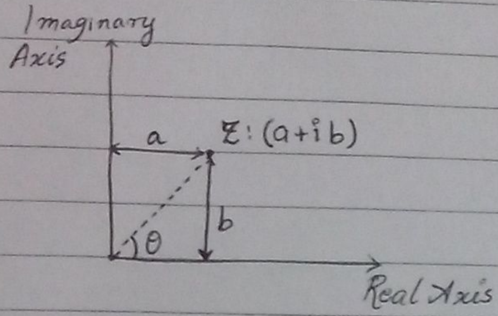
04-08-14

COMPLEX



Argand Diagram
or
Z - Plane

$a, b \in \mathbb{R}$, if $z = a + ib$

$Re(z) = a$, $Im(z) = b$

$|z| = r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\frac{b}{a}$

↑ argument of z (Arg z)

To each z, corresponds $\bar{z}$ (called conjugate of z) $= a - ib$

Note: $Re(z) = a = \frac{z+\bar{z}}{2}$

$Im(z) = b = \frac{z-\bar{z}}{2i}$

Thus if $z = a+ib$ then $\bar{z} = a - ib$ and $z\bar{z} = |z|^2$

Properties

(i) $\bar{\bar{z}} = z$

(ii) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

(iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ $|z_2| \neq 0$

Properties of modulus

(i) $|z| = |\bar{z}|$

(ii) $|z_1 z_2| = |z_1||z_2|$

(iii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad |z_2| \neq 0$

(iv) $Re(z) \leq |z|$, $Im(z) \leq |z|$

(v) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(vi) $||z_1| - |z_2|| \leq |z_1 - z_2|$

Proof: (v)

$$|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$
$$= |z_1|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 + |z_2|^2$$

$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)}$$

$$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$$

$$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$$

$$\therefore |z_1+z_2|^2 \leq (|z_1| + |z_2|)^2$$

$$\therefore |z_1+z_2| \leq |z_1| + |z_2|$$

classmate

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Proof : (vi)

$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 - z_2| + |z_2|$ by triangle inequality.

$\therefore |z_1| - |z_2| \leq |z_1 - z_2| \quad \ldots 1$

$|z_2| - |z_1| \leq |z_2 - z_1| \quad \ldots 2$

$\therefore \big||z_1| - |z_2|\big| \leq |z_1 - z_2|$ (from 1 and 2)

Examples.

(i) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

$(z_1 + z_2)(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)(\overline{z_1 - z_2})$

$\Rightarrow 2(|z_1|^2 + |z_2|^2) + z_1\overline{z_2} + z_2\overline{z_1} - z_2\overline{z_1} - z_1\overline{z_2}$

$\therefore \Rightarrow 2(|z_1|^2 + |z_2|^2)$

(ii) If $\overline{z_1}z_2 \neq 1$ but $|z_1| = 1$ then

$$\left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right| = 1$$

$$\left|\frac{z_1 - z_2}{\overline{z_1}(z_1 - z_2)}\right| \Rightarrow \frac{1}{|z_1|} = 1 \qquad \left[1 = |z_1|^2 = z_1\overline{z_1}\right]$$

Hence Proved.

(iii) If $z+\frac{1}{z}$ is real Prove either z is real or $|z|=1$.

Proof: $\left(z+\frac{1}{z}\right) = \overline{\left(z+\frac{1}{z}\right)}$

$$z+\frac{1}{z} = \bar{z}+\frac{1}{\bar{z}}$$

$$z-\frac{1}{\bar{z}} = \bar{z}-\frac{1}{z}$$

$$\frac{z\bar{z}-1}{(\bar{z})} = \frac{\bar{z}z-1}{z}$$

either $z\bar{z}=1$ or $z=\bar{z}$.

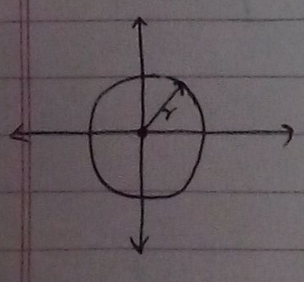
☆ Let $z_1 = a+ib$, $z_2 = c+id$

Then $|z_1-z_2| = |(a+ib)-(c+id)|$

$$= \sqrt{(a-c)^2+(b-d)^2}$$

= Distance b/w z_1 and z_2 on argand plane

☆ $|z|$ = real, z: variable point.



$$|z-0| = r$$

$$|z| = r$$

It is a circle with center at origin and radius r.

☆ $|z-a| = r$ where a is a complex number
A circle with center at (a) and radius r.

