AG-II-Notes

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Contents

Chapter 1

Catgeory Theory Part-0

We begin by recalling some basic notions from category theory which should take some way into the course. This is far from an exhaustive account and focuses on introducing the bare minimum needed for the purposes of these lectures.

1.1 Categories: Definitions and Examples

Recall that a category $\mathscr{C}$ consists of a collection of objects $\mathrm{Ob}(\mathscr{C})$ and a collection of morphisms between these objects. The morphisms are required to satisfy certain properties:

1. For every object A in the category, there is an identity morphism 1_A from A to A.
2. For every pair of morphisms $f : A \to B$ and $g : B \to C$, there is a composite morphism $g \circ f : A \to C$.
3. Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.
4. Composition is unital: $1_B \circ f = f = f \circ 1_A$.

Example 1.1.0.1. The category **Set** has sets as objects and functions as morphisms. The identity morphism on a set A is the identity function $\mathrm{id}_A : A \to A$. The composite of two functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$. The associativity and unitality of composition follow from the corresponding properties of functions.

Example 1.1.0.2. The category **Top** has topological spaces as objects and continuous functions as morphisms. The identity morphism on a topological space X is the identity function $\mathrm{id}_X : X \to X$. The composite of two continuous functions $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f : X \to Z$. The associativity and unitality of composition follow from the corresponding properties of continuous functions.

Example 1.1.0.3. The category $\mathbf{Vect}_k$ has vector spaces over a field k as objects and linear transformations as morphisms. The identity morphism on a vector space V is the identity transformation $\mathrm{id}_V : V \to V$. The composite of two linear transformations $f : V \to W$ and $g : W \to Z$ is the transformation $g \circ f : V \to Z$. The associativity and unitality of composition follow from the corresponding properties of linear transformations.

Example 1.1.0.4. Let S be a scheme. Let $\mathbf{Sch}_S$ be the category whose objects are a pair (X, f), where X is a scheme and $f : X \to S$ a morphism. Morphisms ϕ in this category are commutative diagrams of the form

$$\begin{tikzcd} (X,f) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & (Y,g) \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

An important special case for us is the category $\mathbf{Sch}_k$ of schemes over a field $\mathrm{Spec}(k)$.

Example 1.1.0.5. Let X be a topological space. The category $\mathbf{Op}(X)$ has open sets in X as objects and inclusions as morphisms. The identity morphism on an open set U is the inclusion $U \hookrightarrow U$. The composite of two inclusions $U \hookrightarrow V$ and $V \hookrightarrow W$ is the inclusion $U \hookrightarrow W$. The associativity and unitality of composition follow from the corresponding properties of inclusions. In partiular for any two objects U and V either $\mathrm{Hom}_{\mathbf{Op}(X)}(U, V)$ is either empty or contains a unique morphism.

Example 1.1.0.6. Let $\mathscr{C}$ be a category. The opposite category $\mathscr{C}^{\mathrm{op}}$ has the same objects as $\mathscr{C}$ and morphisms reversed. That is, for every pair of objects A and B in $\mathscr{C}$, we have $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$. The identity morphism on an object A in $\mathscr{C}^{\mathrm{op}}$ is the identity morphism on A in $\mathscr{C}$. The composite of two morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}^{\mathrm{op}}$ is the composite $g \circ f : A \to C$ in $\mathscr{C}$. The associativity and unitality of composition follow from the corresponding properties of composition in $\mathscr{C}$.

1.2 Functors

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $F : \mathscr{C} \to \mathscr{D}$ assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$ and to each morphism $f : A \to B$ in $\mathscr{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathscr{D}$. Functors are required to satisfy the following properties:

1. For every object A in $\mathscr{C}$, we have $F(1_A) = 1_{F(A)}$.

2. For every pair of morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$, we have $F(g \circ f) = F(g) \circ F(f)$.

One can also have what are called as contravariant functors. A contravariant functor $F : \mathscr{C} \to \mathscr{D}$ assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$ and to each morphism $f : A \to B$ in $\mathscr{C}$ a morphism $F(f) : F(B) \to F(A)$ in $\mathscr{D}$. Contravariant functors are required to satisfy properties analogous to those for covariant functors.

Example 1.2.0.1. The forgetful functor F : **Top** $\to$ **Set** assigns to each topological space its underlying set and to each continuous function its underlying function. The identity function on a set is continuous, so the identity morphism on an object in **Top** is sent to the identity morphism on the corresponding object in **Set**. The composite of two continuous functions is continuous, so the composite of two morphisms in **Top** is sent to the composite of the corresponding morphisms in **Set**.

A more non-trivial functor from **Top** to **Set** is the functor Π_0.

Example 1.2.0.2. The functor Π_0 : **Top** $\to$ **Set** assigns to each topological space X the set of connected components $\Pi_0(X)$ of X and to each continuous function $f : X \to Y$ the function $\Pi_0(f) : \Pi_0(X) \to \Pi_0(Y)$ induced by f.

We now state a few properties of functors.

Definition 1.2.0.3. A functor $F : \mathscr{C} \to \mathscr{D}$ is faithful if for every pair of objects A and B in $\mathscr{C}$, the map $F : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ is injective. We say that F is fully faithful if this map is bijective.

Definition 1.2.0.4. A functor $F : \mathscr{C} \to \mathscr{D}$ is essentially surjective if for every object B in $\mathscr{D}$, there is an object A in $\mathscr{C}$ such that $F(A)$ is isomorphic to B.

The examples 1.2.0.1 and 1.2.0.2 are faithful and essentially surjective functors. Next we will discuss an important class of functors called representable functors.

Example 1.2.0.5. Let $\mathscr{C}$ be a category and A an object in $\mathscr{C}$. The representable functor $\mathrm{Hom}_{\mathscr{C}}(A, -) : \mathscr{C} \to$ **Set** assigns to each object B in $\mathscr{C}$ the set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of morphisms from A to B and to each morphism $f : B \to C$ in $\mathscr{C}$ the function $\mathrm{Hom}_{\mathscr{C}}(A, f) : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$ induced by f. The identity morphism on an object B in $\mathscr{C}$ is sent to the identity morphism on $\mathrm{Hom}_{\mathscr{C}}(A, B)$, and the composite of two morphisms $f : B \to C$ and $g : C \to D$ in $\mathscr{C}$ is sent to the composite of the corresponding morphisms $\mathrm{Hom}_{\mathscr{C}}(A, f)$ and $\mathrm{Hom}_{\mathscr{C}}(A, g)$.

Next we discuss natural transformations of functors.

Definition 1.2.0.6. Let F and G be two functors between categories $\mathscr{C}$ and $\mathscr{D}$. A natural transformation $\eta : F \to G$ assigns to each object A in $\mathscr{C}$ a morphism $\eta_A : F(A) \to G(A)$ in $\mathscr{D}$ such that for every morphism $f : A \to B$ in $\mathscr{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes. That is, we have $G(f) \circ \eta_A = \eta_B \circ F(f)$ for every morphism $f : A \to B$ in $\mathscr{C}$.

Example 1.2.0.7. Let $\mathbf{Vect}_k$ be the category of vector spaces over a field k. The double dual functor $\mathbf{Vect}_k \to \mathbf{Vect}_k$ assigns to each vector space V its double dual $V^{\vee\vee}$ and to each linear transformation $f : V \to W$ the linear transformation $f^{\vee\vee} : V^{\vee\vee} \to W^{\vee\vee}$ induced by f. The natural transformation $\eta : \mathrm{id} \to (-)^{\vee}$ assigns to each vector space V the canonical map $\eta_V : V \to V^{\vee\vee}$ and to each linear transformation $f : V \to W$ the commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"] \\ V^{\vee\vee} \arrow[r, "f^{\vee\vee}"'] & W^{\vee\vee} \end{tikzcd} .$$

Note that the vertical arrows are isomorphisms if and only if the vector spaces are finite-dimensional.

Definition 1.2.0.8. A natural transformation $\eta : F \to G$ of functors is a natural equivalence if for every object A in $\mathscr{C}$, the morphism $\eta_A : F(A) \to G(A)$ is an isomorphism in $\mathscr{D}$.

Now we are ready to state the Yoneda Lemma.

Lemma 1.2.0.9 (Yoneda Lemma). *Let $\mathscr{C}$ be a category and A an object in $\mathscr{C}$. Let $F : \mathscr{C} \to$ **Set** be a functor. Then the natural transformations Hom$_{\mathscr{C}}(A, -) \to F$ are in bijection with the elements of $F(A)$.*

Proof. Let $\eta : \mathrm{Hom}_{\mathscr{C}}(A, -) \to F$ be a natural transformation. In particular, $\eta_A : \mathrm{Hom}_{\mathscr{C}}(A, A) \to F(A)$ is a morphism in **Set**. The desired element in $F(A)$ is simply the image of the identity morphism on A under η_A. Conversely, given an element x in $F(A)$, we can define a natural transformation $\eta : \mathrm{Hom}_{\mathscr{C}}(A, -) \to F$ by setting $\eta_B(f) = F(f)(x)$ for every object B in $\mathscr{C}$ and morphism $f : A \to B$ in $\mathscr{C}$. The naturality of η follows from the properties of functors. $\square$

In particular we note the following corollary.

Corollary 1.2.0.10. *Let C be a category. Then the functor Hom$_{\mathscr{C}}(A, -) : \mathscr{C}^{op} \to$ Func($\mathscr{C}$, **Set**) is fully faithful, where Func($\mathscr{C}$, **Set**) is the category whose objects are functors from $\mathscr{C}$ to **Set** and morphisms are natural transformations.*

1.2.1 Limits and Colimits

Let $\mathscr{C}$ be a category and I a category. A functor $F : I \to \mathscr{C}$ is called a diagram in $\mathscr{C}$ indexed by I. A cone over F is an object A in $\mathscr{C}$ together with morphisms $A \to F(i)$ for every object i of I compatible with the functor F. A limit of F is a terminal object[1] in the category of

[1] Meaning it maps uniquely to any other cone

cones over F. Dually, a colimit of F is an initial object[2] in the category of co-cones[3] over F. Note that limits and colimits **maynot** exist in general but when they do they are unique upto unique isomorphism.
Limits are denoted by

$$\lim_{i \in I} F(i)$$

and colimits are denoted by

$$\mathrm{colim}_{i \in I} F(i).$$

Example 1.2.1.1. Let V and W be vector spaces over a field k, and let $f : V \to W$ be a linear transformation. The kernel of f is the limit of the diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ V & \xrightarrow{f} & W \end{array}$$

Here the indexing category is the category with two objects and one non-identity morphism between them.

We have the dual example.

Example 1.2.1.2. Let V and W be vector spaces over a field k, and let $f : V \to W$ be a linear transformation. The cokernel of f is the colimit of the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow & & \\ 0 & & . \end{array}$$

As before the indexing category is the category with two objects and one non-identity morphism between them.

You have seen these before!

Example 1.2.1.3. Let X and Y be schemes over S. The fibre product of X and Y (over S) is the limit of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & S. \end{array}$$

Here the indexing category is the category with three objects and two non-identity morphisms between them.

[2] Meaning it gets an unique map from every cone
[3] Guess its definition!

Here is a basic and important example.

Example 1.2.1.4. Let I be a set. We say a category $\mathscr{C}$ has products (resp. coproducts) indexed by I if every functor indexed by I has limits (resp. colimits). Here I is considered as a category with objects indexed by elements of I and no non-identity morphisms.

Example 1.2.1.5. We say that a category $\mathscr{C}$ has finite limits (resp. colimits) if it has limits (resp. colimits) indexed by any category with finitely many objects and morphisms

Another important class of indexing category for us are the *flitered* ones. Let me give an example first.

Example 1.2.1.6. Let $\mathbb{N}$ be the set of natural numbers. We can consider $\mathbb{N}$ as a category with objects indexed by natural numbers and a unique morphism between any two objects. This is a filtered category.

