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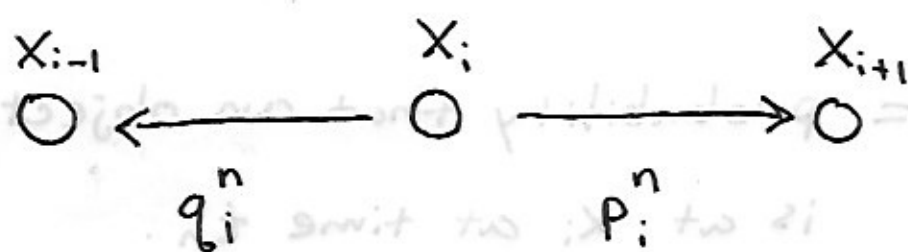
+ Random Processes

Consider a discrete process with states identified by $X_i = \eta i$ where $i \in \mathbb{Z}$ and $\eta > 0$:

$$\overset{X_{i-1}}{\bigcirc} \xleftarrow{\quad} \overset{X_i}{\bigcirc} \xrightarrow{\quad} \overset{X_{i+1}}{\bigcirc}$$

$$q_i^n \qquad P_i^n$$

where transitions between states occur at discrete times $t = \tau n$ where $n \in \mathbb{N}$ and

P_i^n = probability of transition from state i to $i+1$.

q_i^n = Probability of transition from state i to $i-1$.

This process can describe a collection of objects such as atoms, insects computer jobs or microbes. The objects will not move with probability $1 - p_i^n - q_{6i}^n$.

Let u_i^n be the probability density function at time t_n. That is;

$$u_i^n = \text{probability that an object is at } X_i \text{ at time } t_n.$$

Then we have the relationship

$$u_i^{n+1} = P_{i-1}^n u_{i-1}^n + q_{i+1}^n u_{i+1}^n + (1 - p_i^n - q_{6i}^n) u_i^n$$

For example:

$P_{i-1}^n u_{i-1}^n$ = Probability the process was at X_{i-1} at t_n (u_{i-1}^n) times the probability the process transitions from X_{i-1} to X_i.

This equation is called the Chapman–
Kolmogrov equation for our process.

By adding

$$\frac{1}{2} q_{i-1}^n u_{i-1}^n - \frac{1}{2} q_{i-1}^n u_{i-1}^n = 0$$

$$\frac{1}{2} P_i^n u_{i-1}^n - \frac{1}{2} P_i^n u_{i-1}^n = 0$$

and rearranging terms we have

$$u_i^{n+1} - u_i^n = \frac{1}{2}\left(P_{i-1}^n - q_{i-1}^n\right) u_{i-1}^n - \frac{1}{2}\left(P_{i+1}^n - q_{i+1}^n\right) u_{i+1}^n$$

$$\left[\left(P_{i-1}^n + q_{i-1}^n\right) u_{i-1}^n - 2\left(P_i^n + q_i^n\right) u_i^n\right.$$

$$\left. + \left(P_{i+1}^n + q_{i+1}^n\right) u_{i+1}^n\right]$$

As η and τ tend to zero we define
the following functions

$$\sum_{a \leq x_i \leq b} u_i^n \;\longrightarrow\; \int_a^b u(t,x)\,dx$$

$$\frac{P_i^n - q_i^n}{\tau}\,\eta \to c(t,x), \qquad \frac{P_i^n + q_i^n}{2\tau}\,\eta^2 \to d(t,x)$$

Assuming these limits exist we obtain

$$\boxed{\dfrac{\partial u}{\partial t} = -\dfrac{\partial}{\partial x}\big[c(t,x)u\big] + \dfrac{\partial^2}{\partial x^2}\big[d(t,x)u\big]}$$

Fokker - Planck Equation

+ Solutions

Setting $c(t,x) = 0$ we get

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\big[d(t,x)u\big]$$

For now, assume $d(t,x)$ is homogeneous

$$d(t,x) = d(t)$$

So that we now have

$$\frac{\partial u}{\partial t} = d(t) \frac{\partial^2 u}{\partial x^2}.$$

Using the Fourier transform of the above in space we have

$$\hat{u}_t = -d(t) \omega^2 \hat{u}$$

which has the solution

$$\hat{u}(t,\omega) = e^{-\omega^2 \int d(t)\,dt} \, \hat{u}_0(\omega)$$

where $\hat{u}_0(\omega)$ is the Fourier transform of the inital condition $u_0(0,x)$. By the Fourier inversion formula