The circle is $z = a + re^{i\theta}$

Eg. $|(z-3+7i)| = 4$

centre = $(3,-7)$ radius = 4
or
$(3-i7)$

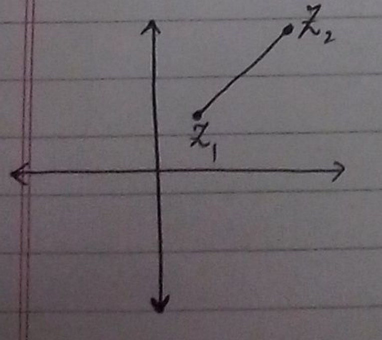
☆ $|z_1 - 1| > 4$

Set of all points in z plane outside the circle with centre 1 and radius 4.

☆ $2 \leq |z| \leq 3$

Set of all points b/w two circles with centre at origin and radius 2 and 3.

☆ $z = \lambda_1 z_1 + (1-\lambda) z_2$, $0 \leq \lambda \leq 1$, z_1 and z_2 are fixed.



This is a line segment joining z_1 and z_2.

$(x+iy) = \lambda_1(x_1+iy_1) + (1-\lambda)(x_2+iy_2)$

$x = \lambda(x_1-x_2) + x_2$

$y = \lambda(y_1-y_2) + y_2$

$(x - x_2) = \lambda(x_1-x_2)$

$(y-y_2) = \lambda(y_1-y_2)$

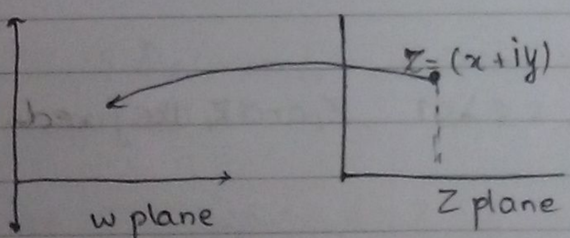
$(y-y_2) = \dfrac{(y_1-y_2)(x-x_2)}{(x_1-x_2)}$

☆ $|z+6i| = |z-1+3i|$

$|z-(-6i)| = |z-(1-3i)|$

z is the perpendicular bisector of line segment joining z_1 and z_2.

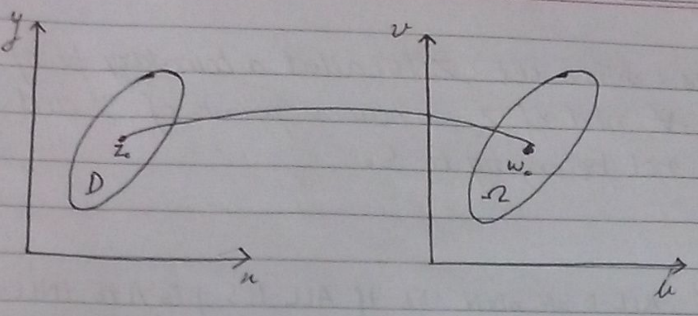
$w = f(z)$



$w = z^2$

if $z = x+iy$

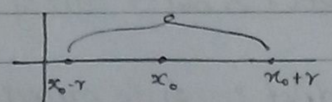
$w = (1+i)^2 = 2i$



We assume $f(z) = w$ is single values, infact we assume one-one mapping of point of a set D.

1. Connected Set. : A set D is said to be connected if any two points of D can be joined by a path made up entirely of the points of D. The path may be a curve or segments of line joined end to end

2. Neighbourhood (nhd).



$N(x_0, r) = \{x : x_0 - r < x < x_0 + r\}$

If $x_0 - r$ and $x_0 + r$ are to be included then it is $\overline{N}(x_0, r) = \{x : x_0 - r \le x \le x_0 + r\}$

$N(z_0, r) = \{z : |z_0 - z| < r\}$, i.e, set of all points within, but not on a center z_0, radius r.

3. Interior point. : A point z_0 is called the interior point of a set S if $\exists$ a nhd of z_0, which is made up entirely of the points of S

4. Boundary Points: S is a set, z_0 is called a boundary pt of S if $\forall$ nhd of z_0 contains points of S and points of not belonging to S.

5 Open Set: S is called an open set if all it's pts are interior pts of S. Eg. $|z|<r$ but $|z|\leq r$ is not open.

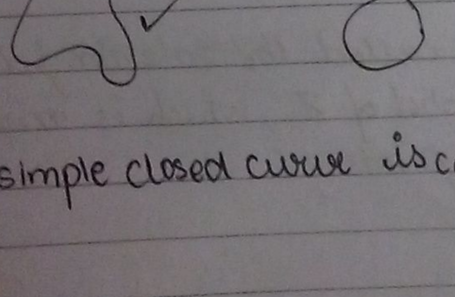
6. Domain: A set which is open and connected is called domain.

7. Limit Point:- S is a set, z_0 is called a limit point of the set, if $\forall$ nhd of z_0 contains inf. points of S.

8. Closed Set: A set S. all of whose limit points belong to the set.

9. Bounded Set:- S is called a bounded set if $\exists\, r \in \mathbb{R}$, s.t. $|z|<r$, $\forall z \in S$.

Simple Curve



It should not cut itself

A simple closed curve is called contour.

