# CHROMATICITY

*A. Verdier*
CERN, Geneva, Switzerland

**Abstract**
The meaning of 'Chromaticity' is taken here as : change of the linear parameters of the transverse motion of a single particle related with a change of the beam energy, for a machine in a fixed status. We give here an overview on general theoretical aspects with a special emphasis on the case of low-$\beta$ insertions.

## 1 INTRODUCTION

Etymologically, chromaticity comes from the Greek word $\chi\rho\omega\mu\alpha$ which means colour. Its use in the accelerator field comes from the fact that, like electro-magnetic waves, the charged particles are deflected according to their energy. However the expression 'Chromaticity correction' has a different meaning depending on the person who uses it. In the old times of accelerators, it meant only correction of the first derivative of the tunes with respect to the momentum. The word chromaticity itself was used for the expression $Q'/Q$ because it has a value of the order of -1 for a simple lattice. Nowadays the word chromaticity is used for 'change of linear optics parameters with beam energy' and will be used below with this meaning. In terms of transfer maps or aberrations, this means considering contributions linear in the transverse coordinates and of arbitrary order in relative momentum deviation. This new meaning of 'chromaticity' arises from the increasing complexity of the machine lattices and correction procedures.

As the chromaticity correction will use non-linear fields, it must be treated with consideration of the non-linear dynamics. Therefore in practice the chromaticity correction will involve non-linear correction too. In this respect, a good method, which can be implemented easily only in large machines, is to construct multipole schemes which are a priori free from low-order transverse non-linearities for any status of the chromaticity correction.

At this point it is worth questioning whether effects which are non-linear both in momentum and coordinates should be considered. From experience, it appears that, if the chromaticity correction is done with multipoles of order below octupole, these effects can be neglected. This means that, as long as the betatron tunes are far from a linear instability condition, the non-linear dynamics are little affected by a momentum change. If the chromaticity correction is done with higher-order multipoles, it may happen that unacceptable non-linearities are created on the off-momentum closed orbit. The most simple example is the creation of off-momentum octupoles from decapoles, which makes an off-momentum anharmonicity. Such effects will not be considered here.

In this course we will give the basic principle which can be used to perform a chromaticity correction in a wide range of machines. The methods differ because all machines do not have the same needs. For instance the correction with sextupole families is very efficient for a large machine like LEP, with low-$\beta$ insertions which make strong non-linear chromatic perturbations. This sort of correction is mandatory to preserve the quantum lifetime of the beam. However, for a proton machine like LHC, which has the same size as, and similar optics to LEP, it is useless because the useful momentum range is far smaller. In a small machine with a small dispersion such as a synchrotron light source, the chromaticity correction with chromatic sextupole families cannot be done, and it is necessary to compensate for the geometric aberrations by means of sextupoles which do not act on the chromaticity.

In what follows, we will first examine why the chromaticity correction has to be made. We will give basic concepts on a simple example. Then, we will give examples of tools used to perform a chromaticity correction. Finally, we will treat in more detail the problem of a machine with low-$\beta$ insertions. For chromaticity correction involving corrections of non-linear transverse motion, the reader should refer to the courses on transverse dynamics or to reference [1].

## 2 WHY CORRECT THE CHROMATICITY ?

The first reason why the derivatives of the tunes with respect to momentum have to be cancelled is to eliminate the dipole mode of the head-tail instability above transition energy. This phenomenon was first observed in Frascati in the storage ring ADONE where a feed-back system was used to cure this instability, as there was no sextupole available. An analysis can be found in Ref. [2].

As this instability is a potential limit to the beam intensity, it is preferable to make the first-order tune derivatives slightly positive in any machine design. This is extremely easy to do, two sextupole families are enough to adjust the two first derivatives of the tunes with respect to momentum. The exact values to which these derivatives have to be set depend on the transverse impedance of the machine. It may happen that, as in LEP, they must not be made too largely positive, otherwise the transverse quadrupole mode m=-1 become unstable.

On top of the cure of this head-tail instability, the first-order chromaticity correction is mandatory in large machines, because the high value of the derivatives results in a betatron instability for a small momentum deviation. For instance a tune derivative of -250 leads surely to a betatron instability for a relative momentum deviation in a one per-mill range. This is indeed what occurs for LEP as shown on Fig. 1.

Once the first derivatives of the tunes are corrected, the variations of the tunes with momentum may still be unacceptable. In order to examine this, we must have a good computer program which calculates these variations. This is not trivial, we will give examples below which illustrate mistakes made in the past. It may then happen that variations with momentum of non-linear parameters like anharmonicity are unacceptable. Clearly chromaticity correction may take us very far! Before going to such complications, we first look at the most simple chromaticity correction we can think of.

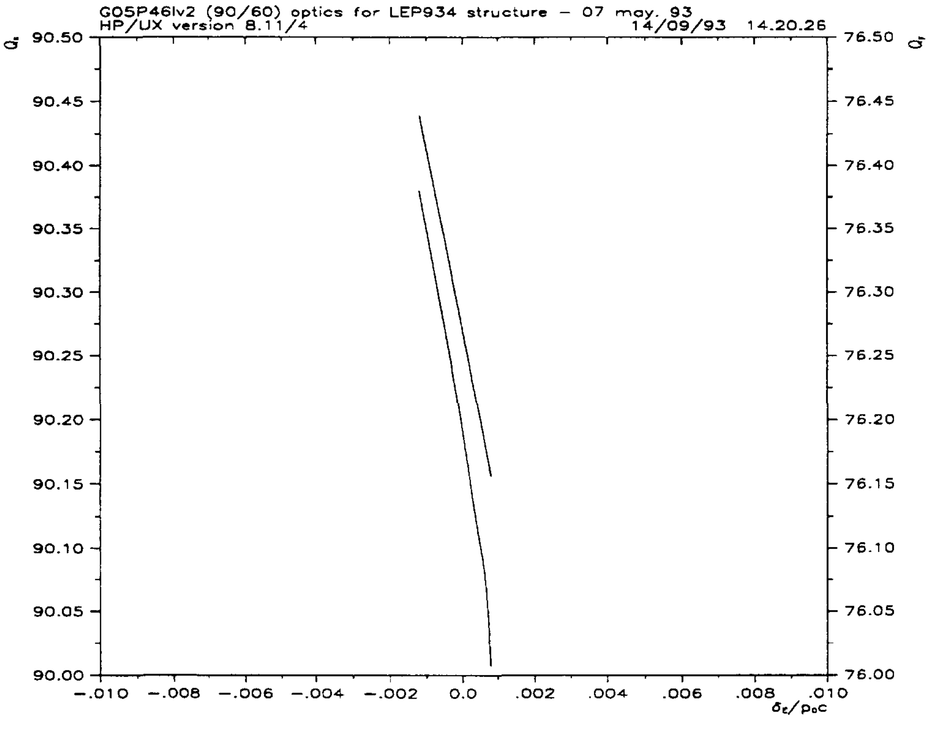



Figure 1: Tunes versus momentum for the LEP 1993 physics optics with no sextupole, G05P46lv2 optics, $Q_h$=90.27, $Q_v$=76.19, $\beta^*$ of 5cm and 1.25m.

## 3 CHROMATICITY CORRECTION OF A FODO CELL

### 3.1 Natural tune derivatives

The most simple FODO-cell model we can think of is made from thin quadrupoles and thin dipoles. Note immediately that using thin dipoles is questionable because the two ends of the dipoles, which produce second-order aberrations, are at the same place and cancel each other. Therefore the simple model is then restricted to the case where the radius of curvature of the dipoles is larger than the length of the dipole in a half cell of length $L$ by at least one order of magnitude. The phase advance $\mu$ in a FODO cell is given by [3]:

$$\sin(\mu/2) = LKl/4. \tag{1}$$

$Kl$ is the integrated normalised gradient $B'l$, which is the same in the F and D quadrupoles, divided by $B\rho$, the magnetic rigidity of the beam. In Eq. (1), the only quantities which depend on $\delta$ are $\mu$ and $K$. The explicit dependence of $K$ is:

$$K = \frac{k}{1+\delta} = k(1 - \delta + \delta^2 - \delta^3 + .....) \tag{2}$$

where $k$ is the value of the normalised gradient for the energy of the reference orbit. From now on, we will use a capital letter for a quantity which depend on $\delta$, as in (2), and small ones for the value of the same quantity for $\delta$=0. Differentiating Eq. (1) with respect to the relative momentum deviation $\delta$, we obtain :

$$\mu' = -\frac{2}{1+\delta} tg(\mu/2)$$

Note that the ratio $\mu'/\mu$ has values close to -1 for phase advances below 90°. Therefore the ratio $Q'/Q$ is of the order of -1 for a machine made of FODO cells. This was the case of the first alternating-gradient machines and this ratio was called 'chromaticity ' (and given the symbol $\xi$).

