Scales In Nature.

Time Scale

Length Scale

Energy Scale.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x(t) = a\cos(\omega t + \phi)$$
$$y(t) = b\sin(\omega t + \phi)$$

It can be written as-

$$(x \quad y)\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \bar{r}^T D \bar{r} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{cxy}{d^2} = 1$$

$$(x \quad y)\begin{bmatrix} \frac{1}{a^2} & -\frac{c}{2d^2} \\ -\frac{c}{2d^2} & \frac{1}{b^2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \bar{r}^T M \bar{r} = 1$$

Now we need to convert M to the form of D.

This can be done by.

$$(x \quad y) = \bar{R}\,(\xi \quad \eta)$$

This can be obtained by rotation of axes.

$$\bar{r} = \bar{R}\bar{q}$$

$$\bar{R}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \qquad \bar{r}^T = \bar{q}^T \bar{R}^T$$

So,

$$(\bar{q}^T R^T) M (R\bar{q}) = 1$$

$$\bar{q}^T \underbrace{(R^T M R)}_{N} \bar{q} = 1$$

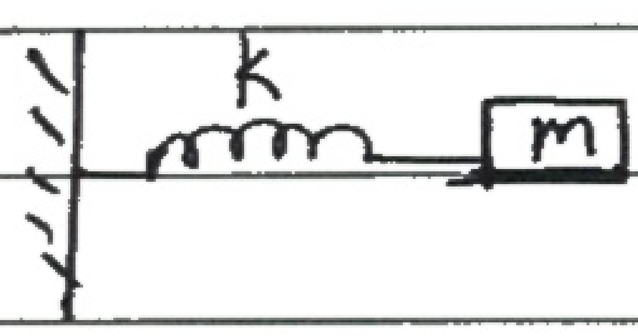
N can be found out as we know R and M.

$$N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\alpha \xi^2 + \delta \eta^2 + (\beta + \gamma) \xi \eta = 1$$

On solving we get,

$$\tan 2\phi = \frac{ab}{a^2 - b^2}$$ where ϕ is the rotation angle.



$$F(x) = -k(x - x_0)$$

$$\frac{dp(t)}{dt} = -kX$$

$$m\frac{d^2X}{dt^2} = -kX$$

$$\frac{d^2X}{dt^2} = -\frac{k}{m}X \qquad X(t) = A\cos(\omega t + \phi)$$

So, it is of the form,

$$\frac{d^2 z}{dy^2} = -\omega_0^2 z$$

$$\therefore \omega_0 = \sqrt{\frac{k}{m}}$$

$$\frac{dX}{dt} = k_1 X - k_2 XY$$

$$\frac{dY}{dt} = -k_3 Y + k_4 XY$$

$$X(t) = a\cos\omega t$$

$$Y(t) = b\sin\omega t$$

$$\frac{dX}{dt} = \alpha X + \beta Y + f(\;)$$

$$\frac{dY}{dt} = \gamma X + \delta Y + f(\;)$$

$$\bar{\gamma} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\frac{d(X)}{dt} = k_1 A X - k_2 XY$$

$$\frac{d}{dt}(Y) = k_2 XY - k_4 Y$$

Dimensionless Equation

$$\frac{dY}{dt} = (k_2 X - k_4) Y = k_4 Y \Big(\underbrace{\frac{k_2}{k_4} X}_{x} - 1\Big)$$

$$\frac{dY}{d(k_4 t)} = Y(x-1)$$

$$\tau = k_4 t$$

$$\frac{dY}{d\tau} = Y(x-1)$$

Substitute x and t in the first eq^n.
we get,

$$\frac{dx}{d\tau} = (a-y)x$$

$\dot{x}(t) = c\,x(t)$ $\qquad\qquad$ $\ddot{x}(t) = c\,x(t)$

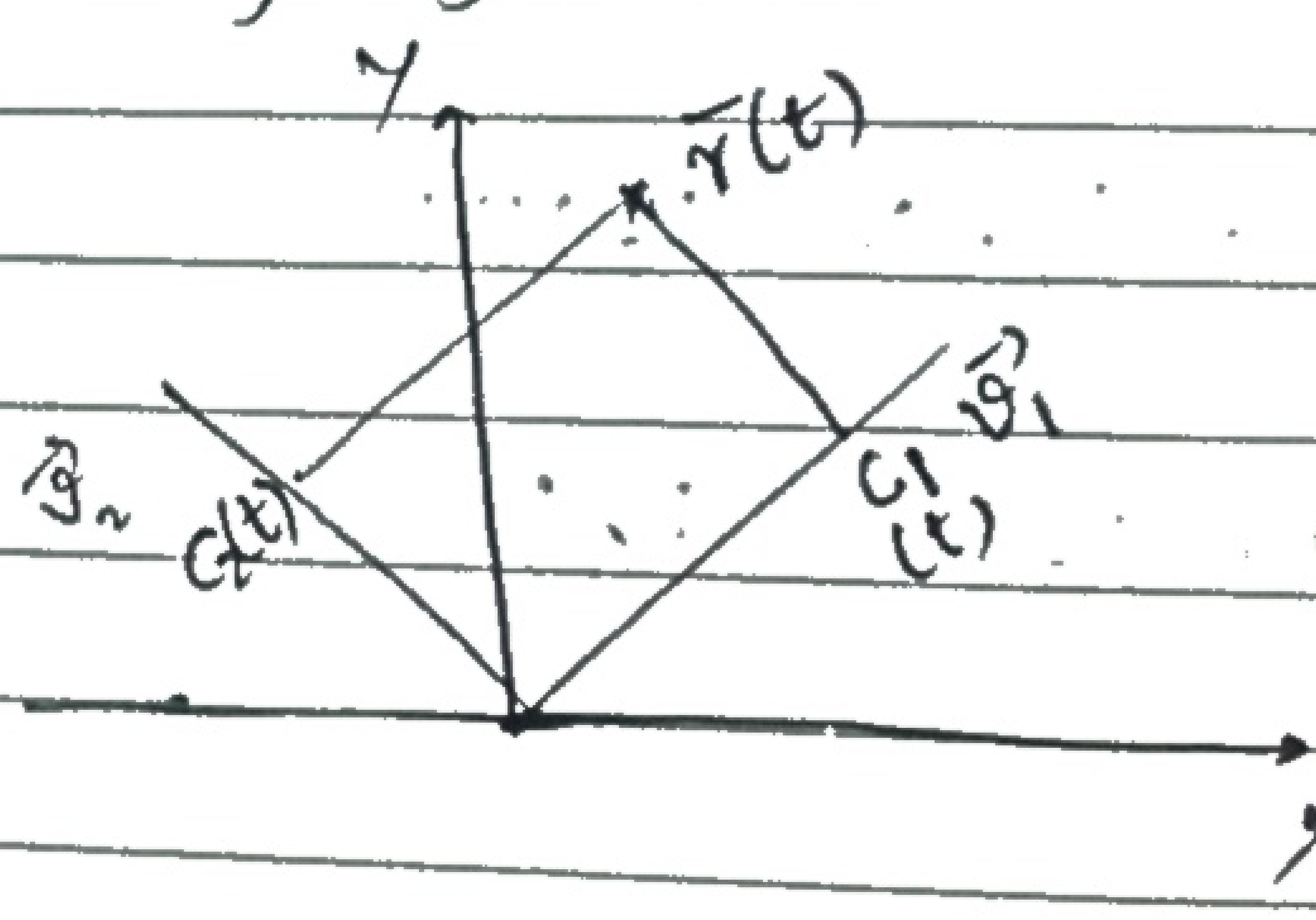
$\dot{\bar{r}} = c\,\bar{r}$ $\qquad\qquad$ $\ddot{\bar{r}} = c\,\bar{r}$

$\dot{\bar{r}} = \bar{M}\bar{r}$ $\qquad\qquad$ $\ddot{\bar{r}} = \bar{M}\bar{r}$

Let $\lambda_1, \lambda_2 \ldots \lambda_N$ be eigen values of $\bar{M}$ and $\bar{v}_1 \ldots \bar{v}_N$ corresponding eigen vectors.

$$\bar{r}(t) = \sum_{i=1}^{N} c_i(t)\,\bar{v}_i$$

$$\bar{r}(t) = \sum_{1}^{N} x_i(t)\,\hat{e}_i$$



$$\frac{dX}{dt} = \alpha X - \beta XY$$

$$\frac{dY}{dt} = \gamma XY - \delta Y$$

At Equilibrium,
$$(X_0, Y_0) = \left\{(0,0),\ \left(\frac{\delta}{\gamma}, \frac{\alpha}{\beta}\right)\right\}$$

$$\frac{dX}{dt} = f_1(X,Y) = f_1(X_0,Y_0) + \left.\frac{\partial f_1}{\partial X}\right|_{X_0,Y_0}(X - X_0) + \left.\frac{\partial f_1}{\partial Y}\right|_{X_0,Y_0}(Y - Y_0).$$