Here is a formal definition.

Definition 1.2.1.7 (Filtered Category)**.** A category I is called filtered if for every pair of objects i and j in I, there is an object k in I and morphisms $f : i \to k$ and $g : j \to k$. Moreover for a pair of morphisms $f, g : i \to j$ in I, there is an object k in I and a morphism $h : j \to k$ such that $h \circ f = h \circ g$.

Example 1.2.1.8. Let R be a ring and M an R-module. Then M is a filtered colimit of its finitely generated submodules. This is often used to reduce statements about arbitrary modules to statements about finitely generated modules.

We have the following *very useful* but formal result.

Lemma 1.2.1.9. *Limits commute with right adjoints and colimits commute with left adjoints.*

1.3 Abelian Categories

We begin with the definition of an additive category.

Definition 1.3.0.1 (Additive Category)**.** An additive category is a category $\mathscr{A}$ with the following properties:

1. For every pair of objects A and B in $\mathscr{A}$, the morphism set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is an abelian group[4].

2. Composition of morphisms is bilinear.

[4] In particular there is a 0 morphism.

3. $\mathscr{A}$ has a zero object, that is, an object that is both initial and terminal[5].

4. $\mathscr{A}$ has finite products and coproducts i.e. the indexing set is finite.

Clearly the opposites of an additive category can also be naturally given a structure of an additive category. We will see additive categories later too, when we discuss cohomology. For now you may think of them as categories where you can add morphisms and have a zero object.

Definition 1.3.0.2 (Additive Functor). Let $\mathscr{C}$ and $\mathscr{D}$ be additive categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is additive if for every pair of objects A and B in sC, the map $F : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ is a group homomorphism. Moreover, F is required to preserve finite products and coproducts[6].

We need few more definitions before we can define an abelian category. In what follows we assume A is an additive category and all functors are additive.

Definition 1.3.0.3 (Kernels and Cokernels). Let $f : A \to B$ be a morphism in a category $\mathscr{A}$. A kernel of f is the limit of the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}.$$

Dually a cokernel of f is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ 0 & \end{tikzcd}.$$

Remark 1.3.0.4. Kernels and cokernels may not exist in general. However when they do they are unique upto unique isomorphism.

We now define monomorphisms and epimorphisms in an additive category.

Definition 1.3.0.5. A morphism $f : A \to B$ in an additive category is a monomorphism if for every object C and morphisms $g, h : C \to A$ such that $f \circ g = f \circ h$, we have $g = h$. A morphism $f : A \to B$ in an additive category is an epimorphism if for every object C and morphisms $g, h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

Now we can state the definition of an abelian category.

[5] This makes it unique upto an unique isomorphism

[6] The product or coproduct indexed by the empty set is the 0-object. Hence F is required to take the zero object to the zero object.

Definition 1.3.0.6 (Abelian Category)**.** An abelian category is an additive category $\mathscr{A}$ with the following properties:

1. Every morphism in $\mathscr{A}$ has a kernel and a cokernel[7].
2. Every monomorphism in $\mathscr{A}$ is the kernel of its cokernel.
3. Every epimorphism in $\mathscr{A}$ is the cokernel of its kernel.

Example 1.3.0.7. The category of abelian groups can be given the structure of an abelian category. The zero object is the trivial group, the product is the direct sum, and the coproduct is the direct product. The kernel of a morphism $f : A \to B$ is the subgroup of elements a in A such that $f(a) = 0$, and the cokernel is the quotient group $B/\text{im}(f)$. The monomorphisms are the injective group homomorphisms, and the epimorphisms are the surjective group homomorphisms.

We can even restrict to the category of finitely generated abelian groups and get an abelian category. Note that finite coproducts are the same as finite products in both these cases. This is not a coincidence. In general in an abelian category finite products and coproducts are the same.
The last thing we need to get us going into geometry is exactness of functors.

Definition 1.3.0.8. Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is left exact if it preserves finite limits, and right exact if it preserves finite colimits. A functor $F : \mathscr{C} \to \mathscr{D}$ is exact if it is both left and right exact.

This coincides with the more usual definition as shown below[8].

Proposition 1.3.0.9. *Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then the following are equivalent:*

1. *F is left exact.*
2. *For every short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

   in $\mathscr{C}$, the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

   is exact in $\mathscr{D}$.

[7] This implies that in addition to have finite products and coproducts by virtue of $\mathscr{A}$ being additive, it also has finite limits and colimits (see Tag 010D)

[8] Ignore on first reading

3. *F preserves kernels.*

Proof. Clearly (2) implies (3). That (1) implies (3) follows from Definition 1.3.0.3. To see that (3) implies (2), it suffices to show that the image of $F(f)$ is the kernel of $F(g)$. Since F preserves monomorphisms, $F(f)$ is a monomorphism. Thus the image of $F(f)$ is naturally isomorphic to $F(A)$, which is the kernel of $F(g)$, since F preserves kernels and A is the kernel of g.
We are left to show that (3) implies (1). For this we use a general result that finite limits can be expressed in terms of kernels and finite products (see Tag 002P for a reference). Since F preserves both we are done.

□

We note here a very useful corollary to Lemma 1.2.1.9

Corollary 1.3.0.10. *Any right adjoint functor is left-exact and any left adjoint functor is right-exact.*

We conclude this section with a couple of examples.

Example 1.3.0.11. Let R be any ring[9] and M an R-module. The tensor product functor $- \otimes M : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is right-exact. This follows from the fact that it is a right adjoint to the Hom functor.

Example 1.3.0.12. Let $f : X \to Y$ be a morphism of schemes. The lower shriek functor f_* is left-exact. This follows from the fact that it is a left adjoint to the pullback functor f^* (which in turn is necessarily right-exact).

[9]All rings in this course are commutative with unity.

Chapter 2

Flatness

Consider the following three maps:

1. $f : \mathrm{Bl}_{(0,0)}\mathbb{A}^2 \to \mathbb{A}^2$,where $\mathrm{Bl}_{(0,0)}\mathbb{A}^2$ is the blow-up of $\mathbb{A}^2$ at the origin and f is the projection map.

2. $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ with $f(z) = z^2$.

3. $f : G_m \to G_m$ with $f(z) = z^2$. Here G_m is $\mathbb{A}^1 \backslash \{0\}$.

The map (1) here is an isomorphism on the complement of the origin, but over the origin the fiber is $\mathbb{P}^1$. The map (2) is nice outside the origin, with the inverse image of any $z \neq 0$ consisting of two points. But at the origin the fiber consists of exactly one point. The map (3) is simply the base change of (1) along the open immersion $G_m \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$, and hence all points have as inverse image exactly two distinct points.

Question 2.0.0.1. How do we capture the discontinuous jump in the fiber dimension at the origin in Example 1? Note that even though Example (2) has a *bad* fiber over the origin, it is still of dimension 0 like every other fiber.

The answer lies in the notion of flatness, a purely algebraic construct!

2.1 Flatness: Definition and Properties

We begin by defining flatness and faithful flatness.

Definition 2.1.0.1. Let A be a ring and M be an A-module. We say that M is **flat** over A if the right-exact functor $- \otimes_A M$ is exact. A map of rings $A \to B$ is said to be **flat** if B is flat as an A-module.

Definition 2.1.0.2. A flat A-module M is said to be **faithfully flat** if the functor $- \otimes_A M$ is faithful.

Let us see some examples of flat and faithfully flat modules.

Example 2.1.0.3.

1. The ring A is flat over itself.
2. Since tensor products are right adjoint, they commute with arbitrary colimits. moreover *filtered* colimits of exact sequences is exact. Combining these two, we get that filtered colimits of flat modules are flat.
3. Combining (1) and (2) we get that filtered colimits of the form $\operatorname{colim}_i M_i$, where each M_i is abstractly isomorphic to A is flat. Note that we dont care what the maps are as long as the indexing category is filtered.

Example 2.1.0.3, (3) has the following corollary.

Corollary 2.1.0.4. *The ring A_f is flat over A. More generally for any multiplicative subset S of A, the ring $A[S^{-1}]$ is flat.*

Proof. The first claim follows from the isomorphism

$$A_f \simeq \operatorname{colim}\{A \to A \to A \cdots\},$$

where the transition maps are multiplication by f. The second part of the claim follows from the isomorphism

$$A[S^{-1}] = \operatorname{colim}_{f \in S} A_f,$$

where the colimit is over the directed set indexed by elements of S, with $f \leqslant g$ if $g = ff'$ for some $f' \in A$. This is directed because S is multiplicative and further the first part of the Corollary implies each of the A_f's are flat. Hence the result.

□

Corollary 2.1.0.5. *For any ring A, arbitrary direct sums of A is a flat A-module. In particular when A is a field, all A-modules are flat.*

Corollary 2.1.0.6. *For any ring R the map $R \to R[x]$ is flat.*

Proof. Direct sums are colimits over an directed set with no non-identity arrows, hence the result. □

Next we list some properties of flatness.

Proposition 2.1.0.7. *We will need the following facts about flatness. Let $\phi : A \to B$ be a map of rings, M be an A-module and N a B-module. Then the following hold*

1. *M is flat over A iff for all finitely generated ideals $\mathfrak{a}$ of A the induced map*
$$\mathfrak{a} \otimes_A M \to M,$$
is injective.

2. *(Base-Change) M is flat over A implies $M \otimes_A B$ is flat over B.*

3. *(Transitivity) B flat over A and N flat over B implies N is flat over A.*

4. *(Local Nature) M is flat over A iff $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of A.*

5. *N is flat over A iff $N_{\mathfrak{q}}$ is flat over $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{q}$ of B, here $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$.*

6. *For a short exact sequence of A-modules*
$$0 \to M' \to M \to M'' \to 0,$$
M is flat if M' and M'' are flat. Also if M and M'' are flat, so is M'.

7. *For a Noetherian local ring A, a finitely generated module M is flat over A iff M is free over A.*

Proof. (1) is proved in Tag 00HD, (2) in Tag 051D, (3) in Tag 051D, (4) and (5) in Tag 051D, (6) in Tag 00HM and finally (7) in Tag 00NZ[1] □

We can now globalize the definition of flatness to schemes.

Definition 2.1.0.8 (Flatness). Let $f : X \to Y$ be a morphism of schemes. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ (resp. f) is flat over Y at a point $x \in X$ if the stalk $\mathscr{F}_x$ (resp. $\mathscr{O}_{X,x}$) is flat as a $\mathscr{O}_{Y,f(y)}$-module. If this holds for all points x in X we say $\mathscr{F}$ is flat over Y (resp. f is a flat morphism).

Remark 2.1.0.9. Note that flatness is local on both the source and the base. Meaning to check a sheaf $\mathscr{F}$ is flat (over Y) it suffices to check this on an open cover of either X or Y or both.

Now we translate Proposition 2.1.0.7 into the language of scheme.

Proposition 2.1.0.10. *Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ a $\mathscr{O}_X$-module of X. Then the following hold.*

[1] If you assume A is Noetherian, the proof can be simplified. As in the proof by Nakayama's Lemma we can pick a surjection $A^n \to M$ where n is the dimension of $\frac{M}{\mathfrak{m}M}$. Here $\mathfrak{m}$ is the unique maximal ideal of A. Suppose K is the kernel of this surjection. Then tensoring this exact sequence with $\frac{A}{\mathfrak{m}}$, we get that $\frac{K}{\mathfrak{m}K}$ is trivial by flatness of M, which by Nakayama implies K is trivial. (Question: Where did we use A is Noetherian?)

1. *If f is an open immersion then it is flat.*

2. *Suppose both X and Y are affine schemes, say $X = \mathrm{Spec}B$ and $Y = \mathrm{Spec}A$. Then $\mathscr{F}$ is flat over Y iff M is flat over A where M is the A-module corresponding to $\mathscr{F}$.*

3. *A base change of a flat quasi-coherent sheaf[2] is flat. That is if we have a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

and assume that $\mathscr{F}$ is flat and quasi-coherent, then the pullback $\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'}$ is flat over Y'.