Differentiating Eq. (1) twice, we obtain:

$$\mu'' = \frac{2\tan(\mu/2)}{(1+\delta)^2}\left[1+\frac{1}{1+tg^2\mu/2}\right].$$

This formula provides only an order of magnitude because the model is already approximate to first-order. However, we see already that the natural second-order derivative of the tunes is numerically of the same order as the natural first-order derivative, but with an opposite sign, for a machine made of FODO cells, neglecting the effect of the dipoles.

It is easy to check that the derivatives of higher order are still of the same order of magnitude. Consequently, the variation of the tunes of a machine made of FODO cells with equal gradients is surely smooth in a range of momentum deviation of some percents.

### 3.2 Cancelling the first derivatives of the tunes

We exploit the fact that off-momentum particles oscillate around an off-centred closed orbit which passes off-centre in the machine magnets. The position of the off-momentum closed orbit is given to first-order in $\delta$ by the product $D\delta$, where $D$ is the dispersion function.

Introducing in the FODO structure a sextupole, which has a quadratic field variation, produces a gradient depending linearly on $\delta$, thanks to the existence of the dispersion function. This can be explained as follows. The normalised components of the sextupole field are $K'(x^2-y^2)/2$ in the horizontal plane and $-K'xy$ in the vertical one. Here $K'$ is the normalised second derivative of the magnetic field : $B''/B\rho$. In order to look for a betatron oscillation around the off-momentum closed orbit, we substitute for $x$ : $D\delta + x_\delta$ (position of the off-momentum closed orbit to first-order in $\delta$, plus betatron oscillation) in the above expressions of the sextupole field. Keeping the terms linear in $x_\delta$ in this expansion, we observe that a gradient equal to $k'D\delta$ in the horizontal plane and $-k'D\delta$ in the vertical plane appears. $k'$ is now the value of $K'$ for $\delta = 0.0$. Choosing $k'D = k$, we compensate exactly the decrease of the quadrupole gradient to first-order in $\delta$. Doing this exercise for the two quadrupoles of the FODO cell makes an exact compensation of the decrease of the quadrupole gradient with the momentum deviation to first-order in $\delta$.

The reader can check for himself that this argument can be extended to higher order: a field varying like $x^n$ can compensate a dependence of the gradient on $\delta^{n-1}$. Doing this compensation, variations of order higher than $\delta^n$ are introduced. Therefore a deeper analysis has to be done. A first approach is given at the end of the course. On top of these higher-order chromatic aberrations, non-linear terms are introduced in the equation of motion : their effect has been analysed in courses on non-linear dynamics (see those in this course or in [1] for instance).

## 4 GENERAL CHROMATICITY CORRECTION

After this introduction concerning basic concepts, we tackle the chromaticity problem for more complicated cases and with more accuracy. In the problem hierarchy, the first one is the computation of the effects. Indeed it is useless to compute beautiful corrections if we are not able to check their efficiency.

The best way to compute chromatic effects is:

- calculate the position of the off-momentum closed orbit by means of an accurate mapping per element
- determine the associated tunes and $\beta$-functions. To this end we must have a correct computation of the transfer matrices around the computed off-momentum closed orbit or an efficient algorithm which can extract the tune values from tracking.

This will provide all variations of the optics parameters with momentum with the maximum accuracy. Many computer codes have been made to solve this problem. An exhaustive review can be found in Ref. [4].

In the case where we are confronted with a non-standard chromaticity problem, the best procedure is to make computations with several different codes. As shown below, this can reveal a mistake. But of course the best thing is to go back to the exact equation of motion in order to make sure of the physics involved in our problem.

Before going into the details of the computation of chromatic effects, it is of prime importance to stress that the numerical computation of the $\delta$-dependence can be performed exactly for machines without synchrotron radiation. This is a consequence of the fact that a mere scaling of the canonical momenta is a canonical transformation. If all momenta are divided by the actual momentum of the particle, the electro-magnetic potential is also divided by this actual momentum $p_s$ so that eventually the momentum dependence disappears from the Hamiltonian of the motion. For a motion with synchrotron losses a similar procedure can be applied, except that $p_s$ is now a sort of momentum averaged over the circumference. Then, in order to describe the synchrotron motion, it is necessary to expand the equations of motion with respect to a variable $p_t$ given by:

$$p_t = \frac{E}{p_s c} - \frac{1}{\beta_s}$$

where $E$ is the energy and $\beta_s$ the relativistic factor associated with $p_s$. As mentioned above, this normalisation to $p_s$ is only applicable to numerical computations, when the explicit dependence with respect to the momentum deviation is not wanted. In what follows we want to describe explicitly the $\delta$-dependence, therefore we will go back to the equation of motion for the computation of chromatic effects.

## 5 EQUATION OF MOTION

Starting from the exact relativistic equation of motion with the Lorentz force:

$$d\vec{p}/dt = e\vec{v} \times \vec{B}$$

we expand it in a curvilinear coordinate system with a curvature $h$ in the horizontal plane and project the motion on the three axes. This has been done by several authors [5, 6]. The exact equation of motion projected on the three axes gives a set of three equations:

$$x'' - h(1+hx) - \frac{x'}{T'^2}[x'x'' + y'y'' + (1+hx)(hx)'] = \frac{T'}{1+\delta}(y'b_s - (1+hx)b_y)$$

$$y'' - \frac{y'}{T'^2}[x'x'' + y'y'' + (1+hx)(hx)'] = \frac{T'}{1+\delta}(-x'b_s + (1+hx)b_x)$$

$$2(hx)' - \frac{(1+hx)}{T'^2}[x'x'' + y'y'' + (1+hx)(hx)'] = \frac{T'}{1+\delta}(x'b_y - y'b_x)$$

The prime denotes here a derivation with respect to the longitudinal coordinate $s$, and $T$ is obtained from :

$$T'^2 = (1+hx)^2 + x'^2 + y'^2.$$

There is clearly a notation problem associated with the prime which means either derivative with respect to the longitudinal coordinate $s$ or with respect to the relative momentum deviation. In fact, for all variables used in this course except the dispersion function, the differentiation is unambiguous and will be specified. For the dispersion function the derivatives with respect to momentum will be noted with indices like $D_2$. The $b$'s with indices are the normalised components of the magnetic field: $b = \mathcal{B}/B\rho$, where $\mathcal{B}$ is the magnetic field itself and $B\rho$ the magnetic rigidity of a particle on the reference orbit. Setting $x, y$ and their derivatives equal to zero, we have :

$$h = b_y(0,0,s)$$

This fixes the sign convention in the field components : a positive $b_y$ means focussing in the horizontal plane. This convention should be applied in any good optics code, it is done in the code MAD [7] that we use at CERN. For right multipoles (i.e. such that the field in the plane of the machine is parallel to the main dipole field), the expansion of $b_y(x, y, s)$ is, up to decapole :

$$b_y(x,y,s) = h + K(s)x + \frac{K'(s)}{2!}[x^2 - y^2] + \frac{K''(s)}{3!}[x^3 - 3xy^2] + \frac{K'''(s)}{4!}[x^4 - 6x^2y^2 + y^4]$$

In order to obtain the chromatic effects, we have to keep all monomials linear in the transverse coordinates and their derivatives. For instance, we show the classical example of the computation of the derivatives of the tunes with respect to momentum. To this end we must expand the equations to second-order in the two transverse planes, because we want to have monomials in $x\delta, x'\delta, y\delta$ and $y'\delta$. This expansion gives (see for instance Ref. [5]) :

$$x'' - h(1+hx) - x'(hx' + h'x) = \frac{T'}{1+\delta}[y'b_s - (1+hx)b_y]$$
$$y'' - y'(hx' + h'x) = \frac{T'}{1+\delta}[-x'b_s + (1+hx)b_x].$$

We consider only the transverse planes and not at all the longitudinal one. It is clear that kinematical terms as well as terms coming from the discontinuity of the curvature $h$ of the reference orbit appear, which were not in the simple model used to establish formula (1).

