PROBLEM SET 2 (DUE 01/29/25)

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1. Prove the following:

Let $\phi : A \to B$ be finite type morphism of Noetherian rings. Let M be a finite module over B. Then the loci

$$\{\mathfrak{q} \in \mathrm{Spec}(B) | M_{\mathfrak{q}} \text{ flat over } A_{\mathfrak{p}}\},$$

is open. Here $\mathfrak{p}$ is the contraction of $\mathfrak{q}$ in A. Fix a prime $\mathfrak{q}$ with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ such that $M_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$. Proceed as follows:

(1) Let I be any ideal of A. Show that there exists a $f \in B$ such that $f \notin \mathfrak{q}$ and

$$I \otimes_A M \to M,$$

is injective after tensoring with B_f.

(2) Show that any ideal of $A_{\mathfrak{p}}$ is the localisation of an ideal of A at $\mathfrak{p}$.

Now globalize the result by showing the following:
For any morphism $f : X \to Y$ of finite type between Noetherian schemes and $\mathscr{F}$ a coherent sheaf on X. The locus in X

$$\{x \in X | \mathscr{F} \text{ is flat over } Y \text{ at } x\},$$

is open.

2. Let $f : X \to Y$ be a morphism from a reduced scheme to a normal scheme Y of dimension 1. Then f is flat iff every generic point of X maps to a generic point of Y.
HINT: Use the fact that the local rings of Y are PID's and that flatness is the same as torsion freeness over a PID.

3. Let X be a scheme and let $\{U_i\}_{1 \leqslant i \leqslant n}$ be a finite open cover. Let $Y := \sqcup_i U_i$ and let $f : Y \to X$ be the natural map. Verify faithfully flat descent of quasi-coherent sheaves for f.

4. In this exercise we will see an application of Galois descent. A rank one torus T over $\mathbb{R}$ is an affine group scheme such that $T \times_{\mathbb{R}} \mathbb{C} \simeq \mathrm{GL}_1$ as affine group schemes over $\mathbb{C}$. Assume

Date: January 21, 2025.

that the endomorphism group of GL_1 (as an affine group scheme) is $\mathbb{Z}$.[1]. Classify all rank one tori over $\mathbb{R}$.
HINT: First come up with a $\mathbb{R}$-torus which is **not** isomorphic to GL_1 as $\mathbb{R}$-group schemes but becomes isomorphic to it over $\mathbb{C}$. For this you may want to use the coordinate ring $\frac{\mathbb{C}[x,y]}{xy-1}$. Then use Galois descent to give an upper bound on the possible number of such tori.

5. Let $\pi : \tilde{X} \to X$ be the normalization of a non-normal one-dimensional integral scheme of finite type over a field. Show that π is never flat.

[1]This is not hard. We will eventually prove this.