Module 2: Mathematical Logic

Content:

Basic Connectives and Truth Tables, Logic Equivalence – The Laws of Logic, Logical Implication Rules of Inference in Python. Fundamentals of Logic contd. The Use of Quantifiers, Quantifiers.

Logic

Logic is the basis of all mathematical reasoning, and of all automated reasoning. It provides rules and techniques for determining whether a given statement/argument is valid. Logical reasoning is used in mathematics to prove theorems.

Proposition/Statement/Argument

- A proposition is the basic building block of logic.
- It is defined as a declarative sentence (i.e. a sentence that declares a fact) that is either True or False, but not both, usually denoted by small alphabet letters p, q, r, s,…
- If the statement is true, then it is denoted by 1 or T and False, then it is denoted by 0 or F.

Examples

Propositions

1. p:This class has 30 students
2. q:4+8=12
3. r:5+3=7

Non propositions

1. p:What time is it?
2. q:Read this carefully?.
3. r: x+2=4

Statements That Are Not Propositions-

Following kinds of statements are not propositions-

1.Command
2.Question
3.Exclamation
4.Inconsistent
5.Predicate or Proposition Function

A predicate is an expression of one or more variables defined on some specific domain

Example:

Following statements are not propositions-

- Close the door. (Command)
- Do you speak French? (Question)
- What a beautiful picture! (Exclamation)
- I always tell lie. (Inconsistent)
- $P(x) : x + 3 = 5$ (Predicate)

Identify which of the following statements are propositions-

1. France is a country.
2. 2020 will be a leap year.
3. Sun rises in the west.
4. $P(x) : x + 6 = 7$
5. $P(5) : 5 + 6 = 2$
6. Apples are oranges.
7. Grapes are black.
8. Two and two makes 4.
9. $x > 10$

1. Proposition (True)
2. Proposition (True)
3. Proposition (False)
4. Not a proposition (Predicate)
5. Proposition (False)
6. Proposition (False)
7. Proposition (False)
8. Proposition (True)
9. Not a proposition (Predicate)

Applications of Propositional Logic

- Propositional Logic plays an important role in computer science as well as in a person's daily life.
- The main benefits of studying and using propositional logic are that it prevents us from making inconsistent inferences and incautious decisions.
- It incorporates reasoning and thinking abilities in one's daily life.
- In the computer science field, propositional logic has a wide variety of applications and hence is very important. Some of the applications are as follows:
  - Circuit designing,
  - System specifications,
  - Logical puzzles,
  - Artificial Intelligence – Fuzzy Logic
  - Boolean searches
  - Inference and Decision making
  - Translating English sentences to mathematical statements and vice-versa, etc.

Logical connectives

A Logical connective is a symbol which is used to connect two or more propositions/statements.

Generally there are five connectives which are −

- OR ($\vee$)
- AND ($\wedge$)
- Negation/ NOT ($\neg$)
- Implication / if-then ($\rightarrow$)
- If and only if ($\Leftrightarrow$).

Simple Statement

The statement which is free from logical connectives is called simple statements.

Eg:

i. p:It is raining

ii. q:It is cold

Compound Statement

Any statement which is obtained by combining two or more statements by logical connectives is called compound statement/ proposition.

Eg:

p:It is raining

q:It is cold

$p \wedge q$: It is raining and it is cold

Truth Table

A truth table shows how the truth or falsity of a compound statement depends on the truth or falsity of the simple statements from which it's constructed.

Negation/ NOT (¬)

- If p be any proposition, then negation of p is read as $not\ p$ and it is denoted by $\neg p$ or $\sim p$.
- If the truth value of p is 1 then truth value of $not\ p$ is 0 & vice versa.

Eg:

Consider the proposition "It is raining today". Then, the negation of the given statement is "It is not raining today". Thus, if the given statement is true, then the negation of the given statement is false.

Truth table: p: It is raining today & $\neg p$: It is not raining today

p	$\neg p$
0	1
1	0

Conjunction/ AND [∧]

- Let p and q be simple propositions, then conjunction of p & q is read as $p\ and\ q$ and is denoted by $p \wedge q$. The truth value of $p \wedge q$ is 1 if both p & q values are 1, otherwise value of $p \wedge q$ is 0.

Eg: p: Today is Friday & q : It is raining today

$p \wedge q$: Today is Friday and it is raining today

- This proposition is true only on rainy Fridays and is false on any other rainy day or on Fridays when it does not rain

Truth table:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Disjunction/ OR [∨]

Let p and q be simple propositions, then disjunction of p & q is read as p *or* q and is denoted by $p \vee q$. The truth value of $p \vee q$ is 1 if any one of p and q is 1 but 0 if both p & q are 0.

Eg: p: Today is Friday & q : It is raining today

$p \vee q$: Today is Friday or it is raining today

This proposition is true on any day that is a Friday or a rainy day(including rainy Fridays) and is false on any day other than Friday when it also does not rain.

Truth table:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Disjunction/ OR [$\vee$]

Let p and q be simple propositions, then disjunction of p & q is read as p *or* q and is denoted by $p \vee q$. The truth value of $p \vee q$ is 1 if any one of p and q is 1 but 0 if both p & q are 0.

Eg: p: Today is Friday & q : It is raining today

$p \vee q$: Today is Friday or it is raining today

This proposition is true on any day that is a Friday or a rainy day(including rainy Fridays) and is false on any day other than Friday when it also does not rain.

Truth table:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Exclusive OR / Exclusive Disjunction [Either or but not both]

Let p and q be simple propositions, then exclusive disjunction of p & q is read as $either\ p\ or\ q\ but\ not\ both$ and is denoted by $p \veebar q$. The truth value of $p \veebar q$ is 1 if any one of value of p & q is different but 0 if both p & q are same.

Eg: p: Today is Friday & q : It is raining today

$p \veebar q$: Either Today is Friday or it is raining today, but not the both

Truth table

p	q	$p \veebar q$
0	0	0
0	1	1
1	0	1
1	1	0

This proposition is true on any day that is a Friday or a rainy day(not including rainy Fridays) and is false on any day other than Friday when it does not rain or rainy Fridays.

Implication /Conditional [If, then (→)]:

Let p and q be simple propositions, then conditional of p & q is read as $If\ p, then\ q$ and is denoted by $p \rightarrow q$. The truth value of $p \rightarrow q$ is 0 if p is 1 & q is 0 and otherwise it is 1.

Eg: p: Today is Friday & q : It is raining today

$p \rightarrow q$: If it is Friday then it is raining today

The above proposition is true if it is not Friday(premise is false) or if it is Friday and it is raining, and it is false when it is Friday but it is not raining.

Truth table:

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Bi-implication /Biconditional [Iff ($\leftrightarrow$)]:

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Let p and q be simple proposition, then biconditional of p & q is read as $p\ iff\ q$ [p if and only if q] and is denoted by $p \leftrightarrow q$.The truth value of $p \leftrightarrow q$ is 1 only when both p & q are same otherwise 0.

Eg: p: Today is Friday & q : It is raining today

$p \leftrightarrow q$: It is raining today if and only if it is Friday today

The above proposition is true if it is not Friday and it is not raining or if it is Friday and it is raining, and it is false when it is not Friday or it is not raining.

Note: The biconditional statement is also read as $If\ p\ then\ q\ and\ if\ q\ then\ p.$

Truth table:

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Precedence of Logical Operators

Precedence of logical operators helps to decide which operator will get evaluated first in a complicated compound proposition.

- We can use parenthesis to specify the order in which logical operators in the compound statement are to be applied.
- To reduce the number of parenthesis, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Eg:

- $\neg p \vee q = (\neg p) \vee q$ students
- $p \vee q \wedge r = p \vee (q \wedge r)$
- $p \wedge q \vee r = (p \wedge q) \vee r$

Syntax

- In linguistics, "syntax" refers to the rules that govern the ways in which words combine to form phrases, clauses, and sentences. The term "syntax" comes from the Greek, meaning "arrange together."
- In computer contexts, the term refers to the proper ordering of symbols and codes so that the computer can understand what instructions are telling it to do.
- The syntax consists of a set of basic items (atomic propositions denoted by symbols: *p*, *q*, *r*, *s*,…; logical connectives and parenthesis) and a set of construction rules (i.e., syntactic rules) which determine the well-formed formula (WFF) in our logical language.

Syntactical Rules

The syntax of propositional logic is easy to learn. It has only three to four rules:

- Any atomic proposition (*p*, *q*, *r*, *s*, *t*..) is itself a well-formed formula (WFF)
- If *p* and *q* are WFF, then the following are also WFFs

  $p \vee q$; $p \wedge q$; $p \rightarrow q$; $p \leftrightarrow q$

- If *p* is a WFF, then $\neg p$ is also a WFF

These rules are also **recursive**. Any WFF determined by these rules can be used in these rules

Examples of WFF

a) p
b) $\neg\neg p$
c) $(p \wedge p)$
d) $\neg(p \vee q)$
e) $(\neg(p \vee q) \wedge p)$
f) $\neg((p \vee q) \wedge p)$
g) $\neg(p \leftrightarrow (r \vee s))$
h) $(p \leftrightarrow \neg(r \vee s))$
i) $((p \wedge q) \vee (s \wedge r))$

Examples of Non WFF

a) pqr
b) $(p$
c) $p\neg$
d) $\vee q$
e) $(\neg p \leftrightarrow r \vee s)$
f) $\rightarrow \vee \wedge$

Tautology [T_0]:

A compound proposition which is true [1] for all possible truth values of its components [Primitive] is called **tautology.**

Contradiction / Absurdity [F_0]:

A compound proposition which is false [0] for all possible truth values of its components is called **contradiction.**

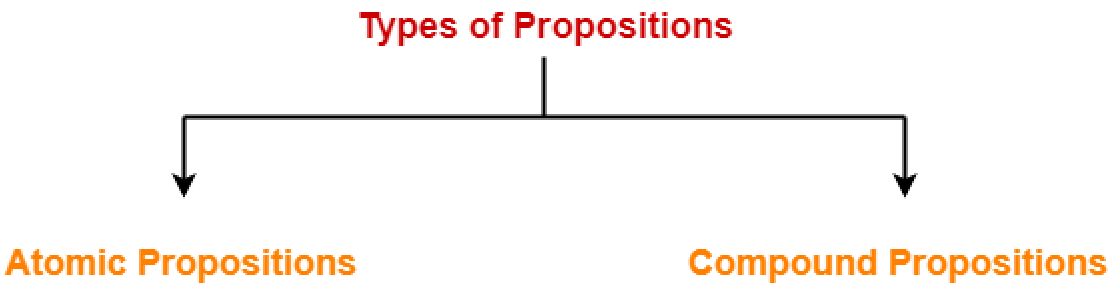
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Contingency Tautology Contradiction

