Lecture January 28:

Notes from previous practice problems

$$N(h) = \frac{e^{x+h} - e^x}{h} \tag{1}$$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \tag{2}$$

$$\lim_{h \to 0} \frac{e^{x+h}}{1} = e^x \tag{3}$$

Using 3 digit chopping arithmetic, $x = 1.00$, and $h = 1.00E-2$,

$$N(h) = \frac{2.71^{1.01} - 2.71}{0.01} = \frac{2.73 - 2.71}{0.01} = 2.00 \tag{4}$$

The relative error is,

$$\frac{|p - p^*|}{|p|} = |(2.00 - 2.73)/2.73| = 0.27 \tag{5}$$

If we approximate the numerator of (1) with its 2nd Taylor series expansion about $h_0 = 0$, and then simplify (1) then:

$$M(h) \equiv e^{x+h} - e^x \tag{6}$$

$$M'(h) = e^{x+h} \quad M''(h) = e^{x+h} \tag{7}$$

$$P_2(h) = M(0) + M'(0)h + M''(0)h^2/2 = e^x h + e^x h^2/2 \tag{8}$$

$$N(h) \approx \frac{e^x h + e^x h^2/2}{h} = e^x + e^x h/2 \tag{9}$$

Using 3 digit chopping arithmetic, $x = 1.00$, and $h = 1.00E-2$, applied to the approximation, we have:

$$N(h) \approx 2.71^{1.00} + (0.0100/2.00)(2.71) = 2.71 + 0.0135 = 2.72 \tag{10}$$

The relative error in this case is,

$$\frac{|p - p^*|}{|p|} = |(2.72 - 2.73)/2.73| = 0.004 \tag{11}$$

If loss of precision cannot be removed, then there is an optimal h when using

$$N(h) = \frac{f(x+h) - f(x)}{h} \tag{12}$$

in order to approximate $f'(x)$. For example, if $f(x) = e^x$, $x = 1$, and I use my TI-30X IIS calculator, the optimal h is about $1/10^6$:

h	$(e^{1+h}-e)/h$	$\lvert(e^{1+h}-e)/h-e\rvert$
10^{-1}	2.858841955	0.1406
10^{-5}	2.7182954	$1.4E-5$
10^{-6}	2.718282	$1.7E-7$
10^{-7}	2.71828	$1.8E-6$
10^{-8}	2.7182	$8.2E-5$

What if numerator cannot be expanded in a Taylor series?

$$N(h) = \frac{f(x+h)-f(x)}{h} \tag{13}$$

$$fl(N(h)) = fl(\frac{fl(f(fl(x+h))) - fl(f(x))}{h}) = \tag{14}$$

$$\frac{f(x+h)+\epsilon_1 - f(x) - \epsilon_2}{h} \tag{15}$$

Estimate the best possible h by expanding the numerator of (13) in a Taylor series (but we do not know a priori $f'(x)$ or $f''(x)$):

$$M(h) = f(x+h) - f(x) \tag{16}$$

$$M'(h) = f'(x+h) \tag{17}$$

$$M''(h) = f''(x+h) \tag{18}$$

$$P_2(h) = 0 + f'(x)h + f''(x)h^2/2 \tag{19}$$

$$\frac{f(x+h)-f(x)}{h} \approx \frac{P_2(h)}{h} = f'(x) + f''(x)\frac{h}{2} \tag{20}$$

$$fl(\frac{f(x+h)-f(x)}{h}) = \frac{f(x+h)+\epsilon_1-f(x)-\epsilon_2}{h} = \tag{21}$$

$$\frac{f(x+h)-f(x)}{h} + \frac{\epsilon_1-\epsilon_2}{h} \approx \tag{22}$$

$$f'(x) + f''(x)\frac{h}{2} + \frac{\epsilon_1-\epsilon_2}{h} \tag{23}$$

$$|N(h) - f'(x)| < M\frac{h}{2} + \frac{2\epsilon}{h} \quad M = \max_{x\le\xi\le x+h}|f''(\xi)| \tag{24}$$

optimal h? Define,

$$g(h) = M\frac{h}{2} + \frac{2\epsilon}{h} \tag{25}$$

$\min_h |g(h)|$ is found by checking critical points:

$$g'(h) = \frac{M}{2} - \frac{2\epsilon}{h^2} = 0 \tag{26}$$

$$\frac{M}{2} = \frac{2\epsilon}{h^2} \tag{27}$$

$$h^2 = \frac{4\epsilon}{M} \tag{28}$$

$$h = \sqrt{\frac{4\epsilon}{M}} \tag{29}$$

inverse problem: predicting ϵ given the optimal h: In order to predict ϵ do the following steps:

1. Make a plot of $\log h$ versus $\log |f'(x) - N(h)|$ where $N(h)$ was found on a computer.
2. identify the value h_c that corresponds to the minimum point on the plot.
3. Solve (29) for ϵ.

in class practice (extra credit) Define,

$$N(h) = \frac{e^{x+h} - e^{x-h}}{2h}. \tag{30}$$

Find an analogous expression, using $N(h)$ from (30), as (29).