In order to have the equation for the betatron motion around the off-momentum closed orbit we then substitute $x$ for $D\delta + x_\delta$ and keep in the expansion the monomials in $x_\delta$ and $x'_\delta$. We obtain :

$$x''_\delta + (h^2 + k)x_\delta - (k - k'D + 2h^2 + h'D' - 2Dh^3 - 4hkD)x_\delta\delta - (hD)'x'_\delta\delta = 0$$
$$y''_\delta - ky_\delta + (k - k'D + h'D' - 2hkD)y_\delta\delta + (hD)'y'_\delta\delta = 0 \qquad (3)$$

We observe that the perturbation in $\delta$ contains two terms linear in $x$ and $x'$ and hence the chromatic effects cannot be estimated by means of the well known first-order tune-shift formula [8] :

$$\Delta Q = \frac{1}{4\pi}\oint \Delta k(s)\beta(s)ds.$$

An estimate of the importance of the error when using this formula can be found in a report on the design of the LEAR machine at CERN [9]. It appears that the ratio $Q'/Q$ can be wrong by a factor as large as 2 if the calculation is not properly done. However such a dramatic effect may only occur in small machines with a radius of the same order as the dispersion function. For instance in LEP, where the dispersion is of the order of a metre and the radius of curvature three kilometres, the error due to neglecting these terms is as the ratio of these numbers, i.e. of a tenth of unit in $Q'$ for a $Q'$ value of the order of -100.

Nowadays accurate transfer maps to large order in transverse coordinates have been computed for all sorts of magnets by means of modern techniques such as LIE algebra [10] or Differential Algebra [11]. Therefore there is little excuse for not computing the chromatic effects properly in any machine.

## 6 COMPUTATIONS OF GRADIENT PERTURBATIONS

Being able now to compute chromatic effects, we wish to compensate the detrimental ones. One possibility is to establish formulae giving the effect of gradient perturbations.

The principle of the method used is rather simple: as we deal with perturbations which modify linear terms in the equation of motion, we can use matrix calculus to estimate their effect. The equation of motion contains chromatic terms (monomials in $\delta$) which can be considered as a perturbation. These terms give exactly the perturbation of $x'$ due to an infinitely thin element. As there is no change in $x$, the transfer matrix of such a perturbation is straightforward. Multiplying the unperturbed Twiss matrix at the place of the perturbation by the transfer matrix of the perturbation makes it possible to compute the Twiss matrix associated with the perturbed machine and then the new tunes. This procedure can be generalised to higher order.

Before examining this in detail, we must stress that such perturbation calculations are the basis of the computation of off-momentum transfer matrices for any sort of magnetic machine elements. Such computations can be found for instance in [12, 13].

## 6.1 First-order focussing perturbation

In equations (3), the chromatic terms can be written as :

$$C_z z\delta + C_{pz} z'\delta$$

where $z$ stands for $x$ or $y$. The transfer matrix associated with such a perturbation of infinitely small length $ds$ is obtained considering that :

- the transverse coordinate $z$ does not change
- the change in $z'$ is merely $z''ds$

The transfer matrix is then:

$$\begin{pmatrix} 1 & 0 \\ -C_z\delta ds & 1 - C_{pz}\delta ds \end{pmatrix} \tag{4}$$

We notice readily that this matrix is not symplectic, i.e. its determinant is not 1. This is due to the term $C_{pz}$ coming from the dipole ends. As an actual dipole has two ends, it is not physical to consider one isolated end. If we compute the off-momentum transfer matrix for a complete dipole, we notice that it is indeed symplectic. Therefore it is legitimate to compute the Twiss matrix of the perturbed machine by multiplying merely the unperturbed Twiss matrix :

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

in which the functions $\beta$ and $\alpha$ are computed at the place of the perturbation, by the matrix of the perturbation. From the trace of the product, we obtain the tune $\mu^*$ of the perturbed machine :

$$\cos\mu^* = \cos\mu - \frac{1}{2}\left[C_z\delta\beta\sin\mu + C_{pz}\delta(\cos\mu - \alpha\sin\mu)\right]ds.$$

The tune $\mu^*$ of the perturbed machine is :

$$\mu^* = \mu + \mu'\delta.$$

Expanding $\cos\mu^*$ and identifying the first-order terms in $\delta$, we obtain

$$\mu' = \frac{1}{4\pi}\oint[C_z\beta(s) + C_{pz}(\cot\mu - \alpha)]ds$$

by a mere integration of all first-order effects. It is worth noting that, as $C_{pz} = (hD)'$, its integral is zero since $hD$ is a function which has a one-turn period. The correct first-order formula for the computation of the tunes derivatives is then :

$$\mu' = \frac{1}{4\pi} \oint [C_z \beta(s) - C_{pz}\alpha)] ds. \tag{5}$$

Integrating the term with $C_{pz}$ by parts gives:

$$\oint -C_{pz}\alpha ds = \frac{1}{2} \oint C'_{pz} \beta ds$$

where $C'_{pz}$ is the derivative of $C_{pz}$ with respect to the longitudinal coordinate $s$. This is the formula given in [14].

### 6.2 First-order chromaticity correction

We apply now the above results to the case of sextupoles. From Eq. (3), we have:

$$C_x = k'D, \;\; C_{px} = 0 \qquad C_y = -k'D, \;\; C_{py} = 0$$

We put these expressions in Eq. (5) to obtain the changes in the first derivative of the tunes with respect to $\delta$ due to sextupoles:

$$\Delta Q'_h = \frac{1}{4\pi} \oint k'(s) D(s) \beta_h(s) ds.$$

and

$$\Delta Q'_v = \frac{1}{4\pi} \oint -k'(s) D(s) \beta_v(s) ds.$$

These expression can be obtain exactly by means of the first-order tune-shift formula.

For practical applications, it is enough to have one set of sextupoles at locations where $\beta_h$ is large to act mainly on $Q'_h$ and another set where $\beta_v$ is large to act mainly on $Q'_v$. The two above equations lead to a set of two linear equations with the two strengths of the two sextupole sets as unknowns. The same system can obviously be used to compute increments of sextupole strengths required to make increments of tune derivatives. This is applied routinely to accelerator control.

### 6.3 Higher-order focussing perturbation

There are two origins of the higher-order terms :

- those appearing in the expansion of the equation of motion
- combined effect of gradient perturbations of lower order.

As an example of higher-order calculations, we give below the main lines of the computation of tune variations in $\delta^2$ in the horizontal plane.

The terms appearing in the equation of motion are obtained from an expansion of this equation up to third order in transverse coordinates and second-order in $\delta$. Then we substitute $x$ for $x_\delta + D\delta + D_2\delta^2$, expand and keep the monomials linear in $x_\beta$ and $x'_\beta$. The monomials in $\delta$ are given in equation (3). The monomials in $\delta^2$ are :

$$
\begin{array}{ll}
x\delta^2 & [-h'D_2' + (2h^3 + 4hk + k')D_2 - 2hh'DD' + 3(kh^2 - h^4/2 + hk' + k''/6)D^2 \\
 & +D'(2h^2 + 3k/2) + D(4hk - k'D - 2Dh^3) + 2h^2 + k] \\
x'\delta^2 & [-h'D_2 - hD_2' - hh'D^2 + (4h^2 + 3k)DD' - 3hD'].
\end{array} \tag{6}
$$

The combined effect of the gradient perturbations of first-order is obtained by generalising the first-order calculation in section 6.1. The unperturbed Twiss matrix is now divided into two sub-matrices A and B, and the matrices of perturbations P1 and P2 similar to that given in Eq. (4) are inserted to make the perturbed Twiss matrix:

$$A \times P1 \times B \times P2. \tag{7}$$

The second-order chromatic term associated with the perturbation of index $i$ in the equation of motion can be written:

$$[Ci_x x\delta + Ci'_x x\delta^2 + Ci_{px} x'\delta + Ci'_{px} x'\delta^2]ds$$

where $Ci_x$ and $Ci_{px}$ are extracted from Eq. (3), and $Ci'_x$ and $Ci'_{px}$ are extracted from Eq. (6). The tune with the chromatic perturbations is obtained from the trace of the matrix given by (7).

