MAD3703, Numerical Analysis I, MWF, 2:30-3:20, room Love 102
Lecture March 3:

Newton's divided differences :

$$f[x_i] = f(x_i)$$
$$f[x_{i_0}, x_{i_1}] = \frac{f[x_{i_1}] - f[x_{i_0}]}{x_{i_1} - x_{i_0}}$$
$$f[x_{i_0}, \ldots, x_{i_n}] = \frac{f[x_{i_1}, \ldots, x_{i_n}] - f[x_{i_0}, \ldots, x_{i_{n-1}}]}{x_{i_n} - x_{i_0}}$$
$$P_n(x) = \sum_{i=0}^{n} f[x_0, \ldots, x_i] \Pi_{j=0}^{i-1}(x - x_j)$$

Properties of Newton's divided differences :

$$f[x_0, \ldots, x_n] = f[x_{i_0}, \ldots, x_{i_n}]$$
$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\eta)}{n!}$$
$$f[x_0, \ldots, x_n, x] = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}$$

example: find $P_n(x)$ if $f(x) = \sqrt{x}$ and x_i given :

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	$f[x_k, x_{k+1}, x_{k+2}]$
1.0	1		
4.0	2	1/3	
9.0	3	1/5	-1/60

$$x_0 = 1, \quad x_1 = 4, \quad x_2 = 9 \tag{1}$$
$$y_0 = 1, \quad y_1 = 2, \quad y_2 = 3 \tag{2}$$
$$P_2(x) = 1 + (1/3)(x-1) - (1/60)(x-1)(x-4) \tag{3}$$

Sanity check:

$$P_2(1) = 1 \tag{4}$$
$$P_2(4) = 1 + 1 = 2 \tag{5}$$
$$P_2(9) = 1 + 8/3 - (1/60)(8)(5) = 11/3 - 8/12 = 11/3 - 2/3 = 3 \tag{6}$$

Error bound ($1 \leq x \leq 9$) :

$$f(x) = P_n + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\Pi_{j=0}^{n}(x - x_j)$$

$$f(x) = x^{1/2} \quad f'(x) = (1/2)x^{-1/2} \quad f''(x) = (-1/4)x^{-3/2} \quad f'''(x) = (3/8)x^{-5/2}$$

$$R_2(x) = \frac{1}{16}(\xi(x)^{-5/2})(x-1)(x-4)(x-9)$$

$$\max_{1\le x\le 9} |R_2(x)| \le$$

$$(\max_{1\le x\le 9} x^{-5/2}/16)(\max_{1\le x\le 9} |(x-1)(x-4)(x-9)|) \le$$

$$(1/16)\max_{1\le x\le 9} |(x-1)(x-4)(x-9)|$$

Let $g(x) = (x-1)(x-4)(x-9)$:

$$g'(x) = (x-1)(x-4) + (x-4)(x-9) + (x-1)(x-9) = 3x^2 - 28x + 49 = 0$$

$$x = \frac{28 \pm \sqrt{28^2 - 4(3)(49)}}{6}$$

$$x = \frac{28 \pm \sqrt{196}}{6} = \frac{28 \pm 14}{6} = 7, 7/3$$

$$\max_{1\le x\le 9} |g(x)| = \max |g(7)|, |g(7/3)| = 36$$

The error bound is,

$$|R_2(x)| \le (1/16)(36) = 18/8 = 9/4$$