MODULE 1

OSCILLATIONS

Introduction:

Oscillations are a fundamental concept in physics that refer to the repetitive motion of a system around a fixed point. In an oscillating system, energy is continuously exchanged between potential and kinetic energy as the system moves back and forth around its equilibrium position. Common examples of oscillating systems are simple pendulum, mass-spring systems, electrical circuits, and waves. The study of oscillations is important in many areas of physics, including mechanics, electromagnetism, and quantum mechanics. Understanding oscillations can help explain phenomena such as sound waves, the behavior of atoms and molecules, and the behavior of astronomical objects such as stars and planets.

The study of oscillations is of great importance in engineering, as many practical systems exhibit oscillatory behavior. A few examples are,

Electrical circuits: The study of oscillations is important in designing and analyzing electrical circuits, such as oscillators, filters, and amplifiers. Oscillations are also used in signal processing and communication systems.

Mechanical systems: Many mechanical systems exhibit oscillatory behavior, including engines, bridges, and buildings. Understanding oscillations is important in designing and analyzing these systems to ensure their stability and safety.

Control systems: Oscillations are often unwanted in control systems, as they can lead to instability and oscillatory behavior. However, the study of oscillations can help engineers design control systems that are stable and minimize oscillatory behavior.

Acoustics: The study of oscillations is important in understanding acoustics, such as the behavior of sound waves and vibrations in materials. This is important in designing and analyzing musical instruments, speakers, and other audio devices.

The study of oscillations is important in many areas of engineering, as it allows engineers to design and analyze systems with predictable and stable behavior.

Periodic Motion:

If we observe the motion of the pendulum of a clock, the motion of the earth around the sun and motion of moon around the earth, we find that the same motion along the same path is repeated again and again after equal intervals of time. Such a type of motion is called a **periodic motion**. It is defined as a motion in which a body describes the same path in the same way again and again in equal intervals of time.

If the body moves in a circular path, it is said to describe **circular periodic motion** and if the motion is repeated along the line, it is said to describe **linear periodic motion**. Periodic motion is also called harmonic motion.

Free Oscillations:

Oscillations of a body or a system in the absence of any external force are called free oscillations.

Simple Harmonic Motion (SHM): Oscillation is a repeating motion that occurs when a time varying force acts on the system. Oscillations are periodic motions. The motion of an object is said to be simple harmonic motion if the restoring force (or acceleration) is directly proportional to the displacement and acts in the direction opposite to that of motion.

• SHM can be linear or angular depending on the path described by the body.
• SHM is a motion in which the acceleration of the body is directly proportional to its displacement from a fixed point and is always directed towards the fixed point.
• SHM is an oscillatory motion of a body where the restoring force is proportional to the negative of the displacement.

Examples of SHM:

• Mass suspended to a spring when pulled down and left free executes simple harmonic motion vertically.
• Excited tuning fork.
• Plucked string in a veena or guitar.
• Shock absorber after being bumped.

Mechanical simple harmonic oscillator and Expression for SHM (Equation for Free Oscillations): (Differential equation of simple harmonic motion (SHM) and its solution)

Consider a mass attached to spring of negligible mass which is attached from a rigid support. The mass is made to oscillate left and right. The oscillation is due to the restoring force developed in the spring and is directly proportional to the displacement of the mass from the equilibrium position. The figure shows the path of the simple harmonic oscillator.

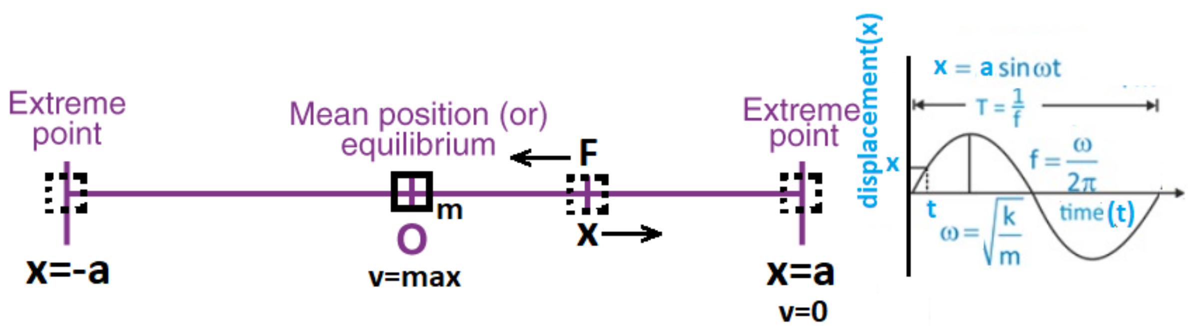

Figure 1

Force acting on the mass is given by,

$F = kx$, where k = Force constant

Vector form

$F = -kx$, (negative sign indicates displacement and force opposite each other)