In order to have the second derivative of the tune, we expand the general expression of the trace of the matrix with the tune expanded to second-order in $\delta$ :

$$\cos(\mu + \mu'\delta + \mu''\frac{\delta^2}{2}) = \cos\mu - \mu'\delta\sin\mu - \frac{1}{2}(\mu'^2 + \mu'')\delta^2\cos\mu \tag{8}$$

and then we identify the terms in $\delta$ and $\delta^2$ in Eq. (8) and in the expression of the trace of the matrix (7). We obtain after lengthy calculations:

$$
\begin{array}{rcl}
2d\mu'/ds & = & C1_x\beta_1 + C1_{px}(\cos\mu - \alpha_1\sin\mu) + C2_x\beta_2 + C2_{px}(\cos\mu - \alpha_2\sin\mu) \\
d^2\mu''/ds1ds2 & = & -d\mu'^2/ds1ds2 \\
 & & +C1'_x\beta_1\tan\mu + C1'_{px}(1 - \alpha_1\tan\mu) + C2'_x\beta_2\tan\mu + C2'_{px}(1 - \alpha_2\tan\mu) \\
 & & +C1_xC2_x\beta_1\beta_2\sin\mu_1\sin\mu_2/\cos\mu \\
 & & +C1_xC2_{px}\beta_1\sin\mu_2(\cos\mu_1 - \alpha_2\sin\mu_1) \\
 & & +C2_xC1_{px}\beta_2\sin\mu_1(\cos\mu_2 - \alpha_1\sin\mu_2) \\
 & & +C2_{px}C1_{px}(\cos\mu_2 - \alpha_2\sin\mu_2)(\cos\mu_1 - \alpha_2\sin\mu_1)
\end{array} \tag{9}
$$

The second-order derivative of the tune with respect to momentum is then obtained from the double integral:

$$Q'' = \oint ds2 \int_0^{s2} \frac{d^2\mu''}{ds1ds2} ds1.$$

In Eq. (9), we recognise in the expression of $2d\mu'/ds$ a mere sum of first-order effects identical to those given by Eq. (5).

In the second-order terms of Eq. 9), we have three sorts of terms:

- the first one is a mere 'first-order product'
- the second one is the contribution of the second-order terms of the equation of motion calculated with the first-order formula
- the third one is made from a series of products of first-order terms with phase factors, i.e. the effect of coupling between first-order terms. These terms generalise the second-order formula established in Ref. [8].

### 6.4 Off-momentum closed orbit to higher order

The position of the off-momentum closed orbit is, to second-order in $\delta$ :

$$D\delta + D_2\delta^2$$

We substitute $x$ for this expression in the equation of motion and factorize in powers of $\delta$. Setting to zero the term in $\delta$ in this equation, we obtain the standard equation of the dispersion [8]:

$$D'' + (k + h^2)D = h$$

Setting to zero the term in $\delta^2$ in this equation, we obtain the equation of $D_2$ :

$$D_2'' + (k + h^2)D_2 = h'DD' - h^3D^2 + hD'^2/2 - 2hkD^2 - k'D^2/2 + kD + h$$

It is worth noting that the right hand side contains both coefficients of the equation of motion and $D$ and $D'$, which come from the solution of the equation of the dispersion, whilst the left hand side is the unperturbed equation of motion. This makes it possible to compute $D_2$ easily from the periodic forced solution of the equation of motion [8]. This recurrence property extends to higher orders, simply because the terms containing the highest-order derivative come from the terms linear in coordinates which are by definition the unperturbed equation of motion.

### 6.5 Higher-order chromaticity correction

The problem of correction is now less straightforward than for the linear case, because we will have to make a compromise between the contributions of all orders, and very often it is not worth having too many variables to do the job. Therefore the best thing is to make a numerical minimisation of all contributions.

Formulae based on a generalisation of the Courant and Snyder second order perturbation formula [8], with contributions of multipoles up to decapole, are old fashioned but relatively easy to implement. For instance the program HARMON [15] is based on such formulae for the chromaticity correction up to third order in $\delta$ and to second-order in betatron amplitude. They are enough to make the third-order correction in a large machine, but not for a machine with a radius of curvature comparable with its dispersion function. For the latter case formulae like (9) should be used.

A promising and elegant alternative method is the Differential Algebra technique [16] which makes it possible to obtain the contribution of variables, e.g. multipole excitations, to the chromatic monomials in the transfer map of a whole machine. From the transfer maps it is possible to compute the expansion of the tunes with respect to momentum, i.e. the tune derivatives. This new method has the advantage that the computation time grows less fast with the expansion order than analytical methods like the above one.

We cannot leave the subject of higher-order chromaticity correction without mentioning how the problem of non-linear terms in transverse coordinates can be solved. This can be done with perturbation theories which make it possible to compute contributions to a given order. Examples can be found for instance in Ref. [1]. The formalism of gradient perturbations can be generalised to this case. For instance the above mentioned HARMON program contains the computation of anharmonicity and resonance effects. Of course the more general Differential Algebra technique can also be applied to this. The correction of non-linearities in the transverse coordinates is important in machines where the fields associated with multipoles are at least of the order of several thousandths of the quadrupole fields at the useful aperture. This occurs for instance in the synchrotron light sources [17] or in superconducting machines where the field defects are much larger than in conventional ones.

## 7 CHROMATICITY OF A MACHINE WITH LOW-$\beta$ INSERTIONS

A low-$\beta$ insertion is a part of a machine in which the envelope functions $\beta$ are made small in both planes in order to reduce the beam size. This is the case in storage rings at the crossing points. Because of the energy spread in the beam, the dispersion functions have also to be made small in order not to destroy the effect of the low-$\beta$ . Usually the dispersion functions are made zero in the whole insertion because the RF cells, which must be put in regions without dispersion, are conveniently placed close to them. In such a region without dispersion, a local correction, as explained in section 3, cannot be made.

In order to make a local correction, we could try to let the dispersion function oscillate in the insertion quadrupoles and make it zero at the crossing point only. However, if there is no bending magnet between the quadrupoles close to the crossing point, the dispersion has opposite values in these quadrupoles because the phase advance is close to $\pi$. The sextupoles for the local correction have then opposite strengths and their non-linear kicks act in phase, which necessitate a compensation with additional sextupoles.

In fact a solution exists for a non-local correction. The problem is the 'off-momentum mismatch' of the insertion, the latter being matched on central orbit, to a lattice which is chromatically corrected. Thus, rearranging the sextupoles in the lattice to force the off-momentum $\beta$'s to take values close to those which make an 'off-momentum matching', is the way to a solution.

We examine here how to estimate the effect of this off-momentum mismatch, which is a particular case of an insertion mismatch due to a gradient perturbation [18].

### 7.1 Global estimation of a gradient perturbation

We consider a perfect machine at the end of which the betatron functions have the values $\beta$ and $\alpha$. By definition the betatron functions have also the values $\beta$ and $\alpha$ at the beginning of the machine.

We introduce in this machine a certain gradient perturbation. The effect of this perturbation can be computed exactly with the transforms of $\beta$ and $\alpha$ through the perturbed machine, which are $\beta^t$ and $\alpha^t$, as well as the associated phase advance $\mu^t$ defined by :

$$\mu^t = \int_0^C \frac{ds}{\beta^{t,s}} \tag{10}$$

where $\beta^{t,s}$ is the transform of $\beta$ at the point of longitudinal coordinate $s$. Indeed these quantities are enough to obtain the perturbed Twiss matrix the expression of which is [19] :

$$\begin{pmatrix} \sqrt{\frac{\beta^t}{\beta}}(\cos\mu^t + \alpha\sin\mu^t) & \sqrt{\beta\beta^t}\sin\mu^t \\ \frac{1}{\sqrt{\beta\beta^t}}\left((1+\alpha\alpha^t)\sin\mu^t + (\alpha^t-\alpha)\cos\mu^t\right) & \sqrt{\frac{\beta}{\beta^t}}(\cos\mu^t - \alpha_t\sin\mu^t) \end{pmatrix}$$

It is important to note that $\beta^t$ and $\alpha^t$ are not true Twiss functions : they have the same meaning as Twiss functions in a transfer line. The actual $\beta$-function at the end of the perturbed machine $\beta^*$ can be obtained from the second element of the first line of this matrix :

$$\beta^* = \sqrt{\beta\beta^t}\sin\mu^t / \sin\mu^*$$

where the new tune $\mu^*$ can be computed from the trace of the perturbed matrix :

$$2\cos\mu^* = \left(\sqrt{\frac{\beta^t}{\beta}} + \sqrt{\frac{\beta}{\beta^t}}\right)\cos\mu^t + \left(\alpha\sqrt{\frac{\beta^t}{\beta}} - \alpha^t\sqrt{\frac{\beta}{\beta^t}}\right)\sin\mu^t \tag{11}$$

Putting :

$$\theta = -\arctan\frac{\alpha\sqrt{\frac{\beta^t}{\beta}} - \alpha^t\sqrt{\frac{\beta}{\beta^t}}}{\sqrt{\frac{\beta^t}{\beta}} + \sqrt{\frac{\beta}{\beta^t}}} = \frac{\alpha^t\beta - \alpha\beta^t}{\beta + \beta^t}$$

We can transform equation (11) into :

$$\cos\mu^* = \cos(\mu^t + \theta)\sqrt{1 + \frac{1}{4}\left(\sqrt{\frac{\beta^t}{\beta}} - \sqrt{\frac{\beta}{\beta^t}}\right)^2 + \frac{1}{4}\left(\alpha\sqrt{\frac{\beta^t}{\beta}} - \alpha^t\sqrt{\frac{\beta}{\beta^t}}\right)^2} \tag{12}$$

In order to obtain this expression, there is a trick consisting of adding 4 to the sum of the squares of the coefficients of the trigonometric functions in equation (11), so that the sign plus in the first one can be changed to minus. As the term under the square root is always larger than 1, there are values of $\mu^t + \theta$ for which $\cos\mu^*$ is larger than 1, even if the unperturbed $\cos\mu$ is smaller than 1 : the gradient perturbation has opened 'gradient stop-bands'.

