$$\begin{bmatrix} a & g & m \\ b & h & n \\ c & i & o \\ d & j & p \\ e & k & q \\ f & l & r \end{bmatrix} \begin{array}{c} \xrightarrow{\text{Step 2}} \\ \xleftarrow{\text{Step 4}} \end{array} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \\ m & n & o \\ p & q & r \end{bmatrix}$$

(a)

$$\begin{bmatrix} a & g & m \\ b & h & n \\ c & i & o \\ d & j & p \\ e & k & q \\ f & l & r \end{bmatrix} \begin{array}{c} \xrightarrow{\text{Step 6}} \\ \xleftarrow{\text{Step 8}} \end{array} \begin{bmatrix} -\infty & d & j & p \\ -\infty & e & k & q \\ -\infty & f & l & r \\ a & g & m & \infty \\ b & h & n & \infty \\ c & i & o & \infty \end{bmatrix}$$

(b)

Figure 2: The operations of even-numbered steps of columnsort. This figure is taken from [Lei85]. For simplicity, this small 6×3 matrix is chosen to illustrate the steps, even though its dimensions fail to obey the columnsort restrictions on r and s. **(a)** The operations of steps 2 and 4. **(b)** The operations of steps 6 and 8.

2 The basic columnsort algorithm

In this section, we review Leighton's columnsort algorithm from [Lei85]. Along the way, we make some observations that will improve the out-of-core implementation.

Columnsort sorts N numbers, which are treated as an $r \times s$ matrix, where $N = rs$, s is a divisor of r, and $r \geq 2(s-1)^2$. When columnsort completes, the matrix is sorted in column-major order. That is, each column is sorted, and the keys in each column are no larger than the keys in columns to the right.

Columnsort proceeds in eight steps. Steps 1, 3, 5, and 7 are all the same: sort each column individually. Each of steps 2, 4, 6, and 8 permutes the matrix entries.

Step 2: Transpose and reshape

As Figure 2(a) shows, we first transpose the $r \times s$ matrix into an $s \times r$ matrix. Then we "reshape" it back into an $r \times s$ matrix by taking each row of r entries and rewriting it as an $r/s \times s$ submatrix. In Figure 2(a), for example, the column with $r = 6$ entries $a\ b\ c\ d\ e\ f$ is transposed into a 6-entry row with entries $a\ b\ c\ d\ e\ f$ and then reshaped into the 2×3 submatrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.

Step 4: Reshape and transpose

As Figure 2(a) shows, we first reshape each set of r/s rows into a single r-element row and then transpose the matrix. In other words, step 4 is the inverse of the permutation performed in step 2.

Step 6: Shift down by $r/2$

As Figure 2(b) shows, we shift each column down by $r/2$ positions, wrapping around into the next column as necessary. This shift operation vacates the first $r/2$ entries of the leftmost column, which are filled with keys of $-\infty$, and it creates a new column on the right, the bottommost $r/2$ entries of which are filled with keys of ∞. Looked at another way, we shift the top half of each column into the bottom half of that column, and we shift the bottom half of each column into the top half of the next column.

Step 8: Shift up by $r/2$

As Figure 2(b) shows, we shift each column up by $r/2$ positions, wrapping around into the previous column as necessary. In other words, we perform the inverse permutation of step 6.

We omit the proof of correctness and refer the reader to [Lei85].