$ma = -kx$, (From Newton's second law of motion)

$ma + kx = 0$,

$m\frac{d^2x}{dt^2} + kx = 0$, (in differential form)

$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$, (by dividing m)

$\frac{d^2x}{dt^2} + \omega^2 x = 0, \dots\dots..(1)\;\; \omega^2 = \frac{k}{m}$ (angular velocity of SHM)

Above is the differential equation of simple harmonic oscillator.

The solution for the above differential equation is

$\boldsymbol{x = \mathbf{a}\sin(\omega t + \phi)} \dots\dots\dots.(3)$

(3) This is the solution of the differential equation and gives the displacement of the object or particle from its mean positive in SHM where ϕ is called initial phase.

<u>Special case 1</u>

$t = 0\;\; x = 0$

$0 = \mathrm{a}\sin\phi$

$\sin\phi = 0$

$\phi = 0°$ (Object starts motion from mean position for which phase angle is zero)

<u>Special case 2</u>

$t = 0\;\; x = a$

$a = \mathrm{a}\sin\phi$

$\sin\phi = 1$

$\phi = \frac{\pi}{2}$ (Object starts motion from extreme position for which phase angle is 90°)

Characteristics of SHM

Amplitude: The value of the displacement x varies from $+a$ to $-a$. Thus, the maximum displacement of the particle during the simple harmonic motion is called amplitude (a).

Phase Angle and Initial Phase: The value $(\omega t + \phi)$ represents the state of the system and is called phase angle. The angle ϕ is called initial phase.

Angular velocity or Frequency $(\boldsymbol{\omega})$: It is the rate of change of angular displacement and is given by $\omega = \sqrt{\frac{k}{m}}$

Frequency $(\boldsymbol{f})$: Frequency of oscillations is defined as the no. of oscillations per second and is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Time Period (T): It is the time taken to complete one oscillation and is given by $T = \frac{1}{f}$

Velocity (v): The velocity of the particle in simple harmonic motion is given by differentiating the equation (3) with respect to time. The velocity of the particle in SHM varies with time. Hence, we get

$$v = \frac{dx}{dt} = a\,\omega \cos(\omega t + \phi) = \omega\sqrt{a^2 - x^2} \ldots\ldots (4)$$

The velocity varies from $+a\,\omega$ to $-\,a\,\omega$. Velocity is of the particle is zero at $x = a \,\&\, x = -a$ and maximum at $x = 0$.

Acceleration (a): The acceleration in SHM varies with time and is given by differentiating the equation velocity (eqn (4)) with respect to time.

$$a = \frac{dv}{dt} = a\,\omega\ \sin(\,\omega t + \phi) = -\omega^2 x \ldots\ldots (5)$$

Thus, the acceleration is proportional to displacement and acts in the direction opposite to displacement. Acceleration varies from $+a\omega^2$ to $-a\omega^2$. Acceleration of the particle is maximum at $x = a \,\&\, x = -a$ and zero at $x = 0$.

Force constant (spring constant) and its physical significance:

"It is the amount of force exerted when a spring is elongated/compressed by unit length." It determines the stiffness of the string. The SI unit of spring constant unit is N/m. The physical significance of k is, if k large then higher force is required for unit extension and if k is small relatively lower force is required for unit extension in the spring.

Natural frequency

If the oscillations occur without the action of an external periodic force, then such oscillations are called *free oscillations*. When a body exhibits free oscillations the frequency with which the oscillations occur is called *Natural Frequency*.

Springs in Series and Parallel

Consider two springs of negligible masses and with force constants $k1$ and $k2$. Let us calculate the effective spring constant when the two springs are connected in series and in parallel, as shown in the figure

