MAD3703, Numerical Analysis I, MWF, 2:30-3:20, room Love 102
Lecture March 30:

1. cubic Hermite interpolation:

$$H_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \tag{1}$$

Interpolating conditions:

$$H_i(x_i) = f(x_i) \tag{2}$$
$$H_i(x_{i+1}) = f(x_{i+1}) \tag{3}$$
$$H_i'(x_i) = f'(x_i) \tag{4}$$
$$H_i'(x_{i+1}) = f'(x_{i+1}) \tag{5}$$

2. example: $f(x) = \tanh(x/0.01)$, $x_0 = -1$, $x_1 = 1$:

z_k	$f[z_k]$	$f[z_k, z_{k+1}]$	$f[z_k, z_{k+1}, z_{k+2}]$	$f[z_k, z_{k+1}, z_{k+2}, z_{k+3}]$
-1.0	-1.00			
-1.0	-1.00	5.54E-85		
1.0	1.00	1.00	0.500	
1.0	1.00	5.54E-85	-0.500	-0.500

$$H_i(x) = -1 + (1/2)(x + 1)^2 - (1/2)(x + 1)^2(x - 1) = \tag{6}$$
$$-1 + (3/2)(x + 1)^2 - (1/2)(x + 1)^3 \tag{7}$$

sanity check:

$$H_i(-1) = -1 \tag{8}$$
$$H_i(1) = -1 + 6 - 4 = 1 \tag{9}$$
$$H_i'(x) = 3(x + 1) - (3/2)(x + 1)^2 \tag{10}$$
$$H_i'(-1) = 0 \tag{11}$$
$$H_i'(1) = 6 - 6 = 0 \tag{12}$$

3. Error bound for piecewise cubic Hermite interpolation:

$$R^{(i)}(x) = \frac{f^{(4)}(\xi(x))}{4!}(x - x_i)^2(x - x_{i+1})^2 \tag{13}$$
$$\max_{a\le x\le b} |f(x) - S(x)| \le \max_{0\le i\le n-1} \max_{x_i\le x\le x_{i+1}} |f(x) - H_i(x)| \le \tag{14}$$
$$\max_{0\le i\le n-1} \max_{x_i\le x\le x_{i+1}} |R^{(i)}(x)| \le \tag{15}$$
$$\frac{1}{4!}(\max_{a\le x\le b} |f^{(4)}(x)|) \max_{0\le i\le n-1} \max_{x_i\le x\le x_{i+1}} |(x - x_i)^2(x - x_{i+1})^2|. \tag{16}$$

Assume the interpolating nodes are evenly spaced,

$$x_i = a + ih \quad h = (b - a)/n, \tag{17}$$

and

$$\frac{1}{4!}\max_{a\le x\le b}|f^{(4)}(x)| = M/24. \tag{18}$$

Then for any i,

$$g(x) \equiv (x-x_i)^2(x-x_{i+1})^2 \tag{19}$$

$$g'(x) = 2(x-x_i)(x-x_{i+1})^2 + 2(x-x_{i+1})(x-x_i)^2 = 0 \tag{20}$$

$$x_{critical} = \frac{x_i + x_{i+1}}{2} \tag{21}$$

$$g(x_{critical}) = (x_{i+1}-x_i)^4/16 = h^4/16 \tag{22}$$

$$\max_{a\le x\le b}|f(x)-S(x)| \le (M/24)(h^4/16) = Mh^4/384 \tag{23}$$

4. What if $f(x_i)$, $f(x_{i+1})$, $f'(x_{i+1})$, and $f''(x_{i+1})$ are gven instead?

z_k	$f[z_k]$	$f[z_k, z_{k+1}]$	$f[z_k, z_{k+1}, z_{k+2}]$	$f[z_k, z_{k+1}, z_{k+2}, z_{k+3}]$
x_i	$f(x_i)$			
x_{i+1}	$f(x_{i+1})$	$f[x_i, x_{i+1}]$		
x_{i+1}	$f(x_{i+1})$	$f'(x_{i+1})$	$\frac{f'(x_{i+1})-f[x_i,x_{i+1}]}{x_{i+1}-x_i}$	
x_{i+1}	$f(x_{i+1})$	$f'(x_{i+1})$	$f''(x_{i+1})/2$	$\frac{f''(x_{i+1})/2-f[x_i,x_{i+1},x_{i+1}]}{x_{i+1}-x_i}$