It is worth noting that the expression under the square root can be used as a measure of mismatch when trying to match an insertion [18]. In the case of an imperfect matching, minimising this term guarantees that the stop-bands associated with the mismatch have the minimum width.

### 7.2 Application to a chromatic perturbation

We expand $\mu^t$ and $\mu*$ in power series of the relative momentum deviation $\delta$ :

$$\mu^t = \mu + {\mu^t}'\delta + \frac{1}{2}{\mu^t}''\delta^2 + ....$$

$\mu$ being the on-momentum phase advance.

$$\mu^* = \mu + {\mu^*}'\delta + \frac{1}{2}{\mu^*}''\delta^2 + .....$$

We expand also $\beta^t$ and $\alpha^t$ :

$$\beta^t = \beta + \beta'\delta + .....$$
$$\alpha^t = \alpha + \alpha'\delta + .....$$

$\beta$ and $\alpha$ being the on-momentum values. We put all these expansions in (11), expand in powers of $\delta$ and identify the terms with the same power of $\delta$ on both sides. We obtain for the first-order :

$${\mu^*}' = {\mu^t}' - \frac{\alpha\beta' - \beta\alpha'}{2\beta}$$

This is already interesting : the first derivative of the tune cannot be obtained simply by the expansion of the integral in (10). To second-order, we obtain after substitution of ${\mu^*}'$ with its above expression :

$${\mu^*}'' = {\mu^t}'' - \frac{1}{4}\cot\mu\left[\left(\frac{\beta'}{\beta}\right)^2 + \left(\frac{\alpha\beta' - \beta\alpha'}{\beta}\right)^2\right] + \frac{\alpha\beta'' - \beta\alpha''}{2\beta} + \frac{\beta'}{2\beta^2}(\alpha'\beta - \beta'\alpha) \qquad (13)$$

In this expression of the second-order derivative of the tune with respect to momentum, ${\mu^t}''$ is a mere contribution of second-order aberrations, and the other terms describe the effect of the off-momentum mismatch of the $\beta$-function to first and second-order. It must be recalled that $\beta'$ is not the derivative of the $\beta$-function with respect to momentum. It is the derivative with respect to momentum of the transform of $\beta$, the on-momentum value, through the machine. The computation of this $\beta'$ can be easily done from the derivative of the expression of the transform of the $\beta$-function through a thin quadrupole of length l, which is :

$$\frac{\beta'}{\beta} = -kl\beta_0 \sin 2[\mu - \mu_0]$$

where the unlabelled optics parameters refer to the point of longitudinal coordinate $s$ where the derivative is computed and the quantities labelled 0 refer to the quadrupole location. Taking the derivative of this expression with respect to $s$ gives :

$$\alpha\frac{\beta'}{\beta} - \alpha' = kl\beta_0 \cos 2[\mu - \mu_0]$$

From these expressions, the first-order contributions are easily obtained by integration over the machine. Such a calculation has been implemented in MAD [7] (command TWISS CHROM) in order to estimate quickly the importance of these basic effects. Substituting the two above expressions in equation (13), it appears that the contribution of a single quadrupole to the off-momentum mismatch effect goes with $(kl\beta_0)^2$. It is worth comparing this with the second-order contribution, which comes from the variation of $K$ with momentum (Eq. (2)), which is $+2kl\beta_0$.

### 7.3 Non-linear chromaticity due to a low-$\beta$ insertion

In the case of a low-$\beta$ insertion, several quadrupoles contribute to the chromatic effects but there is at least a strong one, close to the crossing point, which has a dominant effect. In order to give an idea of the order of magnitude of this effect, we consider the case of LEP under physics conditions, in the vertical plane. The $\beta$-value at the interaction point $\beta^*$ is 5cm. The closest quadrupole is at 3.7m, it has a length of 2m and a strength $k$ of 0.164m$^{-2}$. The $\beta$-value at the quadrupole centre is about 400m, and the expression $kl\beta_0$ has a value of about 130. From this number we deduce that the effect of the off-momentum mismatch induced by this quadrupole, which is proportional to its square, is larger than its second-order effect, due to the expansion of $K$ to second-order in $\delta$, by two orders of magnitude. For the other quadrupoles of the insertion, the product $kl\beta_0$ is always smaller than that of the above one by more than one order of magnitude, and their contributions add-up with phase factors which make their effect even smaller. This is also for the cross terms described in section 6.3.

Under these conditions we can describe the second derivative of the tune with respect to momentum by means of the term which is a sum of squares in equation (13). In order to make this formula easier to use, we express the tunes in units of $2\pi$ and we consider that there are $\mathcal{N}_s$ equal super-periods in the machine with one symmetrical low-$\beta$ insertion each. As said above, the only contributions we have to consider are those of the low-$\beta$ quadrupoles. If we take the origin of the super-period at the symmetry point of the low-$\beta$ insertion, the phase of one low-$\beta$ quadrupole is $\pi/2$ and that of the second one (close to the end of the super-period) is $2\pi Q/\mathcal{N}_s - \pi/2$. Using the above formulae for $\beta'/\beta$ and its derivative, we obtain finally :

$$Q'' \approx -\frac{\mathcal{N}_s \cos^3 \frac{2\pi Q}{\mathcal{N}_s}}{\pi \sin \frac{2\pi Q}{\mathcal{N}_s}} (Kl\beta)^2 \tag{14}$$

Putting the numbers given above for the LEP low-$\beta$ quadrupoles in the vertical plane, we obtain a value of $Q''$ of -6.5$\times 10^4$ with a fractional part of the tune of 0.19 and an integer part multiple of four (tunes used from 1991 to 1994 for physics), the super-periodicity of LEP being four. The parabolic variation of the vertical tune due to this $Q''$ is enough to produce a betatron instability, i.e. make a decrease of the vertical tune of 0.19, for a momentum deviation of $\pm 2.4 \times 10^{-3}$. The calculation done with all the quadrupoles in the machine shows that this instability occurs at about $\pm 3 \times 10^{-3}$ as shown on Fig. 2.

Such a strong non-linear variation of the tune with momentum can be compensated with special sextupole arrangements as shown below, if no parameter entering formula (14) can be changed. For LEP it has been estimated that the tunes with an integer part multiple of the super-periodicity were better for the beam-beam interaction, as the linear part of this interaction makes a decrease of $\beta^*$. This implies that $\sin(\frac{2\pi Q}{\mathcal{N}_s})$ is small and $Q''$ is large in the vertical plane. As a consequence, sextupole families have been foreseen for LEP.