4. *Suppose f was morphism over a base scheme S. If $\mathscr{F}$ is flat over Y and Y is flat over S, then $\mathscr{F}$ is flat over S. In particular composition of flat morphisms is flat.*

5. *Suppose we have a short exact sequence of quasi-coherent sheaves*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

Then $\mathscr{F}$ is flat if $\mathscr{F}'$ and $\mathscr{F}''$ are flat. Also if $\mathscr{F}$ and $\mathscr{F}''$ are flat, so is $\mathscr{F}'$.

6. *Let X be a Noetherian scheme and $\mathscr{F}$ a coherent sheaf. Then $\mathscr{F}$ is flat iff $\mathscr{F}$ is locally free aka a vector bundle.*

Proof. (1) is immediate from the definition since the induced map on local rings is an isomorphism. (2) follows from Proposition 2.1.0.7, (5). The claims (3)-(6) are now a consequence of Remark 2.1.0.9 and Proposition 2.1.0.7. □

Remark 2.1.0.11. 1. Fix a base scheme S. Consider the subcategory of $\mathbf{Sch}_S$ where we only allow morphisms which are flat between the objects. This is a subcategory of $\mathbf{Sch}_S$, and is closed under composition and base change.

2. Thanks to Corollary 2.1.0.6 and Remark 2.1.0.9, for any scheme X, the morphism $\mathbb{A}^n_X \to X$ is flat. More generally for an locally free sheaf $\mathscr{E}$ on a scheme X, the map $\mathbb{A}(\mathscr{E}) \to X$ is flat. Again using Remark 2.1.0.9, we can conclude that $\mathbb{P}(\mathscr{E}) \to X$ is flat.

Recall for any topological space X and a pair of points x and y in X, we have the following:

(a) x is a specialization of y if $x \in \overline{\{y\}}$.

(b) x is a generalisation of y if $y \in \overline{\{x\}}$.

[2]Hartshorne forgets writing quasi-coherent in Chapter III.9, Proposition 9.2 (b).

In particular when $X = \mathrm{Spec}(A)$, the constructible subsets of X which are stable under generalisation are open and those stable under specialization are closed (see [2, Chapter II, Exercise 3.18]

Proposition 2.1.0.12. *Let $f : X \to Y$ be a flat morphism of schemes. Then the image*[3] *of f is stable under generalization.*

Proof. Let y be a point in the image of f. We need to show that any point $y' \in Y$ such that $y \in \overline{\{y'\}}$, also belongs to the image of f. Choose an affine open $V \ni y$ and an affine open $U \ni x$ such that $f(x) = y$ and $f(U) \subseteq V$. It suffices to show that there is a point $x' \in U$ such that $f(x') = y'$. But this is precisely the going down theorem from local algebra (see Tag 00HS). $\square$

Corollary 2.1.0.13 (Openness of flat morphisms)**.** *Let $f : X \to Y$ be a flat morphism, locally of finite presentation*[4]*. Then f is universally open i.e the image of any base change of f is open.*

Proof. Since both flat morphisms and morphisms of finite presentation satisfy BC, we are reduced to showing the openness of f. We have already shown that the image of f is stable under generalizations (without any finite presentation assumptions). As before we can assume that both $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine with the map $A \to B$ bein of finite presentations. By Chevalley's theorem (see Tag 00FE), $\mathrm{Im}(f)$ is constructible and by Prop 2.1.0.12 it is stable under generalizations and hence is open. $\square$

Corollary 2.1.0.14. *Let $f : A \to B$ be a local and flat morphism of local rings. Then the induced maps on Spec is surjective.*

Proof. This is essentially the content of going down theorem. Every point of $\mathrm{Spec}(A)$ is a generalisation of the unique closed point. $\square$

Corollary 2.1.0.15. *Let $f : X \to Y$ be flat and proper morphism of finite presentation such that Y is irreducible. The f is surjective.*

2.2 Flatness and dimension of fibers

The following Proposition tells us that flat morphisms have well behaved fibers. This is mysterious (at least to me) given that flatness itself had a very algebraic definition.

[3] the set theoretic image

[4] For those who want to remain in the Noetherian world, anytime I say finite presentation you may assume that the schemes are Noetherian and that the morphism is of finite type.

Proposition 2.2.0.1. *Let $f : X \to Y$ be a flat morphism of locally Noetherian*[5] *schemes. Then for any point $x \in X$ we have,*

$$\dim(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{Y,f(x)}) + \dim(\mathscr{O}_{X_y,x}).$$

Proof. Since everything is local in x and y we may assume everything is sight is the Spectrum of a Noetherian ring. In which case the result follows from Tag 00ON. □

Example 2.2.0.2. This shows that the morphism (1) in the beginning of the chapter is not flat! The fiber over the origin is of dimension 1, while the fibers over other points are of dimension 0.

We derive one more corollary from Proposition 2.2.0.1.

Corollary 2.2.0.3. *Let $f : X \to Y$ be a flat morphism of schemes finite type over a field k with Y equidimensional*[6]*. Then TFAE*

1. *X is equidimensional of dimension equal to dim$(Y) + n$.*
2. *All fibers (not necessarily over closed points) of f are equidimensional of dimension n.*

In particular if both X and Y are irreducible then dim$(X) \geqslant$ dim(Y) and all the fibers are equidimensional of dimension dim(X) − dim(Y).

Proof. Suppose X if equidimensional of dimension dim$(Y) + n$. Let y be a *closed* point in Y with residue field $k(y)$. We would like to show that $X_y := X \times_{k(y)} Y$ is equidimensional of dimension n. Choose any irreducible component of X_y and in that component choose a closed point x in X_y. Note that x is closed in X (Why?). Then the dimension of X, X_y and Y can be computed using the dimension of the local rings at the points x and y. Thus we are done by Proposition 2.2.0.1.
Reduction the case y a closed point: Now suppose y is a possibly non closed point of Y. Then note that the map $\mathrm{Spec}(k(y)) \to Y$ factors via $Y \times_k k(y)$ and X_y can be considered as a fiber of the map induced between $X \times_k k(y) \to Y \times_k k(y)$ over the closed point $k(y)$ of $Y \times_k k(y)$. Note that both $X \times_k k(y)$ and $Y \times_k k(y)$ continue being equidimensional of dimension dim(X) and dim(Y) respectively (see Tag 00P4).
For the converse, choose a closed point $x \in X$, then $f(x) \in Y$ is a closed point (why?). Then again we are done by Proposition 2.2.0.1. □

But more is true! We have the following *miraculous* result, known colloquially as the *Miracle Flatness Theorem* due to Hironaka.

Theorem 2.2.0.4 (Miracle Flatness Theorem)**.** *Let $R \to S$ be a local morphism of Noetherian local rings. Assume that*

[5] We really need this to ensure dimensions are finite.
[6] Each irreducible component of Y has the same dimension.

1. *R is a regular local ring.*

2. *S is Cohen-Macaulay.*

3. *The dimension formula holds i.e,*

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S),$$

where $\mathfrak{m}$ is the maximal ideal of R.

Then $R \to S$ is flat!

This has the following very useful corollary.

Corollary 2.2.0.5 (Miracle Flatness Theorem for schemes). *Let $f : X \to Y$ be a morphism of locally Noetherian schemes such that X is Cohen-Macaulay and Y is regular. Then f is flat iff the dimension formula holds.*

Example 2.2.0.6. This immediately implies that the examples (2) and (3) in the beginning of the chapter are flat. The fibers are of constant dimension 0.

Chapter 3

Faithful Flatness

3.1 Faithfully flat morphisms

Let $\phi : A \to B$ be a flat morphism of rings. We say ϕ is *faithfully* flat if B is a faithfully flat A-module. Surprisingly faithful flatness can be captured set theoretically!

Lemma 3.1.0.1. *ϕ is faithfully flat iff it is flat and the induced map $\phi^{\#} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.*

Proof. Let $\mathfrak{p}$ be a prime in A, then the induced map $A \to k(\mathfrak{p})$ is non-zero iff $A \otimes_A B \to k(\mathfrak{p}) \otimes_A B$ is non-zero.The latter necessarily implies the fiber over $\mathfrak{p}$ is non-empty. Conversely suppose $\phi^{\#}$ is surjective. We shall prove that for any A-module M, $M \otimes_A B = 0$ iff $M = 0$, a well known criterion for faithful flatness. Let $m \in M$ different from zero inducing an injection

$$0 \longrightarrow \frac{A}{I} \longrightarrow M,$$

here I is the annhilator of $m \in M$. Tensoring the above exact sequence with the flat ring B and knowing that $B \otimes_A \frac{A}{I}$ is non-zero, thanks to surjectivity of $\phi^{\#}$, implies the required result. □

Combining Corollary 2.1.0.14 and Lemma 3.1.0.1 we obtain the following result.

Corollary 3.1.0.2. *Flat and local maps of local rings are faithfully flat.*

Motivated by Lemma 3.1.0.1 we have the following definition.

Definition 3.1.0.3. A morphism of schemes $f : X \to Y$ is said to be faithfully flat if it is flat and surjective.

Example 3.1.0.4. Now we give some examples of faithfully flat morphisms

1. Any extension of fields $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is faithfully flat.

2. Any proper and flat morphism whose target is an irreducible scheme is faithfully flat.

3. Let X be an affine scheme and let $X_{f_i}, 1 \leqslant i \leqslant n$ be a finite cover by basic affines, then

$$\sqcup_i X_{f_i} \to X,$$

   is faithfully flat.

4. Let X be the projective space $\mathbb{P}^n$ and let $D(x_i), 0 \leqslant i \leqslant n$ be the standard affine covering corresponding to a choice of homogeneous coordinates. Then

$$\sqcup_i D(x_i) \to \mathbb{P}^n,$$

   is faithfully flat.

We note the following obvious lemma.

Lemma 3.1.0.5. *Faithfull flatness is stable under base change and composition.*

3.2 Faithfully flat descent

Let X be any scheme and let $\{U_i\}_{1 \leqslant i \leqslant n}$ be an open cover of X. We have the following cartesian diagram

$$\begin{tikzcd} \sqcup_{i,j} U_i \cap U_j \arrow[r, "p_2"] \arrow[d, "p_1"] & \sqcup_i U_i \arrow[d, "f"] \\ \sqcup_j U_j \arrow[r, "f"] & X \end{tikzcd}.$$

Moreover for any schem T giving a morphism $X \to T$ is the same as giving a collection of morphisms $U_i \to T$ which agree on the intersections $U_i \cap U_j$. Put differently the following sequence of sets is exact

$$\mathsf{Hom}(X,T) \xrightarrow{f^*} \prod_i \mathsf{Hom}(U_i,T) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \prod_{i,j} \mathsf{Hom}(U_i \cap U_j, T).$$

There is nothing special about schemes here, one could have done the same starting with any toplogical space X and a cover $\{U_i\}_{1 \leqslant i \leqslant n}$. However doing so obscures the following important fact, the exactness of the above sequence is a consequence of faithfully flatness of f! This is the content of the following theorem.

Theorem 3.2.0.1 (Faithfully Flat descent). *Let X and Y be schemes over S. Let $S' \to S$ be a faithfully flat and quasi-compact morphism*[1]*. Let $S'' := S' \times_S S'$ and we denote by $X_{S'}$ (resp. $X_{S''}$) the base change of X along S' (resp. S''). We use a similar notation for Y. Then the following sequence of sets*

$$Hom_S(X,Y) \longrightarrow Hom_{S'}(X_{S'}, Y_{S'}) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} Hom_{S''}(X_{S''}, Y_{S''}),$$

is exact. Here p_1 and p_2 are induced by the projections $S'' \to S'$.

