Problem Set 5
Due Date 04/27/25

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April 20, 2025

(1) In this exercise you shall prove the following result.

Theorem 0.0.1. *Let $X \subseteq \mathbb{P}^n_A$ be a closed subscheme and $\mathscr{F}$ a coherent sheaf on X with A a* ***reduced*** *Noetherian ring. Then $\mathscr{F}$ is flat over $\mathrm{Spec}(A)$ iff the function taking $s \in \mathrm{Spec}(A) \to P_{\mathscr{F}_s}(X_s)$, the Hilbert polynomial of $\mathscr{F}$ restricted to the fiber $X_s = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k(s))$ is locally constant on $\mathrm{Spec}(A)$.*

In class we have already shown the only if direction (even for proper morphisms, using the Mumford complex). We shall prove the theorem in steps.

(i) Let $\pi_S : \mathbb{P}^n_S \to S$, with S a Noetherian scheme. Let $\phi : T \to S$ be an arbitrary morphism from a Noetherian scheme T. Thus we have a Cartesian diagram

$$\begin{tikzcd} \mathbb{P}^n_T \arrow[r, "\phi'"] \arrow[d, "\pi_T"] & \mathbb{P}^n_S \arrow[d, "\pi_S"] \\ T \arrow[r, "\phi"] & S. \end{tikzcd}$$

Show that for any coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n_S$ and all $r >> 0$, the base change map[1]

$$\phi^*\pi_{S*}(\mathscr{F}(r)) \to \pi_{T*}\phi'^*\mathscr{F}(r),$$

is an isomorphism.

HINT: First by covering S and then T by open affines, reduce to the case where S and T are affine. Next note that when $\mathscr{F} = \mathscr{O}_X$, the isomorphism follows from explicit computation of the global sections of the Serre twists. Finally resolve $\mathscr{F}$ by direct sum of Serre twists, conclude using Serre vanishing.

[1]The selling point here is that ϕ need not be *flat*.

(ii) Let $\pi : \mathbb{P}^n_A \to \mathrm{Spec}(A)$, with A a Noetherian ring. Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n_A$. Suppose $H^0(\mathbb{P}^n_A, \mathscr{F}(r))$ is a flat A-module for all $r \geqslant r_0$. Then $\mathscr{F}$ is flat over $\mathrm{Spec}(A)$.

HINT: Note that $\mathscr{F} = \widetilde{M}$, where M is the graded module, $\oplus_{r \geqslant r_0} H^0(\mathbb{P}^n_A, \mathscr{F}(r))$. Show that M is flat and conclude that $\mathscr{F}$ restricted to each basic affine open set of the form $D_+(X_i)$ is flat and hence is flat over $\mathrm{Spec}(A)$.

(iii) Let M be a finitely generated A-module, with A a reduced Noetherian local ring. Show that M is flat iff the rank function $s \in \mathrm{Spec}(A) \to \dim_{k(s)}(M \otimes_A k(s))$ is constant.

(iv) Finally to prove Theorem 0.0.1, reduce to the case $X = \mathbb{P}^n_A$ and A a reduced Noetherian local ring. By (ii) it suffices to show that $H^0(\mathbb{P}^n_A, \mathscr{F}(r))$ is a flat A-module. Use (iii) together with the base change result (i) to conclude the proof of the Theorem.

(2) In this exercise you shall prove the following result.

Theorem 0.0.2. *Let X/k be a reduced, connected and proper scheme over an algebraically closed field k with $H^1(X, \mathscr{O}_X) = 0$[2]. Let T/k be any connected scheme of finite type. Then there exists a natural isomorphism of abelian groups.*

$$\phi : \mathit{Pic}(X) \times \mathit{Pic}(T) \to \mathit{Pic}(X \times_k T),$$

induced by the pullback map.

We shall do this in steps as follows.

(i) First show that ϕ is *always* injective (by giving a section to ϕ). Hence we are reduced to showing that ϕ is surjective.

(ii) Now let $\mathscr{L}$ be any line bundle on $X \times_k T$ and denote by p_2, the projection to T. Note that p_2 is a proper and flat morphism. Use the vanishing of $H^1(X, \mathscr{O}_X)$ to conclude that $R^0 p_{2*}\mathscr{L}$ commute with arbitrary base change (Use Corollary 7.4.0.3 from the notes). Combine this with Theorem 7.4.0.2 to conclude that $R^0 p_{2*}\mathscr{L} = p_{2*}\mathscr{L}$ must be locally free on T.

(iii) Let $\mathscr{L}$ be a line bundle on any *connected* scheme S which is proper over a field k. Show that $\mathscr{L}$ is trivial iff both $\mathscr{L}$ and $\mathscr{L}^{-1}$ have non-zero sections.

[2] Examples of such schemes are products of projective spaces, over $\mathbb{C}$ any smooth projective variety whose first Betti number is 0....

(iv) Now suppose $\mathscr{L}$ is a line bundle on $X \times_k T$ such that there exists a closed point $t_0 \in T$ such that $\mathscr{L}|_{X_{t_0}}$ is trivial. Show that $\mathscr{L}|_{X_t}$ is trivial for any closed point t.

Hint: Use (ii) to conclude that both $p_{2*}\mathscr{L}$ and $p_{2*}\mathscr{L}^{-1}$ are line bundles on T[3], whose formation commutes with arbitrary base change. Now use (iii).

(v) With $\mathscr{L}$ as in (iv) conclude that the natural map

$$\psi_{\mathscr{L}} : p_2^* p_{2*}\mathscr{L} \to \mathscr{L},$$

is an isomorphism.

Hint: Since $\psi_{\mathscr{L}}$ is a morphism of line bundles, it suffices to show that $\psi_{\mathscr{L}}$ is a surjection or equivalently $\mathscr{M} = \mathrm{coker}(\psi_{\mathscr{L}})$ is trivial. Since $\mathscr{M}$ is coherent and closed points are dense, it suffices to show that $\mathscr{M}_{(x,t)} = 0$ for all closed points $x \in X$ and $t \in T$. This follows from (iv).

(vi) Finally let $\mathscr{F}$ be *any* line bundle on $X \times_k T$. Choose a closed point $t_0 \in T$ and apply (v) to the line bundle $\mathscr{F} \otimes p_1^*\mathscr{F}^{-1}|_{X_{t_0}}$. Here p_1 is the projection from $X \times T$ to X and we identify X_{t_0} with X.

(vii) As an application conclude that $\mathrm{Pic}(\mathbb{P}(\mathscr{E})) \simeq \mathrm{Pic}(T)$ for any vector bundle $\mathscr{E}$ on a connected scheme T of finite type over an algebraically closed field k.

[3] Note all the assumptions about T and X one needs to conclude that $p_{2*}\mathscr{L}$ is indeed a *line bundle*.