Project 4 due by March 16, beginning of class. Analytical work will not be accepted after this time. You must make an appointment to visit my office on or before March 16, 2:30, at a mutually agreeable time, in order to demonstrate the programming part of this project for me. I can make appointments during spring break.

Project 4 involves numerical methods for polynomial interpolation and approximation.

1. find the linear interpolant, $P_1(x)$, of $f(x) = (1+x)^{1/3}$ through the points $x_0 = 0.0$ and $x_1 = 1.0$. Use your interpolant in order to predict $(4/3)^{1/3}$. Find a bound on the error for $0 \leq x \leq 1$ using the formula

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\Pi_{i=0}^{n}(x - x_i). \tag{1}$$

Compare the actual error, $|(4/3)^{1/3} - P_1(1/3)|$, to the error bound.

2. Repeat problem 1 except using quadratic interpolation through the points $x_0 = 0.0$, $x_1 = 1.0$, and $x_2 = 2.0$.

3. Use Newton's divided difference tables in order to construct the cubic interpolant of $f(x) = \cos(x)$ through the points $x_0 = 0$, $x_1 = \pi/8$, $x_2 = \pi/4$, and $x_3 = 3\pi/8$. Use 4 digit rounding arithmetic. From your divided difference table, extract your interpolant in 3 different ways (i.e. find the interpolant using the original ordering of the points, plus using two different permutations of the original ordering). Predict $\cos(\pi/3)$ using your interpolant.

4. a. Show that the cubic polynomials

$$P(x) = 3 - 2(x+1) + 0(x+1)(x) + (x+1)(x)(x-1) \tag{2}$$

and

$$Q(x) = -1 + 4(x+2) - 3(x+2)(x+1) + (x+2)(x+1)(x) \tag{3}$$

both interpolate the data,

$$(x_i, f(x_i)) = (-2,-1), (-1,3), (0,1), (1,-1), (2,3). \tag{4}$$

b. Why does part (a) not violate the uniqueness property of interpolating polynomials?

5. For a function f, the Newton divided difference formula gives the interpolating polynomial,

$$P_3(x) = 1 + 4x + 4x(x-1/4) + \frac{16}{3}x(x-1/4)(x-1/2), \tag{5}$$

on the nodes $x_i = 0, 1/4, 1/2, 3/4$. Find $f(3/4)$.

6. For a function f, the divided difference table is given by:

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	$f[x_k, x_{k+1}, x_{k+2}]$
$x_0 = 0.0$	$f[x_0]$		
$x_1 = 0.4$	$f[x_1]$	$f[x_0, x_1]$	
$x_2 = 0.7$	$f[x_2] = 6$	$f[x_1, x_2] = 10$	$f[x_0, x_1, x_2] = 50/7$

Determine the missing entries in the table.

7. This problem pertains to the sample program `lagrange_polyinterp_sample.cpp` posted online.

a. Add the function,

$$f_{transition}(x) = \tanh(\frac{x}{\epsilon}) \tag{6}$$

to your program and plot the polynomial interpolant, $P_n(x)$, together with the original function for a few different values of n and ϵ. e.g. $n = 5, 10$ and $\epsilon = 1, 0.1, 0.01$. Note: $f_{transition}(x)$ appears often when studying thermal or viscous boundary layers. Note: since ``lagrange_interp'' takes a function parameter, just declare a new function e.g. ``fx_tanh'' and pass it instead of ``fx_smooth''.

b. Write your own polynomial interpolation routine, with the same parameter structure as ``lagrange_interp'' except using the Newton's divided difference table algorithm for finding "y". Test this for different functions, and different n.

c. Write your own "piecewise polynomial" interpolation routine by copying and pasting the routine ``lagrange_interp'' to a new routine ``lagrange_piecewise_interp'' in which the Lagrange interpolating polynomial,

$$L_{n,i}(x) \equiv \Pi_{j=0,j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

is replaced with,

$$L_{n,i}(x) \equiv \begin{cases} 1 & \text{if } x = x_i \\ \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x \text{ is inbetween } x_{i-1} \text{ and } x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x \text{ is inbetween } x_i \text{ and } x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Test your new piecewise polynomial routine for different functions, and different n.