For a proof see Tag 023Q. Here is an application of faithfully flat descent. Let K/k be a finite Galois extension of field with Galois group G. Let X, Y be schemes over k. Let

$$X_K := X \times_k K, Y_K := Y \times_k K.$$

Every element $\sigma \in G$ acts on K while fixing k, thus inducing a morphism of $\mathrm{Spec}(K)$ as k-scheme. By functoriality of the fiber product we get an induced action of σ on $X_K := X \times_k K$ and $Y_K := Y \times_k K$. We denote this action by σ_X and σ_Y. Note that σ_X and σ_Y are *not* morphisms of K-schemes, rather they are only morphisms of k-schemes. Finally we get an action of G on $\mathrm{Hom}_K(X_K, Y_{K)})$ as follows:

$$f \to f^{sigma} := \sigma_Y \circ f \circ \sigma_X^{-1}. \tag{3.1}$$

Corollary 3.2.0.2 (Galois Descent). *The natural map $Hom_k(X,Y) \to Hom_K(X_K, Y_K)$ has image*

$$Hom_K(X_K, Y_K)^G,$$

i.e. precisely those morphisms that are invariant under G.

Proof. Lets start with some basic analysis. Since K/k is Galois we choose an $\alpha \in K$, such that $K = k(\alpha)$ as k-algebras. If $f(x)$ is the minimal polynomial of α, then we have

$$K \simeq \frac{k[x]}{(f(x))},$$

with $x \to \alpha$ under this isomorphism. Using the above isomorphism we identify

$$K \otimes_k K \simeq K \otimes_k \frac{k[x]}{(f(x))} \simeq \frac{K[x]}{(f(x))}.$$

Note that under the above isomorphism $\alpha \otimes 1 \to \alpha$ while $1 \otimes \alpha \to x$. Since K is the splitting field of $f(x)$, we can further identify

$$\psi : K \otimes_k K \simeq \prod_i \frac{K[x]}{(X - \alpha_i)} \simeq \prod_i K,$$

[1] Grothendieck coined the acronym *fpqc* (fidèlement plat et quasi-compact) for such morphisms.

where α_i's are the conjugates of α in K. Note that Ψ is a map of k-algebras and maps $\alpha \otimes 1 \to \alpha$ while $1 \otimes \alpha \to \alpha_i$ along the i^{th}-component. Put differently $1 \otimes \alpha \to \prod_{\sigma \in G} \sigma(\alpha)$. To summarize the diagram

$$K \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} K \otimes_k K$$

is isomorphic to the diagram

$$K \overset{\Delta}{\underset{\prod_{\sigma \in G} \sigma}{\rightrightarrows}} \prod_i K. \tag{3.2}$$

Now we can get back to proving the corollary. Consider the Cartesian diagram

$$\begin{tikzcd} X_K \times_X X_K \arrow[r, "p_2"] \arrow[d, "p_1"] & X_K \arrow[d, "f"] \\ X_K \arrow[r, "f"] & X \end{tikzcd}$$

The morphism f is fpqc and hence by Theorem 3.2.0.1 we have the exact sequence

$$\text{Hom}_k(X, Y) \xrightarrow{f^*} \text{Hom}_K(X_K, Y_K) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \text{Hom}_{K \otimes_k K}(X_{K \otimes_k K}, Y_{K \otimes_k K}).$$

Note that we have isomorphisms $X \times_k (K \otimes_K K) \simeq \sqcup_{\sigma \in G} X_K$ and $Y \times_k (K \otimes_K K) \simeq \sqcup_{\sigma \in G} Y_K$, where the first one comes from properties of fiber product and the last one is the isomorphism ψ above. Further under this identification we may identify p_1 with map which is identity on each of the factors, while p_2 is identified with the map which sends the factor X_K corresponding to σ by σ_X onto X_K. If we start with a morphism $f : X_K \to Y_K$, then it follows from the above isomorphisms that

$$p_1^*(f) = p_2^*(f) \implies f = f^\sigma, \forall \sigma \in G.$$

□

Here is a simple example to see this in action.

Example 3.2.0.3. Let $X = Y = \text{Spec}(\mathbb{R}[x])$. A morphism $f : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is given by $x \to p(x)$, for a complex polynomial $p(x)$. By our criterion this descends iff $\bar{p}(x) = p(x)$, here $\bar{p}(x)$ is the polynomial obtained by applying the uniqe non-trivial element of Gal$(\mathbb{C}/\mathbb{R})$ on the coeffcicients of $p(x)$. In other words $p(x)$ should be a polynomial with real coefficients.

Theorem 3.2.0.1 is the tip of the fpqc descent iceberg. Colloquially Theorem 3.2.0.1 is referred to by saying that morphisms descent along fpqc covers. Here $f : S' \to S$ is thought of as an "cover" of S. We have the following beautiful result.

Theorem 3.2.0.4. *The following properties of morphisms descend along a fpqc cover:*

1. *separatedness,*

2. *properness,*
3. *affineness,*
4. *open immersion,*
5. *closed immersion,*
6. *isomorphism,*
7. *finitness,*
8. *quasi-finiteness.*

For a proof see Tag 02YJ.

Example 3.2.0.5. Suppose $f : X \to Y$ is a morphism of varieties over the rational numbers $\mathbb{Q}$. Let us say you want to prove that f is an isomorphism. Theorem 3.2.0.4 implies that we can base change to $\mathbb{C}$ to prove this. In certain situations this can be quite profitable, for example one can use analytic techniques over $\mathbb{C}$ to prove this which apriori were not accesible over $\mathbb{Q}$.

Before we end this section I would like to state one more result which is a consequence of faithfully flat descent. Let us revisit Example 3.1.0.4 (4). This open covering was crucial in constructing quasi-coherent sheaves on projective space. Well it turns out that all we needed was that the covering was faithfully flat. This is the content of the following theorem.

Theorem 3.2.0.6. *Let $f : Y \to X$ be a fpqc morphism of schemes. Then there is an equivalence of categories between quasi-coherent sheaves on X and those quasi-coherent sheaves $\mathscr{F}$ on Y which satisfy gluing (or more appropriately descend) conditions:*

1. *There exists an isomorphism $\alpha : p_1^*\mathscr{F} \simeq p_2^*\mathscr{F}$ on $Y \times_X Y$.*
2. *α satisfies the cocycle condition on $Y \times_X Y \times_X Y$,*
$$p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha.$$
Here p_{ij} is the projection onto the i^{th} and j^{th} factors.

Moreover the equivalence above respects coherence, local freeness etc..For a proof we refer to Tag 023R

Chapter 4

Smoothness

Recall that a manifold is a topological space that is locally isomorphic to $\mathbb{R}^n$. What we would like is an analogous definition in Algebraic Geometry. Unfortunately a literal analogue would not work. For example, if X is a one-dimensional variety which is Zariski locally isomorphic to $\mathbb{A}^1$, then X is forced to be either $\mathbb{A}^1$ or $\mathbb{P}^1$ (Why?). Even more bizzare things can happen in Algebraic Geometry. Consider the map

$$\phi : \mathbb{A}^1_{\bar{\mathbb{F}}_p} \to \mathbb{A}^1_{\bar{\mathbb{F}}_p},$$

with $\phi(z) = z^p$. Note that every fiber of ϕ is non-reduced. In the language of manifolds every value is a critical value; something not possible in the world of manifolds thanks to Sard's theorem.
The theory of smoothness in Algebraic Geometry has to take into account both the geometric intuition coming from manifolds and the arithmetic complexities arising from various base fields.

4.1 Kähler differentials

Recall that for a smooth manifold X, the tangent vectors at a point x act by derivations on smooth functions around x. In particular if $\mathscr{O}_{X,x}$ is the local ring of smooth functions at x, then to every tangent vector v we can associate a derivation $D_v : \mathscr{O}_{X,x} \to \mathbb{R}$ which satisfies

$$D_v(fg) = fD_v(g) + gD_v(f), \tag{4.1}$$

for any two functions $f, g \in \mathscr{O}_{X,x}$.
Note in particular that Equation (4.1) implies that $D_v(\alpha) = 0, \forall \alpha \in \mathbb{R}$. This motivates the following definition.

Definition 4.1.0.1. Let B be an A-algebra and M a B-module. Then a A-derivation of B with values in M is an A-linear map $D : B \to M$ satisfying the Leibniz rule

$$D(fg) = fD(g) + gD(f), \forall f, g \in B.$$

We denote by $\mathrm{Der}_A(B, M)$ the set of A-derivations from B with values in M.

Remark 4.1.0.2. We note the following obvious properties:

1. For any A derivation D, $D(1.1) = D(1) + D(1) \implies D(1) = 0$. Since D is A-linear, this implies $D(a) = 0, \forall a \in A$.

2. For any $b \in B$ and an A-derivation D, $b.D(f) := bD(f)$ is also an A-derivation. Thus $\mathrm{Der}_A(B, M)$ is a B-module.

3. Let D be an A-derivation of B with values in M. Let $\phi : M \to M'$ be a B-module map. Then $\phi \circ D : B \to M'$ is an A-derivation with values in M'.

Now suppose $D : B \to M$ be any A-module map (derivation or not), then by universal property of tensor products, there exists an unique map of B-modules, $\tilde{D} : B \otimes_A B \to M$ such that $\tilde{D}(b \otimes b') = b'D(b)$. Here $B \otimes_A B$ is thought of as a B-module via the natural map $p_2^* : B \to B \otimes_A B$ given by $b' \to 1 \otimes b'$.
Let $I \subseteq B \otimes_A B$ be the kernel of the multiplication map $m : B \otimes_A B \to B$. We claim I is generated by $b \otimes 1 - 1 \otimes b$. To see this note that $\sum_i (b_i \otimes b_i')$ is in the kernel iff $\sum_i b_i b_i' = 0$. Hence $\sum_i b_i \otimes b_i' = \sum_i (b_i \otimes 1 - 1 \otimes b_i) b_i'$. We now have the following easy lemma.

Lemma 4.1.0.3. *If D in addition is assumed to satisfy Leibniz rule then $\tilde{D}(I^2) = 0$.*

Proof. We can check this on a set of generators of I^2 as a B-module namely elements of the form $(b \otimes 1 - 1 \otimes b)(b' \otimes 1 - 1 \otimes b')$, where this follows from Leibniz rule. □

Thus there exists an unique B-module map

$$\phi : \frac{I}{I^2} \to M,$$

such that $\phi(\bar{\alpha}) = \tilde{D}(\alpha)$, for any $\alpha \in I$ with image $\bar{\alpha} \in \frac{I}{I^2}$. Note here that the B-module structure on $\frac{I}{I^2}$ is the one induced from p_2^*. However it is easy to check that on $\frac{I}{I^2}$, the B-module structure induced by p_1^* is the same as the one induced by p_2^* and moreover there is a natural map

$$d_{B/A} : B \to \frac{I}{I^2},$$

defined by $b \to b \otimes 1 - 1 \otimes b$, which is a A-derivation. Thus we have shown the following.

Proposition 4.1.0.4. *For any A-algebra B there exists an unique B-module $\Omega^1_{B/A} := \frac{I}{I^2}$ together with an universal derivation $d_{B/A} : B \to \Omega^1_{B/A}$ such that for any B-module M*

$$\mathit{Der}_A(B, M) \simeq \mathit{Hom}_B(\frac{I}{I^2}, M).$$

Thank to the canonical nature of our construction it is clear how to globalize this.

Definition 4.1.0.5. Let $f : X \to Y$ be a morphism of schemes. Then the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is an immersion i.e. there exists an open $U \subset X \times_Y X$ such that $X \subseteq U$ is a closed immersion defined by an ideal $\mathscr{I}$. We define the sheaf of relative Kähler differentials of X/Y as $\frac{\mathscr{I}}{\mathscr{I}^2}$[1].

