Floating Point

15-213: Introduction to Computer Systems
4th Lecture, Sept. 8, 2016

Today's Instructor:

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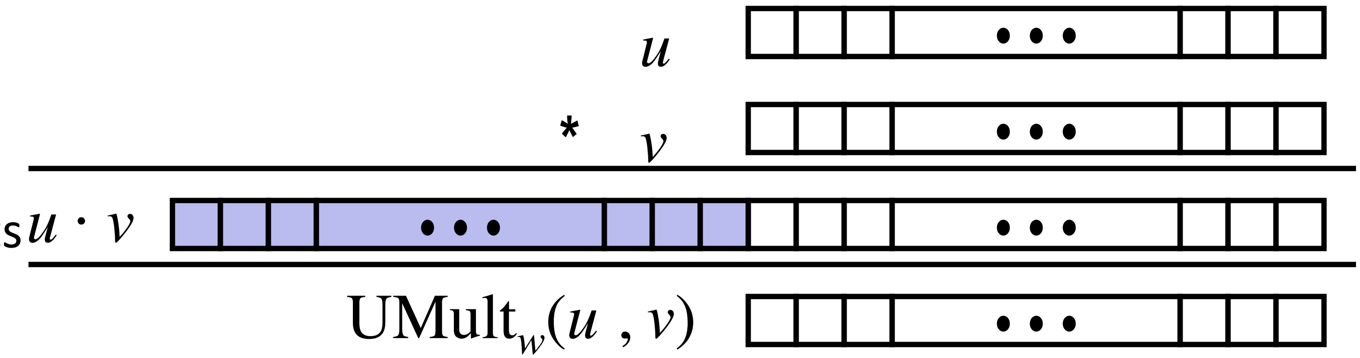
Correction from last time

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2*w bits

Discard *w* bits: *w* bits



- **Standard Multiplication Function**
  - Ignores high order *w* bits
- **Implements Modular Arithmetic**

  $\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$

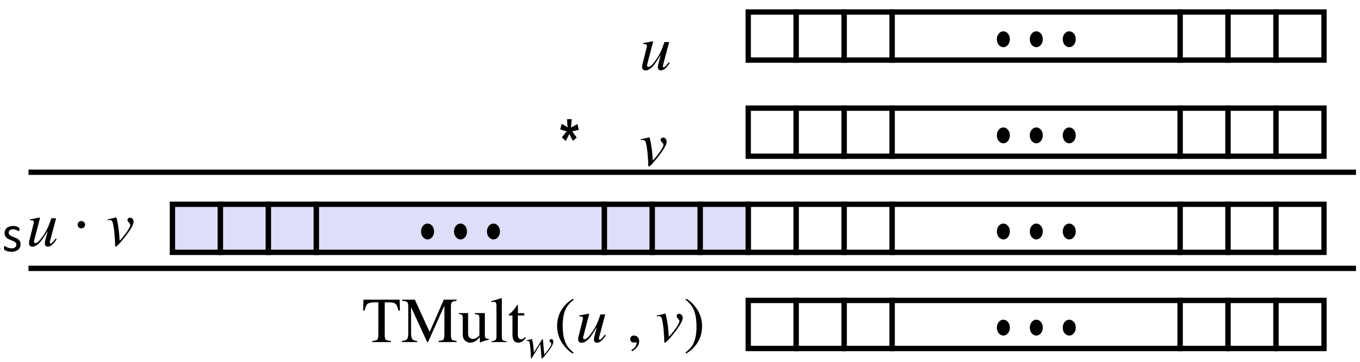
```
           1110 1001       E9        223
*          1101 0101   *   D5   *    213
 1100 0001 1101 1101     C1DD      47499
           1101 1101       DD        221
```

Signed Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



- **Standard Multiplication Function**
  - Ignores high order *w* bits
  - *Some of which are different for signed vs. unsigned multiplication*
  - Lower bits are the same

```
 1111 1111 1110 1001      E9       -23
*1111 1111 1101 0101   *  D5   *   -43
 0000 0011 1101 1101    03DD       989
           1101 1101      DD       -35
```