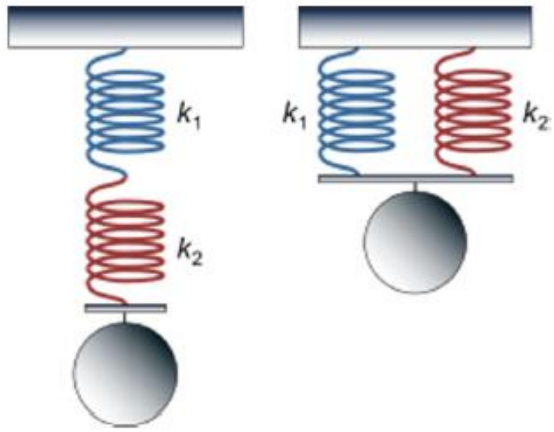

Figure 2

Springs in series

When two springs are connected in series and a mass is attached the force in the springs is same. The net extension x is given by

$$x = x_1 + x_2$$

Here x is the total extension, x_1 is the extension in the spring with force constant k_1 and x_2 is the extension in the spring with force constant k_2. If k_s is the effective force constant of the combination and if F is the force applied on the combination, then we get

$$\frac{F}{k_s} = \frac{F}{k_1} + \frac{F}{k_2}$$

Thus,

$$\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2}$$

Or

$$k_s = \frac{k_1 k_2}{k_1 + k_2}$$

If n number of springs are connected in series, then

$$\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots\ldots + \frac{1}{k_n}$$

Springs in parallel

When springs are in connected in parallel and a mass is attached the displacement in the springs remains same. The net force F on the system of springs is given by

$$F = F_1 + F_2$$

Here F_1 is the force on the spring with force constant k_1 and F_2 is the force on the spring with force constant k_2. If k_p is the effective force constant of the combination and if y is the total extension in the combination then we get,

$$k_p x = k_1 x + k_2 x$$

Thus, the effective force constant of the parallel combination is given by

$$k_p = k_1 + k_2$$

If $'n'$ springs are connected in parallel then the effective spring constant is given by

$$k_p = k_1 + k_2 + k_3 + \ldots\ldots + k_n$$

Damped Oscillations

Free oscillation: Free oscillation is a type of oscillation where there is no opposite force acting on it. The amplitude of the oscillation will stay constant over time. (Refer figure 1)

Damped oscillation: Damped oscillation is a type of oscillatory motion in which the amplitude of the oscillation gradually decreases over time due to the presence of an external force or resistive force or damping force.

In a simple harmonic oscillator, such as a mass-spring system, the oscillation would continue indefinitely if there were no friction or other sources of damping. However, in the presence of damping, the energy of the oscillation is gradually dissipated, causing the amplitude of the oscillation to decrease until it eventually comes to rest.

Damped oscillation is commonly encountered in a variety of physical systems, including electrical circuits (LC circuit), mechanical systems (spring mass system, simple pendulum), and even biological systems. The rate of damping and the frequency of oscillation are important parameters that determine the behaviour of the system.

Examples of damped oscillations

Simple pendulum oscillating in air

Consider a simple pendulum oscillating in air. During the motion the pendulum experiences air resistance which leads to the dissipation of energy. Hence the amplitude of oscillations of the simple pendulum decreases and finally comes to a stop.

Spring mass system with mass immersed in a liquid

Consider a spring mass system in which the oscillating mass is immersed in a viscous fluid. During the oscillations the viscous force acting on the mass reduces the amplitude progressively.

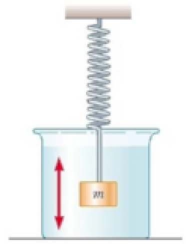

Figure 3

LC oscillations

Let a charged capacitor is connected across an inductor. In this system the capacitor discharges through the inductor and gets charged in the opposite direction. This process continues in setting an oscillatory current in the circuit. Such oscillations are called LC oscillations. If the Inductor and Capacitor are not ideal the energy is dissipated across the components and the amplitude of the oscillatory current decreases continuously and thus leading to damped oscillations.

Expression for the decay of the amplitude in damped oscillations

Let us consider a simple harmonic oscillator system or a mass spring oscillator system damped by executing oscillations in a resistive medium (figure 2). Then the forces acting on the oscillating system are,