Complex function of z i.e $f(z)$ can be written as

(i) Cartesian form : $f(z) = u(x,y) + iv(x,y)$

(ii) Polar form : $f(z) = u(r,\theta) + i\,v(r,\theta)$

Example:- $f(z) = z^2 - z + 2$

(i) Cartesian form: $f(z) = (x+iy)^2 - (x+iy) + 2$

$= (x^2 - y^2 - x + 2) + i(2xy - y)$

(ii) Polar form : $f(z) = z^2 - z + 2$

$= (re^{i\theta})^2 - re^{i\theta} + 2$

$= r^2 e^{2i\theta} - re^{i\theta} + 2$

$= r^2\cos 2\theta - r\cos\theta + 2 + i(r^2\sin 2\theta - r\sin\theta)$.

Example 2.: Find the map of the line segment joining $z = 2+3i$ and $z = 4+5i$ under the mapping $w = \frac{1}{2}z + i$. The line segment is $z = \lambda(2+3i) + (1-\lambda)(4+5i) \quad 0 \le \lambda \le 1$.

$w = u + iv = \frac{1}{2}z + i$. substitute z.

Ex 3. $w = z^{-1}$

$Re^{i\phi} = r^{-1}e^{-i\theta}$ where $z = re^{i\theta}$

$R = r^{-1} \quad \phi = -\theta$

Thus if $r < 1$, then $R > 1$, i.e, inside of unit circle in z-plane is mapped outside of unit circle in w plane

PROBABILITY THEORY

DEF: SAMPLE SPACE: set of outcomes of an experiment.

DEF: EVENT IN S: A subset of S is called an event.

DEF: ELEMENTARY EVENT: Each outcome of the experiment is called an elementary event.

CONDITIONAL PROBABILITY.

Two dice are tossed,

A: Event when the sum of two dice is 6.

$$P(A) = \frac{5}{36}$$

B: Event that sum is 6 and first dice has no. 4.

$$P(B) = \frac{1}{6}$$

$$= \frac{1/36}{6/36}$$

★ $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ ← Conditional Probability.

①. Cards no. 1–10 are placed in a box. 1 card is drawn if the no. is atleast 5 what is the probability that it is 10.

$P(10) = \dfrac{1}{6} = P(10/5)$ or $P(A/B)$ where A: No. is 10

B: atleast 5

Q. Two dice are tossed. What is P. that atleast 1 is 6 if both the faces are diff.

$$P(A|B) = \frac{10}{30} = \frac{1}{3}$$

Q. A box contains 5 defective trans. which donot work, and 10 partially defect. which fail after couple of hours and 25 perfect transistors. What is the Probability that it is an acceptable transistor if a transistor chosen doesn't fail immediately

Ans. $\frac{5}{7}$

Q. An employee of a company has 30% chance that if a new branch opened then he has 60% chance to be the manager. What is the P that he shall be the manager of the new branch of the company.

Ans. $\frac{9}{50}$

BAYES' THEOREM.

$S = A \cup A^c$ with $A \cap A^c = \phi$

$B = B \cap S = B \cap (A \cup A^c)$

$= (B \cap A) \cup (B \cap A^c)$

$$P(B) = P(B \cap A) + P(B \cap A^c)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B \cap A) + P(B \cap A^c)} \quad -2$$

he no.

$$P(A/B) = \frac{P(B/A)\ P(A)}{P(B|A)P(A)+P(B|A^c)\ P(A^c)} \quad -3$$

$$= \frac{P(B|A)\ P(A)}{P(B|A)\ P(A)+P(B|A^c)[1-P(A)]} \quad -4$$

(ii) S is partitioned by A_i, $i=1\ldots n$

i.e, $S=\bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \phi$

$$B = B\cap S = B\cap(\bigcup S_i) = \bigcup_{i=1}^{n}(B\cap A_i)$$

$$P(B) = \sum_{i=1}^{n} P(B\cap A_i)$$

$$P(A_i/B) = \frac{P(B/A_i)\ P(A_i)}{\sum_{i=1}^{n} P(B/A_i)\ P(A_i)}$$

Q. The chance a doctor will diagonose a particular disease is 60%. The probability that the patient will die after correct diagonosis is 40%. The chance of death by wrong diagonosis is 70%.
The patient is treated by the doctor dies. What is P of correct diagonosed

A : Correctly diagoned
B : died

~~0.4 =~~

$$P(A/B) = \frac{P(B/A)\ P(A)}{P(B/A)P(A) + P(B/A^c)\ P(A^c)}$$

$$= \frac{0.4\times 0.6}{0.4\times 0.6 + 0.7\times 0.4}$$

$$= \frac{6}{13}$$

$w = z + \frac{1}{z}$

if $z = re^{i\theta}$

Then $w = re^{i\theta} + \frac{1}{r}e^{-i\theta}$

$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$

$u + iv = \left[r + \frac{1}{r}\right]\cos\theta + i\left[r - \frac{1}{r}\right]\sin\theta$

