AG-II-Notes

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Contents

Chapter 1

Catgeory Theory Part-0

We begin by recalling some basic notions from category theory which should take some way into the course. This is far from an exhaustive account and focuses on introducing the bare minimum needed for the purposes of these lectures.

1.1 Categories: Definitions and Examples

Recall that a category $\mathscr{C}$ consists of a collection of objects $\mathrm{Ob}(\mathscr{C})$ and a collection of morphisms between these objects. The morphisms are required to satisfy certain properties:

1. For every object A in the category, there is an identity morphism 1_A from A to A.

2. For every pair of morphisms $f : A \to B$ and $g : B \to C$, there is a composite morphism $g \circ f : A \to C$.

3. Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.

4. Composition is unital: $1_B \circ f = f = f \circ 1_A$.

Example 1.1.0.1. The category **Set** has sets as objects and functions as morphisms. The identity morphism on a set A is the identity function $\mathrm{id}_A : A \to A$. The composite of two functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$. The associativity and unitality of composition follow from the corresponding properties of functions.

Example 1.1.0.2. The category **Top** has topological spaces as objects and continuous functions as morphisms. The identity morphism on a topological space X is the identity function $\mathrm{id}_X : X \to X$. The composite of two continuous functions $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f : X \to Z$. The associativity and unitality of composition follow from the corresponding properties of continuous functions.

Example 1.1.0.3. The category $\mathbf{Vect}_k$ has vector spaces over a field k as objects and linear transformations as morphisms. The identity morphism on a vector space V is the identity transformation $\mathrm{id}_V : V \to V$. The composite of two linear transformations $f : V \to W$ and $g : W \to Z$ is the transformation $g \circ f : V \to Z$. The associativity and unitality of composition follow from the corresponding properties of linear transformations.

Example 1.1.0.4. Let S be a scheme. Let $\mathbf{Sch}_S$ be the category whose objects are a pair (X, f), where X is a scheme and $f : X \to S$ a morphism. Morphisms ϕ in this category are commutative diagrams of the form

$$\begin{tikzcd} (X,f) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & (Y,g) \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

An important special case for us is the category $\mathbf{Sch}_k$ of schemes over a field $\mathrm{Spec}(k)$.

Example 1.1.0.5. Let X be a topological space. The category $\mathbf{Op}(X)$ has open sets in X as objects and inclusions as morphisms. The identity morphism on an open set U is the inclusion $U \hookrightarrow U$. The composite of two inclusions $U \hookrightarrow V$ and $V \hookrightarrow W$ is the inclusion $U \hookrightarrow W$. The associativity and unitality of composition follow from the corresponding properties of inclusions. In partiular for any two objects U and V either $\mathrm{Hom}_{\mathbf{Op}(X)}(U, V)$ is either empty or contains a unique morphism.

Example 1.1.0.6. Let $\mathscr{C}$ be a category. The opposite category $\mathscr{C}^{\mathrm{op}}$ has the same objects as $\mathscr{C}$ and morphisms reversed. That is, for every pair of objects A and B in $\mathscr{C}$, we have $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$. The identity morphism on an object A in $\mathscr{C}^{\mathrm{op}}$ is the identity morphism on A in $\mathscr{C}$. The composite of two morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}^{\mathrm{op}}$ is the composite $g \circ f : A \to C$ in $\mathscr{C}$. The associativity and unitality of composition follow from the corresponding properties of composition in $\mathscr{C}$.

1.2 Functors

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $F : \mathscr{C} \to \mathscr{D}$ assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$ and to each morphism $f : A \to B$ in $\mathscr{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathscr{D}$. Functors are required to satisfy the following properties:

1. For every object A in $\mathscr{C}$, we have $F(1_A) = 1_{F(A)}$.

2. For every pair of morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$, we have $F(g \circ f) = F(g) \circ F(f)$.

One can also have what are called as contravariant functors. A contravariant functor $F : \mathscr{C} \to \mathscr{D}$ assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$ and to each morphism $f : A \to B$ in $\mathscr{C}$ a morphism $F(f) : F(B) \to F(A)$ in $\mathscr{D}$. Contravariant functors are required to satisfy properties analogous to those for covariant functors.

Example 1.2.0.1. The forgetful functor F : **Top** $\to$ **Set** assigns to each topological space its underlying set and to each continuous function its underlying function. The identity function on a set is continuous, so the identity morphism on an object in **Top** is sent to the identity morphism on the corresponding object in **Set**. The composite of two continuous functions is continuous, so the composite of two morphisms in **Top** is sent to the composite of the corresponding morphisms in **Set**.

A more non-trivial functor from **Top** to **Set** is the functor Π_0.

Example 1.2.0.2. The functor Π_0 : **Top** $\to$ **Set** assigns to each topological space X the set of connected components $\Pi_0(X)$ of X and to each continuous function $f : X \to Y$ the function $\Pi_0(f) : \Pi_0(X) \to \Pi_0(Y)$ induced by f.

We now state a few properties of functors.

Definition 1.2.0.3. A functor $F : \mathscr{C} \to \mathscr{D}$ is faithful if for every pair of objects A and B in $\mathscr{C}$, the map $F : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ is injective. We say that F is fully faithful if this map is bijective.

Definition 1.2.0.4. A functor $F : \mathscr{C} \to \mathscr{D}$ is essentially surjective if for every object B in $\mathscr{D}$, there is an object A in $\mathscr{C}$ such that $F(A)$ is isomorphic to B.

The examples 1.2.0.1 and 1.2.0.2 are faithful and essentially surjective functors. Next we will discuss an important class of functors called representable functors.

Example 1.2.0.5. Let $\mathscr{C}$ be a category and A an object in $\mathscr{C}$. The representable functor $\mathrm{Hom}_{\mathscr{C}}(A, -) : \mathscr{C} \to$ **Set** assigns to each object B in $\mathscr{C}$ the set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of morphisms from A to B and to each morphism $f : B \to C$ in $\mathscr{C}$ the function $\mathrm{Hom}_{\mathscr{C}}(A, f) : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$ induced by f. The identity morphism on an object B in $\mathscr{C}$ is sent to the identity morphism on $\mathrm{Hom}_{\mathscr{C}}(A, B)$, and the composite of two morphisms $f : B \to C$ and $g : C \to D$ in $\mathscr{C}$ is sent to the composite of the corresponding morphisms $\mathrm{Hom}_{\mathscr{C}}(A, f)$ and $\mathrm{Hom}_{\mathscr{C}}(A, g)$.

Next we discuss natural transformations of functors.

Definition 1.2.0.6. Let F and G be two functors between categories $\mathscr{C}$ and $\mathscr{D}$. A natural transformation $\eta : F \to G$ assigns to each object A in $\mathscr{C}$ a morphism $\eta_A : F(A) \to G(A)$ in $\mathscr{D}$ such that for every morphism $f : A \to B$ in $\mathscr{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes. That is, we have $G(f) \circ \eta_A = \eta_B \circ F(f)$ for every morphism $f : A \to B$ in $\mathscr{C}$.

Example 1.2.0.7. Let $\mathbf{Vect}_k$ be the category of vector spaces over a field k. The double dual functor $\mathbf{Vect}_k \to \mathbf{Vect}_k$ assigns to each vector space V its double dual $V^{\vee\vee}$ and to each linear transformation $f : V \to W$ the linear transformation $f^{\vee\vee} : V^{\vee\vee} \to W^{\vee\vee}$ induced by f. The natural transformation $\eta : \mathrm{id} \to (-)^{\vee}$ assigns to each vector space V the canonical map $\eta_V : V \to V^{\vee\vee}$ and to each linear transformation $f : V \to W$ the commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"] \\ V^{\vee\vee} \arrow[r, "f^{\vee\vee}"'] & W^{\vee\vee} \end{tikzcd} .$$

Note that the vertical arrows are isomorphisms if and only if the vector spaces are finite-dimensional.

Definition 1.2.0.8. A natural transformation $\eta : F \to G$ of functors is a natural equivalence if for every object A in $\mathscr{C}$, the morphism $\eta_A : F(A) \to G(A)$ is an isomorphism in $\mathscr{D}$.

Now we are ready to state the Yoneda Lemma.

Lemma 1.2.0.9 (Yoneda Lemma). *Let $\mathscr{C}$ be a category and A an object in $\mathscr{C}$. Let $F : \mathscr{C} \to$ **Set** be a functor. Then the natural transformations $\mathrm{Hom}_{\mathscr{C}}(A, -) \to F$ are in bijection with the elements of $F(A)$.*

Proof. Let $\eta : \mathrm{Hom}_{\mathscr{C}}(A, -) \to F$ be a natural transformation. In particular, $\eta_A : \mathrm{Hom}_{\mathscr{C}}(A, A) \to F(A)$ is a morphism in **Set**. The desired element in $F(A)$ is simply the image of the identity morphism on A under η_A. Conversely, given an element x in $F(A)$, we can define a natural transformation $\eta : \mathrm{Hom}_{\mathscr{C}}(A, -) \to F$ by setting $\eta_B(f) = F(f)(x)$ for every object B in $\mathscr{C}$ and morphism $f : A \to B$ in $\mathscr{C}$. The naturality of η follows from the properties of functors. □

In particular we note the following corollary.

Corollary 1.2.0.10. *Let C be a category. Then the functor $\mathrm{Hom}_{\mathscr{C}}(A, -) : \mathscr{C}^{op} \to \mathrm{Func}(\mathscr{C},$ **Set**$)$ is fully faithful, where $\mathrm{Func}(\mathscr{C},$ **Set**$)$ is the category whose objects are functors from $\mathscr{C}$ to **Set** and morphisms are natural transformations.*

1.2.1 Limits and Colimits

Let $\mathscr{C}$ be a category and I a category. A functor $F : I \to \mathscr{C}$ is called a diagram in $\mathscr{C}$ indexed by I. A cone over F is an object A in $\mathscr{C}$ together with morphisms $A \to F(i)$ for every object i of I compatible with the functor F. A limit of F is a terminal object[1] in the category of

[1] Meaning it maps uniquely to any other cone

cones over F. Dually, a colimit of F is an initial object[2] in the category of co-cones[3] over F. Note that limits and colimits **maynot** exist in general but when they do they are unique upto unique isomorphism.
Limits are denoted by

$$\lim_{i \in I} F(i)$$

and colimits are denoted by

$$\operatorname{colim}_{i \in I} F(i).$$

Example 1.2.1.1. Let V and W be vector spaces over a field k, and let $f : V \to W$ be a linear transformation. The kernel of f is the limit of the diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ V & \xrightarrow{f} & W \end{array}$$

Here the indexing category is the category with two objects and one non-identity morphism between them.

We have the dual example.

Example 1.2.1.2. Let V and W be vector spaces over a field k, and let $f : V \to W$ be a linear transformation. The cokernel of f is the colimit of the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow & & \\ 0 & & . \end{array}$$

As before the indexing category is the category with two objects and one non-identity morphism between them.

You have seen these before!

Example 1.2.1.3. Let X and Y be schemes over S. The fibre product of X and Y (over S) is the limit of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & S. \end{array}$$

Here the indexing category is the category with three objects and two non-identity morphisms between them.

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[2]Meaning it gets an unique map from every cone
[3]Guess its definition!

Here is a basic and important example.

Example 1.2.1.4. Let I be a set. We say a category $\mathscr{C}$ has products (resp. coproducts) indexed by I if every functor indexed by I has limits (resp. colimits). Here I is considered as a category with objects indexed by elements of I and no non-identity morphisms.

Example 1.2.1.5. We say that a category $\mathscr{C}$ has finite limits (resp. colimits) if it has limits (resp. colimits) indexed by any category with finitely many objects and morphisms

Another important class of indexing category for us are the *flitered* ones. Let me give an example first.

Example 1.2.1.6. Let $\mathbb{N}$ be the set of natural numbers. We can consider $\mathbb{N}$ as a category with objects indexed by natural numbers and a unique morphism between any two objects. This is a filtered category.

Here is a formal definition.

Definition 1.2.1.7 (Filtered Category)**.** A category I is called filtered if for every pair of objects i and j in I, there is an object k in I and morphisms $f : i \to k$ and $g : j \to k$. Moreover for a pair of morphisms $f, g : i \to j$ in I, there is an object k in I and a morphism $h : j \to k$ such that $h \circ f = h \circ g$.

Example 1.2.1.8. Let R be a ring and M an R-module. Then M is a filtered colimit of its finitely generated submodules. This is often used to reduce statements about arbitrary modules to statements about finitely generated modules.

We have the following *very useful* but formal result.

Lemma 1.2.1.9. *Limits commute with right adjoints and colimits commute with left adjoints.*

1.3 Abelian Categories

We begin with the definition of an additive category.

Definition 1.3.0.1 (Additive Category)**.** An additive category is a category $\mathscr{A}$ with the following properties:

1. For every pair of objects A and B in $\mathscr{A}$, the morphism set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is an abelian group[4].

2. Composition of morphisms is bilinear.

[4] In particular there is a 0 morphism.

3. $\mathscr{A}$ has a zero object, that is, an object that is both initial and terminal[5].

4. $\mathscr{A}$ has finite products and coproducts i.e. the indexing set is finite.

Clearly the opposites of an additive category can also be naturally given a structure of an additive category. We will see additive categories later too, when we discuss cohomology. For now you may think of them as categories where you can add morphisms and have a zero object.

Definition 1.3.0.2 (Additive Functor). Let $\mathscr{C}$ and $\mathscr{D}$ be additive categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is additive if for every pair of objects A and B in sC, the map $F : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ is a group homomorphism. Moreover, F is required to preserve finite products and coproducts[6].

We need few more definitions before we can define an abelian category. In what follows we assume A is an additive category and all functors are additive.

Definition 1.3.0.3 (Kernels and Cokernels). Let $f : A \to B$ be a morphism in a category $\mathscr{A}$. A kernel of f is the limit of the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd} .$$

Dually a cokernel of f is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ 0 & \end{tikzcd} .$$

Remark 1.3.0.4. Kernels and cokernels may not exist in general. However when they do they are unique upto unique isomorphism.

We now define monomorphisms and epimorphisms in an additive category.

Definition 1.3.0.5. A morphism $f : A \to B$ in an additive category is a monomorphism if for every object C and morphisms $g, h : C \to A$ such that $f \circ g = f \circ h$, we have $g = h$. A morphism $f : A \to B$ in an additive category is an epimorphism if for every object C and morphisms $g, h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

Now we can state the definition of an abelian category.

[5] This makes it unique upto an unique isomorphism

[6] The product or coproduct indexed by the empty set is the 0-object. Hence F is required to take the zero object to the zero object.

Definition 1.3.0.6 (Abelian Category)**.** An abelian category is an additive category $\mathscr{A}$ with the following properties:

1. Every morphism in $\mathscr{A}$ has a kernel and a cokernel[7].
2. Every monomorphism in $\mathscr{A}$ is the kernel of its cokernel.
3. Every epimorphism in $\mathscr{A}$ is the cokernel of its kernel.

Example 1.3.0.7. The category of abelian groups can be given the structure of an abelian category. The zero object is the trivial group, the product is the direct sum, and the coproduct is the direct product. The kernel of a morphism $f : A \to B$ is the subgroup of elements a in A such that $f(a) = 0$, and the cokernel is the quotient group $B/\text{im}(f)$. The monomorphisms are the injective group homomorphisms, and the epimorphisms are the surjective group homomorphisms.

We can even restrict to the category of finitely generated abelian groups and get an abelian category. Note that finite coproducts are the same as finite products in both these cases. This is not a coincidence. In general in an abelian category finite products and coproducts are the same.
The last thing we need to get us going into geometry is exactness of functors.

Definition 1.3.0.8. Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is left exact if it preserves finite limits, and right exact if it preserves finite colimits. A functor $F : \mathscr{C} \to \mathscr{D}$ is exact if it is both left and right exact.

This coincides with the more usual definition as shown below[8].

Proposition 1.3.0.9. *Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then the following are equivalent:*

1. *F is left exact.*
2. *For every short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

   in $\mathscr{C}$, the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

   is exact in $\mathscr{D}$.

[7]This implies that in addition to have finite products and coproducts by virtue of $\mathscr{A}$ being additive, it also has finite limits and colimits (see Tag 010D)
[8]Ignore on first reading

3. *F preserves kernels.*

Proof. Clearly (2) implies (3). That (1) implies (3) follows from Definition 1.3.0.3. To see that (3) implies (2), it suffices to show that the image of $F(f)$ is the kernel of $F(g)$. Since F preserves monomorphisms, $F(f)$ is a monomorphism. Thus the image of $F(f)$ is naturally isomorphic to $F(A)$, which is the kernel of $F(g)$, since F preserves kernels and A is the kernel of g.
We are left to show that (3) implies (1). For this we use a general result that finite limits can be expressed in terms of kernels and finite products (see Tag 002P for a reference). Since F preserves both we are done.

□

We note here a very useful corollary to Lemma 1.2.1.9

Corollary 1.3.0.10. *Any right adjoint functor is left-exact and any left adjoint functor is right-exact.*

We conclude this section with a couple of examples.

Example 1.3.0.11. Let R be any ring[9] and M an R-module. The tensor product functor $- \otimes M : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is right-exact. This follows from the fact that it is a right adjoint to the Hom functor.

Example 1.3.0.12. Let $f : X \to Y$ be a morphism of schemes. The lower shriek functor f_* is left-exact. This follows from the fact that it is a left adjoint to the pullback functor f^* (which in turn is necessarily right-exact).

[9] All rings in this course are commutative with unity.

Chapter 2

Flatness

Consider the following three maps:

1. $f : \mathrm{Bl}_{(0,0)}\mathbb{A}^2 \to \mathbb{A}^2$,where $\mathrm{Bl}_{(0,0)}\mathbb{A}^2$ is the blow-up of $\mathbb{A}^2$ at the origin and f is the projection map.

2. $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ with $f(z) = z^2$.

3. $f : G_m \to G_m$ with $f(z) = z^2$. Here G_m is $\mathbb{A}^1 \backslash \{0\}$.

The map (1) here is an isomorphism on the complement of the origin, but over the origin the fiber is $\mathbb{P}^1$. The map (2) is nice outside the origin, with the inverse image of any $z \neq 0$ consisting of two points. But at the origin the fiber consists of exactly one point. The map (3) is simply the base change of (1) along the open immersion $G_m \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$, and hence all points have as inverse image exactly two distinct points.

Question 2.0.0.1. How do we capture the discontinuous jump in the fiber dimension at the origin in Example 1? Note that even though Example (2) has a *bad* fiber over the origin, it is still of dimension 0 like every other fiber.

The answer lies in the notion of flatness, a purely algebraic construct!

2.1 Flatness: Definition and Properties

We begin by defining flatness and faithful flatness.

Definition 2.1.0.1. Let A be a ring and M be an A-module. We say that M is **flat** over A if the right-exact functor $- \otimes_A M$ is exact. A map of rings $A \to B$ is said to be **flat** if B is flat as an A-module.

Definition 2.1.0.2. A flat A-module M is said to be **faithfully flat** if the functor $-\otimes_A M$ is faithful.

Let us see some examples of flat and faithfully flat modules.

Example 2.1.0.3.
1. The ring A is flat over itself.
2. Since tensor products are right adjoint, they commute with arbitrary colimits. moreover *filtered* colimits of exact sequences is exact. Combining these two, we get that filtered colimits of flat modules are flat.
3. Combining (1) and (2) we get that filtered colimits of the form $\operatorname{colim}_i M_i$, where each M_i is abstractly isomorphic to A is flat. Note that we dont care what the maps are as long as the indexing category is filtered.

Example 2.1.0.3, (3) has the following corollary.

Corollary 2.1.0.4. *The ring A_f is flat over A. More generally for any multiplicative subset S of A, the ring $A[S^{-1}]$ is flat.*

Proof. The first claim follows from the isomorphism

$$A_f \simeq \operatorname{colim}\{A \to A \to A \cdots\},$$

where the transition maps are multiplication by f. The second part of the claim follows from the isomorphism

$$A[S^{-1}] = \operatorname{colim}_{f\in S} A_f,$$

where the colimit is over the directed set indexed by elements of S, with $f \leqslant g$ if $g = ff'$ for some $f' \in A$. This is directed because S is multiplicative and further the first part of the Corollary implies each of the A_f's are flat. Hence the result.

□

Corollary 2.1.0.5. *For any ring A, arbitrary direct sums of A is a flat A-module. In particular when A is a field, all A-modules are flat.*

Corollary 2.1.0.6. *For any ring R the map $R \to R[x]$ is flat.*

Proof. Direct sums are colimits over an directed set with no non-identity arrows, hence the result. □

Next we list some properties of flatness.

Proposition 2.1.0.7. *We will need the following facts about flatness. Let $\phi : A \to B$ be a map of rings, M be an A-module and N a B-module. Then the following hold*

1. *M is flat over A iff for all finitely generated ideals $\mathfrak{a}$ of A the induced map*
$$\mathfrak{a} \otimes_A M \to M,$$
is injective.

2. *(Base-Change) M is flat over A implies $M \otimes_A B$ is flat over B.*

3. *(Transitivity) B flat over A and N flat over B implies N is flat over A.*

4. *(Local Nature) M is flat over A iff $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of A.*

5. *N is flat over A iff $N_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{q}$ of B, here $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$.*

6. *For a short exact sequence of A-modules*
$$0 \to M' \to M \to M'' \to 0,$$
M is flat if M' and M'' are flat. Also if M and M'' are flat, so is M'.

7. *For a Noetherian local ring A, a finitely generated module M is flat over A iff M is free over A.*

Proof. (1) is proved in Tag 00HD, (2) in Tag 051D, (3) in Tag 051D, (4) and (5) in Tag 051D, (6) in Tag 00HM and finally (7) in Tag 00NZ[1] □

We can now globalize the definition of flatness to schemes.

Definition 2.1.0.8 (Flatness). Let $f : X \to Y$ be a morphism of schemes. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ (resp. f) is flat over Y at a point $x \in X$ if the stalk $\mathscr{F}_x$ (resp. $\mathscr{O}_{X,x}$) is flat as a $\mathscr{O}_{Y,f(y)}$-module. If this holds for all points x in X we say $\mathscr{F}$ is flat over Y (resp. f is a flat morphism).

Remark 2.1.0.9. Note that flatness is local on both the source and the base. Meaning to check a sheaf $\mathscr{F}$ is flat (over Y) it suffices to check this on an open cover of either X or Y or both.

Now we translate Proposition 2.1.0.7 into the language of scheme.

Proposition 2.1.0.10. *Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ a $\mathscr{O}_X$-module of X. Then the following hold.*

[1] If you assume A is Noetherian, the proof can be simplified. As in the proof by Nakayama's Lemma we can pick a surjection $A^n \to M$ where n is the dimension of $\frac{M}{\mathfrak{m}M}$. Here $\mathfrak{m}$ is the unique maximal ideal of A. Suppose K is the kernel of this surjection. Then tensoring this exact sequence with $\frac{A}{\mathfrak{m}}$, we get that $\frac{K}{\mathfrak{m}K}$ is trivial by flatness of M, which by Nakayama implies K is trivial. (Question: Where did we use A is Noetherian?)

1. *If f is an open immersion then it is flat.*

2. *Suppose both X and Y are affine schemes, say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Then $\mathscr{F}$ is flat over Y iff M is flat over A where M is the A-module corresponding to $\mathscr{F}$.*

3. *A base change of a flat quasi-coherent sheaf[2] is flat. That is if we have a cartesian diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

and assume that $\mathscr{F}$ is flat and quasi-coherent, then the pullback $\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'}$ is flat over Y'.

4. *Suppose f was morphism over a base scheme S. If $\mathscr{F}$ is flat over Y and Y is flat over S, then $\mathscr{F}$ is flat over S. In particular composition of flat morphisms is flat.*

5. *Suppose we have a short exact sequence of quasi-coherent sheaves*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

Then $\mathscr{F}$ is flat if $\mathscr{F}'$ and $\mathscr{F}''$ are flat. Also if $\mathscr{F}$ and $\mathscr{F}''$ are flat, so is $\mathscr{F}'$.

6. *Let X be a Noetherian scheme and $\mathscr{F}$ a coherent sheaf. Then $\mathscr{F}$ is flat iff $\mathscr{F}$ is locally free aka a vector bundle.*

Proof. (1) is immediate from the definition since the induced map on local rings is an isomorphism. (2) follows from Proposition 2.1.0.7, (5). The claims (3)-(6) are now a consequence of Remark 2.1.0.9 and Proposition 2.1.0.7. □

Remark 2.1.0.11. 1. Fix a base scheme S. Consider the subcategory of $\mathbf{Sch}_S$ where we only allow morphisms which are flat between the objects. This is a subcategory of $\mathbf{Sch}_S$, and is closed under composition and base change.

2. Thanks to Corollary 2.1.0.6 and Remark 2.1.0.9, for any scheme X, the morphism $\mathbb{A}^n_X \to X$ is flat. More generally for an locally free sheaf $\mathscr{E}$ on a scheme X, the map $\mathbb{A}(\mathscr{E}) \to X$ is flat. Again using Remark 2.1.0.9, we can conclude that $\mathbb{P}(\mathscr{E}) \to X$ is flat.

Recall for any topological space X and a pair of points x and y in X, we have the following:

(a) x is a specialization of y if $x \in \overline{\{y\}}$.

(b) x is a generalisation of y if $y \in \overline{\{x\}}$.

[2]Hartshorne forgets writing quasi-coherent in Chapter III.9, Proposition 9.2 (b).

In particular when $X = \mathrm{Spec}(A)$, the constructible subsets of X which are stable under generalisation are open and those stable under specialization are closed (see [3, Chapter II, Exercise 3.18]

Proposition 2.1.0.12. *Let $f : X \to Y$ be a flat morphism of schemes. Then the image*[3] *of f is stable under generalization.*

Proof. Let y be a point in the image of f. We need to show that any point $y' \in Y$ such that $y \in \overline{\{y'\}}$, also belongs to the image of f. Choose an affine open $V \ni y$ and an affine open $U \ni x$ such that $f(x) = y$ and $f(U) \subseteq V$. It suffices to show that there is a point $x' \in U$ such that $f(x') = y'$. But this is precisely the going down theorem from local algebra (see Tag 00HS). $\square$

Corollary 2.1.0.13 (Openness of flat morphisms)**.** *Let $f : X \to Y$ be a flat morphism, locally of finite presentation*[4]*. Then f is universally open i.e the image of any base change of f is open.*

Proof. Since both flat morphisms and morphisms of finite presentation satisfy BC, we are reduced to showing the openness of f. We have already shown that the image of f is stable under generalizations (without any finite presentation assumptions). As before we can assume that both $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine with the map $A \to B$ bein of finite presentations. By Chevalley's theorem (see Tag 00FE), $\mathrm{Im}(f)$ is constructible and by Prop 2.1.0.12 it is stable under generalizations and hence is open. $\square$

Corollary 2.1.0.14. *Let $f : A \to B$ be a local and flat morphism of local rings. Then the induced maps on Spec is surjective.*

Proof. This is essentially the content of going down theorem. Every point of $\mathrm{Spec}(A)$ is a generalisation of the unique closed point. $\square$

Corollary 2.1.0.15. *Let $f : X \to Y$ be flat and proper morphism of finite presentation such that Y is irreducible. The f is surjective.*

2.2 Flatness and dimension of fibers

The following Proposition tells us that flat morphisms have well behaved fibers. This is mysterious (at least to me) given that flatness itself had a very algebraic definition.

[3] the set theoretic image

[4] For those who want to remain in the Noetherian world, anytime I say finite presentation you may assume that the schemes are Noetherian and that the morphism is of finite type.

Proposition 2.2.0.1. *Let $f : X \to Y$ be a flat morphism of locally Noetherian*[5] *schemes. Then for any point $x \in X$ we have,*

$$\dim(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{Y,f(x)}) + \dim(\mathscr{O}_{X_y,x}).$$

Proof. Since everything is local in x and y we may assume everything is sight is the Spectrum of a Noetherian ring. In which case the result follows from Tag 00ON. ☐

Example 2.2.0.2. This shows that the morphism (1) in the beginning of the chapter is not flat! The fiber over the origin is of dimension 1, while the fibers over other points are of dimension 0.

We derive one more corollary from Proposition 2.2.0.1.

Corollary 2.2.0.3. *Let $f : X \to Y$ be a flat morphism of schemes finite type over a field k with Y equidimensional*[6]*. Then TFAE*

1. *X is equidimensional of dimension equal to dim$(Y) + n$.*

2. *All fibers (not necessarily over closed points) of f are equidimensional of dimension n.*

In particular if both X and Y are irreducible then dim$(X) \geqslant$ dim(Y) and all the fibers are equidimensional of dimension dim(X) – dim(Y).

Proof. Suppose X if equidimensional of dimension dim$(Y) + n$. Let y be a *closed* point in Y with residue field $k(y)$. We would like to show that $X_y := X \times_{k(y)} Y$ is equidimensional of dimension n. Choose any irreducible component of X_y and in that component choose a closed point x in X_y. Note that x is closed in X (Why?). Then the dimension of X, X_y and Y can be computed using the dimension of the local rings at the points x and y. Thus we are done by Proposition 2.2.0.1.
Reduction the case y a closed point: Now suppose y is a possibly non closed point of Y. Then note that the map $\operatorname{Spec}(k(y)) \to Y$ factors via $Y \times_k k(y)$ and X_y can be considered as a fiber of the map induced between $X \times_k k(y) \to Y \times_k k(y)$ over the closed point $k(y)$ of $Y \times_k k(y)$. Note that both $X \times_k k(y)$ and $Y \times_k k(y)$ continue being equidimensional of dimension dim(X) and dim(Y) respectively (see Tag 00P4).
For the converse, choose a closed point $x \in X$, then $f(x) \in Y$ is a closed point (why?). Then again we are done by Proposition 2.2.0.1. ☐

But more is true! We have the following *miraculous* result, known colloquially as the *Miracle Flatness Theorem* due to Hironaka.

Theorem 2.2.0.4 (Miracle Flatness Theorem)**.** *Let $R \to S$ be a local morphism of Noetherian local rings. Assume that*

[5]We really need this to ensure dimensions are finite.
[6]Each irreducible component of Y has the same dimension.

1. *R is a regular local ring.*

2. *S is Cohen-Macaulay.*

3. *The dimension formula holds i.e,*

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S),$$

   where $\mathfrak{m}$ is the maximal ideal of R.

Then $R \to S$ is flat!

This has the following very useful corollary.

Corollary 2.2.0.5 (Miracle Flatness Theorem for schemes). *Let $f : X \to Y$ be a morphism of locally Noetherian schemes such that X is Cohen-Macaulay and Y is regular. Then f is flat iff the dimension formula holds.*

Example 2.2.0.6. This immediately implies that the examples (2) and (3) in the beginning of the chapter are flat. The fibers are of constant dimension 0.

Chapter 3

Faithful Flatness

3.1 Faithfully flat morphisms

Let $\phi : A \to B$ be a flat morphism of rings. We say ϕ is *faithfully* flat if B is a faithfully flat A-module. Surprisingly faithful flatness can be captured set theoretically!

Lemma 3.1.0.1. *ϕ is faithfully flat iff it is flat and the induced map $\phi^{\#} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.*

Proof. Let $\mathfrak{p}$ be a prime in A, then the induced map $A \to k(\mathfrak{p})$ is non-zero iff $A \otimes_A B \to k(\mathfrak{p}) \otimes_A B$ is non-zero.The latter necessarily implies the fiber over $\mathfrak{p}$ is non-empty. Conversely suppose $\phi^{\#}$ is surjective. We shall prove that for any A-module M, $M \otimes_A B = 0$ iff $M = 0$, a well known criterion for faithful flatness. Let $m \in M$ different from zero inducing an injection

$$0 \longrightarrow \frac{A}{I} \longrightarrow M,$$

here I is the annhilator of $m \in M$. Tensoring the above exact sequence with the flat ring B and knowing that $B \otimes_A \frac{A}{I}$ is non-zero, thanks to surjectivity of $\phi^{\#}$, implies the required result. $\square$

Combining Corollary 2.1.0.14 and Lemma 3.1.0.1 we obtain the following result.

Corollary 3.1.0.2. *Flat and local maps of local rings are faithfully flat.*

Motivated by Lemma 3.1.0.1 we have the following definition.

Definition 3.1.0.3. A morphism of schemes $f : X \to Y$ is said to be faithfully flat if it is flat and surjective.

Example 3.1.0.4. Now we give some examples of faithfully flat morphisms

1. Any extension of fields $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is faithfully flat.

2. Any proper and flat morphism whose target is an irreducible scheme is faithfully flat.

3. Let X be an affine scheme and let $X_{f_i}, 1 \leqslant i \leqslant n$ be a finite cover by basic affines, then
$$\sqcup_i X_{f_i} \to X,$$
is faithfully flat.

4. Let X be the projective space $\mathbb{P}^n$ and let $D(x_i), 0 \leqslant i \leqslant n$ be the standard affine covering corresponding to a choice of homogeneous coordinates. Then
$$\sqcup_i D(x_i) \to \mathbb{P}^n,$$
is faithfully flat.

We note the following obvious lemma.

Lemma 3.1.0.5. *Faithfull flatness is stable under base change and composition.*

3.2 Faithfully flat descent

Let X be any scheme and let $\{U_i\}_{1 \leqslant i \leqslant n}$ be an open cover of X. We have the following cartesian diagram

$$\begin{array}{ccc} \sqcup_{i,j} U_i \cap U_j & \xrightarrow{p_2} & \sqcup_i U_i \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle f} \\ \sqcup_j U_j & \xrightarrow{f} & X \end{array} .$$

Moreover for any schem T giving a morphism $X \to T$ is the same as giving a collection of morphisms $U_i \to T$ which agree on the intersections $U_i \cap U_j$. Put differently the following sequence of sets is exact

$$\mathsf{Hom}(X, T) \xrightarrow{f^*} \prod_i \mathsf{Hom}(U_i, T) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \prod_{i,j} \mathsf{Hom}(U_i \cap U_j, T).$$

There is nothing special about schemes here, one could have done the same starting with any toplogical space X and a cover $\{U_i\}_{1 \leqslant i \leqslant n}$. However doing so obscures the following important fact, the exactness of the above sequence is a consequence of faithfully flatness of f! This is the content of the following theorem.

