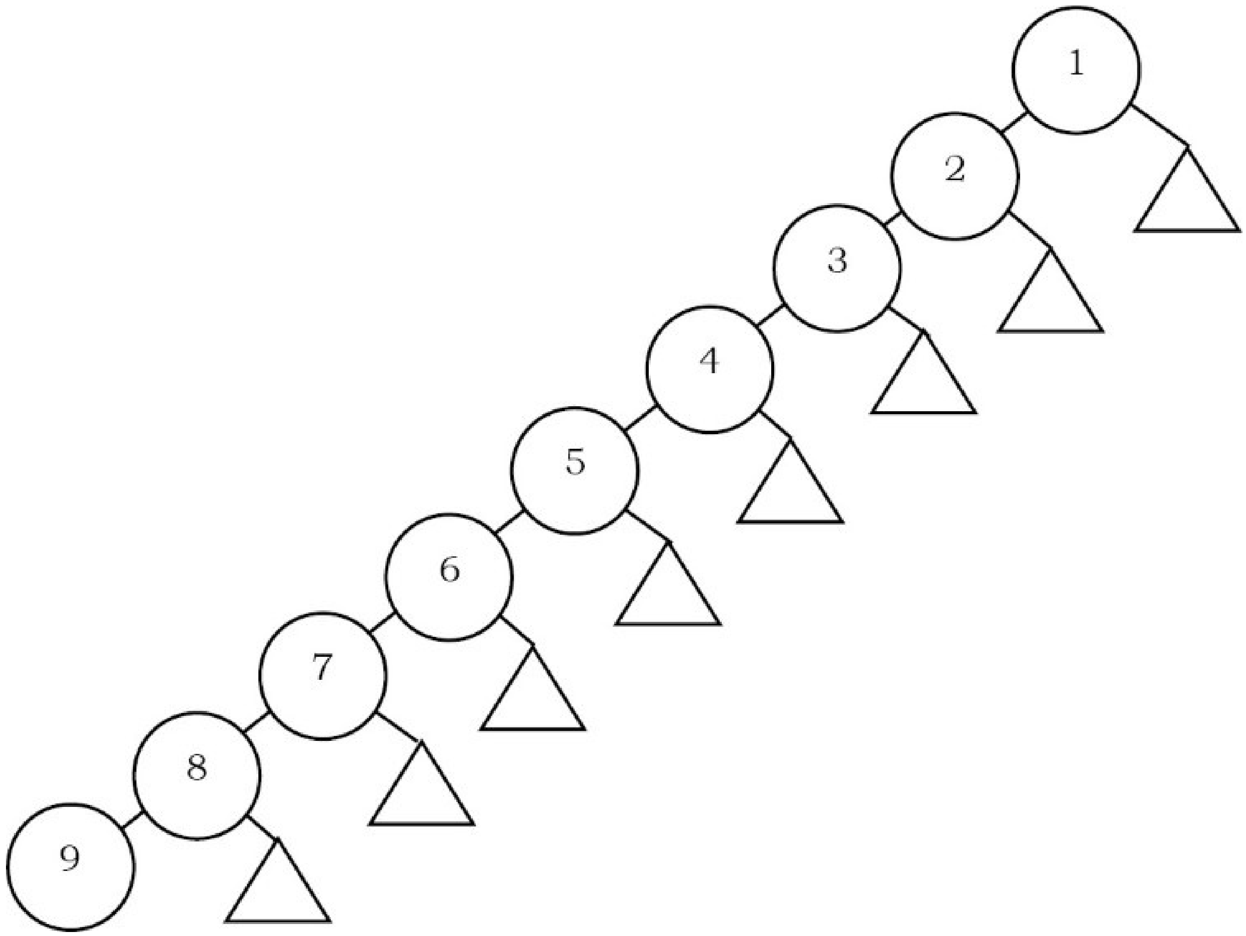


After Insertion of 4:



Problem-36 A complete binary min-heap is made by including each integer in [1, 1023] exactly once. The depth of a node in the heap is the length of the path from the root of the heap to that node. Thus, the root is at depth 0. The maximum depth at which integer 9 can appear is—

Solution: As shown in the figure below, for a given number *i*, we can fix the element *i* at i^{th} level and arrange the numbers 1 to $i - 1$ to the levels above. Since the root is at depth *zero*, the maximum depth of the i^{th} element in a min-heap is $i - 1$. Hence, the maximum depth at which integer 9 can appear is 8.



Problem-37 A *d*-ary heap is like a binary heap, but instead of 2 children, nodes have *d* children. How would you represent a *d*-ary heap with *n* elements in an array? What are the expressions for determining the parent of a given element, *Parent*(*i*), and a j^{th} child of a given element, *Child*(*i, j*), where 1 ≤ j ≤ d?

Solution: The following expressions determine the parent and j^{th} child of element *i* (where 1 ≤ j ≤ d):

$$Parent(i) = \left\lfloor \frac{i+d-2}{d} \right\rfloor$$

$$Child(i,j) = (i-1).d + j + 1$$

CHAPTER 8

DISJOINT SETS ADT

8.1 Introduction

In this chapter, we will represent an important mathematics concept: *sets*. This means how to represent a group of elements which do not need any order. The disjoint sets ADT is the one used for this purpose. It is used for solving the equivalence problem. It is very simple to implement. A simple array can be used for the implementation and each function takes only a few lines of code. Disjoint sets ADT acts as an auxiliary data structure for many other algorithms (for example, *Kruskal's* algorithm in graph theory). Before starting our discussion on disjoint sets ADT, let us look at some basic properties of sets.

8.2 Equivalence Relations and Equivalence Classes

For the discussion below let us assume that 5 is a set containing the elements and a relation *R* is defined on it. That means for every pair of elements in $a, b \in 5$, $a\ R\ b$ is either true or false. If $a\ R\ b$ is true, then we say a is related to b, otherwise a is not related to b. A relation R is called an *equivalence relation* if it satisfies the following properties:

- *Reflexive:* For every element $a \in S$, $a\ R\ a$ is true.
- *Symmetric:* For any two elements $a, b \in S$, if $a\ R\ b$ is true then $b\ R\ a$ is true.
- *Transitive:* For any three elements a, b, c ∈ S, if a R b and $b\ R\ c$ are true then $a\ R\ c$ is true.