- Taylor Series

$$\frac{dX}{dt} = 0 + \alpha(X-0) + 0(Y-0) \qquad X_0 = 0,\ Y_0 = 0.$$

$$\frac{dY}{dt} = 0 + 0(X-0) - \delta(Y-0)$$

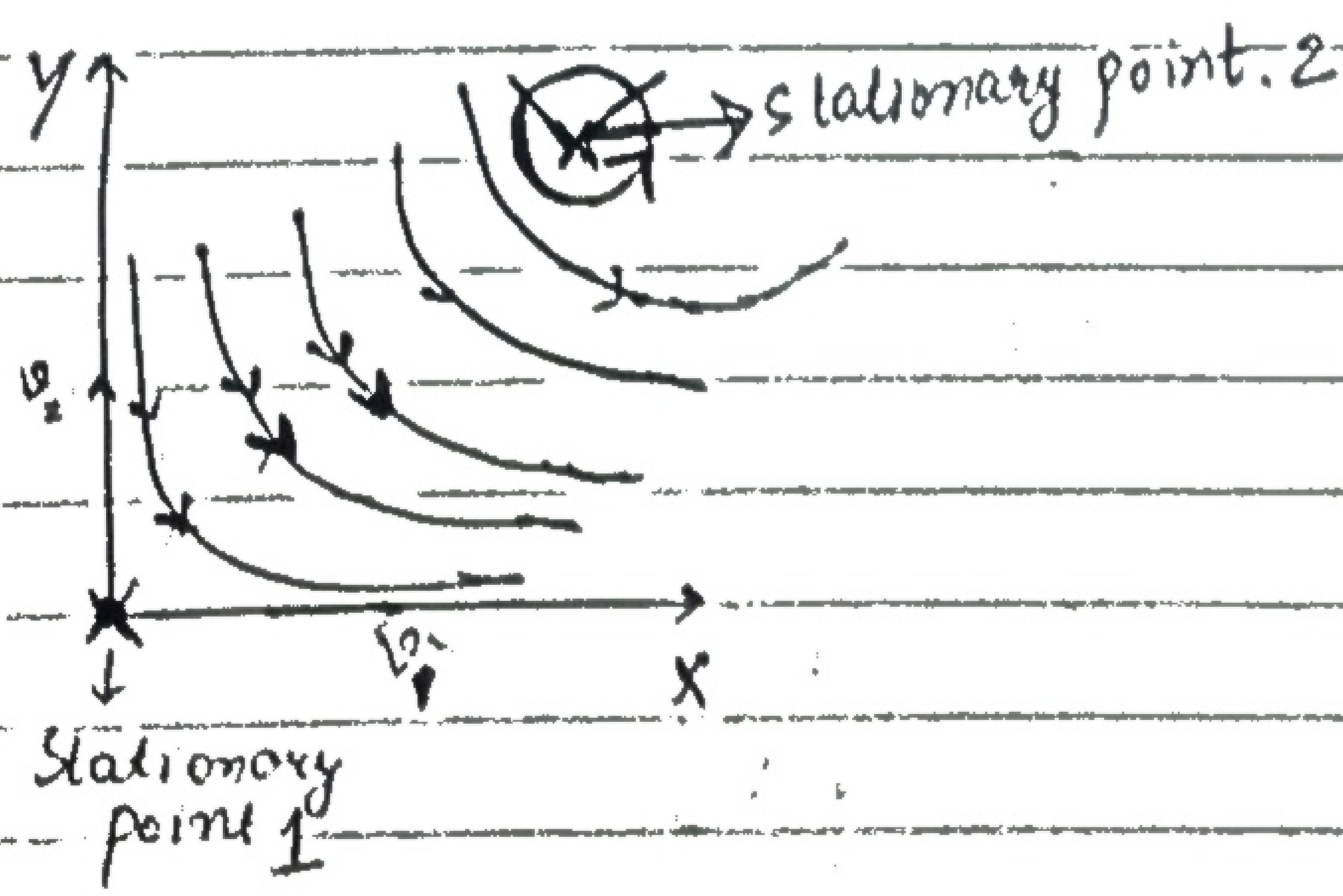
$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \bar{F}_0 + \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}\begin{pmatrix} X-0 \\ Y-0 \end{pmatrix} \longrightarrow \frac{d\bar{r}}{dt} = J_{(0,0)}\bar{r}$$

$$J = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & & \frac{\partial f_m}{\partial x_n} \end{pmatrix}\right|_{(X_0, Y_0)}$$

Case - II will be when $X_0, Y_0 = \left(\frac{\delta}{\gamma}, \frac{\alpha}{\beta}\right)$.

$$J = \begin{bmatrix} 0 & -\frac{\delta\beta}{\gamma} \\ \frac{\alpha\gamma}{\beta} & 0 \end{bmatrix}$$

$$\lambda_{1,2} = \pm i\sqrt{\alpha\delta}$$



$m\ddot{x} = -kx$

Let $x = c e^{\lambda t}$

$m(\lambda^2 x) = -kx$

$\lambda^2 = \frac{-k}{m}$

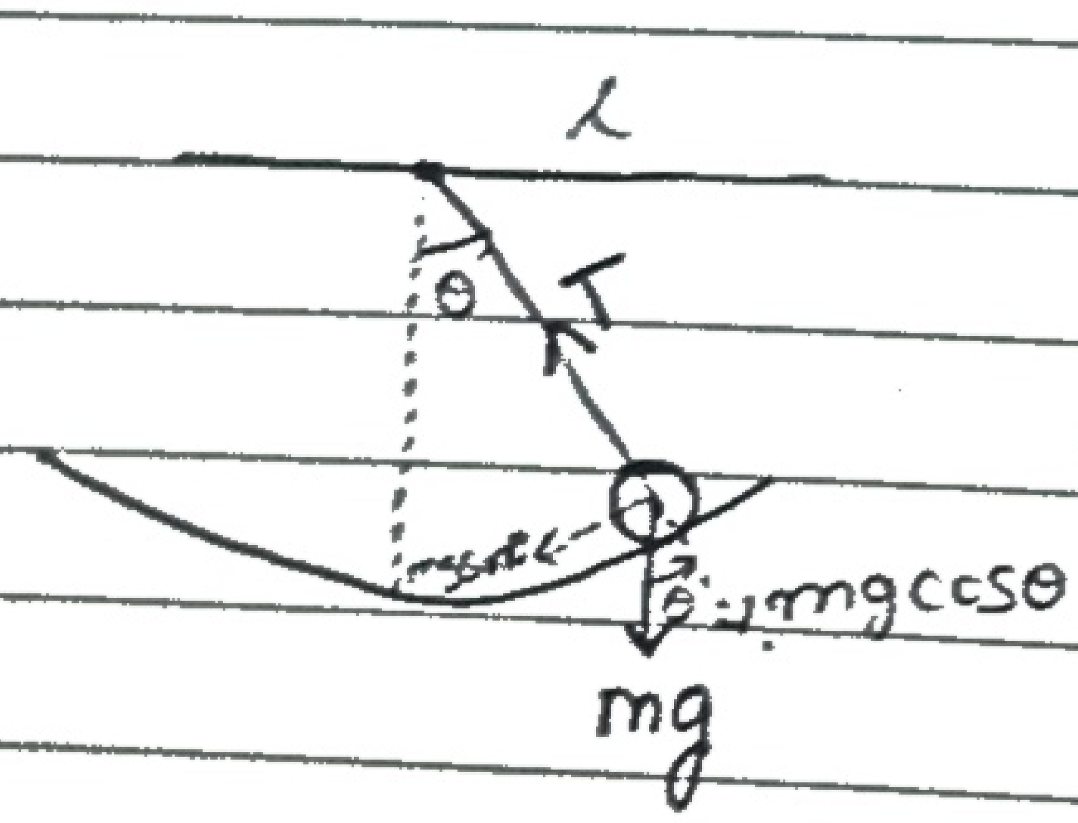
$\lambda = \pm i\omega_0$

$x(t) = c_1 e^{i(\omega t+\phi)} + c_2 e^{-i(\omega t+\phi)}$

$x(t) = (c_1+c_2)\cos(\omega t+\phi) + i(c_1-c_2)\sin(\omega t+\phi)$,

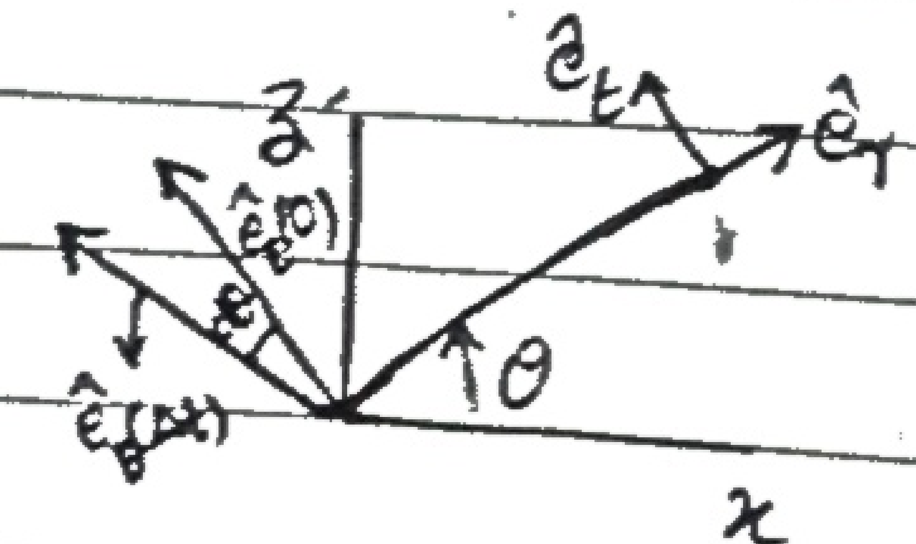
If
$(x, \dot{x})(0) = (0, V_0)$ or $(x_0, 0)$ — $c_1 = c_2$ — $\underline{x(t) = 2c_1\cos(\omega t+\phi)}$

(By initial conditions we can find ϕ)



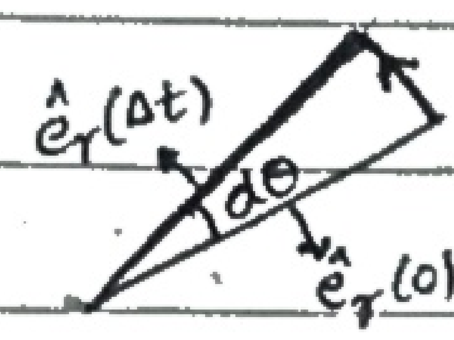
$\overline{r}(x,z) = x\cdot\hat{e}_x + z\,\hat{e}_z$

$\overline{r} = (r,\theta) = r\hat{e}_r + \theta\,\hat{e}_\theta$



$$m\frac{d\ddot{\bar{r}}}{dt} = -mg\sin\theta\,\hat{e}_\theta + \theta\,\hat{e}_r \quad \text{---- I}$$

$$\frac{d\bar{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} + \frac{d\theta}{dt}\hat{e}_\theta + \theta\frac{d\hat{e}_\theta}{dt}$$



$$\frac{d\hat{e}_r}{dt} = \frac{(r\,d\theta)\hat{e}_\theta}{dt} \quad \text{where } r=1.$$

