# Comparing Measures For The Identification Of Partisan Gerrymandering

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# Preliminaries

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Every ten years, each state is divided into congressional districts, each of which elects a representative to the House of Representatives biannually. *Partisan gerrymandering* is when the districts are drawn in such a way to purposely favor one party.

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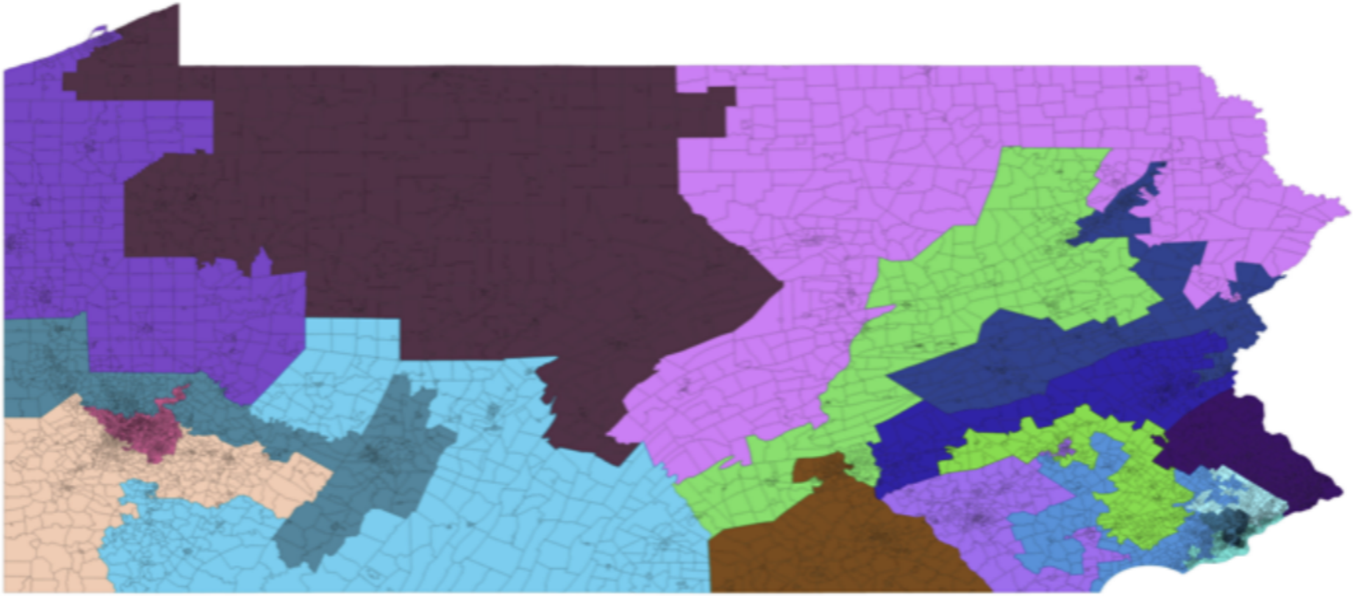

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1. Given a voting plan, quasi-randomly generate a neutral ensemble for the election.
2. Use a mathematical metric to determine if the enacted districting is statistically different from the ensemble.

# Overview

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1. Introduction to metrics and algorithms being tested
2. Comparison of methods
3. Discussion of implications

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## Definition

Each district in a state is divided into a number of *precincts*. Given some precinct $i$, let $V(i)$ be the district $i$ is contained in.

# Flip Algorithm

## Definition

We define a step of the *flip algorithm*:

1. Take a voting plan divided into precincts, and take two precincts $A$ and $B$ such that $A$ and $B$ are geographically adjacent and part of different districts $k = V(A)$ and $l = V(B)$, respectively.

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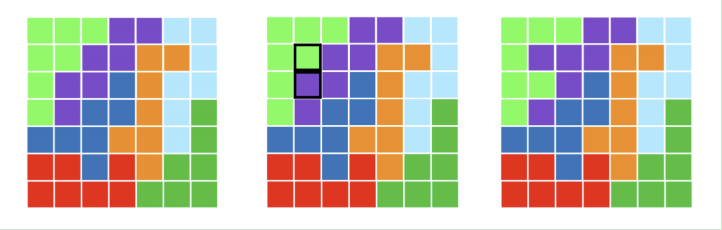
## Example



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4. If possible, cut an edge of $\tau(A, B)$ that splits it into two trees $C$ and $D$ of equal size. These trees correspond to two new districts of roughly equal population, and thus a new voting plan.
5. If no such edge exists, find another spanning tree of $G(A, B)$, or pick two different geographically adjacent districts until such an edge exists.