As an example, relations ≤ (less than or equal to) and ≥ (greater than or equal to) on a set of integers are not equivalence relations. They are reflexive (since $a \le a$) and transitive ($a \le b$ and $b \le c$ implies $a \le$ c) but not symmetric ($a \le b$ does not imply $b \le a$).

Similarly, *rail connectivity* is an equivalence relation. This relation is reflexive because any location is connected to itself. If there is connectivity from city a to city b, then city b also has connectivity to city a, so the relation is symmetric. Finally, if city a is connected to city b and city b is connected to city c, then city a is also connected to city c.

The *equivalence class* of an element $a \in S$ is a subset of S that contains all the elements that are related to a. Equivalence classes create a *partition* of S. Every member of S appears in exactly one equivalence class. To decide if $a\ R\ b$, we just need to check whether a and b are in the same equivalence class (group) or not.

In the above example, two cities will be in same equivalence class if they have rail connectivity. If they do not have connectivity then they will be part of different equivalence classes.

Since the intersection of any two equivalence classes is empty (ϕ), the equivalence classes are sometimes called *disjoint sets*. In the subsequent sections, we will try to see the operations that can be performed on equivalence classes. The possible operations are:

- Creating an equivalence class (making a set)
- Finding the equivalence class name (Find)
- Combining the equivalence classes (Union)

8.3 Disjoint Sets ADT

To manipulate the set elements we need basic operations defined on sets. In this chapter, we concentrate on the following set operations:

- MAKESET(X): Creates a new set containing a single element X.
- UNION(X, Y): Creates a new set containing the elements X and Y in their union and deletes the sets containing the elements X and Y.
- FIND(X): Returns the name of the set containing the element X.

8.4 Applications

Disjoint sets ADT have many applications and a few of them are:

- To represent network connectivity
- Image processing
- To find least common ancestor
- To define equivalence of finite state automata
- Kruskal's minimum spanning tree algorithm (graph theory)
- In game algorithms

8.5 Tradeoffs in Implementing Disjoint Sets ADT

Let us see the possibilities for implementing disjoint set operations. Initially, assume the input elements are a collection of *n* sets, each with one element. That means, initial representation assumes all relations (except reflexive relations) are false. Each set has a different element, so that $S_i \cap S_j = \phi$. This makes the sets *disjoint*.

To add the relation *a R b* (UNION), we first need to check whether *a* and *b* are already related or not. This can be verified by performing FINDs on both *a* and *b* and checking whether they are in the same equivalence class (set) or not.

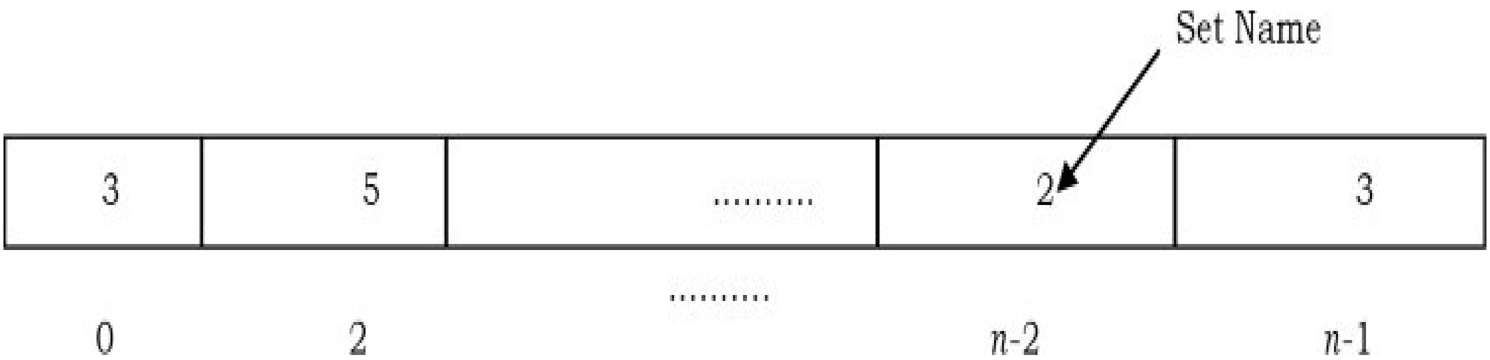
If they are not, then we apply UNION. This operation merges the two equivalence classes containing *a* and *b* into a new equivalence class by creating a new set $S_k = S_i \cup S_j$ and deletes S_i *and* S_j. Basically there are two ways to implement the above FIND/UNION operations:

- Fast FIND implementation (also called Quick FIND)
- Fast UNION operation implementation (also called Quick UNION)

8.6 Fast FIND Implementation (Quick FIND)

In this method, we use an array. As an example, in the representation below the array contains the set name for each element. For simplicity, let us assume that all the elements are numbered sequentially from 0 to $n - 1$.

In the example below, element 0 has the set name 3, element 1 has the set name 5, and so on. With this representation FIND takes only O(1) since for any element we can find the set name by accessing its array location in constant time.



In this representation, to perform UNION(a, *b*) [assuming that *a* is in set *i and b* is in set *j*] we need to scan the complete array and change all *i's to j*. This takes O(*n*).

A sequence of $n - 1$ unions take $O(n^2)$ time in the worst case. If there are $O(n^2)$ FIND operations, this performance is fine, as the average time complexity is O(1) for each UNION or FIND operation. If there are fewer FINDs, this complexity is not acceptable.

8.7 Fast UNION Implementation (Quick UNION)

In this and subsequent sections, we will discuss the faster *UNION* implementations and its variants. There are different ways of implementing this approach and the following is a list of a few of them.

- Fast UNION implementations (Slow FIND)
- Fast UNION implementations (Quick FIND)
- Fast UNION implementations with path compression

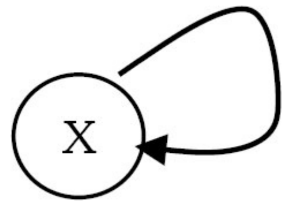
8.8 Fast UNION Implementation (Slow FIND)

As we have discussed, FIND operation returns the same answer (set name) if and only if they are in the same set. In representing disjoint sets, our main objective is to give a different set name for each group. In general we do not care about the name of the set. One possibility for implementing the set is *tree* as each element has only one *root* and we can use it as the set name.