$$\therefore \frac{d\hat{e}_r}{dt} = \dot{\theta}\,\hat{e}_\theta$$

Similarly, $\frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\,\hat{e}_r$

If l is made constant, then.
from I,

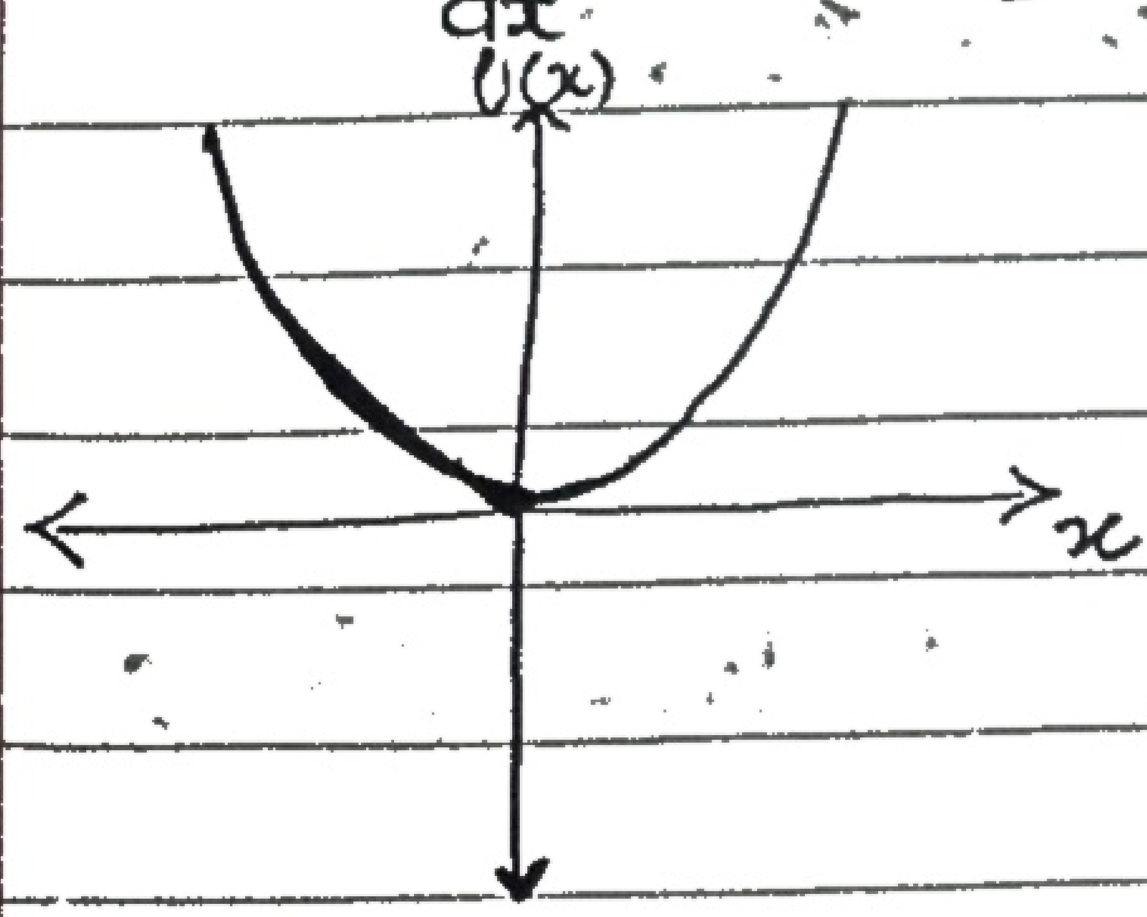
$\hat{e}_\theta$: $ml\ddot{\theta} = -mg\sin\theta$

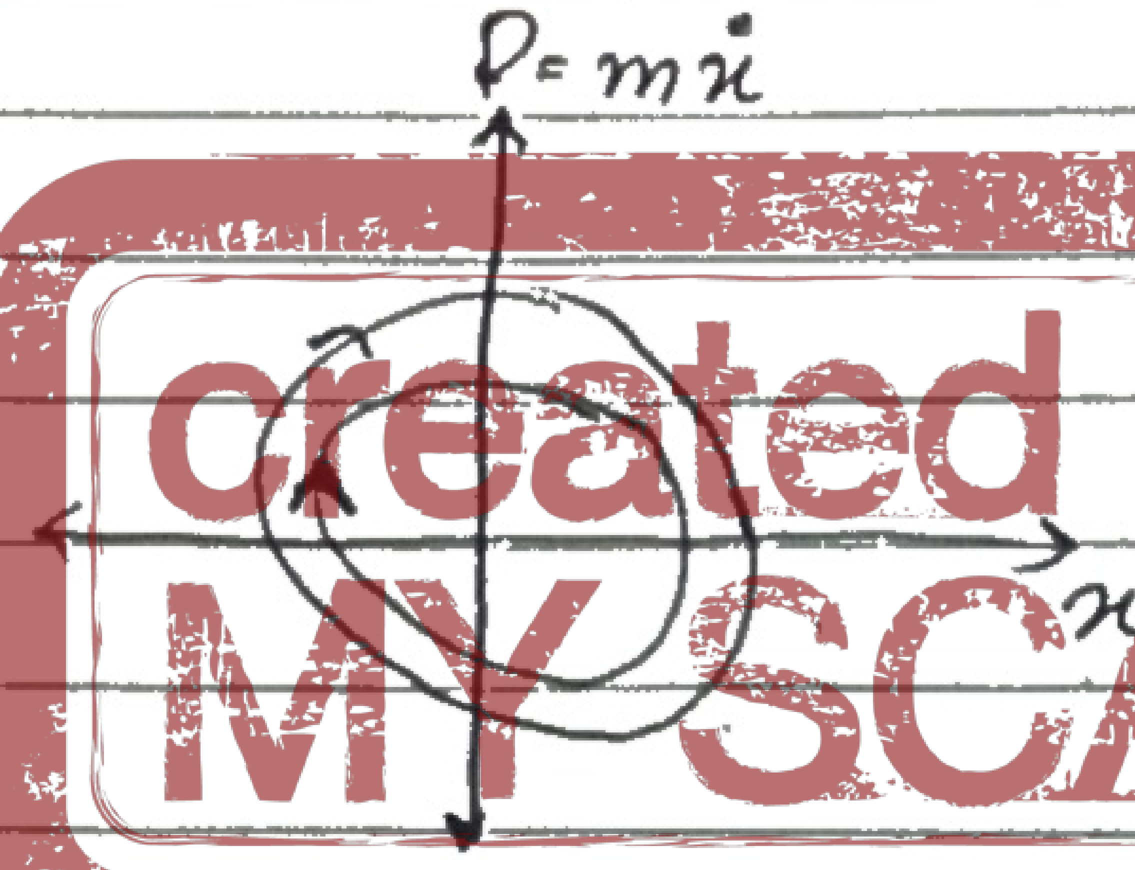
$$\therefore \ddot{\theta} = -\frac{g}{l}(\theta) \qquad \left[\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right]$$

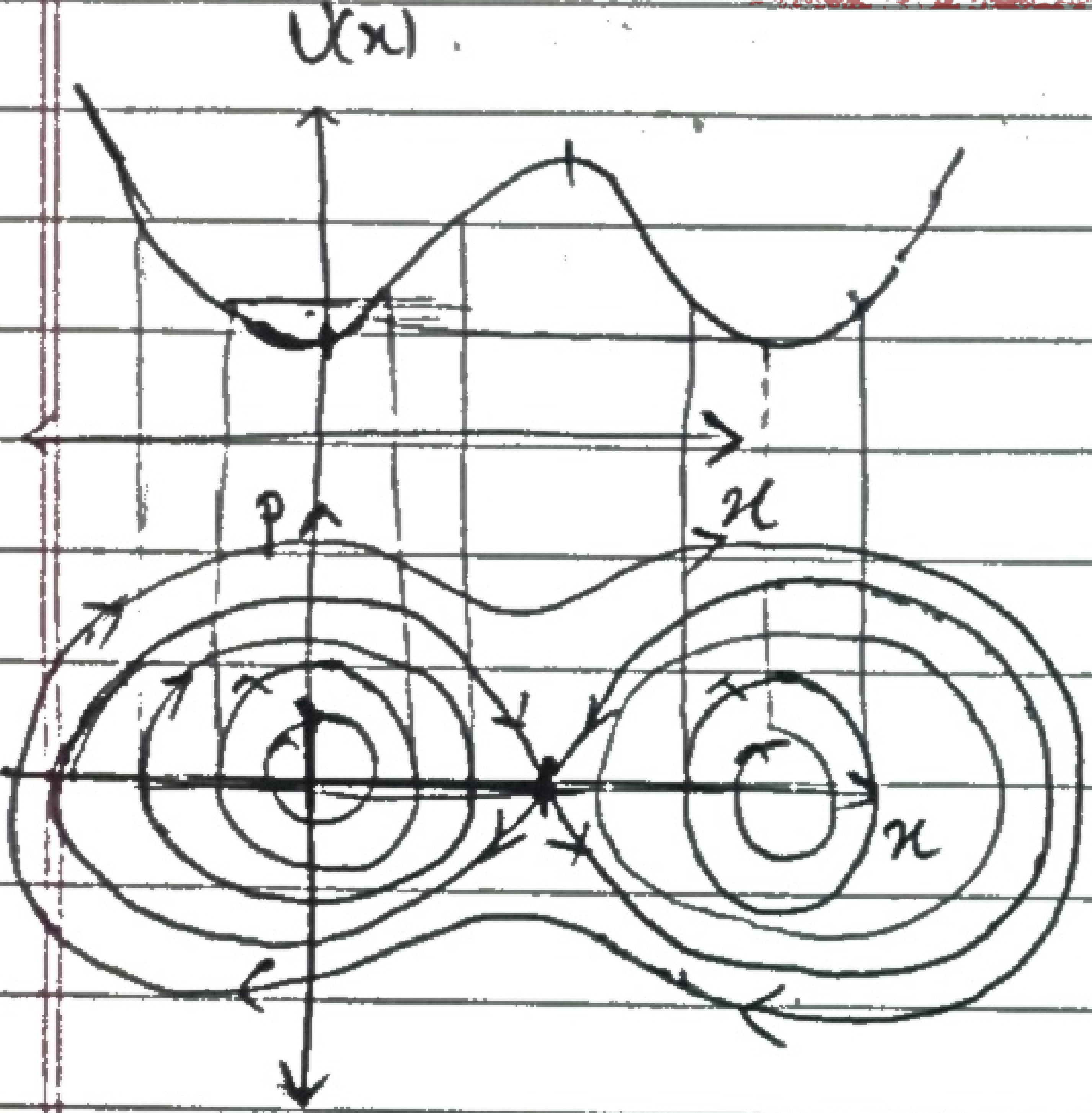
$$\left[\because \text{for small } \theta,\ \sin\theta = \theta\right]$$

$F(x) = -kx$

$F(x) = -\frac{dU(x)}{dx} = \frac{1}{2}kx^2 + C$ where $C=0$.







$$U(x) = U(X_0) + \frac{dU}{dx}\Big|_{X_0}(X - X_0) + \frac{1}{2}\frac{d^2U}{dx^2}(X - X_0)^2 \ldots$$