# ReCom Algorithm Example

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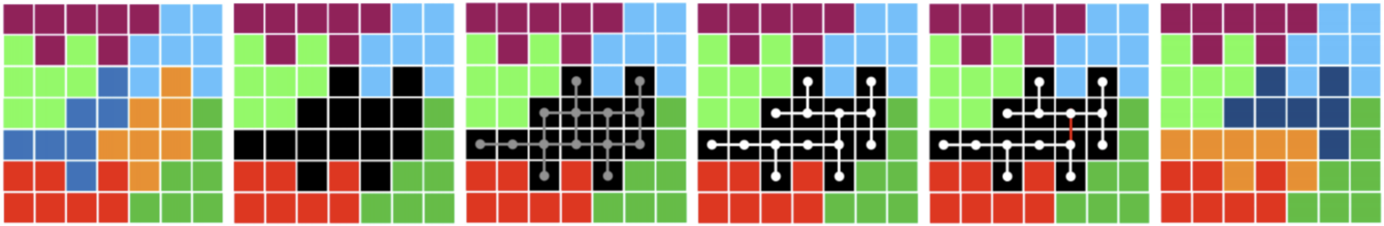

# Efficiency Gap

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Let $V = \{V_1, V_2, \ldots, V_n\}$ be the set of districts in the enacted voting plan of a state. For each $1 \leq i \leq n$, let $D_i$ be the number of Democratic votes in district $V_i$, and $R_i$ be the number of Republican votes in district $V_i$.

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## Definition

A *wasted vote* in a district is a vote for the losing party or a vote for the winning party above the 50% majority needed to win. In a district $V_i$ for some $1 \leq i \leq n$, we let $WD_i$ be the number of wasted Democratic votes in the district, and $WR_i$ be the number of wasted Republican votes.

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## Definition (Stephanopoulos-McGhee, 2015)

The *efficiency gap* of an election on districts $V = \{V_1, V_2, \dots, V_n\}$ is defined as

$$E_V = \frac{\sum_{i=1}^{n}(WD_i - WR_i)}{\sum_{i=1}^{n}(D_i + R_i)}.$$

# Variants of the Efficiency Gap

## Definition

Let $\lambda$ be a fixed positive constant. A *weighted wasted vote* is the same as a wasted vote for the losing party, but is weighted as $\lambda$ of a vote for the winning party. We let $WWD_i(\lambda)$ be the number of weighted wasted votes for the Democratic party in district $V_i$ with weight $\lambda$, and $WWR_i(\lambda)$ be the number of weighted wasted votes for the Republican party in district $V_i$.

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We define the *weighted efficiency gap* of an election with weight $\lambda$ on districts $V = \{V_1, V_2, \ldots, V_n\}$ as

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## Remark

*Note that* $WE_V(1) = E_V$ *for all elections.*

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We define the *relative efficiency gap* of an election with weight $\lambda$ on districts $V = \{V_1, V_2, \ldots, V_n\}$ as

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## Remark

*The commonly used variants in the literature are* $WE_V(1) = E_V, WE_V(2), RE_V(1), RE_V(2)$. *We will focus on these four.*

# Other Metrics

**Definition**

The *mean-median score* of an election on districts $V = \{V_1, V_2, \ldots, V_n\}$ is defined as the median of the list $\{\frac{D_1}{D_1+R_1}, \frac{D_2}{D_2+R_2}, \ldots, \frac{D_n}{D_n+R_n}\}$ minus the mean of this list.

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## Definition

First, uniformly shift the votes in an election so each party gets 50% of the votes. The *partisan bias* is the difference between the Democratic seat share percentage in this hypothetically tied election and 50%.

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## Definition (Warrington, 2018)

Sort the $V_1, V_2, \ldots, V_n$ in order of increasing $D_i$, and draw the points $(\frac{2i-1}{2n}, \frac{D_i}{(D_i+R_i)})$ for each $1 \leq i \leq n$. Let $F$ be the center of mass of the points with $y$-coordinate at least 0.5, $H$ be the center of mass of the points with $y$-coordinate less than 0.5, and $G$ be the center of mass of all points. The *declination* is $\frac{2}{\pi}$ times the angle between $\overline{FG}$ and $\overline{GH}$.

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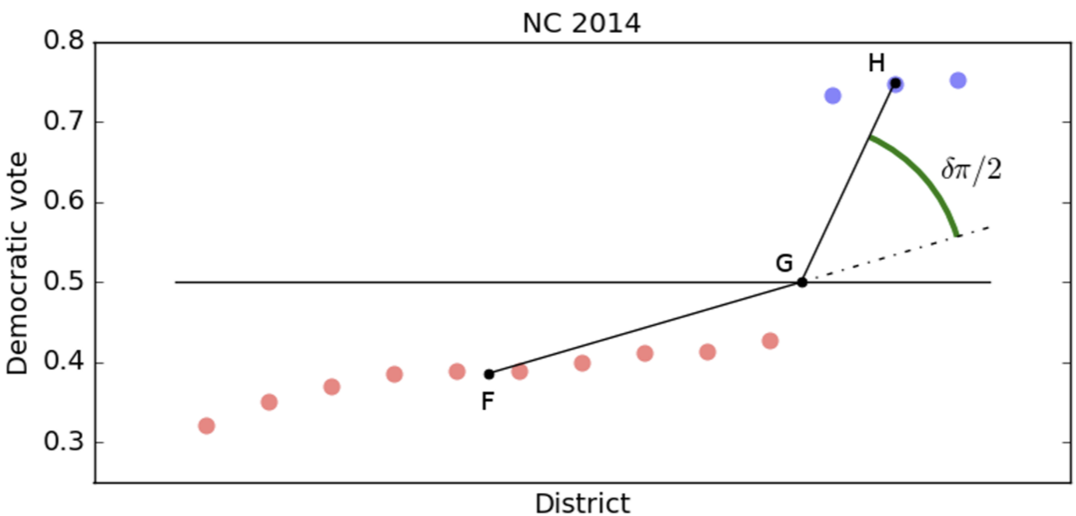