5. For the interpolation algorithm presented in the previous item, what is an error bound analogous to the one in (23)?

6. Error bound for piecewise cubic Hermite interpolation when $f(x_i)$, $f(x_{i+1})$, $f'(x_{i+1})$, and $f''(x_{i+1})$ are given instead of $f(x_i)$, $f(x_{i+1})$, $f'(x_i)$, and $f'(x_{i+1})$:

$$R^{(i)}(x) = \frac{f^{(4)}(\xi_i(x))}{4!}(x-x_i)(x-x_{i+1})^3 \tag{24}$$

$$\max_{a\le x\le b}|f(x)-S(x)| \le \max_{0\le i\le n-1}\max_{x_i\le x\le x_{i+1}}|f(x)-H_i(x)| \le \tag{25}$$

$$\max_{0\le i\le n-1}\max_{x_i\le x\le x_{i+1}}|R^{(i)}(x)| \le \tag{26}$$

$$\frac{1}{4!}(\max_{a\le x\le b}|f^{(4)}(x)|)\max_{0\le i\le n-1}\max_{x_i\le x\le x_{i+1}}|(x-x_i)(x-x_{i+1})^3|. \tag{27}$$

Assume the interpolating nodes are evenly spaced,

$$x_i = a + ih \quad h = (b-a)/n, \tag{28}$$

and

$$\frac{1}{4!}\max_{a\le x\le b}|f^{(4)}(x)| = M/24. \tag{29}$$

Then for any i,

$$g(x) \equiv (x - x_i)(x - x_{i+1})^3 \tag{30}$$

$$g'(x) = (x - x_{i+1})^3 + 3(x - x_{i+1})^2(x - x_i) = \tag{31}$$

$$(x - x_{i+1})^2(x - x_{i+1} + 3(x - x_i)) = 0 \tag{32}$$

$$x_{critical} = x_{i+1} \text{ or } \frac{3x_i + x_{i+1}}{4} \tag{33}$$

$$g(x_{i+1}) = 0 \tag{34}$$

$$g(\frac{3x_i + x_{i+1}}{4}) = \frac{x_{i+1} - x_i}{4}(\frac{3(x_i - x_{i+1})}{4})^3 = -\frac{27}{256}h^4 \tag{35}$$

$$\max_{a \leq x \leq b} |f(x) - S(x)| \leq (M/24)(27h^4/256) = \frac{Mh^4}{384}\frac{27}{16} \tag{36}$$

7. One solution to the group work problem from last class: "find the parameterization for a particle that first moves from $(0,0)$ to $(0,1)$, then moves counter clockwise in a circular arc of radius one until the particle reaches the point $(1,0)$, then moves back to the origin $(0,0)$."

$$x(t) = \begin{cases} 0 & 0 \leq t < 1/2 \\ \cos(\pi t) & 1/2 \leq t < 2 \\ 3 - t & 2 \leq t \leq 3 \end{cases}$$

$$x(t+3) = x(t)$$

$$y(t) = \begin{cases} 2t & 0 \leq t < 1/2 \\ \sin(\pi t) & 1/2 \leq t < 2 \\ 0 & 2 \leq t \leq 3 \end{cases}$$

$$y(t+3) = y(t)$$

8. Trapezoid rule:

$$\int_a^b f(x)dx = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{h^3}{12}f''(\xi)$$

9. Simpson's rule?