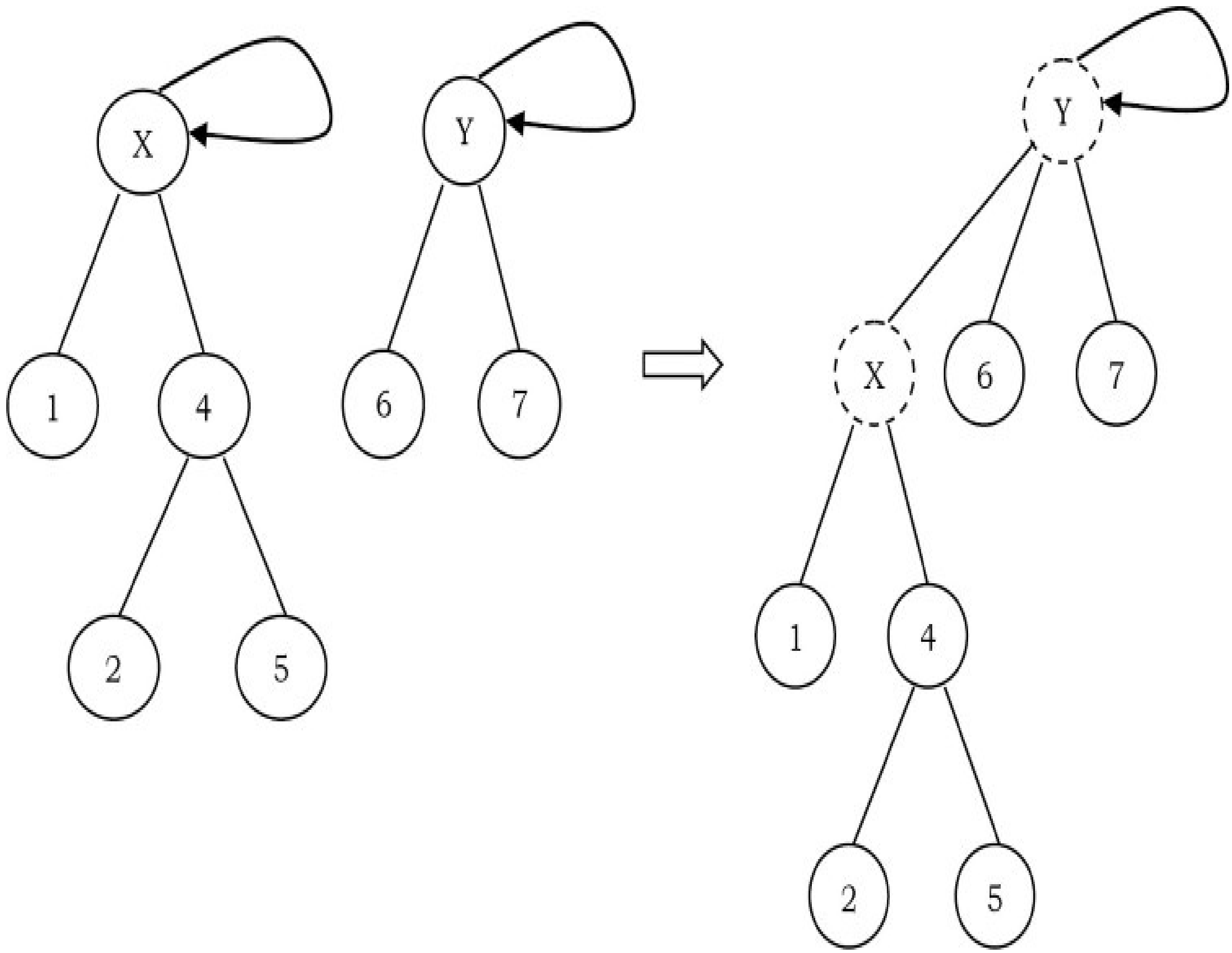
How are these represented? One possibility is using an array: for each element keep the *root* as its set name. But with this representation, we will have the same problem as that of FIND array implementation. To solve this problem, instead of storing the *root* we can keep the parent of the element. Therefore, using an array which stores the parent of each element solves our problem.

To differentiate the root node, let us assume its parent is the same as that of the element in the array. Based on this representation, MAKESET, FIND, UNION operations can be defined as:

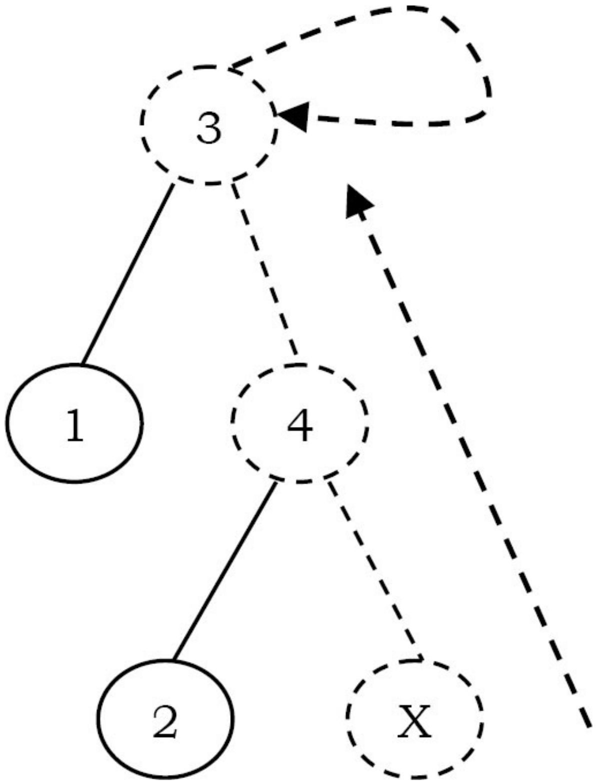
- (X)*:* Creates a new set containing a single element X and in the array update the parent of X as X. That means root (set name) of X is X.