$u = \left[r + \frac{1}{r}\right]\cos\theta \quad v = \left[r - \frac{1}{r}\right]\sin\theta$

Eliminating θ,

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left[r - \frac{1}{r}\right]^2} = 1 \quad \ldots \quad (1)$$

The circle $|z| = r$ is transformed into the elipse given by (1)

Ex - 2

$w = z^2$

$u + iv = (x^2 - y^2) + i2xy$

$u = x^2 - y^2$

$v = 2xy$

If $x = a$, then $u = a^2 - y^2$

$v = 2ay$

$v^2 = 4a^2(a^2 - u) \longrightarrow z$ - plane.

This is a parabola with focus origin and vertex $u = a^2$

Ex-3. Find the image of $|z-3i|=3$ under $z=\frac{1}{w}$.

$$|z-3i|^2=9$$

$$(z-3i)\overline{(z-3i)}=9$$

$$\left(\frac{1}{w}-3i\right)\overline{\left(\frac{1}{w}-3i\right)}=9$$

$$\left(\frac{1-3iw}{w}\right)\overline{\left(\frac{1-3iw}{w}\right)}=9$$

$$(1-3iw)(1+3i\overline{w})=9w\overline{w}$$

$$1+3i\overline{w}-3iw=0$$

$$1+3i(\overline{w}-w)$$

$$1+3i(w-\overline{w})=0 \quad \text{OR } (1+6v=0) \text{ with } w=u+iv$$

Q. If $w=\frac{z+i}{iz+1}$, Prove the region $I(z)\leq 0$ maps with $|w|$.

$$\overline{w}=\overline{\frac{z+i}{iz+1}}$$

$$=\frac{\overline{z}-i}{-i\overline{z}+1}$$

$$w\overline{w}=\frac{(z+i)}{(iz+1)}\times\frac{(\overline{z}-i)}{-i\overline{z}+1}$$

$$|w|^2=\frac{z\overline{z}+1-i(z-\overline{z})}{(z\overline{z}+1)+i(z-\overline{z})}$$

$$=\frac{(|z|^2+1)+2y}{(|z|^2+1)-2y}$$

If $y = Im(z) \leq 0$

$\therefore |w|^2 \leq 1$

Q. Prove $w = \frac{i-z}{i+z}$ maps real axis into the circle $|w| = 1$.

★ $\lim_{z \to z_0} f(z) = L$ means $|L - f(z)| < \epsilon \quad \forall z$ in $|z - z_0| < \delta$

Def : Continuity

$f(z)$ is Continuous at z_0 if $f(z_0) = \lim_{z \to z_0} f(z)$ i.e. $\forall \epsilon > 0$

$|f(z) - f(z_0)| < \epsilon$, $\exists \delta > 0$ s.t $|z - z_0| < \delta$

Q. Prove $f(z) = \frac{xy^3}{x^2 + y^6}$, $z \neq 0$ is $= 0$, $z = 0$ is not continuous at $z = 0$

take $y = mx$

$f(z) = \frac{x(mx)}{x^2 + m^6 x^6}$

$= \frac{m}{1 + m^6 x^4}$

$\therefore$ at $x \to 0$ $f(z) \to m$

$\therefore f(z)$ is discontinuous at $z \to 0$

Differentiability

$f(z)$ is said to be differentiable at z_0,

if $\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = f'(z_0)$

Note: h is Complex

Q. Prove $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

$z = re^{i\theta}$, $z \to 0 \Rightarrow r \to 0$

$$\lim_{z \to 0} f(z) = \frac{re^{i\theta}}{re^{-i\theta}}$$

$$= e^{2i\theta}.$$

Q. Show for $f(z) = \frac{\bar{z}}{Re(z)}$ limit $z \to 0$ doesn't exist.

☆ $f(z) = U(x,y) + iv(x,y)$

if $\lim_{z \to z_0} f(z) = L = L_1 + iL_2$

$\lim_{z \to z_0} [U(x,y) + iv(x,y)] = L_1 + iL_2$

$\lim_{(x,y) \to (x_0,y_0)} U(x,y) = L_1$

$\lim_{(x,y) \to (x_0,y_0)} v = L_2$

Continuity

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Ex 1 $f(z) = i^2 \quad z \neq i$

$f(z) = 0 \quad z = i$

Show $f(z)$ is discontinuous at $z = i$

$f(i) = 0$

$f(i+z) = i^2 = -1$

Hence Proved

Ex. $f(z) = z^2$ is continuous for all z.

Infact all polynomials of z are continuous func^n of z.

We define in complex variable,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \ldots$$

Ex. Prove $f(z) = \bar{z}$ is not differentiable at any point.