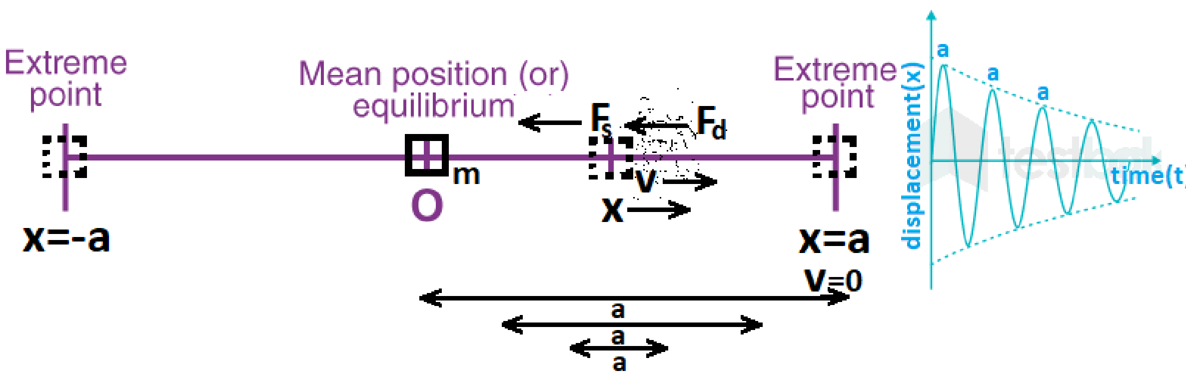

Figure 4

Restoring force (F_s) on mass ‘m’ and is directly proportional to displacement ‘x’ of mass,

$F_s = -kx$, where k - force constant.

Damping force (F_d) on mass ‘m’ and is directly proportional to the velocity ‘v’ of the mass,

$F_d = -bv$, where b - damping constant.

The net force acting on the oscillating mass system is given by

$F_{net} = F_s + F_d$

$F = -kx - bv$

$ma = -kx - bv$, (From Newton's second law of motion)

$ma + bv + kx = 0$

$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$, (in differential form)

$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$, (dividing by 'm')

$\frac{b}{m} = 2\beta \ldots\ldots(1)$ or $\beta = \frac{b}{2m}$ and $\frac{k}{m} = \omega^2 \ldots\ldots(2)$

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x = 0 \ldots\ldots\ldots(3)$$

Equation (3) is the differential equation of motion of damped oscillation. By solving this you will get the solution, using which one can get different possible damped oscillations.

The general solution of the above equation is as below,

$x = Ae^{\alpha t} \ldots\ldots(4)$ where A and α are arbitrary constants

Differentiate 'x' w r to 't'

$$\frac{dx}{dt} = A\alpha e^{\alpha t}$$

Again, differentiate the above equation w r to 't'

$$\frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

Substitute above two equations, $x, \frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in (3)

$(3) \Rightarrow A\alpha^2 e^{\alpha t} + 2\beta A\alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$

$Ae^{\alpha t}(\alpha^2 + 2\beta\alpha + \omega^2) = 0$

Or $(\alpha^2 + 2\beta\alpha + \omega^2) = 0$, It is a kind of quadratic equation, and the solution is.

$$\alpha = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega^2}}{2}$$

$$\alpha = -\beta \pm \sqrt{\beta^2 - \omega^2}$$

Here, α has two possibilities,

$$\alpha_1 = -\beta + \sqrt{\beta^2 - \omega^2}$$

$$\alpha_2 = -\beta - \sqrt{\beta^2 - \omega^2}$$

Now, the general solution of differential equation (3), that is equation (4) becomes.

$x = Ce^{\alpha_1 t} + De^{\alpha_2 t}$

$x = Ce^{\left(-\beta+\sqrt{\beta^2-\omega^2}\right)t} + De^{\left(-\beta-\sqrt{\beta^2-\omega^2}\right)t}$

$x = e^{-\beta t}\left[Ce^{\left(\sqrt{\beta^2-\omega^2}\right)t} + De^{\left(-\sqrt{\beta^2-\omega^2}\right)t}\right] \ldots\ldots\ldots.(5)$

Where *C* and *D* are the constants can be evaluated by applying boundary conditions for oscillating system. Equation (5) is the solution for the differential equation (3) for damped oscillation.

There are four possible cases arise for the above solution,

(i) $\beta^2 < \omega^2$ *or* $b^2 < 4km \rightarrow$ *Oscillatory motion or under(light) damping*
(ii) $\beta^2 = \omega^2$ *or* $b^2 = 4km \rightarrow$ *No oscillation or critical damping*
(iii) $\beta^2 > \omega^2$ *or* $b^2 > 4km \rightarrow$ *No oscillation or over (heavy) damping*

(Case (ii) $\rightarrow \beta^2 = \omega^2 \rightarrow \frac{b^2}{4m^2} = \frac{k}{m} \rightarrow b^2 = 4km,$ *from equation* (1)*and* (2)

Case (i) - $\boldsymbol{\beta^2 < \omega^2}$, *under(light) damping*

Substitute above in (5) and simplify the equation, the equation reduces to

$x = a\, e^{-\beta t}\sin(\sqrt{(\omega^2 - \beta^2}\; t + \phi)$

Above equation is product of two functions, $a\, e^{-\beta t}$- exponential decay of amplitude and $\sin(\sqrt{(\omega^2 - \beta^2}\; t + \phi)$- oscillatory term.

