MAD3703, Numerical Analysis I, MWF, 2:30-3:20, room Love 102
Lecture February 18:

For a given root problem, $f(x) = 0$, there can be many equivalent fixed point problems, $g_1(x) = x$, $g_2(x) = x$, ... The *rate of convergence* for a given $g_i(x)$ varies and is determined by $g'(p)$ where p is the fixed point of g.

Example: suppose we start with the problem $f(x) = 0$ where $f(x) = x^4 - 3$, then the following three fixed point problems have the same fixed point as the root for $f(x)$:

$$\begin{aligned} x &= g_1(x) \quad g_1(x) = x - \frac{x^4-3}{4x^3} \\ x &= g_2(x) \quad g_2(x) = x - \frac{x^4-3}{8x^3} \\ x &= g_3(x) \quad g_3(x) = x - \frac{x^4-3}{16x^3} \end{aligned}$$

analysis of the fixed point problem $x = g_1(x)$:

g_1 maps the interval $[5/4, 2]$ onto itself and is a contraction mapping on the same interval:

$$\begin{aligned} g_1(5/4) &= 1.3215 \in [5/4, 2] \\ g_1(2) &= 1.59375 \in [5/4, 2] \\ g_1'(x_{critical}) = 0 \quad \text{implies} \quad x_{critical} &= 1.31607 \\ g_1(x_{critical}) &= 1.31607 \in [5/4, 2] \\ |g_1'(5/4)| &= 0.1716 \\ |g_1'(2)| &= 0.609375 \\ g_1''(x) = 0 \quad &\text{has no solutions} \\ |g_1'(x)| &\le 0.609375 < 1 \end{aligned}$$

Verification of the relation,

$$\frac{|p_n - p|}{|p_{n-1} - p|} = |g_1'(\xi)| \quad \xi \text{ is inbetween } p_{n-1} \text{ and } p.$$

n	p_n	$\|p_n - p\|$	$\frac{\|p_n-p\|}{\|p_{n-1}-p\|}$
0	1.25	0.066074	
1	1.3215	0.00542599	0.0821198
2	1.31611	$3.33267e-05$	0.00614205
3	1.31607	$1.26584e-09$	$3.79826e-05$

analysis of the fixed point problem $x = g_2(x)$:

g_2 maps the interval $[5/4, 2]$ onto itself and is a contraction mapping on the same interval:

$$\begin{aligned} g_2(5/4) &= 1.28575 \in [5/4, 2] \\ g_2(2) &= 1.79688 \in [5/4, 2] \\ g_2'(x_{critical}) = 0 \quad \text{implies} \quad x_{critical} &= 1.06484 \\ |g_2'(5/4)| &= 0.4142 \\ |g_2'(2)| &= 0.804688 \\ g_2''(x) = 0 \quad &\text{has no solutions} \\ |g_2'(x)| &\leq 0.804688 < 1 \end{aligned}$$

Verification of the relation,

$$\frac{|p_n - p|}{|p_{n-1} - p|} = |g_1'(\xi)| \quad \xi \text{ is inbetween } p_{n-1} \text{ and } p.$$

n	p_n	$\lvert p_n - p\rvert$	$\frac{\lvert p_n - p\rvert}{\lvert p_{n-1} - p\rvert}$
0	1.25	0.066074	
1	1.28575	0.030324	0.45894
2	1.30146	0.0146171	0.482032
3	1.30889	0.00718452	0.491513

analysis of the fixed point problem $x = g_3(x)$:

g_3 maps the interval $[5/4, 2]$ onto itself and is a contraction mapping on the same interval:

$$\begin{aligned} g_3(5/4) &= 1.26788 \in [5/4, 2] \\ g_3(2) &= 1.89844 \in [5/4, 2] \\ g_3'(x_{critical}) = 0 \quad \text{implies} \quad x_{critical} &= 0.880112 \\ |g_3'(5/4)| &= 0.7071 \\ |g_3'(2)| &= 0.902344 \\ g_3''(x) = 0 \quad &\text{has no solutions} \\ |g_3'(x)| &\leq 0.902344 < 1 \end{aligned}$$

Verification of the relation,

$$\frac{|p_n - p|}{|p_{n-1} - p|} = |g_1'(\xi)| \quad \xi \text{ is inbetween } p_{n-1} \text{ and } p.$$

n	p_n	$\lvert p_n - p\rvert$	$\frac{\lvert p_n - p\rvert}{\lvert p_{n-1} - p\rvert}$
0	1.25	0.066074	
1	1.26788	0.048199	0.72947
2	1.28063	0.0354446	0.735379
3	1.28987	0.0262087	0.739428