Project 1 solutions

1.

$$f(x) = 1/(1-x) \quad f'(x) = 1/(1-x)^2 \tag{1}$$

$$f''(x) = 2/(1-x)^3 \quad f'''(x) = 6/(1-x)^3, \ldots, f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \tag{2}$$

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(n)}(0)}{i!} x^i = \sum_{i=0}^{n} \frac{i!}{i!} x^i = \sum_{i=0}^{n} x^i \tag{3}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} x^{n+1} \tag{4}$$

also,

$$xP_n(x) = \sum_{i=0}^{n} x^{i+1} = P_n(x) - 1 + x^{n+1} \tag{5}$$

$$P_n(x) = \frac{1 - x^{n+1}}{1-x} \tag{6}$$

$$R_n(x) = \frac{1}{1-x} - \sum_{i=0}^{n} x^i = \frac{1}{1-x} - \frac{1 - x^{n+1}}{1-x} = \frac{x^{n+1}}{1-x} \tag{7}$$

$$\max_{0 \le x \le 1/2} |R_n(x)| = \frac{1}{2^n} = 10^{-6} \tag{8}$$

$$2^n = 10^6 \tag{9}$$

$$n = \frac{\ln(10^6)}{\ln 2} = 19.9 \quad n \ge 20 \tag{10}$$

2.

$$f(x) = e^x \quad f'(x) = e^x \tag{11}$$

$$f''(x) = e^x \quad f'''(x) = e^x, \ldots, f^{(n)}(x) = e^x \tag{12}$$

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(n)}(1)}{i!}(x-1)^i = \sum_{i=0}^{n} \frac{e}{i!}(x-1)^i = \sum_{i=0}^{n} e\frac{(x-1)^i}{i!} \tag{13}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-1)^{n+1} = \frac{e^{\xi(x)}}{(n+1)!}(x-1)^{n+1} \tag{14}$$

$$\max_{0 \le x \le 1/2} |R_n(x)| = \max_{0 \le x \le 1/2} \left|\frac{e^{\xi(x)}}{(n+1)!}(x-1)^{n+1}\right| \le$$

$$\max_{0 \le x \le 1} \frac{e^x}{(n+1)!} \max_{0 \le x \le 1/2} |(x-1)^{n+1}| \le \frac{e}{(n+1)!}(1)^{n+1}$$

$$|R_n(x)| \le \frac{e}{8!} = 6.7E-5 \quad n = 7$$

$$|R_n(x)| \le \frac{e}{9!} = 7.5E-6 \quad n=8$$
$$|R_n(x)| \le \frac{e}{10!} = 7.5E-7 \quad n=9$$

3.

$$f(x) = \cos(2x) \quad f'(x) = -2\sin(2x) \tag{15}$$
$$f''(x) = -4\cos(2x) \quad f'''(x) = 8\sin(2x), \dots \tag{16}$$
$$f^{(2i+1)}(0) = 0 \quad i = 0,1,2,\dots \tag{17}$$
$$f^{(2i)}(0) = (-1)^i 2^{2i} \quad i = 0,1,2,\dots \tag{18}$$
$$P_{2n}(x) = \sum_{i=0}^{n} \frac{(-1)^i}{(2i)!}(2x)^{2i} \tag{19}$$
$$R_{2n}(x) = \frac{f^{(2n+2)}(\xi(x))}{(2n+2)!}x^{2n+2} = \frac{(-1)^{n+1}\cos(2\xi(x))}{(2n+2)!}(2x)^{2n+2} \tag{20}$$

$$\max_{0\le x\le\pi} |R_{2n}(x)| = \max_{0\le x\le\pi} |\frac{\cos(2\xi(x))}{(2n+2)!}(2x)^{2n+2}| \le$$
$$\max_{0\le x\le\pi} \frac{|\cos(2x)|}{(2n+2)!} \max_{0\le x\le\pi} (2x)^{2n+2} \le \frac{1}{(2n+2)!}(2\pi)^{2n+2}$$

$$|R_{2n}(x)| \le \frac{(2\pi)^{12}}{12!} = 7.9 \quad n=5$$
$$|R_{2n}(x)| \le \frac{(2\pi)^{24}}{24!} = 2.3E-5 \quad n=11$$
$$|R_{2n}(x)| \le \frac{(2\pi)^{26}}{26!} = 1.4E-6 \quad n=12$$
$$|R_{2n}(x)| \le \frac{(2\pi)^{28}}{28!} = 7.3E-8 \quad n=13$$