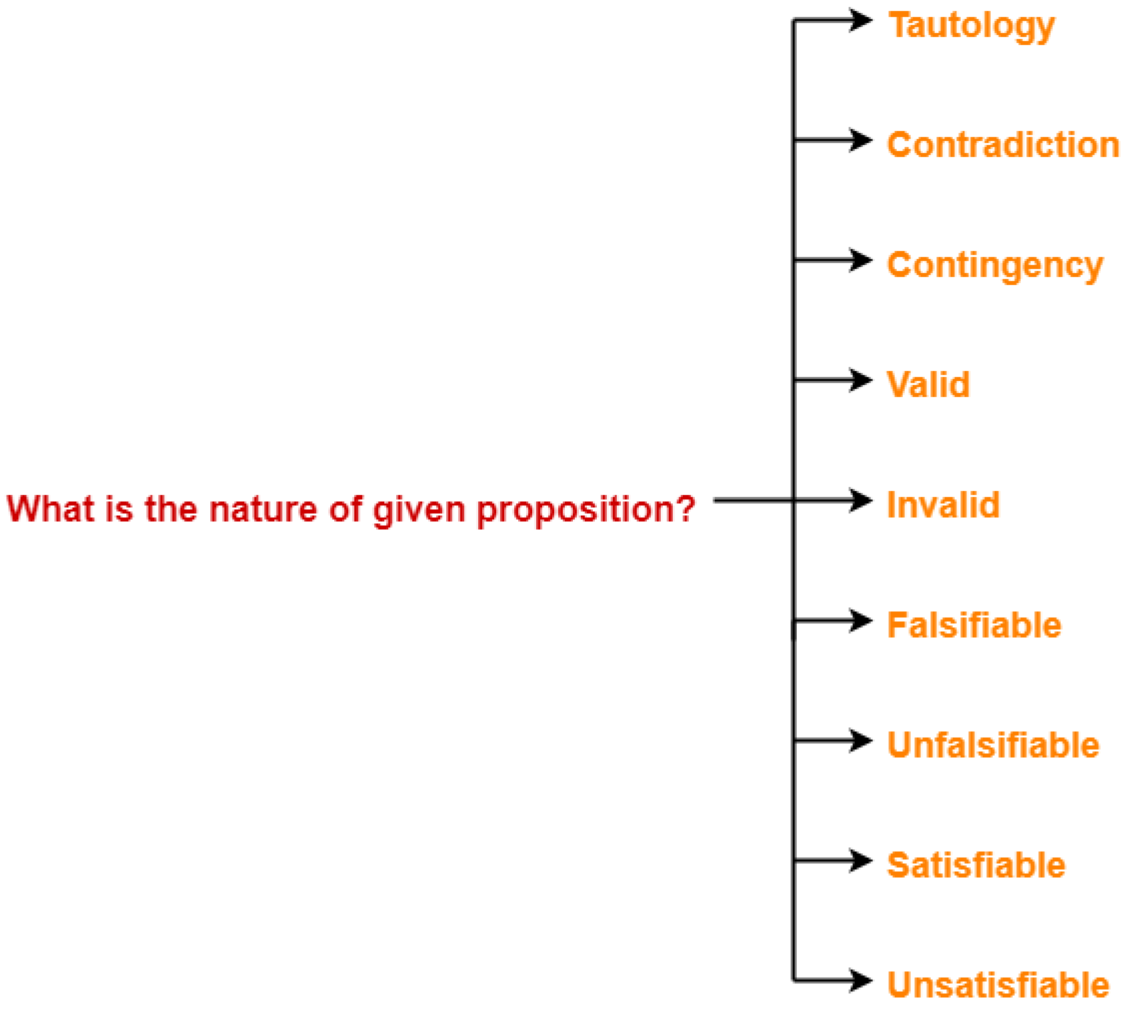
Contingency:

A compound proposition that can be true or false i.e., neither tautology nor contradiction depending upon the truth values of its compound is called **contingency.**

Types of Propositions



Nature of Propositions



Validity/ Valid:

- A compound proposition is called **valid** if and only if it is a tautology.
- It contains only T (Truth) in last column of its truth table.

Invalid:

- A compound proposition is called **invalid** if and only if it is not a tautology.
- It contains either only F (False) or both T (Truth) and F (False) in last column of its truth table.

Falsifiable:

- A compound proposition is called **falsifiable** if and only if it can be made false for some value of its propositional variables.
- It contains either only F (False) or both T (Truth) and F (False) in last column of its truth table.

Unfalsifiable:

- A compound proposition is called **unfalsifiable** if and only if it can never be made false for any value of its propositional variables.
- It contains only T (Truth) in last column of its truth table.

Satisfiable:

- A compound proposition is called **satisfiable** if and only if it can be made true for some value of its propositional variables.
- It contains either only T (Truth) or both T (True) and F (False) in last column of its truth table.

Unsatisfiable:

- A compound proposition is called **unsatisfiable** if and only if it can not be made true for any value of its propositional variables.
- It contains only F (False) in last column of its truth table.

Note

- All contradictions are invalid and falsifiable but not vice-versa.
- All contingencies are invalid and falsifiable but not vice-versa.
- All tautologies are valid and unfalsifiable and vice-versa.
- All tautologies are satisfiable but not vice-versa.
- All contingencies are satisfiable but not vice-versa.
- All contradictions are unsatisfiable and vice-versa.

Examples:

Determine the nature of following propositions-

a) $p \wedge \sim p$
b) $(p \wedge (p \rightarrow q)) \rightarrow \sim q$
c) $[\,(p \rightarrow q) \wedge (q \rightarrow r)\,] \wedge (\,p \wedge \sim r)$ [Practice]
d) $\sim(p \rightarrow q) \vee (\sim p \vee (p \wedge q))$ [Practice]
e) $(p \leftrightarrow r) \rightarrow (\sim q \rightarrow (p \wedge r))$ [Practice]

Solution:

a) We can use different techniques to find the nature such as Truth tables, Laws of logic and digital electronics. As of now we proceed with Truth tables technique

p	$\sim p$	$p \wedge \sim p$
0	1	0
1	0	0

Clearly, last column of the truth table contains only F. Therefore, given proposition is-

- Contradiction
- Invalid
- Falsifiable
- Unsatisfiable

b) $(p \wedge (p \rightarrow q)) \rightarrow \sim q$

Solution:

We proceed with Truth tables technique to determine the nature

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$\sim q$	$(p \wedge (p \rightarrow q)) \rightarrow \sim q$
0	0	1	0	1	1
0	1	1	0	0	1
1	0	0	0	1	1
1	1	1	1	0	0

Clearly, last column of the truth table contains both T and F.

Therefore, given proposition is-

- Contingency
- Invalid
- Falsifiable
- Satisfiable

Logical Equivalence

Let u & v be two compound propositions, which are said to be logically equivalent whenever u & v have same truth value. It is denoted by $u \equiv v$. In other words its biconditional is a tautology. It is denoted by $u \Leftrightarrow v$.

Example1: Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$	$p \rightarrow q \Leftrightarrow \neg p \vee q$
1	1	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

Since $p \rightarrow q$ and $\neg p \vee q$ are having same truth values, thus both are logically equivalent.
In another way, its biconditional is a tautology. Thus, both are logically equivalent.

Example 2: Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent. [Practice].

Example 3: Show that $\neg(p \vee q)$ and $(\neg p \wedge \neg q)$ are logically equivalent. [Practice].

Example 4: Show that $p \underline{\vee} q$ and $(p \vee q) \wedge (\neg p \vee \neg q)$ are logically equivalent. [Practice].

Example 5: Show that $[(p \vee q) \rightarrow r]$ and $[(p \rightarrow r) \wedge (q \rightarrow r)]$ are logically equivalent. [Practice].

or

Prove that for any proposition $[(p \vee q) \rightarrow r] \Leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$

Example 5: Prove that $[(p \vee q) \rightarrow r] \Leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$[Practice].

Duality

Let u be a compound proposition involving the connectives $\wedge$ & $\vee$ a new proposition is obtained by replacing

i. $\wedge$ & $\vee$ by $\vee$ & $\wedge$

ii. T_0 & F_0 by F_0 & T_0 is called dual of given proposition.

NAND & NOR

Let p & q be any two propositions, the compound proposition $not(p \ and \ q)$ is called **NAND** connective & is denoted by $p \uparrow q$. It is given by $p \uparrow q \Leftrightarrow \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$.

Let p & q be any two propositions, the compound proposition $not(p \ or \ q)$ is called **NOR** connective & is denoted by $p \downarrow q$. It is given by $p \downarrow q \Leftrightarrow \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\neg p}$	$\boldsymbol{\neg q}$	$\boldsymbol{p \uparrow q \Leftrightarrow \neg p \vee \neg q.}$	$p \downarrow q \Leftrightarrow \neg p \wedge \neg q$
0	0	1	1	1	1
0	1	1	0	1	0
1	0	0	1	1	0
1	1	0	0	0	0

Solved Examples

1. *Write negation of the following statement.*

    i. $p = 2 + 3 > 1.$

    ii. *It will rain tomorrow, or it will snow tomorrow.*

    iii. p: *It is cold.*

Solution:

i. $\sim p = 2 + 3 \leq 1.$

ii. *It will not rain tomorrow, and it will not snow tomorrow.*

iii. $\sim p$: *It is not cold*

2. *Write conjunction and disjunction of the following statement.*

$$p: num > 10; \quad q: num \leq 15$$

Solution:

Conjunction: $\boldsymbol{p} \wedge \boldsymbol{q}$: $num > 10 \text{ and } num \leq 15.$

disjunction: $\boldsymbol{p} \vee \boldsymbol{q}$: $num > 10 \text{ or } num \leq 15.$

Solved Examples

3. Construct the truth table for the following compound Propositions

$a)\ \ p \vee \neg q$

p	q	$\neg q$	$p \vee \neg q$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	1

b) $p \rightarrow \neg q$

p	q	$\neg q$	$p \rightarrow \neg q$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	0	0

c) $[(p \wedge q) \vee \neg r] \leftrightarrow p$

p	q	r	$p \wedge q$	$\neg r$	(a) $(p \wedge q) \vee (\neg r)$	$a \leftrightarrow p$
0	0	0	0	1	1	0
0	0	1	0	0	0	1
0	1	0	0	1	1	0
0	1	1	0	0	0	1
1	0	0	0	1	1	1
1	0	1	0	0	0	0
1	1	0	1	1	1	1
1	1	1	1	0	1	1

d) $\neg p \veebar \neg q$ [Practice]

e) $(p \vee q) \wedge r$[Practice]

f) $q \wedge [\neg r \rightarrow p]$ [Practice]

4. Show that for any propositions p & q, the compound proposition $p \rightarrow (p \vee q)$ is a tautology and the compound proposition $p \wedge (\neg p \wedge q)$is a contradiction.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\neg p}$	$\boldsymbol{p \vee q}$	$\boldsymbol{\neg p \wedge q}$	$\boldsymbol{p \rightarrow (p \vee q)}$	$\boldsymbol{p \wedge (\neg p \wedge q)}$
0	0	1	0	0	1	0
0	1	1	1	1	1	0
1	0	0	1	0	1	0
1	1	0	1	0	1	0

$\therefore$ $\forall$ possible truth value of compound proposition $p \rightarrow (p \vee q)$is tautology & $p \wedge (\neg p \wedge q)$ is a contradiction.