Thus, resultant equation is product of two functions, It is the oscillatory term which is further modulated by an exponential decay term. This is called under (light) damping.

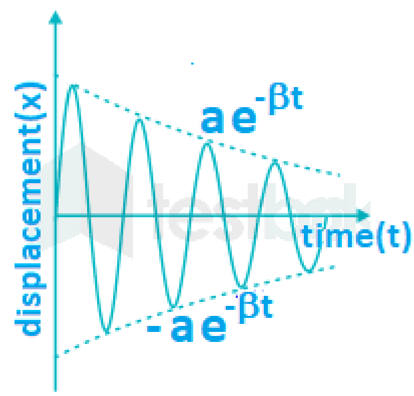

Figure 5

Case (ii) - $\boldsymbol{\beta^2 = \omega^2}$, *citical damping*

Substitute $\beta^2 = \omega^2$ in equation (5) and simplify the equation, the equation reduces to

$x = e^{-\beta t}[(P + Qt]$

Which is the product of two terms. As 't' is present in $e^{-\beta t}$ and also in the term Qt both of them are contributing to the variation x with respect to time 't'. But contribution of Qt to x is very small and therefore x decreases exponentially rapidly to approach the value of zero with respect to time 't' with no oscillation. This is called critical damping.

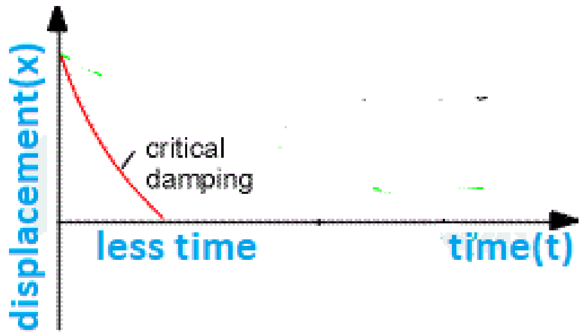

Figure 6

Case (iii) - $\boldsymbol{\beta^2 > \omega^2, over\ (\ heavy)\ damping}$

If $\beta^2 > \omega^2$ in equation (5)

$$(5) \Rightarrow x = e^{-\beta t}\left[Ce^{\left(\sqrt{\beta^2-\omega^2}\right)t} + De^{\left(-\sqrt{\beta^2-\omega^2}\right)t}\right]$$

Or $x = Ce^{\left(-\beta+\sqrt{\beta^2-\omega^2}\right)t} + De^{\left(-\beta-\sqrt{\beta^2-\omega^2}\right)t}$

Since, $(\beta^2 - \omega^2)$ is positive, both the terms $\left(-\beta + \sqrt{\beta^2 - \omega^2}\right)t$ and $\left(-\beta - \sqrt{\beta^2 - \omega^2}\right)t$ are negative indicate the exponential decay of the displacement 'x' of the oscillating system in more time than the earlier case with no oscillations. This is called over damping.

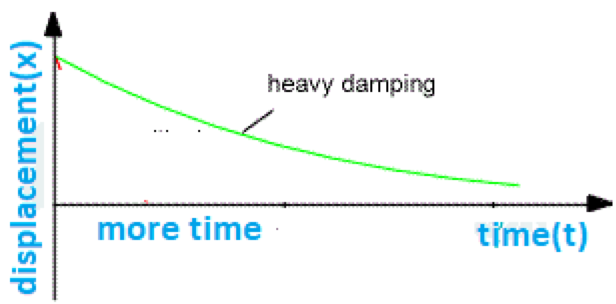

Figure 7

Graphical representation of possible cases of damped oscillations

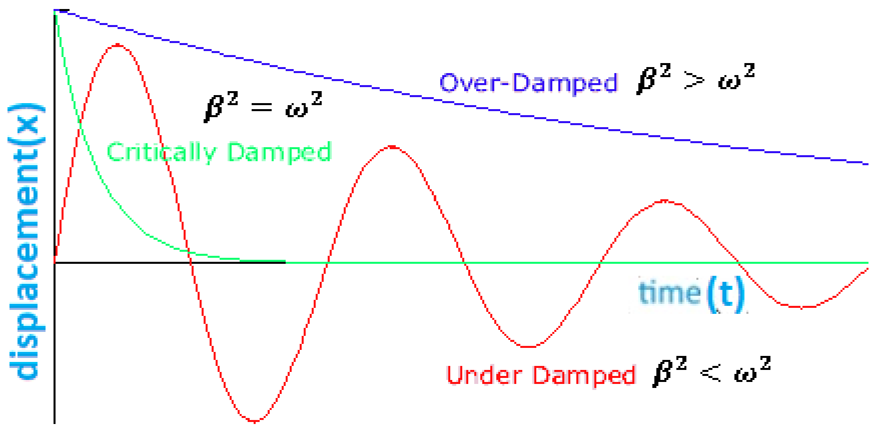

Figure 8

Energy decay in damped oscillations

The total energy of the damped oscillator decreases with respect time. Let us derive an expression for energy decay in damped oscillation.