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \, e^{-\omega^2 b} \, \hat{u}_0(\omega)\,d\omega.$$

⑤

A second formula for $u(t,x)$ can be obtained using the definition of $\hat{u}_0$:

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} e^{-b\omega^2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega y} u_0(y)\, dy \right] d\omega$$

$$= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\omega(x-y)} e^{-b\omega^2}\, d\omega \right) u_0(y)\, dy.$$

Lets evaluate the integral in parentheses:

$$\int_{-\infty}^{\infty} e^{i\omega(x-y)} e^{-b\omega^2}\, d\omega$$

$$= \int_{-\infty}^{\infty} e^{-b\left[\omega^2 - \frac{i\omega(x-y)}{b} \right]}\, d\omega$$

$$= \int_{-\infty}^{\infty} e^{-b\left[\omega^2 - \frac{i\omega(x-y)}{b} + \left(\frac{i(x-y)}{2b} \right)^2 - \left(\frac{i(x-y)}{2b} \right)^2 \right]}\, d\omega$$

$$= \int_{-\infty}^{\infty} e^{-b\left[\omega^2 - \frac{i\omega(x-y)}{b} + \left(\frac{i(x-y)}{2b}\right)^2\right]} e^{-\frac{(x-y)^2}{4b^2}} d\omega$$

$$= e^{-\frac{(x-y)^2}{4b^2}} \int_{-\infty}^{\infty} e^{-b\left[\omega - \frac{i(x-y)}{2b}\right]^2} d\omega$$

$$= \sqrt{\frac{\pi}{b}} \, e^{-\frac{(x-y)^2}{4b}}$$

Hence, the pdf for our random process is:

$$u(t,x) = \frac{1}{\sqrt{4\pi b}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4b}} u_o(y) \, dy$$

where,

$$b = \int_0^t d(t) \, dt .$$

Analytical Probability Distribution
of random process with initial
condition $u_o(x)$.

+ The Diffusion parameter b

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Now we evaluate the integral equation

$$b = \int_0^t d(t)\, dt$$

Its easy to prove the nth repeated integral of a function f based at a:

$$f^{(-n)}(x) = \int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n)\, d\sigma_n \cdots d\sigma_1$$

is given by

$$f^{(-n)}(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt$$

Proof: The proof is by induction. Note that

$$\int_a^x f(\sigma_1)\, d\sigma_1 = \frac{1}{0!} \int_a^x (x-t)^0 f(t)\, dt$$

⑧

Now suppose

$$f^{(-n)}(x) = \int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n)\, d\sigma_n \cdots d\sigma_1$$

for which

$$f^{(-n+1)}(x) = \int_a^x \int_a^{\sigma_1} \cdots \int_a^{\sigma_n} f(\sigma_{n+1})\, d\sigma_{n+1} \cdots d\sigma_2\, d\sigma_1$$
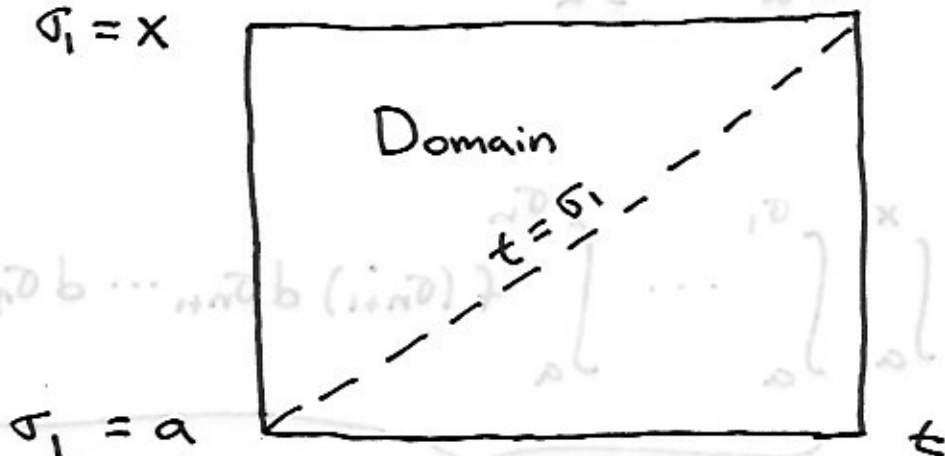