Theorem 3.2.0.1 (Faithfully Flat descent). *Let X and Y be schemes over S. Let $S' \to S$ be a faithfully flat and quasi-compact morphism*[1]*. Let $S'' := S' \times_S S'$ and we denote by $X_{S'}$ (resp. $X_{S''}$) the base change of X along S' (resp. S''). We use a similar notation for Y. Then the following sequence of sets*

$$\mathit{Hom}_S(X,Y) \longrightarrow \mathit{Hom}_{S'}(X_{S'},Y_{S'}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathit{Hom}_{S''}(X_{S''},Y_{S''}),$$

is exact. Here p_1 and p_2 are induced by the projections $S'' \to S'$.

For a proof see Tag 023Q. Here is an application of faithfully flat descent. Let K/k be a finite Galois extension of field with Galois group G. Let X, Y be schemes over k. Let

$$X_K := X \times_k K, Y_K := Y \times_k K.$$

Every element $\sigma \in G$ acts on K while fixing k, thus inducing a morphism of $\mathrm{Spec}(K)$ as k-scheme. By functoriality of the fiber product we get an induced action of σ on $X_K := X \times_k K$ and $Y_K := Y \times_k K$. We denote this action by σ_X and σ_Y. Note that σ_X and σ_Y are *not* morphisms of K-schemes, rather they are only morphisms of k-schemes. Finally we get an action of G on $\mathrm{Hom}_K(X_K, Y_{K)})$ as follows:

$$f \to f^{sigma} := \sigma_Y \circ f \circ \sigma_X^{-1}. \tag{3.1}$$

Corollary 3.2.0.2 (Galois Descent). *The natural map $\mathit{Hom}_k(X,Y) \to \mathit{Hom}_K(X_K,Y_K)$ has image*

$$\mathit{Hom}_K(X_K,Y_K)^G,$$

i.e. precisely those morphisms that are invariant under G.

Proof. Lets start with some basic analysis. Since K/k is Galois we choose an $\alpha \in K$, such that $K = k(\alpha)$ as k-algebras. If $f(x)$ is the minimal polynomial of α, then we have

$$K \simeq \frac{k[x]}{(f(x))},$$

with $x \to \alpha$ under this isomorphism. Using the above isomorphism we identify

$$K \otimes_k K \simeq K \otimes_k \frac{k[x]}{(f(x))} \simeq \frac{K[x]}{(f(x))}.$$

Note that under the above isomorphism $\alpha \otimes 1 \to \alpha$ while $1 \otimes \alpha \to x$. Since K is the splitting field of $f(x)$, we can further identify

$$\psi : K \otimes_k K \simeq \prod_i \frac{K[x]}{(X - \alpha_i)} \simeq \prod_i K,$$

[1]Grothendieck coined the acronym *fpqc* (fidèlement plat et quasi-compact) for such morphisms.

where α_i's are the conjugates of α in K. Note that Ψ is a map of k-algebras and maps $\alpha \otimes 1 \to \alpha$ while $1 \otimes \alpha \to \alpha_i$ along the i^{th}-component. Put differently $1 \otimes \alpha \to \prod_{\sigma \in G} \sigma(\alpha)$. To summarize the diagram

$$K \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} K \otimes_k K$$

is isomorphic to the diagram

$$K \underset{\prod_{\sigma \in G} \sigma}{\overset{\Delta}{\rightrightarrows}} \prod_i K. \tag{3.2}$$

Now we can get back to proving the corollary. Consider the Cartesian diagram

$$\begin{tikzcd}
X_K \times_X X_K \arrow[r, "p_2"] \arrow[d, "p_1"] & X_K \arrow[d, "f"] \\
X_K \arrow[r, "f"] & X
\end{tikzcd}$$

The morphism f is fpqc and hence by Theorem 3.2.0.1 we have the exact sequence

$$\mathrm{Hom}_k(X, Y) \xrightarrow{f^*} \mathrm{Hom}_K(X_K, Y_K) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{K \otimes_k K}(X_{K \otimes_k K}, Y_{K \otimes_k K}).$$

Note that we have isomorphisms $X \times_k (K \otimes_K K) \simeq \sqcup_{\sigma \in G} X_K$ and $Y \times_k (K \otimes_K K) \simeq \sqcup_{\sigma \in G} Y_K$, where the first one comes from properties of fiber product and the last one is the isomorphism ψ above. Further under this identification we may identify p_1 with map which is identity on each of the factors, while p_2 is identified with the map which sends the factor X_K corresponding to σ by σ_X onto X_K. If we start with a morphism $f : X_K \to Y_K$, then it follows from the above isomorphisms that

$$p_1^*(f) = p_2^*(f) \implies f = f^\sigma, \forall \sigma \in G.$$

$\square$

Here is a simple example to see this in action.

Example 3.2.0.3. Let $X = Y = \mathrm{Spec}(\mathbb{R}[x])$. A morphism $f : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is given by $x \to p(x)$, for a complex polynomial $p(x)$. By our criterion this descends iff $\bar{p}(x) = p(x)$, here $\bar{p}(x)$ is the polynomial obtained by applying the uniqe non-trivial element of Gal$(\mathbb{C}/\mathbb{R})$ on the coeffcicients of $p(x)$. In other words $p(x)$ should be a polynomial with real coefficients.

Theorem 3.2.0.1 is the tip of the fpqc descent iceberg. Colloquially Theorem 3.2.0.1 is referred to by saying that morphisms descent along fpqc covers. Here $f : S' \to S$ is thought of as an "cover" of S. We have the following beautiful result.

Theorem 3.2.0.4. *The following properties of morphisms descend along a fpqc cover:*

1. *separatedness,*

2. *properness,*

3. *affineness,*

4. *open immersion,*

5. *closed immersion,*

6. *isomorphism,*

7. *finitness,*

8. *quasi-finiteness.*

For a proof see Tag 02YJ.

Example 3.2.0.5. Suppose $f : X \to Y$ is a morphism of varieties over the rational numbers $\mathbb{Q}$. Let us say you want to prove that f is an isomorphism. Theorem 3.2.0.4 implies that we can base change to $\mathbb{C}$ to prove this. In certain situations this can be quite profitable, for example one can use analytic techniques over $\mathbb{C}$ to prove this which apriori were not accesible over $\mathbb{Q}$.

Before we end this section I would like to state one more result which is a consequence of faithfully flat descent. Let us revisit Example 3.1.0.4 (4). This open covering was crucial in constructing quasi-coherent sheaves on projective space. Well it turns out that all we needed was that the covering was faithfully flat. This is the content of the following theorem.

Theorem 3.2.0.6. *Let $f : Y \to X$ be a fpqc morphism of schemes. Then there is an equivalence of categories between quasi-coherent sheaves on X and those quasi-coherent sheaves $\mathscr{F}$ on Y which satisfy gluing (or more appropriately descend) conditions:*

1. *There exists an isomorphism $\alpha : p_1^*\mathscr{F} \simeq p_2^*\mathscr{F}$ on $Y \times_X Y$.*

2. *α satisfies the cocycle condition on $Y \times_X Y \times_X Y$,*

$$p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha.$$

   Here p_{ij} is the projection onto the i^{th} and j^{th} factors.

Moreover the equivalence above respects coherence, local freeness etc..For a proof we refer to Tag 023R

Chapter 4

Smoothness

Recall that a manifold is a topological space that is locally isomorphic to $\mathbb{R}^n$. What we would like is an analogous definition in Algebraic Geometry. Unfortunately a literal analogue would not work. For example, if X is a one-dimensional variety which is Zariski locally isomorphic to $\mathbb{A}^1$, then X is forced to be either $\mathbb{A}^1$ or $\mathbb{P}^1$ (Why?). Even more bizzare things can happen in Algebraic Geometry. Consider the map

$$\phi : \mathbb{A}^1_{\bar{\mathbb{F}}_p} \to \mathbb{A}^1_{\bar{\mathbb{F}}_p},$$

with $\phi(z) = z^p$. Note that every fiber of ϕ is non-reduced. In the language of manifolds every value is a critical value; something not possible in the world of manifolds thanks to Sard's theorem.
The theory of smoothness in Algebraic Geometry has to take into account both the geometric intuition coming from manifolds and the arithmetic complexities arising from various base fields.

4.1 Kähler differentials

Recall that for a smooth manifold X, the tangent vectors at a point x act by derivations on smooth functions around x. In particular if $\mathscr{O}_{X,x}$ is the local ring of smooth functions at x, then to every tangent vector v we can associate a derivation $D_v : \mathscr{O}_{X,x} \to \mathbb{R}$ which satisfies

$$D_v(fg) = fD_v(g) + gD_v(f), \tag{4.1}$$

for any two functions $f, g \in \mathscr{O}_{X,x}$.
Note in particular that Equation (4.1) implies that $D_v(\alpha) = 0, \forall \alpha \in \mathbb{R}$. This motivates the following definition.

Definition 4.1.0.1. Let B be an A-algebra and M a B-module. Then a A-derivation of B with values in M is an A-linear map $D : B \to M$ satisfying the Leibniz rule

$$D(fg) = fD(g) + gD(f), \forall f, g \in B.$$

We denote by $\mathrm{Der}_A(B, M)$ the set of A-derivations from B with values in M.

Remark 4.1.0.2. We note the following obvious properties:

1. For any A derivation D, $D(1.1) = D(1) + D(1) \implies D(1) = 0$. Since D is A-linear, this implies $D(a) = 0, \forall a \in A$.

2. For any $b \in B$ and an A-derivation D, $b.D(f) := bD(f)$ is also an A-derivation. Thus $\mathrm{Der}_A(B, M)$ is a B-module.

3. Let D be an A-derivation of B with values in M. Let $\phi : M \to M'$ be a B-module map. Then $\phi \circ D : B \to M'$ is an A-derivation with values in M'.

Now suppose $D : B \to M$ be any A-module map (derivation or not), then by universal property of tensor products, there exists an unique map of B-modules, $\tilde{D} : B \otimes_A B \to M$ such that $\tilde{D}(b \otimes b') = b'D(b)$. Here $B \otimes_A B$ is thought of as a B-module via the natural map $p_2^* : B \to B \otimes_A B$ given by $b' \to 1 \otimes b'$.
Let $I \subseteq B \otimes_A B$ be the kernel of the multiplication map $m : B \otimes_A B \to B$. We claim I is generated by $b \otimes 1 - 1 \otimes b$. To see this note that $\sum_i (b_i \otimes b_i')$ is in the kernel iff $\sum_i b_i b_i' = 0$. Hence $\sum_i b_i \otimes b_i' = \sum_i (b_i \otimes 1 - 1 \otimes b_i) b_i'$. We now have the following easy lemma.

Lemma 4.1.0.3. *If D in addition is assumed to satisfy Leibniz rule then $\tilde{D}(I^2) = 0$.*

Proof. We can check this on a set of generators of I^2 as a B-module namely elements of the form $(b \otimes 1 - 1 \otimes b)(b' \otimes 1 - 1 \otimes b')$, where this follows from Leibniz rule. □

Thus there exists an unique B-module map

$$\phi : \frac{I}{I^2} \to M,$$

such that $\phi(\bar{\alpha}) = \tilde{D}(\alpha)$, for any $\alpha \in I$ with image $\bar{\alpha} \in \frac{I}{I^2}$. Note here that the B-module structure on $\frac{I}{I^2}$ is the one induced from p_2^*. However it is easy to check that on $\frac{I}{I^2}$, the B-module structure induced by p_1^* is the same as the one induced by p_2^* and moreover there is a natural map

$$d_{B/A} : B \to \frac{I}{I^2},$$

defined by $b \to b \otimes 1 - 1 \otimes b$, which is a A-derivation. Thus we have shown the following.

Proposition 4.1.0.4. *For any A-algebra B there exists an unique B-module $\Omega^1_{B/A} := \frac{I}{I^2}$ together with an universal derivation $d_{B/A} : B \to \Omega^1_{B/A}$ such that for any B-module M*

$$\mathit{Der}_A(B, M) \simeq \mathit{Hom}_B(\frac{I}{I^2}, M).$$

Thank to the canonical nature of our construction it is clear how to globalize this.

Definition 4.1.0.5. Let $f : X \to Y$ be a morphism of schemes. Then the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is an immersion i.e. there exists an open $U \subset X \times_Y X$ such that $X \subseteq U$ is a closed immersion defined by an ideal $\mathscr{I}$. We define the sheaf of relative Kähler differentials of X/Y as $\frac{\mathscr{I}}{\mathscr{I}^2}$[1].

Note that by construction $\Omega^1_{X/Y}$ is a quasi-coherent sheaf on X. Moreover if we assume that Y is Noetherian and f is of finite type, $X \times_Y X$ is Noetherian and hence so is U and thus the ideal sheaf $\mathscr{I}$ is coherent implying the coherence of $\frac{\mathscr{I}}{\mathscr{I}^2}$. It follows from the construction of $\Omega^1_{X/Y}$ that there is $f^{-1}\mathscr{O}_Y$-linear map

$$d_{X/Y} : \mathscr{O}_X \to \Omega^1_{X/Y},$$

which on local sections is defined by $d_{X/Y}(f) = f \otimes 1 - 1 \otimes f$, and is universal for $f^{-1}\mathscr{O}_Y$-linear derivations from $\mathscr{O}_X \to \mathscr{F}$, here $\mathscr{F}$ is any quasi-coherent $\mathscr{O}_X$-module. Here are some basic properties of Kähler differentials.

Proposition 4.1.0.6. *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

1. *There is a natural morphism of $\mathscr{O}_{X'}$-modules, $g'^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$.*
2. *If $Y' = Y$ and g is the identity map. Then there is an exact sequence of sheaves on X*

$$g'^*\Omega^1_{X/Y} \longrightarrow \Omega^1_{X'/Y} \longrightarrow \Omega^1_{X'/X} \longrightarrow 0$$

3. *If the above diagram is Cartesian then the morphism in (1) induces as isomorphism $g'^*\Omega^1_{X/Y} \simeq \Omega^1_{X'/Y'}$ and $\Omega^1_{X'/Y} \simeq f'^*\Omega^1_{Y'/Y} \oplus g'^*\Omega^1_{X/Y}$.*

Proof. For a proof see Section 00RM ∎

[1] Easy check, this is independent of choice of U

We also have the following important result.

Proposition 4.1.0.7. *Let $f : X \to Y$ be a morphism of schemes. Let Z be a closed subscheme of X. Then*

1. $\Omega^1_{Z/X} \simeq 0$.

2. *The right exact sequence from Proposition 4.1.0.6, (3) can be extended to*

$$\mathscr{I}_Z/\mathscr{I}_Z^2 \xrightarrow{\delta} \Omega^1_{X/Y}|_Z \longrightarrow \Omega^1_{Z/Y} \longrightarrow \Omega^1_{Z/X} = 0,$$

where the map δ is induced by restricting $d_{X/Y} : \mathscr{O}_X \to \Omega^1_{X/Y}$ to $\mathscr{I}_Z$.

Proof. For a proof see Section 00RM ∎

4.1.1 Computing Kähler differentials

In this section we shall comput the sheaf of Kähler differentials in some important cases. Before we start let us make some remarks

Remark 4.1.1.1.
1. We have already seen closed immersions have vanishing relative Kahler differentials. A similar argument also works for open immersions.

2. Let $X := X_1 \sqcup X_2$, then $\Omega^1_{X/Y} \simeq \Omega^1_{X_1/Y} \sqcup \Omega^1_{X_2/Y}$. This follows easily from the universal property or the definition of the sheaf of relative differenrtials.

3. Let B be a directed colimit of A-algebras. Then $\Omega^1_{B/A}$ is colimit of the corresponding Ω^1's. Again this can be checked using the universal property. In particular Ω^1 commutes with localization.

Lemma 4.1.1.2. *Let $X = \mathrm{Spec}(K)$ and $Y = \mathrm{Spec}(k)$ where K/k is a finite separable extension of fields. Then $\Omega^1_{X/Y} \simeq 0$.*

Proof. Let $\bar{k}$ be an algebraic closure of k and let $Y' = \mathrm{Spec}(\bar{k})$. Using Proposition 4.1.0.6, (2) and fpqc descent enough to show that $\Omega^1_{X'/Y'} = 0$ where X' is the base change of X along Y. Since K/k is a finite separable extension, we are done by Remark 4.1.1.1, (2) above. ∎

Corollary 4.1.1.3. *Using Remark 4.1.1.1, (3) it follows that $\Omega^1_{K/k} = 0$, for any separable and algebraic extension K/k.*

Lemma 4.1.1.4. *Let $(B, \mathfrak{m}, k)$ be a local ring containing a copy of k. Then the natural map δ induced from Proposition 4.1.0.7, (2)*

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \to \Omega^1_{B/k} \otimes_B k,$$

is an isomorphism.

Proof. Easy exercise. □

This immediately implies the following corollary.

Corollary 4.1.1.5. *Let X/k be a scheme and $i : \mathrm{Spec}(k) \to X$ be a closed point (denoted by x) and let $\mathfrak{m}_x$ be the maximal ideal of the local ring at the point x. Then the map $\delta : \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \to i_x^*\Omega^1_{X/k}$ is an isomorphism.*

In particular we have the following isomorphism

$$\mathsf{Hom}_k(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}, k) \simeq \mathsf{Hom}_k(\Omega^1_{X/k}, k) \simeq \mathsf{Der}_k(\mathscr{O}_{X,x}, k).$$

This motivates the following definition.

Definition 4.1.1.6 (Zariski Tangent Space)**.** Let X be a scheme and let $x \in X$ be a point with residue field $k(x)$. We define the Zariski tangent space to X at x to be $\mathsf{Hom}_{k(x)}(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}, k(x))$.

We can now combine Lemma 4.1.1.2 and Corollary 4.1.1.5 to obtain the following.

Corollary 4.1.1.7. *Let $X/\mathrm{Spec}(k)$ be finite. Then $\Omega^1_{X/k} \simeq 0$ iff $X \simeq \sqcup \mathrm{Spec}(K_i)$, where K_i/k are finite separable extensions of fields iff X is geometrically reduced.*

Proof. Clearly X/k is geometrically reduced iff X is a finite disjoint union of $\mathrm{Spec}(K_i)$'s with K_i/k finite and separable.
Suppose X/k is geometrically reduced. Then since X/k is finite, $X_{\bar{k}}/\bar{k}$ is a finite reduced scheme. Thus $X_{\bar{k}}$ is a finite disjoint union of $\mathrm{Spec}(\bar{k})$ which in turn implies that $\Omega^1_{X_{\bar{k}}/\bar{k}}$ vanishes and hence $\Omega^1_{X/k}$ vanishes too. Conversely if $\Omega^1_{X/k}$ vanishes then so does $\Omega^1_{X_{\bar{k}}/\bar{k}}$. Thus implies every connected component of $X_{\bar{k}}$ (a spectrum of an Artin local ring with residue field $\bar{k}$) must have maximal ideal 0, thanks to Lemma 4.1.1.5. □

Lemma 4.1.1.8. *Let X be any scheme and $\mathbb{A}^n_X$ be an affine space over X. Then $\Omega^1_{\mathbb{A}^n_X/X} \simeq \oplus_i \mathscr{O}_{\mathbb{A}^n_X} dx_i$. In particular $\Omega^1_{\mathbb{A}^n_X/X}$ is locally free of rank n.*

Proof. Using Proposition 4.1.0.6, (4) we are reduced to the case $n = 1$ and further we may assume $X = \mathrm{Spec}(A)$. In this case the result is obvious using universal property of Kähler differentials. □

We now compute the sheaf of Kähler differentials for projective space.

Proposition 4.1.1.9. *Let* $Y = \mathrm{Spec}(A)$ *and* $X = \mathbb{P}^n_A$. *Then there is an exact sequence of sheaves*[2] *on* X,

$$0 \longrightarrow \Omega^1_{X/Y} \longrightarrow \mathscr{O}_X(-1)^{\oplus(n+1)} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

Remark 4.1.1.10. We already know thanks to Lemma 4.1.1.8 that $\Omega^1_{X/Y}$ is locally free of rank n.

Proof. Consider the sheaf $\mathscr{O}_X(1)$, we know that this is globally generated by its sections, and thus we have a surjection of sheaves

$$\psi : H^0(X, \mathscr{O}(1)_X) \otimes_A \mathscr{O}_X \twoheadrightarrow \mathscr{O}_X(1).$$

We claim:

1. There exists a natural injection

$$\phi : \Omega^1_{X/Y}(1) \to H^0(X, \mathscr{O}(1)_X) \otimes_A \mathscr{O}_X, \tag{4.2}$$

2. with $\mathrm{Im}(\phi) = \ker(\psi)$.

This would give the Euler sequence (upto a twist).
We would like to think of $\mathbb{P}^n_A$ as obtained by gluing $n+1$-copies of $\mathbb{A}^n_A$ denoted by

$$D(x_i) := \mathrm{Spec}(A[\frac{x_0}{x_i}, \frac{x_1}{x_i} \cdots \frac{x_n}{x_i}]).$$

together with the gluing data

$$\theta_{ij} : \mathrm{Spec}(A[\frac{x_0}{x_i}, \frac{x_1}{x_i} \cdots \frac{x_n}{x_i}]_{\frac{x_j}{x_i}}) \simeq \mathrm{Spec}(A[\frac{x_0}{x_j}, \frac{x_1}{x_j} \cdots \frac{x_n}{x_j}]_{\frac{x_i}{x_j}}),$$

given by an A-algebra isomorphism $\theta^*_{ij}(\frac{x_k}{x_j}) = \frac{x_k}{x_j}$. We fix once and for all a basis $e_i, 0 \leqslant i \leqslant n$ for $H^0(X, \mathscr{O}_X(1))$ as an A-module. Restricted to each $D(x_i)$, the morphism Ψ is given by

$$\psi|_{D(x_i)}(e_k \otimes 1) = \frac{x_k}{x_i}, \forall k \neq i \tag{4.3}$$

for $k = i$,

$$\psi|_{D(x_i)}(e_i \otimes 1) = 1.$$

Moreover giving a map ϕ as in (4.2), amounts to giving for each i maps

[2] called the Euler sequence

$$\phi_i : \Omega^1_{D(x_i)/Y} \to H^0(X, \mathscr{O}_X(1)) \otimes_A \mathscr{O}_{D(x_i)},$$

such that

$$\phi_j \circ \frac{x_j}{x_i}\theta_{ij} = \theta_{ij} \circ \phi_i \tag{4.4}$$

on $D(X_i) \cap D(X_j)$, where we have used θ_{ij} to denote the induced map on both Ω^1 and $\mathscr{O}$ and the $\frac{x_j}{x_i}$ factor accounts for the twist by $\mathscr{O}_X(1)$.
We fix once and for all a basis $e_i, 0 \leqslant i \leqslant n$ for $H^0(X, \mathscr{O}_X(1))$ as an A-module. Thanks to Lemma 4.1.1.8, we know how $\Omega^1_{D(x_i)/A}$ looks like and we define

$$\phi_i(d(\frac{x_k}{x_i})) := (e_k \otimes x_i - e_i \otimes x_k)\frac{1}{x_i}. \tag{4.5}$$

It follows from (4.3) that $\ker(\psi|_{D(x_i)}) = \operatorname{Im}(\phi_i)$. Thus we are only left to check the gluing condition for ϕ_i as in equation (4.4). This follows from the identity

$$d(\frac{x_k}{x_i}) - \frac{x_k}{x_j}d(\frac{x_j}{x_i}) = \frac{x_j}{x_i}d(\frac{x_k}{x_j}),$$

on $\operatorname{Spec}(A[\frac{x_0}{x_i}, \frac{x_1}{x_i} \cdots \frac{x_n}{x_i}]_{\frac{x_j}{x_i}})$.

$\square$

4.2 Smoothness

Recall that a smooth manifold X is essentially a topological space with local charts $\{U_i\}$, which are in turn isomorphic to $\mathbb{R}^n$. Unfortunately this model is not good enough to model smoothness in algebraic geometry. For example, if X is a one-dimensional normal variety over $\mathbb{C}$ with an open subset isomorphic to $\mathbb{A}^1$, then in fact X is either $\mathbb{A}^1$ or $\mathbb{P}^1$! So clearly this approach to smoothness is very rigid and needs to modified to account for the so called curves of higher genus. As it turns out even zero dimensional smooth varieties are quite interesting and studying them helps us get to the *correct* definition of smoothness. Before we proceed further let us write down a list of properties we want out of smoothness:

1. We would like to define smoothness in a relative set-up $f : X \to Y$.

2. We would like smooth morphisms to be stable under base change and composition. In particular fibers of smooth morphisms should be smooth schemes over a field.

3. We would like (relative) affine and projective spaces to be smooth.

4. Finally for varieties over an algebraically closed field, one should be able to detect smoothness by the size of its Zariski tangent space (see Definition 4.1.1.6).

Remark 4.2.0.1. Through out this section you may assume either that we are working with Noetherian schemes and finite type morphisms or with arbitrary schemes and morphisms of finite presentation. In particular all relative sheaves of differentials will be coherent sheaves. With a little more effort one can set things up for arbitrary morphisms allowing us to talk about smoothness of say $\mathbb{C}/\mathbb{Q}$!

4.2.1 Étale morphisms

We begin with the definition of étale morphisms.

Definition 4.2.1.1 (étale morphisms). Let $f : X \to Y$ be a morphism. We say f is étale at $x \in X$ if it is flat at x and if the stalk of $\Omega^1_{X/Y}$ vanishes at x. We say f is étale if it is so at every point of X

Remark 4.2.1.2.
1. It immediately follows from Proposition 2.1.0.10, (c) and Proposition 4.1.0.6, (c) that class of étale morphisms is stable under Base Change. Using Proposition 4.1.0.6, (b) it also follows that étale morphisms are stable under composition.

2. Note that by Definition 4.1.0.5 it follows that the immersion $\Delta_{X/Y} : X \to X \times_Y X$ is an *open immersion* when X/Y is étale.

3. Let $f : X \to Y$ be étale at $x \in X$. Since flatness and vanishing of $\Omega^1_{X/Y}$ are both open conditions, so is being étale. Moreover étale morphisms being flat necessarily have an open image.

Let us note down some examples of étale morphisms.

Example 4.2.1.3. Let K/k be a finite separable extension of fields. Then $\mathrm{Spec}(K)/\mathrm{Spec}(k)$ is an étale morphism by Lemma 4.1.1.2. More generally $X = \sqcup_i^n \mathrm{Spec}(K_i)$[3] is étale over $\mathrm{Spec}(k)$ where each K_i/k is a finite separable extension. In Problem Set 3 you will show that $X/\mathrm{Spec}(k)$ a finite morphism is étale iff X is of the above form.

Example 4.2.1.4. Let $j : U \hookrightarrow X$ be an open immersion. Then j is étale.

Here we note down some basic properties of étale morphisms.

Proposition 4.2.1.5. *Let $f : X \to Y$ be an étale morphism of schemes over S. Then the following are true.*

1. *The fibers of f are spectrums of étale algebras. In particular f is quasi-finite.*

2. *The natural map $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism.*

[3] The ring of functions on such an X are called étale algebras over k.

Proof. Since base change of étale morphisms is étale, the fibers of f over any point $y \in Y$ are étale over $\mathrm{Spec}(k(y))$. Quasi-finiteness now follows from Example 4.2.1.3.
For (2), one can use the definition of Ω^1 and that fact that $X \hookrightarrow X \times_Y X$ is an open immersion to conclude the same.

□

We have an converse to Proposition 4.2.1.5

Proposition 4.2.1.6. *Let $f : X \to Y$ be a morphism. Then f is étale iff f is flat and all the fibers are spectrums of étale algebras iff all the geometric fibers are reduced and 0-dimensional.*

Proof. We have already seen that spectrums of étale algebras are geometrically reduced. The converse is easy. So we shall prove that f is étale iff the geometric fibers are reduced and 0-dimensional. $\implies$ direction is clear. For the other direction, we have to show that $\Omega^1_{X/Y}$ vanishes. Since $\Omega^1_{X/Y}$ is a coherent sheaf it suffices to show that for any point $x \in X$, the $k(x)$-vector space $\Omega^1_{X/Y} \otimes k(x)$ vanishes. This follows from Proposition 4.1.0.6, (3) and Example 4.2.1.3 □

Remark 4.2.1.7. Here is another criterion for étaleness which follows from Proposition 4.1.0.6, (2): f is flat and the natural map $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism.

Thanks to Proposition 4.2.1.6 we can now generate a lot of examples of étale morphisms.

Example 4.2.1.8. Let $\psi : \mathbb{C}^* \to \mathbb{C}^*$ be the squaring map $z \to z^2$. We have seen that this is flat. Moreover the fiber over any point is reduced and thus ψ is étale. However the extension if ψ to all of $\mathbb{C}$ is not étale over the origin.

4.2.2 Smooth Morphisms

We are now ready to define smooth morphisms. Again recall that for us either all schemes are Noetherian and morphisms are of finite type or we work in the finite presentation scenario. Our definition of smoothness differs from that of Hartshorne but is closer in spirit to differential geometry.

Definition 4.2.2.1 (Smooth Morphisms). Let $f : X \to Y$ be a morphism. We say f is smooth at $x \in X$ if there exists an open $U \ni x$ and a morphism $g : U \to \mathbb{A}^n_Y$[4] (for some $n \geq 0$), which is étale at x such that following diagram commutes

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{A}^n_Y \arrow[d] \\ & Y \end{tikzcd},$$

here the vertical arrow is the projection map.

[4]One can think of g as giving a local choice of coordinates around the point x.

Remark 4.2.2.2. We remark on same basic properties of smooth morphisms which follow immediately from the definition of smoothness:

1. Smooth morphisms are flat and in particular have open image.
2. Smoothness is an open condition since being étale is (see Remark 4.2.1.2).
3. Base change of a smooth morphism is smooth by stability of étale morphisms under base change.
4. Let $f : X \to Y$ be smooth at x and $f' : Y \to S$ be smooth at $f(x)$. Suppose $g : U \to \mathbb{A}^n_Y$ is a local choice of coordinates around x, and $h : V \to \mathbb{A}^m_S$ is a local choice of coordinates around $f(x) \in Y$. Then $g \times f : U \cap f^{-1}(V) \to \mathbb{A}^{m+n}_S$ give a local choice of coordinates around x for $f' \circ f$.

Before we discuss properties of smooth morphisms let us note down some examples.

Example 4.2.2.3.

1. For any scheme S, $\mathbb{A}^n_S \to S$ is smooth.
2. Open immersions, and more generally étale morphisms are smooth.
3. Smoothness is local in both the source and base. Hence (1), above implies $\mathbb{P}^n_S \to S$ is smooth.

Here is an easy lemma.

Lemma 4.2.2.4. *Let $f : X \to Y$ be a morphism smooth at $x \in X$. Then $\Omega^1_{X/Y}$ is locally free around x. In particular if $f : X \to Y$ is smooth, then $\Omega^1_{X/Y}$ is locally free aka a vector bundle on X.*

Proof. Combine Lemma 4.1.1.8 and Proposition 4.2.1.5. □

Notations 4.2.2.5. Let $f : X \to Y$ be a smooth morphism. The rank of f at a point x is the rank of the locally free sheaf $\Omega^1_{X/Y}$ at x. This is a locally constant function on X.

Following lemma is an easy consequence of quasi-finiteness of étale morphisms.

Lemma 4.2.2.6. *Let $f : X \to Y$ be a smooth morphism. Then for any closed point $x \in X$*

$$\mathit{dim}_x(X_{f(x)}) = \mathit{dim}_{k(x)}(\Omega^1_{X/Y,k(x)}).$$

Proof. Choose a coordinate neighborhood $U \ni x$ with an étale map $g : U \to \mathbb{A}^n_Y$. Since g is quasi-finite (see Proposition 4.2.1.5) and flat, the induced map $U \cap X_{f(y)} \to \mathbb{A}^n_{k(f(y))}$ is quasi-finite and flat. It follows from Corollary 2.2.0.3, that each component of $U \cap X_{f(y)}$ has dimension n which equals $\dim_{k(x)}(\Omega^1_{X/Y,k(x)}$. □

We have the following easy corollary.

Corollary 4.2.2.7. *Let X/k be a smooth equi-dimensional scheme of dimension n. Then $\Omega^1_{X/k}$ is locally free of rank n on X.*

The fundamental exact sequences for the Kähler differentials take a particularly nice form for smooth morphisms. We have the following theorem.

Theorem 4.2.2.8. *Let $f : X \to Y$ be a morphism of schemes over S. Then if f is smooth then the right exact sequence (see Proposition 4.1.0.6, (2))*

$$0 \longrightarrow f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \tag{4.6}$$

is also exact on the left and is locally (on X split).

Proof. Let us prove (1) first. First note that if X/Y is smooth, then $\Omega^1_{X/Y}$ is a locally free coherent sheaf and hence the (apriori) right exact sequence necessarily splits on the right. To show exactness we claim it suffices to prove it for $X = \mathbb{A}^1_Y \to Y$, where it follows from an easy computation. First note that exactness can be checked locally on X, hence we may assume that $f : X \to Y$ factors via an étale map $g : X \to \mathbb{A}^n_Y$, followed by the projection to Y. Suppose we managed to show that

$$0 \longrightarrow f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{\mathbb{A}^n_Y/S} \longrightarrow \Omega^1_{\mathbb{A}^n_Y/Y} \longrightarrow 0\ , \tag{4.7}$$

is exact and locally split. Then applying g^* to the above exact sequence preserves exactness (why?) we obtain the exact sequence (4.6) thanks to Proposition 4.2.1.5, (2). Finally note that the projection $f : \mathbb{A}^n_Y \to Y$ factors as $g : \mathbb{A}^n_Y \to \mathbb{A}^{n-1}_Y$, where the latter projects onto Y (say via h). Suppose we have managed to show the exactness of (4.7) for affine spaces of rank upto $n-1$ (over arbitrary S). Then we have a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & f^*\Omega^1_{Y/S} \arrow[r] \arrow[d, "="] & g^*\Omega^1_{\mathbb{A}^{n-1}_{Y/S}} \arrow[r] \arrow[d] & g^*\Omega^1_{\mathbb{A}^{n-1}_Y/Y} \arrow[r] \arrow[d, "h"] & 0 \\
 & f^*\Omega^1_{Y/S} \arrow[r] \arrow[d] & \Omega^1_{\mathbb{A}^n_Y/S} \arrow[r] \arrow[d] & \Omega^1_{\mathbb{A}^n_Y/Y} \arrow[r] \arrow[d] & 0\ . \\
0 \arrow[r, "="] & 0 \arrow[r] & \Omega^1_{\mathbb{A}^n_Y/\mathbb{A}^{n-1}_Y} \arrow[r, "="] \arrow[d] & \Omega^1_{\mathbb{A}^n_Y/\mathbb{A}^{n-1}_Y} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The left exactness at the middle row is now clear. This proves (1). ☐

In a similar vein we can also strengthen the right exact sequence Proposition 4.1.0.7.