We have a differential equation for damped oscillations.

$$m\frac{d^2x}{dt^2} = -kx - bv$$

$$m\frac{dv}{dx} = -kx - bv$$

$$mv\frac{dv}{dx} = -kxv - bv^2$$

Integrate on both side w. r. to 't'

$$\frac{1}{2}mv^2 = -\frac{1}{2}kx^2 - b\int v^2 dt$$

Here the constant of integration assumes to be zero as we are interested in the decay of energy over time 't'.

Now the total energy of oscillator is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Substitute equation of K.E, $\frac{1}{2}mv^2$ in the above equation

$$E = -\frac{1}{2}kx^2 - b\int v^2 dt + \frac{1}{2}kx^2$$

$$E = -b \int v^2 dt$$

Differentiate total energy of the oscillator w. r. to time 't'

$$\frac{dE}{dt} = -bv^2$$

This shows that the energy of the damped oscillator decreases over time at a rate proportional to the square of the velocity with damping constant 'b'.

Quality factor and its Significance

The energy loss rate of a weakly damped oscillator is conveniently characterized in terms of a parameter Q called Quality factor. The quality factor is defined as 2π times the ratio of energy stored in the oscillator to the energy lost per time period.

$$Q = 2\pi \frac{E}{PT} = \omega\tau \; (for\ low\ damping)$$

Here, P is the power dissipated and T is the time period. The quantity τ is called relaxation time. It is defined as the time taken for total mechanical energy to decay to a value $\frac{1}{e}$ times the original value. Q is also given by

$$Q = \frac{1}{\gamma}\sqrt{\frac{k}{m}}$$

This is because $\tau = \frac{1}{\gamma}$ and $\omega = \sqrt{\frac{k}{m}}$

Significance

If the oscillator is weakly damped, then the energy loss per period is relatively small and the Quality factor is much larger than unity. Quality factor could be considered as the number of oscillations that the oscillator typically completes before its amplitude decays to a negligible value.

Forced oscillations.

It is a steady state of sustained oscillation of a body oscillating in a resistive medium under the action of an external periodic force which acts independently of the restoring force. Examples for forced oscillations are Sonometer wire set to oscillations using a tuning fork or electromagnet. Resonance air column.

Expression for Amplitude and Phase in Forced Oscillations

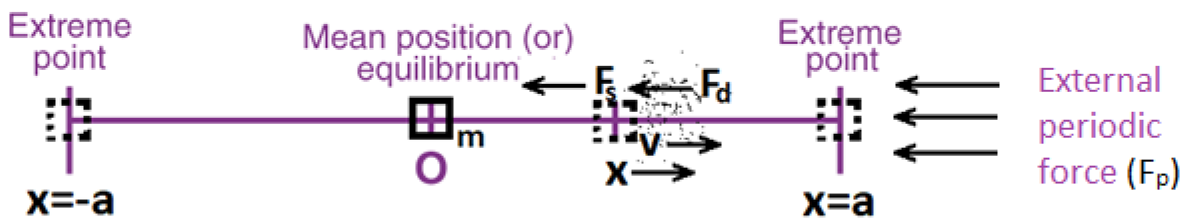

Figure 9

Consider a body of mass 'm' executing vibrations in a damping medium acted upon by an external periodic force F_p, If 'x' is the displacement of the body at any instant of time 't', then the forces acting on the body during forced oscillation.

Restoring force ($\boldsymbol{F_s}$) on mass 'm' and is directly proportional to displacement 'x' of mass,

$F_s = -kx$, where k - force constant.

Damping force ($\boldsymbol{F_d}$) on mass 'm' and is directly proportional to the velocity 'v' of the mass,

$F_d = -bv$, where b - damping constant.