If there is some freedom for acting on the parameters entering formula (14), it is quite easy to find situations where this variation does not occur. For instance, if the tune is close to a quarter integer modulo one half, the chromaticity correction with two sextupole families is quite acceptable. This is what has been done to test LEP with a 90°

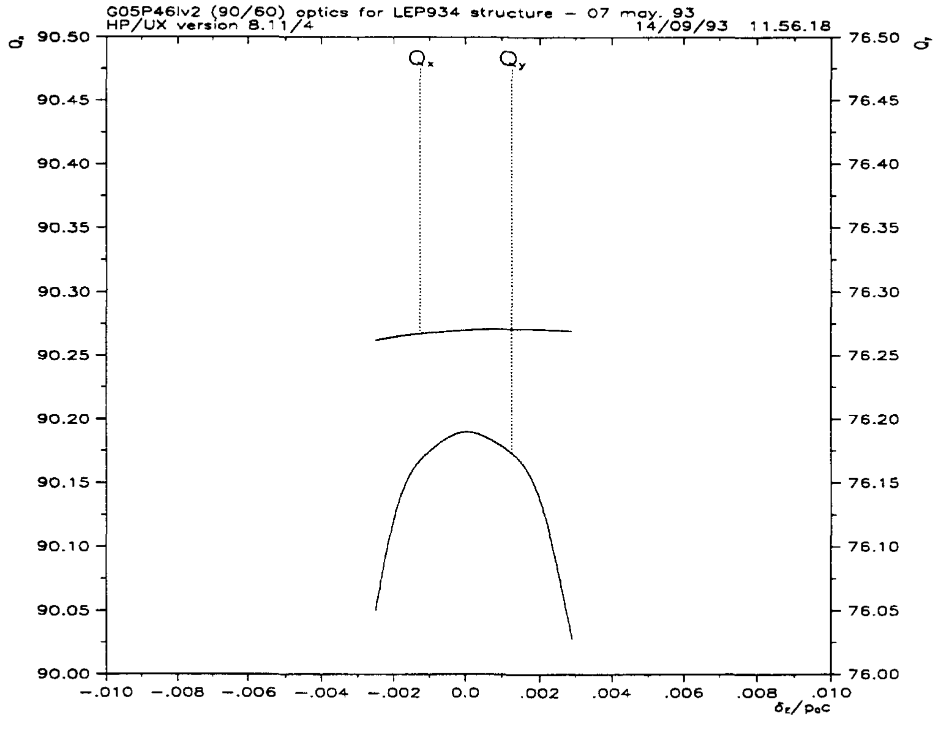



Figure 2: Tunes versus momentum for LEP under physics conditions at 46GeV, G05p46lv2-optics, $Q_h$=90.27, $Q_v$=76.19, $\beta^*$ of 5cm and 1.25m, two sextupole families.

phase advance in the arc cells, whilst the sextupole families were cabled for a 60° phase advance. Choosing the tunes:

$$Q_h = 91.30 \quad Q_v = 97.20$$

made it possible correct the chromaticity with two sextupole families for a $\beta^*$ of 5cm. Another possibility is to increase $\beta^*$ for intermediate optics as shown below.

### 7.4 Scaling law for the second-order chromaticity

We have seen above that the main contribution to the second-order chromaticity is that of the quadrupole close to the crossing point, mainly because the $\beta$-value is large in this quadrupole. This large value is associated directly with the small $\beta$-value at the crossing point. The effect of the quadrupole is indeed to change the sign of the derivative with respect to the longitudinal coordinate of the $\beta$-function so that this function takes smaller values and can be matched to the lattice. This is only approximate as the matching of the insertion consists of a more subtle adjustment of the quadrupole strength, but this estimate makes it possible to obtain an interesting qualitative analysis. This change of sign is expressed, in the thin-lens approximation, by:

$$-2\alpha = Kl\beta$$

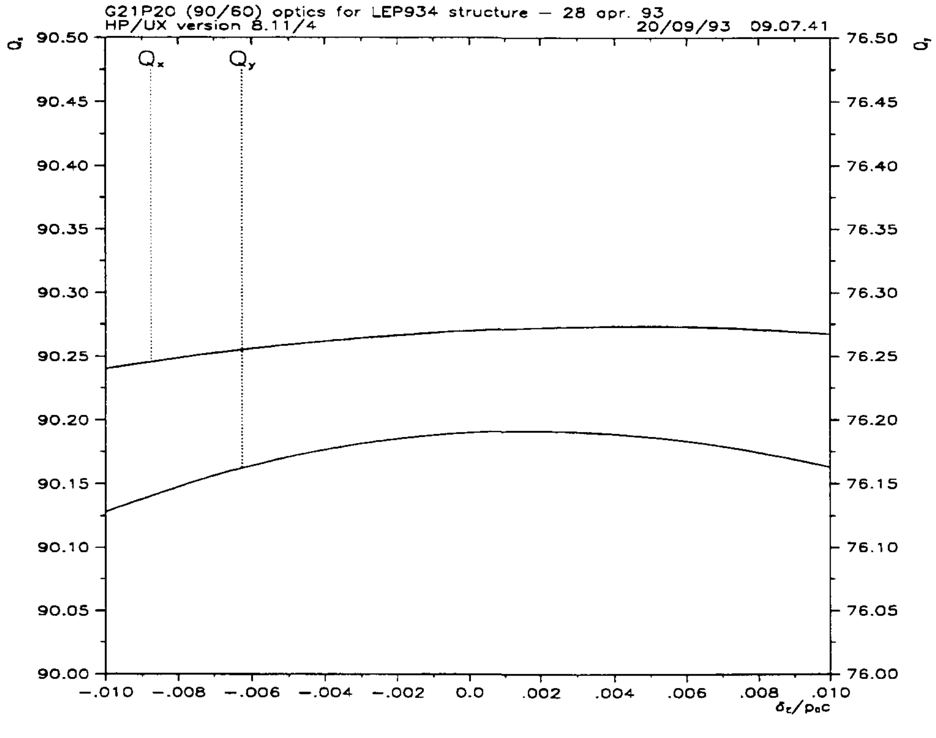



Figure 3: Tunes versus momentum for LEP at injection at 46GeV, G21P20v2-optics, $Q_h$=90.27, $Q_v$=76.19, $\beta^*$ of 0.21m and 2.5m, two sextupole families.

where $\alpha$ is related with the value of the $\beta$-function at the crossing point by:

$$\alpha = -\frac{L}{\beta^*}$$

$L$ being the distance between the crossing point and the centre of the quadrupole. From these two equations, we obtain the estimate of the value of the term $kl\beta$ entering formula (14):

$$kl\beta \approx \frac{L^2}{\beta^{*2}}$$

from characteristic quantities of the insertion. For the case of LEP the $\beta^*$'s are reduced by a factor 3 at injection, which reduces $Qy''$ by a factor 9 (and mainly reduces the maximum $\beta$-value in the machine). Thus for the injection optics, the chromaticity correction with two sextupole families is quite acceptable, as shown on Fig. 3.

## 8 CHROMATICITY CORRECTION WITH SEXTUPOLE FAMILIES

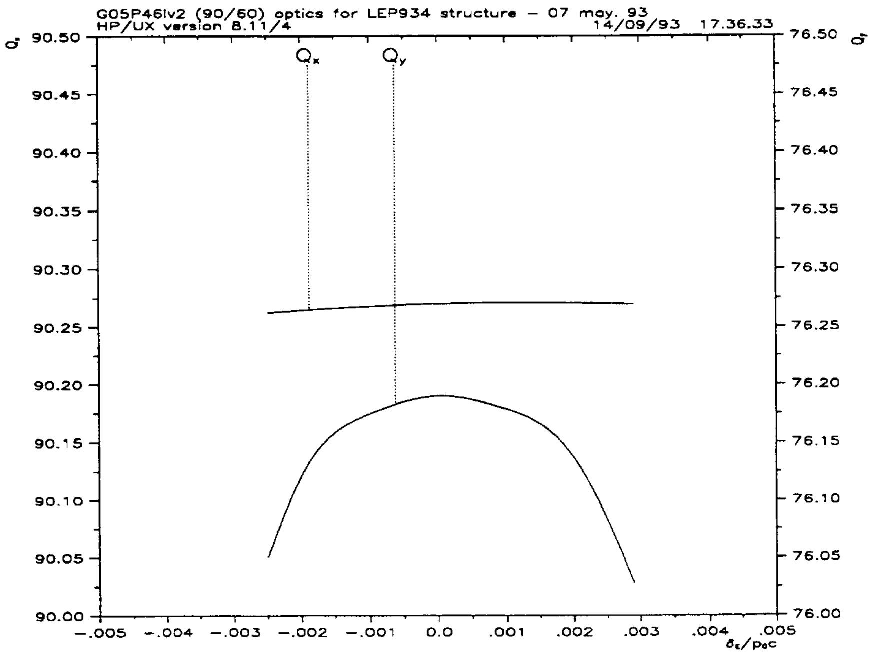
The off-momentum mismatch due to a low-$\beta$ insertion is so evident that even for the first large machine with tunes suitable for the beam-beam interaction, it was proposed to make an off-momentum matching of the insertions by means of sextupole families

| Family label | $\beta_v$ at $\delta$=-0.002 | $\beta_v$ at $\delta$=+0.002 |
|---|---|---|
| 2 | 343 | 99 |
| 1 | 57 | 355 |
| 3 | 282 | 173 |
| 2 | 347 | 102 |
| 1 | 58 | 356 |
| 3 | 277 | 169 |
| 2 | 351 | 106 |
| 1 | 60 | 357 |
| 3 | 271 | 164 |

Table 1: Off-momentum $\beta_v$'s at the nine first SD sextupoles for the present LEP. G05P46lv2 optics, two sextupole families. The sextupoles are placed in the arc FODO cells with a vertical phase advance of 60° . Thus there is a $\pi$ vertical phase advance between two successive sextupoles of each family (sextupoles with the same label). Each family has its own power supply, there are 32 sextupoles per octant. As this number is not an even multiple of 3, this ensemble of sextupoles does not constitute a second-order achromat (two sextupoles are in excess). The on-momentum $\beta_v$'s are about 150m.