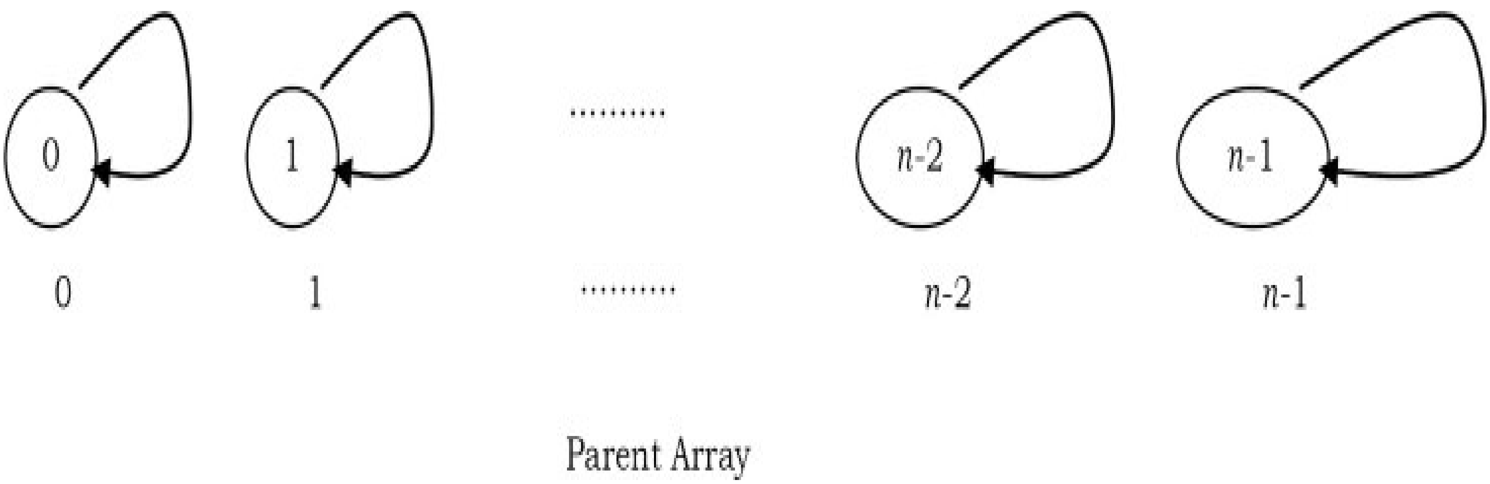
- UNION(*X*, *Y*)*:* Replaces the two sets containing *X* and *Y* by their union and in the array updates the parent of *X* as *Y*.



- FIND(X): Returns the name of the set containing the element *X*. We keep on searching for *X's* set name until we come to the root of the tree.

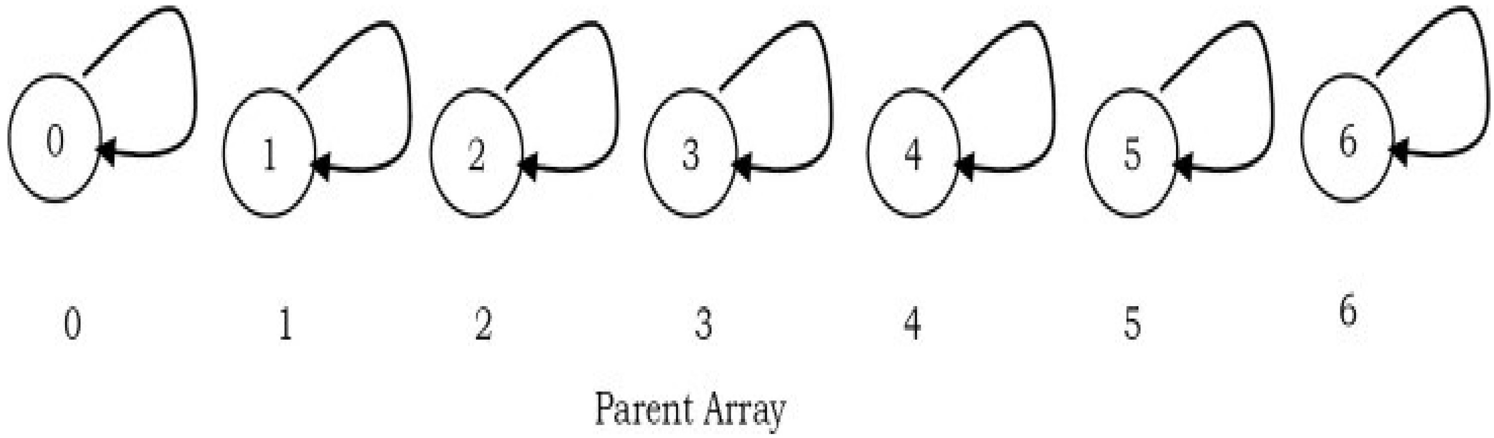


For the elements 0 to $n-1$ the initial representation is:

