

Integral Problems

Question 1:

$$\int \sin^2 x \cos^3 x \, dx$$

Solution 1: We use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite the integral:

$$\begin{aligned} \int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x \cos^2 x \cos x \, dx \\ &= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx \end{aligned}$$

Let $u = \sin x$, so $du = \cos x \, dx$. Then the integral becomes:

$$\int u^2(1 - u^2) \, du = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

Substituting back $u = \sin x$, we get:

$$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Question 2:

$$\int \sin^3 \theta \cos^4 \theta \, d\theta$$

Solution 2: We use the identity $\sin^2 \theta = 1 - \cos^2 \theta$ to rewrite the integral:

$$\begin{aligned} \int \sin^3 \theta \cos^4 \theta \, d\theta &= \int \sin^2 \theta \sin \theta \cos^4 \theta \, d\theta \\ &= \int (1 - \cos^2 \theta) \cos^4 \theta \sin \theta \, d\theta \end{aligned}$$

Let $u = \cos \theta$, so $du = -\sin \theta \, d\theta$. Then the integral becomes:

$$\begin{aligned} \int (1 - u^2)u^4(-du) &= \int (u^6 - u^4) \, du \\ &= \frac{u^7}{7} - \frac{u^5}{5} + C \end{aligned}$$

Substituting back $u = \cos \theta$, we have:

$$\frac{\cos^7 \theta}{7} - \frac{\cos^5 \theta}{5} + C$$

Question 3:

$$\int_0^{\pi/2} \sin^7 \theta \cos^5 \theta \, d\theta$$

Solution 3: Let $u = \cos \theta$, so $du = -\sin \theta d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \pi/2$, $u = 0$. Then

$$\begin{aligned}
 \int_0^{\pi/2} \sin^7 \theta \cos^5 \theta d\theta &= \int_0^{\pi/2} \sin^6 \theta \cos^5 \theta \sin \theta d\theta \\
 &= \int_1^0 (1 - \cos^2 \theta)^3 \cos^5 \theta (-\sin \theta d\theta) \\
 &= \int_0^1 (1 - u^2)^3 u^5 du \\
 &= \int_0^1 (1 - 3u^2 + 3u^4 - u^6) u^5 du \\
 &= \int_0^1 (u^5 - 3u^7 + 3u^9 - u^{11}) du \\
 &= \left[\frac{u^6}{6} - \frac{3u^8}{8} + \frac{3u^{10}}{10} - \frac{u^{12}}{12} \right]_0^1 \\
 &= \frac{1}{6} - \frac{3}{8} + \frac{3}{10} - \frac{1}{12} = \frac{20 - 45 + 36 - 10}{120} = \frac{1}{120}
 \end{aligned}$$

Question 4:

$$\int_0^{\pi/2} \sin^5 x dx$$

Solution 4: We use the reduction formula for $\int \sin^n x dx$:

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

For $n = 5$,

$$\begin{aligned}
 \int_0^{\pi/2} \sin^5 x dx &= \left[-\frac{1}{5} \sin^4 x \cos x \right]_0^{\pi/2} + \frac{4}{5} \int_0^{\pi/2} \sin^3 x dx \\
 &= 0 + \frac{4}{5} \int_0^{\pi/2} \sin^3 x dx \\
 &= \frac{4}{5} \left[\left(-\frac{1}{3} \sin^2 x \cos x \right)_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \sin x dx \right] \\
 &= \frac{4}{5} \left[0 + \frac{2}{3} (-\cos x)_0^{\pi/2} \right] \\
 &= \frac{4}{5} \left(\frac{2}{3} (0 - (-1)) \right) = \frac{8}{15}
 \end{aligned}$$

Question 5:

$$\int \sin^2(\pi x) \cos^5(\pi x) dx$$

Solution 5: Let $u = \sin(\pi x)$, so $du = \pi \cos(\pi x) dx$. Then $\cos^2(\pi x) = 1 - \sin^2(\pi x) = 1 - u^2$.

$$\begin{aligned} \int \sin^2(\pi x) \cos^5(\pi x) dx &= \int \sin^2(\pi x) \cos^4(\pi x) \cos(\pi x) dx \\ &= \int u^2(1-u^2)^2 \frac{1}{\pi} du \\ &= \frac{1}{\pi} \int u^2(1-2u^2+u^4) du \\ &= \frac{1}{\pi} \int (u^2 - 2u^4 + u^6) du \\ &= \frac{1}{\pi} \left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right) + C \\ &= \frac{1}{\pi} \left(\frac{\sin^3(\pi x)}{3} - \frac{2\sin^5(\pi x)}{5} + \frac{\sin^7(\pi x)}{7} \right) + C \end{aligned}$$

Question 6:

$$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}} dx$$

Solution 6: Let $u = \sqrt{x}$, so $u^2 = x$ and $2u du = dx$. Then

$$\begin{aligned} \int \frac{\sin^3(\sqrt{x})}{\sqrt{x}} dx &= \int \frac{\sin^3 u}{u} 2u du \\ &= 2 \int \sin^3 u du \\ &= 2 \int \sin^2 u \sin u du \\ &= 2 \int (1 - \cos^2 u) \sin u du \end{aligned}$$

Let $v = \cos u$, so $dv = -\sin u du$. Then

$$\begin{aligned} 2 \int (1 - v^2)(-dv) &= -2 \int (1 - v^2) dv \\ &= -2\left(v - \frac{v^3}{3}\right) + C \\ &= -2 \cos u + \frac{2}{3} \cos^3 u + C \\ &= -2 \cos(\sqrt{x}) + \frac{2}{3} \cos^3(\sqrt{x}) + C \end{aligned}$$

Question 7:

$$\int_0^{\pi/2} \cos^2 \theta d\theta$$

Solution 7: We use the identity $\cos^2 \theta = \frac{1+\cos(2\theta)}{2}$:

$$\begin{aligned} \int_0^{\pi/2} \cos^2 \theta d\theta &= \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta \\ &= \frac{1}{2} \left[\theta + \frac{\sin(2\theta)}{2} \right]_0^{\pi/2} \\ &= \frac{1}{2} \left[\left(\frac{\pi}{2} + 0 \right) - (0 + 0) \right] = \frac{\pi}{4} \end{aligned}$$

Question 8:

$$\int_0^{2\pi} \sin^2\left(\frac{\theta}{3}\right) d\theta$$

Solution 8: We use the identity $\sin^2 x = \frac{1 - \cos(2x)}{2}$:

$$\begin{aligned} \int_0^{2\pi} \sin^2\left(\frac{\theta}{3}\right) d\theta &= \int_0^{2\pi} \frac{1 - \cos\left(\frac{2\theta}{3}\right)}{2} d\theta \\ &= \frac{1}{2} \left[\theta - \frac{3}{2} \sin\left(\frac{2\theta}{3}\right) \right]_0^{2\pi} \\ &= \frac{1}{2} \left[\left(2\pi - \frac{3}{2} \sin\left(\frac{4\pi}{3}\right) \right) - \left(0 - \frac{3}{2} \sin 0 \right) \right] \\ &= \frac{1}{2} \left[2\pi - \frac{3}{2} \left(-\frac{\sqrt{3}}{2} \right) \right] = \pi + \frac{3\sqrt{3}}{8} \end{aligned}$$