5. Show that the following compound propositions are tautology

i. $\{p \to (q \to r)\} \to \{(p \to q) \to (p \to r)\}$[Practice]

ii. $\{(p \vee q) \wedge [(p \to r) \wedge (q \to r)]\} \to r$

p	q	r	(a) $p \vee q$	(b) $p \to r$	(c) $q \to r$	(d) $b \wedge c$	(e) $a \wedge d$	$e \to r$
0	0	0	0	1	1	1	0	1
0	0	1	0	1	1	1	0	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

6. Determine the true or false for the following statement.

i. $2 < 3\ and\ 3\ is\ a + ve\ integer.$

ii. $2 \geq 3\ and\ 3\ is\ not\ a + ve\ interger.$

iii. $2 < 3\ and\ 3\ is\ not\ a + ve\ integer[Practice].$

Solution:

i. $p: 2 < 3.$

$q: 3\ is + ve\ inetger.$

$p \wedge q = T \wedge T = T$

Hence given compound proposition is true.

ii. $p: 2 \geq 3$

$q: 3\ is\ not\ a\ + ve\ integer$

$p \wedge q = F \wedge F = F\ .$

Hence given compound proposition is false.

7. Find the truth vale of each proposition if p & r is 1 and q is 0.

i. $\sim p \wedge \sim q$.

ii. $(\sim p \vee q) \wedge r$.

iii.$p \vee q \vee r$.

iv.$\sim(p \vee q) \wedge r$.

v. $\neg p \wedge (q \vee r)$.

vi.$p \wedge [\neg(q \vee \neg r)]$.[homework]

vii.$(r \wedge \neg q) \vee (p \vee r)$. [homework]

viii.$\neg(p \wedge q) \vee \neg(q \leftrightarrow p)$.

ix.$(p \to q) \vee \neg(p \leftrightarrow q)$.

Solution:

Given $p = 1, q = 0, r = 1$. Thus $\sim p = 0, \sim q = 1, \sim r = 0$.

i. $\sim p \wedge \sim q = 0 \wedge 1 = 0$.

ii. $(\sim p \vee q) \wedge r = (0 \vee 0) \wedge 1 = 0 \wedge 1 = 0$.

iii. $p \vee q \vee r = 1 \vee 0 \vee 1 = 1 \vee 1 = 1$.

iv. $\sim(p \vee q) \wedge r = \sim(1 \vee 0) \wedge r = \sim(1) \wedge 1 = 0 \wedge 1 = 0$.

v. $\neg p \wedge (q \vee r) = 0 \wedge (0 \vee 1) = 0 \wedge 1 = 0$.

vi. $p \wedge [\neg(q \vee \neg r)] = 1$.

vii. $(r \wedge \neg q) \vee (p \vee r) = 1$.

viii. $\neg(p \wedge q) \vee \neg(q \leftrightarrow p) = \neg(1 \wedge 0) \vee \neg(0 \leftrightarrow 1) = \neg(0) \vee \neg(0) = 1$.

ix. $(p \rightarrow q) \vee \neg(p \leftrightarrow \neg q) = (1 \rightarrow 0) \vee \neg(1 \leftrightarrow 1) = 0 \vee \neg(1) = 0 \vee 0 = 0$.

8. *Indicate how many rows are needed in the truth table for the compound proposition* $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \rightarrow t]$. *Find the truth value of proposition if* p & r *are true and* $q, s,$ & t *are false.*

Solution:

The give compound statement contains $n = 5$ propositions.

$\therefore$ the number rows required in the truth table is $2^n = 2^5 = 32$.

Consider $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \rightarrow t] =$

$$= (1 \vee 1) \leftrightarrow [(0 \wedge 0) \rightarrow 0]$$

$$= 1 \leftrightarrow [0 \rightarrow 0]$$

$$= 1 \leftrightarrow 1$$

$$= 1$$

9. *If a proposition q has truth value* 1 *determine all truth value for primitive* $p, r, \&\ s$ *for which the truth value of the following compound proposition is 1*

$$\{q \to [(\neg p \vee r) \wedge \sim s]\} \wedge \{\neg s \to (\neg r \wedge q)\}.$$

Solution: Given truth value of the compound proposition is 1.

Thus, the truth value of

$\Rightarrow q \to \{(\neg p \vee r) \wedge \sim s\}$ is 1 $-------(1)$

And $\sim s \to (\sim r \wedge q)$ is also 1 $-------(2)$

Consider (1) and

since it is true also given that q is also 1

$(\neg p \vee r) \wedge \neg s$ must be 1.

$\therefore \neg p \vee r$ is 1 and $\neg s$ is 1. Hence s is 0

Consider (2)

and its truth value is 1 and $\neg s$ is 1 given q is 1

$\therefore \neg r \wedge q$ must be 1. Hence $r = 0$

But $\neg p \vee r$ is 1 and r is 0

$\therefore \neg p$ must be 1 and hence p is 0.

Logical Equivalences: Laws of Logic

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<u>The Laws of logic:</u>

Let p, q & r be any proposition, T_0 denotes tautology and F_0 denotes contradiction

Law of double negation:

$$\neg\neg p \Leftrightarrow p$$

Idempotent law:

$$p \vee p \Leftrightarrow p$$

$$p \wedge p \Leftrightarrow p$$

Identity law:

$$p \vee F_0 \Leftrightarrow p$$

$$p \wedge T_0 \Leftrightarrow p$$

Inverse Law:

$$p \vee \neg p \Leftrightarrow T_0$$

$$p \wedge \neg p \Leftrightarrow F_0$$

Domination Laws:

$$p \vee T_0 \Leftrightarrow T_0$$

$$p \wedge F_0 \Leftrightarrow F_0$$

Commutative law:

$$p \vee q \Leftrightarrow q \vee p$$

$$p \wedge q \Leftrightarrow q \wedge p$$

Associate law:

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$$

$$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

Distributive law:

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

DeMorgan's law:

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

Absorption law:

$$[p \vee (p \wedge q)] \Leftrightarrow p$$

$$[p \wedge (p \vee q)] \Leftrightarrow p$$

Law of negation of conditional:

$$\neg(p \rightarrow q) \Leftrightarrow p \wedge \sim q$$

$$p \rightarrow q \Leftrightarrow \sim p \vee q$$

Note: By using these laws, we can prove two propositions are logical equivalent.

Logical Equivalences Involving Conditional Statements

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical Equivalences Involving Biconditional Statements

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Solved Examples:

Ex-1: Prove following logical equivalence without using Truth table.

a. $p \vee [\, p \wedge (p \vee q)] \Leftrightarrow p$

b. $[p \vee q \vee (\neg\, p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

Solution:

a. $p \vee [\, p \wedge (p \vee q)] \Leftrightarrow p$

$\Rightarrow p \vee [\, p \wedge (p \vee q)] \equiv p \vee [(p \wedge p) \vee (p \wedge q)]$ -------- Distributive law

$\Rightarrow\ p \vee [p \vee (p \wedge q)]$ -------- Idempotent Law

$\Rightarrow\ p \vee p$ -------- Absorption Law

p -------- Idempotent Law

b. $[p \vee q \vee (\neg\, p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

$\Rightarrow\ [p \vee q \vee (\neg\, p \wedge \neg q \wedge r)] \equiv \{p \vee q \vee [\neg\,(p \vee q) \wedge r\}$ de Morgan's Laws

$\Rightarrow\ \equiv \{(p \vee q) \vee [\neg\,(p \vee q) \wedge r\}$ Associative Laws

$\Rightarrow\ \equiv [(p \vee q) \vee \neg\,(p \vee q)] \wedge [(p \vee q) \vee r]$ Distributive law

$\Rightarrow\ \equiv [T \wedge [(p \vee q) \vee r]$

$\Rightarrow\ \equiv [(p \vee q) \vee r]$

Ex-2: Prove following logical equivalence without using Truth table.

$[(p \lor q) \land (p \lor \neg q)] \lor q \Leftrightarrow p \lor q$

Solution: $[(p \lor q) \land (p \lor \neg q)] \lor q \equiv [p \lor (q \land p) \lor (q \land \neg q)] \lor q$ **Distributive**

$\Rightarrow \equiv [p \lor (q \land \neg q)] \lor q$ **Commutative**

$\Rightarrow \equiv [p \lor (F)] \lor q$ **inverse law**

$\Rightarrow \equiv p \lor q$ **Absorption Law and identity law (pVF)=p**

Ex-3: Prove by using laws of logic. $(p \to q) \land [\neg q \land (r \lor \neg q)] \Leftrightarrow \neg(q \lor p)$

Solution: $(p \to q) \land [\neg q \land (r \lor \neg q)] \Leftrightarrow (p \to q) \land [\neg q \land (\neg q \lor r)$ **(Commutative law)**

$\Leftrightarrow (\neg p \lor q) \land \neg q$ **(Absorption law)**

$\Leftrightarrow \neg q \land (\neg p \lor q)$ **(Commutative law)**

$\Leftrightarrow (\neg q \land \neg p) \lor (\neg q \land q)$ **(Distributive law)**

$\Leftrightarrow (\neg q \land \neg p) \lor F$ **($\because u \land \neg u \Leftrightarrow F$)**

$\Leftrightarrow (\neg q \land \neg p)$ **($\because u \lor F \Leftrightarrow u$)**

$\Leftrightarrow \neg(q \lor p)$ **(De Morgan's law)**

Ex-4: Prove without using Truth table. $[\sim p \wedge (\sim q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$

Solution $[\sim p \wedge (\sim q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)]$

$\Leftrightarrow [(\sim p \wedge \sim q) \wedge r)] \vee [(r \wedge q) \vee (r \wedge p)]$ (commutative)