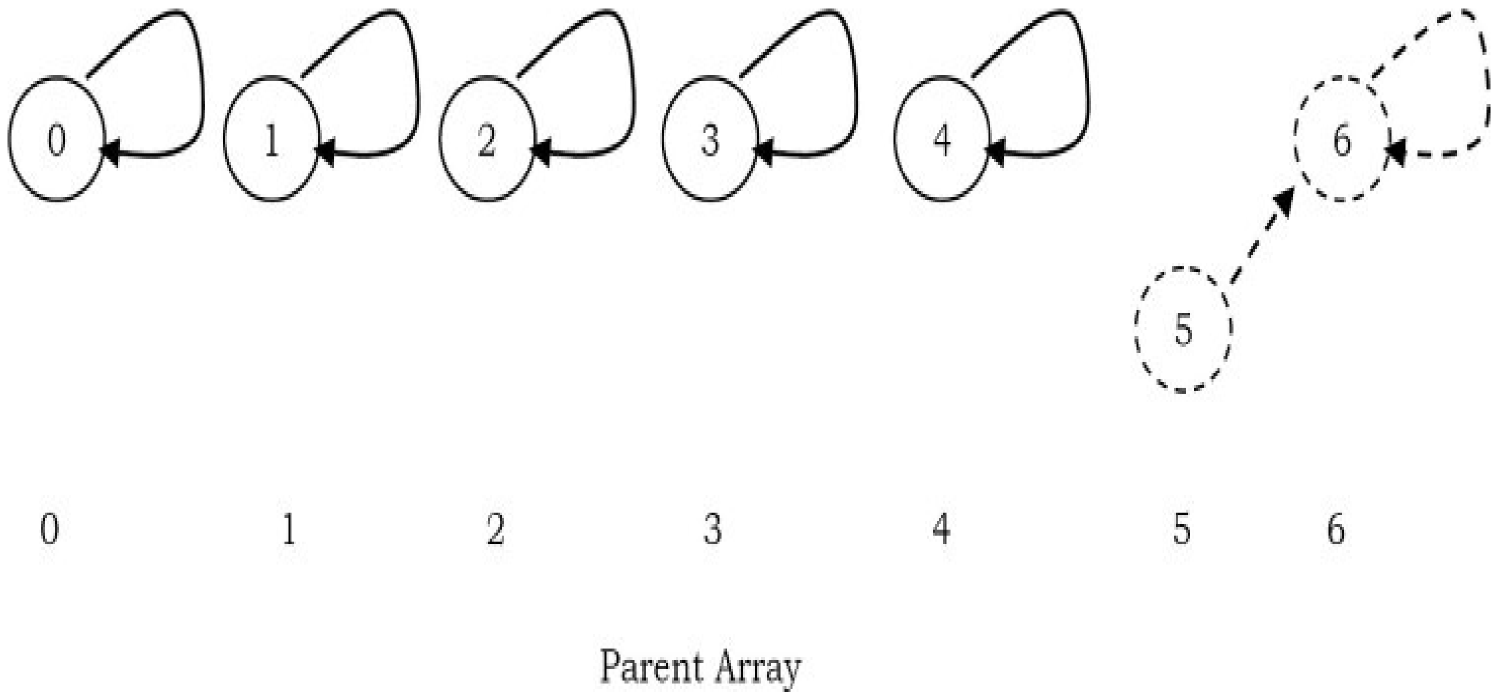


To perform a UNION on two sets, we merge the two trees by making the root of one tree point to the root of the other.

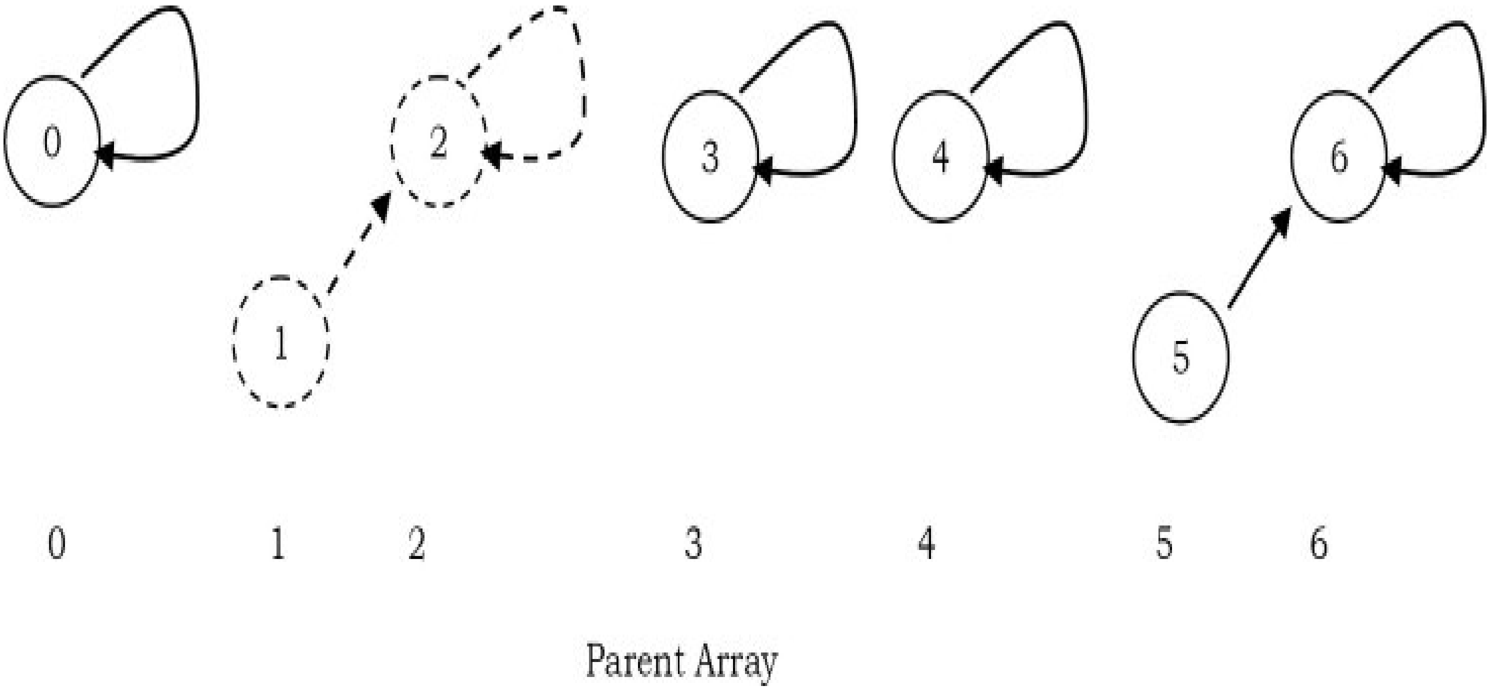
Initial Configuration for the elements 0 to 6



