Lecture January 30 and February 2:

Comparison between $N(h)$ and $N_2(h)$

$$N(h) = \frac{e^{x+h} - e^x}{h} \tag{1}$$

$$N_2(h) = \frac{e^{x+h} - e^{x-h}}{2h} \tag{2}$$

Analysis of $N(h)$

$$N(h) = \frac{e^{x+h} - e^x}{h} \tag{3}$$

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \tag{4}$$

$$\lim_{h\to 0} \frac{e^{x+h}}{1} = e^x \tag{5}$$

Using 3 digit chopping arithmetic, $x = 1.00$, and $h = 1.00E-2$,

$$N(h) = \frac{2.71^{1.01} - 2.71}{0.01} = \frac{2.73 - 2.71}{0.01} = 2.00 \tag{6}$$

The relative error is,

$$\frac{|p - p^*|}{|p|} = |(2.00 - 2.73)/2.73| = 0.27 \tag{7}$$

If we approximate the numerator of (1) with its 2nd Taylor series expansion about $h_0 = 0$, and then simplify (1) then:

$$M(h) \equiv e^{x+h} - e^x \tag{8}$$

$$M'(h) = e^{x+h} \quad M''(h) = e^{x+h} \tag{9}$$

$$P_2(h) = M(0) + M'(0)h + M''(0)h^2/2 = e^x h + e^x h^2/2 \tag{10}$$

$$N(h) \approx \frac{e^x h + e^x h^2/2}{h} = e^x + e^x h/2 \tag{11}$$

Using 3 digit chopping arithmetic, $x = 1.00$, and $h = 1.00E-2$, applied to the approximation, we have:

$$N(h) \approx 2.71^{1.00} + (0.0100/2.00)(2.71) = 2.71 + 0.0135 = 2.72 \tag{12}$$

The relative error in this case is,

$$\frac{|p - p^*|}{|p|} = |(2.72 - 2.73)/2.73| = 0.004 \tag{13}$$

If loss of precision cannot be removed, then there is an optimal h when using

$$N(h) = \frac{f(x+h) - f(x)}{h} \tag{14}$$

h	$(e^{1+h}-e)/h$	$\lvert(e^{1+h}-e)/h-e\rvert$
10^{-1}	2.858841955	0.1406
10^{-5}	2.7182954	$1.4E-5$
10^{-6}	2.718282	$1.7E-7$
10^{-7}	2.71828	$1.8E-6$
10^{-8}	2.7182	$8.2E-5$

Table 1: The optimal h when using $N(h)$ is about $h = 10^{-6}$

in order to approximate $f'(x)$. For example, if $f(x) = e^x$, $x = 1$, and I use my TI-30X IIS calculator, the optimal h is about $1/10^6$ (see table 1).

Analysis of $N_2(h)$

$$N_2(h) = \frac{e^{x+h} - e^{x-h}}{2h} \tag{15}$$

$$\lim_{h\to 0} \frac{e^{x+h} - e^{x-h}}{2h} = \tag{16}$$

$$\lim_{h\to 0} \frac{e^{x+h} + e^{x-h}}{2} = e^x \tag{17}$$

For $N_2(h)$, Using 3 digit chopping arithmetic, $x = 1.00$, and $h = 1.00E{-2}$,

$$N_2(h) = \frac{2.71^{1.01} - 2.71^{0.99}}{0.02} = \frac{2.73 - 2.68}{0.02} = 2.50 \tag{18}$$

The relative error is,

$$\frac{|p - p^*|}{|p|} = |(2.50 - 2.72)/2.72| = 0.08 \tag{19}$$

If we approximate the numerator of (2) with its 3rd Taylor series expansion about $h_0 = 0$, and then simplify (2) then:

$$M(h) \equiv e^{x+h} - e^{x-h} \tag{20}$$

$$M'(h) = e^{x+h} + e^{x-h} \tag{21}$$

$$M''(h) = e^{x+h} - e^{x-h} \tag{22}$$

$$M'''(h) = e^{x+h} + e^{x-h} \tag{23}$$

$$P_3(h) = M(0) + M'(0)h + M''(0)h^2/2 + M'''(0)h^3/6 = 2e^x h + 2e^x h^3/6 \tag{24}$$

$$N_2(h) \approx \frac{2e^x h + 2e^x h^3/6}{2h} = e^x + e^x h^2/6 \tag{25}$$

Using 3 digit chopping arithmetic, $x = 1.00$, and $h = 1.00E - 2$, applied to the approximation, we have:

$$N_2(h) \approx 2.71^{1.00} + (0.000100/6.00)(2.71) = 2.71 + 0.000045167 = 2.71 \tag{26}$$

The relative error in this case is,

$$\frac{|p-p^*|}{|p|} = |(2.71-2.72)/2.72| = 0.004 \tag{27}$$

If loss of precision cannot be removed, then there is an optimal h when using

$$N_2(h) = \frac{f(x+h)-f(x-h)}{2h}, \tag{28}$$

in order to approximate $f'(x)$. For example, if $f(x) = e^x$, $x = 1$, and I use my TI-30X IIS calculator, the optimal h is about 10^{-4} (see table 2).

h	$(e^{1+h}-e^{1-h})/(2h)$	$\lvert(e^{1+h}-e^{1-h})/(2h)-e\rvert$
10^{-1}	2.722814564	0.005
10^{-2}	2.718327133	$5.0E-5$
10^{-3}	2.718282282	$5.0E-7$
10^{-4}	2.71828183	$2.0E-9$
10^{-5}	2.71828185	$2.0E-8$