Proposition 4.2.2.9. *Let $f : X \to Y$ be a morphism of schemes. Let Z be a closed subscheme of X. Then the right exact sequence*

$$\mathscr{I}_Z/\mathscr{I}_Z^2 \xrightarrow{\delta} \Omega^1_{X/Y}|_Z \longrightarrow \Omega^1_{Z/Y} \longrightarrow 0, \tag{4.8}$$

is exact and locally split if Z/Y is smooth[5]

Proof. For a proof of the first part we refer to Tag 06A8. Note that locally split follows from the fact that under the assumptions $\Omega^1_{Z/Y}$ is a locally free coherent sheaf on Z. ☐

One can do better if one assumes X/Y is smooth. In fact in that case one has the following *intuitive* characterization of sub-schemes smooth over Y.

Theorem 4.2.2.10. *Let $f : X \to Y$ be a smooth morphism and let Z be a closed sub scheme of X. Then TFAE*

1. *Z/Y is smooth.*
2. *The right exact sequence*

$$\mathscr{I}_Z/\mathscr{I}_Z^2 \xrightarrow{\delta} \Omega^1_{X/Y}|_Z \longrightarrow \Omega^1_{Z/Y} \longrightarrow 0, \tag{4.9}$$

   is exact and locally split.

3. *For any point $z \in Z$, there exists an open $U \hookrightarrow X$ containing x and an étale map $g : U \to \mathbb{A}^n_Y$ and a Cartesian diagram*

$$\begin{tikzcd} U \cap Z \arrow[rr] \arrow[d, "g'"] & & U \arrow[d, "g"] \\ \mathbb{A}^r_Y \simeq Z(t_1, t_2 \cdots t_{n-r}) \arrow[rr] & & \mathbb{A}^n_Y = \operatorname{Spec}(\mathscr{O}_Y[t_1, t_2 \cdots t_n]). \end{tikzcd}$$

Proof. For a proof we refer to [2, Exposé II, Théorème 4.10]. The case when $Y = \operatorname{Spec}(k)$ is handled in [3, Chapter II, Theorem 8.17] ☐

Intuitively Theorem 4.2.2.10 tells us that just as étale locally smooth schemes are like affine spaces, similarly smooth subschemes are like linear subspaces of affine spaces.

[5] We do not need X/Y to be smooth!

Conormal exact sequence when $Y = \mathrm{Spec}(k)$

Suppose $Y = \mathrm{Spec}(k)$ in Theorem 4.2.2.10 and lets assume both Z and X are smooth varieties over $\mathrm{Spec}(k)$. Then in that case we have a short exact sequence of vector bundles on Z

$$0 \longrightarrow \mathscr{I}_Z/\mathscr{I}_Z^2 \xrightarrow{\delta} \Omega^1_{X/k}|_Z \longrightarrow \Omega^1_{Z/k} \longrightarrow 0. \tag{4.10}$$

Dualizing this and recalling that the dual of Ω^1 is the tangent space gives us a familiar exact sequence from differential geometry

$$0 \longrightarrow \mathrm{T}^*Z \longrightarrow \mathrm{T}^*X|_Z \longrightarrow N_{Z/X} \longrightarrow 0. \tag{4.11}$$

Here $N_{Z/X}$ is the normal bundle of Z inside X. Thus it makes sense to call $\mathscr{I}_Z/\mathscr{I}_Z^2$ the *conormal sheaf* of Z in X (even when Z and X are possibly non-smooth). The corresponding exact sequence is called the *conormal exact sequence*.
Finally combining Corollary 4.2.2.7 and Theorem 4.2.2.10 shows us that the conormal sheaf is a vector bundle of rank equal to the codimension of Z in X and that Z is locally cut out by its codimension-many equations.
We end this section with the familiar Jacobian criterion for smoothness which is a corollary to Theorem. 4.2.2.10.

Corollary 4.2.2.11 (Jacobian criterion: Smooth form). *Let Z be a closed sub scheme of $\mathbb{A}^n_k$. Then Z is smooth over k at a point $z \in Z$ iff there exists an open $U \hookrightarrow \mathbb{A}^n_k$ containing z such that $Z \cap U$ is defined by the vanishing $f_1, f_2 \cdots f_r \in \mathscr{O}(U)$ satisfying the Jacobian criterion i.e.*

$$rk_{k(z)}(\{\frac{\partial f_i}{\partial x_j}\}_{i,j}) = r.$$

This form of the Jacobian criterion is well adapted to check for smoothness of subvarieties of $\mathbb{A}^n_k$. We also have a form which can be used to check for singularities.

Corollary 4.2.2.12 (Jacobian criterion: Singular form). *Let Z be a closed sub scheme of $\mathbb{A}^n_k$. Then Z is singular over k at a point $z \in Z$ iff there exists an open $U \hookrightarrow \mathbb{A}^n_k$ containing z such that $Z \cap U$ is defined by the vanishing $f_1, f_2 \cdots f_r \in \mathscr{O}(U)$ such that the images of f_i form a basis for $\frac{\mathscr{I}}{\mathscr{I}^2} \otimes k(z)$[6] but*

$$rk_{k(z)}(\{\frac{\partial f_i}{\partial x_j}\}_{i,j}) < r.$$

Let us see this in action.

[6] Note that if $r = 1$, then this is automatically satisfied for $f_1 \neq 0$.

Example 4.2.2.13.
1. Let $Z := Z(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2_k$ be the nodal curve. Then Z is globally defined by $f(x,y) = y^2 - x^2(x+1)$. Its Jacobian matrix is given by $[3x^2+2x \;\; 2y]$. Thus a point $(x,y) \in Z$ is singular (i.e. not smooth) iff $f(x,y) = 2y = 3x^2 + 2x = 0$. Clearly this only happens when $x = y = 0$. The unique nodal singularity of Z.

2. Consider the Fermat cubic $Z := Z(x^3 + y^3 + z^3 + w^3) \subseteq \mathbb{P}^3_k$. Then on each affine chart Z is given by vanishing of $f(x,y,z) := 1 + x^3 + y^3 + z^3$. The Jacobian matrix of f is given by $[3x \;\; 3y \;\; 3z]$. Thus Z is smooth iff it is smooth on each affine chart iff there are no common solutions to $f(x,y,z) = 3x = 3y = 3z = 0$. Thus Z is smooth away from char$(k) = 3$. But in char$(k) = 3$ every point is a singular point!

4.2.3 More computations with Kähler differentials

In this section we shall use the results from Sections 4.1.1 and 4.2.2 to compute some examples. Before we do so we need a defintion.

Definition 4.2.3.1 (Canonical Sheaf)**.** Let $f : X \to Y$ be a smooth morphism of relative dimension n. We define the *relative canonical sheaf* $\omega_{X/Y} := \bigwedge^n \Omega^1_{X/Y}$. Thus $\omega_{X/Y}$ is a line bundle on X.

Example 4.2.3.2.
1. Let $X = \mathbb{A}^n_A$ and $Y = \mathrm{Spec}(A)$. Then it follows from Lemma 4.1.1.8 that $\omega_{X/Y} \simeq \mathscr{O}_X dx_1 \wedge dx_2 \cdots dx_n$.

2. If $X = \mathbb{P}^n_A$ and $Y = \mathrm{Spec}(A)$. Then it follows from the Euler exact sequence (Proposition 4.1.1.9) that $\omega_{\mathbb{P}^n_A/A} \simeq \mathscr{O}(-n-1)_{\mathbb{P}^n_A}$. In particular when $n = 1$, $\Omega^1_{\mathbb{P}^1_A/A} = \omega_{\mathbb{P}^1_A/A} = \mathscr{O}(-2)_{\mathbb{P}^1_A}$.

3. Let X and Y be smooth varieties over a field k. Then Proposition 4.1.0.6, (3) and [3, Chapter II, Ex. 5.16d] imply that $\omega_{X\times_k Y} \simeq p_X^*\omega_{X/k} \otimes p_Y^*\omega_{Y/k}$.

Here is an easy consequence of Theorem 4.2.2.10.

Proposition 4.2.3.3. *Let $Z \subseteq X$ be a smooth subvariety of a smooth variety X/k. Then*

$$\omega_X|Z = \omega_Z \otimes \bigwedge^r \mathscr{I}_Z/\mathscr{I}_Z^2,$$

here r is the codimension of Z in X. In particular if Z is given by the zero section of a line bundle $\mathscr{L}$ (and hence a divisor on X). Then

$$\omega_Z = (\omega_X \otimes \mathscr{L})|_Z$$

Proof. The first formula is an immediate consequence of Equation (4.10) and [3, Chapter II, Ex. 5.16d]. For the second one we simply observe that $\mathscr{I}_Z \simeq \mathscr{L}^{-1}$. $\square$

Example 4.2.3.4. Let $X \subseteq \mathbb{P}^n_k$ be a smooth hypersurface of degree d. Then $\omega_{X/k} = \mathscr{O}_X(-n-1+d)$. In particular $\omega_{X/k}$ is ample iff $d \geqslant n+2$

More generally for any smooth variety X/k of dimension n we have:

1. locally free sheaves $\Omega^i_{X/k} := \bigwedge^i \Omega^1_{X/k}$ of rank $n-i$.
2. The *de Rham complex*:

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega^1_{X/k} \qquad \cdots \xrightarrow{d} \Omega^n_{X/k} \longrightarrow 0$$

   Here the differentials $d : \Omega^i_{X/k} \to \Omega^{i+1}_{X/k}$ satisfy the usual Leibniz rule and when $i = 0$ correspond the universal differential from $\mathscr{O}_X \to \Omega^1_{X/k}$.

4.2.4 Regularity and Smoothness

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. Recall that A is said to be *regular* if any of the following equivalent conditions are satisfied:

1. $\dim_k(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = \dim(A)$.
2. $\mathfrak{m}$ is generated by d elements, where $d = \dim(A)$.

We need the following basic results about regular local rings.

Proposition 4.2.4.1. *Let A be a regular local ring as above. Then*

1. $\oplus_{n \geqslant 0} \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \simeq k[t_1, t_2 \cdots t_n]$.
2. *A collection of elements $(x_1, x_2, \cdots x_d)$ generate $\mathfrak{m}$ iff they form a regular system of parameters i.e. x_i is a non zero-divisor in $A/(x_1, x_2, \cdots x_{i-1})$*
3. *Let $I \subseteq A$ be an ideal. The ring $B = A/I$ is regular local iff $I = (x_1, x_2 \cdots x_r)$ with $(x_i)_{1 \leqslant i \leqslant r}$ part of a regular system of parameters for A.*
4. *$A_\mathfrak{p}$ is also regular local for any prime ideal $\mathfrak{p}$.*

Proof. These are shown in Tag 00NO, Tag 00NQ, Tag 00NR and Tag 0AFS. □

Remark 4.2.4.2. We note the following about regular local rings.

1. Dimension 0 regular[7] local rings are precisely fields and dimension 1 regular local rings are dvr's.

[7] Henceforth anytime we mention regularity we shall always be in the Noetherian setting.

2. Thanks to Proposition 4.2.4.1, (1) implies that A is domain (see Tag 00NP).

Thanks to Proposition 4.2.4.1, (4) it makes sense to have the following definition.

Definition 4.2.4.3. Let X be a locally Noetherian scheme. We say X is regular iff for any point $x \in X$, the local ring $\mathscr{O}_{X,x}$ is regular[8].

Here are a couple of simple corollaries to Remark 4.2.4.2.

Corollary 4.2.4.4. *Let X be a regular scheme. Then*

1. *X is normal.*

2. *In addition if X is Noetherian then every irreducible component of X is open in it and hence also a connected component of X.*

Proof. Normality follows from Serre's criterion of Normality[9] as in Tag 0567.
For the latter we simply note that every local ring of X is an integral domain and thus has an unique minimal prime ideal. This in particular implies that every point lies in an unique irreducible component (else the local ring at that point would have at least two minimal prime ideals). □

The key result relating smoothness and regularity is the following.

Theorem 4.2.4.5. *Let X/k be a scheme of finite type. Then*

1. *X/k smooth implies X is a regular scheme. In particular every irreducible component of X is also a connected component.*

2. *Conversely, if k is perfect then X regular implies X/k is smooth.*

Proof. Since regularity is a local property, we may assume X is affine and in particular we choose an embedding $X \hookrightarrow \mathbb{A}^n_k$ as a closed sub scheme. Moreover it suffices to check for regularity at closed points of X. Let $x \in X$ be a closed point.
Now suppose $\mathscr{I}$ and $\mathfrak{m}_x$ be the ideals defining X and x respectively. Then

$$\mathscr{I} \subset \mathfrak{m}_x \subset \mathscr{O}_{\mathbb{A}^n_k} \mapsto \Omega^1_{\mathbb{A}^n_k/k},$$

induces

$$\begin{tikzcd} \frac{\mathscr{I}}{\mathscr{I}^2} \otimes k(x) \arrow[r, "\delta_X"] \arrow[dr] & \Omega^1_{\mathbb{A}^n_k/k} \otimes k(x) \\ & \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}. \arrow[u, "\delta_x"] \end{tikzcd}$$

[8] Thanks to Proposition 4.2.4.1, (4) it suffices to check this at closed points!
[9] Thanks Cheng and Fuxiang for pointing the error in an earlier argument

Choose elements $f_1, f_2 \cdots f_r \in \mathscr{I}$ whose images form a basis of $\frac{\mathscr{I}}{\mathscr{I}^2} \otimes k(x)$. Since X/k is smooth, δ_X is injective and hence the images of these elements in $\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$ is also span a r-dimensional subspace and hence can be extended to a basis of $\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$. Thus by Proposition 4.2.4.1, (3) $\mathscr{O}_{X,x}$ is a regular local ring.
The converse follows from the Jacobian criterion (the key point is vanishing of $\Omega^1_{k(x)/k}$ which of course uses k being perfect). For a proof see [2, Exposé II, Corollaire 5.3]

□

We have the following corollary.

Corollary 4.2.4.6. *Let X/k be of finite type. Then the following are equivalent*

1. *X/k is smooth.*
2. *$X_{k'}$ is regular for any field extension k'/k.*
3. *dim$_{k(x)}(\Omega^1_{X/k} \otimes k(x)) = n$, where n is the dimension of the component of X containing x.*

Proof. Since smoothness is preserved under base change (1) implies (2) by Theorem 4.2.4.5. For the converse note that it suffices to show X/k is smooth iff $X_{\bar{k}}/\bar{k}$ is smooth for an algebraic closure $\bar{k}$ of k. This is because regularity of $X_{\bar{k}}$ implies $X_{\bar{k}}/\bar{k}$ is smooth.
Since the question is local on X, we may assume we have a closed embedding $X \hookrightarrow \mathbb{A}^n_k$. Let $x \in X$ and let $\bar{x} \in X_{\bar{k}}$ be a point on $X_{\bar{k}}$ mapping to x under the flat and surjective map $\pi : X_{\bar{k}} \to X$. We denote by $\mathscr{I}_X$ (resp. $\mathscr{I}_{X_{\bar{k}}}$) the ideal sheaves of X (resp. $X_{\bar{k}}$) in $\mathbb{A}^n_k$ (resp. $\mathbb{A}^n_{\bar{k}}$). Then flatness of π implies that

$$\pi^*(\mathscr{I}_X) = \mathscr{I}_{X_{\bar{k}}}$$

,

$$\pi^*(\frac{\mathscr{I}_X}{\mathscr{I}_X^2}) = \frac{\mathscr{I}_{X_{\bar{k}}}}{\mathscr{I}^2_{X_{\bar{k}}}}.$$

In particular

$$\frac{\mathscr{I}_{X_{\bar{k}}}}{\mathscr{I}^2_{X_{\bar{k}}}} \otimes \bar{k}(\bar{x}) = (\frac{\mathscr{I}_x}{\mathscr{I}_x^2} \otimes k(x)) \otimes k(\bar{x}).$$

Now since $X_{\bar{k}}/\bar{k}$ is smooth, the induced map $\frac{\mathscr{I}_{X_{\bar{k}}}}{\mathscr{I}^2_{X_{\bar{k}}}} \otimes \bar{k}(\bar{x}) \to \Omega^1_{\mathbb{A}^n_{\bar{k}}/\bar{k}} \otimes \bar{k}(\bar{x})$ is injective which by the isomorphism above implies that the induced map $\frac{\mathscr{I}_X}{\mathscr{I}_X^2} \otimes k(x) \to \Omega^1_{\mathbb{A}^n_k/k} \otimes k(x)$ is injective, and hence by the Jacobian criterion we are done.
Clearly (1) implies (3) by Lemma 4.2.2.6. It suffices to show (3) implies (2). This can be argued as above using Corollary 4.1.1.5. For a proof refer to Tag 01V9.

□

Remark 4.2.4.7. Note that Theorem 4.2.4.5 is the best possible result one can hope for in general. For example $X = \mathrm{Spec}(\mathbb{F}_p(t^{1/p}))$ is not smooth over $Y = \mathrm{Spec}(\mathbb{F}_p(t))$ (owing to $\Omega^1_{X/Y}$ being larger than expected i.e 0), however it is regular. Corollary 4.2.4.6 tells us that smoothness is the same as geometric regularity.

Corollary 4.2.4.8 (Generic Smoothness over a perfect field). *Let X/k be a reduced scheme of finite type over a perfect field k. Then there exists a dense open subset $U \hookrightarrow X$ such that U/k is smooth.*

Proof. Since X/k is reduced and of finite type, it has finitely many irreducible components and the local ring at any generic point is a field. Let $\eta \in X$ be one such generic point in a component of dimension n. Then $\Omega^1_{X/k} \otimes k(\eta) = \Omega^1_{k(\eta)/k}$ (Why?) and by [3, Theorem 8.6A] $\dim_{k(\eta)}\Omega^1_{k(\eta)/k} = n$. Since X is reduced, there exists an irreducible open containing η in X such that $\Omega^1_{X/k}$ is locally free of rank n. By Corollary 4.2.4.6, (3) this open subset is smooth over k. Since we can do this around every generic point, we win. ☐

We now state a very important Bertini theorem. This is frequently (and freely!) used in induction arguments. We do not prove it here but I strongly recommend reading the proof in [3, Chapter II, Theorem 8.18].

Theorem 4.2.4.9 (Bertini Theorem). *Let $X \hookrightarrow \mathbb{P}_k$ be a smooth projective variety over an algebraically closed field k. Let $\mathbb{P}^\vee_k$ be the projective variety parametrizing linear homogeneous polynomials on $\mathbb{P}_k$ or equivalently they parametrize hyperplane sections of $\mathbb{P}_k$. Then there exists a dense open $U \hookrightarrow \mathbb{P}^\vee_k$ such that for any closed point $x \in U$, the scheme $X \cap H$ is also smooth over k.*

Now we compare our notion of smoothness to the one in Hartshorne.

Theorem 4.2.4.10. *Let $f : X \to Y$ be a morphism of finite type between Noetherian schemes of relative dimension n[10]. Then f is smooth iff*

1. *f is flat.*
2. *The fibers X_y are smooth for all points $y \in Y$ or equivalently by Corollary 4.2.4.6, (3) $\dim_{k(x)}(\Omega^1_{X/Y} \otimes k(x)) = n$ for any point $x \in X$.*

Proof. This follows from Tag 00TF. ☐

But more is true! We have the following *miraculous* result, known colloquially as the *Miracle Flatness Theorem* due to Hironaka.

Theorem 4.2.4.11 (Miracle Flatness Theorem). *Let $R \to S$ be a local morphism of Noetherian local rings. Assume that*

[10] Every fiber of f is equidimensional of dimension n

1. *R is a regular local ring.*
2. *S is Cohen-Macaulay (ex. regular).*
3. *The dimension formula holds i.e,*

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S),$$

where $\mathfrak{m}$ is the maximal ideal of R.

Then $R \to S$ is flat!

This has the following very useful corollary.

Corollary 4.2.4.12 (Miracle Flatness Theorem for schemes). *Let $f : X \to Y$ be a morphism of locally Noetherian schemes such that X is Cohen-Macaulay (for ex. regular) and Y is regular. Then f is flat iff the dimension formula holds.*

Here is corollary to the above theorem which recovers the classical notion of smoothness for morphisms of smooth varieties.

Corollary 4.2.4.13. *Let $f : X \to Y$ be a morphism of smooth varieties over a field k. Then f is smooth iff for any closed point $x \in X$, the vector space $\Omega^1_{X/Y} \otimes k(x)$ is of dimension dim(X) − dim(Y) iff the induced map*

$$df_x : T_xX \to T_{f(x)}Y,$$

between their Zariski tangent spaces in surjective.

Proof. Miracle flatness gives you flatness for free. Once you have flatness the rest follows from Theorem 4.2.4.10. □

Chapter 5

A crash course in derived categories

In this chapter we shall give a crash course on derived categories. We aim to have a working understanding of what these are and more importantly (over time) appreciate their utility. Throughout $\mathscr{A}$ will denote an abelian category (see Definition 1.3.0.6). To fix ideas it is best to think of $\mathscr{A}$ as category of R-modules for a ring R or as the abelian category of $\mathscr{O}_X$-modules for a scheme X.

5.0.1 What is our goal?

Very often in algebraic geometry (and allied topics) one comes across the following situation; One has an additive functor $F : \mathscr{A} \to \mathscr{B}$ between two abelian categories of interest. On a good day F would preserve exact sequences, but more often than not F would only be either left exact or right exact. The typical examples are $\mathrm{Hom}_R(M, -)$ and $\otimes_R M$ for an R-module M. The natural question then is:

Question:

Given a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

in $\mathscr{A}$ and a left exact functor $F : \mathscr{A} \to \mathscr{B}$, how do we understand the *defect* of right exactness? Put differently how do continue the exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z).$$

To begin with one can make a definition:

Definition 5.0.1.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. A cohomological δ-functor extending F is a sequence of additive functors $F^i : \mathcal{A} \to \mathcal{B}$ such that $F^0 = F$, together with boundary maps (natural in the following short exact sequences)

$$\delta : F^i(Z) \to F^{i+1}(X)$$

for all short exact sequences

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$, such that for all such short exact sequences we obtain the following complex:

$$0 \to F^0(X) \to F^0(Y) \to F^0(Z) \xrightarrow{\delta} F^1(X) \to F^1(Y) \to F^1(Z) \xrightarrow{\delta} F^2(X) \to \cdots ,$$

which is exact.
Moreover, such a δ is called *universal* if it is initial in the category of cohomological δ-functors extending F.

Having made this definition, a natural question then is when do universal δ functors exist? Note that by definition once they exist, they are unique upto an unique isomorphism. Following theorem was one of the important results in the famous Tohoku article of Grothendieck. Before we can state it we need a couple of definitions.

Definition 5.0.1.2 (Injective Object). An object I in an abelian category is said to be injective if the following equivalent conditions are satisfied:

1. $\mathrm{Hom}_{\mathscr{A}}(-, I)$ is an exact functor.

2. Every injection $I \hookrightarrow X$ is a split injection[1].

Example 5.0.1.3. An abelian group M is injective iff for any integer n, multiplication by n is surjective on M. Such groups are called divisible. See Tag 01D7.

Definition 5.0.1.4. An abelian category $\mathscr{A}$ is said to have *enough injectives* if for every object X there exists an injection of X inside an injective object $\tilde{X}$.

Example 5.0.1.5. Following abelian categories have enough injectives:

- Category of R-modules.
- Sheaves of abelian groups on a topological space X.
- Sheaves of $\mathscr{O}_X$-modules modules on a ringed space $(X, \mathscr{O}_X)$.

Following abelian category **do not** have enough injectives in general:

- Category of finite R-modules.
- The category of coherent sheaves $\mathrm{Coh}(X)$ on a Noetherian scheme X.

[1]This should tell you that injective objects in $\mathscr{A}$ are dual to projective objects i.e they correspond to projective objects in $\mathscr{A}^{\mathrm{op}}$

Proof. This is standard. First embed X inside an injective say I^0, then take the quotient I^0/X, embed that inside an injective I^1 so on and so forth. □

Now we are ready to state the promised theorem.

Theorem 5.0.1.6 (Grothendieck). *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor from an abelian category $\mathscr{A}$ with enough injectives. Then there exists an universal δ-functor extending F. The F^i's are called the right derived functors of F.*

Remark 5.0.1.7. By symmetry if F is right exact and $\mathscr{A}$ has enough projectives (Guess the definition!) then we get left derived functors of F.

Here are some examples:

Example 5.0.1.8. Here we list some examples of derived functors:

1. For any R-module M, the left derived functors of $\mathrm{Hom}(M, -)$ are denoted by $\mathrm{Ext}^i(M, -)$.

2. For any continuous map of topological spaces $f : X \to Y$, we denote by $R^i f_* \mathscr{F}$ the derived functors of the left exact functor $f_* \mathscr{F}$. These are also called the higher direct images.

3. Let $f : X \to Y$ be a morphism of Noetherian schemes. In particular f is a map of ringed spaces and hence it make sense to talk about the derived functors $R^i f_* \mathscr{F}$ for any $\mathscr{O}_X$-module $\mathscr{F}$. It is apriori not clear (or even true) that $R^i f_* \mathscr{F}$ have any additional structure that $\mathscr{F}$ may have like being coherent or quasi-coherent. These are important results and we shall discuss them later in this course

5.1 Injective Resolutions: Turning the crank

Thanks to Theorem 5.0.1.6 we know that it makes sense to talk about the derived functors of a left (or right) exact functor. But then how do we actually compute these? For simplicity we work with only left exact functors unless otherwise stated. Everything we say works well in $\mathscr{A}^{\mathrm{op}}$, and hence with projectives replaced by injectives.
What is amazing is that, once an abelian category has enough injectives, there is an *uniform* way to compute these for all possible left exact functors.
Here is an easy lemma.

Lemma 5.1.0.1. *Let $\mathscr{A}$ be an abelian category with enough injectives. Then every object X in $\mathscr{A}$ can be resolved using injectives, i.e there exists an exact complex*[2]

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots,$$

where each I^j is an injective object.

[2] Our complexes have cohomological indexing

But one can do better. Before we state the result we need a few definitions.

Definition 5.1.0.2. We denote by $C(\mathscr{A})$, the category of chain complexes with values in $\mathscr{A}$, that is objects in $\mathscr{A}$ are complexes $\mathbf{X}^{\cdot}$

$$\cdots X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \cdots,$$

here X^i's are objects in $\mathscr{A}$ and $d^{i+1} \circ d^i = 0$ for all i[3]. Morphisms of complexes $f : \mathbf{X}^{\cdot} \to \mathbf{Y}^{\cdot}$ are given by maps $f^i : X^i \to Y^i$ for all i commuting with the differentials.

Definition 5.1.0.3. For any complex $\mathbf{X}^{\cdot}$ and any integer n, we have an exact functor called **shift by n**

$$[n] : C(\mathscr{A}) \to C(\mathscr{A}),$$

which sends $\mathbf{X}^{\cdot}$ to a complex $\mathbf{X}^{\cdot}[n]$ whose i^{th}-term is X^{i+n} and the differentials are one induced from $X^{\cdot}$.

For any i there exists a functor

$$H^i : C(\mathscr{A}) \to \mathscr{A},$$

obtained by sending

$$\mathbf{X}^{\cdot} \to \frac{\ker(d^i)}{\operatorname{Im}(d^{i-1})}.$$

We note the following easy lemma.

Lemma 5.1.0.4. *The category $C(\mathscr{A})$ can be given the structure of an abelian category with termwise kernels and cokernels. A sequence of complexes is exact iff it is termwise exact. Moreover there is an exact functor from $\mathscr{A}$ to $c(\mathscr{A})$ which sends any object X to the complex with only one non-zero object X at degree 0.*
Finally any short exact sequence of complexes induces a long exact sequence of cohomologies.

Proof. The first part is an easy exercise. For the last claim use snake lemma (see in Tag 0117). ☐

Definition 5.1.0.5. Two maps $f, g : \mathbf{X}^{\cdot} \to \mathbf{Y}^{\cdot}$ are said to be chain homotopic[4] if there exists maps $\partial^i : X^i \to Y^{i-1}$ for all i such that

$$f - g = d \circ \delta + \delta \circ d.$$

Moreover a map $f : \mathbf{X}^{\cdot} \to \mathbf{Y}^{\cdot}$ is said to be *homotopy equivalence* if there exists $g : \mathbf{Y}^{\cdot} \to \mathbf{X}^{\cdot}$ such that $g \circ f \sim 1_{\mathbf{X}^{\cdot}}$. Finally f is said to be *quasi-isomorphism* if it induces an isomorphism on the cohomology groups.

[3] In what follows we will simply denote these maps called *differentials* by d, if at all.
[4] denoted by $f \sim g$.

Here is a standard fact about chain homotopic maps.

Lemma 5.1.0.6. *Let $f, g : \mathbf{X}^{\cdot} \to \mathbf{Y}^{\cdot}$ be chain homotpic maps. Then they induce identical maps on cohomology. Moreoever if f is an homotopy equivalence then f is a quasi-isomorphism.*

Proof. Easy exercise. □

Thanks to Lemma 5.1.0.6 it makes sense to talk about the *homotopy category* of $C(\mathscr{A})$.

Definition 5.1.0.7. The homotopy category $K(\mathscr{A})$ is the category whose objects are the same as $C(\mathscr{A})$ but morphisms are given by

$$\mathrm{Hom}_{K(\mathscr{A})}(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot}) := \mathrm{Hom}_{c(\mathscr{A})}(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})/\sim,$$

where $\sim$ is the equivalence relation coming from homotopy equivalence.

Question: So why do we care about $K(\mathscr{A})$?

The easy answer is because the functors $H^i : C(\mathscr{A}) \to \mathscr{A}$ factor via $K(\mathscr{A})$ thanks to Lemma 5.1.0.6. But why care about H^i's at all?

Proposition 5.1.0.8. *Let $\mathcal{A}$ be an abelian category, and consider the following solid diagram in $\mathcal{A}$:*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "d"] \arrow[d, "f"] & I^0 \arrow[r, "d^0"] \arrow[d, dotted, "f_0"] & I^1 \arrow[r, "d^1"] \arrow[d, dotted, "f_1"] & \cdots \\
0 \arrow[r] & Y \arrow[r, "e"] & J^0 \arrow[r, "e^0"] & J^1 \arrow[r, "e^1"] & \cdots
\end{tikzcd}$$

where $I^\bullet$ and $J^\bullet$ are injective resolutions for X and Y, respectively.
Then the dotted arrows f_i exist such that the whole diagram commutes, and between any two choices of f_i and f_i', there exists a chain homotopy between them.

This allows us define for any left exact functor $F : \mathscr{A} \to \mathscr{B}$, the derived functor F^i as follows:

1. For any object X in $\mathscr{A}$ take an injective resolution $\mathbf{I}^\bullet$ as above. This amounts to replacing X by a complex of injectives which is *quasi-isomorphic* to X.

2. Define $F^i(X) := H^i(F(\mathbf{I}^\bullet))$. The latter is well defined and functorial (in X) thanks to Proposition 5.1.0.8.

We claim that these $F^i(X)$ are δ-functors and that they are universal. The universality follows from Tag 010T. For the former we need the following proposition (together with Lemma 5.1.0.4).

Proposition 5.1.0.9. *For any short exact sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

one can choose injective resolutions $\boldsymbol{I}^\bullet$, $\boldsymbol{J}^\bullet$ *and* $\boldsymbol{K}^\bullet$ *of* X, Y *and* Z *respectively such that*

1. *There exists an exact sequence*

$$0 \longrightarrow \boldsymbol{I}^\bullet \longrightarrow \boldsymbol{J}^\bullet \longrightarrow \boldsymbol{K}^\bullet \longrightarrow 0,$$

   such that the obvious diagrams commute.

2. *Moreover the above exact sequence of complexes is termwise split, and hence remains exact after applying any additive functor* F.

Proof. For the former see Tag 013T. The latter follows from the fact that terms of these resolutions by choice are injective. □

Definition 5.1.0.10. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor with an universal δ-functor extending it. We say an object M is F-acyclic if $F^i(M) = 0$ for any $i > 0$.

Clearly if $\mathscr{A}$ has enough injectives then every injective is F-acyclic. Though you can have F-acyclics which are not injective, they serve the same purpose as injectives as the following lemma shows.

Lemma 5.1.0.11. *Consider an exact complex*

$$0 \longrightarrow X \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \cdots,$$

where M^i*'s are* F*-acyclics. Then* $H^i(F(\boldsymbol{M}^\bullet)) \simeq F^i(X)$.

Proof. □

Sheaf Cohomology

Let us discuss a very important case of derived functors: The case of sheaf cohomology. Here $\mathscr{A}$ is the abelian category of the sheaf of $\mathscr{O}_X$-modules on a ringed space $(X, \mathscr{O}_X)$. Note that we may simply choose $\mathscr{O}_X$ to be $\mathbb{Z}_X$, so this covers the case of sheaves of abelian groups too. As we have noted earlier $\mathscr{A}$ has enough injectives hence for any $\mathscr{O}_X$-module $\mathscr{F}$ we define by

$$H^i(X, \mathscr{F}),$$

the i^{th}-derived functor of the left exact functor $\mathscr{F} \to \Gamma(X, \mathscr{F})$. With this notation $H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

A very important class of sheaves for us would be flasque sheaves.

Definition 5.1.0.12. A sheaf $\mathscr{F}$ is said to be flasque if for any $U \subseteq V$ open subsets in X, the natural map

$$\mathscr{F}(V) \to \mathscr{F}(U),$$

is surjective.

The reason they are interesting is the following:

Proposition 5.1.0.13. *There are plenty of flasque sheaves. In fact*

1. *Injective sheaves are flasque.*

2. *Flasque sheaves are F-acyclic for the global sections functor.*

3. *For any sheaf $\mathscr{F}$, the pre-sheaf God($\mathscr{F}$) defined as follows*

$$\mathit{God}(\mathscr{F})(U) := \prod_{x \in U} \mathscr{F}_x,$$

is a flasque sheaf. Moreover the natural map $\mathscr{F} \to$ God($\mathscr{F}$) is an injection. Thus every sheaf can be embedded canonically inside a flasque sheaf and the associated resolution is called the Godement resolution of $\mathscr{F}$.