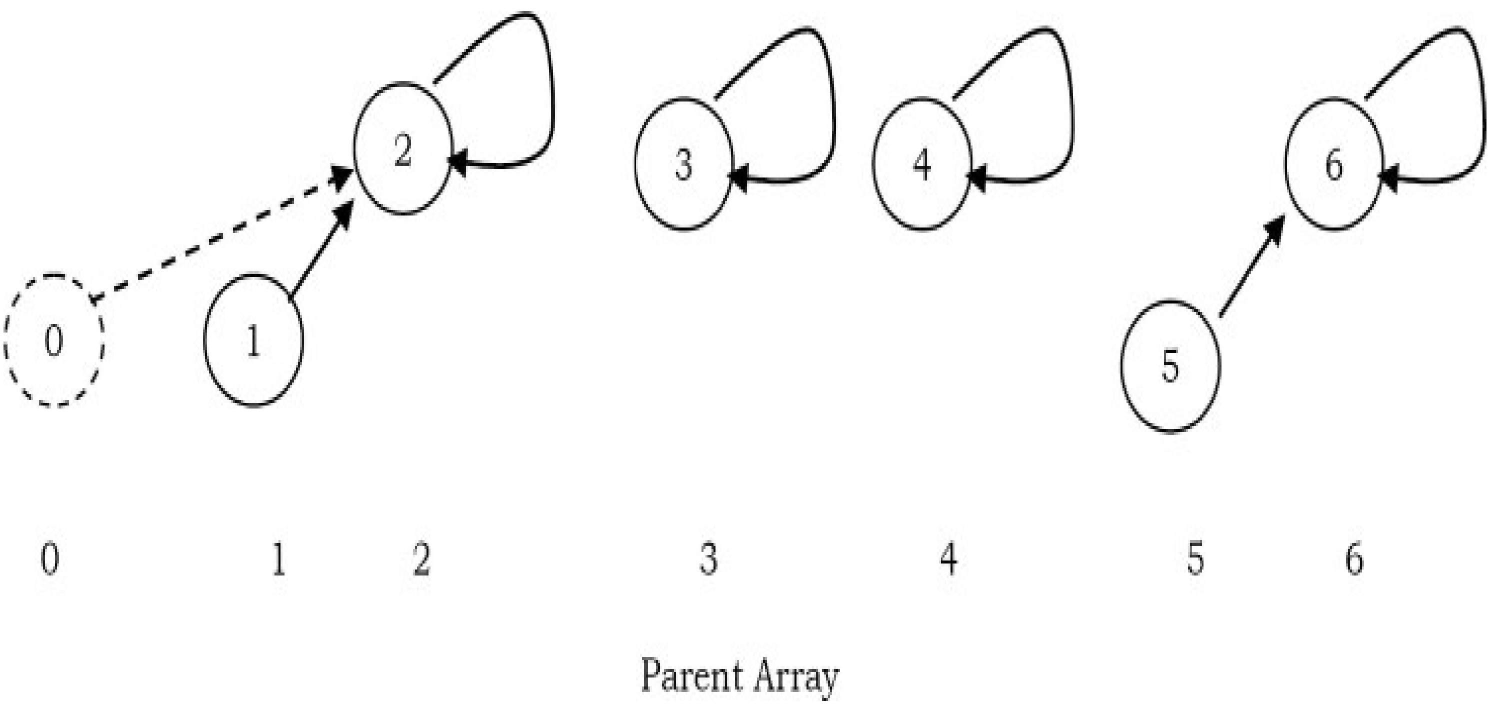
After UNION(5,6)



After UNION(1,2)



After UNION(0,2)



One important thing to observe here is, UNION operation is changing the root's parent only, but not for all the elements in the sets. Due to this, the time complexity of UNION operation is O(1).

A FIND(X) on element *X* is performed by returning the root of the tree containing *X*. The time to perform this operation is proportional to the depth of the node representing *X*.

Using this method, it is possible to create a tree of depth n – 1 (Skew Trees). The worst-case running time of a FIND is O(*n*) and *m* consecutive FIND operations take O(*mn*) time in the worst case.

```
public class  DisjointSet {
      public int[] S;
      public int size;                              // Number of elements in set
      public MAKESET(int size)                      { //Refer Below sections }
      public int FIND(int X)                        { //Refer Below sections }
      public int UNION(int root1, int root2)        {//Refer Below sections }
      .........
}
```

MAKESET

```
public void MAKESET(int size){
        S = new int[size];
        for(int i = size-1; i >=0; i-- )
              S[i] = i;
}
```

FIND

```
public int FIND(int X){
        if(! (X >= 0 && X < size))
              return;
        if( S[X] == X )
              return X;
        else
              return FIND(S, S[X]);
}
```

UNION

```
public void UNION(int root1, int root2){
        if(FIND(root1) == FIND(root2))
              return;
        if(! ((root1 >= 0 && root1 < size) && (root2 >= 0 && root2 < size)))
              return;
        S[root1] = root2;
}
```

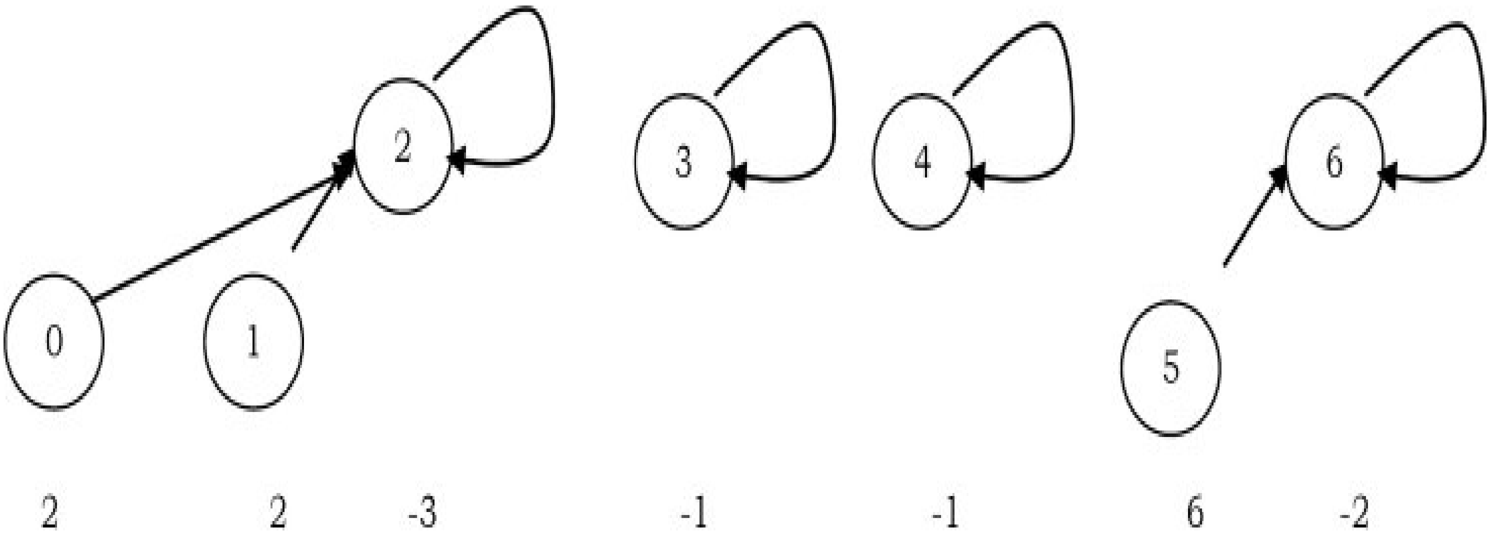
8.7 Fast UNION Implementations (Quick FIND)

The main problem with the previous approach is that, in the worst case we are getting the skew trees and as a result the FIND operation is taking O(n) time complexity. There are two ways to improve it:

- UNION by Size (also called UNION by Weight): Make the smaller tree a subtree of the larger tree
- UNION by Height (also called UNION by Rank): Make the tree with less height a subtree of the tree with more height

UNION by size

In the earlier representation, for each element i we have stored i (in the parent array) for the root element and for other elements we have stored the parent of i. But in this approach we store negative of the size of the tree (that means, if the size of the tree is 3 then store –3 in the parent array for the root element). For the previous example (after UNION(0,2)), the new representation will look like:



Parent Array

Assume that the size of one element set is 1 and store –1. Other than this there is no change.