H – H

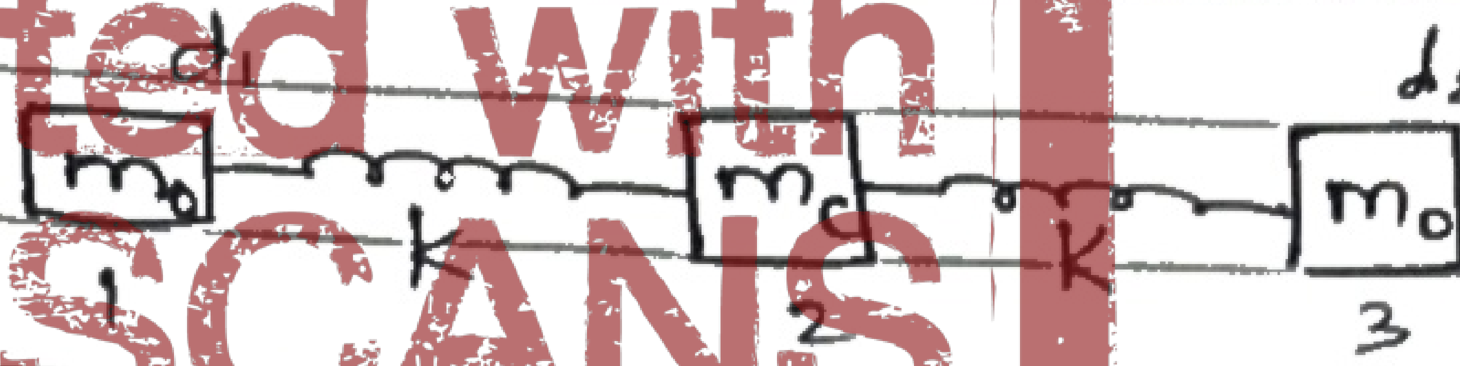
$$U(d) = \frac{1}{2}k(d - d_0)^2$$

$$\overset{\quad d_1 \quad\; d_2}{O = C = O}$$
$$x_1 \quad x_2 \quad x_3$$

$$U(x_1, x_2, x_3) = \frac{1}{2}k\left((x_2 - x_1) - d_0\right)^2 + \frac{1}{2}k\left((x_3 - x_2) - d_0\right)^2$$

$$U(d_1, d_2) = \frac{1}{2}k(d_1 - d_0)^2 + \frac{1}{2}k(d_2 - d_1)^2$$

$$U(\bar{r}_1, \bar{r}_2 \ldots, \bar{r}_N) = U(|r_1 - r_2|, |r_1 - r_3|, |r_2 - r_3|) \text{ for 1 to 3}$$



$$m_0 \frac{d^2 x_1}{dt^2} = k(d_1 - d_0)$$

$$m_c \frac{d^2 x_3}{dt^2} = k(d_2 - d_1)$$

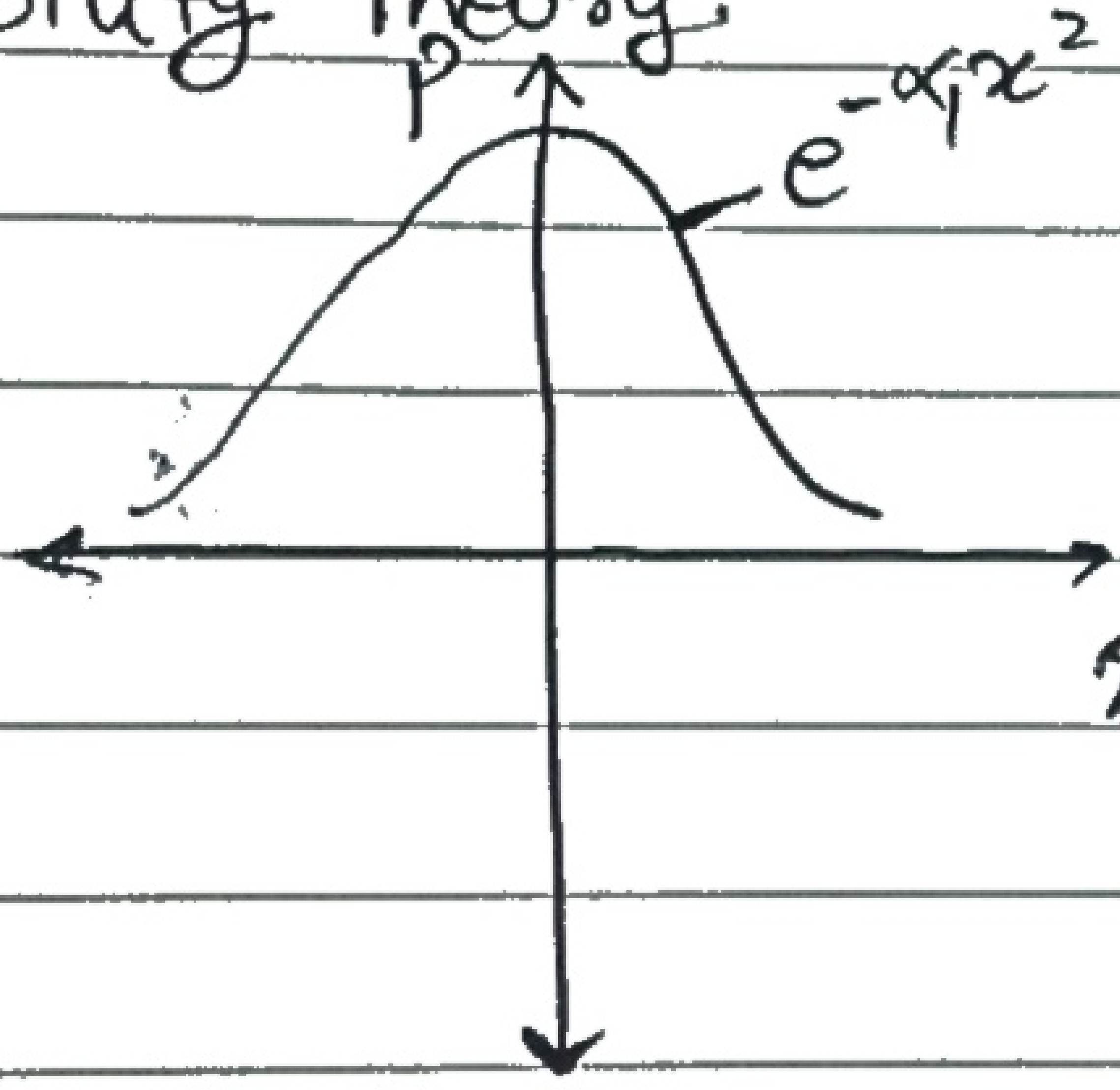
⋮ ⋮

$$\bar{r} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$\frac{d^2 \bar{r}}{dt^2} = \bar{M} \vec{r}$$