$\Leftrightarrow [(\sim p \wedge \sim q) \wedge r)] \vee [(r \wedge (q \vee p)]$ (Distributive)

$\Leftrightarrow [(\sim (p \vee q) \wedge r)] \vee [(r \wedge (p \vee q)]$ (De Morgan's and commutative)

$\Leftrightarrow [r \wedge \sim (p \vee q)] \vee [r \wedge (p \vee q)]$ (commutative)

$\Leftrightarrow [r \wedge \{\sim (p \vee q) \vee (p \vee q)\}]$ (Distributive)

$\Leftrightarrow r \wedge T$ $(\because \sim u \vee u \Leftrightarrow T.$

Here $u : (p \vee q) \Leftrightarrow r$ $(\because v \wedge T \Leftrightarrow v)$

Ex-5: Prove that $(p \vee q) \wedge ((p \wedge (p \wedge q)) \Leftrightarrow (p \wedge q)$, by using rules of Logic.(Home work)

Ex-6: Prove the following using laws of logic : $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

Solution $p \rightarrow (q \rightarrow r) \Leftrightarrow p \rightarrow (\neg q \vee r)$ (Equivalence of Conditional)

$\Leftrightarrow \neg p \vee (\neg q \vee r)$ (Equivalence of Conditional)

$\Leftrightarrow (\neg p \vee \neg q) \vee r$ (Associative law)

$\Leftrightarrow \neg(p \wedge q) \vee r$ (De Morgan's law)

$\Leftrightarrow (p \wedge q) \rightarrow r$ (Equivalence of Conditional

Ex-7: Simplify $\neg [\neg\{(p \vee q) \wedge r\} \vee \neg q]$

Solution $\neg [\neg\{(p \vee q) \wedge r\} \vee \neg q]$ $\equiv [\neg\neg\{(p \vee q) \wedge r\} \vee \neg\neg q]$ using Dem organ Law

$\equiv \{(p \vee q) \wedge r)\} \wedge q$, using the Law of double negation

$\equiv \{q \wedge (p \vee q)\} \wedge r$, using Associative law

$\equiv \{q \wedge (q \vee p)\} \wedge r$, Commutative law

$\equiv q \wedge r$, using Absorption law

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Ex-8: Prove that $(\neg p \land q) \lor \neg (p \lor q) \Leftrightarrow \neg p$.

Solution : $(\neg p \land q) \lor \neg(p \lor q) \Leftrightarrow (\neg p \land q) \lor (\neg p \land \neg q)$ (DeMorgan0 slaw)

$\Leftrightarrow \neg p \land (q \lor \neg q)$ (Distributive law)

$\Leftrightarrow \neg p \land T0$ (Inverse law)

$\Leftrightarrow \neg p$ (Identity law)

Ex-9: Prove that $[(\neg p \lor \neg q) \rightarrow (p \land q \land r) \Leftrightarrow p \land q$

Solution: $(\neg p \lor \neg q) \rightarrow (p \land q \land r) \Leftrightarrow \neg(\neg p \lor \neg q) \lor (p \land q \land r)$, (Equivalence of conditional)

$\Leftrightarrow (\neg\neg p \lor \neg\neg q) \lor (p \land q \land r)$, (De Morgans law)

$\Leftrightarrow (p \land q) \lor [(p \land q) \land r]$, Double Negation and Associative Law

$\Leftrightarrow p \land q$, by Absorption Law

Ex-10: Using Laws of Logic Prove that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.

Solution: $\neg(p \lor (\neg p \land q)) \Leftrightarrow \neg p \land \neg(\neg p \land q)$, (by the De Morgan law)

$\Leftrightarrow \neg p \land [\neg(\neg p) \lor \neg q]$ (De Morgan law)

$\Leftrightarrow \neg p \land (p \lor \neg q)$ (double negation law)

$\Leftrightarrow (\neg p \land p) \lor (\neg p \land \neg q)$ (distributive law)

$\Leftrightarrow F0 \lor (\neg p \land \neg q)$ $(\because \neg p \land p \Leftrightarrow F0) \Leftrightarrow (\neg p \land \neg q)$ $(\because F0 \lor u \Leftrightarrow u)$

Ex-11: Establish the following logical equivalence. $(p \vee q) \vee (\neg p \wedge \neg q \wedge r) \Leftrightarrow (p \vee q \vee r)$

Solution:

$(p \vee q) \vee (\neg p \wedge \neg q \wedge r) \Leftrightarrow (p \vee q) \vee (\neg p \wedge \neg q) \wedge r,$ by Associative Law

$\Leftrightarrow (p \vee q) \vee [\neg(p \vee q) \wedge r]$ by De Morgan Law.

$\Leftrightarrow [(p \vee q) \vee \neg(p \vee q)] \wedge [(p \vee q) \vee r]$ (Distributive)

$\Leftrightarrow T0 \wedge [(p \vee q) \vee r]$ ($\because u \vee \neg u \Leftrightarrow T$, Here, $u : (p \vee q)$)

$\Leftrightarrow (p \vee q) \vee r$ ($\because T \wedge v \Leftrightarrow v$, Here $v : (p \vee q) \vee r$)

Ex-12: Prove that $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Solution:

$(p \rightarrow r) \vee (q \rightarrow r) \Leftrightarrow (\neg p \vee r) \vee (\neg q \vee r)$ (Equivalence of conditional)

$\Leftrightarrow (\neg p \vee \neg q) \vee (r \vee r)$ (Associative)

$\Leftrightarrow (\neg p \vee \neg q) \vee r$ ($\because r \vee r \Leftrightarrow r$)

$\Leftrightarrow \neg(p \wedge q) \vee r$ (De Morgan’s law)

$\Leftrightarrow (p \wedge q) \rightarrow r$ (Equivalence of conditional

Converse, Inverse and Contrapositive; Logical implication

Consider a conditional $p \to q$ then:

- $q \to p$ is called the converse of $p \to q$
- $\neg p \to \neg q$ is called the inverse of $p \to q$
- $\neg q \to \neg p$ is called the contrapositive of $p \to q$.

For Ex, Let
p: 2 is an integer
q: 9 is a multiple of 3.
Then,
$p \to q$: if 2 is an integer, then 9 is a multiple of 3

Converse of this condition is
$q \to p$: If 9 is a multiple of 3, then 2 is an integer

inverse of this condition is
$\neg p \to \neg q$: if 2 is not an integer, then 9 is not a multiple of 3

contrapositive of this condition is
$\neg q \to \neg p$: If 9 is not a multiple of 3, then 2 is not an integer

Truth Table for Converse, Inverse and Contrapositive

p	q	¬p	¬q	p → q	q → p	¬p → ¬q	¬q → ¬p
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

$p \to q$ and $\neg q \to \neg p$ have same truth values
Also $q \to p$ and $\neg p \to \neg q$ have same truth values in all possible situation. And we have following two important results:

1. $p \to q \Leftrightarrow \neg q \to \neg p$ (Conditional and its contrapositive are logically equivalent)
2. $q \to p \Leftrightarrow \neg p \to \neg q$ (Converse and the inverse are logically equivalent)

Logical implication

let say, p: 6 is a multiple of 2 , q: 3 is a prime number, and we get conditional p → q : if 6 is a multiple of 2, then 3 is a prime number

p is true q is true

hence p → q is true But question is does this conditional p → q make any sense ?

The Answer is NO

Because there is no consistency in the statement p → q
(all tough it is logically true!

Ex- consider the proposition p: 4 is a odd number, q: Bangalore is not in Karnataka

p is False q is False

But p → q is true: if 4 is odd no, then Bangalore is not in Karnataka is true

which is logically True but makes no sense!

We do not deal with conditional as stated above,
Our interest lies in conditional p → q where p and q are related in some way so that truth values of q depends upon truth values of p or vice-versa. Such Condition are called hypothetical (implicative) statements

When hypothetical statement p → q is such that q is true whenever p is true, we say that p (logically) implies q.
Symbolically written as p => q

Q-1- Write down the contrapositive of $[p \to (q \to r)]$ with

a) Only one occurrence of the connective $\to$

b) No occurrence of connective $\to$

Sol: contrapositive of $[p \to (q \to r)]$ is $[\sim(q \to r) \to (\sim p)]$

$$[\sim(q \to r) \to (\sim p)] \Leftrightarrow \sim [\sim(q \to r)] \vee \sim p$$

$$\Leftrightarrow (q \to r) \vee \sim p \text{ ------(a)}$$

$$\Leftrightarrow (\sim q \vee r) \vee \sim p \text{ -----(b)}$$

Q-2Write inverse, converse and contrapositive of “If you do your homework, you will not be punished

Sol: The inverse of the given statement is $(\neg p \to \neg q)$: “If you do not do your homework, you will be punished.”

Converse $(q \to p)$: “If you will not be punished, then you do your homework”.

contrapositive $(\neg q \to \neg p)$: “If you will be punished, then you do not your homework”

Q-3 Replace the following statement with its contrapositive: ”If x and y are rational, then x + y is rational.”

Solution : p : x is rational,

q : y is rational,

r: x + y is rational.

This statement is in the form $(p \wedge q) \to r$.

Its contrapositive statement is $(\neg q \to \neg p)$: $\sim r \to \sim (p \wedge q)$

Using De Morgans law, this can be written as $\sim r \to (\sim p \vee \sim q)$

Hence contrapositive is the statement: ” If x + y is irrational, then either x is irrational or y is irrational

Q-4: Write converse, inverse and contrapositive of
(1) If a quadrilateral is a parallelogram, then its diagonals bisect each other.
(2) If a real number x^2 is greater than zero, then x is not equal to zero.
(3) If a triangle is not isosceles, then it is not equilateral. (4) If two lines are parallel, then they are equidistant.

Solution :

(1) converse : If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.
inverse : If a quadrilateral is not a parallelogram, then its diagonals do not bisect each other.
contrapositive : If the diagonals of a quadrilateral do not bisect each other, then it is not a parallelogram.

(2) Converse : If a real number x is not equal to zero, then x^2 is greater than zero.
inverse : If a real number x^2 is not greater than zero, then x is equal to zero.
contrapositive : If a real number x is equal to zero, then x^2 is not greater than zero.