MAKESET

```
public void MAKESET(int size) {
        for(int i = size-1; i >= 0; i-- )
            S[i] = -1;
}
```

FIND

```
public int FIND(int X) {
        if(!(X >= 0 && X < size)) return;
        if( S[X] == -1 )
            return X;
        else return FIND(S, S[X]);
}
```

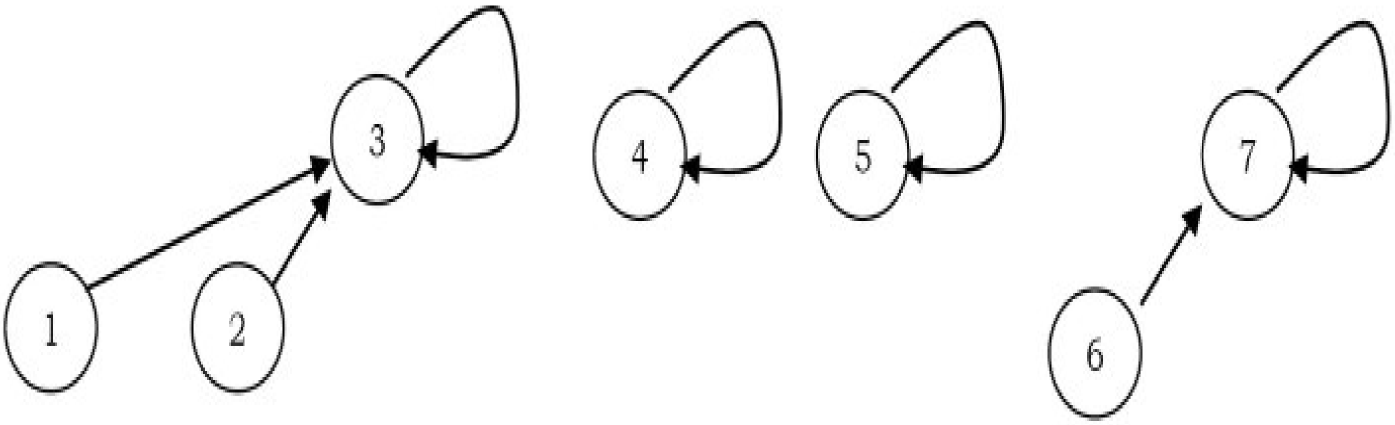
UNION by Size

```
public void UNIONBySize(int root1, int root2) {
    if((FIND(root1) == FIND(root2)) && FIND(root1) != -1)
        return;
    if( S[root2] < S[root1] )   {
            S[root1] = root2;
            S[root2] += S[root1];
    }
    else {
            S[root2] = root1;
            S[root1] += S[root2];
    }
}
```

Note: There is no change in FIND operation implementation.

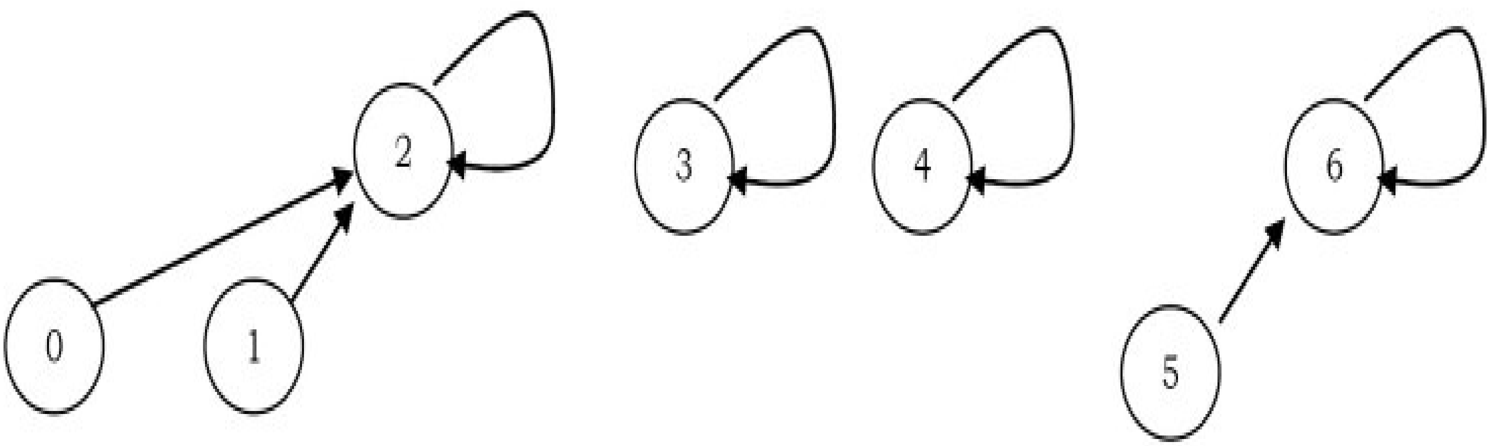
UNION by Height (UNION by Rank)

As in UNION by size, in this method we store negative of height of the tree (that means, if the height of the tree is 3 then we store –3 in the parent array for the root element). We assume the height of a tree with one element set is 1. For the previous example (after UNION(0,2)), the new representation will look like:



2 2 -2 -1 -1 6 -2

Parent Array



2 2 -2 -1 -1 6 -2

Parent Array

```
public void UNIONByHeight(int root1, int root2) {
    if((FIND(root1) == FIND(root2)) && FIND(root1) != -1)
            return;
    if( S[root2] < S[root1] )
            S[root1] = root2;
    else {
          if( S[root2] == S[root1] )
                  S[root1]--;
            S[root2] = root1;
    }
}
```

Note: For FIND operation there is no change in the implementation.