Proof. For a proof see Tag 01EA and Tag 09SY. □

Here is a corollary to Proposition5.1.0.13.

Corollary 5.1.0.14. *Let X be a topological space with a sheaf of rings $\mathscr{O}_X$. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. Then we can also look at $\mathscr{F}$ as a sheaf of abelian groups (forgetting the $\mathscr{O}_X$-module) structure, we call this sheaf $\mathscr{F}^{ab}$. Then there exists a canonical isomorphism*

$$H^i(X, \mathscr{F}) \simeq H^i(X, \mathscr{F}^{ab}), \forall i.$$

Proof. Thanks to Proposition 5.1.0.13 and Lemma 5.1.0.11, either side of the isomorphism can be computed using Godement resolutions. □

Higher direct images

One should view sheaf cohomology as an absolute theory. There is relative version which as you will see will turn out to be equally if not more important. Given any map $f : X \to Y$ of ringed space, there is an induced left exact functor $f_* : \mathrm{Mod}_{\mathscr{O}_X} \to \mathrm{Mod}_{\mathscr{O}_Y}$.
Note that we can take Y to be a point and $\mathscr{O}_Y$ to be the unique sheaf whose global sections on the point are $H^0(X, \mathscr{O}_X)$. In this case f_* is simply the global sections functor. As before thanks to existence of enough injectives in $\mathrm{Mod}_{\mathscr{O}_X}$, we can take the right derived functors $R^i f_*$ of f_*, these are often called the *higher direct images*. Here is a simple lemma.

Lemma 5.1.0.15. *Let $f : X \to Y$ be morphism of ringed spaces which induces a homeomorphism onto a closed subset. Then $R^i f_*$ vanishes for $i > 0$.*

Proof. First note that the stalk of $f_*\mathscr{F}$ at any point $y \in Y$ is 0 if $y \notin f(X)$ and is equal to $\mathscr{F}_y$ for $y \in f(X)$. Thus $f_*\mathscr{F}$ is an exact functor. Hence the result. $\square$

5.2 Derived Categories and Derived Functors

We can do better. Recall that $\mathscr{A}$ is realized as an abelian sub category of $C(\mathscr{A})$ (Lemma 5.1.0.4). Even when begin with an object X in $\mathscr{A}$, it's injective resolution lives not in $\mathscr{A}$ but rather in $C(\mathscr{A})$. One can try to address this asymmetry.

Lemma 5.2.0.1. *Let $\mathbf{X}^\bullet$ be a complex which is exact[5] for all $i \leqslant n$ for some $n \in \mathbb{Z}$. Then there exists a complex $\mathbf{I}^\bullet$ of injectives and a quasi-isomoprphism (qis henceforth) $\alpha : \mathbf{X}^\bullet \to \mathbf{I}^\bullet$.*

Proof. For a proof see Tag 013K. $\square$

Given the importance of complexes whose cohomology vanishes in sufficiently small degrees, we introduce a notation

Notations 5.2.0.2. We denote by $C^+(\mathscr{A})$ the full abelian sub category of bounded below chain complexes i.e. complexes $\mathbf{X}^\bullet$ such that $H^i(\mathbf{X}^\bullet) = 0$ for all $i \leqslant n$ and some $n \in \mathbb{Z}$.
In a similar vein we denote by $C^b(\mathscr{A})$ the full abelian sub category of bounded chain complexes i.e. complexes $\mathbf{X}^\bullet$ such that $H^i(\mathbf{X}^\bullet) = 0$ for all $|i| \geqslant n$ and some $n \in \mathbb{N}$.

One can now ask for analogues of Proposition 5.1.0.8 and this turns out to be true verbatim.

Proposition 5.2.0.3. *Given complexes*

1. *$\mathbf{X}^\bullet, \mathbf{Y}^\bullet$ in $C^+(\mathscr{A})$*

2. *a morphism $f : \mathbf{X}^\bullet \to \mathbf{Y}^\bullet$ and,*

3. *quasi-isomorphisms $\alpha : \mathbf{X}^\bullet \to \mathbf{I}^\bullet$, $\beta : \mathbf{Y}^\bullet \to \mathbf{J}^\bullet$ to a complex of injectives.*

There exists a morphism $\tilde{f} : \mathbf{I}^\bullet \to \mathbf{J}^\bullet$ making the obvious diagram commute. Moreover $\tilde{f}$ is unique upto homotopy.

Thanks to Proposition 5.2.0.3 we can not define derived functors not just for objects in $\mathscr{A}$ but also for bounded below complexes in $C(\mathscr{A})$. As before start with a left exact functor $F : \mathscr{A} \to \mathscr{B}$. Then we define functors $F^i : C^+(\mathscr{A}) \to \mathscr{B}$ which restrict to the usual derived functors on $\mathscr{A}$ as follows:

[5] A complex is said to be exact at an index i if it's cohomology vanishes at that index.

1. Resolve this complex $\mathbf{X}^\bullet$ by a complex of injectives (say $\mathbf{I}^\bullet$).
2. Define $F^i(\mathbf{X}^\bullet) := H^i(F(\mathbf{I}^\bullet))$. Thanks to Proposition 5.2.0.3 and Lemma 5.1.0.6 this is a functor.
3. As before using Tag 013T together with Lemma 5.1.0.4 short exact sequence of complexes give long exact sequences of their derived functors.
4. We also have an analogue of Lemma 5.1.0.11.

The upshot of all this being we have a diagram,

$$\begin{tikzcd} \mathscr{A} \arrow[r] \arrow[drr, dashed, bend right, "F^i"'] & C^+(\mathscr{A}) \arrow[r] & K^+(\mathrm{Inj}_{\mathscr{A}}) \arrow[d] & \\ & & K^+(\mathscr{B}) \arrow[r, "H^i"] & \mathscr{B}. \end{tikzcd} \tag{5.1}$$

Here $K^+(\mathrm{Inj}_{\mathscr{A}})$ is the full subcategory of $K^+(\mathscr{C})$ consisting of complexes all of whose terms are injectives in $\mathscr{A}$. Moreover note that injective resolutions of complexes are only defined *upto quasi-isomorphisms*.This motivates the following definition. Let $\mathscr{A}$ be any abelian category.

Definition 5.2.0.4 (Derived Category)**.** Define by $D(\mathscr{A})$, the derived category of $\mathscr{A}$ as the localisation of $K(\mathscr{A})$ at quasi-isomorphisms i.e. , the objects of $D(\mathscr{A})$ are the same as those of $K(\mathscr{A})$ (or equivalently $C(\mathscr{A})$). A morphism $f : \mathbf{X}^\bullet \to \mathbf{Y}^\bullet$ in $D(\mathscr{A})$ is a triple $(\mathbf{X'}^\bullet, t, f')$ where

1. $\mathbf{X'}^\bullet$ is a complex in $K(\mathscr{A})$.
2. $t : \mathbf{X'}^\bullet \to \mathbf{X}^\bullet$ and $f' : \mathbf{X'}^\bullet \to \mathbf{Y}$ are morphisms in $K(\mathscr{A})$ (One should think of $f = \frac{f'}{t}$)
3. t is a quasi-isomorphism.

On can analogously define $D^+(\mathscr{A})$ as a localisation of $K^+(\mathscr{A})$. Of course one needs to check a few things here which we state as a theorem.

Theorem 5.2.0.5. *$D(\mathscr{A})$ is a well defined category i.e. composition is well defined. Moreover $D(\mathscr{A})$ is an additive category and the natural functor from $K(\mathscr{A}) \to D(\mathscr{A})$ is universal among those where quasi-isomorphisms are sent to isomorphisms. Finally when $\mathscr{A}$ has enough injectives then*

$$K^+(\mathit{Inj}_{\mathscr{A}}) \to D^+(\mathscr{A}),$$

is an equivalence of categories.

Proof. For a proof see Tag 05RT. For the second part it suffices to show that quasi-isomorphisms of complexes all of whose terms are injective is in fact an isomorphism in the homotopy category. This follows from Tag 013P. □

Corollary 5.2.0.6. *For any i the cohomology functor $H^i : C(\mathscr{A}) \to \mathscr{A}$ factors via $D(\mathscr{A})$.*

Proof. We have already seen that H^i factors via $K(\mathscr{A})$. Since H^i maps quasi-isomorphisms to isomorphisms, we are done by the universal property. □

Finally we have the following result.

Theorem 5.2.0.7. *The natural functor $\mathscr{D}^+(\mathscr{A}) \to D(\mathscr{A})$ is fully faithful and essentially surjective on the subcategory consisting of objects $X \in D(\mathscr{A})$[6] with $H^i(X) = 0, \forall i << 0$. Moreover, the functor $\mathscr{A} \to K(\mathscr{A}) \to D(\mathscr{A})$ sending an object X to a complex with exactly one non-zero term in degree 0, is also fully faithful and essentially surjective onto to the subcategory whose objects are $X \in D(\mathscr{A})$ with $H^i(X) = 0, \forall i \neq 0$.*

Proof. For a proof see [4, Proposition 1.7.2]. □

Now suppose $\mathscr{A}$ is an abelian category with enough injectives. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Then we can define an additive functor called the derived functor of F,

$$RF : D^+(\mathscr{A}) \to D^+(\mathscr{B}), \tag{5.2}$$

as the composition of the following functors,

$$D^+(\mathscr{A}) \xrightarrow{\simeq} K^+(\mathrm{Inj}_{\mathscr{A}}) \xrightarrow{F} K^+(\mathscr{B}) \longrightarrow D^+(\mathscr{B}).$$

Here is a simple consequence of the definition.

Lemma 5.2.0.8. *There exists an equivalence of functors*

$$H^i \circ RF \simeq R^i F.$$

Proof. The is an immediate consequence of the definition of RF and Corollary 5.2.0.6. □

We end with the following characterization of an isomorphism in $D(\mathscr{A})$.

Lemma 5.2.0.9. *A morphism $f : X \to Y$ in $D(\mathscr{A})$ is an isomorphism iff $H^i(f)$ is an isomorphism for all i, i.e. f is an isomorphism iff it is an quasi-isomorphism.*

Proof. Clearly if f is an isomorphism, then $H^i(f)$ is an isomorphism for all i, since H^i is a functor from $D(\mathscr{A})$ to $\mathscr{A}$. Conversely suppose f corresponds to a triple (X', t, f'). Then f is an quasi-isomorphism iff f' is an quasi-isomorphism. Thus the inverse of f is given by the morphism $g := (X', f', t)$ from Y to X[7]. □

[6] Henceforth we shall not use boldfont for complexes i.e. we shall treat objects and complexes in the same footing.

[7] Of course hidden in all this is the well definedness of composition of morphisms in $D(\mathscr{A})$.

5.2.1 Spectral Sequences

Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be two left exact functors between abelian categories. Assume that $\mathscr{A}$ and $\mathscr{B}$ have enough injectives. Thus we can talk about

$$RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$$

and

$$RG : D^+(\mathscr{B}) \to D^+(\mathscr{C}).$$

Under what conditions do we have

$$R(G \circ F) = RG \circ RF?$$

Let us first work out a necessary condition. Suppose X is an injective object in $\mathscr{A}$. Then both $RF(X)$ and $R(G \circ F)(X)$ are complexes with no cohomology outside degree 0. So for an equality as above to hold, we must have that $RG(F(X))$, is also a complex with no cohomology outside degree 0. Put differently F maps injective objects to G-acyclic objects. Turns out this is all that we need.

Theorem 5.2.1.1 (Grothendieck). *Let F and G be abelian functors as above. If F maps injective objects to G-acyclic objects then $RG \circ RF$ and $R(G \circ F)$ are naturally equivalent.*

Proof. Tag 015M □

Before we move onto applications, here is a simple criterion which ensures that injectives are mapped to injectives (and hence to acyclics, provided there are enough injectives!).

Lemma 5.2.1.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Suppose F has a left adjoint which is exact, then F maps injectives to injectives.*

Proof. Let I be an injectives object in $\mathscr{A}$. We need to show that the functors $Y \to \mathrm{Hom}_{\mathscr{B}}(Y, F(I))$ is exact. Let (G, F) be an adjoint pair. Then there exists a natural (in Y) isomorphism $\mathrm{Hom}(G(Y), I) \simeq \mathrm{Hom}(Y, F(I))$. By assumption G is exact and $\mathscr{I}$ is injective, hence the functor $Y \to \mathrm{Hom}_{\mathscr{B}}(Y, F(I))$ is exact and thus $F(I)$ is injective. □

Here are some useful applications.

Proposition 5.2.1.3. *Let $f : X \to Y$ be a map of ringed spaces and let $\mathscr{F}$ be an $\mathscr{O}_X$ module. Then the sheaf $R^i f_* \mathscr{F}$ is canonically isomorphic to the sheafification of the presheaf $U \mapsto H^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$ for U open in Y.*

Proof. Let $\mathscr{A}$ be the abelian category sheaves of $\mathscr{O}_X$-modules, let $\mathscr{B}$ be the pre-sheaves of $\mathscr{O}_Y$-modules and $\mathscr{C}$ be the sheaves of $\mathscr{O}_Y$-modules. Let F be the composition of the forgetful functor from $\mathscr{A}$ to presheaves of $\mathscr{O}_X$-modules on X followed by f_*. Note that f_* is an exact functor on the category of presheaves, while the forgetful functor is only left exact. Thus F is left exact and G being the sheafification functor is exact. Hence by Theorem 5.2.1.1 $R^i f_* \mathscr{F} = G \circ F^i \mathscr{F}$.
We claim $F^i \mathscr{F}$ is the required pre sheaf. To see this note that to compute $\mathscr{F}$, we replace $\mathscr{F}$ by an injective resolution $\mathscr{I}^\bullet$and then apply the forgetful functor to presheaves followed by the exact functor f_*. Thus on any open set U of Y, $F^i\mathscr{F}(U)$ is the same as $H^i(\mathscr{I}(f^{-1}(U)))$ (Why?). The latter computes $H^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$ because injective sheaves are flasque (see Proposition 5.1.0.13). □

Corollary 5.2.1.4. *Let $f : X \to Y$ and $g : Y \to Z$ be morphism of ringed spaces. Then f_* takes injectives to g_*-acyclic sheaves. Hence $R(g \circ f)_* = Rg_* \circ Rf_*$*

Proof. Let $\mathscr{F}$ be an injective $\mathscr{O}_X$-module. Then Proposition 5.1.0.13 implies that $\mathscr{F}$ is flasque. Thus $f_*\mathscr{F}$ is flasque and hence for any open $U \hookrightarrow Y$, $f_*\mathscr{F}|_U$ is flasque. Thus by Proposition 5.2.1.3 we are done. □

So how do we apply Corollary 5.2.1.4?

1. Suppose f_* is exact (for example a closed immersion see Lemma 5.1.0.15). Then $R^i(g \circ f)_*\mathscr{F} = R^i g_*(f_*\mathscr{F})$ for any sheaf $\mathscr{F}$. Put differently we can compute the cohomology of $\mathscr{F}$ after pushing it to the ambient space. Similarly if g_* is exact then $R^i(g \circ f)_*\mathscr{F} = g_* R^i f_*$.
2. Suppose $\mathscr{F}$ is a sheaf on X with no higher direct images. Then we claim $R^i(g \circ f)_*\mathscr{F} = R^i g_*(f_*\mathscr{F})$.

In general though it is not so easy to relate $R^i g_*(R^j f_*\mathscr{F})$ and $R^{i+j}(g \circ f)_*\mathscr{F}$. They are related by what are called spectral sequences which can be thought of as a collection of many long exact sequences whose cohomologies compute what we want. Our spectral sequence has an E_2-page indexed by two non-negative integers p and q, which looks like

$$E_2^{p,q} = R^p f_*\Big(R^q g_* M\Big).$$

The E_2-page of our spectral sequence has maps

$$d_2\colon E_2^{p,q} \longrightarrow E_2^{p+2,q-1},$$

which are called differentials, since any composition of two of these maps is zero. Since we have differentials, we can take cohomology, and the E_3-page is defined exactly as that,

$$E_3^{p,q} = \frac{\ker\Big(d_2\colon E_2^{p,q} \to E_2^{p+2,q-1}\Big)}{\operatorname{im}\Big(d_2\colon E_2^{p-2,q+1} \to E_2^{p,q}\Big)}.$$

Again we have differentials, called

$$d_3\colon E_3^{p,q} \longrightarrow E_3^{p+3,q-2},$$

and this process continues to the E_4-page. In general, we have an E_r-page of our spectral sequence defined in the obvious way. This is a first quadrant spectral sequence, so we can see that since these differentials grow larger as r increases, eventually an element $E_r^{p,q}$ with $r > p, q$ cannot be hit or receive nonzero differentials, and hence

$$E_r^{p,q} \cong E_k^{p,q} \quad \text{for all } k \geqslant r.$$

In this situation we define

$$E_r^{p,q} = E_\infty^{p,q},$$

where the position has stabilised.
All of this information leads us to the following theorem, which is also true in the generality of Proposition 11.11.

Theorem 5.2.1.5. *The sheaf $R^i(f \circ g)_*\mathscr{F}$ has a decreasing filtration*

$$F^p R^i(f \circ g)_*\mathscr{F} \subset R^i(f \circ g)_*\mathscr{F},$$

with

$$F^{-1} = R^i(f \circ g)_*\mathscr{F} \quad \textit{and} \quad F^i = 0,$$

such that the associated graded object is

$$\operatorname{gr}^p R^i(f \circ g)_*\mathscr{F} = E_\infty^{p,i-p}$$

in the spectral sequence defined above. In the usual language of spectral sequences we may write

$$E_2^{p,q} = R^p f_*\Big(R^q g_*\mathscr{F}\Big) \quad \Longrightarrow \quad R^{p+q}(f \circ g)_*\mathscr{F}.$$

A specific case of the above spectral sequence is the so called *Leray* spectral sequence which is obtained by taking $Z = \{pt\}$. Then we get a spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*\mathscr{F}) \implies H^{p+q}(X, \mathscr{F})$$

These spectral sequences are special cases of filtered complex spectral sequence, which we shall state now. Let $\mathcal{A}$ be an abelian category. A *filtered complex* in $\mathcal{A}$ is a complex

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots,$$

together with a *decreasing* filtration $\{F^pC^\bullet\}_{p\in\mathbb{Z}}$ by subcomplexes, that is, for each $p \in \mathbb{Z}$ and for all n,

$$F^pC^n \subseteq C^n,$$

and

$$d^n\big(F^pC^n\big) \subseteq F^pC^{n+1}.$$

These filtrations satisfy

$$\cdots \supseteq F^pC^\bullet \supseteq F^{p+1}C^\bullet \supseteq \cdots .$$

The Filtered Complex Spectral Sequence

Given the filtered complex $(C^\bullet, F^\bullet)$, one forms the *associated graded complex* by setting

$$\mathrm{Gr}^p\, C^n := \frac{F^pC^n}{F^{p+1}C^n}.$$

Since the differential d is compatible with the filtration, it induces a differential

$$d_0^{p,q}\colon\ \mathrm{Gr}^p\, C^{p+q} \longrightarrow \mathrm{Gr}^p\, C^{p+q+1}.$$

Thus, we define the E_0-page of the spectral sequence by

$$E_0^{p,q} := \mathrm{Gr}^p\, C^{p+q}.$$

Taking cohomology with respect to d_0 gives the E_1-page:

$$E_1^{p,q} := H^{p+q}\Big(\mathrm{Gr}^p\, C^\bullet\Big).$$

In general, one obtains a spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ with differentials

$$d_r^{p,q}\colon E_r^{p,q} \longrightarrow E_r^{p+r,\,q-r+1},$$

and, under appropriate boundedness conditions, the spectral sequence converges (or *abuts*) to $H^{p+q}(C^\bullet)$ and thus one has

$$E_\infty^{p,q} \cong \mathrm{Gr}^p\, H^{p+q}(C^\bullet) = \frac{F^pH^{p+q}(C^\bullet)}{F^{p+1}H^{p+q}(C^\bullet)}.$$

Example: Two Spectral Sequences Associated to a Total Complex

Suppose we are given a first quadrant *double complex* $C^{\bullet,\bullet}$ in $\mathcal{A}$, with horizontal differentials d_h and vertical differentials d_v satisfying

$$d_h^2 = d_v^2 = d_hd_v + d_vd_h = 0.$$

The *total complex* $\mathrm{Tot}(C)$ is defined by

$$\mathrm{Tot}^n(C) = \bigoplus_{p+q=n} C^{p,q},$$

with differential

$$d = d_h + d_v.$$

There are two natural filtrations on $\mathrm{Tot}(C)$:

(a) The Column Filtration

Define the filtration by columns:

$$F^p_{\mathrm{col}}\,\mathrm{Tot}^n(C) := \bigoplus_{\substack{p' \geqslant p \\ p'+q=n}} C^{p',q}.$$

The associated spectral sequence has:

$$E_0^{p,q} = C^{p,q},$$

with the vertical differential d_v yielding

$$E_1^{p,q} = H^q\Big(C^{p,\bullet}, d_v\Big).$$

The differential $d_1^{p,q}$ on the E_1-page is induced by the horizontal differential d_h:

$$d_1^{p,q}\colon H^q\Big(C^{p,\bullet}\Big) \longrightarrow H^q\Big(C^{p+1,\bullet}\Big).$$

This spectral sequence converges to the cohomology $H^{p+q}(\mathrm{Tot}(C))$.

(b) The Row Filtration

Alternatively, define the filtration by rows:

$$F^q_{\mathrm{row}}\,\mathrm{Tot}^n(C) := \bigoplus_{\substack{q' \geqslant q \\ p+q'=n}} C^{p,q'}.$$

Then the associated spectral sequence has:

$$E_0^{p,q} = C^{p,q},$$

and the horizontal differential d_h gives

$$E_1^{p,q} = H^p\Big(C^{\bullet,q}, d_h\Big).$$

The differential on the E_1-page, $d_1^{p,q}$, is induced by the vertical differential d_v:

$$d_1^{p,q}\colon H^p\Big(C^{\bullet,q}\Big) \longrightarrow H^p\Big(C^{\bullet,q+1}\Big).$$

This spectral sequence also converges to $H^{p+q}(\mathrm{Tot}(C))$.

Both of these spectral sequences provide computational tools for analyzing the cohomology of the total complex, by first computing cohomology along one of the two directions (columns or rows) and then taking the cohomology of the resulting complex.

Chapter 6

Cohomology: Basic Computations

In this chapter we will compute the cohomology of *all* affine schemes and the projective space. The former is particularly simple and the latter (unsurprisingly) uses the former.

6.1 A fundamental exact sequence

We fix a base ring R (for example $\mathbb{Z}$). By a sheaf we shall me sheaves of R-modules[1]. Let X be any topological space. Let $j : U \hookrightarrow X$ be an open subset and let $i : Z \hookrightarrow X$ be the closed complement.

Definition 6.1.0.1. Let $\mathscr{F}$ be a pre sheaf on U. The extension by zero $j_!\mathcal{F}$ is defined as follows

$$j_!(\mathscr{F})(V) = \begin{cases} \mathscr{F}(V) & \text{if } V \subseteq U, \\ 0 & \text{otherwise.} \end{cases}$$

As before we let $j : U \hookrightarrow X$ be an open immersion and we denote by $i : Z \hookrightarrow X$ the corresponding closed immersion.

Lemma 6.1.0.2. *The functor $j_!$ takes sheaves to sheaves. Moreover*

1. *For any $x \in X$ and sheaf $\mathscr{F}$ on X,*

$$(j_!\mathscr{F})_x = \begin{cases} \mathscr{F}_x & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

2. *$j_!$ is an exact functor.*

[1]Alternatively you can think of every topological space as a ringed space with the sheaf of rings given by the constant sheaf with values in R

3. *For any sheaf $\mathscr{F}$ there exists an exact sequence of sheaves*

$$0 \longrightarrow j_!j^*\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow i_*i^*\mathscr{F} \longrightarrow 0.$$

Proof. That $j_!$ takes sheaves to sheaves is clear from the definition. So it the claim about its stalk. This implies (2) and (3). □

Here is simple observation about $j_!$ which will be pretty handy.

Lemma 6.1.0.3. *There exists an adjoint triple $(j_!, j^*, j_*)$.*

Proof. We only need to show that $(j_!, j^*)$ form an adjoint pair or equivalently that there exists a natural isomorphism,

$$\mathsf{Hom}(j_!\mathscr{F}, G) \simeq \mathsf{Hom}(\mathscr{F}, j^*\mathscr{G}).$$

This is clear from the definition of $j_!$. □

Here is a simple corollary to the adjointness.

Corollary 6.1.0.4. *For any sheaf $\mathscr{F}$ there exists a natural isomorphism*[2]

$$\mathit{Hom}(j_!R_U, \mathscr{F}) \simeq \mathscr{F}(U).$$

In particular there exists a natural surjection (even of pre-sheaves)

$$\bigoplus_{(j:U\hookrightarrow X, s\in\mathscr{F}(U))} j_!R_U \to \mathscr{F},$$

where the sum is over all open subsets $U \hookrightarrow X$ and sections $s \in \mathscr{F}(U)$.

Proof. By adjunction $\mathsf{Hom}(j_!R_U, \mathscr{F})$ is isomorphic to $\mathsf{Hom}(R_U, j^*\mathscr{F})$ which is isomorphic to $\mathscr{F}(U)$. □

We end this section with an useful proposition. To state it we need a few notations. Let $\mathscr{U} := \{U_i\}_{i\in I}$ be a finite open cover of X indexed by $I := \{0, 1 \cdots n\}$. For any subset $J \subseteq I$, we denote by U_J the intersection of open subsets indexed elements in J and by j_J the corresponding open immersion. Moreover for any two subsets $K \subseteq J$, we have a natural map

$$r_{JK} : j_{K!}R \to j_{J!}R.$$

With these notations we have the following, inclusion-exclusion principle.

[2]For any topological space X, by R_X we mean the constant sheaf with values in R. We shall drop the subscript X, when there is no scope of confusion.

Proposition 6.1.0.5. *The following sequence of sheaves is exact*

$$0 \longrightarrow j_{I!}R \longrightarrow \bigoplus_{J\subseteq I,|J|=|I|-1} j_{J!}R \longrightarrow \cdots \qquad \bigoplus_{J\subseteq I,|J|=1} j_{J!}R \longrightarrow R_X \longrightarrow 0,$$

where the map

$$j_{K!}R \to \bigoplus_{J\subseteq I,|J|=|K|-1} j_{J!}R,$$

is given by $\sum(-1)^{K\setminus J} r_{JK!}$. *Moreover if* $X \in \mathscr{U}$, *then the above sequence of sheaves in null homotopic to* 0.

Proof. Exactness of a sequence of sheaves can be checked on an open cover. Hence by replacing X by open sets in $\mathscr{U}$ we may assume $X \in \mathscr{U}$, say $X = U_i$ for some $0 \leqslant i \leqslant n$. In particular it suffices to prove the second part of the proposition.
Denote the complex of sheaves by $\mathscr{C}_{\mathscr{U}}$. To prove null-homotopy we need to give a map

$$\partial : \mathscr{C}_{\mathscr{U}} \to \mathscr{C}_{\mathscr{U}}[-1],$$

inducing an homotopy equivalence between the zero map and identity. The key point is to note that for any subset $J \subseteq I$ not containing i, $J' := \{i\} \cup J$ satisfies

1. $|J'| = |J| + 1$.
2. $U_{J'} = U_J$.

We consider the sheaf R_X to be in degree 0. Thus we define

$$\partial_0 : R_X \to \bigoplus_{J,|J|=1} r_{J!}R,$$

via the isomorphism $R_X \simeq r_{\{i\}!}R$. For the lower ∂'s (upto a sign) they are given by identifying $r_{J!}R \simeq r_{J'!}R$ for $i \notin J$. We leave it to the reader to work out the signs. □

6.2 Čech Cohomology

In this section we will define and study some properties of a very useful tool called Čech Cohomology, this can be thought of as a sheaf theoretic inclusion-exclusion principle. We begin by defining the Čech complex.
As before we fix an open covering $\mathscr{U} = \{U_i\}_{i\in I}$ of our topological space X indexed by a finite set $I = \{0, 1 \cdots n\}$. We continue using the notations from Section 6.1. The Čech complex $(_{\mathscr{U}})$ is a functor from pre sheaves of abelian groups on X to complex of abelian groups.

Definition 6.2.0.1. The Čech complex of $\mathscr{F}$ with respect to $\mathscr{U}$ is the following complex of abelian groups

$$\prod_{J\subseteq I,|J|=1} \mathscr{F}(U_J) \longrightarrow \prod_{J\subseteq I,|J|=2} \mathscr{F}(U_J) \longrightarrow \cdots \qquad \mathscr{F}(\textstyle\bigcap_{i\in J} U_i),$$

where the map

$$\mathscr{F}(U_J) \to \prod_{K\subseteq I,|K|=|J|+1} \mathscr{F}(U_K),$$

is given by sending a section $s \to \prod_{J\subseteq K\subseteq I,|K|=|J|+1}((-1)^{K\setminus J} s|_{U_K})$.

We need one more notation before we can state an important corollary to Proposition 6.1.0.5.

Notations 6.2.0.2. Denote by $\check{H}^p(\mathscr{U},\mathscr{F})$ the p^{th} cohomology of the Čech complex of $\mathscr{F}$ with respect to $\mathscr{U}$.

Corollary 6.2.0.3. *Let $\mathscr{F}$ be a sheaf on X. Then we have the following*

1. *$\check{H}^p(\mathscr{U},\mathscr{F})$ vanishes for $p > |I|$.*

2. *$\check{H}^0(\mathscr{U},\mathscr{F}) \simeq \mathscr{F}(X)$.*

3. *$\check{H}^p(\mathscr{U},\mathscr{F})$ vanishes for $p > 0$ if $\mathscr{F}$ is an injective sheaf.*

4. *$\check{H}^p(\mathscr{U},\mathscr{F})$ vanishes for $p > 0$ if $X \in \mathscr{U}$.*

Proof. (1) is clear from the definition of the Čech complex, since there are no terms in the complex of degree greater than $|I|$. (2) follows from the definition of the Čech complex and the fact that $\mathscr{F}$ is a sheaf. For (3) and (4) simply note that, thanks to Corollary 6.1.0.4, Čech complex can be obtained by applying the functor Hom$(-,\mathscr{F})$ to the exact sequence of sheaves in (**??**) and ignoring the degree 0 term (which as we just saw is $\mathscr{F}(X)$). If $\mathscr{F}$ is injective, then the functor Hom$(-,\mathscr{F})$ is exact and hence implies (3). On the other hand if $X \in \mathscr{U}$, the complex in (**??**) is null homotopic and thus continues to be null homotopic after applying the additive functor Hom$(-,\mathscr{F})$. Hence the result. □

Now suppose $\mathscr{F}$ is an arbitrary sheaf. Then what do we do? Of course as always we first resolve $\mathscr{F}$ by injectives. Thus get a complex of injectives

$$\mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots$$

quasi-isomorphic to $\mathscr{F}$. Each of those $\mathscr{I}^j$'s has its own Čech complex. Thus we obtain a double complex

$$\begin{tikzcd}
\prod_{\substack{J\subseteq I\\|J|=1}} \mathscr{I}^0(U_J) \arrow[r, "\check{\delta}^0"] \arrow[d, "d^0"'] & \prod_{\substack{J\subseteq I\\|J|=2}} \mathscr{I}^0(U_J) \arrow[r, "\check{\delta}^1"] \arrow[d, "d^0"'] & \cdots \arrow[r] & \mathscr{I}^0\Big(\bigcap_{i\in J} U_i\Big) \arrow[d, "d^0"] \\
\prod_{\substack{J\subseteq I\\|J|=1}} \mathscr{I}^1(U_J) \arrow[r, "\check{\delta}^0"] \arrow[d, "d^1"'] & \prod_{\substack{J\subseteq I\\|J|=2}} \mathscr{I}^1(U_J) \arrow[r, "\check{\delta}^1"] \arrow[d, "d^1"'] & \cdots \arrow[r] & \mathscr{I}^1\Big(\bigcap_{i\in J} U_i\Big) \arrow[d, "d^1"] \\
\vdots \arrow[r] \arrow[d, dotted] & \vdots \arrow[r] \arrow[d, dotted] & {} & \vdots \arrow[d, dotted] \\
\prod_{\substack{J\subseteq I\\|J|=1}} \mathscr{I}^q(U_J) \arrow[r, "\check{\delta}^0"] \arrow[d, dotted] & \prod_{\substack{J\subseteq I\\|J|=2}} \mathscr{I}^q(U_J) \arrow[r, "\check{\delta}^1"] \arrow[d, dotted] & \cdots \arrow[r] & \mathscr{I}^q\Big(\bigcap_{i\in J} U_i\Big) \arrow[d, dotted] \\
{} \arrow[r, dotted] & {} \arrow[r, dotted] & \cdots \arrow[r, dotted] & {}
\end{tikzcd}$$

Here are a few remarks about this double complex:

1. Thanks to Corollary 6.2.0.3, each row is exact outside degree 0 and at degree 0 its cohomology is the global sections of the corresponding injective sheaf.

2. On a fixed column the cohomology is simply the product of the cohomologies of $\mathscr{F}|_{U_J}$, thanks to restriction of injective sheaf to open subsets being flasque (see Proposition 5.1.0.13).

In short we have the following,

Proposition 6.2.0.4. *There exists a first quadrant E_1 spectral sequence*

$$E_1^{p,q} := \prod_{\varnothing\subsetneq J\subseteq I, |J|=p+1} H^q(U_J, \mathscr{F}|_{U_J}) \implies H^{p+q}(X, \mathscr{F}), \tag{6.1}$$

such that the differentials $E_1^{p,0}$ are the one's corresponding to the Čech complex.

Here is a quick corollary.

Corollary 6.2.0.5. *Suppose $\mathscr{U}$ is an open cover such that $H^q(U_J, \mathscr{F}|_{U_J}) = 0$ for any $J \subseteq I$ with $|J| \geqslant 1$. Then the spectral sequence in (6.1) degenerates in the E_1-page and*

$$H^p(X, \mathscr{F}) \simeq \check{H}^p(\mathscr{U}, \mathscr{F}).$$

6.3 Cohomology of affine schemes

The aim of this section is to show that affine schemes have no higher cohomology with coefficients in a quasi-coherent sheaf. As we shall see this uniquely characterizes affine schemes among quasi compact and (quasi) separated schemes.