Induct here

we have

$$f^{(-n+1)}(x) = \int_a^x \left(\frac{1}{(n-1)!} \int_a^{\sigma_1} (\sigma_1 - t)^{n-1} f(t)\, dt \right) d\sigma_1$$

$$= \frac{1}{(n-1)!} \int_a^x \left(\int_a^{\sigma_1} (\sigma_1 - t)^{n-1} f(t)\, dt \right) d\sigma_1$$

$$= \frac{1}{(n-1)!} \int_a^x \left(\int_t^x (\sigma_1 - t)^{n-1} f(t)\, d\sigma_1 \right) dt$$

Where we note the change of order of integration:

$$\sigma_1 = x$$

$$\text{Domain}$$

$$t = \sigma_1$$

$$\sigma_1 = a$$

$$t$$

$$\int_a^x \int_a^{\sigma_1} (*) \, dt \, d\sigma_1 \qquad \begin{array}{c} a \leq \sigma_1 \leq x \\ a \leq t \leq \sigma_1 \end{array}$$

Our domain can also be described by

$$\int_a^x \int_t^x (*) \, d\sigma_1 \, dt \qquad \begin{array}{c} a \leq t \leq x \\ t \leq \sigma_1 < x \end{array}$$

Hence

$$f^{(-n+1)}(x) = \frac{1}{(n-1)!} \int_a^x \left(\int_t^x (\sigma_1 - t)^{n-1} f(t) \, d\sigma_1 \right) dt$$

So that

$$f^{(-n+1)}(x) = \frac{1}{(n-1)!} \int_a^x \frac{1}{n} (x-t)^n - (t-t)^n \, dt$$

$$= \frac{1}{n!} \int_a^x (x-t)^n \, dt$$

The proof is complete. This result is known as **Cauchy's** formula for repeated integration. Note that we can extend the above formula by replacing $n!$ with $\Gamma(\alpha)$ and n with some $\alpha \in \mathbb{R}^+$ to get

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u) \, du$$

which is the Riemann-Liouville Operator.

Next, we pick some $\alpha \in \mathbb{R}^+$, let n be the nearest integer greater than α. The Riemann-Liouville fractional derivative of order q of a function $f(t)$ is

$$D^\alpha f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^t (t-u)^{n-\alpha-1} f(u)\, du$$

For example, the fractional derivative of order 1.5 of a function $f(t)$ is

$$D^{1.5} f(t) = D^2 J^{.5} f(t).$$

To understand the parameter b recall we defined $d(x,t)$ such that

$$(p_i^n + q_i^n) \frac{n^2}{2t} \rightarrow d(t,x)$$

Where for our random process, a particle does not move (i.e it waits) with probability

$$\mathbb{P} = 1 - (P_i^n + q_i^n)$$

We can specifically define $P_i^n + q_i^n$ using our power-law waiting time pdf as

$$P_i^n + q_i^n = \frac{1}{\Gamma(\alpha)}\left(\frac{\tau_0}{t-t'}\right)^{1-\alpha} \quad (0 < \alpha < 1)$$

where t' denotes the arrival time of the particle at its current location.

Thus, we set

$$\frac{1}{\Gamma(\alpha)}\left(\frac{\tau_0}{t-t'}\right)^{1-\alpha}\left(\frac{n^2}{2\tau}\right) = d(t)$$

So that we see the diffusion parameter
b is the fractional integral of order α
given as

$$b = \int_0^t d(t)\, dt' = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{\tau_0}{t-t'}\right)^{1-\alpha} \frac{n^2}{2\tau}\, dt'$$

$$= \frac{\tau_0^{1-\alpha}}{\Gamma(\alpha)} \frac{n^2}{2\tau} \int_0^t \left(\frac{1}{t-t'}\right)^{1-\alpha} dt'$$

$$= \frac{D_0}{\Gamma(\alpha)\, \tau_0^{\alpha-1}} \int_0^t (t-t')^{\alpha-1}\, dt'$$

$$= \frac{D_0}{\Gamma(\alpha)\, \tau_0^{\alpha-1}} \cdot \frac{t^\alpha}{\alpha} = \frac{D_0\, t^\alpha}{\alpha\, \Gamma(\alpha)\, \tau_0^{\alpha-1}}$$

$$= \frac{D_0\, t^\alpha}{\Gamma(\alpha+1)\, \tau_0^{\alpha-1}}$$

Again,

$$b = \frac{D_0 \, t^{\alpha}}{\Gamma(\alpha+1) \, \tau_0^{\alpha-1}}$$

Hence, the diffusion parameter b is precisely our diffusion coefficent for anomalous diffusion. In the strictest sense its derivative defines the fractional diffussion coefficent for our Fokker-Planck equation.