ADAPTIVE REFINEMENT WITH THE MINIMIZATION SOLVER, NUMERICAL RESULTS, REPORT

Abstract. Adaptive mesh refinement is studied for constrained first-order system least-squares (CFOSLS) formulations for three-dimensional space-time problems. The central idea of the considered framework is to minimize the least-squares functional under the divergence constraint which makes the method conservative. Following this idea, we use local functional values as a natural measure of the local errors.

Two test problems are considered, one is the Laplace equation in an L-shaped domain and the second is the transport equation in $H(\mathrm{div}) - L^2$ formulation.

Efficiency of the proposed refinement strategy is compared numerically to the uniform refinement. In addition to the traditional refinement strategy based on local element errors, a modified strategy which noticeably smoothes out the error distribution is implemented by exploiting supplementary face error indicators.

For solving the arising linear systems, an efficient multilevel algorithm is proposed which minimizes the eneregy functional over a hierarchy of mesh levels. The multilevel algorithm outperforms the simpler block-diagonal preconditioners in terms of iteration count significantly.

The results obtained show that ...

1. Introduction

2. Problem statement

Description of the setup in general terms, applicable to both Laplace an transport.

3. Refinement strategy

Description of the element error indicators, and a modification with the face error indicators.

Definition of threshold in MFEM:

$$\text{threshold} = \max\left\{\text{total error} \cdot \text{total error fraction} \cdot n_{el}^{-\frac{1}{p}}, \text{local error goal}\right\}.$$

Total error is defined by

$$\text{total error} = \begin{cases} \left(\sum_i (\text{local error i})^p\right)^{1/p}, & p < \infty \\ \max_i \text{local error i}, & p = \infty \end{cases}$$

Default parameter values are:

- total error fraction $= 0.5$;
- $p = \infty$;
- local error goal $= 0.0$.

Key words and phrases. CFOSLS, space-time, adaptive mesh refinement, multilevel algorithms, multigrid.

4. Linear solvers

4.1. Block-diagonal preconditioners.

4.2. Multilevel minimization solver.

5. Results

5.1. Refinement stats?

5.2. Comparison to the uniform refinement.
Laplace

Transport

Pictures

5.3. Solver comparison.
Block-diagonal preconditioner

Multilevel solver

For example, for the first two tables, total error fraction for the adaptive mesh refinement was 0.95. This means, that with default values for the rest of parameters, a refinement criteria was

$$\text{local error i} \geq 0.95 \max_j \text{local error j}$$

For the L-shaped domain, this strategy leads to the following refinement statistics:

Table 1. AMR stats, L-shaped domain in $\mathbb{R}^3$, $H(\text{div})-H^1$ formulation for the Laplace equation

#dofs	$n_{\text{el, marked}}$	marked el,%	$n_{\text{el, new}}$	new el,%
63677	6	0.03	362	1.86
64861	1	0.005	85	0.43
65139	6	0.03	694	3.49
67381	2	0.01	88	0.43
67678	2	0.01	114	0.55
68053	2	0.01	174	0.84
68626	-	-	-	-

Table 2. AMR stats, L-shaped domain in $\mathbb{R}^3$, $H(\text{div})-H^1$ formulation for the Laplace equation

#dofs	#iter1	#iter2	$\varepsilon_{\boldsymbol{\sigma}}$	ε_u	funct	funct for $\pi exsol$
63677	1	1	0.073	0.008	0.00029	0.00034
64861	1+5	2	0.068	0.0069	0.00025	0.00028
65139	1+5+4	2	0.067	0.0068	0.00024	0.00027
67381	1+5+4+7	3	0.062	0.0056	0.0002	0.00023
67678	1+5+4+7+3	2	0.061	0.0054	0.00019	0.00022
68053	1+5+4+7+3+5	2	0.060	0.0053	0.00019	0.00022

Notations: #iter1 is for the setup with re-solving from coarsest to finest level each time, #iter2 is for simple minimization solver at the finest level. The numbers are the numbers of V-cycles so that the first number (1) in column 2 (#iter1) corresponds to the single iteration (solution of the global system for a correction) at the coarsest level.

Next, the same test but with total error fraction equal to 0.8.

TABLE 3. AMR stats, L-shaped domain in $\mathbb{R}^3$, $H(\mathrm{div})-H^1$ formulation for the Laplace equation

#dofs	$n_{\text{el, marked}}$	marked el,%	$n_{\text{el, new}}$	new el,%
63677	9	0.05	623	3.2
65709	15	0.07	859	4.2
68501	50	0.24	3290	15.7
79092	42	0.17	2272	9.4
86468	10	0.04	949	3.6
89541	111	0.4	6584	23.9
110743	-	-	-	-

TABLE 4. AMR stats, L-shaped domain in $\mathbb{R}^3$, $H(\mathrm{div})-H^1$ formulation for the Laplace equation

#dofs	#iter	$\varepsilon_{\boldsymbol{\sigma}}$	ε_u	funct	funct for $\pi exsol$
63677	1	0.073	0.008	0.00029	0.00034
65709	1+6	0.066	0.0065	0.00023	0.00026
68501	1+6+7	0.059	0.0052	0.00019	0.00022
79092	1+6+7+5	0.052	0.0036	0.00013	0.00015
86468	1+6+7+5+8	0.048	0.0031	0.00010	0.00013
89541	1+6+7+5+8+4	0.047	0.0031	0.00009	0.00012

TABLE 5. Uniform refinement, L-shaped domain in $\mathbb{R}^3$, $H(\mathrm{div}) - H^1$ formulation for the Laplace equation

#dofs	#iter	$\varepsilon_{\boldsymbol{\sigma}}$	ε_u
63677	9	0.073	0.008
501049	11	0.047	0.0032

Here a MG preconditioner was used, #levels = 2 and 3.

6. Possible things for the paper

- Minimization solver
- AMR scheme for the minimization solver, three approaches
- Refinement strategies including beta

- Laplace equation in the L-shaped domain and transport in the cube (pictures, comparison with uniform refinement and comparison between different approaches)

What would be the main concept (idea) of the paper?

We can say "we suggest a minimization solver for AMR in the considered CFOSLS setting".

I'd like to say that we can efficiently reuse the previous iterations in terms of the iteration count, but the results don't show it.

7. To-do list

- Write the draft for theoretical sections
- Numerical results: tables
- Numerical resuls: pictures
- Introduction

8. Questions:

- How to include results by Paulina?

References

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Portland State University