Theorem 6.3.0.1. *Let X be an affine scheme and $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $H^p(X, \mathcal{F})$ vanishes for all $p > 0$.*

Proof. Thanks to Corollary 5.1.0.14, we can work in the category of abelian sheaves and thus use results from Section 6.2.

Claim 1: Given any topological space X, a sheaf $\mathcal{F}$ on it and any class $\alpha \in H^q(X, \mathcal{F})$, $q > 0$, there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\alpha|_{U_i} \in H^q(U_i, \mathcal{F}|_{U_i})$ vanishes. Moreover if X is quasi-compact we can choose this cover to be finite.

Claim 2: Let X be an affine scheme and $\mathcal{U}$ be a finite open cover of X by basic affine opens. Then $\check{H}^q(X, \mathcal{F})$ vanishes for $q > 0$ and any quasi-coherent sheaf $\mathcal{F}$.

Taking these for granted we complete the proof of the theorem. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open affine cover of X indexed by $I = \{0, 1, 2 \ldots n\}$. Then we have an E_1 spectral sequence (6.2) computing $H^p(X, \mathcal{F})$. This is a first quadrant spectral sequence and we note that the complex $E_1^{\bullet,0}$ is simply the Čech complex of the cover which by claim (2) is exact in positive degrees. Thus $E_\infty^{p,0}$ vanishes for $p > 0$. Further the filtration of $H^1(X, \mathcal{F})$ induced by this is particularly simple, it is a two-step filtration i.e

1. $F_i H^1(X, \mathcal{F}) = H^1(X, \mathcal{F})$ for all $i \leqslant 0$.
2. $F_i H^1(X, \mathcal{F}) = 0$ for all $i \geqslant 2$.

Thus we have an exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1(X, \mathcal{F}) \longrightarrow E_\infty^{0,1} \longrightarrow 0.$$

Notice also that $E_\infty^{0,1} = \ker(\prod_{i \in I} H^1(U_i, \mathcal{F}|_{U_i}) \longrightarrow \prod_{i_0 < i_1} H^1(U_{i_0} \cap U_{i_1}, \mathcal{F}|_{U_{i_0} \cap U_{i_1}}))$ and that $E_\infty^{0,1} = \check{H}^1(U, \mathcal{F})$. Since $E_\infty^{1,0}$ vanishes, this implies that

$$H^1(X, \mathcal{F}) \cong E_\infty^{0,1} \hookrightarrow \prod_i H^1(U_i, \mathcal{F}|_{U_i}).$$

Note that U_i's are any finite open affine cover of X. Since any non-zero element of α can be (locally) killed by choosing an appropriate open affine cover, we must have $H^1(X, \mathcal{F}) = 0$ for any affine scheme X and a quasi-coherent sheaf $\mathcal{F}$.
Now suppose we have proved the vanishing of $H^p(X, \mathcal{F})$ for $1 \leqslant p \leqslant q-1$ and any affine X with a quasi-coherent sheaf $\mathcal{F}$. Assuming this we shall prove the vanishing at $H^q(X, \mathcal{F})$. By the induction step our E_1 spectral sequence will have vanishing $E_1^{p,q'}$ for all $1 \leqslant p \leqslant q-1$ and any q'. Thus $E_\infty^{p,q'} = 0$ for all $1 \leqslant p \leqslant q-1$ and any q'. Together with the vanishing of $E_\infty^{p,0}$, $p \geqslant 1$ this implies as before that

$$H^q(X, \mathcal{F}) \cong E_\infty^{0,q} \hookrightarrow \prod_i H^q(U_i, \mathcal{F}|_{U_i}).$$

Arguing as before we show that $H^q(X, \mathcal{F})$ cannot have any non-zero classes. Thus proving the theorem modulo the claim, which we shall prove now.
Proof of Claim 1: This is simple. By definition to compute $H^q(X, \mathcal{F})$, we take an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$, and thus any element $\alpha \in H^q(X, \mathcal{F})$ begins life in $\mathcal{I}^q(X)$ as the kernel of the map to $\mathcal{I}^{q+1}(X)$. Since $\mathcal{I}^\bullet$ is an injective resolution, of sheaves, there exists an open cover $\mathcal{U}$ of X such that $\alpha|_{U_i}$ is necessarily in the image of the map from $\mathcal{I}^{q-1}(U_i) \longrightarrow \mathcal{I}^q(U_i)$. Since injective sheaves are flasque (see Proposition 5.1.0.13), this means that as a class in cohomology, α vanishes when restricted to the U_i's.
Proof of Claim 2: Suppose $X = \mathrm{Spec}(A)$, $\mathcal{F} = \tilde{M}$ and $U_i = \mathrm{Spec}(A_{f_i})$ we need to show that the following complex is exact:

$$0 \longrightarrow M \longrightarrow \oplus_i M_{f_i} \longrightarrow \oplus_{i_0 < i_1} M_{f_{i_0} f_{i_1}} \longrightarrow \dots$$

We may check exactness after a faithfully flat base change $\coprod_i \mathrm{Spec}(A_{f_i}) \longrightarrow \mathrm{Spec}(A)$, and since finite product of exact sequences is exact it suffices to check this after tensoring this with A_{f_i}. In which case the result follows from Corollary 6.2.0.3.

□

We now deduce some corollaries.

Corollary 6.3.0.2. *Let X be a separated (over $\mathbb{Z}$) scheme and $\mathcal{U}$ be a finite affine cover of X. Then $\check{H}^p(X, \mathcal{F}) \cong H^p(X, \mathcal{F})$ for any quasi-coherent sheaf $\mathcal{F}$ on X. Moreover let I be the indexing set of such a cover. Then $H^p(X, \mathcal{F})$ vanishes for $p > |I| - 1$.*

Proof. The first part follows from Corollary 6.2.0.5. The second part is a consequence of the fact that the Čech complex has non-zero terms only upto degree $|I| - 1$. □

Corollary 6.3.0.3. *Let $f : X \longrightarrow Y$ be an affine morphism and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $R^q f_* \mathcal{F}$ vanishes for $q > 0$ and hence f_* is exact on the abelian category of quasi-coherent sheaves. In particular $H^p(X, \mathcal{F}) \cong H^p(Y, f_* \mathcal{F})$.*

Proof. Combine Theorem 6.3.0.1 and Proposition 5.2.1.3. □

We end this section with Serre's affineness criterion, which is a cohomological criterion to detect affineness.

Theorem 6.3.0.4 (Serre's criterion for affineness). *Let X be a quasi-compact scheme. Then TFAE*

1. *X is affine.*

2. *$H^p(X, \mathcal{F}) = 0$ for all $p > 0$ and any quasi-coherent sheaf $\mathcal{F}$ on X.*

3. *$H^1(X, \mathcal{I}) = 0$ for any quasi-coherent sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$.*

Proof. Thanks to Theorem 6.3.0.1 it suffices to show (3) implies (1). To do so we shall use [3, Chapter II, Exercise 2.17 (b)]. Thus we need to find f_i, $1 \leqslant i \leqslant n$ such that 1. the ideal generated by the f_i's is $H^0(X, \mathcal{O}_X)$. 2. The X_{f_i}'s are affine
Let $x \in X$ be a closed point of X[3]. Choose an affine open $U \ni x$ and let Z be the complement of U with the reduced induced structure. Let $Z_1 = Z \backslash \{x\}$, also with the reduced induced structure. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_{Z_1} \longrightarrow \mathcal{I}_Z \longrightarrow \frac{\mathcal{I}_Z}{\mathcal{I}_{Z_1}} \longrightarrow 0.$$

Since $Z \cap (X \backslash \{x\}) = Z_1 \cap (X \backslash \{x\})$, the quotient $\frac{\mathcal{I}_Z}{\mathcal{I}_{Z_1}}$ is supported only at x, further restricting to the affine open U we conclude that $\frac{\mathcal{I}_Z}{\mathcal{I}_{Z_1}} = i_{x*}k(x)$[4], where $i_x : x \longrightarrow X$ is the corresponding closed immersion. Taking the long exact sequence in cohomology of the above short exact sequence of sheaves together with the vanishing condition (3) we get a surjection

$$H^0(X, \mathcal{I}_Z) \twoheadrightarrow k(x).$$

Let $f \in H^0(X, \mathcal{I}_Z) \subset H^0(X, \mathcal{O}_X)$ be a lift of the $1 \in k(x)$. Then $X_f = U_f$ and hence is affine since U was affine by choice. We can do this around every closed point. Let $\tilde{X}$ be the union of open affines X_f obtained as above. Then their complement if non-empty is a non-empty quasi-compact scheme with no closed points. This is not possible (Why?). Thus $\tilde{X} = X$ and by quasi-compactness we can assume that X is covered by finitely many X_{f_i}'s (say $f_1, f_2 \cdots f_n$. Now we shall show that the f_i's generate $H^0(X, \mathcal{O}_X)$. By assumption there exists a surjection of sheaves

$$\psi : \mathscr{O}_X^{\oplus n} \to \mathscr{O}_X,$$

given by $(f_1, f_2 \cdots f_n)$. Let $\mathscr{F}$ be the kernel of this map. It suffices to show that $H^1(X, \mathscr{F}) = 0$, since this would imply the map induced by ψ after applying H^0 is surjective.

[3] Question: Why does this exist?

[4] Here is an argument: Suppose $U = \operatorname{Spec}(A)$ and let I and $\mathfrak{m}$ be the ideals of Z and x respectively. Then $\frac{\mathscr{I}_Z}{\mathscr{I}_{Z_1}}|_U$ is given by $\frac{I}{\sqrt{I\mathfrak{m}}}$. Since $I \subseteq A$, the latter is a non-zero $A/\mathfrak{m}$-sub module of $A/\mathfrak{m}$ and hence the result!

Though $\mathscr{F}$ is not necessarily an ideal sheaf in $\mathscr{O}_X$, however it has a finite decreasing filtration $\mathscr{F}_i \subseteq \mathscr{F}$, where each quotient is an ideal sheaf and hence has vanishing H^1, which in turn implies $H^1(X, \mathscr{F})$ vanishes. This filtration is obtained by intersecting the standard filtration[5] on $\mathscr{O}_X^{\oplus n}$ with $\mathscr{F}$.

□

6.4 Finiteness theorems in cohomology

In this section we shall show that properness is the *correct* condition which ensures finiteness of cohomology. This is already evident in the one dimensional case:

1. $H^0(\mathbb{A}^1_k, \mathscr{O}_{\mathbb{A}^1_k})$ is infinite dimensional over k.
2. $H^0(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k})$ on the other hand is finite dimensional.

We begin with the following simple lemma.

Lemma 6.4.0.1. *Let A be any ring. Then $H^i(\mathbb{P}^n_A, \mathscr{O}_{\mathbb{P}^n_A})$ vanishes for $i > n$.*

Proof. $\mathbb{P}^n_A$ is separated and has an affine cover with $n+1$ elements. Hence we are done by Corollary. □

Next we can use the standard open affine cover of $\mathbb{P}^n_A$ to compute the cohomology of $\mathscr{O}_{\mathbb{P}^n_A}(d), d \in \mathbb{Z}$. As can be seen from the case of global sections the answer depends on the sign of d. We now assume that

Proposition 6.4.0.2. *Let $X = \mathbb{P}^n_A$ and $d > 0$. Then the following hold*

$$H^i\big(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-d)\big) = \begin{cases} 0, & \text{if } i \neq n \text{ or } d \leqslant n, \\ \dfrac{1}{x_0 \cdots x_n} A[x_0^{-1}, \dots, x_n^{-1}]_d, & \text{if } i = n \text{ and } d > n, \end{cases}$$

and

$$H^i(\mathbb{P}^n_A, \mathscr{O}_{\mathbb{P}^n_A}(d)) = \begin{cases} A[x_0, x_1 \cdots x_n]_d \text{ if } i = 0. \\ 0 \text{ otherwise} \end{cases}$$

where each x_i has degree 1. In particular, the cohomology modules are finitely generated as A-modules.

Assuming Proposition 6.4.0.2 for the moment we shall derive some consequences.

[5] The i$^{\text{th}}$ piece of this filtration is $\mathscr{O}_X^{\oplus i}$, by setting the last $n-i$ factors to 0.

Theorem 6.4.0.3. *Let A be a Noetherian ring and let $i : X \hookrightarrow \mathbb{P}^n_A$ be a projective scheme*[6]*. Then $H^p(X, \mathscr{F})$ is a finitely generated A-module for any coherent sheaf $\mathscr{F}$ on X.*

Proof. By Corollary 6.3.0.3 $H^p(X, \mathscr{F}) = H^p(\mathbb{P}^n_A, i_*\mathscr{F})$. Since $i_*\mathscr{F}$ is coherent on $\mathbb{P}^n_A$, we may assume $X = \mathbb{P}^n_A$. Since coherent sheaves are globally generated on projective space [3, Chapter II, Theorem 5.17], there exists a surjection

$$\mathscr{O}^r_X \twoheadrightarrow \mathscr{F}(m).$$

Let $\mathscr{I}$ be the kernel of this surjection. Then we have a short exact sequence of sheaves

$$0 \to \mathscr{I}(-m) \to \mathscr{O}^r_X(-m) \to \mathscr{F} \to 0.$$

By taking the long exact sequence in cohomology we see that $H^p(X, \mathscr{F})$ is sandwiched between two terms

$$\cdots \to H^p(X, \mathscr{O}^r_X(-m)) \to H^p(X, \mathscr{F}) \to H^{p+1}(X, \mathscr{I}(-m)) \cdots .$$

The claim of the theorem is trivially true for $p > n$ by Corollary 6.3.0.2. We prove the result by descending induction on p. Since A is assumed to be Noetherian, the above long exact sequence implies that it suffices to prove finiteness for sheaves of the form $\mathscr{O}_X(-m)$, for which we use Proposition 6.4.0.2. □

In particular when $A = k$ a field, Theorem 6.4.0.3 implies that $H^p(X, \mathscr{F})$ are finite dimensional vector spaces. Hence it makes sense to keep track of dim$(H^p(X, \mathscr{F}))$ (often denoted by $h^p(X, \mathscr{F})$.) We will comeback to this in a later section. Now we would like to localize Theorem 6.4.0.3.

Before doing so we first prove quasi-coherence of higher direct images.

Proposition 6.4.0.4. *Let $f : X \to Y$ be a quasi-compact and separated*[7] *and let $\mathscr{F} \in$ QCoh(X). Then $R^p f_*\mathscr{F}$ is quasi-coherent for all p.*

Proof. The statement is local on Y, hence we may assume $Y = \mathrm{Spec}(A)$. We claim that the natural map induced by adjunction [3, Chapter II, Exercise 5.3]

$$\widetilde{H^p(X, \mathscr{F})} \to R^p f_*\mathscr{F},$$

is an isomorphism of sheaves. Thanks to Proposition 5.2.1.3 it suffices to show that

$$H^p(X, \mathscr{F})_g \simeq H^p(f^{-1}(D(g)), \mathscr{F}_{f^{-1}(D(g))}).$$

[6] Here and in what follows i will denote the associated closed embedding

[7] quasi-separated is good enough, but we need to argue a little more. See Tag 01XJ.

Choose a finite affine cover $\mathscr{U} := \{U_i\}_{i\in I}$ of X. Then $\mathscr{U}_g := \{f^{-1}(D(g)) \cap U_i\}_{i\in I}$ is also an affine open cover of $f^{-1}(D(g))$. Since X and $f^{-1}(D(f)$ are separated, Corollary 6.3.0.2 implies that there is a commutative square whose horizontal arrows are isomorphisms

$$\begin{tikzcd} H^p(X, \mathscr{F}) \arrow[r, "\simeq"] \arrow[d] & H^p(\mathscr{U}, \mathscr{F}) \arrow[d] \\ H^p(f^{-1}(D(g)), \mathscr{F}|_{f^{-1}(D(f))}) \arrow[r, "\simeq"] & H^p(\mathscr{U}_g, \mathscr{F}|_{f^{-1}(D(g))}). \end{tikzcd}$$

The result now follows from the fact that $\mathscr{F}$ is quasi-coherent and hence

$$H^p(\mathscr{U}_g, \mathscr{F}|_{f^{-1}(D(g))}) \simeq H^p(\mathscr{U}, \mathscr{F})_g.$$

□

Here is a corollary.

Corollary 6.4.0.5. *Let $f : X \to \mathrm{Spec}(A)$ be a quasi-compact and separated morphism. Then $H^0(\mathrm{Spec}(A), R^p f_* \mathscr{F}) \simeq H^p(X, \mathscr{F})$.*

Proof. We have a spectral sequence

$$E_2^{p,q} = H^p(\mathrm{Spec}(A), R^q f_* \mathscr{F}) \implies H^{p+q}(X, \mathscr{F}).$$

Thanks to Proposition 6.4.0.4 and Theorem 6.3.0.1, this spectral sequence degenerates in the E_2 page and hence the result. □

Now we state and prove a generalization of Theorem 6.4.0.3.

Theorem 6.4.0.6. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes and let $\mathscr{F} \in Coh(X)$. Then $R^p f_* \mathscr{F}$ is coherent for all p.*

Proof. We already know by Proposition 6.4.0.4 that $R^p f_* \mathscr{F}$ is quasi-coherent. We first prove the assertion for projective morphisms. Since the assertion is local on Y, we may assume Y is affine in which case we are done by combining Corollary 6.4.0.5 and Theorem 6.4.0.3.
For a general f we proceed by Noetherian induction. Clearly when $\mathrm{Supp}(\mathscr{F}) = \varnothing$, then $\mathscr{F} = 0$ and the statement of the theorem is vacuously true. Hence we may use Noetherian induction and assume the theorem for any coherent sheaf whose support is a proper closed subset of X. Now by Chow's lemma, there exists a birational projective morphism $g : \tilde{X} \to X$, such that $f \circ g : \tilde{X} \to \mathrm{Spec}(A)$ is projective. Moreover $g^* \mathscr{F}$ is a coherent sheaf on $\tilde{X}$. Thus $R^p(f \circ g)_*(g^* \mathscr{F})$ are coherent sheaves on $\mathrm{Spec}(A)$.
Moreover there exists an E_2-spectral sequence

$$E_2^{p,q} = R^p f_* R^q g_*(g^* \mathscr{F}) \implies R^{p+q}(f \circ g)_*(g^* \mathscr{F}).$$

By induction hypothesis $E_2^{p,q}$ are coherent sheaves for $q \geqslant 1$ (Since g is birational). Moreover since $f \circ g$ is projective, $R^{p+q}(f \circ g)_*(g^*\mathscr{F})$ are coherent and thus so are the E_∞ terms (being sub-quotients of a coherent sheaf). Thus we have a first quasdrant spectral sequence whose E_∞ are coherent and all but the terms on the X-axis are coherent. This implies that the terms on the X-axis are also coherent.
Finally note that the natural map

$$\mathscr{F} \to g_* g^* \mathscr{F},$$

has kernel and cokernel supported along proper closed subsets of X. Thus have coherent higher direct images. Thus we are done. $\square$

Here is a corollary.

Corollary 6.4.0.7. *Let $f : X \to Y$ be a morphism of Noetherian schemes. Then f is finite iff it is affine and proper.*

Proof. Clearly finite morphisms are both affine and proper. For the converse, we reduce to the case when both X and Y are affine. The result then follows from Theorem 6.4.0.6 (We just need the coherence of R^0, in any case there are no higher direct images!). $\square$

6.5 Vanishing theorems

In this section we shall prove some vanishing theorems in the projective setting. The proofs are similar to those from earlier sections. Before we proceed we need an useful result, the projection formula.

Lemma 6.5.0.1. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of finite rank. Then the natural map*

$$f_*\mathscr{F} \otimes \mathscr{E} \to f_*(\mathscr{F} \otimes f^*\mathscr{E}),$$

induced by adjunction, is an isomorphism.

Proof. We can check this locally on Y. Hence we may assume $\mathscr{E}$ is trivial, in which case the statement is obvious. $\square$

Let A be a Noetherian ring.

Theorem 6.5.0.2. *Let $X/\mathrm{Spec}(A)$ be a proper scheme and $\mathscr{L}$ an ample line bundle on X. Then for any coherent sheaf $\mathscr{F}$ on X, there exists an $n_0 \geqslant 0$ such that $H^p(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $p > 0$ and any $n \geqslant n_0$.*

Proof. Since X is proper, by replacing $\mathscr{L}$ by a sufficiently large twist, we may assume that $X \subseteq \mathbb{P}^r_A$ (via a closed immersion i) and that $\mathscr{L}$ is the pullback of $\mathscr{O}(1)_{\mathbb{P}^r_A}$. Thus we need to show that $H^p(\mathbb{P}^r_A, i_*(\mathscr{F} \otimes i^*\mathscr{O}(n)))$ vanishes for $n >> 0$. By Lemma 8.2.0.8, we reduce to the case $X = \mathbb{P}^r_A$ and $\mathscr{F}$ a coherent sheaf on X. There exists a n_0, such that for any $n \geqslant n_0$, we have a short exact sequence

$$0 \to \mathscr{I} \to \mathscr{O}_X^m \to \mathscr{F}(n) \to 0.$$

As before we shall use a descending induction argument on the index p. Thanks to Corollary 6.3.0.2, the statement of the theorem is true for $p \geqslant r + 1$ (and any n_0). Now suppose we have proved the statement for all $p \geqslant p' + 1$. Then the long exact sequence in cohomology gives us an exact sequence

$$\cdots \to H^{p'}(X, \mathscr{O}_X^m(n')) \to H^{p'}(X, \mathscr{F}(n + n') \to H^{p'+1}(X, \mathscr{I}(n')) \to \cdots,$$

for any integer n'. We may now choose $n' >> 0$ such that $H^{p+1}(X, \mathscr{I}(n'))$ vanishes. The result then follows from Proposition 6.4.0.2. □

Now we give a cohomological criterion for ampleness, similar to Serre's criterion for affineness not just in spirit but also in the nature of it's proof.

Proposition 6.5.0.3. *Let Let $X/\mathrm{Spec}(A)$ be proper and $\mathscr{L}$ an invertible sheaf on X. Then TFAE*

1. *$\mathscr{L}$ is ample.*
2. *For any coherent sheaf $\mathscr{F}$, there exists an n, such that the cohomology groups $H^p(X, \mathscr{F} \otimes \mathscr{L}^n)$ vanish for $p > 0$.*
3. *For any sheaf of ideals $\mathscr{I}$, there exist an n, such that the cohomology groups $H^1(X, \mathscr{I} \otimes \mathscr{L}^n)$ vanish.*

Proof. Thanks to Theorem 6.5.0.2 it suffices to show $(3) \implies (1)$. To do so we claim the following:
Claim[8]: An invertible sheaf $\mathscr{L}$ on a separated scheme X finite type over $\mathrm{Spec}(A)$ is ample iff there there exists an $n \geqslant 1$ and a finite cover X_{f_i} of X with $f_i \in H^0(X, \mathscr{L}^{\otimes n})$.
We complete the proof of the result assuming the claim. As before we choose a closed point x, an open affine $U \ni x$ such that $\mathscr{L}|_U$ is trivial. Denote by Z the complement of U with the reduced induced structure. Let $Z' = Z \cup \{x\}$. Then as before we have an exact sequence

$$0 \to \mathscr{I}_{Z'} \to \mathscr{I}_Z \to i_{x*}k(x) \to 0,$$

where i_x is the closed immersion $x \hookrightarrow X$. Tensoring the above exact sequence with a suitable power of $\mathscr{L}$, we get a surjection

[8] This is an exercise

$$H^0(X, \mathscr{I}_Z \otimes \mathscr{L}^{\otimes n}) \to k(x).$$

As before we lift $1 \in k(x)$ to a section of $\mathscr{I}_Z \otimes \mathscr{L}^{\otimes n}$ and hence to a section f of ${}^{\otimes n}$. As before $U_f = X_f$ is affine and by doing this across all closed points, using quasi-compactness of X we get a finite cover.

□

6.6 Cohomology of Projective space

Finally we compute the cohomology of projective space

Theorem 6.6.0.1. *Let A be a Noetherian ring and let $X = \mathbb{P}^n_A$. We let $S = A[x_0, x_1 \cdots x_n]$. Then the following hold.*

1. *The natural map $S \to \oplus_{d \in \mathbb{Z}} H^0(X, \mathscr{O}_X(d))$ is an isomorphism of rings.*
2. *$H^p(X, \mathscr{O}_X(d))$ vanishes for $0 < i < n$.*
3. *There exists a functorial (in A) isomorphism*
$$\text{Tr}: H^n(X, \mathscr{O}_X(-n-1)) \simeq A.$$
4. *There exists a functorial (in A) perfect pairing of finite free A-modules*
$$H^0(X, \mathscr{O}_X(d)) \otimes_A H^n(X, \mathscr{O}_X(-n-d-1)) \to H^n(X, \mathscr{O}_X(-n-1) \simeq A$$

Proof. Let $\mathscr{F} := \oplus_{d \in \mathbb{Z}} \mathscr{O}_X(d)$. Then $\mathscr{F}$ is a quasi-coherent sheaf of graded rings. Moreover since X is Noetherian[9], cohomology commutes with direct sums and hence there exists a bi-graded[10] isomorphism

$$H^*(X, \mathscr{F}) \simeq \oplus_{d \in \mathbb{Z}} H^*(X, \mathscr{O}_X(d)).$$

Moreover we can compute the cohomology of X with values in $\mathscr{F}$ using Čech cohomology with respect to the standard cover $\mathscr{U} := \{D(x_i)\}_{0 \leqslant i \leqslant n}$. Moreover it follows from the definition of these twists that the Čech complex with respect to $\mathscr{U}$ is given by

$$0 \to \prod_i S_{x_i} \to \prod_{i<j} S_{x_i x_j} \cdots \prod_{i_0 < i_i \cdots < i_{n-1}} S_{x_{i_0} x_{i_1} \cdots x_{i_{n-1}}} \to S_{x_0 x_1 x_2 \cdots x_n} \to 0. \tag{6.2}$$

This is a complex of S-modules. We have already shown that there exists a graded isomorphism

$$H^0(X, \mathscr{F}) \simeq S.$$

[9] This can be relaxed, it suffices to observe that X is quasi-compact and quasi-separated. In particular we may remove the hypothesis that A is Noetherian.

[10] One for the degree in cohomology, the other from $\mathscr{F}$.

Now lets compute $H^n(\mathscr{U}, \mathscr{F})$. This is the cokernel of the map

$$\phi_n : \prod_{i_0<i_i\cdots<i_{n-1}} S_{x_{i_0}x_{i_1}\cdots x_{i_{n-1}}} \to S_{x_0x_1x_2\cdots x_n},$$

in (6.2). We think of $S_{x_0x_1x_2\cdots x_n}$ as the graded free A-module generated by

$$x_0^{l_0}x_1^{l_1}\cdots x_n^{l_n}, l_i \in \mathbb{Z}.$$

Hence the cokernel of ϕ_n is a free A-module generated by

$$x_0^{l_0}x_1^{l_1}\cdots x_n^{l_n}, l_i \in \mathbb{Z}^{<0}.$$

In particular, the degree $-n-1$ part is a rank 1 free A-module generated by $\frac{1}{x_0x_1\cdots x_n}$. This allows us to define the Trace map as the composite of

$$H^n(X, \mathscr{O}_X(-n-1)) \simeq H^n(\mathscr{U}, \mathscr{O}_X(-n-1)) \simeq A,$$

where the last isomorphism sends $\frac{1}{x_0x_1\cdots x_n} \to 1$. This proves (3). Now we shall prove (4). This is also an easy consequence of the following

(a) $H^0(X, \mathscr{O}_X(d))$ and $H^n(X, \mathscr{O}_X(-n-d-1)$ vanish for $d < 0$.

(b) For $d > 0$, there exists a functorial in A isomorphism of

$$H^0(X, \mathscr{O}_X(d)) \simeq S_{(d)} \simeq \oplus_{\substack{l_i \in \mathbb{Z}^{\geq 0} \\ \sum_i l_i = d}} x_0^{l_0}x_1^{l_1}\cdots x_n^{l_n}A$$

, and

$$H^n(X, \mathscr{O}_X(-n-d-1)) \simeq \oplus_{\substack{l_i \in \mathbb{Z}^{<0} \\ \sum_i l_i = -n-d-1}} x_0^{l_0}x_1^{l_1}\cdots x_n^{l_n}A.$$

(c) The pairing is simply multiplication.

Finally we are left to prove (2). First note that if we localize the Čech complex with respect to x_i, we get the Čech complex of $\mathscr{F}|_{D(x_i)}$, which has trivial cohomology due to Theorem 6.3.0.4 and Corollary 6.3.0.2. Since localization is an exact functor we conclude that $\check{H}^r(\mathscr{U}, \mathscr{F})_{x_i} \simeq 0$ for $r > 0$ and any i or equivalently that every element of $\check{H}^r(\mathscr{U}, \mathscr{F})$ is killed[11] by some power of x_i for all i and any $r > 0$. Thus it suffices to show that multiplication by x_i is injective on $\check{H}^r(\mathscr{U}, \mathscr{F})$ for any i and $n > r > 0$.
We have a short exact sequence of graded S-modules

$$0 \longrightarrow S(-1) \xrightarrow{x_i} S \longrightarrow S/x_i \longrightarrow 0. \tag{6.3}$$

[11] Note how computing cohomology of all twists at a time via $\mathscr{F}$ allows for such an argument.

Let $S' := S/x_i$, note that this is a projective space of dimension $n-1$. Let $\mathscr{F}(-1)$ and $\mathscr{F}|_H$ be the coherent sheaves obtained respectively by twisting $\mathscr{F}$ by $\mathscr{O}(-1)$ and restricting it to $x_i = 0$. Note also that $\mathscr{F}(-1) = \widetilde{S(-1)}$ and $\mathscr{F}|_H = \widetilde{S/x_i}$. Thanks to (6.3), we obtain a short exact sequence of Čech complexes

$$0 \longrightarrow \check{C}(\mathscr{U}, \mathscr{F}(-1)) \xrightarrow{x_i} \check{C}(\mathscr{U}, \mathscr{F}) \longrightarrow \check{C}(\mathscr{U}, \mathscr{F}|_H) \longrightarrow 0.$$

Taking the long exact sequence in cohomology and inducting on n, we conclude that $\check{H}^r(\mathscr{U}, \mathscr{F})$ vanishes for $i \neq 0, 1, n-1, n$. Thus we are left to show $\check{H}^1(\mathscr{U}, \mathscr{F})$ and $\check{H}^{n-1}(\mathscr{U}, \mathscr{F})$ vanish. Equivalently it suffices to show that the maps

$$\check{H}^1(\mathscr{U}, \mathscr{F}(-1)) \to \check{H}^1(\mathscr{U}, \mathscr{F}),$$

and

$$\check{H}^{n-1}(\mathscr{U}, \mathscr{F}_H) \to \check{H}^n(\mathscr{U}, \mathscr{F}(-1))$$

are injective. The first injectivity is a consequence of (1) and the commutative diagram

$$\begin{tikzcd}
\check{H}^0(\mathscr{U}, \mathscr{F}) \arrow[r, "\simeq"] \arrow[d] & S \arrow[d, two heads] \\
\check{H}^0(\mathscr{U}, \mathscr{F}_H) \arrow[r, "\simeq"] & S/x_i.
\end{tikzcd}$$

The second injectivity is a consequence of the trace isomorphism (3) and the commutative diagram

$$\begin{tikzcd}
\check{H}^{n-1}(\mathscr{U}, \mathscr{F}|_H) \arrow[r, "\simeq"] \arrow[d] & \oplus_{l_i \in \mathbb{Z}^{<0}} x_0^{l_0} x_1^{l_1} \cdots x_{i-1}^{l_{i-1}} x_{1+1}^{l_{i+1}} \cdots x_n^{l_n} A \arrow[d, "1/x_i"] \\
\check{H}^n(\mathscr{U}, \mathscr{F}(-1)) \arrow[r, "\simeq"] & \oplus_{l_i \in \mathbb{Z}^{<0}} x_0^{l_0} \cdots x_n^{l_n} A(-1).
\end{tikzcd}$$

□

6.6.1 Cohomology of complete intersections

Now we shall compute the cohomology of *complete intersections* in $\mathbb{P}^n_k$ where k is a field. We begin by recalling the definition of a complete intersection

Definition 6.6.1.1 (Complete Intersection). A closed subscheme $X \subseteq \mathbb{P}^n_k$ is said to be a complete intersection if $X = H_1 \cap H_2 \cdots \cap H_r$ where each $H_i \subseteq \mathbb{P}^n_k$ is a hypersurface and $r = \operatorname{codim}(X, \mathbb{P}^n_k)$.

Let us begin by making some simple observations.

Lemma 6.6.1.2. *A complete intersection $X \subseteq \mathbb{P}^n_k$ is equidimensional.*

Proof. We can check this on affine patches, say $D(X_0)$. Suppose that $X \cap D(X_0)$ is defined by $f_1, f_2 \cdots f_r$. We induct on r. For $r = 1$ this follows from Krull's haputidealsatz Tag 00KV. Suppose we have verifies this for $H_1 \cap H_2 \cdots H_i$. Now intersecting this scheme with H_{i+1} by Tag 00KV, the dimension can drop atmost by 1. This means every irreducible component has dimension at most 1 less than that of the equidimensional scheme $H_1 \cap H_2 \cdots H_i$. However since we know by definition that the codimension increase by 1, all the irreducible components are necessarily of the same dimension. □

We continue using the notation from the proof of Lemma 6.6.1.2.