(3) converse : If a triangle is not equilateral, then it is not isosceles.
inverse : If a triangle is isosceles, then it is equilateral.
contrapositive : If a triangle is equilateral, then it is isosceles

Q-5: Prove the Following :

i. $[p \wedge (p \rightarrow q)] \Rightarrow q$
ii. $[(p \rightarrow q) \wedge \sim q] \Rightarrow \sim p$
iii. $[(p \vee q) \wedge \sim p] \Rightarrow q$

p	q	¬p	¬q	p V q	p → q
0	0	1	1	0	1
0	1	1	0	1	1
1	0	0	1	1	0
1	1	0	0	1	1

i. From table we find that both p and p->q is true then q is true [p Λ (p→q)] =>q
ii. From table we find that both p→q and ~q is true then ~p is true [(p→q) Λ ~q]=> ~p
iii. From table we find that both p V q and ~p is true then q is true [(p V q) Λ~p]=> q

Q-6: Prove the Following :

i. [p Λ (p→q) Λ r] => [(p V q)→r]
ii. {[p V (q V r)] Λ~q }=> p V r

p	q	r	p →q	p V q	(p V q) →r
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

i. From table we find that both p, p->q and r is true then [(p V q)→r is true:
p Λ (p→q) Λ r] => [(p V q)→r]

p	q	r	p V (q V r)	~q	[p V (q V r)] Λ~q	p V r
0	0	0	0	1	0	0
0	0	1	1	1	1	1
0	1	0	1	0	0	0
0	1	1	1	0	0	1
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	0	0	1
1	1	1	1	0	0	1

ii From table we find that {[p V (q V r)] Λ~q } is true then p V r is true

Rules of Inference

Premise

- It is a proposition on the basis of which we would able to draw a conclusion. We can think of premise as an evidence or an assumption.
- Therefore, initially we assume something is true and on the basis of that assumption, we draw some conclusion.

Conclusion

- It is a proposition that is reached from the given set of premises.

If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a **deduction** or a **formal proof** and the argument is called a valid argument or conclusion is called a valid conclusion.

Note: Premises means set of assumptions, axioms, hypothesis.

Rules of Inference

Consider a set of propositions $p_1, p_2, \dots, p_n$ and q then the compound proposition

$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q [\Rightarrow q]$ is called an argument

Here $p_1, p_2, \dots, p_n$ are called premises q is called conclusion.

Note: An argument is generally written as follows

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \qquad \therefore q \end{array}$$

The argument is valid when all $p_1, p_2, \dots, p_n$ are true and likewise q is true.

i.e., The conclusion is true only in the case of valid argument

We use the following rules known as the rules of inference to check the validity of an argument

Types of Inference Rule

Rule	Tautology	Name
$p \to q$ p $\therefore q$	$((p \to q) \land p) \Rightarrow q$	Modus Ponens (Law of Detachment)
$p \to q$ $\neg q$ $\therefore \neg p$	$((p \to q) \land \neg q) \Rightarrow \boxed{\neg p}$	Modus Tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \Rightarrow (p \to r)$	Hypothetical Syllogism (Transitivity)
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \Rightarrow q$	Disjunctive Syllogism

Disjunctive amplification

p $\therefore p \lor q$	$p \Rightarrow p \lor q$	Addition
$p \land q$ $\therefore p$	$(p \land q) \Rightarrow p$	Simplification
p q $\therefore p \land q$	$(p) \land (q) \Rightarrow (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \Rightarrow (q \lor r)$	Resolution

How to Build Arguments by using Rule of Inference

Ex-1 Test Whether following is a valid argument

Premises : If Sachin hits a century then he gets a free car

Premises : Sachin hits a century

Conclusion: sachin gets a free car

If Sachin hits a century then he gets a free car

Sachin hits a century

∴ Conclusion: sachin gets a free car

p: If Sachin hits a century then he gets a free car, q: Sachin hits a century

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

In view of modus Pones Rule this is valid argument

Q-1 Test Whether following is a valid argument

If Sachin hits a century then he gets a free car

sachin does not get a free car.

∴ Sachin has not hit century

p: Sachin hits a century, q: Sachin gets a free car

Then argument reads

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

In view of modus Tollens Rule this is valid argument

Q-2 Test Whether following is a valid argument
If Sachin hits a century he gets a free car
sachin get a free car.

∴ Sachin has hit a century

p: Sachin hits a century, q: Sachin gets a free car

Then argument reads

$$\begin{array}{r} p \rightarrow q \\ q \\ \hline \therefore\ p \end{array}$$

But p can be F when p → q are true

Thus $[(p \rightarrow q) \wedge q] \rightarrow q$ is not a tautology, and given statement is not a valid argument

p	q	p →q	(p →q) ∧ q
0	1	1	1

Q-3 Test Whether following is a valid argument
If I drive to Work, Then I will arrive tired
I am not tired (when I arrive at work)

I do not drive to Work

p: I drive to Work q: I arrived tired

Then argument reads

$$\begin{array}{r} p \rightarrow q \\ \sim q \\ \hline \therefore\ \sim p \end{array}$$

In view of modus Tollens
Rule this is valid argument

Q-4 Test Whether following is a valid argument

If I Study, then I do not fail in the exam

If I do not fail in the exam, then my father gifts a car to me

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if I study then my father gift me a car

p: I Study
q: I do not fail in the exam
r : my father gifts a car to me

Then argument reads

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow \mathbf{r} \\ \hline \therefore\ p \rightarrow \mathbf{r} \end{array}$$

In view of Rule of Syllogism this is valid argument

Q-5 Test Whether following is a valid argument

If Ravi goes out with friends, he will not study.

If Ravi does not study, his father becomes angry

His father is not angry

Ravi has not gone out with friends

p : Ravi goes out with friends
q: Ravi will not study
r : his father becomes angry

Then argument reads

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow \mathbf{r} \\ \sim r \\ \hline \therefore\ \sim p \end{array}$$

Here we have three premises, so we club two premises to simplify by using rues of inference to make conclusion

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow \mathbf{r} \\ \hline \therefore\ p \rightarrow \mathbf{r} \end{array}$$

By Rule of Syllogism

$$\begin{array}{l} p \rightarrow r \\ \sim \mathbf{r} \\ \hline \therefore\ \sim p \end{array}$$

By Rule of Modus Tollens

Q-6 Test Whether following is a valid argument

If I study, I will not fail in exam
If I do not watch TV in evenings, I will study
I failed in exam

I must have watched TV in the Evenings

p : I study
q: I fail in exam
r : I watch TV in evenings

Then argument reads

$$\begin{array}{l} p \rightarrow \sim q \\ \sim r \rightarrow p \\ \quad q \\ \hline \therefore \quad r \end{array}$$

This argument is logically equivalent to

$$\begin{array}{l} q \rightarrow \sim p \\ \sim p \rightarrow r \\ \quad q \\ \hline \therefore \quad r \end{array}$$

As by contrapositive

$p \rightarrow \sim q \Leftrightarrow (\sim\sim q \rightarrow \sim p)$

$\sim r \rightarrow p \Leftrightarrow (\sim p \rightarrow r)$

This argument is logically equivalent to

$$\begin{array}{l} q \rightarrow r \\ \quad q \\ \hline \therefore \quad r \end{array}$$

By Modus Ponens

Q-7 Test Whether following is a valid argument

I will get grade A in this course or I will not graduate
if I do not graduate, I will join the army
I got grade A

I will not join Army

p : I will get grade A in this course
q: I will not graduate
r : I will join the army

Then argument reads

$$\begin{array}{l} p \vee q \\ q \rightarrow r \\ \quad p \\ \hline \therefore \quad \sim r \end{array}$$

By contrapositive

$p \vee q \Leftrightarrow \sim p \to q \Leftrightarrow \sim q \to p$

$q \to r \Leftrightarrow \sim q \to \sim r$

This argument is logically equivalent to

$$\begin{array}{l} \sim q \to p \\ \sim q \to \sim r \\ \hline \therefore p \to \sim r \end{array}$$

This argument is logically equivalent to

$$\begin{array}{l} p \to \sim r \\ p \\ \hline \therefore \sim r \end{array}$$

By MP

given statement is not a valid argument But ~r can be F when p→ ~r are true

Q-8: Test the validity of following arguments:

i.

$$\begin{array}{l} P \wedge q \\ p \to (q \to r) \\ \hline \therefore r \end{array}$$

ii.

$$\begin{array}{l} p \\ p \to \sim q \\ \sim q \to \sim r \\ \hline \therefore \sim r \end{array}$$

iii.

$$\begin{array}{l} P \to r \\ q \to r \\ \hline \therefore (p \vee q) \to r \end{array}$$

i. Since P ∧ q is true, both p and q are True.
Since p is true and p → (q →r) is true, (q →r) is true, and since q is true (q →r) is true, r has to be true hence given argument is valid

ii. The Premises p →~q and ~q →~r together yields the premises p →~r. (by Rule of syllogism)
Since p is true this premises p →~r establish that ~r is true. Hence given statement is true

iii. $(P \to r) \wedge (q \to r) \Leftrightarrow (\sim P \vee r) \wedge (\sim q \vee r)$

$\Leftrightarrow (r \vee \sim P) \wedge (r \vee \sim q)$ by Commutative

$\Leftrightarrow r \vee (\sim p \wedge \sim q)$ by distributive

$\Leftrightarrow r \vee \sim(p \vee q)$ By de Morgan's

$\Leftrightarrow \sim(p \vee q) \vee r$ by Commutative

$\Leftrightarrow (p \vee q) \to r$

This logical Equivalence shows that the given argument is valid

Q-9: Prove that the following are valid arguments:

i.

$$\begin{array}{l} P \rightarrow (q \rightarrow r) \\ \sim q \rightarrow \sim p \\ p \\ \hline \therefore r \end{array}$$

ii.

$$\begin{array}{l} \sim P \leftrightarrow q \\ q \rightarrow r \\ \sim r \\ \hline \therefore p \end{array}$$

Q-10: Test the validity for following arguments:

i.

$$\begin{array}{l} (\sim P \vee q) \rightarrow r) \\ r \rightarrow (s \vee t) \\ \sim s \wedge \sim u \\ \sim u \rightarrow \sim t \\ \hline \therefore p \end{array}$$

ii.

$$\begin{array}{l} P \rightarrow r \\ r \rightarrow s \\ t \vee \sim s \\ \sim t \vee u \\ \sim u \\ \hline \therefore \sim p \end{array}$$

Q-11 Establish the validity of the following Argument

$$\begin{array}{l} p \\ p \rightarrow q \\ s \lor r \\ r \rightarrow \sim q \\ \hline \therefore s \lor t \end{array}$$

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline q \end{array}$$ Modus Ponens

$$\begin{array}{c} r \rightarrow \sim q \\ q \\ \hline \sim r \end{array}$$ Modus Tollens

$$\begin{array}{c} s \lor r \\ \sim r \\ \hline s \end{array}$$ Disjunctive syllogism

$$\begin{array}{c} s \\ \hline s \lor t \end{array}$$ Addition

Q-12: Establish the validity of the following Argument. $(\sim p \lor \sim q) \rightarrow (r \land s)$, $r \rightarrow t$, $\sim t$, $\therefore p$

$$\begin{array}{l} (\sim p \lor \sim q) \rightarrow (r \land s) \\ r \rightarrow t \\ \sim t \\ \hline \therefore p \end{array}$$

$$\begin{array}{l} r \rightarrow t \\ \sim t \\ \hline \therefore \sim r \end{array}$$, Modus Tollens Rule

$$\begin{array}{l} \sim r \\ \hline \therefore \sim r \lor \sim s \end{array}$$ disjunctive amplification

$\therefore \sim r \lor \sim s \Rightarrow \sim(r \land s)$ De Morgan's Law

$$\begin{array}{l} (\sim p \lor \sim q) \rightarrow (r \land s) \\ \sim(r \land s) \\ \hline \therefore \sim (\sim p \lor \sim q) \end{array}$$ Modus Tollens Rule

$\therefore (p \land q)$ De Morgan's Law

$\therefore (p \land q) \Rightarrow p$ Simplification

Open Statements

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- In Mathematical discussions, declarative sentences such as those given below are encountered previously:

Ex: 1) x+3=6, 2) $x^2 < 10$ 3) x divides 4 4) $x = \sqrt{2}$, These sentences are not propositions unless symbol x is specified.