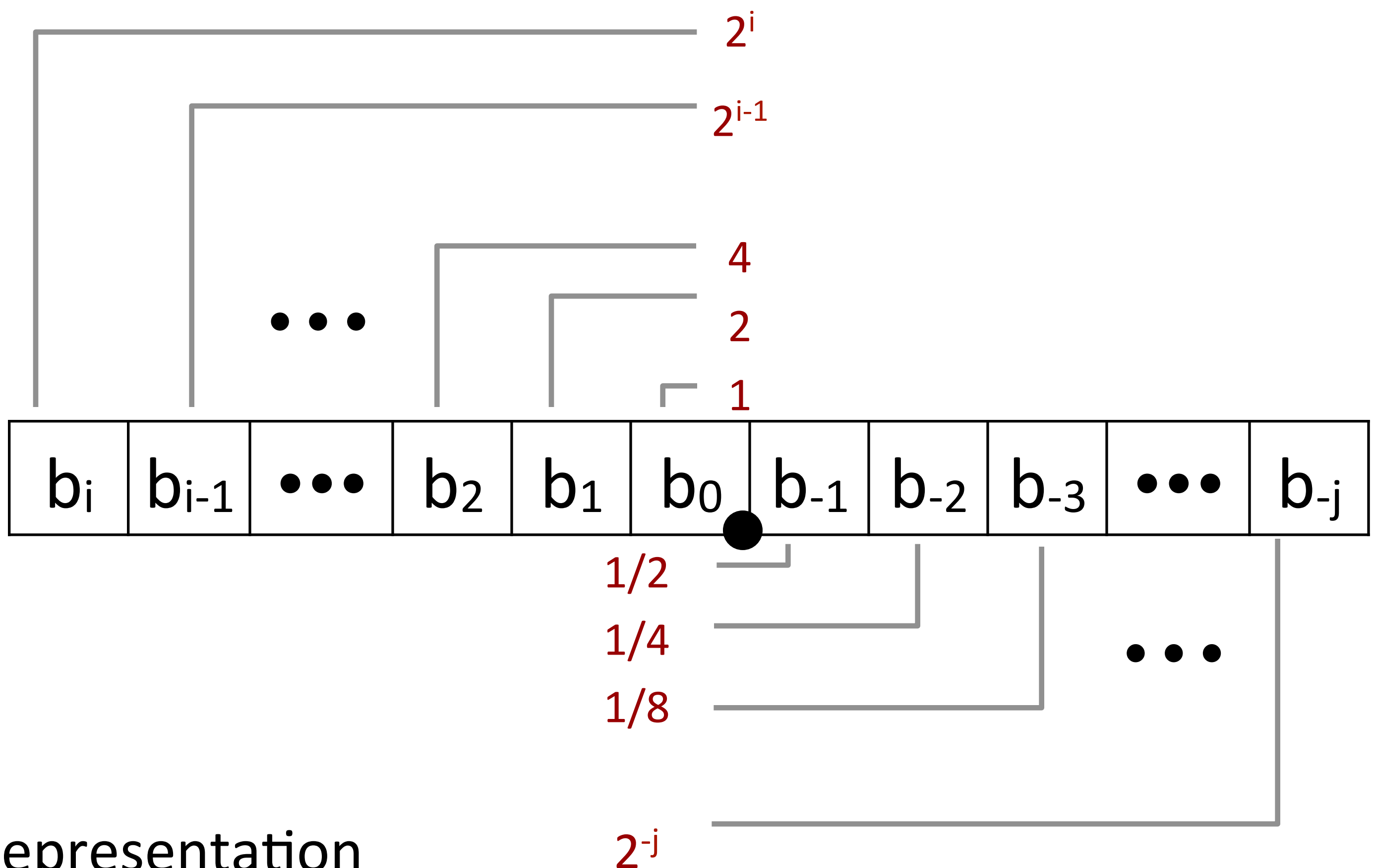
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101_2?

Fractional Binary Numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

- **Value** / **Representation**

Value		Representation	
5 3/4	= 23/4	`101.11`$_2$	= 4 + 1 + 1/2 + 1/4
2 7/8	= 23/8	`10.111`$_2$	= 2 + 1/2 + 1/4 + 1/8
1 7/16	= 23/16	`1.0111`$_2$	= 1 + 1/4 + 1/8 + 1/16

- Observations
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- Limitation #1
  - Can only exactly represent numbers of the form $x/2^k$
    - Other rational numbers have repeating bit representations
  - Value Representation
    - 1/3 `0.0101010101[01]`$\ldots_2$
    - 1/5 `0.001100110011[0011]`$\ldots_2$
    - 1/10 `0.0001100110011[0011]`$\ldots_2$
- Limitation #2
  - Just one setting of binary point within the *w* bits
    - Limited range of numbers (very small values? very large?)

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IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
  - Some CPUs don't implement IEEE 754 in full
    e.g., early GPUs
- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

Example:
$15213_{10} = (-1)^0 \times 1.1101101101101_2 \times 2^{13}$

- Numerical Form:

$$(-1)^s \, M \;\, 2^E$$

  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0).
  - Exponent E weights value by power of two

- Encoding
  - MSB S is sign bit s
  - exp field encodes E (but is not equal to E)
  - frac field encodes M (but is not equal to M)

s	exp	frac

Precision options

- Single precision: 32 bits
  ≈ 7 decimal digits, $10^{\pm 38}$

- Double precision: 64 bits
  ≈ 16 decimal digits, $10^{\pm 308}$

- Other formats: half precision, quad precision

"Normalized" Values

$$v = (-1)^s M\ 2^E$$

- When: exp ≠ 000…0 and exp ≠ 111…1

- Exponent coded as a biased value: E = Exp – Bias
  - Exp: unsigned value of exp field
  - Bias = 2^{k-1} - 1, where k is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- Significand coded with implied leading 1: M = $1.xxx…x_2$
  - xxx…x: bits of frac field
  - Minimum when frac=000…0 (M = 1.0)
  - Maximum when frac=111…1 (M = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$v = (-1)^s\, M\, 2^E$
$E = \text{Exp} - \text{Bias}$

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**
  M = `1.`$\underline{\texttt{1101101101101}}_2$
  `frac=` $\underline{\texttt{1101101101101}}\texttt{0000000000}_2$

- **Exponent**
  E = 13
  $Bias$ = 127
  Exp = 140 = $\texttt{10001100}_2$

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

$v = (-1)^s M 2^E$
$E = 1 - \text{Bias}$

- Condition: exp = 000…0

- Exponent value: E = 1 – Bias (instead of 0 – Bias)
- Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = **000…0**, `frac` = **000…0**
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - `exp` = **000…0**, `frac` ≠ **000…0**
    - Numbers closest to 0.0
    - Equispaced