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

We take $\Delta z = \Delta x + i0$ then $\Delta z \to 0 \Rightarrow \Delta x \to 0$

$$\lim_{\Delta z \to 0} \frac{\Delta x}{\Delta x} = 1$$

Take $\Delta z = 0 + i\Delta y$

$$\lim_{\Delta z \to 0} \frac{-\Delta y}{\Delta y} = -1$$

Hence Proved

Q $f(z) = \frac{z^5}{|z|^4}, z \neq 0$

$= 0, z = 0$

is not differentiable at $z = 0$.

R.H.L

$\frac{df(z)}{dz}_{z\to 0} \frac{z^5}{z^4} = \frac{dz}{dz} = 1$

L.H.L

$\frac{df(z)}{dz}_{z\to 0} = \frac{d}{dz}\frac{-z^5}{z^4} = \frac{d}{dz} - z = -1$

Hence Proved

Another method.

$\lim_{z\to 0} \frac{f(0+z) - f(0)}{z} = \frac{\frac{z^5}{|z^4|}}{z} = \frac{z^4}{|z^4|} = e^{4i\theta}$

It depends on θ $\therefore$ Limit doesn't exist

Hence Proved

Def. Analytic func^n : $f(z)$ is said to be analytic at z_0 if

(i) $f'(z_0)$ exists,

(ii) $\exists$ a domain of z_0, in which $f(z)$ has derivative.

Cauchy - Riemann Conditions: If $f(z) = u + iv$ is analytic at z then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Proof: Since $f(z)$ is analytic at z, Therefore $f'(z)$ exists.

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}, \text{ where } h = h_1 + ih_2$$

if we take $h = h_1 + i0$

then, $$f'(z) = \lim_{h\to 0} \frac{u(x+h_1, y) + iv(x+h_1, y) - u(x,y) - iv(x,y)}{h_1}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \ldots 1$$

Again, if we take $h = 0 + ih_2$

$$f'(z) = \lim_{h_2\to 0} \frac{u(x, y+h_2) + iv(x, y+h_2) - u(x,y) - iv(x,y)}{ih_2}$$

$$= \frac{1}{i}\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right] = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \ldots 2$$

Note :- Cauchy-Riemann eqn are only necessary conditions for $f(z)$ to be analytic.

(ii) If Cauchy-Riemann eqn are satisfied and the first order derivative of u and v is a continuous funcn then $f(z)$ is analytic

Ex. Prove $f(z) = |z|$ is not analytic funcn for any z.

$$u + iv = \sqrt{x^2+y^2}$$

$$u = \sqrt{x^2+y^2} \quad v = 0$$

Ex. If $f(z)$ and $\overline{f(z)}$ are both analytic func^n then $f(z) =$ Constant

$f(z) = u + iv$ is analytic

$u_x = v_y$, $u_y = -v_x$ ... 1

Again $\overline{f(z)} = u - iv$ is analytic.

$u_x = -v_y$, $u_y = v_x$ ... 2.

From 1 and 2, $v_y = -v_y$ $u_y = -u_y$

$$\left.\begin{aligned} v_y = 0 \text{ and } u_y = 0 \\ u_x = 0 \text{ and } v_x = 0 \end{aligned}\right\} \begin{aligned} u = \text{cons.} \\ v = \text{const.} \end{aligned}$$

§ From Cauchy-Riemann eq^n two Important results follow:

Lemma 1: If $f(z) = u(x,y) + iv(x,y)$ is an analytic func^n then $u(x,y) =$ Const. and $v(x,y) =$ Const form an orthogonal set of curves.

Proof: $u(x,y) =$ Const

$du(x,y) = 0$

$u_x dx + u_y dy = 0$

$$\left(\frac{dy}{dx}\right)_{u = const} = -\frac{u_x}{u_y} \quad \dots 1$$

Similarly,

$$\left.\frac{dy}{dx}\right)_{v = const} = -\frac{v_x}{v_y} \quad \dots 2$$

1×2

$$\left(\frac{dy}{dx}\right)_{u = \text{const}} \times \left(\frac{dy}{dx}\right)_{v = \text{const}} = \frac{u_x \, v_x}{u_y \, v_y} = -1 \quad \left(\because \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right)$$

Lemma 2: If $f(z) = u + iv$ is an analytic func^n
Then u and v satisfy Laplace's eq^n

$$\left.\begin{array}{l} \nabla_u^2 = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ \nabla_v^2 = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0 \end{array}\right\} \; v, u \text{ are harmonic funs}$$

Cauchy-Riemann Eq^n in Polar coordinates.

$$u + iv = f(z) = f(re^{i\theta})$$

$$u_r + iv_r = f' = f'(re^{i\theta}) . e^{i\theta} \quad \ldots 1$$

$$u_\theta + iv_\theta = f'(re^{i\theta}) \cdot rie^{i\theta} \quad \ldots 2$$

$$u_\theta + iv_\theta = ir(u_r + iv_r)$$

$$u_\theta = -rv_r$$

$$v_\theta = ru_r$$

$$u_r = \frac{1}{r} v_\theta$$

$$v_r = -\frac{1}{r} u_\theta$$

$u_x = v_y$

$v_x = -u_y$

b) $f(z) = u + iv$ is an analytic func^n
and $u = x^2 - y^2$
Find $f(z)$.