Note that by construction $\Omega^1_{X/Y}$ is a quasi-coherent sheaf on X. Moreover if we assume that Y is Noetherian and f is of finite type, $X \times_Y X$ is Noetherian and hence so is U and thus the ideal sheaf $\mathscr{I}$ is coherent implying the coherence of $\frac{\mathscr{I}}{\mathscr{I}^2}$. It follows from the construction of $\Omega^1_{X/Y}$ that there is $f^{-1}\mathscr{O}_Y$-linear map

$$d_{X/Y} : \mathscr{O}_X \to \Omega^1_{X/Y},$$

which on local sections is defined by $d_{X/Y}(f) = f \otimes 1 - 1 \otimes f$, and is universal for $f^{-1}\mathscr{O}_Y$-linear derivations from $\mathscr{O}_X \to \mathscr{F}$, here $\mathscr{F}$ is any quasi-coherent $\mathscr{O}_X$-module. Here are some basic properties of Kähler differentials.

Proposition 4.1.0.6. *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

1. *There is a natural morphism of $\mathscr{O}_{X'}$-modules, $g'^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$.*
2. *If $Y' = Y$ and g is the identity map. Then there is an exact sequence of sheaves on X*

$$g'^*\Omega^1_{X/Y} \longrightarrow \Omega^1_{X'/Y} \longrightarrow \Omega^1_{X'/X} \longrightarrow 0$$

3. *If the above diagram is Cartesian then the morphism in (1) induces as isomorphism $g'^*\Omega^1_{X/Y} \simeq \Omega^1_{X'/Y'}$ and $\Omega^1_{X'/Y} \simeq f'^*\Omega^1_{Y'/Y} \oplus g'^*\Omega^1_{X/Y}$.*

Proof. For a proof see Section 00RM ∎

[1] Easy check, this is independent of choice of U

We also have the following important result.

Proposition 4.1.0.7. *Let $f : X \to Y$ be a morphism of schemes. Let Z be a closed subscheme of X. Then*

1. $\Omega^1_{Z/X} \simeq 0$.

2. *The right exact sequence from Proposition 4.1.0.6, (3) can be extended to*

$$\mathscr{I}_Z/\mathscr{I}_Z^2 \xrightarrow{\delta} \Omega^1_{X/Y}|_Z \longrightarrow \Omega^1_{Z/Y} \longrightarrow \Omega^1_{Z/X} = 0,$$

where the map δ is induced by restricting $d_{X/Y} : \mathscr{O}_X \to \Omega^1_{X/Y}$ to $\mathscr{I}_Z$.

Proof. For a proof see Section 00RM

□

4.1.1 Computing Kähler differentials

In this section we shall comput the sheaf of Kähler differentials in some important cases. Before we start let us make some remarks

Remark 4.1.1.1.
1. We have already seen closed immersions have vanishing relative Kahler differentials. A similar argument also works for open immersions.

2. Let $X := X_1 \sqcup X_2$, then $\Omega^1_{X/Y} \simeq \Omega^1_{X_1/Y} \sqcup \Omega^1_{X_2/Y}$. This follows easily from the universal property or the definition of the sheaf of relative differenrtials.

3. Let B be a directed colimit of A-algebras. Then $\Omega^1_{B/A}$ is colimit of the corresponding Ω^1's. Again this can be checked using the universal property. In particular Ω^1 commutes with localization.

Lemma 4.1.1.2. *Let $X = \operatorname{Spec}(K)$ and $Y = \operatorname{Spec}(k)$ where K/k is a finite separable extension of fields. Then $\Omega^1_{X/Y} \simeq 0$.*

Proof. Let $\bar{k}$ be an algebraic closure of k and let $Y' = \operatorname{Spec}(\bar{k})$. Using Proposition 4.1.0.6, (2) and fpqc descent enough to show that $\Omega^1_{X'/Y'} = 0$ where X' is the base change of X along Y. Since K/k is a finite separable extension, we are done by Remark 4.1.1.1, (2) above. □

Corollary 4.1.1.3. *Using Remark 4.1.1.1, (3) it follows that $\Omega^1_{K/k} = 0$, for any separable and algebraic extension K/k.*

Lemma 4.1.1.4. *Let $(B, \mathfrak{m}, k)$ be a local ring containing a copy of k. Then the natural map δ induced from Proposition 4.1.0.7, (2)*

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \to \Omega^1_{B/k} \otimes_B k,$$

is an isomorphism.

Proof. Easy exercise. □

This immediately implies the following corollary.

Corollary 4.1.1.5. *Let X/k be a scheme and $i : \mathrm{Spec}(k) \to X$ be a closed point (denoted by x) and let $\mathfrak{m}_x$ be the maximal ideal of the local ring at the point x. Then the map $\delta : \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \to i_x^*\Omega^1_{X/k}$ is an isomorphism.*

In particular we have the following isomorphism

$$\mathrm{Hom}_k(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}, k) \simeq \mathrm{Hom}_k(\Omega^1_{X/k}, k) \simeq \mathrm{Der}_k(\mathscr{O}_{X,x}, k).$$

This motivates the following definition.

Definition 4.1.1.6 (Zariski Tangent Space)**.** Let X be a scheme and let $x \in X$ be a point with residue field $k(x)$. We define the Zariski tangent space to X at x to be $\mathrm{Hom}_{k(x)}(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}, k(x))$.

We can now combine Lemma 4.1.1.2 and Corollary 4.1.1.5 to obtain the following.

Corollary 4.1.1.7. *Let $X/\mathrm{Spec}(k)$ be finite. Then $\Omega^1_{X/k} \simeq 0$ iff $X \simeq \sqcup \mathrm{Spec}(K_i)$, where K_i/k are finite separable extensions of fields iff X is geometrically reduced.*

Proof. Clearly X/k is geometrically reduced iff X is a finite disjoint union of $\mathrm{Spec}(K_i)$'s with K_i/k finite and separable.
Suppose X/k is geometrically reduced. Then since X/k is finite, $X_{\bar{k}}/\bar{k}$ is a finite reduced scheme. Thus $X_{\bar{k}}$ is a finite disjoint union of $\mathrm{Spec}(\bar{k})$ which in turn implies that $\Omega^1_{X_{\bar{k}}/\bar{k}}$ vanishes and hence $\Omega^1_{X/k}$ vanishes too. Conversely if $\Omega^1_{X/k}$ vanishes then so does $\Omega^1_{X_{\bar{k}}/\bar{k}}$. Thus implies every connected component of $X_{\bar{k}}$ (a spectrum of an Artin local ring with residue field $\bar{k}$) must have maximal ideal 0, thanks to Lemma 4.1.1.5. □

Lemma 4.1.1.8. *Let X be any scheme and $\mathbb{A}^n_X$ be an affine space over X. Then $\Omega^1_{\mathbb{A}^n_X/X} \simeq \oplus_i \mathscr{O}_{\mathbb{A}^n_X} dx_i$. In particular $\Omega^1_{\mathbb{A}^n_X/X}$ is locally free of rank n.*

Proof. Using Proposition 4.1.0.6, (4) we are reduced to the case $n = 1$ and further we may assume $X = \mathrm{Spec}(A)$. In this case the result is obvious using universal property of Kähler differentials. □

We now compute the sheaf of Kähler differentials for projective space.

Proposition 4.1.1.9. *Let* $Y = \mathrm{Spec}(A)$ *and* $X = \mathbb{P}^n_A$. *Then there is an exact sequence of sheaves*[2] *on* X,

$$0 \longrightarrow \Omega^1_{X/Y} \longrightarrow \mathscr{O}_X(-1)^{\oplus(n+1)} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

Remark 4.1.1.10. We already know thanks to Lemma 4.1.1.8 that $\Omega^1_{X/Y}$ is locally free of rank n.

Proof. Consider the sheaf $\mathscr{O}_X(1)$, we know that this is globally generated by its sections, and thus we have a surjection of sheaves

$$\psi : H^0(X, \mathscr{O}(1)_X) \otimes_A \mathscr{O}_X \twoheadrightarrow \mathscr{O}_X(1).$$

We claim:

1. There exists a natural injection

$$\phi : \Omega^1_{X/Y}(1) \to H^0(X, \mathscr{O}(1)_X) \otimes_A \mathscr{O}_X, \tag{4.2}$$

2. with $\mathrm{Im}(\phi) = \ker(\psi)$.

This would give the Euler sequence (upto a twist).
We would like to think of $\mathbb{P}^n_A$ as obtained by gluing $n+1$-copies of $\mathbb{A}^n_A$ denoted by

$$D(x_i) := \mathrm{Spec}(A[\frac{x_0}{x_i}, \frac{x_1}{x_i} \cdots \frac{x_n}{x_i}]).$$

together with the gluing data

$$\theta_{ij} : \mathrm{Spec}(A[\frac{x_0}{x_i}, \frac{x_1}{x_i} \cdots \frac{x_n}{x_i}]_{\frac{x_j}{x_i}}) \simeq \mathrm{Spec}(A[\frac{x_0}{x_j}, \frac{x_1}{x_j} \cdots \frac{x_n}{x_j}]_{\frac{x_i}{x_j}}),$$

given by an A-algebra isomorphism $\theta^*_{ij}(\frac{x_k}{x_j}) = \frac{x_k}{x_j}$. We fix once and for all a basis $e_i, 0 \leqslant i \leqslant n$ for $H^0(X, \mathscr{O}_X(1))$ as an A-module. Restricted to each $D(x_i)$, the morphism Ψ is given by

$$\psi|_{D(x_i)}(e_k \otimes 1) = \frac{x_k}{x_i}, \forall k \neq i \tag{4.3}$$

for $k = i$,

$$\psi|_{D(x_i)}(e_i \otimes 1) = 1.$$

Moreover giving a map ϕ as in (4.2), amounts to giving for each i maps

[2] called the Euler sequence

$$\phi_i : \Omega^1_{D(x_i)/Y} \to H^0(X, \mathscr{O}_X(1)) \otimes_A \mathscr{O}_{D(x_i)},$$

such that

$$\phi_j \circ \frac{x_j}{x_i}\theta_{ij} = \theta_{ij} \circ \phi_i \tag{4.4}$$

on $D(X_i) \cap D(X_j)$, where we have used θ_{ij} to denote the induced map on both Ω^1 and $\mathscr{O}$ and the $\frac{x_j}{x_i}$ factor accounts for the twist by $\mathscr{O}_X(1)$.
We fix once and for all a basis $e_i, 0 \leqslant i \leqslant n$ for $H^0(X, \mathscr{O}_X(1))$ as an A-module. Thanks to Lemma 4.1.1.8, we know how $\Omega^1_{D(x_i)/A}$ looks like and we define

$$\phi_i(d(\frac{x_k}{x_i})) := (e_k \otimes x_i - e_i \otimes x_k)\frac{1}{x_i}. \tag{4.5}$$

It follows from (4.3) that ker$(\psi|_{D(x_i)})$ = Im(ϕ_i). Thus we are only left to check the gluing condition for ϕ_i as in equation (4.4). This follows from the identity

$$d(\frac{x_k}{x_i}) - \frac{x_k}{x_j}d(\frac{x_j}{x_i}) = \frac{x_j}{x_i}d(\frac{x_k}{x_j}),$$

on $\mathrm{Spec}(A[\frac{x_0}{x_i}, \frac{x_1}{x_i} \cdots \frac{x_n}{x_i}]_{\frac{x_j}{x_i}})$.

□

4.2 Smoothness

Recall that a smooth manifold X is essentially a topological space with local charts $\{U_i\}$, which are in turn isomorphic to $\mathbb{R}^n$. Unfortunately this model is not good enough to model smoothness in algebraic geometry. For example, if X is a one-dimensional normal variety over $\mathbb{C}$ with an open subset isomorphic to $\mathbb{A}^1$, then in fact X is either $\mathbb{A}^1$ or $\mathbb{P}^1$! So clearly this approach to smoothness is very rigid and needs to modified to account for the so called curves of higher genus. As it turns out even zero dimensional smooth varieties are quite interesting and studying them helps us get to the *correct* definition of smoothness. Before we proceed further let us write down a list of properties we want out of smoothness:

1. We would like to define smoothness in a relative set-up $f : X \to Y$.

2. We would like smooth morphisms to be stable under base change and composition. In particular fibers of smooth morphisms should be smooth schemes over a field.

