Project 4 solutions

1.

$$f(x) = (1+x)^{1/3}$$
$$P_1(x) = f(0.0)\frac{x-1.0}{0.0-1.0} + f(1.0)\frac{x-0.0}{1.0-0.0} = (1-x) + 2^{1/3}x$$
$$f'(x) = \frac{1}{3}(1+x)^{-2/3}$$
$$f''(x) = -\frac{2}{9}(1+x)^{-5/3}$$
$$f'''(x) = \frac{10}{27}(1+x)^{-8/3}$$
$$\max_{0\le x\le 1}|f(x) - P_1(x)| = \max_{0\le x\le 1}|\frac{f''(\xi(x))}{2}x(x-1)| \le$$
$$\max_{0\le x\le 1}|\frac{1}{9}(1+x)^{-5/3}| \max_{0\le x\le 1}|x(x-1)|$$
$$\max_{0\le x\le 1}|\frac{1}{9}(1+x)^{-5/3}| = 1/9$$
$$g(x) = x(x-1)$$
$$g'(x) = (x-1) + x = 2x - 1 = 0 \quad x = 1/2 \quad g(1/2) = -1/4$$
$$\max_{0\le x\le 1}|f(x) - P_1(x)| \le (1/9)(1/4) = 1/36$$
$$|(4/3)^{1/3} - P_1(1/3)| = |1.10064 - 1.08664| = 0.014002$$

2.

$$f(x) = (1+x)^{1/3}$$
$$P_2(x) = \frac{(x-1)(x-2)}{2} + 2^{1/3}\frac{(x)(x-2)}{-1} + 3^{1/3}\frac{(x)(x-1)}{2} =$$
$$\frac{(x-1)(x-2)}{2} - 2^{1/3}x(x-2) + 3^{1/3}\frac{x(x-1)}{2}$$
$$f'''(x) = \frac{10}{27}(1+x)^{-8/3}$$
$$\max_{0\le x\le 2}|f(x) - P_2(x)| = \max_{0\le x\le 2}|\frac{f'''(\xi(x))}{6}x(x-1)(x-2)| \le$$
$$\max_{0\le x\le 2}|\frac{5}{81}(1+x)^{-8/3}| \max_{0\le x\le 2}|x(x-1)(x-2)|$$
$$\max_{0\le x\le 2}|\frac{5}{81}(1+x)^{-8/3}| = 5/81$$
$$g(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$$
$$g'(x) = 3x^2 - 6x + 2 = 0$$
$$x = \frac{6 \pm \sqrt{36 - 4(3)(2)}}{6} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{1}{\sqrt{3}}$$

$$g(1+1/\sqrt{3}) = -0.385$$
$$g(1-1/\sqrt{3}) = 0.385$$
$$\max_{0\le x\le 2} |f(x) - P_2(x)| \le (5/81)(0.385) = 0.024$$
$$P_2(1/3) = \frac{(-2/3)(-5/3)}{2} - 2^{1/3}(1/3)(-5/3) + 3^{1/3}\frac{(1/3)(-2/3))}{2} =$$
$$10/18 + 2^{1/3}5/9 - 3^{1/3}/9 = 1.09526$$
$$|(4/3)^{1/3} - P_2(1/3)| = |1.10064 - 1.09526| = 0.0054$$

3. Note: the numbers in the table below were the result of IEEE double precision arithmetic (only 4 digits are shown though). $f(x) = \cos(x)$, $x_0 = 0$, $x_1 = \pi/8$, $x_2 = \pi/4$, and $x_3 = 3\pi/8$:

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	$f[x_k, x_{k+1}, x_{k+2}]$	$f[x_k, x_{k+1}, x_{k+2}, x_{k+3}]$
0.0	1.0			
$\pi/8$	0.9239	-0.1938		
$\pi/4$	0.7071	-0.5520	-0.4560	
$3\pi/8$	0.3827	-0.8261	-0.3490	0.09083

$$P_3(x) = 1.0 - 0.1938x - 0.4560x(x-\pi/8) + 0.09083x(x-\pi/8)(x-\pi/4)$$
$$Q_3(x) = 0.9239 - 0.1938(x-\pi/8) - 0.4560(x-\pi/8)x + 0.09083(x-\pi/8)x(x-\pi/4)$$
$$R_3(x) = 0.7071 - 0.5520(x-\pi/4) - 0.4560(x-\pi/4)(x-\pi/8) + 0.09083(x-\pi/4)(x-\pi/8)x$$

$$\cos(\pi/3) = 0.5$$
$$P_3(\pi/3) = 0.5007$$

4.

$$P(-2) = 3 + 2 - 6 = -1$$
$$P(-1) = 3$$
$$P(0) = 3 - 2 = 1$$
$$P(1) = 3 - 4 = -1$$
$$P(2) = 3 - 6 + 6 = 3$$
$$P(x) = 3 - 2x - 2 + x^3 - x = x^3 - 3x + 1$$
$$Q(x) = -1 + 4x + 8 - 3x^2 - 9x - 6 + x^3 + 3x^2 + 2x = x^3 - 3x + 1$$

After simplification $P(x)$ and $Q(x)$ are the same polynomial so if P interpolates the data, so too must Q. Part (a) does not violate the uniqueness property since although P and Q look different (superficially), after simplification, one derives that P and Q are identical.

5. Since $f(x_i) = P_3(x_i)$, $i = 0, \dots, 3$, and $x_3 = 3/4$, then

$$f(3/4) = P_3(3/4) = 1 + 4(3/4) + 4(3/4)(3/4 - 1/4) + (16/3)(3/4)(3/4 - 1/4)(3/4 - 1/2) = 6$$

6. For a function f, the divided difference table is given by:

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	$f[x_k, x_{k+1}, x_{k+2}]$
$x_0 = 0.0$	$f[x_0]$		
$x_1 = 0.4$	$f[x_1]$	$f[x_0, x_1]$	
$x_2 = 0.7$	$f[x_2] = 6$	$f[x_1, x_2] = 10$	$f[x_0, x_1, x_2] = 50/7$

$$50/7 = \frac{10 - f[x_0, x_1]}{7/10} \tag{1}$$

$$5 = 10 - f[x_0, x_1] \tag{2}$$

$$f[x_0, x_1] = 5 \tag{3}$$

$$10 = \frac{6 - f[x_1]}{3/10} \tag{4}$$

$$3 = 6 - f[x_1] \tag{5}$$

$$f[x_1] = 3 \tag{6}$$

$$5 = \frac{3 - f[x_0]}{4/10} \tag{7}$$

$$2 = 3 - f[x_0] \tag{8}$$

$$f[x_0] = 1 \tag{9}$$