These sentences of these kind are called open sentences or open statements

And symbol x which is unspecified is called free variable

Let us consider open sentence (1) and set of Real number "R".

This sentence becomes a proposition if x is replaced by and=y element of R

For Ex: if x is replaced as 3,

The sentence

x+3=6 becomes true Proposition

And if x =5 it becomes False Proposition

Here we say R is Universe (universe of discourse) for variable x

- Open statement containing variable x are denoted by p(x), q(x) etc..
- If U is universe for variable x in an open statement p(x) and if a ∈ U, then proposition got by replacing a by a in p(x) is denoted by p(a)

p(x): x+3 where x ∈ set of integers

p(2): is false proposition " 2+3=6"

q(x) is open statement , q(x): $x^2 < 10$ where x ∈ set of real numbers as the universe for x then

p($\sqrt{2}$): is false proposition "2<10" which is true

Note: open statement p(x) becomes proposition only when x is replaced by chosen element of the universe

Note: Like compound proposition studied earlier, compound open statement p(x), q(x)… Can be formed by using logical connective's

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Thus ~ p(x) is negation of open statement p(x)

Also for open statement p(x) and q(x)

i. p(x) ∧ q(x) is conjunction
ii. p(x) V q(x) is disjunction
iii. p(x) → q(x) is conditional
iv. p(x) ↔ q(x) is bi conditional for given universe

Q-1: suppose the universe is consists of all integers. Consider the following open statements:

1. $\mathrm{p}(\mathrm{x}): x \leq 3$ 2. $\mathrm{q}(\mathrm{x}): \mathrm{x}+1$ is odd 3. $\mathrm{r}(\mathrm{x}): \mathrm{x}>0$

Write down the truth values of the following:

1. $\mathrm{p}(2)$ 2. ~q(4) 3. p(-1) ∧ q(1) 4. ~p(3) V r(0) 5. p(0) →q(0) 6. p(1) ↔~q(2) 7. p(4) V ((q(1) ∧ r(2))
8. p(2) ∧((q(0) V ~r(2))

1. $\mathrm{p}(2)$ is proposition " $2 \leq 3$" which is true
2. q(4) is proposition " 4+1" is odd which is true, therefore ~q(4) is false
3. p(-1) is proposition "$-1 \leq 3$" which is true, and q(1) is proposition "1+1" is odd which is false. Therefore p(-1) ∧ q(1) is false
4. p(3) is true so that ~p(3) is false and r(0) is false. ~p(3) V r(0) is false.
5. p(0) is true and q(0) is true. Therefore p(0) →q(0) is true.
6. p(1) is true and q(2) is true. therefore p(1) ↔~q(2) is false.
7. p(4) is false, q(1) is false and r(2) is true. Hence ((q(1) ∧ r(2)) is false, so that p(4) V ((q(1) ∧ r(2)) is false.
8. p(2) is true, q(0) is true and r(2) true. Therefore, q(0) V ~r(2) is true, so that p(2) ∧((q(0) V ~r(2)) is true.

Consider the following propositions:

1. **All** squares are rectangles.
2. **For every** integer x, x^2 is non-negative integer.
3. **Some** determinants are equal to zero
4. **There exists** a real number whose square is equal to itself

In these propositions
"All" ∀
"For Every" ∀ Universal quantifiers
"Some" ∃
"there exists" ∃ Existential quantifiers
Are associated with idea of quantity. Such words are called quantifiers

The proposition (1)-(4) considered above may be re written in alternative forms as explained below.

Let S denote set of all squares. Then the proposition (1) may be rewritten as:

1. for **All** x ∈ S , x is rectangles. Symbolically, $\forall x \in S$, p(x)

∀ denotes for all, p(x) stands for open statement x is rectangles

2. $\forall x \in Z$, q(x) , Z is set of all integers

1. **D denotes set of all** determinants, $\exists\ x \in D$, p(x)
2. **R denotes a set of** a real number , $\exists\ x \in R$, q(x)

A proposition involving Universal or existential quantifiers is called quantified statement

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Q-2 For the universe of all integers, let p(x), q(x), r(x), s(x) and t(x) denote the following open statements

p(x) : $x > 0$,

q(x): x is even,

r(x): x is a perfect square,

s(x) : x is divisible by 3,

t(x) : x is divisible by 7.

Write the following statements in symbolic form:

i) At least one integer is even.

ii) There exists a positive integer that is even.

iii) Some even integers are divisible by 3

iv) If x is even and a perfect square, then x is not divisible by 3.

v) If x is odd or is not divisible by 7, then x is divisible by 3.

Solution :

(i) $\exists x, q(x)$

(ii) $\exists x, [p(x) \wedge q(x)]$

(iii) $\exists x, [q(x) \wedge s(x)]$

(iv) $\forall x, [q(x) \wedge r(x)] \rightarrow \neg s(x)$

(v) $\forall x, [\sim q(x) \vee \sim t(x)] \rightarrow s(x)$

Truth values of quantified statement

The following Rules are employed for determining the truth value of a quantified statement
Rule-1: The statement '$\forall x \in S$, $p(x)$' is true only when $p(x)$ is true for each $x \in S$.
Rule-2: The statement '$\forall x \in S$, $p(x)$' is False only when $p(x)$ is False for every $x \in S$.

Accordingly,

- To infer that proposition of the form '$\forall x \in S$, $p(x)$' is false, it is enough to exhibit one element a of S such that p(a) is false. The element a is called counter example.
- To infer that proposition of the form '$\forall x \in S$, $p(x)$' is true, it is enough to exhibit one element a of S such that p(a) is true.

From these quantified statement

1. **All** squares are rectangles.
2. **For every** integer x, x^2 is non-negative integer.
3. **Some** determinants are equal to zero
4. **There exists** a real number whose square is equal to itself

All propositions are true propositions

But it is obvious that following propositions are false

1. **All** rectangles are squares
2. **For every** integer x, x^2 is positive integer.
3. The square of real numbers are negative

Two rules of Inference

As a consequence of rule 1 and 2 indicated above, we obtain the following rules of inference
Rule-3: If an open statement **p(x) is known to be true** for all x in a universe S and if $a \in S$, then p(a) is true [known as rule of universal specification)
Rule-4: If an open statement **p(x) is proved to be true** for any arbitrary x chosen from set S, then quantified statement "$\forall x \in S$, $p(x)$' is true [known as rule of universal generalization]

Logical Equivalence

1. ∀x, [p(x) ∧ q(x)] ⇔ (∀x, p(x)) ∧ (∀x, q(x))
2. ∃x, [p(x) V q(x)] ⇔ (∃x, p(x)) V (∃x, q(x))
3. ∃x, [p(x) → q(x)] ⇔ ∃x, [p(x) V q(x)]
4. **∀x, ~p(x) ⇔ for no x, p(x)**

For ex: the statement " for every integer x, x^2 is non-negative" is ⇔ **statement " for no integer x, x^2 is negative**

Rules for Negation of a quantified statement

Rule 5:

~ [∀x, p(x)] ⇔ ∃x, [~p(x)]

~ [∃x, p(x)] ⇔ ∀ x, [~p(x)]

For ex: let us consider the quantified statement **" All equilateral triangle are isosceles"**

Symbolic form: **∀x** ∈ T, p(x)" T is set of all equilateral triangle, p(x) is open statement "x is isosceles"

As per rule of negation
~ [∀x, p(x)] ⇔∃x, [~p(x)]
i.e. ∃x ∈ T, [~p(x)]
" for some equilateral triangle x, x is not isosceles.
Or, " some equilateral triangle are not isosceles

Q-1 For the universe of all integers, let p(x), q(x), r(x), s(x) and t(x) denote the following open statements

$p(x) : x > 0$,

q(x): x is even,

r(x): x is a perfect square,

s(x) : x is divisible by 3,

t(x) : x is divisible by 7.

Write the following symbolic statements in words and indicate its truth value:

i) $\forall x, [r(x) \rightarrow p(x)]$ ii) $\exists x, [s(x) \wedge \sim q(x)]$ iii) $\forall x, \sim[r(x)]$ iv) $\forall x,[r(x) \vee t(x)]$

i) $\forall x, [r(x) \rightarrow p(x)]$: For any integer x, if x is a perfect square then x > 0 [false take x=0]

ii) $\exists x, [s(x) \wedge \sim q(x)]$: For some integer, x is divisible by 3 and x is not even [true take x=9]

iii) $\forall x, \sim[r(x)]$: for any integer, x is not a perfect square [false]

iv) $\forall x,[r(x) \vee t(x)]$: For any integer x, x is a perfect square or divisible by 7 [false take x=8]

Q-2 Consider the Following open statements with the set of all real numbers as universe

$p(x) : |x| > 3$, $q(x): x>3$.