3. We would like (relative) affine and projective spaces to be smooth.

4. Finally for varieties over an algebraically closed field, one should be able to detect smoothness by the size of its Zariski tangent space (see Definition 4.1.1.6).

Remark 4.2.0.1. Through out this section you may assume either that we are working with Noetherian schemes and finite type morphisms or with arbitrary schemes and morphisms of finite presentation. In particular all relative sheaves of differentials will be coherent sheaves. With a little more effort one can set things up for arbitrary morphisms allowing us to talk about smoothness of say $\mathbb{C}/\mathbb{Q}$!

4.2.1 Étale morphisms

We begin with the definition of étale morphisms.

Definition 4.2.1.1 (étale morphisms). Let $f : X \to Y$ be a morphism. We say f is étale at $x \in X$ if it is flat at x and if the stalk of $\Omega^1_{X/Y}$ vanishes at x. We say f is étale if it is so at every point of X

Remark 4.2.1.2.

1. It immediately follows from Proposition 2.1.0.10, (c) and Proposition 4.1.0.6, (c) that class of étale morphisms is stable under Base Change. Using Proposition 4.1.0.6, (b) it also follows that étale morphisms are stable under composition.

2. Note that by Definition 4.1.0.5 it follows that the immersion $\Delta_{X/Y} : X \to X \times_Y X$ is an *open immersion* when X/Y is étale.

3. Let $f : X \to Y$ be étale at $x \in X$. Since flatness and vanishing of $\Omega^1_{X/Y}$ are both open conditions, so is being étale. Moreover étale morphisms being flat necessarily have an open image.

Let us note down some examples of étale morphisms.

Example 4.2.1.3. Let K/k be a finite separable extension of fields. Then $\mathrm{Spec}(K)/\mathrm{Spec}(k)$ is an étale morphism by Lemma 4.1.1.2. More generally $X = \sqcup_i^n \mathrm{Spec}(K_i)$[3] is étale over $\mathrm{Spec}(k)$ where each K_i/k is a finite separable extension. In Problem Set 3 you will show that $X/\mathrm{Spec}(k)$ a finite morphism is étale iff X is of the above form.

Example 4.2.1.4. Let $j : U \hookrightarrow X$ be an open immersion. Then j is étale.

Here we note down some basic properties of étale morphisms.

Proposition 4.2.1.5. *Let $f : X \to Y$ be an étale morphism of schemes over S. Then the following are true.*

1. *The fibers of f are spectrums of étale algebras. In particular f is quasi-finite.*

2. *The natural map $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism.*

[3] The ring of functions on such an X are called étale algebras over k.

Proof. Since base change of étale morphisms is étale, the fibers of f over any point $y \in Y$ are étale over $\mathrm{Spec}(k(y))$. Quasi-finiteness now follows from Example 4.2.1.3.
For (2), one can use the definition of Ω^1 and that fact that $X \hookrightarrow X \times_Y X$ is an open immersion to conclude the same.

□

We have an converse to Proposition 4.2.1.5

Proposition 4.2.1.6. *Let $f : X \to Y$ be a morphism. Then f is étale iff f is flat and all the fibers are spectrums of étale algebras iff all the geometric fibers are reduced and 0-dimensional.*

Proof. We have already seen that spectrums of étale algebras are geometrically reduced. The converse is easy. So we shall prove that f is étale iff the geometric fibers are reduced and 0-dimensional. $\implies$ direction is clear. For the other direction, we have to show that $\Omega^1_{X/Y}$ vanishes. Since $\Omega^1_{X/Y}$ is a coherent sheaf it suffices to show that for any point $x \in X$, the $k(x)$-vector space $\Omega^1_{X/Y} \otimes k(x)$ vanishes. This follows from Proposition 4.1.0.6, (3) and Example 4.2.1.3 □

Remark 4.2.1.7. Here is another criterion for étaleness which follows from Proposition 4.1.0.6, (2): f is flat and the natural map $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism.

Thanks to Proposition 4.2.1.6 we can now generate a lot of examples of étale morphisms.

Example 4.2.1.8. Let $\psi : \mathbb{C}^* \to \mathbb{C}^*$ be the squaring map $z \to z^2$. We have seen that this is flat. Moreover the fiber over any point is reduced and thus ψ is étale. However the extension if ψ to all of $\mathbb{C}$ is not étale over the origin.

4.2.2 Smooth Morphisms

We are now ready to define smooth morphisms. Again recall that for us either all schemes are Noetherian and morphisms are of finite type or we work in the finite presentation scenario. Our definition of smoothness differs from that of Hartshorne but is closer in spirit to differential geometry.

Definition 4.2.2.1 (Smooth Morphisms). Let $f : X \to Y$ be a morphism. We say f is smooth at $x \in X$ if there exists an open $U \ni x$ and a morphism $g : U \to \mathbb{A}^n_Y$[4] (for some $n \geqslant 0$), which is étale at x such that following diagram commutes

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{A}^n_Y \arrow[d] \\ & Y \end{tikzcd},$$

here the vertical arrow is the projection map.

[4]One can think of g as giving a local choice of coordinates around the point x.

Remark 4.2.2.2. We remark on same basic properties of smooth morphisms which follow immediately from the definition of smoothness:

1. Smooth morphisms are flat and in particular have open image.
2. Smoothness is an open condition since being étale is (see Remark 4.2.1.2).
3. Base change of a smooth morphism is smooth by stability of étale morphisms under base change.
4. Let $f : X \to Y$ be smooth at x and $f' : Y \to S$ be smooth at $f(x)$. Suppose $g : U \to \mathbb{A}^n_Y$ is a local choice of coordinates around x, and $h : V \to \mathbb{A}^m_S$ is a local choice of coordinates around $f(x) \in Y$. Then $g \times f : U \cap f^{-1}(V) \to \mathbb{A}^{m+n}_S$ give a local choice of coordinates around x for $f' \circ f$.

Before we discuss properties of smooth morphisms let us note down some examples.

Example 4.2.2.3.
1. For any scheme S, $\mathbb{A}^n_S \to S$ is smooth.
2. Open immersions, and more generally étale morphisms are smooth.
3. Smoothness is local in both the source and base. Hence (1), above implies $\mathbb{P}^n_S \to S$ is smooth.

Here is an easy lemma.

Lemma 4.2.2.4. *Let $f : X \to Y$ be a morphism smooth at $x \in X$. Then $\Omega^1_{X/Y}$ is locally free around x. In particular if $f : X \to Y$ is smooth, then $\Omega^1_{X/Y}$ is locally free aka a vector bundle on X.*

Proof. Combine Lemma 4.1.1.8 and Proposition 4.2.1.5. □

Notations 4.2.2.5. Let $f : X \to Y$ be a smooth morphism. The rank of f at a point x is the rank of the locally free sheaf $\Omega^1_{X/Y}$ at x. This is a locally constant function on X.

Following lemma is an easy consequence of quasi-finiteness of étale morphisms.

Lemma 4.2.2.6. *Let $f : X \to Y$ be a smooth morphism. Then for any closed point $x \in X$*

$$\dim_x(X_{f(x)}) = \dim_{k(x)}(\Omega^1_{X/Y,k(x)}).$$

Proof. Choose a coordinate neighborhood $U \ni x$ with an étale map $g : U \to \mathbb{A}^n_Y$. Since g is quasi-finite (see Proposition 4.2.1.5) and flat, the induced map $U \cap X_{f(y)} \to \mathbb{A}^n_{k(f(y))}$ is quasi-finite and flat. It follows from Corollary 2.2.0.3, that each component of $U \cap X_{f(y)}$ has dimension n which equals $\dim_{k(x)}(\Omega^1_{X/Y,k(x)}$. □

We have the following easy corollary.

Corollary 4.2.2.7. *Let X/k be a smooth equi-dimensional scheme of dimension n. Then $\Omega^1_{X/k}$ is locally free of rank n on X.*

The fundamental exact sequences for the Kähler differentials take a particularly nice form for smooth morphisms. We have the following theorem.

Theorem 4.2.2.8. *Let $f : X \to Y$ be a morphism of schemes over S. Then if f is smooth then the right exact sequence (see Proposition 4.1.0.6, (2))*

$$0 \longrightarrow f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \tag{4.6}$$

is also exact on the left and is locally (on X split).

Proof. Let us prove (1) first. First note that if X/Y is smooth, then $\Omega^1_{X/Y}$ is a locally free coherent sheaf and hence the (apriori) right exact sequence necessarily splits on the right. To show exactness we claim it suffices to prove it for $X = \mathbb{A}^1_Y \to Y$, where it follows from an easy computation. First note that exactness can be checked locally on X, hence we may assume that $f : X \to Y$ factors via an étale map $g : X \to \mathbb{A}^n_Y$, followed by the projection to Y. Suppose we managed to show that

$$0 \longrightarrow f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{\mathbb{A}^n_Y/S} \longrightarrow \Omega^1_{\mathbb{A}^n_Y/Y} \longrightarrow 0\ , \tag{4.7}$$

is exact and locally split. Then applying g^* to the above exact sequence preserves exactness (why?) we obtain the exact sequence (4.6) thanks to Proposition 4.2.1.5, (2). Finally note that the projection $f : \mathbb{A}^n_Y \to Y$ factors as $g : \mathbb{A}^n_Y \to \mathbb{A}^{n-1}_Y$, where the latter projects onto Y (say via h). Suppose we have managed to show the exactness of (4.7) for affine spaces of rank upto $n-1$ (over arbitrary S). Then we have a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & f^*\Omega^1_{Y/S} \arrow[r] \arrow[d, "="] & g^*\Omega^1_{\mathbb{A}^{n-1}_Y/S} \arrow[r] \arrow[d] & g^*\Omega^1_{\mathbb{A}^{n-1}_Y/Y} \arrow[r] \arrow[d, "h"] & 0 \\
 & f^*\Omega^1_{Y/S} \arrow[r] \arrow[d] & \Omega^1_{\mathbb{A}^n_Y/S} \arrow[r] \arrow[d] & \Omega^1_{\mathbb{A}^n_Y/Y} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, "="] & 0 \arrow[r] & \Omega^1_{\mathbb{A}^n_Y/\mathbb{A}^{n-1}_Y} \arrow[r, "="] \arrow[d] & \Omega^1_{\mathbb{A}^n_Y/\mathbb{A}^{n-1}_Y} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \ .$$

The left exactness at the middle row is now clear. This proves (1). □

In a similar vein we can also strengthen the right exact sequence Proposition 4.1.0.7.

Proposition 4.2.2.9. *Let $f : X \to Y$ be a morphism of schemes. Let Z be a closed subscheme of X. Then the right exact sequence*

$$\mathscr{I}_Z/\mathscr{I}_Z^2 \xrightarrow{\delta} \Omega^1_{X/Y}|_Z \longrightarrow \Omega^1_{Z/Y} \longrightarrow 0, \tag{4.8}$$

is exact and locally split if Z/Y is smooth[5]

Proof. For a proof of the first part we refer to Tag 06A8. Note that locally split follows from the fact that under the assumptions $\Omega^1_{Z/Y}$ is a locally free coherent sheaf on Z.

□

One can do better if one assumes X/Y is smooth. In fact in that case one has the following *intuitive* characterization of sub-schemes smooth over Y.

Theorem 4.2.2.10. *Let $f : X \to Y$ be a smooth morphism and let Z be a closed sub scheme of X. Then TFAE*

1. *Z/Y is smooth.*

2. *The right exact sequence*

$$\mathscr{I}_Z/\mathscr{I}_Z^2 \xrightarrow{\delta} \Omega^1_{X/Y}|_Z \longrightarrow \Omega^1_{Z/Y} \longrightarrow 0, \tag{4.9}$$

   is exact and locally split.