External periodic force ($\boldsymbol{F_p}$) acting on the body,

$x = a \sin \omega t$

$F_p = F\, sin\, pt$

$F_p = F\, sin\, pt$, where F – maximum force and P – angular frequency of external periodic force

Then the net force acting on the oscillating body is given by

$F_{net} = F_s + F_d + F_p$

$F = -kx - bv + F\, sin\, pt$

$ma = -kx - bv + F\, sin\, pt$, (From Newton's second law of motion)

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F\, sin\, pt \qquad \text{(differential form)}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F\, sin\, pt$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}\, sin\, pt$$

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x = \frac{F}{m}\, sin\, pt \ldots (1) \qquad \frac{b}{m} = 2\beta \text{ and } \frac{k}{m} = \omega^2$$

Equation (1) is the differential equation for forced oscillation.

Solution: The general solution of forced oscillation is

$x = A\sin\ (pt - \theta)$, where x – displacement, A- amplitude, p – frequency and θ – initial phase.

Differentiate x w. r. to 't' twice

$$\frac{dx}{dt} = A\,p\cos\ (pt - \theta)$$

$$\frac{d^2x}{dt^2} = -A\,p^2\sin\ (pt - \theta)$$

Substitute above three equations in (1)

$$2 \Rightarrow -A\,p^2\sin\ (pt - \theta) + 2\beta A\,p\cos\ (pt - \theta) + \omega^2 A\sin\ (pt - \theta) = \frac{F}{m}\ sin\ pt$$

$$-A\,p^2\sin\ (pt - \theta) + 2\beta A\,p\cos\ (pt - \theta) + \omega^2 A\sin\ (pt - \theta) = \frac{F}{m}\ sin\ [(pt - \theta) + \theta]$$

$$-A\,p^2\sin\ (pt - \theta) + 2\beta A\,p\cos\ (pt - \theta) + \omega^2 A\sin\ (pt - \theta) = \frac{F}{m}\sin(pt - \theta)\cos\theta + \frac{F}{m}\cos(pt - \theta)\sin\theta$$

$$A\,(\omega^2 - p^2)\sin\ (pt - \theta) + 2\beta A\,p\cos\ (pt - \theta) = \frac{F}{m}\sin(pt - \theta)\cos\theta + \frac{F}{m}\cos(pt - \theta)\sin\theta$$

Equating the coefficients of $\sin\ (pt - \theta)$

$$A\,(\omega^2 - p^2) = \frac{F}{m}\cos\theta \ldots\ldots (2)$$

Equating the coefficients of $\cos\ (pt - \theta)$

$$2\beta A\,p = \frac{F}{m}\sin\theta \ldots\ldots (3)$$

Squaring and adding equations (2) and (3)

$$4\,\beta^2 A^2\,p^2 + \ A^2\,(\omega^2 - p^2)^2 = \frac{F^2}{m^2}$$

$$A^2[4\,\beta^2\,p^2 + \ (\omega^2 - p^2)^2] = \frac{F^2}{m^2}$$

$$A^2 = \frac{F^2/m^2}{[4\,\beta^2\,p^2 + \ (\omega^2 - p^2)^2]}$$

$$A = \frac{F/m}{\sqrt{[4\,\beta^2\,p^2 + \ (\omega^2 - p^2)^2]}} \ldots\ldots (4)$$

Dividing equation (2) by equation (1)

$$\frac{(2)}{(1)} \Rightarrow \tan\theta = \frac{2\beta\, p}{\omega^2 - p^2}$$

$$\theta = \tan^{-1}\left(\frac{2\beta\, p}{\omega^2 - p^2}\right) \dots\dots (5)$$

Equation (4) and (5) are the expression for amplitude and phase of the forced oscillation.

As per the equation (4) the frequency of oscillating body is ‘p’. But the frequency of the applied periodic force also ‘p’. Hence, it means that due to the application of external periodic force, the body adopts the frequency of the external force as its own in the steady state, no matter with what frequency it was oscillating earlier. Provided the value of F in $F\, sin\, pt$ is large enough.

Dependence of amplitude and phase on the frequency of applied periodic force.

As frequency of applied force ‘p’ is not a constant, when it is varied, following are the three broadly classified responses of the oscillating system.

(i) $\boldsymbol{p \ll \omega}$ where ω is natural frequency of the oscillating system.
For $p \ll \omega$, p^2 will be very small
ie., $\omega^2 - p^2 \approx \omega^2$ and $2\beta\, p \approx 0$

$$(4) \Rightarrow A = \frac{F/m}{\omega^2}$$

Though the system oscillates with frequency ‘p’ its amplitude depends on F/m and is not sensitive to variation in ‘p’. That is for the applied periodic force of constant amplitude ‘F’ ‘A’ will be a constant.