[20, 21, 22]. One of the proposals was to introduce a 'sextupole insertion' to perform this matching. Unfortunately this leads to the use of strong localised sextupoles which result in important non-linear geometric aberrations, so condemning the project or implicating a complicated scheme for the correction of the non-linear geometric aberrations. An analysis of such a failure can be found in [23].

Rather than trying to make a sextupole insertion, it is much more efficient to change the off-momentum $\beta$'s in the machine cells by breaking the off-momentum cell periodicity via re-shuffling of the sextupoles into families. However it is of prime importance to make sure that the arrangement so obtained is efficient, as the tolerance of the phase advance per cell is tight. Therefore the best procedure to make sextupole families is first to adjust the first tune derivatives with two sextupole families and then to inspect the beating of the $\beta$-functions on the off-momentum closed orbits. This beating gives a simple solution to the compensation of the higher-order tune derivatives, because it makes the off-momentum $\beta$-values different at each sextupole. In order to show how this can be exploited, it is best to give an example. The off-momentum $\beta$'s at some sextupole locations in the beginning of a LEP arc are shown in Table 1.



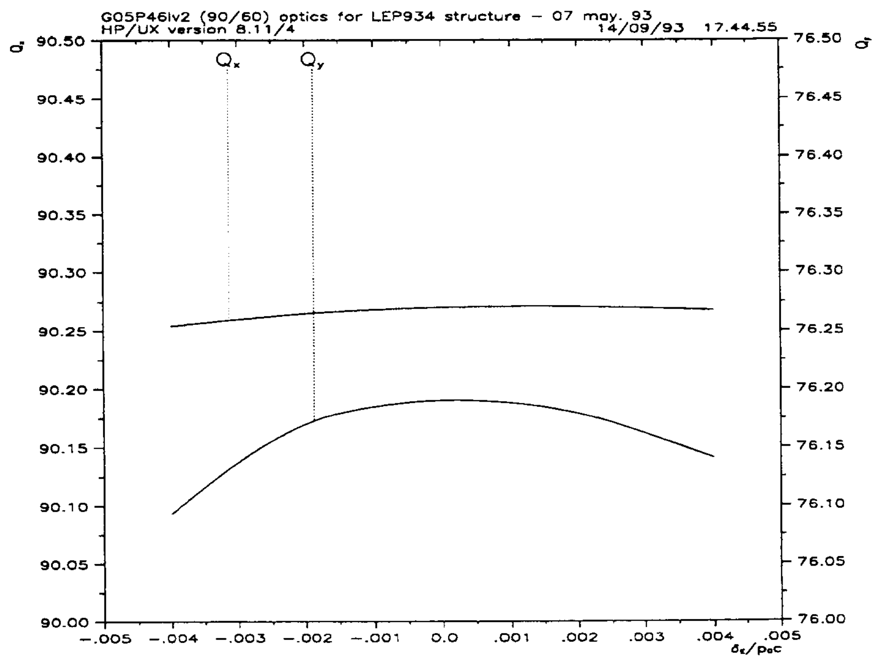




Figure 4: Tunes versus momentum for the LEP physics optics G05P46lv2. The upper graph is the same as Fig. 2 with the horizontal scale changed. For the lower graph the $K'$ of the SD2 family has been incremented by +0.04 and that of the SD1 family has been incremented by -0.04.

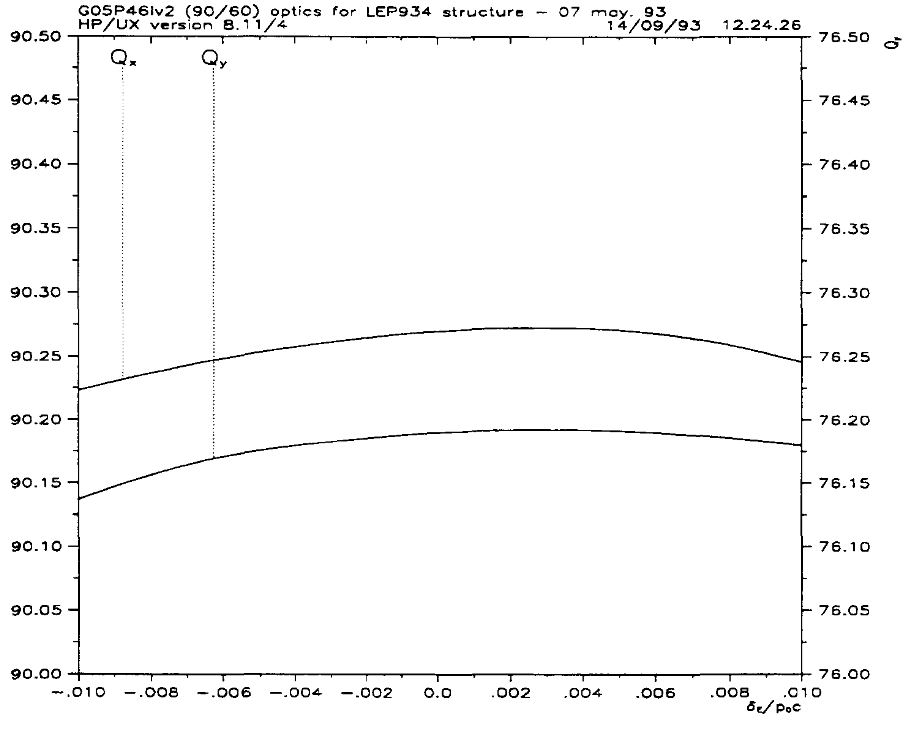



Figure 5: Tunes versus momentum for LEP under physics conditions at 46GeV, G05p46lv2-optics, $Q_h$=90.27, $Q_v$=76.19, $\beta^*$ of 5cm and 1.25m, five sextupole families.

Thanks to the difference between the $\beta$-values at the sextupole locations, it is possible to increase certain sextupole strengths and decrease others, whilst keeping $Q'$ constant, and hence make a tune-shift on an off-momentum orbit. As $Q'$ is kept constant, this tune-shift is due to tune derivatives of order higher than, or equal to, two. Taking the numbers in Table 1, if a positive increment is applied to the sextupoles labelled 1 and the opposite increment applied to the sextupoles labelled 2, positive tune shifts are made on both negative $\delta$ and positive $\delta$, which corrects a little the variation of the tunes with momentum. The efficiency of this action is shown on Fig. 4. A complete correction has been done by means of the HARMON program, which uses a third-order perturbation theory. The associated variation of the tunes with momentum is shown on Fig. 5. The improvement is impressive, comparing this figure and Fig. 2.

For this LEP example the vertical phase advance per cell is exactly 60° and the extrema of the off-momentum modulation of the vertical $\beta$-function are well located at the SD sextupoles. For a vertical phase advance per cell of 55°, the modulation would be dephased by 10° at each cell since it goes with twice the phase advance. This makes a change of sign of the modulation after 18 cells. As there are 30 cells per arc, it is not possible to use regular families. A more complicated distribution of sextupoles into three families has to be done, as for the first LEP project [24].

Regular sextupole families associated with a phase advance per cell which is a simple fraction of $\pi$ allow chromaticity correction schemes free from second-order aberrations [25]. It is possible to devise similar schemes completely free from geometrical aberrations on the reference orbit by grouping the sextupoles in pairs separated by a $\pi$ phase advance. Under this condition, the non-linear kicks due to two sextupoles of a given pair are equal because of the quadratic variation of the sextupole field, and do not produce any effect outside the pair. This sort of arrangement is referred to as 'non-interleaved sextupole schemes'. Their drawback is obvious : for the off-momentum trajectories the compensation of the non-linear kicks does not hold any more and the dynamic aperture decreases very quickly with the momentum deviation. An example extracted from the LEP design study can be found in [26], it is shown in this paper how to optimise such schemes. For the present LEP they have been discarded because of the existence of multipole components in the arc dipoles which destroy the non-interleaved condition. However they are potentially interesting because of the large dynamic aperture they guaranty for small momentum deviations.