3. *For any point $z \in Z$, there exists an open $U \hookrightarrow X$ containing x and an étale map $g : U \to \mathbb{A}^n_Y$ and a Cartesian diagram*

$$\begin{tikzcd} U \cap Z \arrow[r] \arrow[d, "g'"] & U \arrow[d, "g"] \\ \mathbb{A}^r_Y \simeq Z(t_1, t_2 \cdots t_{n-r}) \arrow[r] & \mathbb{A}^n_Y = \operatorname{Spec}(\mathscr{O}_Y[t_1, t_2 \cdots t_n]). \end{tikzcd}$$

Proof. For a proof we refer to [1, Exposé II, Théorème 4.10]. The case when $Y = \operatorname{Spec}(k)$ is handled in [2, Chapter II, Theorem 8.17] □

Intuitively Theorem 4.2.2.10 tells us that just as étale locally smooth schemes are like affine spaces, similarly smooth subschemes are like linear subspaces of affine spaces.

[5]We do not need X/Y to be smooth!

Conormal exact sequence when $Y = \mathrm{Spec}(k)$

Suppose $Y = \mathrm{Spec}(k)$ in Theorem 4.2.2.10 and lets assume both Z and X are smooth varieties over $\mathrm{Spec}(k)$. Then in that case we have a short exact sequence of vector bundles on Z

$$0 \longrightarrow \mathscr{I}_Z/\mathscr{I}_Z^2 \xrightarrow{\delta} \Omega^1_{X/k}|_Z \longrightarrow \Omega^1_{Z/k} \longrightarrow 0. \tag{4.10}$$

Dualizing this and recalling that the dual of Ω^1 is the tangent space gives us a familiar exact sequence from differential geometry

$$0 \longrightarrow \mathrm{T}^*Z \longrightarrow \mathrm{T}^*X|_Z \longrightarrow N_{Z/X} \longrightarrow 0. \tag{4.11}$$

Here $N_{Z/X}$ is the normal bundle of Z inside X. Thus it makes sense to call $\mathscr{I}_Z/\mathscr{I}_Z^2$ the *conormal sheaf* of Z in X (even when Z and X are possibly non-smooth). The corresponding exact sequence is called the *conormal exact sequence*.
Finally combining Corollary 4.2.2.7 and Theorem 4.2.2.10 shows us that the conormal sheaf is a vector bundle of rank equal to the codimension of Z in X and that Z is locally cut out by its codimension-many equations.
We end this section with the familiar Jacobian criterion for smoothness which is a corollary to Theorem. 4.2.2.10.

Corollary 4.2.2.11 (Jacobian criterion: Smooth form). *Let Z be a closed sub scheme of $\mathbb{A}^n_k$. Then Z is smooth over k at a point $z \in Z$ iff there exists an open $U \hookrightarrow \mathbb{A}^n_k$ containing z such that $Z \cap U$ is defined by the vanishing $f_1, f_2 \cdots f_r \in \mathscr{O}(U)$ satisfying the Jacobian criterion i.e.*

$$rk_{k(z)}(\{\frac{\partial f_i}{\partial x_j}\}_{i,j}) = r.$$

This form of the Jacobian criterion is well adapted to check for smoothness of subvarieties of $\mathbb{A}^n_k$. We also have a form which can be used to check for singularities.

Corollary 4.2.2.12 (Jacobian criterion: Singular form). *Let Z be a closed sub scheme of $\mathbb{A}^n_k$. Then Z is singular over k at a point $z \in Z$ iff there exists an open $U \hookrightarrow \mathbb{A}^n_k$ containing z such that $Z \cap U$ is defined by the vanishing $f_1, f_2 \cdots f_r \in \mathscr{O}(U)$ such that the images of f_i form a basis for $\frac{\mathscr{I}}{\mathscr{I}^2} \otimes k(z)$[6] but*

$$rk_{k(z)}(\{\frac{\partial f_i}{\partial x_j}\}_{i,j}) < r.$$

Let us see this in action.

[6] Note that if $r = 1$, then this is automatically satisfied for $f_1 \neq 0$.

Example 4.2.2.13. 1. Let $Z := Z(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2_k$ be the nodal curve. Then Z is globally defined by $f(x,y) = y^2 - x^2(x+1)$. Its Jacobian matrix is given by $[3x^2+2x \ \ 2y]$. Thus a point $(x,y) \in Z$ is singular (i.e. not smooth) iff $f(x,y) = 2y = 3x^2 + 2x = 0$. Clearly this only happens when $x = y = 0$. The unique nodal singularity of Z.

2. Consider the Fermat cubic $Z := Z(x^3 + y^3 + z^3 + w^3) \subseteq \mathbb{P}^3_k$. Then on each affine chart Z is given by vanishing of $f(x,y,z) := 1 + x^3 + y^3 + z^3$. The Jacobian matrix of f is given by $[3x \ \ 3y \ \ 3z]$. Thus Z is smooth iff it is smooth on each affine chart iff there are no common solutions to $f(x,y,z) = 3x = 3y = 3z = 0$. Thus Z is smooth away from char$(k) = 3$. But in char$(k) = 3$ every point is a singular point!

4.2.3 More computations with Kähler differentials

In this section we shall use the results from Sections 4.1.1 and 4.2.2 to compute some examples. Before we do so we need a defintion.

Definition 4.2.3.1 (Canonical Sheaf). Let $f : X \to Y$ be a smooth morphism of relative dimension n. We define the *relative canonical sheaf* $\omega_{X/Y} := \bigwedge^n \Omega^1_{X/Y}$. Thus $\omega_{X/Y}$ is a line bundle on X.

Example 4.2.3.2. 1. Let $X = \mathbb{A}^n_A$ and $Y = \operatorname{Spec}(A)$. Then it follows from Lemma 4.1.1.8 that $\omega_{X/Y} \simeq \mathscr{O}_X dx_1 \wedge dx_2 \cdots dx_n$.

2. If $X = \mathbb{P}^n_A$ and $Y = \operatorname{Spec}(A)$. Then it follows from the Euler exact sequence (Proposition 4.1.1.9) that $\omega_{\mathbb{P}^n_A/A} \simeq \mathscr{O}(-n-1)_{\mathbb{P}^n_A}$. In particular when $n = 1$, $\Omega^1_{\mathbb{P}^1_A/A} = \omega_{\mathbb{P}^1_A/A} = \mathscr{O}(-2)_{\mathbb{P}^1_A}$.

3. Let X and Y be smooth varieties over a field k. Then Proposition 4.1.0.6, (3) and [2, Chapter II, Ex. 5.16d] imply that $\omega_{X \times_k Y} \simeq p_X^* \omega_{X/k} \otimes p_Y^* \omega_{Y/k}$.

Here is an easy consequence of Theorem 4.2.2.10.

Proposition 4.2.3.3. *Let $Z \subseteq X$ be a smooth subvariety of a smooth variety X/k. Then*

$$\omega_X|Z = \omega_Z \otimes \bigwedge^r \mathscr{I}_Z/\mathscr{I}_Z^2,$$

here r is the codimension of Z in X. In particular if Z is given by the zero section of a line bundle $\mathscr{L}$ (and hence a divisor on X). Then

$$\omega_Z = (\omega_X \otimes \mathscr{L})|_Z$$

Proof. The first formula is an immediate consequence of Equation (4.10) and [2, Chapter II, Ex. 5.16d]. For the second one we simply observe that $\mathscr{I}_Z \simeq \mathscr{L}^{-1}$. $\square$

Example 4.2.3.4. Let $X \subseteq \mathbb{P}^n_k$ be a smooth hypersurface of degree d. Then $\omega_{X/k} = \mathscr{O}_X(-n-1+d)$. In particular $\omega_{X/k}$ is ample iff $d \geqslant n+2$

More generally for any smooth variety X/k of dimension n we have:

1. locally free sheaves $\Omega^i_{X/k} := \bigwedge^i \Omega^1_{X/k}$ of rank $n-i$.
2. The *de Rham complex*:

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega^1_{X/k} \qquad \cdots \xrightarrow{d} \Omega^n_{X/k} \longrightarrow 0$$

   Here the differentials $d : \Omega^i_{X/k} \to \Omega^{i+1}_{X/k}$ satisfy the usual Leibniz rule and when $i = 0$ correspond the universal differential from $\mathscr{O}_X \to \Omega^1_{X/k}$.

4.2.4 Regularity and Smoothness

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. Recall that A is said to be *regular* if any of the following equivalent conditions are satisfied:

1. $\dim_k(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = \dim(A)$.
2. $\mathfrak{m}$ is generated by d elements, where $d = \dim(A)$.

We need the following basic results about regular local rings.

Proposition 4.2.4.1. *Let A be a regular local ring as above. Then*

1. $\bigoplus_{n\geqslant 0} \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \simeq k[t_1, t_2 \cdots t_n]$.
2. *A collection of elements $(x_1, x_2, \cdots x_d)$ generate $\mathfrak{m}$ iff they form a regular system of parameters i.e. x_i is a non zero-divisor in $A/(x_1, x_2, \cdots x_{i-1})$*
3. *Let $I \subseteq A$ be an ideal. The ring $B = A/I$ is regular local iff $I = (x_1, x_2 \cdots x_r)$ with $(x_i)_{1\leqslant i\leqslant r}$ part of a regular system of parameters for A.*
4. *$A_{\mathfrak{p}}$ is also regular local for any prime ideal $\mathfrak{p}$.*

Proof. These are shown in Tag 00NO, Tag 00NQ, Tag 00NR and Tag 0AFS. □

Remark 4.2.4.2. We note the following about regular local rings.

1. Dimension 0 regular[7] local rings are precisely fields and dimension 1 regular local rings are dvr's.

[7] Henceforth anytime we mention regularity we shall always be in the Noetherian setting.

2. Thanks to Proposition 4.2.4.1, (1) implies that A is domain (see Tag 00NP).

Thanks to Proposition 4.2.4.1, (4) it makes sense to have the following definition.

Definition 4.2.4.3. Let X be a locally Noetherian scheme. We say X is regular iff for any point $x \in X$, the local ring $\mathscr{O}_{X,x}$ is regular[8].

Here are a couple of simple corollaries to Remark 4.2.4.2.

Corollary 4.2.4.4. *Let X be a regular scheme. Then*

1. *X is normal.*
2. *In addition if X is Noetherian then every irreducible component of X is open in it and hence also a connected component of X.*

Proof. Normality follows from Serre's criterion of Normality[9] as in Tag 0567.
For the latter we simply note that every local ring of X is an integral domain and thus has an unique minimal prime ideal. This in particular implies that every point lies in an unique irreducible component (else the local ring at that point would have at least two minimal prime ideals). □

The key result relating smoothness and regularity is the following.

Theorem 4.2.4.5. *Let X/k be a scheme of finite type. Then*

1. *X/k smooth implies X is a regular scheme. In particular every irreducible component of X is also a connected component.*
2. *Conversely, if k is perfect then X regular implies X/k is smooth.*

Proof. Since regularity is a local property, we may assume X is affine and in particular we choose an embedding $X \hookrightarrow \mathbb{A}^n_k$ as a closed sub scheme. Moreover it suffices to check for regularity at closed points of X. Let $x \in X$ be a closed point.
Now suppose $\mathscr{I}$ and $\mathfrak{m}_x$ be the ideals defining X and x respectively. Then

$$\mathscr{I} \subset \mathfrak{m}_x \subset \mathscr{O}_{\mathbb{A}^n_k} \mapsto \Omega^1_{\mathbb{A}^n_k/k},$$

induces

$$\begin{tikzcd} \frac{\mathscr{I}}{\mathscr{I}^2} \otimes k(x) \arrow[r, "\delta_X"] \arrow[dr] & \Omega^1_{\mathbb{A}^n_k/k} \otimes k(x) \\ & \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \arrow[u, "\delta_x"] \end{tikzcd}.$$

[8]Thanks to Proposition 4.2.4.1, (4) it suffices to check this at closed points!
[9]Thanks Cheng and Fuxiang for pointing the error in an earlier argument

Choose elements $f_1, f_2 \cdots f_r \in \mathscr{I}$ whose images form a basis of $\frac{\mathscr{I}}{\mathscr{I}^2} \otimes k(x)$. Since X/k is smooth, δ_X is injective and hence the images of these elements in $\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$ is also span a r-dimensional subspace and hence can be extended to a basis of $\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$. Thus by Proposition 4.2.4.1, (3) $\mathscr{O}_{X,x}$ is a regular local ring.