Lemma 6.6.1.3. *$(f_1, f_2 \cdots f_r)$ form a regular sequence in $A := k[D(X_0)]$ and $A/(f_1, f_2 \cdots f_r)$ is Cohen-Macualay.*

Proof. Clearly f_1 is a non-zero divisor and hence we may assume that we have verifies $I_i := (f_1, f_2 \cdots f_i)$ forms a regular sequence for some $i \geqslant 1$. Then A/I_i is Cohen-Macaulay by Tag 02JN. The same result also implies that f_{i+1} is a non-zero divisor in A/I_i and hence verifying the claim □

Given any complete intersection $X = H_1 \cap H_2 \cdots \cap H_r$ as above we obtain a sequence of complete intersections
$$\mathbb{P}^n_k \supset H_1 \supset H_1 \cap H_2 \cdots \supset X.$$

Lemma 6.6.1.4. *The ideal sheaf of $H_1 \cap H_2 \cdots \cap H_i$ inside $H_1 \cap H_2 \cdots \cap H_{i-1}$ is given by the restriction of $\mathscr{O}(-\deg(H_i))$ to $H_1 \cap H_2 \cdots \cap H_{i-1}$.*

Proof. Let $Y = H_1 \cap H_2 \cdots \cap H_{i-1}$ and $X = H_1 \cap H_2 \cdots \cap H_i$. We can think of X as the base change of $H_i \subseteq \mathbb{P}^n_k$ along $Y \subseteq \mathbb{P}^n_k$. Since the ideal sheaf of H_i in $\mathbb{P}^n_k$ is isomorphic to $\mathscr{O}(-\deg(H_i))$, we get a surjection
$$\mathscr{O}(-\deg(H_i))|_Y \twoheadrightarrow \mathscr{I}_X \subseteq \mathscr{O}_Y,$$
where $\mathscr{I}_X$ is the ideal sheaf of X in Y. Thus it suffices to show that the above surjection is injective, which can be checked on affine opens of the form $D(X_p), 0 \leqslant p \leqslant n$ where the result follows from Lemma 6.6.1.3.

□

For the next proposition we assume $\dim(X) \geqslant 1$ and we denote by $d_i := \deg(H_i)$.

Proposition 6.6.1.5. *Let $X \subseteq \mathbb{P}^n_k$ be a complete intersection. Then*

(a) The natural map
$$H^0(\mathbb{P}^n_k, \mathscr{O}(i)) \to H^0(X, \mathscr{O}(i)|_X),$$
is surjective for any $i \in \mathbb{Z}$.

(b) *The cohomology groups $H^p(X, \mathscr{O}(i)|_X)$ vanish for $0 < p <$ dim(X) and any $i \in \mathbb{Z}$.*

Proof. We induct on the co dimension r. For $r = 1$, both (a) and (b) follow from Theorem 6.6.0.1 and Lemma 6.6.1.4. Now assume the result for all complete intersections of codimension $r-1$, we prove the result for a complete intersection X of codimension $r \geqslant 2$. Thus by Lemma 6.6.1.4 there exists a complete intersection Y of codimension $r-1$ and a closed embedding $X \subseteq Y$ defined by the ideal sheaf $\mathscr{O}(-d_r)|_Y$. Thus we have a short exact sequence

$$0 \to \mathscr{O}(-d_r)|_Y \to \mathscr{O}_Y \to \mathscr{O}_X \to 0.$$

The result now follows from the induction hypothesis. □

Here is an useful corollary.

Corollary 6.6.1.6. *Let $X \subseteq \mathbb{P}^n_k$ be a complete intersection. Then X is geometrically connected.*

Proof. By definition $X_{\bar{k}}$ will also be a complete intersection for any algebraic closure $\bar{k}$ of k. Using Proposition 6.6.1.5 (a), we obtain a surjection

$$\bar{k} \simeq H^0(\mathbb{P}^n_{\bar{k}}, \mathscr{O}_{\mathbb{P}^n_{\bar{k}}}) \twoheadrightarrow H^0(X_{\bar{k}}, \mathscr{O}_{X_{\bar{k}}}),$$

which implies $\dim_{\bar{k}}(H^0(X_{\bar{k}}, \mathscr{O}_{\bar{k}})) = 1$ and thus the connectedness of $X_{\bar{k}}$. □

6.6.2 Euler characteristic and Hilbert polynomial

Through this section A will be an Artinan ring. Let M be a finite A-module. Then there exists a finite filtration

$$M = M_0 \supseteq M_1 \cdots \supseteq M_\ell \supseteq 0,$$

such that each of the graded quotients M_i/M_{i+1} is a simple A-module. The number ℓ and the simple modules that appear in this filtration are unique (see Tag 0FCK). We shall call ℓ the length of M as an A-module and denote it by $\ell_A(M)$.
Now suppose X/A be a proper scheme and $\mathscr{F}$ a coherent sheaf on X. Then by Theorem 6.4.0.6, the A-modules $H^i(X, \mathscr{F})$ are finite A-modules.

Definition 6.6.2.1. The *Euler characteristic* of $\mathscr{F}$, $\chi(X, \mathscr{F})$ is the integer $\sum_i (-1)^i \ell_A(H^i(X, \mathscr{F}))$.

Here is a standard lemma.

Lemma 6.6.2.2. *Let $0 \to \mathscr{F}'' \to \mathscr{F} \to \mathscr{F}' \to 0$ be a short exact sequence of coherent sheaves on X. Then $\chi(X, \mathscr{F}) = \chi(X, \mathscr{F}') + \chi(X, \mathscr{F}'')$.*

Now suppose $X \subseteq \mathbb{P}^n_A$ is a closed immersion and let $\mathscr{F}$ be a coherent sheaf on X. For any $i \in \mathbb{Z}$ we denote by $\mathscr{F}(i)$ the Serre twist $\mathscr{F} \otimes \mathscr{O}(i)|_X$. In your assignment you shall prove the following theorem.

Theorem 6.6.2.3. *There exists a polynomial $P_{\mathscr{F}}(z) \in \mathbb{Q}[z]$ such that*

(a) deg$(P_{\mathscr{F}}(z)) =$ dim(Supp$(\mathscr{F})$).

(b) $P(i) = \chi(\mathscr{F}(i))$ for all $i \in \mathbb{Z}$.

The polynomial in Theorem 6.6.2.3 is called the *Hilbert polynomial* of $\mathscr{F}$ (with the respect to the embedding).
We can now define the degree of closed subschemes of $\mathbb{P}^n_k$ for a field k.

Definition 6.6.2.4. Let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme of dimension $n-r$. The *degree* of X with respect to the embedding is defined to be $(n-r)!$ times the leading coefficient of $P_{\mathscr{O}_X}(z)$.

Following is an easy consequence of the definition of degree.

Lemma 6.6.2.5. *The degree of a complete intersection $X = H_1 \cap H_2 \cap \cdots H_r$ is deg(H_1)deg$(H_2) \cdots$ deg(H_r)*[12].

Proof. Induct on r and use Lemma 6.6.1.4. □

Here is a nice application. Recall that the *twisted cubic* $X \subseteq \mathbb{P}^3_k$ is the rational curve $\mathbb{P}^1_k$ embedded inside $\mathbb{P}^3_k$ via the 3-uple embedding.

Corollary 6.6.2.6. *X is not a complete intersection.*

Proof. First note that the pullback of $\mathscr{O}(1)_{\mathbb{P}^3_k}$ under the 3-uple embedding is $\mathscr{O}_{\mathbb{P}^1_k}(3)$. This immediately implies that under this embedding deg$(X) = 3$. By lemma 6.6.2.5 if X were to be a complete intersection in $\mathbb{P}^3_k$, then it would have to be an intersection of two surfaces H_1 and H_2 in $\mathbb{P}^3_k$ one of which (say H_1) has degree 1. Thus $X \subsetneq H_1 \subsetneq \mathbb{P}^3$, where $H_1 \simeq \mathbb{P}^2$. This in turn implies X is a smooth cubic inside $\mathbb{P}^2$, but thanks to the adjunction formula 4.2.3.4, this would imply that ω_X is trivial, contradicting our computation in Proposition 4.1.1.9. □

[12] Here by deg(H_i) we mean the degree of the polynomial defining H_i, which thanks to the lemma will also equal the degree of H_i.

Chapter 7

Cohomology and base change

In algebraic geometry one is often dealing with not just one scheme, but rather a family of them parameterized by another scheme. Geometrically this means one has a map $f : X \to S$ and we would like to think of S as a parameter space for schemes $\{X_s\}_{s \in S}$. From this point of view we would like to study how various topological and cohomological properties of X_s change as s varies. Questions of these kind can often be translated to studying the behaviour of $R^i f_* \mathscr{F}$ for appropriate sheaves $\mathscr{F}$ on X. A recurring theme is what can one say about the map

$$H^i(X_s, \mathscr{F}_S) \to R^i f_* \mathscr{F} \otimes k(s)?$$

We begin by revisiting flatness.

7.1 Flatness revisited

Recall that for a morphism $f : X \to Y$ of schemes, a quasi-coherent sheaf $\mathscr{F}$ on X, is said to be flat over Y if for all points $x \in X$, $\mathscr{F}_x$ is flat over $\mathscr{O}_{Y,f(x)}$. We begin by proving the following lemma of Grothendieck.

Lemma 7.1.0.1. *Let A be a Noetherian domain and let B be a finite type A-algebra. Let M be a finite B-module. Then there exists a non-zero $f \in A$ such that M_f is a free (and hence flat) A_f-module.*

Proof. Let K be the quotient field of A. Then $M \otimes_A K$ is a finite module over $B \otimes_A K$, which in turn is a finitely generated algebra over the field K. We prove by inducting on the dimension d of support of $M \otimes_A K$. If $d = -1$, then $M \otimes_A K = 0$. Since M is finitely generated as a B-module. There exists a $f \in A \backslash \{0\}$ such that $M_f = 0$ and we are done. Now we assume $d \geqslant 0$ and that the result has been verified for all B-modules M with dimension of support less than d.
Now note that if we have a short exact sequence

$$0 \to M'' \to M \to M' \to 0,$$

the validity of the statement for M' and M'' implies the same for M. Finally using [3, Chapter 1, Proposition 7.4] we are reduced to the case where $M = B/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$. Thus we may assume B is a domain and that $M = B$.
Now $B \otimes_A K$ is a finitely generated A algebra and $B \otimes_A K$ is non-zero finitely generated domain over a field K. In particular A injects inside B. By Noether normalization lemma (see Tag 00OY), there exists a injective finite map of K-algebras

$$K[y_1, y_2 \cdots y_d] \to B \otimes_A K.$$

Let $\alpha_i, 1 \leqslant i \leqslant r$ be a finite set of generators of B as a A-algebra. By clearing denominators we may ensure that the y_i's map to elements in B. Each α_i is integral over $K[y_1, y_2 \cdots y_d]$. Thus there exists a non-zero $g \in A$ such that each α_i is integral over $A_g[y_1, y_2 \cdots y_d]$. Thus the natural injective map

$$B' := A_g[y_1, y_2 \cdots y_d] \to B_g$$

is both integral and finitely generated and thus finite. Let K' be the quotient fiels of B', then $B_g \otimes K'$ is finite over a field K' and is a domain, thus is itself a field. Hence we get an exact sequence of B' modules

$$0 \to B'^{\oplus r} \to B_g \to T \to 0,$$

where T is not supported at the generic point of B'. In particular the dimension of support of $T \otimes_{A_g} K$ is strictly smaller than d and hence we are done by induction.

□

Here is a nice corollary

Corollary 7.1.0.2. *Let S be an integral Noetherian scheme. Let $f : X \to S$ be a morphism of finite type and $\mathscr{F}$ a coherent sheaf on X. Then there exists an open dense $U \hookrightarrow S$, such that $\mathscr{F}|f^{-1}(U)$ is flat over U.*

A more difficult result which we shall not prove here is the openness of flat locus.

Theorem 7.1.0.3. *Let $f : X \to S$ be a finite type morphism of Noetherian schemes and let $\mathscr{F}$ be a coherent sheaf on X. Then the locus of points $x \in X$ such that $\mathscr{F}$ is flat at x is open.*

Proof. For a proof see Tag 0399. □

7.1.1 Base change morphism

Consider a commutative diagram of ringed spaces

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Lemma 7.1.1.1. *For any sheaf $\mathscr{F}$ on X there exists a functorial base change map*

$$g^* R^i f_* \mathscr{F} \to R^i f'_* g'^* \mathscr{F},$$

for all $i \in \mathbb{Z}^{\geqslant 0}$.

Proof. Lets work out the $i = 0$ case first. By adjunction we know there exists a natural map

$$\mathscr{F} \to g'_* g'^* \mathscr{F},$$

applying f_* to both sides we get a natural map

$$f_* \mathscr{F} \to f_* g'_* g'^* \mathscr{F} \simeq g_* f'_* g'^* \mathscr{F},$$

which by adjunction again gives us the required base change. For $i > 0$ we have to work a little harder. As before we get by adjunction

$$R^i f_* \mathscr{F} \to R^i f_* (g'_* g'^* \mathscr{F}).$$

Now we need to slide the $R^i f_*$ on the left through the bracket. We claim there exists *edge* maps (constructed below)

$$R^i f_* (g'_* g^* \mathscr{F}) \to R^i (f \circ g')_* (g^* \mathscr{F}),$$

and

$$R^i (f \circ g')_* (g^* \mathscr{F}) = R^i (g \circ f')_* (g'^* \mathscr{F}) \to g_* R^i f'_* g'^* \mathscr{F}.$$

Assuming these exist we are done by adjunction.

□

Let $f : X \to Y$ and $g : Y \to Z$ be a map of ringed spaces. Then for any sheaf $\mathscr{F}$ there exists a first quadrant E_2-spectral sequence

$$E_2^{p,q} = R^p g_* R^q f_* \mathscr{F} \implies R^{p+q} (g \circ f)_* \mathscr{F}.$$

Now for any $n \geqslant 0$, we have

$$E_\infty^{n,0} \simeq F^n R^n (g \circ f)_* \mathscr{F} \subseteq R^n (g \circ f)_* \mathscr{F} \twoheadrightarrow E_\infty^{0,n}.$$

Moreover by construction

$$E_2^{n,0} = R^n g_* f_* \mathscr{F} \twoheadrightarrow E_\infty^{n,0},$$

and

$$E_2^{0,n} = g_* R^n f_* \mathscr{F} \hookleftarrow E_\infty^{0,n}.$$

Thus we get edge maps

$$R^n g_* f_* \mathscr{F} \to R^n (g \circ f)_* \mathscr{F}$$

and

$$R^n (g \circ f)_* \mathscr{F} \to g_* R^n f_* \mathscr{F}.$$

7.2 Base Change and Cohomology

This is a very important section answering one of the simplest questions;
Question: Consider a *Cartesian* diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Under what conditions on f, g and a sheaf $\mathscr{F}$ on X is the base change map

$$g^* R^i f_* \mathscr{F} \to R^i f'_* g'^* \mathscr{F}, \tag{7.1}$$

an isomorphism?
We begin with the following easy result.

Proposition 7.2.0.1. *Assume that f is quasi-compact and separated and that g is flat. Then (7.1) is an isomorphism for any quasi-coherent sheaf $\mathscr{F}$ on X.*

Proof. By first covering Y by affines and then refining the corresponding cover of Y' by affines, we reduce to the case where $Y = \operatorname{Spec}(B)$ and $Y' = \operatorname{Spec}(B')$ with $B \to B'$ being a flat morphism of rings. Since f is quasi-compact and separated there exists a finite open affine cover $\mathscr{U} := \{U_i\}_{1 \leqslant i \leqslant n}$ of X such that $U_i \cap U_j$ are also affines for any i, j. Moreover the base change of this cover $\mathscr{U}' := \{U_i \times_Y Y'\}_{1 \leqslant i \leqslant n}$ is a similar cover for X'. Hence by Corollary 6.3.0.2 we have isomorphisms

$$H^p(X, \mathscr{F}) \simeq \check{H}^p(\mathscr{U}, \mathscr{F})$$

and

$$H^p(X', \mathscr{F}') \simeq \check{H}^p(\mathscr{U}', \mathscr{F}')$$

where $\mathscr{F}' := g'^* \mathscr{F}$. It follows from the flatness of g that

$$\check{H}^p(\mathscr{U}', \mathscr{F}') = \check{H}^p(\mathscr{U}, \mathscr{F}) \otimes_B B'.$$

The result now follows from Corollary 6.4.0.5.

□

Here is a nice corollary.

Corollary 7.2.0.2. *Let X/k be a quasi-compact and separated scheme. Then for any field extension k'/k and quasi-coherent sheaf $\mathscr{F}$ on X, there exists an isomorphism of k'-vector spaces $H^i(X, \mathscr{F}) \otimes_k k' \simeq H^i(X_{k'}, \mathscr{F}_{k'})$.*

A careful look at the proof underlying Proposition 7.2.0.1 tells us that when $Y = \operatorname{Spec}(B)$ and $Y' = \operatorname{Spec}(B')$, our argument produces a bounded[1] complex $K^\bullet = \check{C}(\mathscr{U}, \mathscr{F})$ of B-modules which *universally computes* the cohomology of $\mathscr{F}$. We restate Proposition 7.2.0.1 as follows.

Proposition 7.2.0.3. *Let $f : X \to \operatorname{Spec}(B)$ be a quasi-compact and separated morphism of schemes. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then there exists a* ***bounded*** *complex $K^\bullet$ of B-modules and a functorial isomorphism*

$$H^p(X', \mathscr{F}') \simeq H^p(K^\bullet \otimes_B B') \simeq H^p(K^\bullet) \otimes_B B',$$

for any flat B-algebra B'.

Using derived categories this can be made even more precise. Let $f : X \to \operatorname{Spec}(B)$ be morphism of schemes. Then we know that there exists an additive functor[2]

$$R\Gamma(X, \ ) : D^+(\mathscr{O}_X - \text{mod}) \to D^+(B).$$

Thus given any quasi-coherent sheaf $\mathscr{F}$ we obtain a complex $R\Gamma(X, \mathscr{F})$ in $D^+(B)$ whose cohomologies compute $H^i(X, \mathscr{F})$. In addition if f is quasi-compact and separated and $\mathscr{U} := \{U_i\}_{1 \leqslant i \leqslant n}$ be a finite open affine cover of X. Then we can upgrade Corollary 6.3.0.2 to the following statement in derived category.

Corollary 7.2.0.4. *There exists a functorial in $\mathscr{F}$ isomorphism, $\check{C}(\mathscr{U}, \mathscr{F}) \simeq R\Gamma(X, \mathscr{F})$ in $D^+(B)$.*

Proof. For a proof see Tag 0FLH. □

In light of Corollary 7.2.0.4 we can restate Proposition 7.2.0.1 as follows.

Proposition 7.2.0.5. *Let $f : X \to \operatorname{Spec}(B)$ be a quasi-compact and separated morphism. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then there exists a bounded complex $K^\bullet$ of B-modules such that for flat B-algebra B' there exists an isomorphism $R\Gamma(X', \mathscr{F}') \simeq K^\bullet \otimes_B B'$ in $D^+(B')$.*

[1]That is a complex for which $K^n = 0$ for all $|n| >> 0$.
[2]See Tag 01DI for a proof of $\mathscr{O}_X$-modules have enough injectives.

Now let us come back to the original question we were trying to answer. A special case of the base change map is when $Y' = \mathrm{Spec}(k(y))$ for a point $y \in Y$. In this case the map (7.1) reduces to

$$\psi_y^i : R^i f_* \mathscr{F} \otimes k(y) \to H^i(X_y, \mathscr{F}_y).$$

Unless y is a generic point, we cannot answer this question using Proposition 7.2.0.1. We will see later that these maps need not be isomorphisms in general. First we shall prove some positive results.

Proposition 7.2.0.6. *Let $f : X \to Y = \mathrm{Spec}(B)$ be a quasi-compact and separated morphism of schemes and let $\mathscr{F}$ be a quasi-coherent sheaf on X which is flat over Y. Then there exists a* ***bounded*** *complex $K^\bullet$ of* ***flat*** *B-modules such that for any ring B-algebra B' we have a functorial isomorphism*

$$H^p(X', \mathscr{F}') \simeq H^p(K^\bullet \otimes_B B').$$

Proof. The proof is analogous to that of Proposition 7.2.0.1. Indeed we can choose a finite open affine cover $\mathscr{U} = \{U_i\}_i$ of X and set $K^\bullet := \check{C}(\mathscr{U}, \mathscr{F})$. □

Remark 7.2.0.7. Note that if B' is not a flat B-algebra, then $H^p(K^\bullet \otimes_B B')$ may be different than $H^p(K^\bullet) \otimes_B B'$. For the curious reader there is a *gcd* of Propositions 7.2.0.1 and 7.2.0.6 using the notion of tor-independence. For a statement see Tag 0AA7.

Before we proceed further we need to introduce the Tor-functor.

7.2.1 Tor Functor

Let A be a rings . Then for any A-module N we know that there exists a *right* exact functor $\otimes_A N$ from the abelian category $\mathscr{A}$ of A-modules to itself.

1. Every complex in $\mathscr{C}^-(\mathscr{A})$ is quasi-isomorphic to a complex of free (and hence projective) A-modules (see Tag 05T7).

2. Given two resolutions we can complete them into an obvious commutative diagram.

Thus analogous to 5.2 we can define a left derived functor

$$\otimes_A^L : D^-(\mathscr{A}) \to D^-(\mathscr{A}),$$

and we define for any bounded above complex of A-modules $M^\bullet$

$$\mathrm{Tor}_i(M^\bullet, N) := H^{-i}(M^\bullet \otimes^L N).$$

More practically what this means is, resolve $M^\bullet$ be free objects and then tensor the resulting complex by N and compute cohomology. Note that negative sign in the definition, which accounts for the homological nature of Tor. Here are some basic properties of the Tor functor.

Lemma 7.2.1.1. *Let N be an A-module. Then*

1. *$\mathrm{Tor}_0(M, N) = M \otimes_A N$, for any A-module M.*
2. *For any short exact sequence of A-modules*
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'',$$
there exists a long exact sequence of A-modules
$$\cdots \mathrm{Tor}_1(M', N) \to \mathrm{Tor}_1(M, N) \to \mathrm{Tor}_1(M'', N) \to M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0.$$
3. *N is flat iff $\mathrm{Tor}_i(M, N) = 0$ for any $i > 0$ and A-module M.*

Finally we have an analogue of Lemma 5.1.0.11.

Lemma 7.2.1.2. *Let $M^\bullet$ be a bounded above complex of A-modules. Let $K^\bullet$ be a **flat** resolution of $N^\bullet$. Then for any A-module N, there exists a functorial isomorphism*

$$M^\bullet \otimes_A^L N \simeq K^\bullet \otimes_A N$$

in $D^-(\mathscr{A})$. Thus we can compute Tor using flat resolutions.

Now we can state a cleaner version of Proposition 7.2.0.6.

Proposition 7.2.1.3. *Let $f : X \to Y = \mathrm{Spec}(B)$ be a quasi-compact and separated morphism. Let $\mathscr{F}$ be a quasi-coherent sheaf on X flat over Y. Then there exists a functorial isomorphism*

$$R\Gamma(X', \mathscr{F}') \simeq R\Gamma(X, \mathscr{F}) \otimes_B^L B',$$

in $D(B')$, for any B-algebra B'.

Proof. Combine Corollary 7.2.0.4 and Lemma 7.2.1.2. □

Here is a corollary.

Corollary 7.2.1.4. *There is a **2nd** quadrant E_2 spectral sequence*

$$E_2^{p,q} = \mathrm{Tor}_{-p}(H^q(X, \mathscr{F}), B') \implies H^{p+q}(X', \mathscr{F}').$$

Proof. Combine Proposition 7.2.1.3 and Tag 0662. □

Here is a geometric corollary.

Corollary 7.2.1.5. *Let $f : X \to Y$ be a proper morphism of schemes finite type over a field*[3]*. Let $\mathscr{F}$ be a coherent sheaf on X. Then $R^i f_* \mathscr{F}$ vanishes for $i > dim(f)$. Here $dim(f)$ is defined to be the supremum of the dimension of all fibers of f.*

Proof. We may assume $Y = \mathrm{Spec}(B)$. By results from assignment 4, we know $H^i(X, \mathscr{F}) = 0$ for $i >> 0$. Choose i largest such that $H^i(X, \mathscr{F}) \neq 0$ and choose a point $y \in Y$ such that the finitely generated B-module $H^i(X, \mathscr{F})$ is supported at y. Then by definition $H^{i+1}(X, \mathscr{F}) = 0$ and $H^i(X, \mathscr{F}) \otimes_B k(y) \neq 0$. Thus Corollary 7.2.1.4 implies $H^i(X_y, \mathscr{F}_y) \simeq H^i(X, \mathscr{F}) \otimes k(y) \neq 0$. Thus we are again done by the result from assignment 4. □

7.3 Base change and Cohomology: Theorems

Now we are ready to state our meta theorem about computing cohomology universally using complexes.

Theorem 7.3.0.1. *Let $f : X \to \mathrm{Spec}(B)$ be a proper morphism of Noetherian schemes. Let $\mathscr{F}$ be a coherent sheaf on X flat over $\mathrm{Spec}(B)$. Then there exists a* ***bounded*** *complex of* ***finite flat*** *B-modules such that for any B-algebra B' there exists a functorial isomorphism*

$$R\Gamma(X', \mathscr{F}') \simeq K^\bullet \otimes_B B' = K^\bullet \otimes^L_B B',$$

in $D(B')$.

Proof. We already know that $R\Gamma(X, \mathscr{F})$ is quasi-isomorphic to a bounded complex of flat B-modules by Proposition 7.2.1.3. Since f is proper, this complex has cohomology which is finitely generated by Theorem 6.4.0.6. Thus it suffices to show that a bounded complex of flat B-modules whose cohomology is finitely generated is quasi-isomorphic to a bounded complex of finitely generated flat B-module. One can prove this *explicitly*[4]. □

Let us derive some corollaries to the above result. Before we do so we list some basic facts about finitely generated flat modules over a Noetherian ring and of morphisms between them.

Lemma 7.3.0.2. *Let K be finitely generated module over a Noetherian ring B. Then*

(a) Then function $y \to dim_{k(y)}(M \otimes k(y))$ is upper semi continuous i.e may jump under specialization.

(b) If K is flat then $y \in \mathrm{Spec}(B)$ to $dim_{k(y)}(M \otimes k(y))$ is locally constant. If in addition B is reduced then the converse is also true.

(c) K is flat iff it is locally free iff it is projective.

[3] The proof also works for Y a finite dimensional Noetherian scheme.

[4] Or use the full power of the triangulated structure of $D(B)$ as in Tag 066U to give a slick argument.

Proof. (a) follows from Nakayama. For (b) and (c) we refer to Tag 00NX and Tag 0FWG. □

We will also need the following.

Lemma 7.3.0.3. *Let $\phi : K \to K'$ be a morphism of finitely generated and flat B-modules. Then*

1. *the function $y \to \dim_{k(y)}(\mathrm{Im}(\phi_y))$ is* ***lower*** *semi-continuous.*

2. *Coker(ϕ) is locally free then Im(ϕ) is locally free. Moreover under these assumptions formation of Im(ϕ) commutes with all base changes.*

3. *If B is reduced and the function $y \to \dim_{k(y)}(\mathrm{Im}(\phi_y))$ is constant then coker(ϕ) and Im(ϕ) are locally free.*

Proof. Thanks to Lemma 7.3.0.2, (c) we may assume K and K' are both free. We need to show that for any $r \in \mathbb{Z}$, that the set of points $y \in \mathrm{Spec}(B)$ where $\dim_{k(y)}(\mathrm{Im}(\phi_y)$ is less than r is closed. Since ϕ induces a map $\bigwedge^r K \to \bigwedge^r K'$, we reduce to showing that locus $y \in \mathrm{Spec}(B)$ where $\phi_y = 0$ is closed. This is clear since ϕ is represented by a matrix and $\phi_y = 0$ iff all the entries of the matrix vanish which is a closed condition.
We have a short exact sequence

$$0 \to \mathrm{Im}(\phi) \to K' \to \mathrm{Coker}(\phi) \to 0,$$

which implies that if Coker(ϕ) is locally free then the short exact sequence above splits and hence implying the result.
Now we prove (3). By (2) since B is reduced, it suffices to show that $y \to \mathrm{coker}(\phi_y)$ is locally constant. Since $\mathrm{Coker}(\phi) \otimes k(y) = \mathrm{coker}(\phi_y)$, this follows from the short exact sequence

$$0 \to \mathrm{Im}(\phi_y) \to K' \otimes k(y) \to \mathrm{Coker}(\phi_y) \to 0.$$

□

In what follows we start with a proper morphism $f : X \to \mathrm{Spec}(B)$ and a coherent sheaf $\mathscr{F}$ on X flat over B. Moreover we choose a bounded complex of finitely generated flat B-modules, $K^\bullet$ with maps $\phi^i : K^i \to K^{i+1}$, quasi-isomorphic to $R\Gamma(X, \mathscr{F})$ in $D(B)$.

Corollary 7.3.0.4. *Under the assumptions of Theorem 7.3.0.1, the function*

$$y \to \chi(\mathscr{F}_y)$$

is locally constant.

Proof. Since $H^i(X_y, \mathscr{F}_y) = H^i(K^\bullet \otimes k(y))$, we have $\chi(\mathscr{F}_y) = \sum_i (-1)^i \dim(H^i(K^\bullet \otimes k(y))) = \sum_i (-1)^i \dim(K^i \otimes k(y))$. The latter is locally constant by Lemma 7.3.0.2, (b). □

Corollary 7.3.0.5. *Under the assumptions of Theorem 7.3.0.1, the functions $y \to h^i(X_y, \mathscr{F}_y)$*[5] *are upper semi-continuous, i.e. if y' specializes to y then $h^i(X_{y'}, \mathscr{F}_{y'}) \leqslant h^i(X_y, \mathscr{F}_y)$*

Proof. By definition $h^i(X_y, \mathscr{F}_y) = \dim_{k(y)}(\ker(\phi^i_y)) - \dim_{k(y)}(\mathrm{Im}(\phi^{i-1}_y))$. However note that $\dim_{k(y)}(\ker(\phi^i_y) = \dim_{k(y)}(K^i \otimes k(y)) - \dim_{k(y)}(\mathrm{Im}(\phi^i_y))$. Hence

$$h^i(X_y, \mathscr{F}_y) = \dim_{k(y)}(K^i \otimes k(y)) - \dim_{k(y)}(\mathrm{Im}(\phi^i_y)) - \dim_{k(y)}(\mathrm{Im}(\phi^{i-1}_y)). \quad (7.2)$$

The result now follows from Lemmas 7.3.0.2, (c) and7.3.0.3. □

Now we are ready to prove Grauert's theorem.

Theorem 7.3.0.6 (Grauert's Theorem). *Let $f : X \to Y$ be a proper morphism of Noetherian schemes with Y* ***reduced****. Let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y. Suppose that the function $y \to h^i(X_y, \mathscr{F}_y)$ is locally constant. Then*

1. *$R^i f_* \mathscr{F}$ is locally free coherent sheaf on Y.*

2. *Moreover the base change map (7.1) is an isomorphism for* ***any*** *morphism $g : Y' \to Y$.*

Proof. First we reduce to the case where $Y = \mathrm{Spec}(B)$ and represent $R\Gamma(X, \mathscr{F})$ by $K^\bullet$ in $D(B)$, a bounded complex of finitely generated flat B-modules. We are given that the map $y \to H^i(K^\bullet \otimes k(y))$ is locally constant and we would like to conclude that $H^i(K^\bullet)$ is locally free.
Using (7.2) and Lemmas 7.3.0.3 and 7.3.0.2, (b) we conclude that the functions $y \to \dim_{k(y)}(\mathrm{Im}(\phi^i_y))$ and $\dim_{k(y)}(\phi^{i-1}_y)$ are locally constant. Thus by Lemma 7.3.0.3 we have that $\mathrm{Coker}(\phi^{i-1})$ and $\mathrm{Im}(\phi^i)$ are locally free. Thanks to the short exact sequence

$$0 \to H^i(K^\bullet) \to \mathrm{Coker}(\phi^{i-1}) \to \mathrm{Im}(\phi^i) \to 0,$$

we are done with (1).
For (2) we may reduce to the case $Y' = \mathrm{Spec}(B')$ in which we case we are done by Lemma 7.3.0.3, (2). □

Now we are ready to state a very important cohomology and base change result which works even without the reducedness assumption in Grauert's theorem.

Theorem 7.3.0.7. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes and let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y. Suppose the natural map*

$$R^i f_* \mathscr{F} \otimes k(y) \to H^i(X_y, \mathscr{F}_y)$$

is surjective for some i and some $y \in Y$. Then

[5] As is commone, we shall use lower-case letter to denote the dimensions of vector spaces.

1. *There exists an open neighbourhood U, around $y \in Y$ such that for any $g : Y' \to U \hookrightarrow Y$, the base change map (7.1) is an isomorphism.*

2. *Moreover $R^i f_* \mathscr{F}$ is locally free in a neighbourhood of $y \in Y$ iff the natural map*

$$R^{i-1} f_* \mathscr{F} \otimes k(y) \to H^{i-1}(X_y, \mathscr{F}_y)$$

is an isomorphism.

The proof is along the lines of Grauert's theorem, but a little more involved. This will be part of Problem set 5. Now we will apply these theorems to prove some beautiful results.

7.4 Application of Cohomology and base change theorems

In this section we present some corollaries to cohomology and base change theorems.

Corollary 7.4.0.1. *Let $X \subseteq \mathbb{P}^n_A$ be a projective scheme over a Noetherian ring A. Let $\mathscr{F}$ be a coherent sheaf on X. Then the set of Hilbert polynomials $y \to P_{\mathscr{F}_y}(z)$ for $y \in \operatorname{Spec}(A)$ is finite.*

Proof. Let $Y = \operatorname{Spec}(A)$. Let $Z \subseteq Y$ be a closed subset and suppose for every proper closed subset of Z, there exists only finitely many Hilbert polynomials of the fibers with $y \in Z$. We would like to show the same for Z. If Z was reducible we would be done. Now assume Z is irreducible, since the map $\operatorname{Spec}(k(y)) \to Z$ factors through Z_{red}, we may also assume Z is reduced. Thus it suffices to prove the corollary under the assumption that

(a) A is a Noetherian integral domain.

(b) The result is valid for all rings of the form A/I with $I \neq 0$.

Hence it suffices to show that the Hilbert polynomial is constant on a non-empty open subset of $\operatorname{Spec}(A)$. This follows from generic flatness (Corollary 7.1.0.2) and Corollary 7.3.0.4. □

In a similar vein we also have the following result.

Theorem 7.4.0.2. *Let $f : X \to Y$ be a proper morphism of schemes finite type over a field. Let $\mathscr{F}$ be a coherent sheaf on X. Then the set $y \in Y$ such that $h^i(X_y, \mathscr{F}_y) = p$ is constructible*[6] *for any $p \in \mathbb{Z}$.*

[6]A constructible set on a Noetherian topological space is a finite union of locally closed subsets i.e sets which are intersection of an open and closed subset

Proof. As before we use Noetherian induction on Y. Since finite unions of constructible sets are constructible, we may assume Y is irreducible. Since the function $y \to h^i(X_y, \mathscr{F}_y)$ only depends on Y_{red} we may assume Y is integral. Finally by generic flatness (Corollary 7.1.0.2) we may also assume $\mathscr{F}$ is flat over Y. Under this assumption it suffices to show that $h^i(X_y, \mathscr{F}_y)$ is locally constant for every i. In fact we claim that for any i, there exists a non-empty open subset $U_i \hookrightarrow Y$ such that

(a) $R^i f_* \mathscr{F}$ is locally free on U_i.