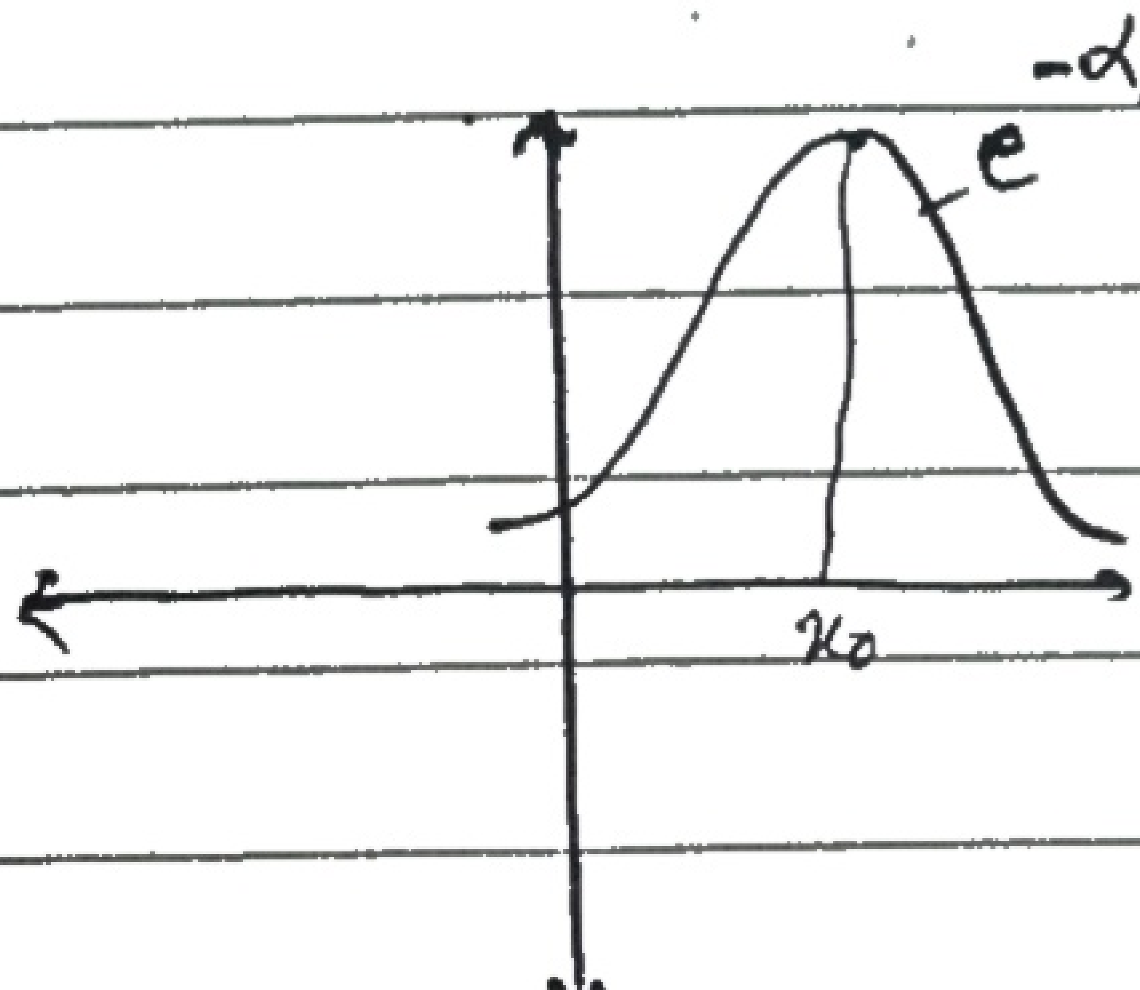
Probability Theory

Fair coin



H: Heads
T: Tails

$$x = \frac{\#H - \#T}{N} \qquad N \to \#H + \#T.$$

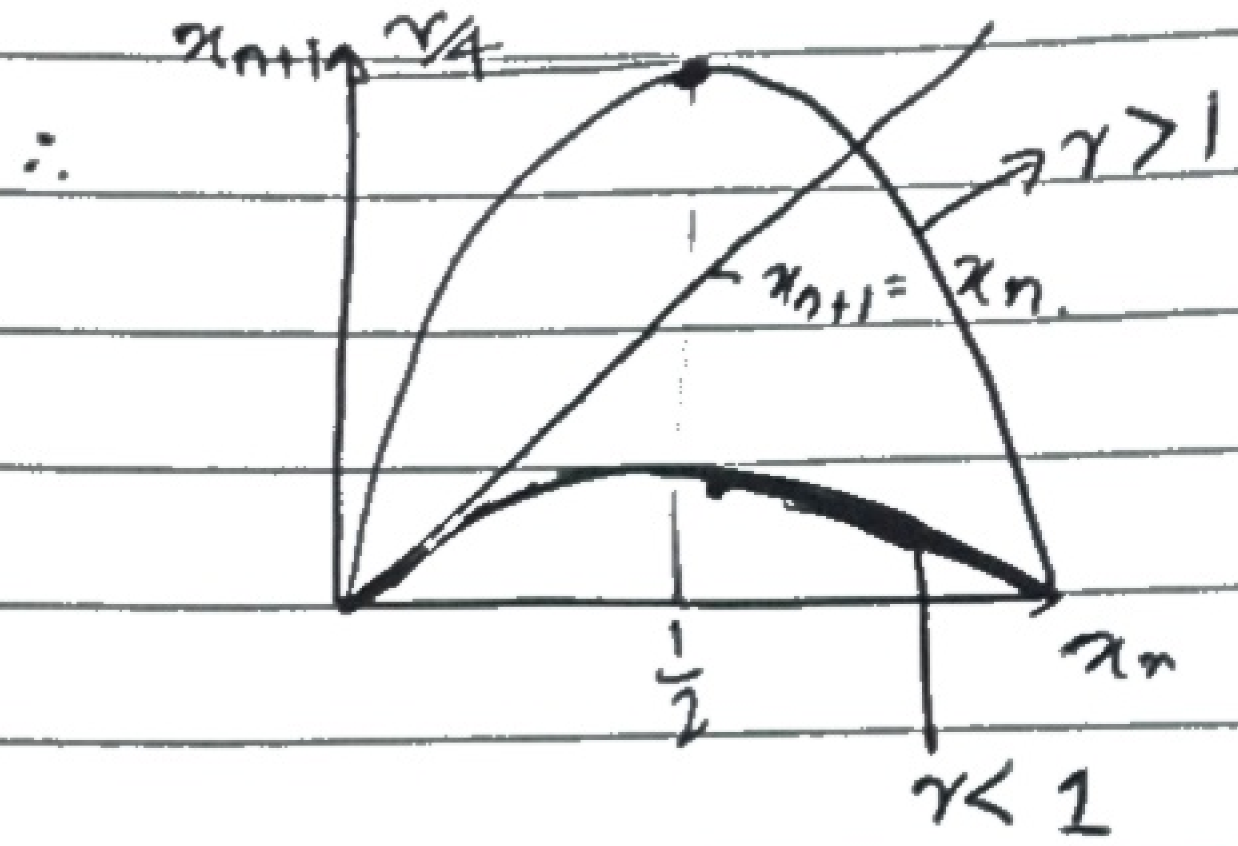


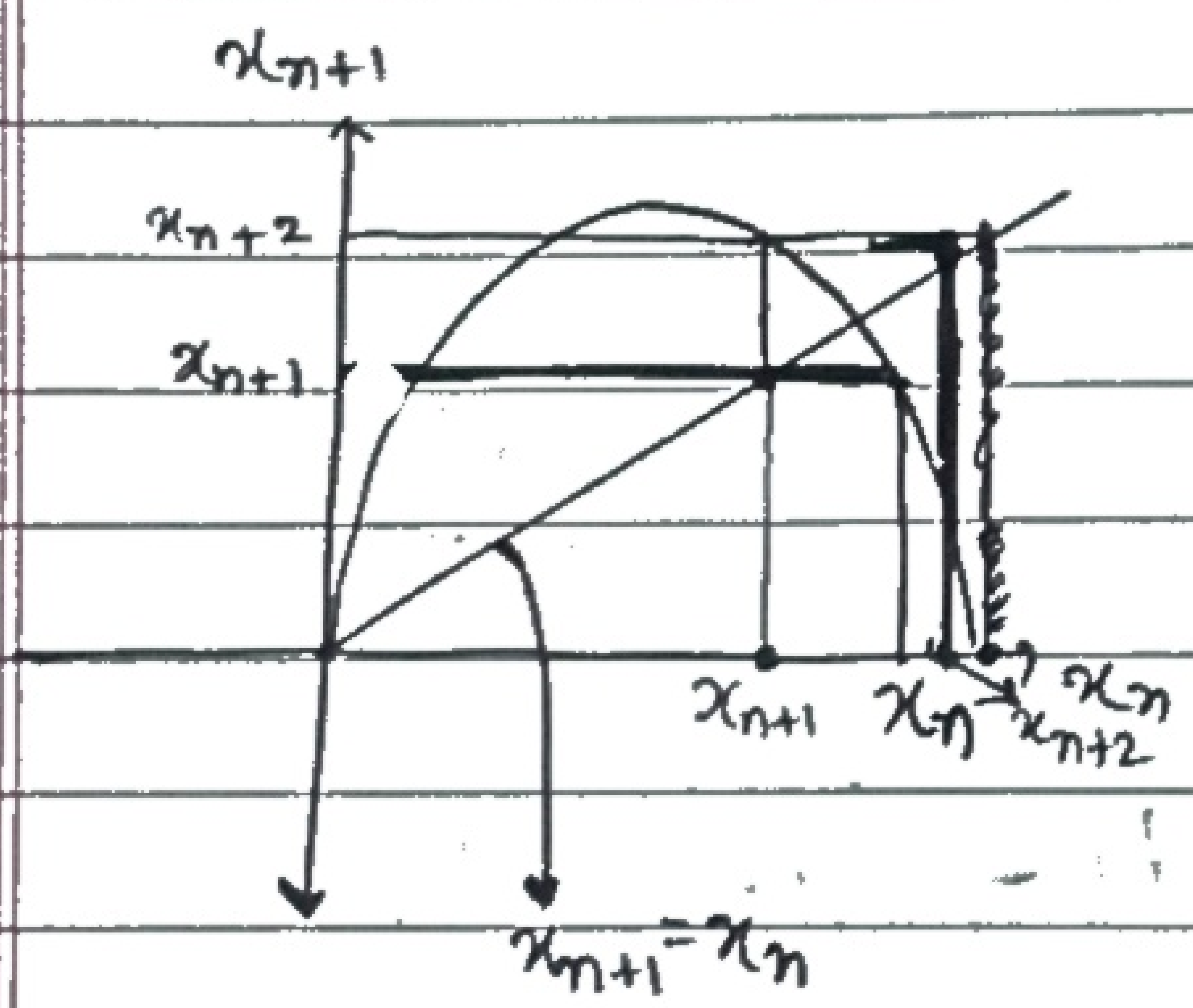
Unfair Coin

$x_{n+1} = r\, x_n(1-x_n)$

$r \in [0,4]$

$(x_{n+1})_{max} = \frac{r}{4}$ for $x_n \in [0,1]$.

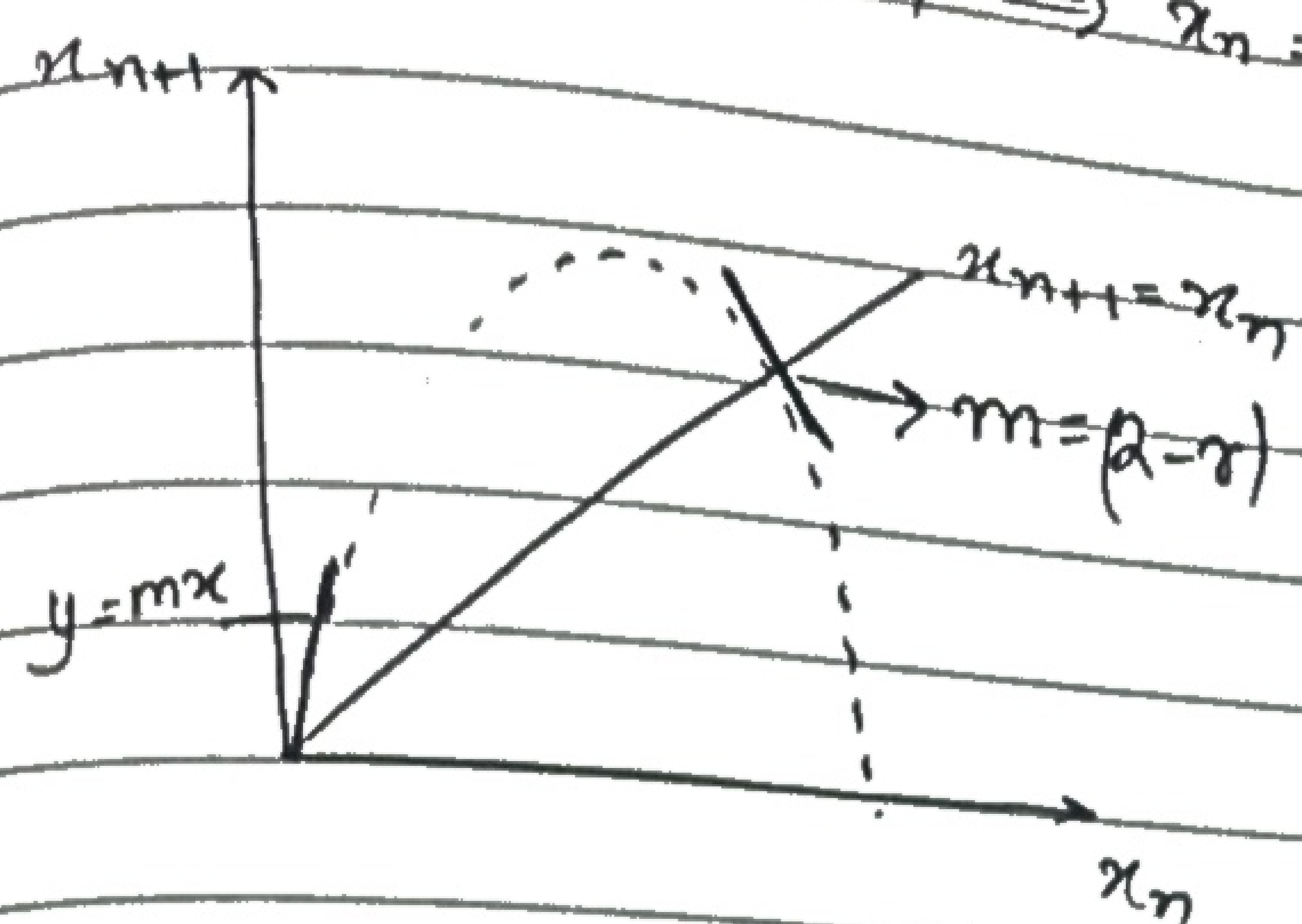




CHECK THE LAST CLASS NOTES IN DB NB.