## 9 CHROMATICITY CORRECTION BY MEANS OF MULTIPOLES

We have seen in section 3 that a variation of the tunes with $\delta^n$ can be compensated with a $2(n+2)-pole$. In the equation of motion, a right $2(n+2)$-pole introduces a term :

$$\frac{K^{(n)}}{(n+1)!}\left[x^{n+1}-(n+1)x^{n-1}y^2+.....\right]$$

substituting $x$ for $D\delta + x_\beta$ and keeping only the linear terms in $x_\beta$ we obtain

$$\frac{K^{(n)}}{n!}(D\delta)^n x_\beta$$

The contribution of a term like this to the $n^{th}$ derivative of the tunes with respect to momentum is simply obtained using the first-order formula for gradient perturbations. If there are many such terms, their contributions are merely added. However, for a correct computation of the chromatic effects to a given order, all the contributions of multipoles of lower order have to be added by means of the formulae in section 6.3 which describe the coupling between gradient perturbations. For instance if we want to have chromatic effects in $\delta^3$, we must add contributions of :

- decapoles which are merely summed
- octupoles and sextupoles which are computed with the second-order formulae given in section 6.3
- sextupoles which are computed with a third-order formula, i.e. coupling between three gradient perturbations. Such a formula is obtained by a generalisation of what is done in section 6.3.

The implementation of the decapole and octupole contributions are quite straightforward once the second-order formalism applied to sextupoles is available. For instance they have been included in a negligible time in the HARMON program, used as a MAD routine.

There is another very simple way of making such corrections. It consists of making successive corrections in increasing order in $\delta$ by means of multipoles of increasing order.The variation of the tunes with momentum has to be recomputed after each correction. This poor man's method has the advantage of taking into account all the higher-order effects mentioned above and is extremely accurate thanks to the iteration process.

Rather than correcting the higher-order tune derivatives due to parasitic multipoles, it is possible to reduce these derivatives by merely decreasing the dispersion function. This is why 90° lattices are attractive. It is probably possible to go further if sextupole families are not needed, i.e. if there is no need of a phase advance per cell which is a simple fraction of $\pi$. However we must keep in mind that a phase advance per cell of 120° is probably forbidden because the systematic sextupole components act in phase for the excitation of the third-order non-linear resonances. In conclusion, using a phase advance per cell substantially larger than the magic value of 60° , could be a solution to the compensation of higher-order chromatic effects due to parasitic multipoles, as long as the increase of the sextupole strength needed to correct the first-order chromaticity does not ruin the effort.

## 10 CHROMATICITY MEASUREMENTS

At fixed machine settings, changing the RF frequency changes the beam energy, i.e. the beam goes on an off-momentum closed orbit. Indeed the RF frequency $F_{RF}$ is a harmonic of the revolution frequency $f_{rev}$ :

$$F_{RF} = H f_{rev} \tag{15}$$

where $H$ is the harmonic number. The revolution frequency is, by definition :

$$f_{rev} = \beta c / C$$

where $\beta c$ is the velocity of the particles, $c$ being the speed of light. The lengthening of the off-momentum closed orbit with respect to the central one is, by definition :

$$\Delta C / C = \alpha_c \delta.$$

Taking the logarithmic derivative of equation (15), we obtain :

$$\frac{\Delta F_{RF}}{F_{RF}} = \frac{\Delta f_{rev}}{f_{rev}} = \frac{\Delta\beta}{\beta} - \frac{\Delta C}{C}$$

From the relativistic relationship between $\beta$ and the momentum, we deduce:

$$\frac{\Delta\beta}{\beta} = \frac{1}{\gamma^2}\delta$$

Putting this in the previous equation, we obtain the relationship between a change of the RF frequency and a change of the beam momentum :

$$\frac{\Delta F_{RF}}{F_{RF}} = \delta\left(\frac{1}{\gamma^2} - \alpha_c\right)$$

This formula is used to compute $\delta$ from a given change of the RF frequency. Measuring the tunes for various RF frequencies makes it possible to compute the variation of the tunes with momentum, which is the most important part of the chromaticity.

If the RF frequency is kept constant and the dipole field is varied, the momentum of the beam changes but it stays on the same closed orbit. For a separated function machine, this is equivalent to changing the normalised field strengths of the focussing magnets. Doing this on the central orbit produces a tune change proportional to the momentum change, which is the measure of the derivative of the tunes with respect to momentum without any effect coming from the non-linear magnets. This is a means to obtain the 'natural' chromaticity of the linear machine without any correction.

## References

[1] B. Autin, Non-linear betatron oscillations and A. Ropert, Dynamic aperture. Both lectures in the Proceedings of the CAS third advanced accelerator physics course, Ed. S. Turner, CERN 90-04 (1990).

[2] M. Sands, The head-tail effect : an instability mechanism in storage rings, SLAC-TN-69-8.

[3] E. Keil, Single-particle dynamics, Linear machine lattices, Proceedings of the first course of the International School of Particle Accelerators of the Ettore Majorana Centre for Scientific Culture, Erice ,November 1976, CERN 77-13 (1977).

[4] Computer codes for particle accelerator design and analysis : a compendium, LA-UR-90-1766. Second edition, May 1990.

[5] K. Brown, A first- and second-order matrix theory for the design of beam transport systems and particle spectrometers. SLAC report 75, or Advances in Particle Physics 1, 71-134 (1967).

[6] K. Steffen, High energy beam optics, Inter-science Publishers, New-York, 1965.

[7] H. Grote and F.C. Iselin. The MAD program (Methodical Accelerator Design), version 8.10, User's reference manual, CERN/SL/90-13(AP), (Rev. 3, January 19, 1993).

[8] E. D. Courant and H. S. Snyder, Theory of the alternating-gradient synchrotron, Annals of Physics, vol. 3, 1-48 (1958).

[9] J. Jaeger and D. Moehl, Comparisons of methods to evaluate the chromaticity in LEAR, PS/DL/LEAR/Note 81-7, (1981).

[10] A.J. Dragt, R.D. Ryne, L.H. Healy and F. Neri, MARYLIE 3.0 (1985), University of Maryland.

[11] M. Berz, Differential Algebraic description of beam dynamics to very high orders. Particle Accelerators Vol. 24, Number 2 ( 1989 ). ISSN 0031-2460.

[12] B. Autin and A. Verdier, Focussing perturbations in alternating gradient structures, CERN ISR-LTD/76-14, (1976).

[13] M. Bassetti, Perturbative method for chromatic and sextupolar effects, $2^{nd}$ order solution. DESY report PET-77/35, (1977).

[14] W. Hardt, J. Jaeger and D. Moehl, A general analytical expression for the chromaticity of accelerators rings, PS/DL/LEAR/Note 82-5, (1982).

[15] M. Donald, A users guide to the HARMON Programs, PEP note 311.

[16] M. Berz, Private communication (Seminar on Differential Algebra given at CERN in 1990).

[17] A. Ropert, Sextupole correction scheme for the ESRF, second advanced ICFA beam dynamics workshop, Lugano, April 1988. CERN 88-04 (1988).

[18] A. Verdier, Linear optics design of the superconducting low-$\beta$ insertion in the Intersecting Storage Rings at CERN, Nuclear Instruments and Methods, 207 (1983) 301-307.

[19] K.G. Steffen, H. Hultschig and W. Kern, Use of generalised amplitude and phase functions in designing beam transport systems, DESY/A 2-70 (1960).

[20] E. Keil, The behaviour of betatron oscillations in the vicinity of half-integral structure resonances, PEP-121 (1975).

[21] B. Autin, Sextupole and octupole schemes in very high energy electron rings, CERN ISR-LTD/76-37, (1976).

[22] B. Montague, Linear optics for improved chromaticity correction, LEP Note 165 (1979).

[23] J. Hagel, A local chromaticity correction scheme applied to LEP, LEP Note 552 (1985).

[24] J.R.J. Bennett et al., Design concept for a 100GeV $e^+e^-$ storage ring (LEP), CERN 77-14, (1977).

[25] K. Brown. SLAC-PUB 2257.

[26] A. Verdier, La luminosité à haute énergie pour le LEP11 avec $\beta^*$ nominaux selon les arrangements hexapolaires. LEP Note 396 (October 3 1982).