Find the truth value of quantified statement : $\forall x, [p(x) \rightarrow q(x)]$

Also write the converse, inverse and contrapositive of this statement and find their truth values

We note that: $p(-4) \Leftrightarrow |-4|>3 \Leftrightarrow 4>3$ is True, $q(-4) \Leftrightarrow -4>3$ is false

Thus $[p(x) \rightarrow q(x)]$ is false for x=-4.

i. The converse statement of $\forall x, [p(x) \rightarrow q(x)]$ is $\forall x, [q(x) \rightarrow p(x)]$ which reads : *for Every real number x, if x>3 then |x|>3*

ii. The Inverse statement of $\forall x, [p(x) \rightarrow q(x)]$ is $\forall x, [\sim p(x) \rightarrow \sim q(x)]$ which reads : *for Every real number x, if |x|<=3, then x<=3.*

Since converse $\Leftrightarrow$ Inverse hence truth value of (ii) is true

iii. The Contrapositive is $\forall x, [\sim q(x) \rightarrow \sim p(x)]$ which reads : *for Every real number x, if x<=3 then |x|<=3. [False]*

Q-3Write the following sentences in symbolic form, and find its negation:
(i) If all triangles are right angled, then no triangle is equiangular. (ii) All integers are rational numbers and some rational numbers are not integers

Solution :
(i) Let T denote the set of all triangles.
Also, let p(x) : x is right-angled,
q(x) : x is equiangular.
Then the given proposition can be written in symbolic form as $\{\forall x \in T , p(x)\} \rightarrow \{\forall x \in T , \neg q(x)\}$
whose negation is $\{\forall x \in T , p(x)\} \wedge \{\exists x \in T , q(x)\}$
This reads as “All triangles are right-angled and some triangles are equiangular”
(ii) Let p(x) : x is a rational number.
q(x) : x is an integer. Z : Set of all integers. Q : Set of all rational numbers.
Then the given proposition can be written in symbolic form as $\{\forall x \in Z, p(x)\} \wedge \{\exists x \in Q, \neg q(x)\}$.
The negation of this is : $\neg\{\forall x \in Z, p(x)\} \vee \neg\{\exists x \in Q, \neg q(x)\}$ which is logically equivalent to $\{\exists x \in Z, \neg p(x)\} \vee \{\forall x \in Q, q(x)\}$
i.e. “Some integers are not rational numbers or every rational number is an integer”.

Q-4: Write the negation of each of the following statements for (i) and (ii) the universe consists of all integers and for (iii) the universe consists of all real numbers.
(i) For all integers n, if n is not divisible by 2, then n is odd.
(ii) If k, m, n are any integers where $k–m$ and $m - n$ are odd, then $k - n$ is even.
(iii) For all real numbers x, if $|x - 3| < 7$, then $-4 < x < 10$.

(Let Z denote the set of all integers and R denote the set of all real numbers.

(i) The given statement can be written as $\forall n \in Z, \neg p(n) \rightarrow q(n)$ where

$p(n)$: n is divisible by 2,

$q(n)$: n is odd.

Its negation is : $\exists n \in Z, \neg p(n) \wedge \neg q(n)$ i.e. For some integer n, n is not divisible by 2 and n is not odd.

(ii) The given statement can be written as : $\forall k, m, n \in Z, [p(x) \wedge q(x)] \rightarrow r(x)$

Its negation is : $\exists k, m, n \in Z, [p(x) \wedge q(x)] \wedge \neg r(x)$

i.e. There exist integers k, m, n such that $k - m$, $m - n$ are odd and $k - n$ is not even.

(iii) The given statement can be written as : $\forall x \in R, p(x) \rightarrow q(x)$ where $p(x) : |x - 3| < 7$ and $q(x) : -4 < x < 10$

Its negation is : $\exists x \in R, \neg[p(x) \rightarrow q(x)]$ i.e. $\exists x, [p(x) \wedge \sim q(x)]$ i.e. For some real number x, $|x - 3| < 7$ and $x /\in (-4, 10)$

Q-5: Let the universe comprise of all integers i) Given p(x) : x is odd, q(x) : x 2 − 1 is even. Express the statement “If x is odd then x 2 − 1 is even” in symbolic form using quantifiers and negate it

Solution :

(i) The given statement can be written in symbolic form as “$\forall x \in Z, [p(x) \rightarrow q(x)]$”

where Z is the set of all integers. Its negation is $\exists x \in Z, [p(x) \wedge \neg q(x)]$.

i.e. For some integer x, x is odd and x 2 − 1 is not even

Statements with more than one variable

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Consider the following statements:

(1) $x-2y$ is a positive integer.

(2) $x+y-z=0$.

These are open statements which contain more than one free variable.

These become propositions if each variable is replaced by an element of a certain Universe.

For example, if ***W*** take the set of all integers as the Universe and replace x and y in the statement **(1)** by 5 and -3 respectively, then this statement becomes the proposition

"5-2(-3) is a positive integer" **(which is true).**

Similarly, if we take the set of all rational numbers as the Universe and replace x, y, z in the statement **(2)** by 1/2, 1/4, 1/4, then the statement becomes the proposition

"1/2+1/4-1/4= 0" **(which is false).**

Open statements containing two variables x and y are usually denoted by **p(x, y), q(x,y)** etc and those with three variables x, y, z are denoted by **p(x, y, z), q(x, y, z),** etc.

For an open statement with more than one variable, the Universe can be the same for all variables **or** can be different for different variables.

Example: in the case of the open statement "x - 2y is a positive integer“

the set of all integers can be the Universe for both x and y, **or** the set of **all integers** can be the Universe for x and the set of **all positive integers** can be the Universe for y .

If U is the universe for x and V is the Universe for y in an open statement p(x, y) and if a € U and b € V, then the proposition got by replacing x by a and y by b in p(x, y) is denoted by p(a, b).

Thus,

if p(x, y) is the open statement **"x-2y is a positive integer**" with Z as the Universe for both x and y,

then p(6, 4) is the proposition "**6-(2x4) is a positive integer**".

Similarly, p(-4, 2) is the proposition **"-4-(2 x 2) is a positive integer**".

Example 1: Let p(x, y) and q(x, y) denote the following open statements.

p(x, y): $x^2 \geq y$, **q(x, y): (x+2) < y**

If the universe for both of x, y is the set of all real numbers, determine the truth value of each of the following statements:

$(\mathbf{i})\mathbf{p}(\mathbf{2},\mathbf{4})$ $(\mathbf{ii})\mathbf{q}(\mathbf{1},\boldsymbol{\pi})$ $(\mathbf{iii})\mathbf{p}(-\mathbf{3},\mathbf{8}) \wedge \mathbf{q}(\mathbf{1},\mathbf{3})$

$(\mathbf{iv})\ \mathbf{p}(\mathbf{1/2},\mathbf{1/3})\ \mathbf{V} \sim \mathbf{q}(-\mathbf{2},-\mathbf{3})$ $(\mathbf{v})\ \mathbf{p}(\mathbf{2},\mathbf{2}) \rightarrow\ \mathbf{q}(\mathbf{1},\mathbf{1})$ $(\mathbf{vi})\ \mathbf{p}(\mathbf{1},\mathbf{2})\ \leftrightarrow\ \sim \mathbf{q}(\mathbf{3},\mathbf{8})$.

Solution:

$(\mathrm{i})\ \ \mathrm{p}(2,4)\ =\ 22\ \geq\ 4$, which is true.

$(\mathrm{ii})\ \mathrm{q}(1,л)\ =\ (1\ +\ 2)\ <\ л$, which is true.

$(\mathrm{iii})(\mathrm{p}(-3{,}8)\ \wedge\ \mathrm{q}(1{,}3))\ =\ [(-3)^2 \geq\ 8]\Lambda[(1\ +\ 2)\ <\ 3]$, which is false.

$(\mathrm{iv})(\mathrm{p}(1/2,1/3)\ \mathrm{V} \neg\ \mathrm{q}(-2,-3))\ =\ [(1/2)^2\ \geq\ (1/3)|\ \mathrm{V}\ [(-2\ +\ 2)\ \geq\ -3]$ which is true

$(\mathrm{v})\ (\mathrm{p}(2{,}2) \rightarrow\ \mathrm{q}(1,1))\ =\ (2^2\ \geq\ 2) \rightarrow ((1+2)\ <\ 1)$, which is false.

$(\mathrm{vi})(\mathrm{p}(1,2) \leftrightarrow {\sim}\mathrm{q}(3{,}8))\ =\ (1^2 \geq\ 2) \leftrightarrow (3+2\ \geq\ 8)$, which is true.

When an open statement contains more than one free variable, quantification may be applied to each of the variables.

Thus, if p(x, y) is an open statement with variables x, y, we can have quantified statements of the following form:

(1) $\forall x, \forall y, p(x, y)$ **(2)** $\exists x, \exists y, p(x, y)$

(3) $\forall x, \exists y, p(x, y)$ **(4)** $\exists x, \forall y, p(x, y)$

In the above statements, x and y can have the same universe or different universes.

When x and y have the same universe, the statements (1) and (2) are respectively rewritten as

(1) $\forall x, y, p(x, y)$, **(2)** $\exists x, y, p(x, y)$.

From the meaning of the quantifiers, the following results are readily obtained:

$$\forall x, \forall y, p(x, y) \Leftrightarrow \forall y, \forall x, p(x, y).$$
$$\exists x, \exists y, p(x, y) \Leftrightarrow \exists y, \exists x, p(x, y).$$

Let us analyse the quantified statement (3) in some detail.

Let us consider the open statement $\mathbf{p(x, y): \ x + y = 1}$

with the set of all integers as the universe.

Then, the statement $\forall\ \mathbf{x}, \exists\ \mathbf{y}, \mathbf{p(x, y)}$ reads:

For every (each) integer x, there exists an integer y such that x+y=1".

This statement carries the same meaning as the statement:

"Given any integer, we can find a corresponding integer y such that x+y=1". This is a true statement;

because once we select any x, there does exist y = 1 - x which meets the requirement x+y=1.

On the other hand, the statement $\exists\, y, \forall\, x, p(x, y)$

Reads: **"For some integer y and for all integers x, we have $\mathbf{x + y = 1}$"**.This is a false statement;

because if this statement were to be true, then every integer would be equal to 1−y for some (fixed) integer y.

The above example illustrates the following important result:

$$\forall\, \mathbf{x}, \exists\, \mathbf{y}, \mathbf{p}(\mathbf{x}, \mathbf{y}) \;\; <\neq> \;\; \exists\, \mathbf{y}. \forall\, \mathbf{x}, \mathbf{p}(\mathbf{x}, \mathbf{y}).$$

$$\exists\, \mathbf{x}, \forall\, \mathbf{y}, \mathbf{p}(\mathbf{x}, \mathbf{y}) \;\; <\neq> \;\; \forall\, \mathbf{y}, \exists\, \mathbf{x}, \mathbf{p}(\mathbf{x}, \mathbf{y}).$$

Similarly.

Quantified statements involving more than two variables can be analysed similarly.

All rules applicable to quantified statements with one variable can be extended in a natural way to those involving more than one variable.