$$x_{n+1} = r x_n (1 - x_n)$$

fixed point: $x_{n+1} = x_n \Rightarrow x_n = \left[0, 1 - \frac{1}{r}\right]$



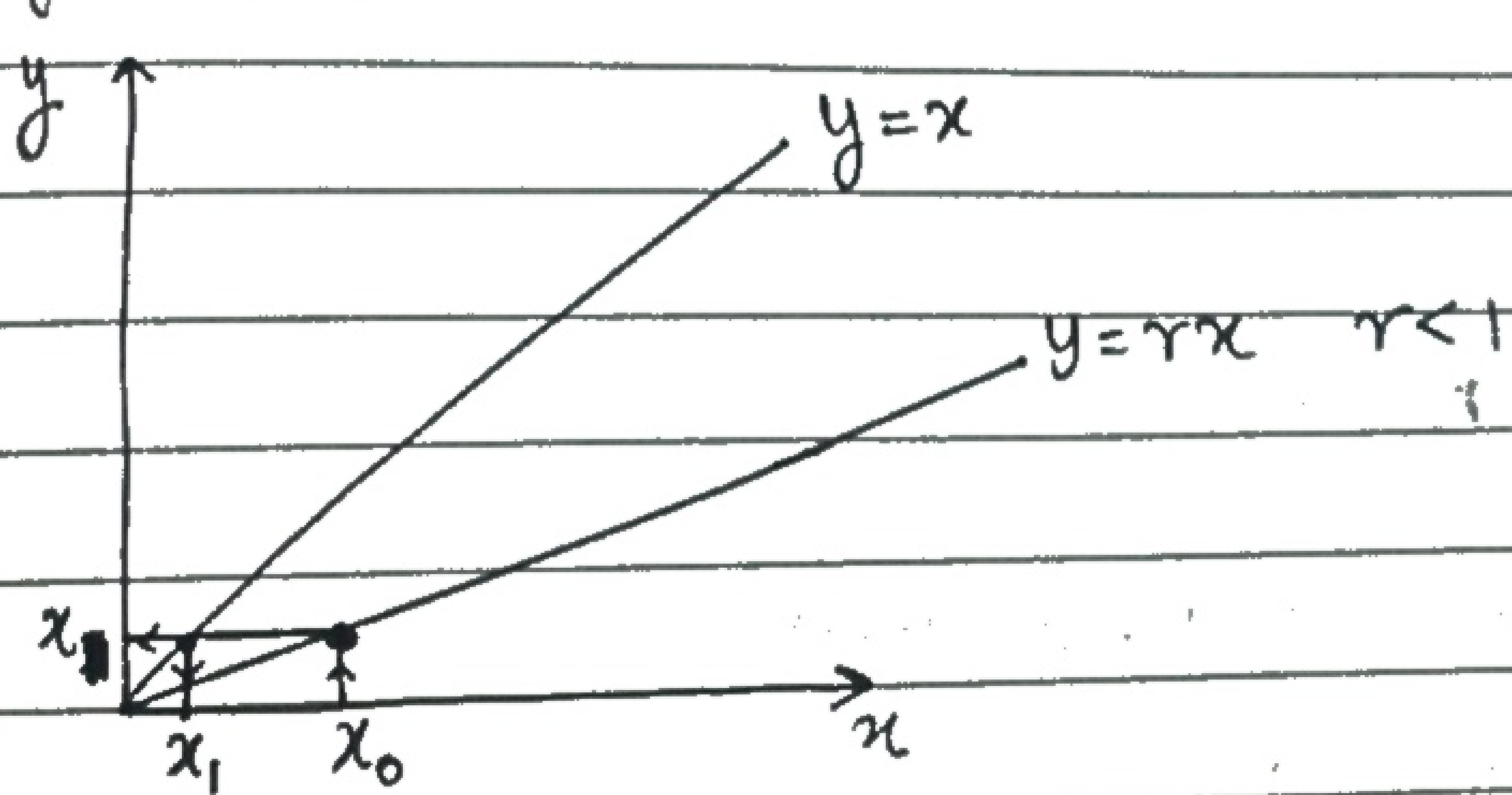
$$y = rx - rx^2$$

$$\frac{dy}{dx} = r[1-2x]$$

$x = 0 \Rightarrow r$

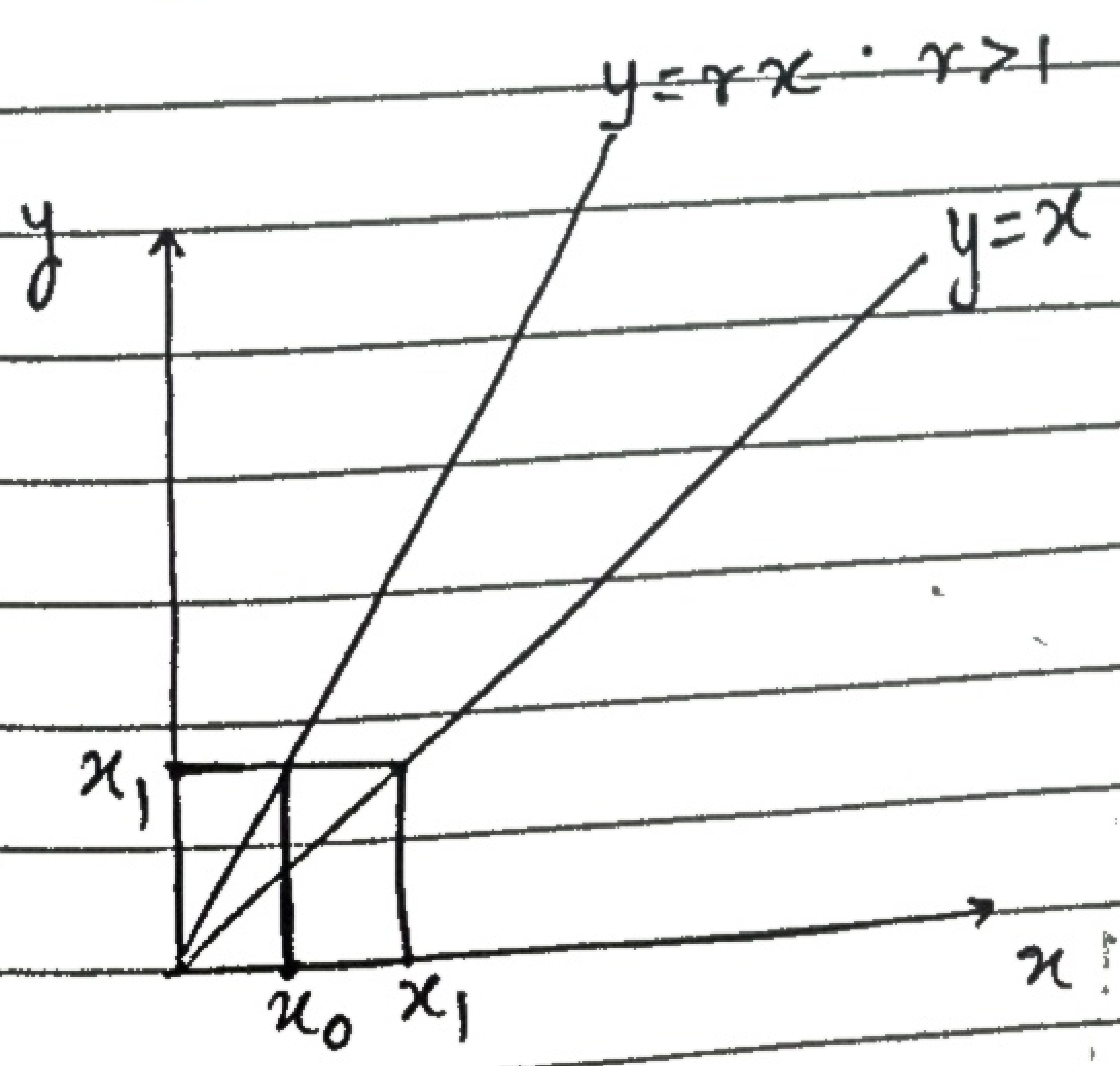
$x = 1 - \frac{1}{r} \Rightarrow r\left[-1 + \frac{2}{r}\right] = (2-r)$

