Integral Exercises

Question 1:

$$\int \sin^2 x \cos^3 x \, dx$$

Solution 1: We use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite the integral:

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

Let $u = \sin x$, so $du = \cos x \, dx$. Then the integral becomes:

$$\int u^2 (1 - u^2) \, du = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

Substituting back $u = \sin x$, we get:

$$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Question 2:

$$\int \sin^3 \theta \cos^4 \theta \, d\theta$$

Solution 2: We rewrite the integral using the identity $\sin^2 \theta = 1 - \cos^2 \theta$:

$$\int \sin^3 \theta \cos^4 \theta \, d\theta = \int \sin^2 \theta \cos^4 \theta \sin \theta \, d\theta = \int (1 - \cos^2 \theta) \cos^4 \theta \sin \theta \, d\theta$$

Let $u = \cos \theta$, so $du = -\sin \theta \, d\theta$. Then the integral becomes:

$$\int (1 - u^2) u^4 (-du) = \int (u^6 - u^4) \, du = \frac{u^7}{7} - \frac{u^5}{5} + C$$

Substituting back $u = \cos \theta$, we have:

$$\frac{\cos^7 \theta}{7} - \frac{\cos^5 \theta}{5} + C$$

Question 3:

$$\int_0^{\pi/2} \sin^7 \theta \cos^5 \theta \, d\theta$$

Solution 3: Let $u = \cos \theta$, so $du = -\sin \theta \, d\theta$. When $\theta = 0$, $u = 1$, and when $\theta = \pi/2$, $u = 0$. Then we have:

$$\begin{aligned}
\int_0^{\pi/2} \sin^7\theta\cos^5\theta\,d\theta &= \int_0^{\pi/2} \sin^6\theta\cos^5\theta\sin\theta\,d\theta \\
&= \int_1^0 (1-\cos^2\theta)^3\cos^5\theta(-\sin\theta\,d\theta) \\
&= \int_0^1 (1-u^2)^3u^5\,du \\
&= \int_0^1 (1-3u^2+3u^4-u^6)u^5\,du \\
&= \int_0^1 (u^5-3u^7+3u^9-u^{11})\,du \\
&= \left[\frac{u^6}{6}-\frac{3u^8}{8}+\frac{3u^{10}}{10}-\frac{u^{12}}{12}\right]_0^1 \\
&= \frac{1}{6}-\frac{3}{8}+\frac{3}{10}-\frac{1}{12} = \frac{20-45+36-10}{120} = \frac{1}{120}
\end{aligned}$$

Question 4:

$$\int_0^{\pi/2} \sin^5 x\,dx$$

Solution 4: We use reduction formula for $\int \sin^n x\,dx$:

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

For $n=5$:

$$\begin{aligned}
\int_0^{\pi/2} \sin^5 x\,dx &= \left[-\frac{1}{5}\sin^4 x\cos x\right]_0^{\pi/2} + \frac{4}{5}\int_0^{\pi/2}\sin^3 x\,dx \\
&= 0 + \frac{4}{5}\left(\left[-\frac{1}{3}\sin^2 x\cos x\right]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2}\sin x\,dx\right) \\
&= \frac{4}{5}\left(0+\frac{2}{3}[-\cos x]_0^{\pi/2}\right) \\
&= \frac{4}{5}\left(\frac{2}{3}(0-(-1))\right) = \frac{8}{15}
\end{aligned}$$

Question 5:

$$\int \sin^2(\pi x)\cos^5(\pi x)\,dx$$

Solution 5: Let $u = \sin(\pi x)$, so $du = \pi \cos(\pi x)\, dx$. Then $\cos^2(\pi x) = 1 - \sin^2(\pi x) = 1 - u^2$.

$$\begin{aligned}
\int \sin^2(\pi x)\cos^5(\pi x)\, dx &= \int \sin^2(\pi x)\cos^4(\pi x)\cos(\pi x)\, dx \\
&= \int u^2(1-u^2)^2 \frac{1}{\pi} du \\
&= \frac{1}{\pi}\int u^2(1-2u^2+u^4)\, du \\
&= \frac{1}{\pi}\int (u^2-2u^4+u^6)\, du \\
&= \frac{1}{\pi}\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C \\
&= \frac{1}{\pi}\left(\frac{\sin^3(\pi x)}{3} - \frac{2\sin^5(\pi x)}{5} + \frac{\sin^7(\pi x)}{7}\right) + C
\end{aligned}$$

Question 6:

$$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\, dx$$

Solution 6: Let $u = \sqrt{x}$, so $u^2 = x$ and $2u\, du = dx$. Then

$$\begin{aligned}
\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\, dx &= \int \frac{\sin^3 u}{u} 2u\, du \\
&= 2\int \sin^3 u\, du \\
&= 2\int \sin^2 u \sin u\, du \\
&= 2\int (1-\cos^2 u)\sin u\, du
\end{aligned}$$

Let $v = \cos u$, so $dv = -\sin u\, du$. Then

$$2\int (1-v^2)(-dv) = 2\int (v^2-1)\, dv = 2\left(\frac{v^3}{3} - v\right) + C = 2\left(\frac{\cos^3 u}{3} - \cos u\right) + C$$

Substituting back $u = \sqrt{x}$:

$$2\left(\frac{\cos^3(\sqrt{x})}{3} - \cos(\sqrt{x})\right) + C$$

Question 7:

$$\int_0^{\pi/2} \cos^2\theta\, d\theta$$

Solution 7: Using the reduction formula for $\int \cos^n x\, dx$ or the identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$:

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}+0\right) = \frac{\pi}{4}$$

Question 8:

$$\int_0^{2\pi} \sin^2\left(\frac{\theta}{3}\right)\, d\theta$$

Solution 8: Using the identity $\sin^2 x = \frac{1-\cos(2x)}{2}$:

$$\begin{aligned}
\int_0^{2\pi} \sin^2\left(\frac{\theta}{3}\right) d\theta &= \int_0^{2\pi} \frac{1-\cos\left(\frac{2\theta}{3}\right)}{2} d\theta \\
&= \frac{1}{2}\left[\theta - \frac{3}{2}\sin\left(\frac{2\theta}{3}\right)\right]_0^{2\pi} \\
&= \frac{1}{2}\left(2\pi - \frac{3}{2}\sin\left(\frac{4\pi}{3}\right) - 0 + 0\right) \\
&= \frac{1}{2}\left(2\pi - \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right)\right) = \pi + \frac{3\sqrt{3}}{8}
\end{aligned}$$