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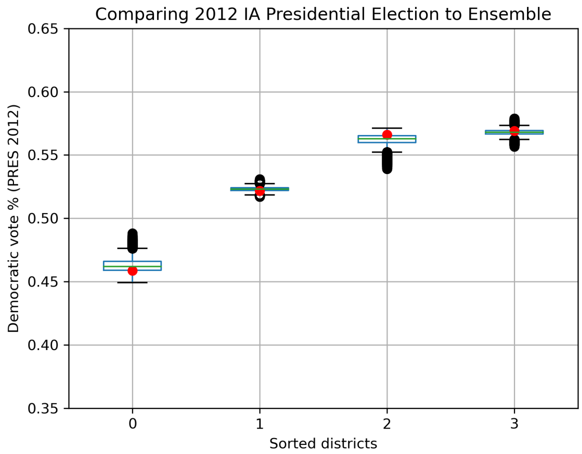


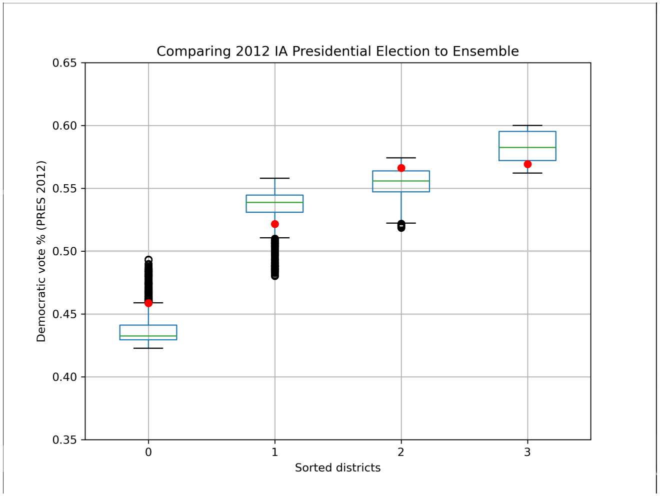


The left figure shows the Flip algorithm, and the right shows the ReCom algorithm.

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| | Flip Algorithm (1,000,000 steps) | ReCom Algorithm (10,000 steps) |
|---|---|---|
| 6 seats | 0 | 38 |
| 7 seats | 4601 | 968 |
| 8 seats | 221392 | 4731 |
| 9 seats | 773998 | 3466 |
| 10 seats | 9 | 744 |
| 11 seats | 0 | 53 |
| Mean | 8.769 | 8.407 |
| Standard Deviation | 0.432 | 0.799 |

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2. For each mathematical metric, evaluate each generated voting plan.
3. Use a $t$-test to determine if the enacted plan is statistically different from the neutral ensemble (under that metric).
4. Track accuracy for different metrics by doing this for different elections.

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We use ten different elections from 9 different states, all taking place between 2012 and 2018. Data is taken from the MGGG Redistricting Lab.

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| Metric | # False Results |
|---|---|
| Efficiency Gap | 3 |
| Mean-Median | 4 |
| Partisan Bias | 5 |
| Weighted Efficiency Gap (2) | 3 |
| Relative Efficiency Gap (1) | 3 |
| Relative Efficiency Gap (2) | 3 |
| Declination | 2 |

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# Discussion

1. The declination performs the best out of all the metrics, with the least false results; this agrees with the analysis in Warrington's 2019 paper.
2. All variants of the efficiency gap performed the same on this dataset, but the relative efficiency gap is preferred, as in Tapp, 2019.
3. The mean-median score and partisan bias are the most unreliable and should not be used in real-world scenarios.

# Sources of False Results

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2. If elections have two districts, the mean-median score fails.
3. Mean-median score and partisan bias favor elections that are not very competitive.
4. Large numbers of independent votes create inaccurate results.

# Future Work

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1. Perform this survey with a larger suite of elections to notice and confirm general trends.
2. Take compactness scores into account.
3. Create a new metric to identify partisan gerrymandering, using the sources of false results as inspiration.

# Acknowledgements

We would like to thank the following people for their support on this project:

1. Dr. Kevin Crowthers, Massachusetts Academy of Math and Science at WPI
2. Dr. Diana Davis, Philips Exeter Academy