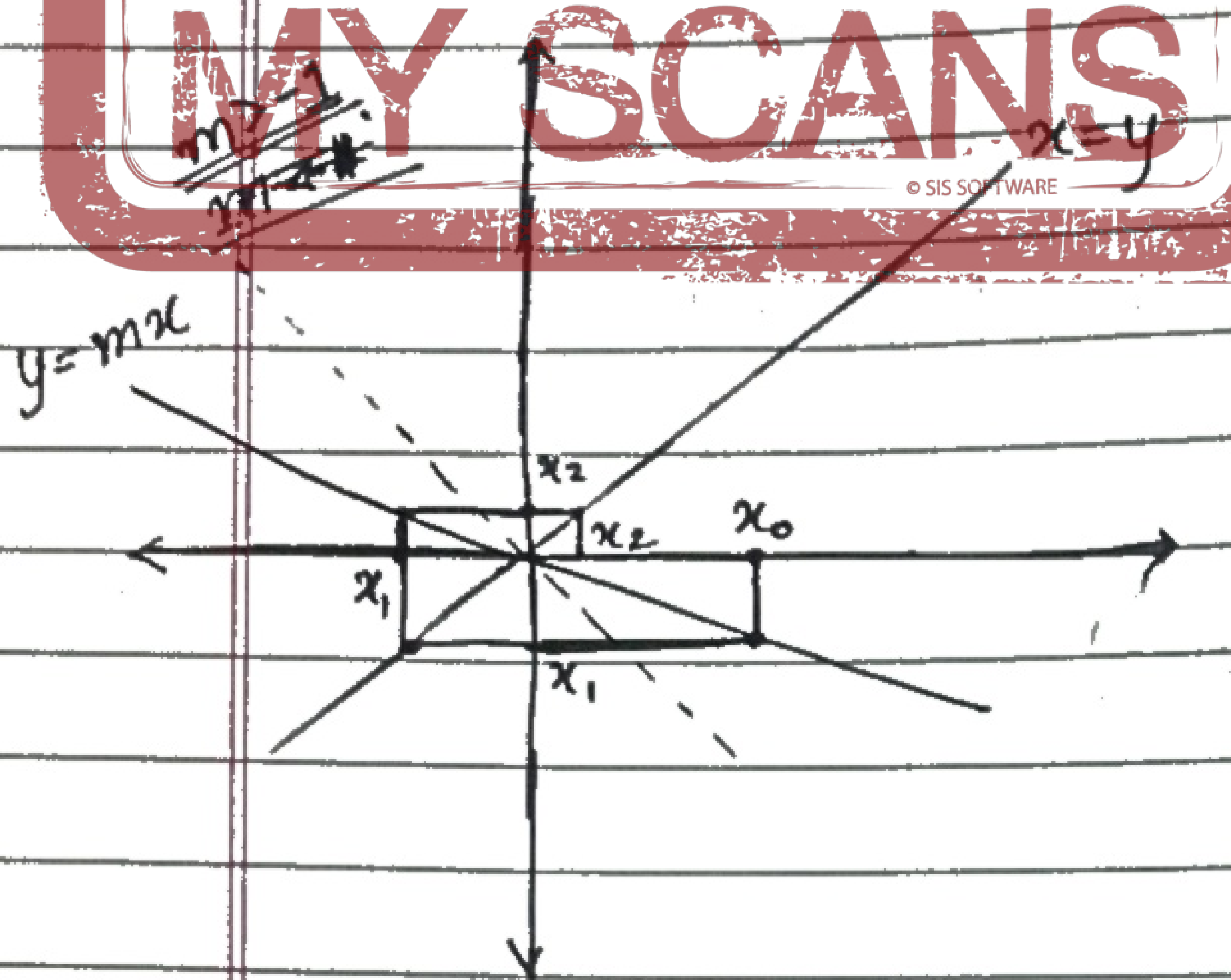
$y = mx + c$



$0 < m < 1$ stable

$m > 1$ unstable





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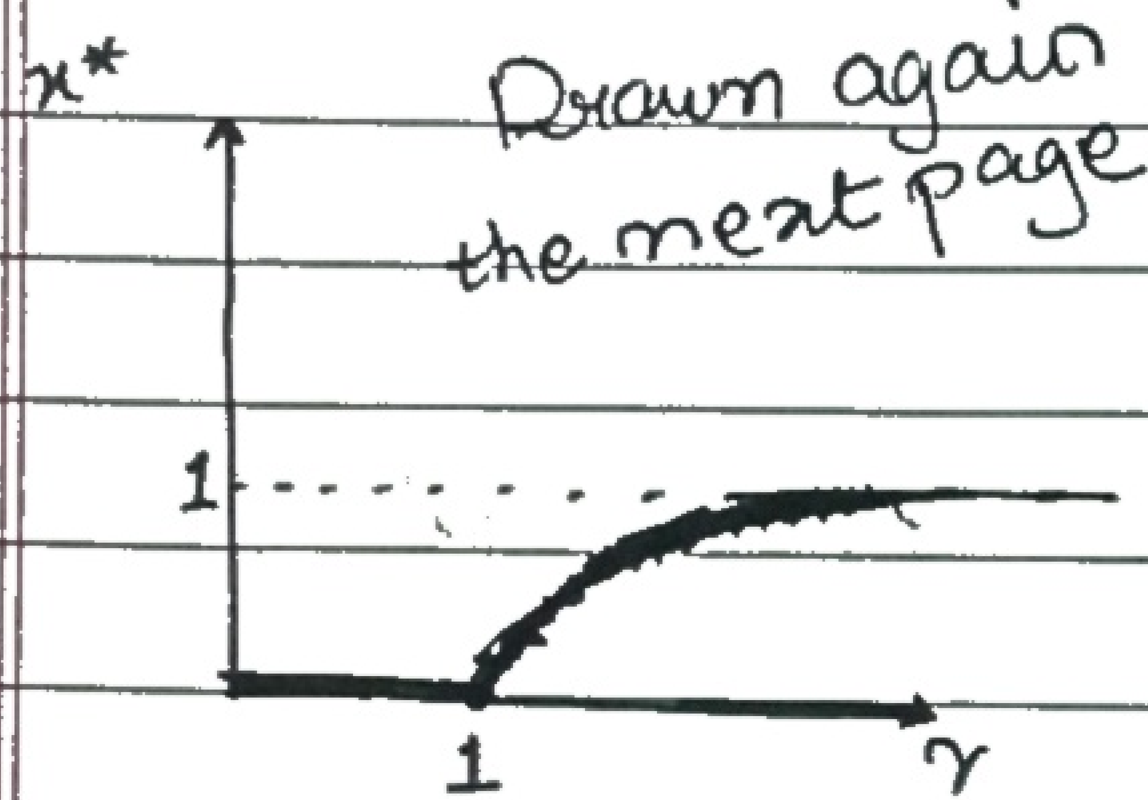


x^* is the stable point.
As we know,
if $r < 1$, then $x^* = 0$
if $r > 1$ $\quad x^* = 0$ or $1 - \frac{1}{r}$

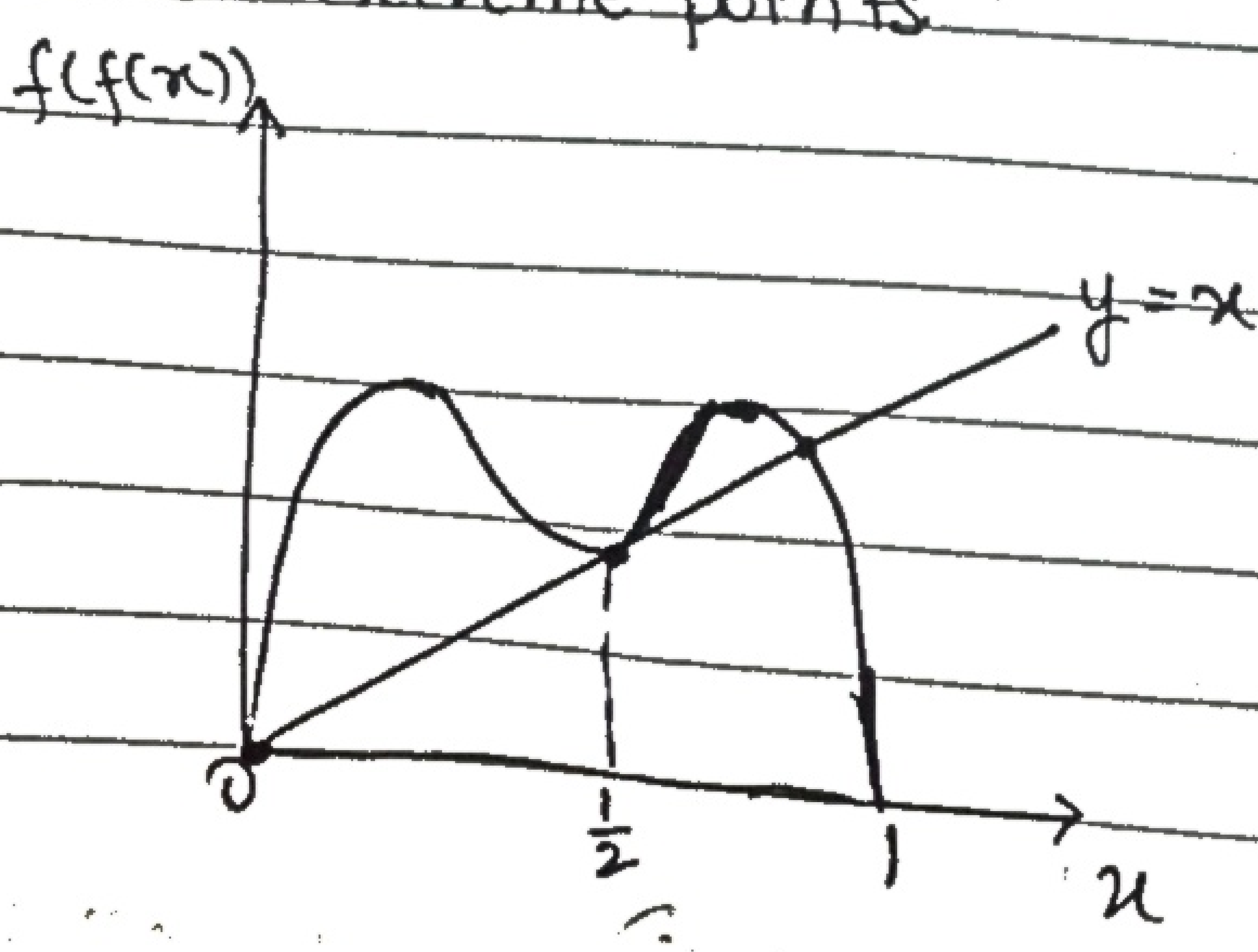
$$f(f(x)) = f(r(x)(1-x))$$

$$= r[r(x)(1-x)][1 - r(x)(1-x)]$$

= biquadratic

$\therefore f'(f(x))$ = cubic.

$\therefore$ three extreme points



Stable point.

$x = f(f(x))$

$$S_1 = 1 - \frac{1}{r}$$

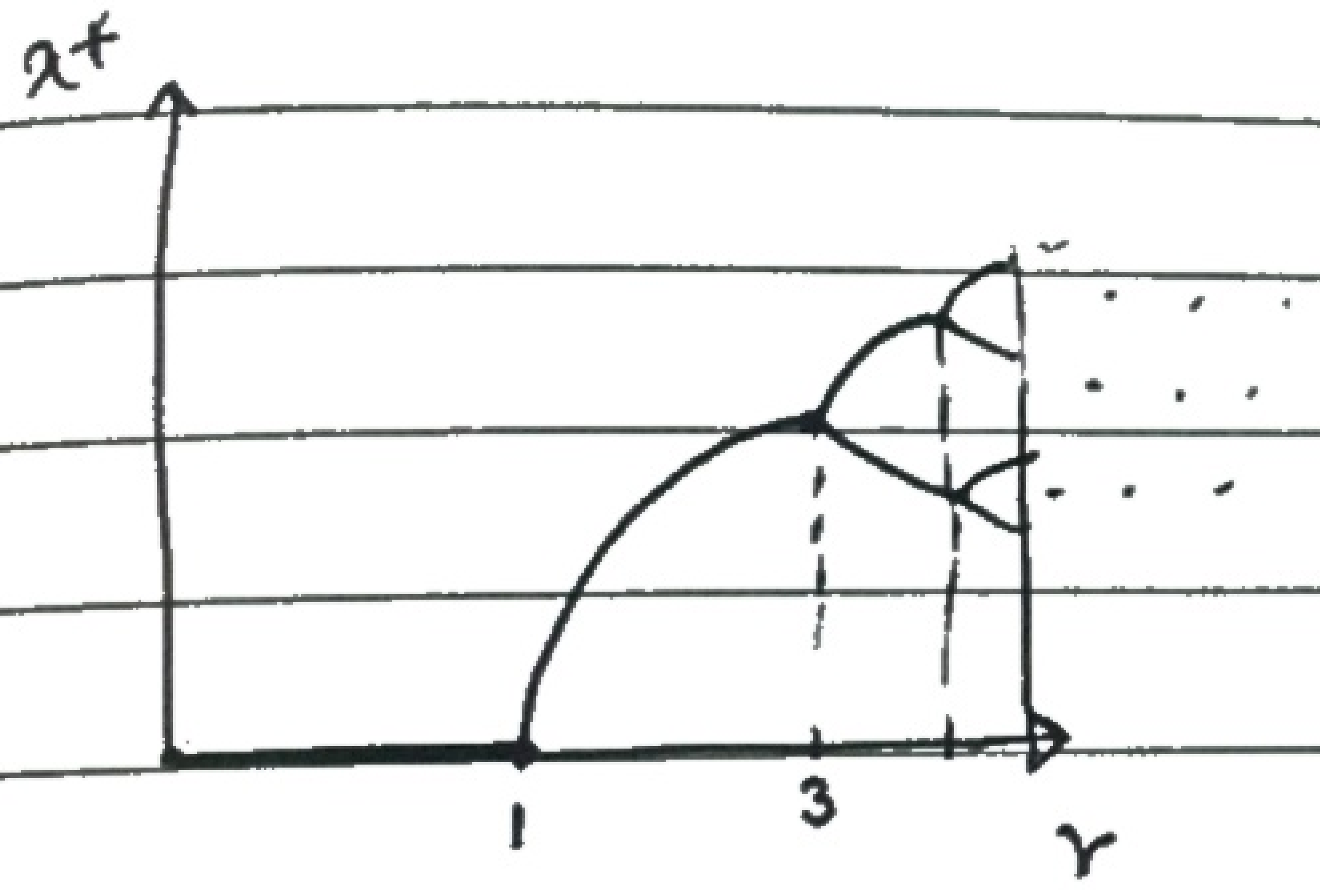
$$S_2 = \frac{(r+1) \pm \sqrt{(r+1)(r-3)}}{2r}$$

S_2,

for $r > 3$, two roots (cuts $x=y$ at 2 points)

$r < 3$ imaginary (doesn't cut)

$r = 3$ 1 root. (cuts at one point).



x^* stable point.