(b) $R^i f_* \mathscr{F}$ satisfies base change along arbitrary maps $Y' \to Y$ which factor through U_i.

(a) is obvious. For (b) note that with y as the generic point of Y, by flat base change (see Proposition 7.2.0.1)

$$R^i f_* \mathscr{F} \otimes k(y) \to H^i(X_y, \mathscr{F}_y)$$

is an isomorphism. Thus there exists an open subset (and hence containing the generic point) where (b) holds. Clearly (a)+(b) implies $h^i(X_y, \mathscr{F}_y)$ is constant for $y \in U_i$.

□

Corollary 7.4.0.3. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Let $y \in Y$ be such that $H^i(X_y, \mathscr{F}_y) = 0$. Then there exists an open neighbourhood U containing y such that*

1. *$R^i f_* \mathscr{F}|_U$ vanishes.*
2. *$R^i f_* \mathscr{F}$ satisfies base change for any morphism $Y' \to Y$ factoring through $U \hookrightarrow Y$.*
3. *$R^{i-1} f_* \mathscr{F}$ satisfies base change for any morphsim $Y' \to Y$ factoring through $U \hookrightarrow Y$*

Proof. Since $H^i(X_y, \mathscr{F}_y) = 0$, we have by Theorem 7.3.0.7, the coherent sheaf $R^i f_* \mathscr{F}$ commutes with arbitrary base changes in a neighbourhood of y and in particular at y. Further since $R^i f_* \mathscr{F}$ is coherent, we must have that $R^i f_* \mathscr{F}$ vanishes in a neighbourhood if y and hence is locally free in a neighbourhood of y. This in turn implies $R^{i-1} f_* \mathscr{F}$ satisfies base change in a neighbourhood of y.

□

Here is an application of the above corollary.

Proposition 7.4.0.4. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $\mathscr{F}$ be a coherent sheaf on X, flat over S. Let $d := \{\text{sup}(\text{dim})(f^{-1}(y)); y \in Y\}$. Then $R^d f_* \mathscr{F}$ satisfies base change along all morphisms $Y' \to Y$.*

Proof. By Grothendieck's theorem on vanishing of cohomology above dimension for Noetherian schemes, $H^{d+1}(X_y, \mathscr{F}_y) = 0$ for all $y \in Y$. Thus we are done by Corollary 7.4.0.3. □

We will need the following lemma.

Lemma 7.4.0.5. *Let S/k be a scheme of finite type over a field k which is* ***proper***, ***geometrically*** *reduced and* ***geometrically*** *connected. Then the natural map $k \to H^0(S, \mathscr{O}_S)$ is an isomorphism.*

Proof. Using Corollary 7.2.0.2, it suffices to show that $H^0(S, \mathscr{O}_S) \simeq k$ for any connected, reduced and proper schemes over an algebraically closed field k. Clearly $H^0(S, \mathscr{O}_S)$ is a finite algebra over k (Theorem 6.4.0.6 + Corollary 6.4.0.5), it is also reduced since S is reduced. Thus $H^0(S, \mathscr{O}_S) \simeq \prod_{i \in I} k$ for some finite set I. If $|I| > 1$, we will have *non-zero* global functions e_1 and e_2 such that $e_1 e_2 = 0$ and $e_1 + e_2 = 1$, thus contradicting the connectedness of S.

□

This is a *local* analogue of Lemma 7.4.0.5.

Proposition 7.4.0.6. *Let $f : X \to Y$ be a proper and flat morphism of Noetherian schemes with geometrically connected and reduced fibers. Then the natural map $\mathscr{O}_Y \to f_* \mathscr{O}_X$ is an isomorphism.*

Proof. Note that flatness of f is the same as flatness of $\mathscr{O}_X$ over Y. First we claim that

$$f_* \mathscr{O}_X \otimes k(y) = R^0 f_* \mathscr{O}_X \otimes k(y) \to H^0(X_y, \mathscr{O}_{X_y}), \tag{7.3}$$

is a surjection. By Lemma 7.4.0.5, the latter is isomorphic to $k(y)$. Since $1 \otimes 1 \in f_* \mathscr{O}_X \otimes k(y) \to 1$ under (7.3), the claim follows[7].

Thus by Theorem 7.3.0.7, the map in (7.3) is an isomorphism and $f_* \mathscr{O}_X$ satisfies base change along arbitrary morphisms $Y' \to Y$. Moreover (and even more magically) since $h^{-1}(X_y, \mathscr{O}_{X_y}) = 0$ for all y, Theorem 7.3.0.7 implies that $f_* \mathscr{O}_X$ is locally free (necessarily of rank 1)!

Now the natural map $\phi : \mathscr{O}_Y \to f_* \mathscr{O}_X$ is a morphism of line bundles such that ϕ_y is an isomorphism for all $y \in Y$. Thus $\operatorname{coker}(\phi_y) = \operatorname{coker}(\phi)_y$ vanishes for all y and hence so does $\operatorname{coker}(\phi)$. Thus ϕ is a surjective map of line bundles and hence is necessarily an isomorphism (Check this!).

□

7.5 Theorem on formal functions and applications

In this section we shall state (though not prove!) the theorem on formal functions. We shall then focus on applications of the formal function theorem.

[7] Note that we have used the fact that $\mathscr{O}_X$ is a sheaf of algebras here and that everything is compatible with this algebra structure.

Let A be a Noetherian local ring complete with respect to an ideal $I \subseteq A$. Note that this means the natural map

$$A \to \varprojlim_n A/I^{n+1}, \tag{7.4}$$

is an isomorphism. Since A is Noetherian, completeness with any non-zero ideal I implies completeness with respect to any other ideal I'.

Example 7.5.0.1. Here are some examples of A to keep in mind. $A = k[[x]]$, the ring of formal power series in one-variable. An arithmetic analogue of this is $A = \mathbb{Z}_p$, the p-adic completion of $\mathbb{Z}$ along a prime p.

One way to restate (7.4) is as follows. Let $X = \operatorname{Spec}(A)$ and denote by $X_n := \operatorname{Spec}(A/I^{n+1})$. Then we have an isomorphism

$$H^0(X, \mathscr{O}_X) \simeq \varprojlim_n H^0(X_n, \mathscr{O}_{X_n}).$$

Now we are ready to state a version of the formal function theorem.

Theorem 7.5.0.2. *Let $X \to \operatorname{Spec}(A)$ be a proper morphism, with A a Noetherian ring complete with respect to an ideal $I \subseteq A$. Let $X_n := X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A/I^{n+1})$ and suppose $\mathscr{F}$ is a coherent sheaf on X. Then the natural map*

$$H^i(X, \mathscr{F}) \to \varprojlim_n H^i(X_n, \mathscr{F}|_{X_n})$$

induced by pull-back is an isomorphism.

Proof. For a proof see Tag 087U. □

Here is a local version of the same result.

Corollary 7.5.0.3. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $y \in Y$ be a point and denote by $X_n := X \times_{\operatorname{Spec}(\mathscr{O}_{y,y})} \operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y}^{n+1})$, here $\mathfrak{m}_{Y,y} \subseteq \mathscr{O}_{Y,y}$ be the maximal ideal corresponding to the point y. Then for any[8] coherent sheaf $\mathscr{F}$, the natural map*

$$\varprojlim_n (R^i f_* \mathscr{F} \otimes \mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y}^{n+1}) \to \varprojlim_n H^i(X_n, \mathscr{F}_{X_n}),$$

is an isomorphism of $\widehat{\mathscr{O}_{Y,y}}$-modules, where $\widehat{\mathscr{O}_{Y,y}}$ is the completion of $\mathscr{O}_{Y,y}$ along $\mathfrak{m}_{Y,y}$.

[8]Note that there are no flatness assumptions!

Proof. Let $A = \widehat{\mathscr{O}_{Y,y}}$, then A is a Noetherian local ring which is complete along it's maximal ideal $\mathfrak{m}$. Note that since $R^i f_* \mathscr{F}$ is coherent (Theorem 6.4.0.6) we have an isomorphsim (see Tag 0912)

$$\varprojlim_n (R^i f_* \mathscr{F} \otimes \mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y}^{n+1}) \simeq R^i f_* \mathscr{F} \otimes A.$$

Since the morphism $\mathrm{Spec}(A) \to Y$ is flat (see Tag 0912), using Proposition 7.2.0.1 the RHS above can be identified with $H^i(X_A, \mathscr{F}_{X_A})$, where X_A (resp. $\mathscr{F}_A$) is the base change of X (resp. $\mathscr{F}$) along $\mathrm{Spec}(A) \to Y$.
Finally note that the natural map $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_{Y,y}^{n+1}) \to Y$, factors through $\mathrm{Spec}(A) \to Y$, hence we are done by Theorem 7.5.0.2. $\square$

Here is a converse to Lemma 7.4.0.5.

Proposition 7.5.0.4. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes such that the natural map $\mathscr{O}_Y \to f_* \mathscr{O}_X$ is an isomorphism. Then the fibers of f are connected*[9].

Proof. Note that by Proposition 7.2.0.1, the condition $\mathscr{O}_Y \to f_* \mathscr{O}_X$ is an isomorphisms is preserved under arbitrary flat base changes $Y' \to Y$. So we may assume $Y = \mathrm{Spec}(A)$, where $A = \widehat{\mathscr{O}_{Y,y}}$, the completion of the local ring $\mathscr{O}_{Y,y}$ along its maximal ideal $\mathfrak{m}_{Y,y}$. As before we denote by $X_n := X \times A/\mathfrak{m}^{n+1}$. Note that the schemes X_n are all *homeomorphic* to $X_0 = X_Y$. In particular, they are disconnected iff X_y is so.
Now suppose $X_y = Z_0 \sqcup Z_1$, where Z_0 is a connected component and $Z_1 \neq \varnothing$. Then a similar decomposition holds for all X_n's and if we denote by e_0^n (resp. e_1^n) the section of $\mathscr{O}_{X_n}$ which is 1 on Z_0 (resp. Z_1) and 0 on the complement then we have

$$e_0^n + e_1^n = 1, e_0^n e_1^n = 0,$$

for all n and $e_j^n \to e_j^{n+1}$ under the natural map

$$H^0(X_n, \mathscr{O}_{X_n}) \to H^0(X_{n+1}, \mathscr{O}_{X_{n+1}}).$$

This implies there exists $e_0, e_1 \in \varprojlim_n H^0(X_n, \mathscr{O}_{X_n}$ *non-zero* elements such that

$$e_0 + e_1 = 1, e_0 e_1 = 0.$$

Note however that by the theorem on formal function and owing to $f_* \mathscr{O}_X = \mathscr{O}_Y$

$$\varprojlim_n H^0(X_n, \mathscr{O}_{X_n}) \simeq A.$$

Since A is a local ring, $\mathrm{Spec}(A)$ is necessarily connected, contradicting the exists of e_0 and e_1 with the above properties. $\square$

[9]One can prove geometric connectedness, perhaps I will set this as an exercise.

Now we can prove a version of Zariski's main theorem.

Theorem 7.5.0.5 (Zariski's main theorem). *Let $f : X \to Y$ be a proper and birational morphism of integral Noetherian schemes with Y* ***normal****. Then X_y is connected for all $y \in Y$.*

Proof. Thanks to Proposition 7.5.0.4, it suffices to show that the natural map $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is an isomorphism. Since the statement is local on Y, we may assume $Y = \mathrm{Spec}(A)$. Thus it suffices to prove that

$$A \simeq H^0(X, \mathscr{O}_X).$$

We argue as in Lemma 7.4.0.5. Let $\tilde{A} = H^0(X, \mathscr{O}_X)$. Since f is proper $\tilde{A}$ is *finite* over A. Moreover if K is the function field of A, then by birationality of f and Proposition 7.2.0.1 we must have that the natural map

$$K \to \tilde{A} \otimes_A K,$$

is an isomorphism. Since the schemes are all integral, the natural map

$$A \to \tilde{A},$$

is an injection. Thus we have $A \subseteq \tilde{A} \subseteq K$ with $\tilde{A}/A$ finite and A a normal domain. Thus $A = \tilde{A}$ as desired.

□

Here is another application of the theorem of formal functions (via Proposition 7.5.0.4).

Theorem 7.5.0.6 (Stein Factorizations). *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Then there exists an unique (upto an unique isomorphism) factorization of f as $f' : X \to Y'$ and $g : Y' \to Y$ such that*

1. *g is finite.*

2. *$f'_*\mathscr{O}_X = \mathscr{O}_{Y'}$ and hence the fibers of f' are connected.*

3. *$f = g \circ f'$*

Proof. The uniqueness of such a factorization can be checked locally on Y. Hence we may assume $Y = \mathrm{Spec}(A)$. Then Y' is also forced to be affine (by condition (1)) and $Y' = \mathrm{Spec}(H^0(Y', \mathscr{O}_{Y'})) \simeq H^0(X, \mathscr{O}_X)$ (by condition (2)). Hence the uniqueness.
By uniqueness, it suffices to give a construction of Y'. We choose $Y' = \mathbf{Spec}(f_*\mathscr{O}_X)$ and $g : Y' \to Y$, $f : X \to Y'$ are the natural maps. This clearly does the required job. □

We are not doing justice to the full power of Zariksi's main theorem here. We relegate some applications to the assignments!

Chapter 8

Curves

In this chapter we will discuss the simplest (and yet sufficiently complex!) class of algebraic varieties, namely those of dimension 1. Our aim will be to apply the full force of whatever we have learnt so far to understand algebraic curves.
This is a very rich subject; For example smooth projective algebraic curves over complex numbers are precisely compact Riemann surfaces i.e. compact complex manifold of complex dimension 1[1]. In particular smooth algebraic curves over complex numbers are a *layer* on top of connected compact orientable surfaces, which in turn are classified by their genus. This is just the story over complex numbers...

8.1 Curves: Definition and basic properties

Through this section, we denote by k an arbitrary field, by $\bar{k}$ an algebraic closure of k. All schemes (unless other wise states) will be of finite type and separated over k.

Definition 8.1.0.1. A curve X/k is a geometrically connected scheme that is equi dimensional of dimension 1.

Remark 8.1.0.2. Let $\mathscr{P}$ be a property of schemes over k, by a $\mathscr{P}$ curve we mean a curve over k which satisfies the property $\mathscr{P}$. Thus a smooth curve is one which is smooth over k, an integral curve is a curve which is an integral scheme etc...

Here is a simple lemma about smooth curves

Lemma 8.1.0.3. *Let X/k be a smooth curve. Then X is geometrically integral.*

Proof. Geometric integrality follows from 4.2.4.4. □

Here is a special case of Theorem 4.2.4.5.

[1] Note that the latter is apriori *analytic*!

Proposition 8.1.0.4. *Let X/k be a smooth curve. Then X is normal. The converse holds when k is perfect.*

Recall for any integral scheme X, there exists an unique-upto-unique isomorphism scheme X^ν, together with a map $\pi : X^\nu \to X$ such that π is initial among *dominant* morphisms to X from normal schemes. The scheme X^ν is called the normalization of X and π, the normalization map. Moreover the normalization map π is a finite[2], birational morphism. See Tag 035E for a proof.
Specializing to curves we have the following lemma.

Lemma 8.1.0.5. *Let X/k be an integral curve. Then the normalization map $\pi : X^\nu \to X$ is finite and an isomorphism outside finitely many closed points of X, which are necessarily the singular or non-regular points of X.*

Proof. π is finite and birational, hence an isomorphism on an open dense subset of X, whose complement is necessarily finitely many points. □

Example 8.1.0.6. Let $X \subseteq \mathbb{A}^2_k$ be the vanishing locus of $f(x,y) = y^2 - x^2(x+1)$. Since $f(x,y)$ is irreducible, X is geometrically integral. Moreover using Jacobian criterion (see Example 4.2.2.13) X is smooth outside $(0,0)$. Consider the map $\pi : \mathbb{A}^1 \to X$ sending $t \to (t^2-1, t(t^2-1))$ and $\phi : X \backslash (0,0) \to \mathbb{A}^1$ sending $(x,y) \to \frac{y}{x}$. Then $\pi \circ \phi = 1_{X\backslash(0,0)}$ and $\phi \circ \pi|_{\mathbb{A}^1\backslash\{-1,1\}}$ are both identity. Thus π is birational and by construction is finite. Thus $\mathbb{A}^1$ is the normalization of X.

Here is a fundamental result about curves which makes their birational classification a very simple problem.

Theorem 8.1.0.7. *The following categories are equivalent*

1. *The category of fields finitely generated over k of transcendence degree 1, with morphisms being k-algebra morphisms.*

2. *The category of integral curves, with morphisms being rational and dominant morphisms over k, i.e. morphisms defined on an open subset of the source which are dominant.*

3. *The category of proper and regular (or normal) curves, with morphisms being dominant morphisms as schemes over k.*

4. *The category of projective and regular (or normal) curves, with morphisms being dominant morphisms as schemes over k*

In particular every proper (or equivalently projective) regular curve over k is determined upto isomorphism by its function field.

[2]Recall that our schemes are finite type over a field, in general finiteness of the normalization map is false.

Proof. For a proof see Tag 0BY1. □

Morphisms between curves are also very nice. In fact we have the following.

Proposition 8.1.0.8. *Let $\phi : X \to Y$ be a non-constant morphism between regular curves. Then*

1. *ϕ is dominant and flat.*
2. *Suppose ϕ is finite, then $\phi_* \mathscr{O}_X$ is a vector bundle of rank equal to $[k(Y) : k(X)]$, which we call the degree of ϕ and denote by deg(ϕ).*
3. *If ϕ is finite and $\mathscr{F}$ a vector bundle on X, then $\phi_* \mathscr{F}$ is a vector bundle on Y of rank equal to rk$(\mathscr{F})$deg(ϕ).*
4. *If ϕ if finite, then for any point $y \in Y$*

$$\textit{dim}_{k(y)}(H^0(X_y, \mathscr{O}_{X_y})) = \textit{deg}(\phi).$$

Proof. If ϕ is not dominant i.e misses the generic point, then $\phi(X)$ is an irreducible closed subset of Y not containing the generic point and hence has to be a closed point, contradicting non-constancy of ϕ. Since ϕ is dominant and Y is regular (and hence normal), f is flat. This proves (1). For (2) and (3) we may assume $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Then B/A is a finite and flat algebra, and hence locally free. Suppose this rank is n, then $B \otimes k(y)$ is a n-dimensional vector space over $k(y)$ for any point $y \in Y$. Since the only point over generic point of Y is the generic point of X, thus (2) follows. (3) clearly follows from (2). For (4) we can use Corollary 7.4.0.3 applied to $i = 1$ (or argue directly). □

8.1.1 Line bundles on a curve

Let X/k be a regular curve. We denote by Pic(X), the Picard group of X and by CaCl(X), the Cartier class group of X. Since X is integral we have an isomorphism (see [3, Chapter II, Proposition 6.15])

$$\mathrm{Pic}(X) \simeq \mathrm{CaCl}(X).$$

We also have the divisor class group of X which is defined as follows: First we look at $Z^1(X)$[3], the free abelian group generated by the closed points of X. A 0-cycle (resp. effective 0-cycle) is formal linear sum of the form $\sum_i n_i p_i$ (resp. $\sum_i n_i p_i, n_i \geqslant 0$) where p_i's are finitely many closed points of X (not necessarily k-points).

Definition 8.1.1.1. Let $f \in k(X)^\times$ and let $x \in X$ be a closed point. Suppose $f = a/b$ for $a, b \in \mathscr{O}_{X,x}$, we define $v_x(f) := \ell(\mathscr{O}_{X,x}/a) - \ell(\mathscr{O}_{X,x}/b)$, here by ℓ we mean the length of the Artinian ring.

[3]The superscript 1 tells us that we are looking at co dimension 1 subvarieties of X.

Recall that dimension 1 regular local rings are dvr's. In particular for any closed points $x \in X$, the function field $k(X)$ comes equipped with an unique discrete valuation (say ν_x) for which $\mathscr{O}_{X,x}$ is the valuation ring. Here is a simple lemma.

Lemma 8.1.1.2. *For any $f \in k(X)^\times$ and any closed point $x \in X$, we have $\nu_x(f) = v_x(f)$*[4].

Proof. Clearly we may assume $f \in \mathscr{O}_{X,x}$. Let $f = a\pi_x^i, i \geqslant 0$, where $a \in \mathscr{O}_{X,x}^\times$ and π_x is an uniformizer of $\mathscr{O}_{X,x}$. Then by definition $\nu_x(f) = i$. It suffices to show that $\ell(\mathscr{O}_{X,x}/\pi_x^i) = i$, which is obvious. □

We say a 0-cycle $\sum_i n_i p_i$ is rationally equivalent to 0 and denote by

$$\sum_i n_i p_i \sim_{\mathsf{rat}} 0$$

if there exists a $f \in k(X)$ such that

$$\mathsf{div}(f) := \sum_{x \in |X|} v_x(f)x = \sum_i n_i p_i,$$

here $v_x(f)$ is the *valuation* of f at x. Clearly the cycles which are rationally equivalent to 0 form a subgroup (say $\mathsf{Rat}^1(X)$) and we denote by $A^1(X) := \frac{Z^1(X)}{\mathsf{Rat}^1(X)}$ (called the *Chow group* of codimension 1 cycles). Since X/k is a regular curve we have by [3, Chapter II, Proposition 6.11]:

Proposition 8.1.1.3. *There exists a canonical isomorphism (denoted by c_1; the 1st Chern class), between Pic(X) and $A^1(X)$. Moreover under this isomorphism if $s \in H^0(X, \mathscr{L})$, then $\mathscr{L} \to$ div(s).*

Next we would like to define pull-back and push-forward of divisor classes. Let $\phi : X \to Y$ be a *finite* (and hence dominant) map of regular curves over k. Let $x \in X \to \phi(x) \in Y$. Then $y = f(x)$ is also a closed point and $\phi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is a map of dvr's. We denote by $v_\phi(x, y) := v_x(\phi^\#(\pi_y))$, where π_y is any uniformizer at $y \in Y$.

1. **Pullback:** We have a pullback homomorphism $\phi^* : Z^1(Y) \to Z^1(X)$ defined (on generators) by
$$\phi^*(p) := \sum_{q_i \in f^{-1}(p)} v_f(p, q_i).$$

2. **Pushforward:** We have a pushforward homomorphism $\phi_* : Z^1(X) \to Z^1(Y)$ defines (on generators) by
$$\phi_*(p) := [k(p) : k(f(p))]f(p).$$

[4] The advantage of the definition we gave is that it makes sense even when X is *not* regular. This the definition in Fulton's Intersection theory [1, Section 1.3].

The key result about ϕ^* and ϕ_* are as follows. We do not prove them here, though they are pretty standard and easy to prove.

Proposition 8.1.1.4. *Let $\phi : X \to Y$ be a finite morphism of regular curves (over k). Then*

(a) For any $f \in k(Y)^\times$, $\phi^*(\mathit{div}(f)) = \mathit{div}(\phi^*(f))$, here $\phi^*(f) \in k(X)$, is the image of $f \in k(Y)$ under the map $\phi^* : k(Y) \to k(X)$.

(b) For any $f \in k(X)^\times$, $\phi_*(\mathit{div}(f)) = \mathit{div}(\mathit{Nm}(f))$, here $\mathit{Nm}(f) \in k(Y)$ is the determinant of $f \in k(X)^\times \subseteq \mathit{Aut}_{k(Y)-\mathit{linear}}(k(X))$.

Proof. (a) is easy to see from the definition. For a proof of (b) see [1, Proposition 1.4]. □

In particular for ϕ_* and ϕ^* induce maps at the level of Chow group. Moreover the pullback map ϕ^* is compatible with isomorphism in Proposition 8.1.1.3.

Next we would like to define the *degree* of a zero-cycle[5].

Definition 8.1.1.5. Let $\alpha = \sum_i n_i p_i \in Z^1(X)$, then we define $\deg(\alpha) = \sum_i n_i [k(p_i) : k] \in \mathbb{Z}$.

We have the following easy lemma.

Lemma 8.1.1.6. *Let X be a proper regular curve. Then $\mathit{deg} : Z^1(X) \to \mathbb{Z}$ factors through the Chow group $A^1(X)$. Moreover for $\phi : X \to Y$ a non-constant map of proper regular curves*

$$\mathit{deg}(\phi^*(\alpha)) = \mathit{deg}(\phi)\mathit{deg}(\alpha).$$

Proof. For a proof see [3, Chapter II, Proposition 6.9, Corollary 6.10]. □

Example 8.1.1.7. Properness is necessary in Lemma 8.1.1.6. For example consider the 0-cycle $[0] \in Z^1(\mathbb{A}^1_k)$. Clearly $\deg([0]) = 1$, however $[0] = \mathrm{div}(x)$, where x is the coordinate in $\mathbb{A}^1_k$.

Notations 8.1.1.8. Let $\mathscr{L} \in \mathrm{Pic}(X)$ where X/k is a proper regular curve. By *degree* of $\mathscr{L}$ we mean $\deg(c_1(\mathscr{L}))$ (see Proposition 8.1.1.3). In particular if $H^0(X, \mathscr{L}) \neq 0$ then $\deg(c_1(\mathscr{L})) \geqslant 0$.

For a smooth curve X/k, the sheaf of Kähler differentials, $\Omega^1_{X/k}$ is a line bundle (see Lemma 4.2.2.4) and hence is the same as the canonical sheaf $\omega_{X/k}$.

Definition 8.1.1.9. Let X/k be a smooth proper curve. The genus of X/k is defined to be $h^0(X, \omega_{X/k})$ and denoted by $p_g(X)$ (or g, when X is clear from the context.

[5] Now we like to think of $Z^1(X)$ as the free abelian group on subvarieties of dimension 0.

We end this section by justifying the definition of genus. Suppose X is a smooth projective curve over $\mathbb{C}$. Then $X(\mathbb{C})$ underlies a compact Riemann surface (and hence a compact connected oriented surface) and we have the Hodge decomposition[6]

$$H^1_{\mathsf{sing}}(X(\mathbb{C}), \mathbb{C}) \simeq H^1_{\mathsf{sing}}(X(\mathbb{C}), \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq H^1(X, \mathscr{O}_X) \oplus H^0(X, \Omega^1_{X/\mathbb{C}}).$$

Moreover the unique non-trivial element $\sigma \in \mathsf{Gal}(\mathbb{C}/\mathbb{R})$ which acts on the left by $\mathbb{R}$-linear automorphisms induces an automorphism on the right via the Hodge decomposition. Under this action the summands get exchanged. In particular both of them are of the same real (and hence complex) dimension. Thus

$$\mathsf{dim}(H^0(X, \Omega^1_{X/\mathbb{C}})) = \frac{1}{2}\mathsf{dim}_{\mathbb{C}}(H^1_{\mathsf{sing}}(X(\mathbb{C}), \mathbb{C})) = g,$$

where g is the genus of the underlying compact connected oriented surface $X(\mathbb{C})$.

8.2 Serre duality and Riemann Roch for curves

In this section we shall state two very important theorems about cohomology of curves. As you shall see, much of what we can say about curves boils down to extracting as much juice as possible from these results. Both these statements are true in much more generality, but even stating them in the correct generality requires some effort.

Theorem 8.2.0.1. *Let X/k be a smooth projective curve. Then*

1. *There exists a trace isomorphism*

$$\mathsf{Tr}_{X/k} : H^1(X, \omega_X) \simeq k.$$

2. *For any vector bundle $\mathscr{E}$ on X, there exists a perfect pairing*

$$H^i(X, \mathscr{E}) \otimes_k H^{1-i}(X, \mathscr{E}^\vee \otimes \omega_{X/k}) \to H^1(X, \omega_{X/k}) \simeq k.$$

In particular for any vector bundle $\mathscr{E}$ on X we have an equality

$$h^0(X, \mathscr{E}) = h^1(X, \mathscr{E}^\vee \otimes \omega_{X/k}).$$

Proof. For a proof see [3, Chapter III, Section 7]. □

Corollary 8.2.0.2. *For any smooth projective curve X/k, we have*

$$p_a(X) := 1 - \chi(\mathscr{O}_X) = h^1(X, \mathscr{O}_X) = p_g(X).$$

[6] For the careful reader, in addition to the Hodge decomposition we are also invoking Serre's GAGA!

Next we state and prove the Riemann-Roch theorem for curves.

Theorem 8.2.0.3. *Let X/k be a smooth projective curve. Then for any line bundle $\mathscr{L}$*

$$\chi(\mathscr{L}) = \deg(\mathscr{L}) - p_g(X) + 1.$$

Proof. Corollary 8.2.0.2 implies Riemann-Roch is true for $\mathscr{L} = \mathscr{O}_X$. Thanks to Proposition 8.1.1.3, it suffices to show that Riemann-Roch is true for a line bundle $\mathscr{L}$ iff it is true for the line bundle $\mathscr{L}(-p) := \mathscr{L} \otimes \mathscr{O}_X(-p)$ for any closed point $p \in X$.
Let $p \in X$ be a closed point and denote by $\mathscr{O}_p := i_* k(p)$, the structure sheaf of the closed subscheme $p \subseteq X$. Then we have an exact sequence

$$0 \to \mathscr{O}_X(-p) \to \mathscr{O}_X \to \mathscr{O}_p \to 0.$$

Tensoring with sL and taking Euler-characteristic we obtain

$$\chi(\mathscr{L}) = \chi(\mathscr{L}(-p)) + \chi(\mathscr{L} \otimes \mathscr{O}_p).$$

However note that $\mathscr{L} \otimes \mathscr{O}_p \simeq \mathscr{O}_p$ and thus $\chi(\mathscr{L} \otimes \mathscr{O}_p) = [k(p) : k]$. Hence

$$\chi(\mathscr{L}) - \chi(\mathscr{L}(-p)) = [k(p) : k],$$

since $\deg(\mathscr{L}) = \deg(\mathscr{L}(-p)) + [k(p) : k]$, we are done. □

Notations 8.2.0.4. By a divisor on a smooth curve X/k we mean an element of $Z^1(X)$. We shall often write D to denote its image in $A^1(X)$ (or Pic(X)). We use K_X (or K, when X is clear) to denote the divisor in $A^1(X)$ corresponding to $\omega_{X/k}$.

Combining Riemann-Roch and Serre-duality we have for any divisor D

$$\chi(D) = h^0(X, D) - h^0(X, K - D) = \deg(D) - g + 1. \tag{8.1}$$

Corollary 8.2.0.5. *Let X/k be a smooth projective curve of genus g. Then $\deg(K) = 2g - 2$.*

Proof. Substitute $D = 0$ in Equation 8.1. □

Corollary 8.2.0.6. *Let D be divisor on a smooth projective curve X. Then*

(a) If $\deg(D) < 0$, then $h^0(X, D) = 0$.

(b) If $\deg(D) = 0$ and $h^0(X, D) > 0$, then $D = 0$.

(c) if $\deg(D) > 2g - 2$, then $h^1(X, D) = 0$.

(d) If $\deg(D) = 2g - 2$ and $h^1(X, D) > 0$, then $D = K$.

Proof. By Serre duality (a)+(b) implies (c)+(d). (a) is obvious since effective divisors have positive degrees. Let $f \in H^0(X, D)$. To show $D = 0$, it suffices to show f is constant. This follows from the fact that $v_x(f) = 0$ for all $x \in X$. □

Corollary 8.2.0.7. *Let D be a divisor on a smooth projective curve of genus g.*

1. *If deg$(D) > g - 1$, then $h^0(X, D) > 0$.*

2. *If deg$(D) > 2g - 2$, then $h^0(X, D) =$ deg$(D) - g + 1$.*

3. *If deg$(D) = 2g - 2$, then $h^0(X, D) = g - 1$ if $D \neq K$ else $h^0(K, D) = g$.*

Proof. For (1), by Equation (8.1), $h^0(X, D) \geqslant \chi(D) = \deg(D) - g + 1 > 0$. For (2), by Corollary 8.2.0.6, (c), $h^0(X, D) = \deg(D) - g + 1$. (3) follows from Corollary 8.2.0.6, (d). □

Existence of curves of arbitrary genus

Here is an useful lemma, which is a baby version of the projection formula

Lemma 8.2.0.8. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and $\mathscr{G}$ an $\mathscr{O}_Y$-module. Then there exists a natural map*

$$f_*\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{G} \to f_*(\mathscr{E} \otimes f^*\mathscr{G}).$$

Moreover when $\mathscr{G}$ is a locally free sheaf of finite rank, this is an isomorphism.

Proof. It suffices to show that such a natural map exists. Since, once we show the existence and $\mathscr{G}$ is locally free of finite rank, to check this is an isomorphism we may assume $\mathscr{G}$ is free, in which case, it is obvious. The existence of the map follows by adjunction from the sequence of maps,

$$f^*(f_*\mathscr{F} \otimes \mathscr{G}) \simeq f^*f_*\mathscr{F} \otimes f^*\mathscr{G} \to \mathscr{F} \otimes f^*\mathscr{G},$$

where the last map is induced by adjunction between (f^*, f_*).

□

We will need the following corollary of Lemma 8.2.0.8.