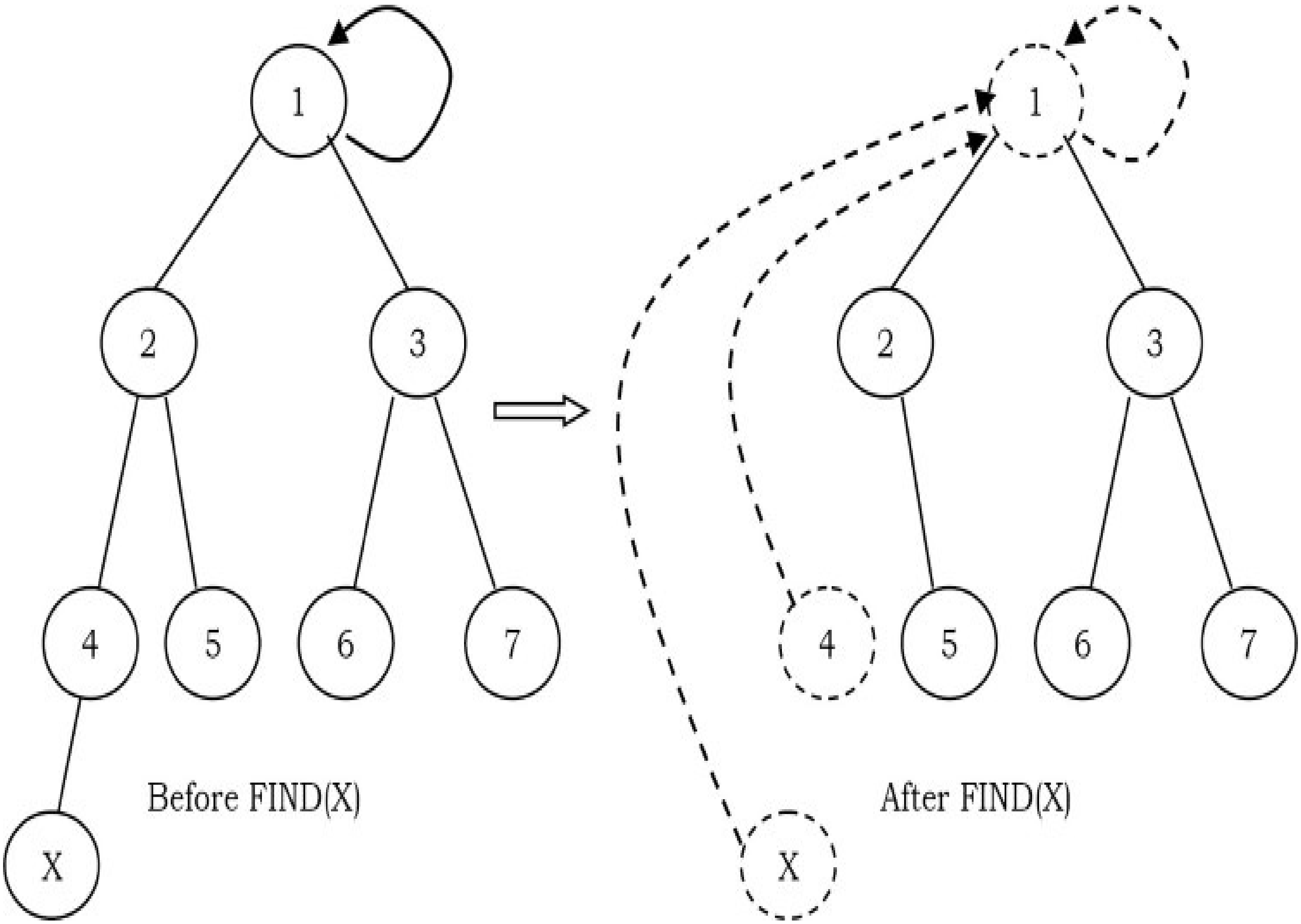
Comparing UNION by Size and UNION by Height

With UNION by size, the depth of any node is never more than *logn*. This is because a node is initially at depth 0. When its depth increases as a result of a UNION, it is placed in a tree that is at least twice as large as before. That means its depth can be increased at most *logn* times. This means that the running time for a FIND operation is O(*logn*), and a sequence of *m* operations takes O(*m logn*).

Similarly with UNION by height, if we take the UNION of two trees of the same height, the height of the UNION is one larger than the common height, and otherwise equal to the max of the two heights. This will keep the height of tree of *n* nodes from growing past O(*logn*). A sequence of *m* UNIONs and FINDs can then still cost O(*m logn*).

8.8 Path Compression

FIND operation traverses a list of nodes on the way to the root. We can make later FIND operations efficient by making each of these vertices point directly to the root. This process is called *path compression*. For example, in the FIND(*X*) operation, we travel from *X* to the root of the tree. The effect of path compression is that every node on the path from *X* to the root has its parent changed to the root.



With path compression the only change to the FIND function is that $S[X]$ is made equal to the value returned by FIND. That means, after the root of the set is found recursively, X is made to point directly to it. This happen recursively to every node on the path to the root.

```
public int FIND(int X) {
        if(! (X >= 0 && X < size))
            return;
        if( S[X] <= 0 ) return X;
        else  return( S[X] = FIND(S[X]));
}
```

Note: Path compression is compatible with UNION by size but not with UNION by height as there is no efficient way to change the height of the tree.

8.9 Summary

Performing *m* union-find operations on a set of *n* objects.

Algorithm	Worst-case time
Quick-find	mn
Quick-union	mn
Quick-Union by Size/Height	$n + m \log n$
Path compression	$n + m \log n$
Quick-Union by Size/Height + Path Compression	$(m + n) \log n$

8.10 Disjoint Sets: Problems & Solutions

Problem-1 Consider a list of cities $c_1; c_2, \ldots, c_n$. Assume that we have a relation R such that, for any i, j, $R(c_i, c_j)$ is 1 if cities c_i and c_j are in the same state, and 0 otherwise. If R is stored as a table, how much space does it require?

Solution: R must have an entry for every pair of cities. There are $\Theta(n^2)$ of these.

Problem-2 For Problem-1, using a Disjoint sets ADT, give an algorithm that puts each city in a set such that c_i and c_j are in the same set if and only if they are in the same state.

Solution:

```
for (i = 1; i<= n; i++) {
    MAKESET(ci);
    for (j = 1; j <= i-1; j++) {
        if(R(cj, ci)) {
            UNION(cj, ci);
            break;
        }
    }
}
```

Problem-3 For Problem-1, when the cities are stored in the Disjoint sets ADT, if we are given two cities c_i and c_j, how do we check if they are in the same state?

Solution: Cities c_i and c_j are in the same state if and only if $\text{FIND}(c_i) = \text{FIND}(c_j)$.

Problem-4 For Problem-1, if we use linked-lists with UNION by size to implement the