Table 2: The optimal h when using $N_2(h)$ is about $h = 10^{-4}$

What if numerator cannot be expanded in a Taylor series? ($N(h)$)

$$N(h) = \frac{f(x+h)-f(x)}{h} \tag{29}$$

$$fl(N(h)) = fl(\frac{fl(f(fl(x+h))) - fl(f(x))}{h}) = \tag{30}$$

$$\frac{f(x+h)+\epsilon_1 - f(x) - \epsilon_2}{h} \tag{31}$$

Estimate the best possible h by expanding the numerator of (29) in a Taylor series (but we do not know a priori $f'(x)$ or $f''(x)$):

$$M(h) = f(x+h) - f(x) \tag{32}$$

$$M'(h) = f'(x+h) \tag{33}$$

$$M''(h) = f''(x+h) \tag{34}$$

$$P_2(h) = 0 + f'(x)h + f''(x)h^2/2 \tag{35}$$

$$\frac{f(x+h)-f(x)}{h} \approx \frac{P_2(h)}{h} = f'(x) + f''(x)\frac{h}{2} \tag{36}$$

$$fl(\frac{f(x+h)-f(x)}{h}) = \frac{f(x+h)+\epsilon_1 - f(x) - \epsilon_2}{h} = \tag{37}$$

$$\frac{f(x+h)-f(x)}{h} + \frac{\epsilon_1 - \epsilon_2}{h} \approx \tag{38}$$

$$f'(x) + f''(x)\frac{h}{2} + \frac{\epsilon_1-\epsilon_2}{h} \tag{39}$$

$$|N(h) - f'(x)| < M\frac{h}{2} + \frac{2\epsilon}{h} \quad M = \max_{x \leq \xi \leq x+h} |f''(\xi)| \tag{40}$$

In order to find the optimal h for $N(h)$, we define,

$$g(h) = M\frac{h}{2} + \frac{2\epsilon}{h} \tag{41}$$

$\min_h |g(h)|$ is found by checking critical points:

$$g'(h) = \frac{M}{2} - \frac{2\epsilon}{h^2} = 0 \tag{42}$$

$$\frac{M}{2} = \frac{2\epsilon}{h^2} \tag{43}$$

$$h^2 = \frac{4\epsilon}{M} \tag{44}$$

$$h_c = \sqrt{\frac{4\epsilon}{M}} \tag{45}$$

If we plug h_c into (41) then we have

$$g(h_c) = M\frac{h_c}{2} + \frac{2\epsilon}{h_c} = 2\sqrt{M\epsilon}$$

What if numerator cannot be expanded in a Taylor series? ($N_2(h)$)

$$N_2(h) = \frac{f(x+h) - f(x-h)}{2h} \tag{46}$$

$$fl(N(h)) = \ldots = N_2(h) + \epsilon/h \tag{47}$$

$$\tag{48}$$

Estimate the best possible h by expanding the numerator of (46) in a Taylor series (but we do not know a priori $f'(x)$ or $f''(x)$):

$$M(h) = f(x+h) - f(x-h) \tag{49}$$

$$M'(h) = f'(x+h) + f'(x-h) \tag{50}$$

$$M''(h) = f''(x+h) - f''(x-h) \tag{51}$$

$$M'''(h) = f'''(x+h) + f'''(x-h) \tag{52}$$

$$P_3(h) = 0 + 2f'(x)h + 2f'''(x)h^3/6 \tag{53}$$

$$\frac{f(x+h) - f(x-h)}{2h} \approx \frac{P_3(h)}{2h} = f'(x) + f'''(x)\frac{h^2}{6} \tag{54}$$

$$fl(\frac{f(x+h) - f(x)}{h}) = \frac{f(x+h) + \epsilon_1 - f(x-h) - \epsilon_2}{2h} = \tag{55}$$

$$\frac{f(x+h) - f(x-h)}{2h} + \frac{\epsilon_1 - \epsilon_2}{2h} \approx \tag{56}$$

$$f'(x) + f'''(x)\frac{h^3}{6} + \frac{\epsilon_1 - \epsilon_2}{2h} \tag{57}$$

$$|N_2(h) - f'(x)| < M\frac{h^2}{6} + \frac{\epsilon}{h} \quad M = \max_{x \le \xi \le x+h} |f'''(\xi)| \tag{58}$$

In order to find the optimal h for $N_2(h)$, we define,

$$g_2(h) = M\frac{h^2}{6} + \frac{\epsilon}{h} \tag{59}$$

$\min_h |g_2(h)|$ is found by checking critical points:

$$g_2'(h) = \frac{Mh}{3} - \frac{\epsilon}{h^2} = 0 \tag{60}$$

$$\frac{Mh}{3} = \frac{\epsilon}{h^2} \tag{61}$$

$$h^3 = \frac{3\epsilon}{M} \tag{62}$$

$$h_c = (\frac{3\epsilon}{M})^{1/3} \tag{63}$$

If we plug h_c into (59) then we have

$$g_2(h_c) = M\frac{h_c^2}{6} + \frac{\epsilon}{h_c} = \frac{3^{2/3}}{2}M^{1/3}\epsilon^{2/3}$$

inverse problem: predicting ϵ given the optimal h: In order to predict ϵ do the following steps:

1. Make a plot of $\log h$ versus $\log |f'(x) - N(h)|$ ($\log |f'(x) - N_2(h)|$) where $N(h)$ ($N_2(h)$) was found on a computer.
2. identify the value h_c that corresponds to the minimum point on the plot.
3. Solve (45) (for $N_2(h)$ solve (63)) for ϵ.