Project 6 due April 27 (or before). Computer lab parts must be demonstrated for me (make an appointment). Solutions to analytical portions of projects 5 and 6 will be posted on April 27 at 5:00pm in preparation for the final exam (May 1, 12:30).

1. Project 6 involves numerical methods for approximating $\int_a^b f(x)dx$ or approximating $\int_0^\infty e^{-x} f(x)dx$.

2. Approximate the following integral using the Trapezoid rule and then the Simpson's rule:

$$\int_{-1/4}^{1/4} (\cos(x))^2 dx \tag{1}$$

For trapezoid rule and Simpson rule, find a bound on the error using the error formula and compare this to the actual error.

3. For the previous problem but analyzing the *Composite* Simpson's and *Composite* Trapezoid rule instead, determine the values of n and h in order to approximate (1) to within 10^{-8}.

4. Show that the formula $Q(P) = \sum_{i=1}^n c_i P(x_i)$ cannot have degree of precision greater than $2n-1$, regardless of the choice of $c_1, \ldots, c_n$ and $x_1, \ldots, x_n$. (hint: construct a polynomial that has a double root at each of the x_i's.

5. The Laguerre polynomials $L_0(x)$, $L_1(x)$, ...form an orthogonal set on $[0, \infty)$ with respect to the weight function e^{-x}. They satisfy,

$$\int_0^\infty e^{-x} L_i(x) L_j(x) dx = 0 \qquad i \neq j.$$

The polynomial $L_n(x)$ has n distinct zeros x_1, x_2, ..., x_n in $[0, \infty)$. Let

$$c_{n,i} = \int_0^\infty e^{-x} \Pi_{j=1, j\neq i}^n \frac{x - x_j}{x_i - x_j} dx.$$

Show that the quadrature formula,

$$\int_0^\infty f(x) e^{-x} dx = \sum_{i=1}^n c_{n,i} f(x_i)$$

has degree of precision $2n-1$. (hint: follow the same steps as in the proof of showing that regular Gaussian quadrature (where x_1, x_2, ..., x_n are the roots of the nth Legendre polynomial $P_n(x)$) has degree of precision $2n-1$).

6. The Laguerre polynomials are $L_0(x) = 1$, $L_1(x) = 1 - x$, and then

$$L_{k+1}(x) = \frac{(2k+1-x)L_k(x) - kL_{k-1}(x)}{k+1} \qquad k = 1, 2, 3, \ldots$$

As shown in the previous problem, these polynomials are useful in approximating integrals of the form $\int_0^\infty f(x)e^{-x}dx$:

$$\int_0^\infty f(x)e^{-x}dx \approx \sum_{i=1}^{n} c_{n,i} f(x_i) \tag{2}$$

(a) Derive the quadrature formula using $n = 2$ and the zeros of $L_2(x)$.

(b) Derive the quadrature formula using $n = 3$ and the zeros of $L_3(x)$.

7. Write a computer program that implements the composite Simpson's rule and the method given by (2) with $n = 3$. Use your program to approximate,

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx. \tag{3}$$

Compare the two methods (investigate accuracy and cost). Some important notes:

(a) $\Gamma(t+1) = t!$ when t is an integer; this way, you can check the error of the respective approximations.

(b) sample code for calculating the integral $\int_a^b f(x)dx$ using the composite midpoint rule, `midpoint.cpp`, is posted online.

(c) In order to approximate (3) using the composite Simpson's rule, you will have to replace (3) with,

$$\Gamma(t) = \int_0^N x^{t-1}e^{-x}dx.$$

where N is sufficiently large.

(d) You might want to do a sanity check on your codes in order to verify that your respective methods (the composite simpson's rule, and (2) with $n = 3$) satisfy the expected degree of precision requirement.