$u = x^2 - y^2$

$u_x = 2x$

$u_y = -2y$

$v_x = -u_y = 2y$

$v_y = u_x = 2x$

$\therefore\ v = 2yx + f(y)$

$v_y = 2x + f'(y)$

$\therefore\ 2x + f'(y) = 2x$

$f'(y) = 0$

$\therefore f(y) = c$ [c is ~~purely~~ ~~real~~ ~~imaginary~~ complex no.]

$v = 2xy + c$.

$f(z) = (x^2 - y^2) + i(2xy) + iC$. [C is purely real]
because if it has a imaginary part then after multiplying it with i we get an extra real part which is not present in the given eq^n.

Q. $f(z) = u + iv$ is an analytic function
and $u = 3x^2y - y^3$
Find $f(z)$.

Answer :- $f(z) = u + iv = (3x^2y - y^3) + i(3y^2x - x^3) + ic$
$= -iz^3$.

Lemma:- $f(z)$ is an analytic function.

$$f(z) = u(x,y) + iv(x,y)$$
$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

This is true for $z = \bar{z}$ also, i.e. $x = z$, $y = 0$

In $u(x,y) + iv(x,y)$, Put $x = z$ and $y = 0$ we get $f(z)$

So, $f(z) = -i[x^3 - 3y^2x + 3x^2y - y^3] + ic$.
Put $x = z$ and $y = 0$
$= -iz^3$

$f(z) = u + iv$ is an analytic function

and $u = e^x(x\cos y - y\sin y)$
Find $f(z)$.

$$u_x = e^x(x\cos y) + e^x\cos y - e^x y\sin y.$$

$$u_y = -e^x x\sin y - e^x\sin y - e^x y\cos y.$$

$f = u + iv$.

$$\frac{df}{dz} = u_x + iv_x = u_x + i(-u_y)$$
$$= u_x - iu_y$$

On substituting,
u_x and u_y.
and put $x = z$ and $y = 0$
we get, $\frac{df}{dz} = ze^z + e^z$

$$f = \int (z+1)e^z \, dz$$
$$= ze^z + iC$$

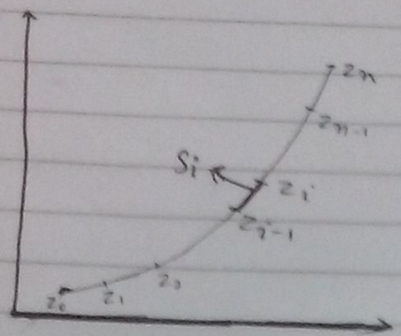
Q. $f(z) = u + iv$ is an analytic function.

Find $f(z)$ if.
i) $u = e^{-x}(\sin y + \cos y)$
(ii) $u = x^2 - y^2 - 2xy - 2x - y - 1$

Q. $f(z) = u + iv$ is analytic function
Find $f(z)$ if.

$u - v = (x-y)(x^2 + 4xy + y^2)$

Integration :



Given a curve C in z-plane and $f(z)$ is defined ~~at~~ C.
Divide C by points $z_0, z_1, \ldots z_n$
Form the sum,
$S_n = \sum f(s_i)\,\Delta z_i$
where s_i is a pt. on the curve between z_{i-1} and z_i.

In the limit $n \to \infty$, if S_n tends to a unique limit for arbitrary choice of z_i and s_i
then

$$\lim_{n\to\infty} S_n = \int_C f(z)\,dz .$$

PROBABILITY

Q A box 1 contains 3 w and 4 red balls box 2 5 w and 6 r. A ball is taken from one of the box. A ball is white find the box is 1

$$\frac{\left(\frac{3}{7}\right)\frac{1}{2}}{\left(\frac{3}{7}+\frac{5}{11}\right)\frac{1}{2}} = P(W/B1)$$

$$= \frac{3\times 11}{33+35} = \frac{33}{68}$$

Q. A person has in it's pocket a fair coin and a 2 headed coin he selects one of them at random. and when he flips it shows head. Find P that the coin is fair.

a.) $P = \dfrac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times 1} = \dfrac{1}{3}$

b. Suppose he flips the same coin again and it again shows head. Now what is the prob it's a fair coin.