FRACTALS :

Dimensions:

Euclidean - d_e

Topological - d_T

Fractal - d_F

$d_T < d_F < d_e$

Seemingly irregular objects have certain symmetry.

Prediction of spatial distribution of matter over wide ranges of l.

Self Similarity

length = $N \cdot l$ → unit

area = $N l^2$

volume = $N l^3$

$V = N l^{d_e}$

where d_e : dimensions.

$\log V = \log N + d_e \log l$

In general, d may not $\in \mathbb{Z}$ (Integers).

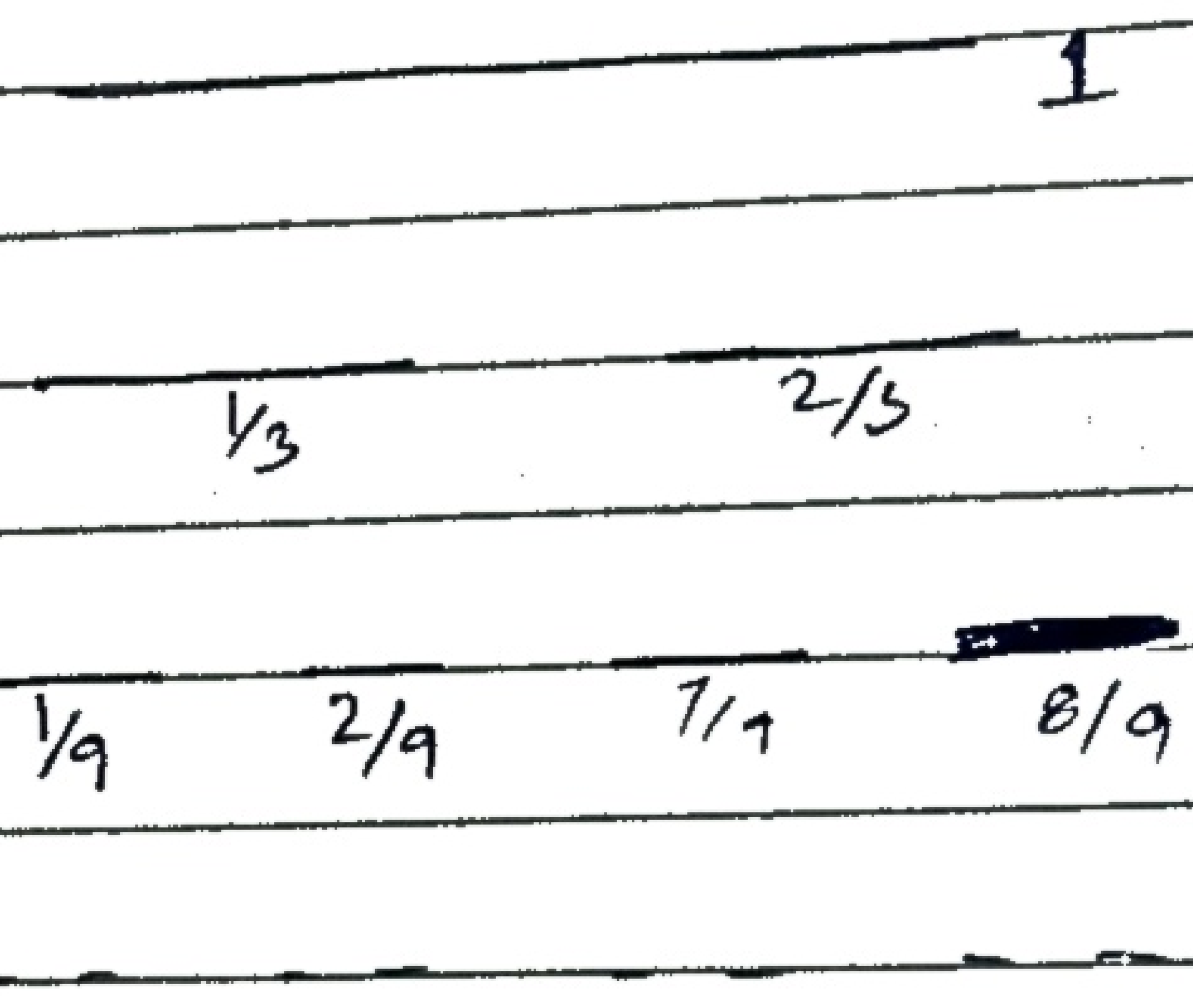
To Find dimension of a fractal

$N = S^d$

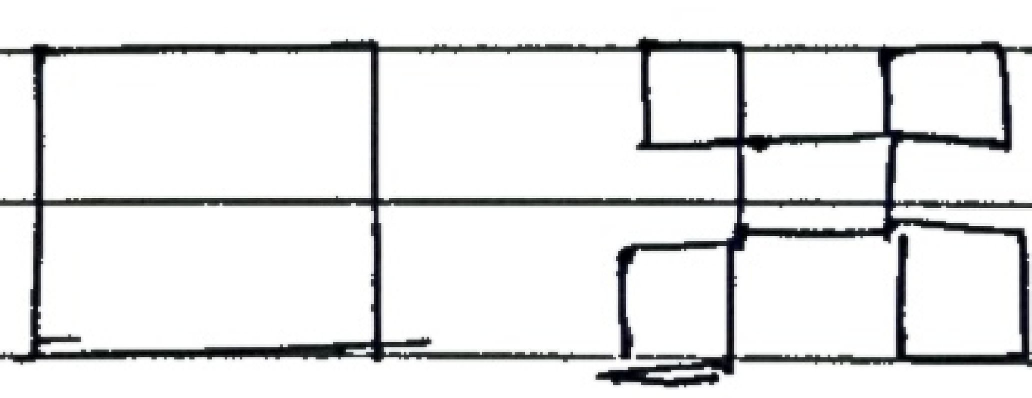
where N is no of pieces

S : Scaling factor

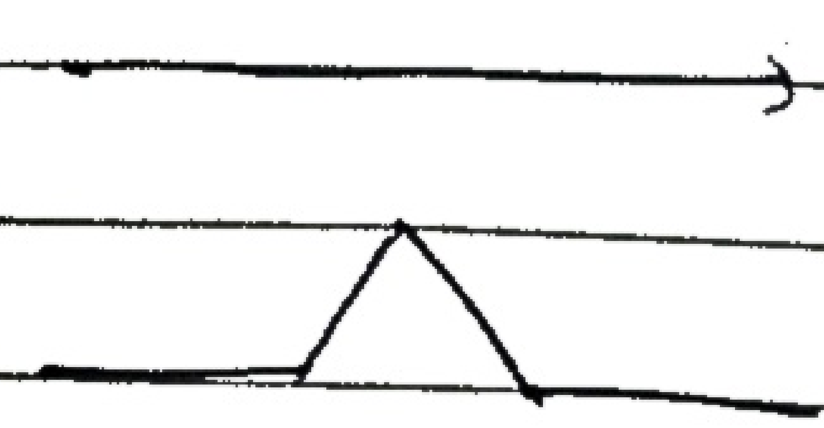
d : dimension



$N = 2 \quad S = 3 \quad D = \log_3 2$



$N = 5 \quad S = 3 \quad D = \log_3 5$



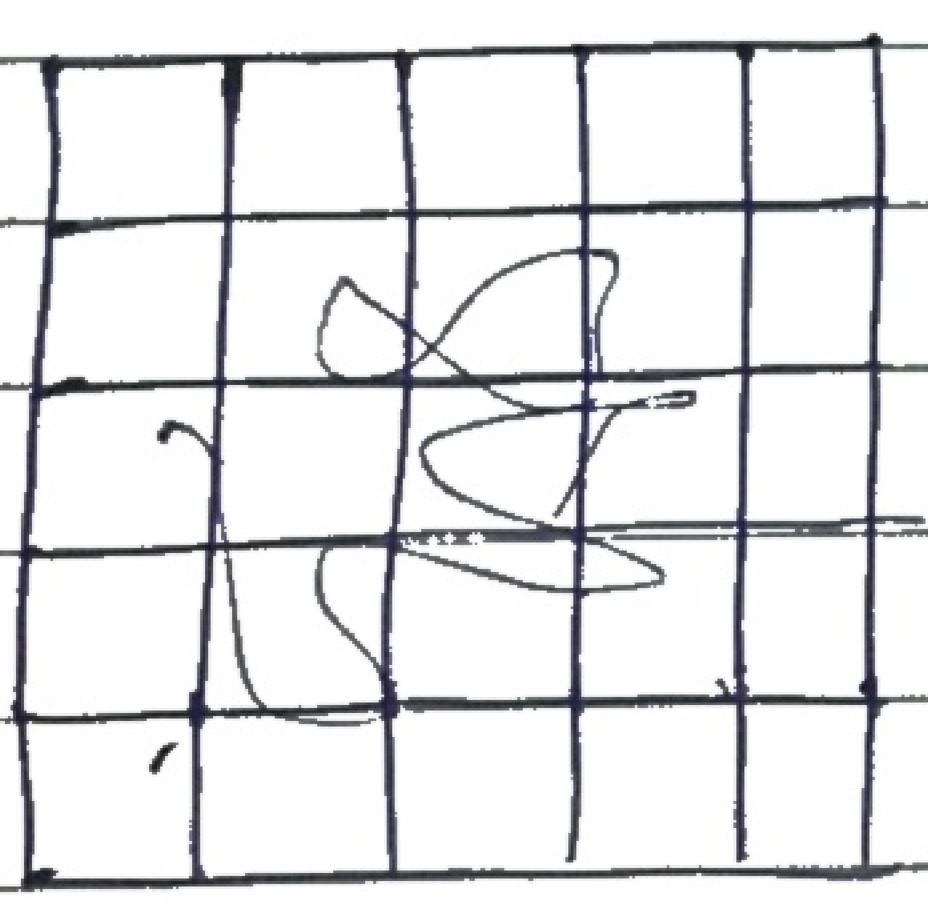
$N = 4 \quad S = 3 \quad D = \log_3 4$

$N = 8 \quad S = 4 \quad d = \log_4 8 = \frac{3}{2}$

Fractal Dimension: Box counting

Consider a grid of square boxes of side λ.

- Mark the cells that contain part of the boxes.
- The length of the boundary measured at this resolution is n.
- Then $d_F = \log n / \log(1/\lambda)$



λ