Lecture January 23:

Notes from last class Last class we started investigating loss-of-precision error when solving,

$$f(x) = \epsilon x^2 + x - 1 = 0, \tag{1}$$

where $0 < \epsilon << 1$ and when evaluating,

$$N(h) = \frac{f(x+h) - f(x)}{h}, \tag{2}$$

where x is a constant in (2) and $0 < h << 1$.

example

$$N(h) = \frac{\log(x+h) - \log(x)}{h}. \tag{3}$$

$$\lim_{h \to 0} N(h) = \lim_{h \to 0} \frac{\frac{1}{x+h}}{1} = \frac{1}{x}. \tag{4}$$

evaluate $N(h)$ using 4 digit chopping and $x = 2$

h	p^*	p	$\lvert(p - p^*)/p\rvert$
0.01	$(0.6981 - 0.6931)/0.01 = 0.5000$	0.4987542	0.00249
0.001	$(0.6936 - 0.6931)/0.001 = 0.5000$	0.499875	0.00025
0.0001	$(0.6931 - 0.6931)/0.001 = 0.0$	0.49999	1.0

replace numerator of $N(h)$ with $P_2(h)$ and simplify; Define $M(h)$ to be the numerator of $N(h)$:

$$M(h) = \log(x+h) - \log(x) \tag{5}$$

$$M'(h) = \frac{1}{x+h} \tag{6}$$

$$M''(h) = \frac{-1}{(x+h)^2} \tag{7}$$

expand about $h_0 = 0$:

$$P_2(h) = 0 + \frac{1}{x}h - \frac{1}{2x^2}h^2 \tag{8}$$

$$N(h) \approx \frac{P_2(h)}{h} = \frac{1}{x} - \frac{1}{2x^2}h \tag{9}$$

evaluate $P_2(h) = \frac{1}{x} - \frac{1}{2x^2}h$ using 4 digit chopping and $x = 2$

h	p^*	p	$\|(p - p^*)/p\|$
0.01	$0.5000 - 0.00125 = 0.4987$	0.4987542	0.00011
0.001	$0.5000 - 0.000125 = 0.4998$	0.499875	0.00015
0.0001	$0.5000 - 0.0000125 = 0.4999$	0.49999	0.00018

What if numerator cannot be expanded in a Taylor series?

$$N(h) = \frac{f(x+h) - f(x)}{h} \tag{10}$$

$$fl(N(h)) = fl(\frac{fl(f(fl(x+h))) - fl(f(x))}{h}) = \tag{11}$$

$$\frac{f(x+h) + \epsilon_1 - f(x) - \epsilon_2}{h} \tag{12}$$

Estimate the best possible h by expanding the numerator of (10) in a Taylor series (but we do not know a priori $f'(x)$ or $f''(x)$):

$$M(h) = f(x+h) - f(x) \tag{13}$$

$$M'(h) = f'(x+h) \tag{14}$$

$$M''(h) = f''(x+h) \tag{15}$$

$$P_2(h) = 0 + f'(x)h + f''(x)h^2/2 \tag{16}$$

$$\frac{f(x+h) - f(x)}{h} \approx \frac{P_2(h)}{h} = f'(x) + f''(x)\frac{h}{2} \tag{17}$$

$$fl(\frac{f(x+h) - f(x)}{h}) = \frac{f(x+h) + \epsilon_1 - f(x) - \epsilon_2}{h} = \tag{18}$$

$$\frac{f(x+h) - f(x)}{h} + \frac{\epsilon_1 - \epsilon_2}{h} \approx \tag{19}$$

$$f'(x) + f''(x)\frac{h}{2} + \frac{\epsilon_1 - \epsilon_2}{h} \tag{20}$$

$$|N(h) - f'(x)| < M\frac{h}{2} + \frac{2\epsilon} {h} \quad M = \max_{x \le \xi \le x+h} |f''(\xi)| \tag{21}$$

optimal h? Define,

$$g(h) = M\frac{h}{2} + \frac{2\epsilon}{h} \tag{22}$$

$\min_h |g(h)|$ is found by checking critical points:

$$g'(h) = \frac{M}{2} - \frac{2\epsilon}{h^2} = 0 \tag{23}$$

$$\frac{M}{2} = \frac{2\epsilon}{h^2} \tag{24}$$

$$h^2 = \frac{4\epsilon}{M} \tag{25}$$

$$h = \sqrt{\frac{4\epsilon}{M}} \tag{26}$$

inverse problem: predicting ϵ given the optimal h: In order to predict ϵ do the following steps:

1. Make a plot of $\log h$ versus $\log |f'(x) - N(h)|$ where $N(h)$ was found on a computer.
2. identify the value h_c that corresponds to the minimum point on the plot.
3. Solve (26) for ϵ.

in class practice (extra credit) Define,

$$N(h) = \frac{e^{x+h} - e^x}{h}. \tag{27}$$

1. Find $\lim_{h\to 0} N(h)$ using L'Hospital's rule.
2. Evaluate $N(h)$ using 3 digit chopping, $x = 1.0$, and $h = 0.01$. The exact value is 2.7319187. Find the absolute and relative errors.
3. replace the numerator of $N(h)$ (27) with $P_2(h)$, then simplify, then evaluate the resulting expression again using 3 digit chopping, $x = 1.0$, and $h = 0.01$. The exact value is 2.7319187. Find the absolute and relative errors.