$P = \dfrac{1}{5}$

Q. A box contains b black balls and r red balls. A ball is drawn at random and when it is put back in the box c additional balls of the same colour are put. Now we draw another ball. Show that the p. of first ball was black given that the second ball drawn was red.

$$P = \frac{\frac{b}{b+r} \times \frac{r}{b+r+c}}{\frac{b}{b+r} \times \frac{r}{b+r+c} + \frac{r}{b+r} \times \frac{r+c}{b+r+c}}$$

$$= \frac{b}{b+r+c}$$

Q. There are 3 true coins and 1 fake coin with head as both sides. A coin is chosen at random and is tossed 4 times If head turns up all the four times, what is the prob that false coin is chosen.

$$P = \frac{\left(\frac{1}{4}\right)\left[1 \times 1 \times 1 \times 1\right]}{\frac{1}{4}(1) + \frac{3}{4}\left(\frac{1}{2}\right)^4}$$

$$P = \frac{1}{1+\frac{3}{16}} = \frac{16}{19} \checkmark$$

Q. A and B shoot at a target at the same time. The prob of A and B hitting the target are 0.7 and 0.4. Given exactly one shot hit the target What is the probabity it was hit by B

$$P(B) = \frac{\frac{1}{\cancel{2}} \times 0.4 \times 0.3}{\frac{0.3}{\cancel{2}} \times 0.4 + \frac{0.6}{\cancel{2}} \times 0.7}$$

$$= \frac{0.4}{1.8} = \frac{2}{9}$$

Q. For a certain binary channel '0' and '1' are transmitted. Due to disturbances in the channel sometimes 0 is transmitted but recieved as 1 and vice versa. The probability that zero is recieved as 0 is 0.95 and P that 1 is recieved as 1 is 0.90

If the probability that 0 is transmitted is 0.4 find the P that 1 was transmitted when 1 was recd

$$P = \frac{0.9 \times \cancel{0.6}^{\,3}}{0.9 \times \cancel{0.6}_{\,3} + 0.05 \times \cancel{0.4}_{\,2}}$$

$$= \frac{0.9 \times 3}{0.9 \times 3 + 0.1}$$

$$= \frac{27}{28}$$

Q. A lab test is 99% accurate in detecting a disease when it is present. However, if a healthy person is tested the test shows in 1% cases. If 0.5% of the population has the dis. P that the person has disease given that test result was +ve

$$P = \frac{\frac{0.5}{\cancel{100}} \times 99}{\frac{0.5}{\cancel{100}} \times 99 + \frac{99.5}{\cancel{100}} \times \frac{1}{\cancel{100}}}$$

$$= \frac{99 \times \cancel{5}}{99 \times \cancel{5} + \cancel{995}_{\,199}}$$

$$= \frac{99}{298}$$

Q. In answering a question on a MCQ test a student knows or guesses. If P is the prob. that he knows the answer and 1-P is guessing the ans. Assume that a student who guesses the answer will be correct is $\frac{1}{M}$ where M is no. of MCQ options ~~questions~~. What is prob. that the student knew the answer to a quest. given that he answered it correctly.

$$P = \frac{P}{P+\frac{(1-P)}{M}}$$

$$= \frac{MP}{P(M-1)+1}$$

Q. A plane is missing and it is equally missing that it is in one of the 3 regions. Let $(1-\alpha_i)$ is the probability it is found in the region i, when it is infact in that region. What is ? that the plane is in the i^{th} region given that the search in region 1 is unsuccessful.

$$P_2 = \frac{(1-\alpha_2)(\alpha_2)(\alpha_1) + \alpha_1\alpha_2(1-\alpha_3)}{\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2(1-\alpha_3) + \alpha_1\alpha_3(1-\alpha_2)} = P_3$$

o — o — o — o

R_i : Event Plane is in the region i; $i = 1, 2, 3$.

B : Event that search in R_i is unsuccessful.

$$P(R_1/B) = \frac{P(B/R_1)\,P(R_1)}{P(B/R_1)P(R_1) + P(B/R_2)P(R_2) + P(B/R_3)P(R_3)}$$

$$= \frac{\alpha_1 \times \frac{1}{3}}{\alpha_1 \times \frac{1}{3} + 1 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{\alpha_1}{\alpha_1 + 2}$$

Q Suppose $\frac{3}{4}$ % of the P have a disease and the test to detect this disease is 99 % in identifying those. Test is [illegible] % accurate to find those not with the disease. Compute the P that one has the disease given that the test is +ve

$$P = \frac{\frac{\cancel{0.75}^{\,0.15}}{100} \times \frac{99}{100}}{\frac{\cancel{0.75}^{\,0.15}}{100} \times \frac{99}{100} + \frac{5}{100} \times \frac{99.25}{100}}$$

$$= \frac{\cancel{75}^{\,3} \times 99}{\cancel{75}_{\,3} \times 99 + \cancel{9925}_{\,1985}}$$

$$= \frac{297}{2282}$$

Q An insurance company insures 2000 scooter drivers 4000 car and 6000 truck. The probability of meeting an accident are 0.1, 0.03, 0.15.

