Integral Exercises

Question 1:

$$\int \sin^2 x \cos^3 x \, dx$$

Solution 1: We use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite the integral:

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

Let $u = \sin x$, then $du = \cos x \, dx$. The integral becomes:

$$\int u^2 (1 - u^2) \, du = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

Substituting back $u = \sin x$:

$$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Question 2:

$$\int \sin^3 \theta \cos^4 \theta \, d\theta$$

Solution 2: We use the identity $\sin^2 \theta = 1 - \cos^2 \theta$ to rewrite the integral:

$$\int \sin^3 \theta \cos^4 \theta \, d\theta = \int \sin^2 \theta \cos^4 \theta \sin \theta \, d\theta = \int (1 - \cos^2 \theta) \cos^4 \theta \sin \theta \, d\theta$$

Let $u = \cos \theta$, then $du = -\sin \theta \, d\theta$. The integral becomes:

$$\int -(1 - u^2) u^4 \, du = \int (u^6 - u^4) \, du = \frac{u^7}{7} - \frac{u^5}{5} + C$$

Substituting back $u = \cos \theta$:

$$\frac{\cos^7 \theta}{7} - \frac{\cos^5 \theta}{5} + C$$

Question 3:

$$\int_0^{\pi/2} \sin^7 \theta \cos^5 \theta \, d\theta$$

Solution 3: Let $u = \cos \theta$, then $du = -\sin \theta \, d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \pi/2$, $u = 0$. Also, $\sin^2 \theta = 1 - \cos^2 \theta = 1 - u^2$. The integral becomes:

$$\begin{aligned}
\int_1^0 (1 - u^2)^3 u^5 (-du) &= \int_0^1 (1 - 3u^2 + 3u^4 - u^6) u^5 \, du \\
&= \int_0^1 (u^5 - 3u^7 + 3u^9 - u^{11}) \, du \\
&= \left[\frac{u^6}{6} - \frac{3u^8}{8} + \frac{3u^{10}}{10} - \frac{u^{12}}{12} \right]_0^1 \\
&= \frac{1}{6} - \frac{3}{8} + \frac{3}{10} - \frac{1}{12} = \frac{20 - 45 + 36 - 10}{120} = \frac{1}{120}
\end{aligned}$$

Question 4:

$$\int_0^{\pi/2} \sin^5 x \, dx$$

Solution 4: We use the reduction formula for $\int \sin^n x \, dx$:

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

For $n = 5$:

$$\begin{aligned}
\int_0^{\pi/2} \sin^5 x \, dx &= \left[-\frac{1}{5} \sin^4 x \cos x \right]_0^{\pi/2} + \frac{4}{5} \int_0^{\pi/2} \sin^3 x \, dx \\
&= 0 + \frac{4}{5} \left(\left[-\frac{1}{3} \sin^2 x \cos x \right]_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \sin x \, dx \right) \\
&= \frac{4}{5} \left(0 + \frac{2}{3} [-\cos x]_0^{\pi/2} \right) = \frac{4}{5} \left(\frac{2}{3} (0 - (-1)) \right) = \frac{8}{15}
\end{aligned}$$

Question 5:

$$\int \sin^2(\pi x) \cos^5(\pi x) \, dx$$

Solution 5: Let $u = \sin(\pi x)$, then $du = \pi \cos(\pi x) \, dx$. Also, $\cos^2(\pi x) = 1 - \sin^2(\pi x) = 1 - u^2$.

$$\int \sin^2(\pi x) \cos^5(\pi x) \, dx = \int u^2 (1 - u^2)^2 \cos(\pi x) \, dx = \frac{1}{\pi} \int u^2 (1 - 2u^2 + u^4) \, du$$

$$= \frac{1}{\pi} \int (u^2 - 2u^4 + u^6) \, du = \frac{1}{\pi} \left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right) + C$$

Substituting back $u = \sin(\pi x)$:

$$\frac{1}{\pi} \left(\frac{\sin^3(\pi x)}{3} - \frac{2 \sin^5(\pi x)}{5} + \frac{\sin^7(\pi x)}{7} \right) + C$$

Question 6:

$$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}} \, dx$$

Solution 6: Let $u = \sqrt{x}$, then $u^2 = x$ and $2u \, du = dx$.

$$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}} \, dx = \int \frac{\sin^3 u}{u} 2u \, du = 2 \int \sin^3 u \, du = 2 \int \sin^2 u \sin u \, du = 2 \int (1 - \cos^2 u) \sin u \, du$$

Let $v = \cos u$, then $dv = -\sin u \, du$.

$$2 \int -(1 - v^2) \, dv = -2 \left(v - \frac{v^3}{3} \right) + C = -2 \cos u + \frac{2}{3} \cos^3 u + C$$

Substituting back $u = \sqrt{x}$:

$$-2 \cos(\sqrt{x}) + \frac{2}{3} \cos^3(\sqrt{x}) + C$$

Question 7:

$$\int_0^{\pi/2} \cos^2 \theta \, d\theta$$

Solution 7: Use the identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$:

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} + 0 - 0\right) = \frac{\pi}{4}$$

Question 8:

$$\int_0^{2\pi} \sin^2\left(\frac{\theta}{3}\right)\, d\theta$$

Solution 8: Use the identity $\sin^2 u = \frac{1-\cos(2u)}{2}$ with $u = \frac{\theta}{3}$:

$$\begin{aligned}
\int_0^{2\pi} \sin^2\left(\frac{\theta}{3}\right)\, d\theta &= \int_0^{2\pi} \frac{1-\cos\left(\frac{2\theta}{3}\right)}{2}\, d\theta \\
&= \frac{1}{2}\left[\theta - \frac{3}{2}\sin\left(\frac{2\theta}{3}\right)\right]_0^{2\pi} \\
&= \frac{1}{2}\left(2\pi - \frac{3}{2}\sin\left(\frac{4\pi}{3}\right) - 0\right) \\
&= \frac{1}{2}\left(2\pi - \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right)\right) = \pi + \frac{3\sqrt{3}}{8}
\end{aligned}$$