Problem 1(a)

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Show that the estimator can be written as

$$\hat{m}_n(X) = LY$$

where $\hat{m}_n(x) = (\hat{m}_n(x_1) \dots \hat{m}_n(x_n))^T$ and

$$Y = (y_1, \dots y_n)^T$$

For smoothing kernel,

we can assume $Y_i = m(x_i) + \epsilon_i$

where $m(x)$ is a smooth function of x.

we want to take a local average of Y_i's that are within a distance of h of some point x.

So we can write.

$$\hat{m}_h(x) = \sum_{i=1}^{n} Y_i \, l_i(x)$$

where $l_i(x) = \dfrac{K\left(\frac{X_i - x}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)}$

where K can be any kernel that you choose.

$$\hat{m}_n(x) = \frac{\sum_{i=1}^{n} Y_i \, K\left(\frac{X_i - x}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)}$$

or

$$= LY$$

where $L = n \times n$ matrix where $l_{ij} = l_j(X_i) = \dfrac{K\left(\frac{X_j - X_i}{h}\right)}{\sum_{j=1}^{n} K\left(\frac{X_j - X_i}{h}\right)}$

and $Y = (y_1, \dots, y_n)^T$.

1b) leave-one-out Cross Validation (LOOCV)

show that $\hat{R}(h) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i - \hat{m}_n(x_i)}{1-L_{ii}}\right)^2$

The cross validation estimate of risk is defined as

$$\hat{R}(h) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

for LOOCV,

$$\hat{R}(h) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{Y}_{-i})^2$$

where $\hat{Y}_{-i}$ = estimator of Y after removing i-th observation

we can write $\hat{Y}_i$ in terms of LY.

$$\hat{Y} = LY.$$

where $\hat{Y}_i = \sum_j L_{ij} Y_j$

now, let's find an expression of $\hat{Y}_{-i}$ in terms of L and Y.

when we delete i-th column, i-th row should sum to $1 - l_{ii}$

so, $$\hat{Y}_{-i} = \frac{1}{1-l_{ii}}\sum_{\substack{j=1 \\ j\neq i}}^{n} l_{ij} y_j$$

& $$\hat{y}_i = \sum_{j=1}^{n} l_{ij} y_j$$

multiply each side by $1-l_{ii}$

$$\hat{y}_{-i} - \hat{y}_{-i} l_{ii} = \sum_{\substack{j=1 \\ j\neq i}}^{n} l_{ij} y_j$$

$$\hat{y}_{-i} = \sum_{\substack{j=1 \\ j \neq i}}^{n} l_{ij} y_j + l_{ii} \hat{y}_{-i}$$

$j \neq i \Rightarrow$ take this out

$$= \sum_{j=1}^{n} l_{ij} y_j + l_{ii} \hat{y}_{-i} - l_{ii} y_i$$

$$= \hat{y}_i + l_{ii} \hat{y}_{-i} - l_{ii} y_i$$

so, $$y_i - \hat{y}_{-i} = y_i - \left[\hat{y}_i + l_{ii}(\hat{y}_{-i} - y_i)\right]$$

$$= y_i - \hat{y}_i + l_{ii}(y_i - \hat{y}_{-i})$$

$$y_i - \hat{y}_{-i} - l_{ii}(y_i - \hat{y}_{-i}) = y_i - \hat{y}_i$$

$$(1 - l_{ii})(y_i - \hat{y}_{-i}) = y_i - \hat{y}_i$$

$$y_i - \hat{y}_{-i} = \frac{y_i - \hat{y}_i}{1 - l_{ii}}$$

$$\boxed{\begin{aligned} \hat{R}(h) &= \frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{Y}_{-i})^2 \\ &= \frac{1}{n} \sum_{i=1}^{n} \left(\frac{Y_i - \hat{Y}_i}{1 - L_{ii}} \right)^2 \quad \text{or} \quad \frac{1}{n} \sum_{i=1}^{n} \left(\frac{Y_i - \hat{m}_n(x_i)}{1 - L_{ii}} \right)^2 \end{aligned}}$$