Phase: since p^2 is very small, $\omega^2 - p^2 \approx \omega^2$ and $\frac{2\beta\, p}{\omega^2} \approx 0$

$$(5) \Rightarrow \theta = \tan^{-1} 0 = 0$$

Since $\boldsymbol{\theta = 0}$, the displacement and applied force will be in same phase.

(ii) $\boldsymbol{p = \omega}$
For $p = \omega$, $(\omega^2 - p^2) = 0$

$$(4) \Rightarrow A = \frac{F/m}{2\beta\, p} \quad \text{or} \quad A = \frac{F/m}{2\left(\frac{b}{2m}\right)\omega} \quad \text{or } A = \frac{F}{b\,\omega}$$

Here amplitude of the oscillation is maximum. This condition is called resonance.

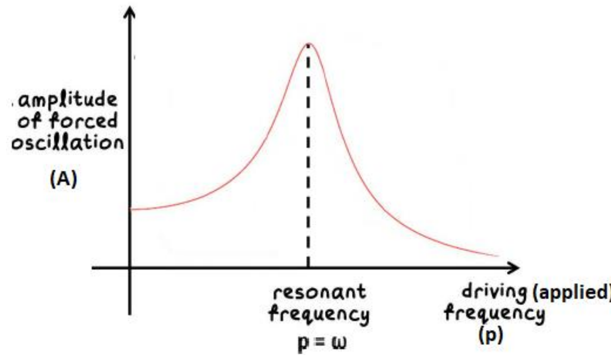

Figure 10

Resonance: When the frequency of applied periodic force is varied, if the frequency of applied force 'p' is matches with the natural frequency 'ω' of the oscillating system, then the amplitude of the oscillation will be maximum and maximum transfer of energy occurs from the driving system to oscillating system. This condition is called resonance.

Phase:

$\theta = \tan^{-1}\left(\frac{2\beta\, p}{0}\right)$ or $\theta = \tan^{-1}(\infty)$ or $\boldsymbol{\theta = \frac{\pi}{2}}$

$\theta = \frac{\pi}{2}$ indicates the displacement has a phase lag of $\frac{\pi}{2}$ with respect to phase of the applied periodic force.

(iii) $\boldsymbol{p \gg \omega}$ (This case is significance only when β is small)

For $p \gg \omega$, $(\omega^2 - p^2)^2 \approx (-p^2)^2 \approx p^4$

$A = \frac{F/m}{\sqrt{p^4}}$ or $A = \frac{F/m}{p^2}$ ($4\,\beta^2\,p^2 \approx 0$ as β is very small)

Phase:

$$\theta = \tan^{-1}\left(-\frac{2\beta\, p}{p^2}\right) = \tan^{-1}\left(-\frac{2\beta}{p}\right) = \tan^{-1}(-0) \quad \text{(Since } \beta \text{ is small } \frac{2\beta}{p} \approx 0)$$

$\boldsymbol{\theta = \pi}$

As the applied periodic frequency 'p' becomes larger, the displacement develops a phase lag that approaches the value 'π' with respect to phase of the applied periodic force.

Sharpness of resonance

The process of varying the frequency of applied periodic force to match the resonant frequency of an oscillating body is called tuning. During the tuning of oscillating system, based on the equation for resonance $p = \omega$, $A = \frac{F}{b\,\omega}$ the rate at which the amplitude varies near the resonance depends on damping constant. Smaller the damping sharper will be the resonance and larger the damping flatter will be resonance. It is as shown in the below figure.

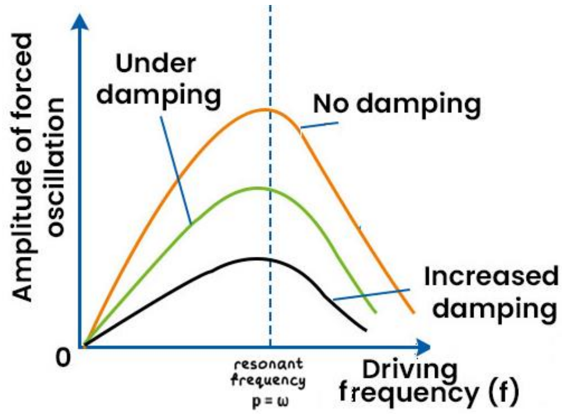
Amplitude of forced oscillation
Under damping
No damping
Increased damping
0
resonant frequency
p = ω
Driving frequency (f)

Figure 11