Corollary 8.2.0.9. *Let X and Y be schemes over a field k. Let $\mathscr{F}$ and $\mathscr{G}$ be finite rank vector bundles on X and Y respectively. Denote by p_X (resp. p_Y) the projection from $X \times_k Y$ to X (resp. Y). Then*

$$H^0(X \times_k Y, p_X^*\mathscr{F} \otimes p_Y^*\mathscr{G}) \simeq H^0(X, \mathscr{F}) \otimes_k H^0(Y, \mathscr{G}).$$

Proof. Let $\pi_X : X \to \operatorname{Spec}(k)$ and $\pi_Y \to Spec(k)$ be the structural maps. Then by flat base change

$$\pi_Y^* H^0(X, \mathscr{F}) \simeq \pi_Y^* \pi_{X*} \mathscr{F} \simeq p_{Y*} p_X^* \mathscr{F}. \tag{8.2}$$

By the Leray spectral sequence for $X \times_k Y \to \operatorname{Spec}(k)$ factored as $\pi_Y \circ p_Y$, we have

$$H^0(X \times_k Y, p_X^* \mathscr{F} \otimes p_Y^* \mathscr{G}) \simeq R^0 \pi_{Y*} R^0 p_{Y*} (p_X^* \mathscr{F} \otimes p_Y^* \mathscr{G}).$$

By the projection formula (Lemma 8.2.0.8) along p_Y,

$$R^0 p_{Y*} (p_X^* \mathscr{F} \otimes p_Y^* \mathscr{G}) \simeq p_{Y*} p_X^* \mathscr{F} \otimes \mathscr{G}.$$

Combining this with Equation (8.2),

$$H^0(X \times_k Y, p_X^* \mathscr{F} \otimes p_Y^* \mathscr{G}) \simeq \pi_{Y*}(\pi_Y^* H^0(X, \mathscr{F}) \otimes \mathscr{G}).$$

By another appplication of the projection formula along π_Y, we get

$$\pi_{Y*}(\pi_Y^* H^0(X, \mathscr{F}) \otimes \mathscr{G}) \simeq H^0(X, \mathscr{F}) \otimes_k H^0(Y, \mathscr{G}).$$

□

Before we proceed further, lets show that there exist curves of every possible genus over an algebraically closed field.

Proposition 8.2.0.10. *For any integer $g \geqslant 0$, there exists a smooth projective curve X/k with genus g.*

Proof. Let $Y = \mathbb{P}_k^1 \times_k \mathbb{P}_k^1$. We know that

(a) Pic$(Y) = \mathbb{Z} \oplus \mathbb{Z}$, one corresponding to each factor.

(b) $Y \subseteq \mathbb{P}_k^3$ is a quadric hypersurface under the Segre embedding, thus the line bundle $\mathscr{O}(1,1)$ is the pullback of $\mathscr{O}(1)$ from $\mathbb{P}_k^3$.

Consider the line bundle $\mathscr{L} = \mathscr{O}(g+1, 2)$ on Y. Since $g \geqslant 0$, $\mathscr{O}(g+1)$ is very ample and thus $\mathscr{L}$ is very ample on Y. By Bertini theorem (see Theorem 4.2.4.9), there exists a smooth section $X \subseteq Y$ of this very ample line bundle. By the adjunction formula (see Proposition 4.2.3.3)

$$\omega_X = (\omega_Y \otimes \mathscr{L})|_X.$$

We have an exact sequence of sheaves

$$0 \to \omega_Y \to \omega_Y \otimes \mathscr{L} \to (\omega_Y \otimes \mathscr{L})|_X \to 0.$$

Using Example 4.2.3.2, (3) we have $\omega_Y = \mathscr{O}(-2, -2)$, and thus by (b) above is the pullback of $\mathscr{O}(-2)$ on $\mathbb{P}^3$. Thus by Proposition 6.6.1.5

$$h^0(Y, \omega_Y) = h^1(Y, \omega_Y) = 0.$$

Thus we conclude that $h^0(Y, \omega_Y \otimes \mathscr{L}) = h^0(Y, \omega_Y \otimes \mathscr{L}|_X) = h^0(X, \omega_X)$. Finally note that $\omega_Y \otimes \mathscr{L} = \mathscr{O}(g-1, 0)$, and thus we are done by Corollary 8.2.0.9. □

Genus 0 curves

Let us first look at genus 0-curves.

Proposition 8.2.0.11. *Let X/k be a smooth projective curve. Then $X \simeq \mathbb{P}^1_k$ iff*

(a) $|X(k)| \geqslant 2$.

(b) For $P, Q \in X(k)$, we have $[P] \sim_{rat} [Q]$

Proof. The only if direction is clear. Choose any two closed points $P, Q \in X(k)$. Then there exists a $f \in k(X)^\times$ such that $\mathrm{div}(f) = [P] - [Q]$.
Choose an affine open $U \hookrightarrow X$ such that $f \in H^0(U, \mathscr{O}_U)$. Thus there exist a map $\phi_f : U \to \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ such that the pullback of t is f. By Theorem 8.1.0.7, we can extend ϕ_f to a map of their compactifications $\bar{\phi}_f : X \to \mathbb{P}^1$. By Proposition 8.1.1.4, (a)

$$\mathrm{div}(f) = [P] - [Q] = \bar{\phi}_f^*(\mathrm{div}(t)).$$

In particular, $\bar{\phi}_f^*([0]) = [P]$ and thus by Lemma 8.1.1.6, $\deg(\bar{\phi}_f) = 1$ and hence by Theorem 8.1.1.4 $\bar{\phi}_f$ is an isomorphism.

□

Here is a corollary.

Corollary 8.2.0.12. *Let X/k be a smooth projective curve of genus 0 over a field k such that $X(k) \neq \varnothing$. Then $X \simeq \mathbb{P}^1_k$. In particular over an algebraically closed field k, there exists an unique smooth projective curve of genus 0.*

Proof. Let $P \in X(k)$ and let $D = [P]$. Then by Corollary 8.2.0.7, (2), $h^0(X, D) = 2$. Hence there exists a section $s \in H^0(X, D)$ such that $\deg(\mathrm{div}(s)) = 1$ (by Lemma 8.1.1.6) and $\mathrm{div}(s) \neq [P]$. Hence $\mathrm{div}(s) = [Q]$ with $Q \neq P$. Thus we are done by Proposition 8.2.0.11. □

Following example show that the assumption $X(k) \neq \varnothing$ is essential.

Example 8.2.0.13. Let k be a field of characteristic different from 2 such that $X \subseteq \mathbb{P}^2_k$ defined by $f(x, y, z) = x^2 + y^2 + z^2$ has *no* rational points (for example $k = \mathbb{Q}, \mathbb{R}$...). Then $p_g(X) = 0$ but $X \neq \mathbb{P}^1_k$. That $p_g(X) = 0$ is clear from Example 4.2.3.4 (since it implies $\omega_X = \mathscr{O}(-1)|_X$, and hence $p_g(X) = h^0(X, K) = h^0(X, \mathscr{O}(-1)|_X) = 0$) and since $X(k) = \varnothing$, $X \neq \mathbb{P}^1_k$.

Genus 1 curves

Let X/k be a genus 1 curve.

Lemma 8.2.0.14. *The canonical bundle of a genus 1 curve is trivial.*

Proof. Since deg$(K) = 0$ and $h^0(X, K) = g = 1$, this follows from Corollary 8.2.0.6, (b). □

let $O \in X(k)$. Let Pic$^0(X)$ be the sub-group of Pic$(X) \simeq A^1(X)$ consisting of degree 0 line bundles.

Lemma 8.2.0.15. *The map $X(k) \to$ Pic$^0(X)$ given by sending $P \to [P] - [O]$ is a bijection.*

Proof. To see this it suffices to show that for any divisor D of degree 0 on X we must have $h^0(X, D + [O]) = 1$. For then $D + [0]$ will be a degree 1 divisor with an unique (upto scaling) section s and we can take $[P] =$ div(s). Note that deg$(D + [0]) > 2p_g(X) - 2 = 0$ and hence we are done by Corollary 8.2.0.7, (2). □

The upshot of Lemma 8.2.0.15 is that for genus 1 curves X with $X(k) \neq \varnothing$, the set of rational points $X(k)$ carry a group structure! These are the so called Elliptic curves.

8.3 Criterion for base-point freeness and very ampleness

In this section we shall state and prove numerical criterions for base-point freeness and very ampleness of line bundles on smooth projective curves.

Proposition 8.3.0.1. *Let X/k be a smooth projective curve over an algberaically closed field k. Let $\mathscr{L} \in$ Pic(X). Then*

(a) $|\mathscr{L}|$ is base point free iff

$$h^0(X, \mathscr{L}) = h^0(X, \mathscr{L}(-P)) + 1,$$

for all closed points P.

(b) $\mathscr{L}$ is very ample iff

$$h^0(X, \mathscr{L}) = h^0(X, \mathscr{L}(-P - Q)) + 2,$$

for all closed points $P, Q \in X$ (not necessarily distinct).

Proof. For a proof see [3, Chapter IV, Proposition 3.1]. □

Here is the numerical criterion that we promised.

Corollary 8.3.0.2. *Let X/k be a smooth projective curve. Let $\mathscr{L} \in$ Pic(X). If*

(a) deg$(\mathscr{L}) \geqslant 2g$, then $\mathscr{L}$ is base-point free and hence defines a map $\phi_{\mathscr{L}} : X \to \mathbb{P}(H^0(X, \mathscr{L})^\vee)$.

(b) deg$(\mathscr{L}) \geqslant 2g + 1$, then $\phi_{\mathscr{L}}$ above is a closed embedding.

Proof. First note that a line bundle $\mathscr{L}$ on a scheme X is base point free iff the map

$$H^0(X, \mathscr{L}) \otimes \mathscr{O}_X \to \mathscr{L},$$

is surjective. By flat base change (Proposition 7.2.0.1) and faithful flatness of the base change $X_{\bar{k}} \to X$, we can check this after going to $\bar{k}$. Also a map of schemes $\phi : Z \to Y$ finite type over k is a closed immersion iff it is so after base changing to $\bar{k}$. Hence we may assume k is algebraically closed.
Then (a) and (b) both follow from Proposition 8.3.0.1 and Corollary 8.2.0.7. □

Corollary 8.3.0.3. *Let X/k be a smooth projective curve. Then $\mathscr{L} \in$ Pic(X) is ample iff deg$(\mathscr{L}) > 0$.*

Proof. A line bundle $\mathscr{L}$ is ample iff $\mathscr{L}^n$ is very ample for $n >> 0$. Since very ample line bundles have sections the only if directions is clear. For the if direction use Corollary 8.3.0.2. □

Here is an application of the ampleness criterion.

Proposition 8.3.0.4. *Let X/k be an irreducible curve. Then either X/k is proper or it is affine.*

Proof. Suppose X/k is not proper. We may assume X is reduced and hence that X is integral. Thus it suffices to prove that a non-proper integral curve is affine. Since the normalization map is finite, we may also assume X is normal (and hence regular). By Theorem 8.1.0.7, there exists an unique proper regular compactification $\bar{X}$ of X. Denote by $j : X \hookrightarrow \bar{X}$, the associated open immersion. By assumption $X \subsetneq \bar{X}$. Suppose $\{p_1, p_2 \cdots p_r\}$ be the complement of X in $\bar{X}$. Note that the p_i's are closed points of X.
Let $D = \sum_i n_i p_i$ be a divisor on X with $n_i >> 0$. By Corollary 8.3.0.2, (b) the line bundle $\mathscr{L}(D)$ is very ample. Let $\mathscr{I}_{\bar{X}}$, be the ideal sheaf of $\bar{X}$ under this embedding. By Theorem 6.5.0.2, there exists $m >> 0$ such that $\mathscr{I}_X(m)$ has no cohomology in positive degrees. Thus the natural map

$$H^0(\mathbb{P}(H^0(\bar{X}, \mathscr{L}(D))^\vee), \mathscr{O}(m)) \to H^0(\bar{X}, \mathscr{L}(D)^{\otimes m}),$$

is surjective. Thus there exists a section $f \in H^0(\mathbb{P}(H^0(\bar{X}, \mathscr{L}(D))^\vee), \mathscr{O}(m))$, such that div$(f) \cap \bar{X} = \sum_i m n_i p_i$. Thus $X = \bar{X} \backslash \{p_1, p_2 \cdots p_r\}$ is a closed subscheme of the *affine scheme*, div$(f)^c$. Hence the result. □

Now let us do a genus-by-genus analysis.

Example 8.3.0.5. (a) Let X/k be a smooth projective curve of genus 0. The a line bundle $\mathscr{L}$ on X is ample (equivalently of positive degree) iff it is very ample. In particular $\omega_X^\vee$ is very ample since $\deg(\omega_X^\vee) = 2$. Moreover by Corollary 8.2.0.7, (2) we must have $h^0(X, \omega_X^\vee) = 3$. Thus $\phi_{\omega_X^\vee} : X \to \mathbb{P}(H^0(X, \omega_X^\vee) \simeq \mathbb{P}^2$ is a closed embedding. Finally since X is necessarily defined by an hyperplanse section, by Example 4.2.3.4, we see that X is indeed a quadric hypersurface.

(b) Let X/k be a smooth projective curve of genus 1. Then a line bundle $\mathscr{L}$ on X is very ample iff $\deg(\mathscr{L}) \geqslant 3$. The if direction follows from Corollary 8.3.0.2, (b). For the only if direction note that if $\deg(\mathscr{L}) = 1$, then by Corollary 8.2.0.7, $h^0(X, \mathscr{L}) = 1$ and hence every section of $\mathscr{L}$ vanishes at the unique point P such that $\mathscr{L}([P]) = \mathscr{L}$. Hence $\mathscr{L}$ is not even base-point free. If $\deg(\mathscr{L}) = 2$ and $\mathscr{L}$ is very ample, then $\phi_{\mathscr{L}}$ would given an embedding of X inside $\mathbb{P}^1$, a contradiction. Finally arguing as in (a), we can see that if $P \in X(k)$, then the embedding $\phi_{\mathscr{O}_X(3P)} : X \hookrightarrow \mathbb{P}(H^0(X, \mathscr{O}_X(3P))) \simeq \mathbb{P}^2$ is a closed embedding given by a smooth cubic.

The above examples have a straightforward generalization in the following result which we shall not prove.

Proposition 8.3.0.6. *Let X/k be a smooth projective curve over an algebraically closed*[7] *field k. Then there exists an embedding $X \hookrightarrow \mathbb{P}^3_k$.*

Proof. For a proof see [3, Chapter IV, Corollary 3.6]. □

Proposition 8.3.0.6 can be thought of as an analogue of the Whitney embedding theorem. In particular the embedding in Proposition 8.3.0.6 is not explicit, that is involves various choices and is not *canonical*. One way to remedy this is to use a very ample line bundle on the curve which is intrinsic to it. This leads us to the canonical embedding of the curve.

8.3.1 Canonical embedding of a curve

Let X/k be a smooth projective curve and let K denote the canonical bundle. We have already shown that

1. When $p_g(X) = 0$, the anti-canonical bundle, $-K$ is very ample and ϕ_{-K} realizes X as a conic in $\mathbb{P}^2_k$.

2. When $p_g(X) = 1$, the canonical bundle is trivial.

Next we show that for $p_g(X) \geqslant 2$, the canonical bundle behaves very nicely.

[7] An infinite field should suffice.

Lemma 8.3.1.1. *Let $p_g(X) \geqslant 2$. Then K is base-point free.*

Proof. As before we may assume k is algebraically closed. Note that we cannot use Corollary 8.3.0.2, (a) directly since $\deg(K) = 2g-2 \leqslant 2g$. So we need to argue directly using Proposition 8.3.0.1. Let $P \in X(k)$. We need to show that

$$h^0(X, K) = h^0(X, K(-P)) + 1.$$

Equivalently by Serre duality it suffices to show that $h^1(X, \mathscr{O}_X(P)) = h^0(X, K(-P)) = g - 1$. Now by Riemann-Roch,

$$h^1(X, \mathscr{O}_X(P)) = h^0(X, \mathscr{O}_X(P)) + g - 2.$$

Since $p_g(X) \geqslant 2$, thanks to Proposition 8.2.0.11 we have $h^0(X, \mathscr{O}_X(P)) = 1$ and hence we are done. □

In the rest of this section we assume $p_g(X) \geqslant 2$. We denote by $\phi_K : X \to \mathbb{P}^{g-1}$, the morphism induced by the canonical bundle. Note that the key step in Lemma 8.3.1.1 is that under the conditions, there is no degree 1 divisor on X with $h^0 \geqslant 2$. This motivates the following definition.

Definition 8.3.1.2. A curves X/k is said to be hyperelliptic if $p_g(X) \geqslant 2$ and there exists a map $\phi : X \to \mathbb{P}^1$ of degree 2.

Lemma 8.3.1.3. *If $p_g(X) = 2$, then X is hyperelliptic.*

Proof. By Lemma 8.3.1.1, we have a morphism $\phi_K : X \to \mathbb{P}^{2-1} = \mathbb{P}^1$ induced by the canonical divisor. Since $\phi^*(\mathscr{O}(1)) = K$ and $\deg(K) = 2$ (Corollary 8.2.0.5), ϕ_K realized X as a hyperlliptic curve. □

Here is a non-example.

Lemma 8.3.1.4. *Let $X \subseteq \mathbb{P}^2_k$ be a plane curve defined by a smooth section of $\mathscr{O}(d), d \geqslant 4$. Then X/k is not hyperlliptic.*

Proof. By Example 4.2.3.4, we have

1. $K_X \simeq \mathscr{O}_X(d-3)$ and hence is very ample.
2. $p_g(X) = h^0(X, \mathscr{O}_X(d-3)) = \frac{(d-1)(d-2)}{2}$ and hence $p_g(X) \geqslant 2$.

Now suppose $\mathscr{L}$ be a line bundle of degree 2 on X. We claim $h^0(X, \mathscr{L}) = h^1(X, K \otimes \mathscr{L}^{-1}) \leqslant 1$, and hence there cannot be a degree 2 map to $\mathbb{P}^1$. We may assume k is algebraically closed. Suppose $s \neq 0 \in H^0(X, \mathscr{L})$ (if no such s exists, there is nothing to prove) and let $D = \mathrm{div}(s)$. Then $D \subseteq X$ is a divisor of degree 2, then we have a long exact sequence

$$0 \to H^0(X, K-D) \to H^0(X, K) \to H^0(X, K \otimes \mathscr{O}_D) \to H^1(X, K-D) \to H^1(X, K) \to 0.$$

Since $h^1(X, K) = h^0(X, \mathscr{O}_X) = 1$, it suffices to show that the natural map

$$H^0(X, K) \to H^0(X, K \otimes \mathscr{O}_D) = H^0(D, K \otimes \mathscr{O}(D)) \tag{8.3}$$

is surjective or equivalently that the kernel is $g - 2$ dimensional. Since D is a divisor of degree 0 and K is very ample, this follows from Proposition 8.3.0.1.

□

We introduce the following notation.

Notations 8.3.1.5. Let X/k be a smooth projective curve. By an element of g^r_d we mean a pair $(\mathscr{L}, V)$ where $\mathscr{L}$ is a line bundle of degree d and $V \subseteq H^0(X, \mathscr{L})$ is a vector space of dimension $r + 1$.

We have the following easy lemma.

Lemma 8.3.1.6. *Given any $(\mathscr{L}, V) \in g^r_d$. Then there exists a morphism $\phi : X \to \mathbb{P}^r$ such that*

(a) $\phi^*\mathscr{O}(1) \to \mathscr{L}$.

(b) The induced map $H^0(\mathbb{P}^r, \mathscr{O}(1)) \to H^0(X, \mathscr{L})$ is an isomorphism onto V.

Finally if $r = 1$, then deg$(\phi) \leqslant d$.

Proof. Let $V \subseteq H^0(X, \mathscr{L})$ be a vector space of dimension $r + 1$. We think of V as coherent sheaf over $\mathrm{Spec}(k)$ and $H^0(X, \mathscr{L}) = R^0\pi_*\mathscr{L}$, where $\pi : X \to \mathrm{Spec}(k)$ is the structural morphism. Thus by adjunction we get a map

$$\mathscr{O}_X^{r+1} \simeq \pi^*V \to \mathscr{L}.$$

Let $\mathscr{L}'$ be the image of this morphism. Then $\mathscr{L}'$ is a coherent sheaf on X which is a sub-sheaf of a locally free sheaf of rank 1. Hence by structure theorem for finitely generated modules over a PID, $\mathscr{L}'_x$ is a finitely generated free $\mathscr{O}_{X,x}$-module of rank 1 for all $x \in X$. Hence $\mathscr{L}'$ is a line bundle and by definition is globally generated. also by definition $H^0(X, \mathscr{L}') \subset H^0(X, \mathscr{L})$ is precisely the subspace $V \subseteq H^0(X, \mathscr{L})$. Hence the result.
Now suppose $r = 1$, then we have an short exact sequence

$$0 \to \mathscr{L}' \to \mathscr{L} \to \mathscr{L}/\mathscr{L}' \to 0,$$

where the quotient $\mathscr{L}/\mathscr{L}'$ is supported at finitely many points of X. Hence

$$\chi(\mathscr{L}) = \chi(\mathscr{L}') + \chi(\mathscr{L}/\mathscr{L}') \geqslant \chi(\mathscr{L}'),$$

and thus by Riemann-Roch, we must have deg$(\mathscr{L}) = r \geqslant$ deg$(\mathscr{L}')$. □

Example 8.3.1.7. A curve X/k is hyperelliptic iff $p_g(X) \geqslant 2$ and it has a non-empty g_2^1. If g_2^1 is non-empty, by Lemma 8.3.1.6, there exists a morphism of degree atmost 2 to $\mathbb{P}^1$. Since $p_g(X) \geqslant 2$, the degree of the morphism must be 2. The converse is obvious.

Now we shall prove an important embedding result.

Proposition 8.3.1.8. *Let X/k be a smooth projective curve of genus at least 2. Then either X is hyperelliptic or the morphism ϕ_K is a closed embedding.*

Proof. Suppose g_2^1 is empty. To show ϕ_K is a closed embedding we can assume k is algebraically closed. Using Proposition 8.3.0.1, it suffices to show that for any $P, Q \in X(k)$

$$h^0(X, K(-P-Q)) = h^0(X, K) - 2 = g - 2,$$

or equivalently by Riemann-Roch

$$h^0(X, \mathscr{O}_X(P+Q)) = 1.$$

This is trivially true by the definition of g_2^1. We are done by Example 8.3.1.7. □

Definition 8.3.1.9. For any non-hyperelliptic curve X/k of genus atleast 2, we denote by ϕ_K the canonical embedding of $X \subseteq \mathbb{P}^{g-1}$. The image of X under thus embedding is called the *canonical curve*.

We can easily compute the degree (see Definition 6.6.2.4) of the canonical curve.

Lemma 8.3.1.10. *The degree of the canonical curve is $2g-2$.*

Proof. By definition deg(X) is the leading coefficient in the Hilbert Polynomial $P_{\mathscr{O}_X}(z)$. By definition

$$P_{\mathscr{O}_X}(n) = \chi(\mathscr{O}_X(n)) = \chi(nK) = (2g-2)n - g + 1.$$

Hence the result. □

Now we can continue with our genus-by-genus analysis. So far we have dealt with curves of genus at most 2.

Example 8.3.1.11. Let X/k be a non-hyperelliptic curve of genus 3, then its canonical embedding in $\mathbb{P}^2$, realizes it as a quartic curve. Moreover under this embedding K_X is the pullback of $\mathscr{O}(1)$. Thus for genus 3 curves we have just two choices, either they are hyperelliptic or they are smooth quartics in $\mathbb{P}^2$.

Example 8.3.1.12. Let X/k be a non-hyperelliptic curve of genus 4, then its canonical embedding in $\mathbb{P}^3$, realizes it as a curve of degree 6. We claim X is contained in an unique irreducible quadric surface Q and that X is the complete intersection of Q with an irreducible cubic surface and hence is a complete intersection.

Let $\mathscr{I} \subseteq \mathscr{O}_{\mathbb{P}^3}$ be the ideal sheaf of X. Then we have an exact sequence

$$0 \to \mathscr{I} \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_X \to 0. \tag{8.4}$$

Twisting this with $\mathscr{O}(2)$ and noticing that $\mathscr{O}_X(2) = 2K$, we get a long exact sequence

$$0 \to H^0(\mathbb{P}^3, \mathscr{I}(2)) \to H^0(\mathbb{P}^3, \mathscr{O}(2) \to H^0(X, 2k) \to \cdots .$$

Since $h^0(\mathbb{P}^3, \mathscr{O}(2)) = 10$ and $h^0(X, 2K) = 9$ by Corollary 8.2.0.7 we must have $h^0(\mathbb{P}^3, \mathscr{I}(2)) \geqslant 1$. Thus there exists a degree 2 homogeneous polynomial $f \in H^0(\mathbb{P}^3, \mathscr{O}(2))$ such that $X \subseteq Z(f)$. If f is reducible, then $X \subseteq \mathbb{P}^2$, which is not possible since we cannot have a plane curve of degree 4. Thus $Q := Z(f)$ is an irreducible quadric surface (possibly singular). Further such a Q is unique, for if $X \subseteq Q' \neq Q$ for another quadric surface Q', then $X \subseteq Q \cap Q'$, but $\deg(X) = 6$ while $\deg(Q \cap Q') = 4$.
Finally twisting (8.4) by $\mathscr{O}(3)$, we get $h^0(\mathbb{P}^3, \mathscr{I}(3)) \geqslant 5$. As before any cubic $f \in H^0(\mathbb{P}^3, \mathscr{I}(3))$ must be irreducible and hence $X \subseteq F \cap Q$.

We now describe the canonical map ϕ_K for a hyperelliptic curve.

Proposition 8.3.1.13. *Let X/k be a hyperelliptic curve and let $\mathscr{L} \in g_2^1$ (see Example 8.3.1.7). Then*

1. *$h^0(X, \mathscr{L}) = 2$.*
2. *$\mathscr{L}^{\otimes(g-1)} = K$.*
3. *$(\mathscr{L}, H^0(X, \mathscr{L}))$ is (upto isomorphism) the only element in g_2^1.*

Proof. Let $(\mathscr{L}, V) \in g_2^1$, then as before we obtain a non-zero map

$$V \otimes \mathscr{O}_X \to \mathscr{L},$$

and denote by $\mathscr{L}'$ its image. Then we have

(a) $H^0(X, \mathscr{L}') = V$ as subspaces of $H^0(X, \mathscr{L})$.

(b) $\chi(\mathscr{L}) = \chi(\mathscr{L}') + H^0(X, \mathscr{L}/\mathscr{L}')$, and hence $\deg(\mathscr{L}') \leqslant \deg(\mathscr{L}) = 2$.

Since X is hyperelliptic, $\deg(\mathscr{L}) \neq 1$, for then it would define a degree 1 map (and hence an isomorphism) to $\mathbb{P}(H^0(X, \mathscr{L}')^\vee) \simeq \mathbb{P}^1$. Thus $\chi(\mathscr{L}) = \chi(\mathscr{L}')$ and by (b) above $H^0(X, \mathscr{L}/\mathscr{L}') = 0$. Since $\mathscr{L}/\mathscr{L}'$ is supported at finitely many points of X, this implies $\mathscr{L}/\mathscr{L}' = 0$ and hence that $\mathscr{L} \simeq \mathscr{L}'$. This proves (1).
Next we prove (b), to see this it suffices by Corollary 8.2.0.7, (3) to show that $h^0(X, \mathscr{L}^{\otimes(g-1)}) = g$, for $\deg(\mathscr{L}^{\otimes(g-1)}) = 2g-2$. Let $\phi_{\mathscr{L}} : X \to \mathbb{P}(H^0(X, \mathscr{L})^\vee) \simeq \mathbb{P}^1$ be the morphism induced by the complete linear system on $\mathscr{L}$. Let $\iota_{g-1} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$ be the $(g-1)^{\text{th}}$-uple embedding

and denote by $\phi : X \to \mathbb{P}^{g-1}$, the composite map. Then by construction $\phi^*(\mathscr{O}(1)) = \mathscr{L}^{\otimes(g-1)}$. Again by Corollary 8.2.0.7, (3) since $h^0(X, \mathscr{L}^{\otimes(g-1)}) \geqslant g-1$, it suffices to show that pullback map

$$H^0(\mathbb{P}^{g-1}, \mathscr{O}(1)) \to H^0(X, \mathscr{L}^{\otimes(g-1)}), \tag{8.5}$$

is an injection. By the construction of ϕ, the map in (8.5) factors as

$$H^0(\mathbb{P}^{g-1}, \mathscr{O}(1)) \simeq H^0(\mathbb{P}^1, \mathscr{O}(g-1)) \to H^0(X, \mathscr{L}^{\otimes(g-1)}).$$

Thus it suffices to show that

$$H^0(\mathbb{P}^1, \mathscr{O}(g-1)) \to H^0(X, \mathscr{L}^{\otimes(g-1)}), \tag{8.6}$$

is injective (and hence an isomorphism). First note that by finiteness (and hence affineness) of $\phi_{\mathscr{L}}$, we have (see 6.3.0.3)

$$H^0(X, \mathscr{L}^{\otimes(g-1)}) \simeq H^0(\mathbb{P}^1, \phi_{\mathscr{L}*}(\mathscr{L})) \simeq H^0(\mathbb{P}^1, \phi_{\mathscr{L}*}\mathscr{O}_X(g-1)),$$

where the last isomorphism follows from the projection formula (see Lemm 8.2.0.8). Hence it suffices to show that the natural map

$$H^0(\mathbb{P}^1, \mathscr{O}(g-1)) \to H^0(\mathbb{P}^1, \phi_{*\mathscr{L}}(g-1)),$$

is injective. This is an immediate consequence of the injectivity of $\mathscr{O}_{\mathbb{P}^1} \to \phi_{\mathscr{L}*}\mathscr{O}_X$, which in turn is a consequence of $\phi_{\mathscr{L}}$ being a dominant map of integral schemes. Note that our proof also implies that ϕ is isomorphic to ϕ_K.
Now we are left to prove (3). Now suppose $(\mathscr{L}', V') \in g_2^1$. Then as in (2) we can use this pair to produce a map $\phi' : X \to \mathbb{P}^{g-1}$, which as before is isomorphic to ϕ_K. In particular, there exists a an isomorphism of ϕ with ϕ' and hence of $\phi_{\mathscr{L}}$ with $\phi_{\mathscr{L}'}$ (since the ι_{g-1}'s are closed embeddings). Hence the result.

□

The upshot of Proposition 8.3.1.13 is that every hyperelliptic curve is so in an *unique* way. We conclude this section with the following Corollary.

Corollary 8.3.1.14. *Let X/k be a hyperelliptic curve and let $\mathscr{L} \in g_2^1$ be the unique element. Then $H^0(\mathbb{P}^1, \mathscr{O}(r)) \to H^0(X, \mathscr{L}^{\otimes r})$ is an isomorphism for $0 \leqslant r \leqslant g-1$.*

Proof. We have already shown this for $r = g-1$. We prove by descending induction on r. Assuming the result to be true for r, we show the result for $r-1$. Choose a point $P \in \mathbb{P}^1(k)$. Let D be the scheme $\phi_{\mathscr{L}}^{-1}(P)$. Then we have a commutative diagram

$$\begin{tikzcd}
H^0(X, \mathscr{L}^{\otimes(r)}) \arrow[r] & H^0(X, \mathscr{L}^{\otimes r} \otimes \mathscr{O}_D) \\
H^0(\mathbb{P}^1, \mathscr{O}(r)) \arrow[r] \arrow[u, "\simeq"] & H^0(\mathbb{P}^1, \mathscr{O}_P(r)) \simeq k \arrow[u]
\end{tikzcd}$$

In particular the image of $H^0(X, \mathscr{L}^{\otimes r}) \to H^0(X, \mathscr{L}^{\otimes r} \otimes \mathscr{O}_D)$ is 1-dimensional. Let $s \in H^0(X, \mathscr{L})$ corresonding to the section of $H^0(\mathbb{P}^1, \mathscr{O}(1))$ given by the point P. Then by definition, the Cartier divisior defined by s is D and hence we have a left exact sequence

$$0 \to H^0(X, \mathscr{L}^{\otimes(r-1)}) \to H^0(X, \mathscr{L}^{\otimes r}) \to H^0(X, \mathscr{L}^{\otimes r} \otimes \mathscr{O}_D) \to \cdots$$

Thus $h^0(X, \mathscr{L}^{\otimes(r-1)}) = h^0(X, \mathscr{L}^{\otimes r}) - 1 = r - 1$. □

8.4 Hurwitz Theorem

Let $\phi : X \to Y$ be a morphism of smooth curves over a field k. Then we have a right exact sequence of coherent sheaves on X (see Proposition 4.1.0.6, (2))

$$\phi^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0.$$

Since X and Y are assumed to be smooth over k, both $\phi^*\Omega^1_{Y/k}$ and $\Omega^1_{X/k}$ are line bundles on X, an integral scheme of dimension 1. Let $\mathscr{W}_\phi := \ker(\phi^*\Omega^1_{Y/k} \to \Omega^1_{X/k})$. Then

1. $\mathscr{W}_\phi$ is a coherent and torsion free sheaf.
2. $\mathscr{W}_\phi = 0$ iff $\Omega^1_{X/Y} \otimes k(X) = 0$.
3. if $\mathscr{W}_\phi \neq 0$, then the map $d_\phi : \phi^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ is 0 and hence $\mathscr{W}_\phi = \phi^*\Omega^1_{Y/k}$.

(1) is clear since $\mathscr{W}_\phi$ is a sub-sheaf of a line bundle on the integral scheme X. By structure theorem for PID, $\mathscr{W}_\phi$ is then itself a line bundle on X. In particular $\mathscr{W}_\phi$ is 0 iff $\mathscr{W}_\phi \otimes k(X) = 0$. But $\mathscr{W}_\phi \otimes k(X) = 0$ iff $d_\phi \otimes k(X)$ is injective and hence an isomorphism. This proves (2). For (3) note that d_ϕ is a map of vector bundles on an integral scheme. Hence $d_\phi = 0$ iff $d_\phi \otimes k(X) = 0$, the latter holds true iff $\mathscr{W}_\phi \otimes k(X) \neq 0$.
Thus we have the following proposition.

Proposition 8.4.0.1. *Let $\phi : X \to Y$ be a map of smooth curves over k. Then one of the following two possibilities are true*

1. *The right exact sequence*

   $$\phi^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0,$$

   is also left exact and hence $\Omega^1_{X/Y}$ is supported at finitely many points of X called the points of ramification of ϕ.

2. *The map d_ϕ is the 0-map.*

Proof. Suppose d_ϕ is not the 0-map. Then by the above discussion $\mathscr{W}_\phi = 0$ and hence the result. □

Corollary 8.4.0.2. *Let $\phi : X \to Y$ be a morphism of smooth curves over k. Suppose $d_\phi \neq 0$. Then ϕ is étale outside the ramification locus.*

Proof. Since $d_\phi \neq 0$, ϕ cannot be the constant map and hence ϕ is dominant and thus flat. Étaleness is now obvious. □

Finally we give a simple criterion for $d_\phi \neq 0$.

Proposition 8.4.0.3. *Let $\phi : X \to Y$ be a dominant morphism of smooth curves. Then $d_\phi \neq 0$ iff the map $k(Y) \to k(X)$ of functions fields is separable.*

Proof. □

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