Example 2: Let x and y denote integers. Consider the statement

$$\mathbf{p(x, y):\ x\ +\ y\ is\ even}$$

Write down the following statements in words:

$$\mathbf{(i)\ \forall\ x, \exists\ y, p(x, y) \qquad (ii)\ \exists\ x, \forall\ y, p(x, y).}$$

Solution: In words, the required statements are

(i) With every integer x, there exists an integer y such that x + y is even

(ii) There exists an integer x such that x+y is even for every integer y.

Example 3: Write down the following statements in symbolic form using quantifiers:

(1) Every real number has an additive inverse

(2) The set of real numbers has a multiplicative identity.

(3) The integer 58 is equal to the sum of two perfect squares.

(1) The statement
"Every real number has an additive inverse" is the same as:
"Given any real number x, there is a real number y such that x + y = y + x = 0".
In symbols, this reads $\forall$ **x, $\exists$ y, (x + y = y + x = 0].**
Here, the set of all real numbers is the universe.
(2) The statement
"The set of real numbers has a multiplicative identity" is the same as:
"There exists a real number x such that xy= yxy for every y".
In symbols, this reads $\exists\, x, \forall\, y, \quad [xy = yx = y]$
Here, the set of all real numbers is the universe.
(3) The given statement is the same as **"There exist integers $\mathbf{m}$ and $\mathbf{n}$ such that $\mathbf{58 = m^2 + n^2}$**
"In symbols, this reads **$\exists$ m, $\exists$ n, $58 = m^2 + n^2$.**
Here, the set of all integers is the universe.

Example 4 : Determine the truth value of each of the following quantified statements , the universe being the set of all non-zero integers.

(i) $\exists\, \mathbf{x}, \exists\, \mathbf{y}, [\mathbf{xy} = \mathbf{1}]$

(ii) $\exists\, \mathbf{x}, \forall\, \mathbf{y}, [\mathbf{xy} = \mathbf{1}]$

(iii) $\forall\, \mathbf{x}, \exists\, \mathbf{y}, [\mathbf{xy} = \mathbf{1}]$

(iv) $\exists\, \mathbf{x}, \exists\, \mathbf{y}, [(\mathbf{2x} + \mathbf{y} = \mathbf{5}) \wedge (\mathbf{x} - \mathbf{3y} = -\mathbf{8})]$

(v) $\exists\, \mathbf{x}, \exists\, \mathbf{y}, [(\mathbf{3x} - \mathbf{y} = \mathbf{17}) \wedge (\mathbf{2x} + \mathbf{4y} = \mathbf{3})]$

Solution:

(i) True. (Take x = 1,y= 1).

(ii) False. (For a specified x, xy= 1 for every y is not true).

(iii) False. (For x = 2, there is no integer y such that xy = 1).

(iv) True. (Take x = 1,y= 3).

(v) False. (Equations 3x - y = 7 and 2x+4y = 3 do not have a common integer solution).

Mathematical Logic using Python

Propositional Logic

A proposition is a sentence that declares a fact that is either True or False. In Python, we can use boolean variables (typically p and q) to represent propositions and define functions for each propositional rule. Each rule can be implemented using the boolean operators (and, or, not, etc).

Logical Expressions in Sympy

SymPy is a symbolic mathematics Python package. Its goal is to develop into a completely featured computer algebra system while keeping the code as basic as possible to make it understandable and extendable. The package is entirely written in python language. Logical expressions in sympy are expressed by using boolean functions. **sympy.basic.booleanarg** module of sympy contains boolean functions.

The common Python operators **& (And), | (Or),** and **~ (Not)** can be used to create Boolean expressions. **>>(Implies)** and **(Equivalent)** can be used for to create implication and bi-implication. Other boolean operations or gates are **NAND, NOR, XOR, etc**.

Topics :

1. Negation
2. Conjunction
3. Disjunction
4. Exclusive Disjunction
5. Implification
6. Bi-implification
7. Python program for the compound proposition
8. Rules of inference using Python

Negation

The negation is a statement that has the opposite truth value. The negation of a proposition p, denoted by ¬p, is the proposition "It is not the case , that p".
For example, the negation of the proposition "Today is Friday." would be "It is not the case that, today is Friday." or more succinctly "Today is not Friday".

```
from sympy.logic.boolalg import Not
def negation(p):
    return Not(p)
print("p    ans")
for p in [True, False]:
    ans = negation(p)
    print(p, ans)
```

```
 p   ans
True False
False True
```

Sample Python Program based on negation / NOT [~]

Python program to check whether the input number 10 is divisible by 5 or 3.

```
a = float(input("Enter the number: "))
if not (a%3==0 or a%5==0):
  print("10 is not divisible by either 3 or 5")
else:
   print("10 is divisible by either 3 or 5")
```

```
Enter first number: 10
10 is divisible by either 3 or 5
```

Conjunction

Let p and q be propositions. The conjunction of p and q, denoted in mathematics by p∧q, is True when both p and q are True, False otherwise.

sympy.logic.boolalg.And

It analyzes each of its arguments in sequence, it returns true if all of the arguments are true. if at least one argument is false, false is returned.

```
from sympy.logic.boolalg import And
def conjunction(p,q):
    return And(p,q)
print("p     q     ans")
for p in [True, False]:
  for q in [True, False]:
    ans = conjunction(p,q)
    print(p,q, ans)
```

```
p     q     ans
True True True
True False False
False True False
False False False
```

Sample Python Program based on Conjunction/ AND [∧]

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Python program to find the largest number among the three input numbers.

```
num1 = float(input("Enter first number: "))
num2 = float(input("Enter second number: "))
num3 = float(input("Enter third number: "))
if (num1 >= num2) and (num1 >= num3):
   largest = num1
elif (num2 >= num1) and (num2 >= num3):
   largest = num2
else:
   largest = num3
print("The largest number is", largest)
```

```
Enter first number: 10
Enter second number: 12
Enter third number: 8
The largest number is 12.0
```

Disjunction

Let p and q be propositions. The disjunction of p and q, denoted in mathematics by p∨q, is True when at least one of p and q are True, False otherwise.

sympy.logic.boolalg.Or

It analyzes each of its arguments in sequence, it returns true if any of the arguments is true otherwise the argument is false.

```
from sympy.logic.boolalg import Or
def disjunction(p,q):
    return Or(p,q)
print("p     q     ans")
for p in [True, False]:
  for q in [True, False]:
    ans = disjunction(p,q)
    print(p,q, ans)
```

```
p     q     ans
True True True
True False True
False True True
False False False
```

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Sample Python Program based on Disjunction/ OR [V]

Python program to check whether all the two input numbers are positive.

```
a = float(input("Enter first number: "))
b = float(input("Enter second number: "))
if a>0 or b>0:
  print("Atleast one of the numbers is greater than zero")
else:
   print("No number is greater than zero")
```

```
Enter first number: 8
Enter second number: -10
Atleast one of the numbers is greater than zero
```

Exclusive Disjunction

Let p and q be propositions. The exclusive disjunction of p and q (also known as xor), denoted in mathematics by $p \oplus q$, is True when exactly one of p and q are True, False otherwise.

```
from sympy.logic.boolalg import Or, And, Not
def exclusive_disjunction(p,q):
    return Or(And(p,Not(q)),And(Not(p),q))
print("p     q     ans")
for p in [True, False]:
  for q in [True, False]:
    ans = exclusive_disjunction(p,q)
    print(p,q, ans)
```

```
p     q     ans
True True False
True False True
False True True
False False False
```

Implication

Let p and q be propositions. The implication of p and q, denoted in mathematics by $p \Longrightarrow q$, is short hand for the statement "if p then q".

sympy.logic.boolalg.Implies

In other words, implication fails (is False) when p is True and q is False otherwise the argument is True.

```
from sympy.logic.boolalg import Implies
def implication(p,q):
    return Implies(p,q)
print("p     q     ans")
for p in [True, False]:
  for q in [True, False]:
    ans = implication(p,q)
    print(p,q, ans)
```

```
p     q     ans
True True True
True False False
False True True
False False True
```

Bi-Implication

Let p and q be propositions. The bi-implication of p and q, denoted in mathematics by $p \Leftrightarrow q$, is short hand for the statement "p if and only if q".

sympy.logic.boolalg.Equivalent

As such, bi-implication requires q to be True only when p is True. In other words, bi-implication fails (is False) when p is True and q is False or when p is False and q is True.

```
from sympy.logic.boolalg import Equivalent
def bi_implication(p,q):
    return Equivalent(p,q)
print("p     q     ans")
for p in [True, False]:
  for q in [True, False]:
    ans = bi_implication(p,q)
    print(p,q, ans)
```

```
p     q     ans
True True True
True False False
False True False
False False True
```

Python program for the compound proposition

Construct the Python program for the followowing compound proposition

$$[(p \wedge q) \vee \neg r] \leftrightarrow p$$

```
from sympy.logic.boolalg import Or, And, Not, Equivalent
def compound_prop(p,q,r):
      return Equivalent(Or(And(p,q),Not(r)),p)
print("p     q      r   ans")
for p in [True, False]:
 for q in [True, False]:
  for r in [True, False]:
      ans=compound_prop(p,q,r)
      print(p,q,r, ans)
```

```
p     q      r   ans
True True True True
True True False True
True False True False
True False False True
False True True True
False True False False
False False True True
False False False False
```

Rules of Inference using Python

Test Whether following is a valid argument by using Python code:

If Sachin hits a century then he gets a free car

sachin does not get a free car.

∴Sachin has not hit century

p: Sachin hits a century, q: Sachin gets a free car

$$\begin{array}{r} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

We need to prove $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ is tautology

```
from sympy.logic.boolalg import Or, And, Not, Implies
def rules_of_inference(p,q):
    return Implies(And(Implies(p,q),Not(q)),Not(p))
print("p     q     ans")
for p in [True, False]:
  for q in [True, False]:
    ans = rules_of_inference(p,q)
    print(p,q, ans)
```

```
p     q     ans
True True True
True False True
False True True
False False True
```

Exercise question

1. Write the python code for the following compound proposition:

(i) $p \vee \neg q$

(ii) $p \rightarrow \neg q$

(iii) $p \rightarrow (p \vee q)$

(iv) $p \wedge (\neg p \ \wedge q)$

(v) $(p \vee q) \rightarrow r$

(v) $\{(p \vee q) \wedge [(p \rightarrow r) \wedge (q \rightarrow r)]\} \rightarrow r$

2. Test whether following is a valid argument or not by using python code

If Sachin hits a century then he gets a free car

Sachin hits a century

$\therefore$ sachin gets a free car

3. Test whether following is a valid argument or not by using python code

If I Study, then I do not fail in the exam

If I do not fail in the exam, then my father gifts a car to me

---

$\therefore$ If I study then my father gift me a car