$$\frac{0.1 \times \frac{1}{6}}{0.1 \times \frac{1}{6} + 0.03 \times \frac{1}{3} + 0.15 \times \frac{1}{2}}$$

$$\frac{\frac{1}{6\cancel{0}}}{\frac{1}{6\cancel{0}} + \frac{1}{10\cancel{0}} + \frac{3}{4\cancel{0}}} = \frac{\frac{1}{6}}{\frac{10+6+45}{60}}$$

$$= \frac{10}{61}$$

Independent Events

In general, $P(A) \neq P(A/B)$
In some cases, the occurence of B does not change the Prob of
i.e, $P(A) = P(A/B)$

So, $P(A) = P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$P(A \cap B) = P(A) \cdot P(B).$$

Q. A: Sum of two dice is 7.
B: first dice shows 4.

$P(A \cap B) = \dfrac{1}{36}$

$P(A) = \dfrac{1}{6}$ A and B are independent.

$P(B) = \dfrac{1}{6}$

Lemma: If A and B are independent events. Then A and $\overline{B}$ are also independent.

Proof: $A = A \cap (B \cup \overline{B})$
$= (A \cap B) \cup (A \cap \overline{B})$

$P(A) = P(A \cap B) + P(A \cap \overline{B})$ $\because$ $P(A \cap B)$ and $P(A \cap \overline{B})$ are mutually exclusive
$P(A) - P(A \cap B) = P(A \cap \overline{B})$

$P(A)(1 - P(B)) = $ "

$\therefore P(A)\, P(\overline{B}) = P(A \cap \overline{B})$

$P(A)[1-P(B)] = P(A \cap \bar{B}) \Rightarrow P(A \cap \bar{B}) = P(A)P(\bar{B})$

Random Variable (R.V)

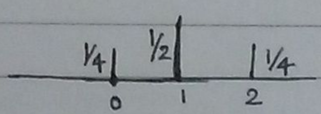
$S = \{\underset{(1)}{H}, \underset{(0)}{T}\}$

X: R.V denoting the head.

Two coins

$S = \{\underset{2}{HH}, \underset{1}{HT}, \underset{1}{TH}, \underset{0}{TT}\}$

R.V 'X' denotes no. of heads



Probability distribution.

Def: R.V.

It is a function that associates a real No. with each element of Sample Space S. It is denoted a Capital letter X (or Y) and the corresponding letter x (say) for one of it's value. The range of R.V may be Discrete or continuous depending on the samplespace being Discrete or Continuous.

A discrete R.V assumes each of it's value with certain Prob. This is denoted by $P(X=x)$ or $f(x)$. The set $(x, f(x))$ is called 'Prob distribution' or 'Prob. mass func^n' if

- $P(X=x) = f(x)$
- $f(x) \geq 0$
- $\sum_x f(x) = 1$

eg.

Distribution Function: $F(x)$

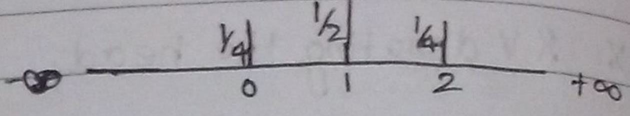
Cumulative Probability Distribution.

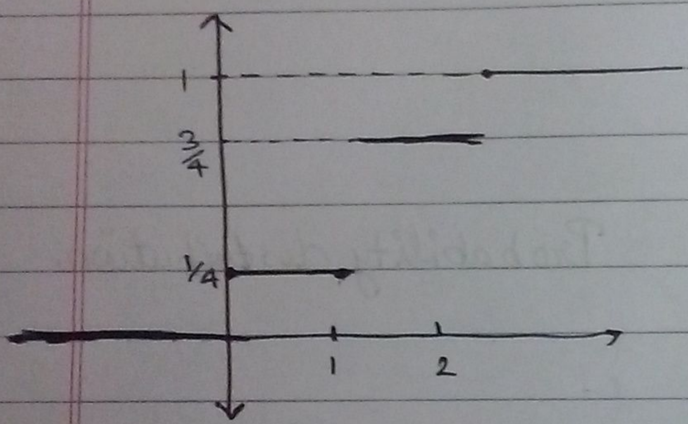
$F(x) = 0 \quad -\infty < x < 0$

$= \frac{1}{4} \quad 0 \le x < 1$

$= \frac{3}{4} \quad 1 \le x < 2$

$= 1 \quad 2 \le x < +\infty$





Sometimes, we may be interested to compute Prob for X, will be less than or equal to x, i.e, $P(X \le x)$ Then it is denoted by

Continuous Probability Distribution

Prob at any x is zero and we talk of Prob. in an interval i.e,

$P[a < x \le b] = P[a < x < b] = P[a \le x < b] = P[a \le x \le b]$, i.e

end points inclusion or otherwise does n't matter.

The func^n $f(x)$ is prob. density func^n (pdf) Satisfies.

- $f(x) \ge 0$
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

- $P[a<x<b] = \int_a^b f(x)\,dx$