The converse follows from the Jacobian criterion (the key point is vanishing of $\Omega^1_{k(x)/k}$ which of course uses k being perfect). For a proof see [1, Exposé II, Corollaire 5.3]

□

We have the following corollary.

Corollary 4.2.4.6. *Let X/k be of finite type. Then the following are equivalent*

1. *X/k is smooth.*
2. *$X_{k'}$ is regular for any field extension k'/k.*
3. *dim$_{k(x)}(\Omega^1_{X/k} \otimes k(x)) = n$, where n is the dimension of the component of X containing x.*

Proof. Since smoothness is preserved under base change (1) implies (2) by Theorem 4.2.4.5. For the converse note that it suffices to show X/k is smooth iff $X_{\bar{k}}/\bar{k}$ is smooth for an algebraic closure $\bar{k}$ of k. This is because regularity of $X_{\bar{k}}$ implies $X_{\bar{k}}/\bar{k}$ is smooth.

Since the question is local on X, we may assume we have a closed embedding $X \hookrightarrow \mathbb{A}^n_k$. Let $x \in X$ and let $\bar{x} \in X_{\bar{k}}$ be a point on $X_{\bar{k}}$ mapping to x under the flat and surjective map $\pi : X_{\bar{k}} \to X$. We denote by $\mathscr{I}_X$ (resp. $\mathscr{I}_{X_{\bar{k}}}$) the ideal sheaves of X (resp. $X_{\bar{k}}$) in $\mathbb{A}^n_k$ (resp. $\mathbb{A}^n_{\bar{k}}$). Then flatness of π implies that

$$\pi^*(\mathscr{I}_X) = \mathscr{I}_{X_{\bar{k}}}$$

,

$$\pi^*(\frac{\mathscr{I}_X}{\mathscr{I}^2_X}) = \frac{\mathscr{I}_{X_{\bar{k}}}}{\mathscr{I}^2_{X_{\bar{k}}}}.$$

In particular

$$\frac{\mathscr{I}_{X_{\bar{k}}}}{\mathscr{I}^2_{X_{\bar{k}}}} \otimes \bar{k}(\bar{x}) = (\frac{\mathscr{I}_x}{\mathscr{I}^2_x} \otimes k(x)) \otimes k(\bar{x}).$$

Now since $X_{\bar{k}}/\bar{k}$ is smooth, the induced map $\frac{\mathscr{I}_{X_{\bar{k}}}}{\mathscr{I}^2_{X_{\bar{k}}}} \otimes \bar{k}(\bar{x}) \to \Omega^1_{\mathbb{A}^n_{\bar{k}}/\bar{k}} \otimes \bar{k}(\bar{x})$ is injective which by the isomorphism above implies that the induced map $\frac{\mathscr{I}_X}{\mathscr{I}^2_X} \otimes k(x) \to \Omega^1_{\mathbb{A}^n_k/k} \otimes k(x)$ is injective, and hence by the Jacobian criterion we are done.

Clearly (1) implies (3) by Lemma 4.2.2.6. It suffices to show (3) implies (2). This can be argued as above using Corollary 4.1.1.5. For a proof refer to Tag 01V9.

□

Remark 4.2.4.7. Note that Theorem 4.2.4.5 is the best possible result one can hope for in general. For example $X = \mathrm{Spec}(\mathbb{F}_p(t^{1/p}))$ is not smooth over $Y = \mathrm{Spec}(\mathbb{F}_p(t))$ (owing to $\Omega^1_{X/Y}$ being larger than expected i.e 0), however it is regular. Corollary 4.2.4.6 tells us that smoothness is the same as geometric regularity.

Corollary 4.2.4.8 (Generic Smoothness over a perfect field). *Let X/k be a reduced scheme of finite type over a perfect field k. Then there exists a dense open subset $U \hookrightarrow X$ such that U/k is smooth.*

Proof. Since X/k is reduced and of finite type, it has finitely many irreducible components and the local ring at any generic point is a field. Let $\eta \in X$ be one such generic point in a component of dimension n. Then $\Omega^1_{X/k} \otimes k(\eta) = \Omega^1_{k(\eta)/k}$ (Why?) and by [2, Theorem 8.6A] $\dim_{k(\eta)}\Omega^1_{k(\eta)/k} = n$. Since X is reduced, there exists an irreducible open containing η in X such that $\Omega^1_{X/k}$ is locally free of rank n. By Corollary 4.2.4.6, (3) this open subset is smooth over k. Since we can do this around every generic point, we win. □

We now state a very important Bertini theorem. This is frequently (and freely!) used in induction arguments. We do not prove it here but I strongly recommend reading the proof in [2, Chapter II, Theorem 8.18].

Theorem 4.2.4.9 (Bertini Theorem). *Let $X \hookrightarrow \mathbb{P}_k$ be a smooth projective variety over an algebraically closed field k. Let $\mathbb{P}^\vee_k$ be the projective variety parametrizing linear homogeneous polynomials on $\mathbb{P}_k$ or equivalently they parametrize hyperplane sections of $\mathbb{P}_k$. Then there exists a dense open $U \hookrightarrow \mathbb{P}^\vee_k$ such that for any closed point $x \in U$, the scheme $X \cap H$ is also smooth over k.*

Now we compare our notion of smoothness to the one in Hartshorne.

Theorem 4.2.4.10. *Let $f : X \to Y$ be a morphism of finite type between Noetherian schemes of relative dimension n[10]. Then f is smooth iff*

1. *f is flat.*
2. *The fibers X_y are smooth for all points $y \in Y$ or equivalently by Corollary 4.2.4.6, (3) $\dim_{k(x)}(\Omega^1_{X/Y} \otimes k(x)) = n$ for any point $x \in X$.*

Proof. This follows from Tag 00TF. □

But more is true! We have the following *miraculous* result, known colloquially as the *Miracle Flatness Theorem* due to Hironaka.

Theorem 4.2.4.11 (Miracle Flatness Theorem). *Let $R \to S$ be a local morphism of Noetherian local rings. Assume that*

[10] Every fiber of f is equidimensional of dimension n

1. *R is a regular local ring.*

2. *S is Cohen-Macaulay (ex. regular).*

3. *The dimension formula holds i.e,*

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S),$$

where $\mathfrak{m}$ is the maximal ideal of R.

Then $R \to S$ is flat!

This has the following very useful corollary.

Corollary 4.2.4.12 (Miracle Flatness Theorem for schemes). *Let $f : X \to Y$ be a morphism of locally Noetherian schemes such that X is Cohen-Macaulay (for ex. regular) and Y is regular. Then f is flat iff the dimension formula holds.*

Here is corollary to the above theorem which recovers the classical notion of smoothness for morphisms of smooth varieties.

Corollary 4.2.4.13. *Let $f : X \to Y$ be a morphism of smooth varieties over a field k. Then f is smooth iff for any closed point $x \in X$, the vector space $\Omega^1_{X/Y} \otimes k(x)$ is of dimension dim(X) − dim(Y) iff the induced map*

$$df_x : T_xX \to T_{f(x)}Y,$$

between their Zariski tangent spaces in surjective.

Proof. Miracle flatness gives you flatness for free. Once you have flatness the rest follows from Theorem 4.2.4.10. □

Chapter 5

A crash course in derived categories

In this chapter we shall give a crash course on derived categories. We aim to have a working understanding of what these are and more importantly (over time) appreciate their utility. Throughout $\mathscr{A}$ will denote an abelian category (see Definition 1.3.0.6). To fix ideas it is best to think of $\mathscr{A}$ as category of R-modules for a ring R or as the abelian category of $\mathscr{O}_X$-modules for a scheme X.

5.0.1 What is our goal?

Very often in algebraic geometry (and allied topics) one comes across the following situation; One has an additive functor $F : \mathscr{A} \to \mathscr{B}$ between two abelian categories of interest. On a good day F would preserve exact sequences, but more often than not F would only be either left exact or right exact. The typical examples are $\mathrm{Hom}_R(M, -)$ and $\otimes_R M$ for an R-module M. The natural question then is:

Question:

Given a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

in $\mathscr{A}$ and a left exact functor $F : \mathscr{A} \to \mathscr{B}$, how do we understand the *defect* of right exactness? Put differently how do continue the exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z).$$

To begin with one can make a definition:

Definition 5.0.1.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. A cohomological δ-functor extending F is a sequence of additive functors $F^i : \mathcal{A} \to \mathcal{B}$ such that $F^0 = F$, together with boundary maps (natural in the following short exact sequences)

$$\delta : F^i(Z) \to F^{i+1}(X)$$

for all short exact sequences

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$, such that for all such short exact sequences we obtain the following complex:

$$0 \to F^0(X) \to F^0(Y) \to F^0(Z) \xrightarrow{\delta} F^1(X) \to F^1(Y) \to F^1(Z) \xrightarrow{\delta} F^2(X) \to \cdots,$$

which is exact.
Moreover, such a δ is called *universal* if it is initial in the category of cohomological δ-functors extending F.

Having made this definition, a natural question then is when do universal δ functors exist? Note that by definition once they exist, they are unique upto an unique isomorphism. Following theorem was one of the important results in the famous Tohoku article of Grothendieck. Before we can state it we need a couple of definitions.

Definition 5.0.1.2 (Injective Object). An object I in an abelian category is said to be injective if the following equivalent conditions are satisfied:

1. $\mathrm{Hom}_{\mathscr{A}}(-, I)$ is an exact functor.

2. Every injection $I \hookrightarrow X$ is a split injection[1].

Example 5.0.1.3. An abelian group M is injective iff for any integer n, multiplication by n is surjective on M. Such groups are called divisible. See Tag 01D7.

Definition 5.0.1.4. An abelian category $\mathscr{A}$ is said to have *enough injectives* if for every object X there exists an injection of X inside an injective object $\tilde{X}$.

Example 5.0.1.5. Following abelian categories have enough injectives:

- Category of R-modules.

- Sheaves of abelian groups on a topological space X.

- Sheaves of $\mathscr{O}_X$-modules modules on a ringed space $(X, \mathscr{O}_X)$.

Following abelian category **do not** have enough injectives in general:

- Category of finite R-modules.

- The category of coherent sheaves $\mathrm{Coh}(X)$ on a Noetherian scheme X.

[1]This should tell you that injective objects in $\mathscr{A}$ are dual to projective objects i.e they correspond to projective objects in $\mathscr{A}^{\mathrm{op}}$

Proof. This is standard. First embed X inside an injective say I^0, then take the quotient I^0/X, embed that inside an injective I^1 so on and so forth. □

Now we are ready to state the promised theorem.

Theorem 5.0.1.6 (Grothendieck). *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor from an abelian category $\mathscr{A}$ with enough injectives. Then there exists an universal δ-functor extending F. The F^i's are called the right derived functors of F.*

Remark 5.0.1.7. By symmetry if F is right exact and $\mathscr{A}$ has enough projectives (Guess the definition!) then we get left derived functors of F.

Here are some examples:

Example 5.0.1.8. Here we list some examples of derived functors:

1. For any R-module M, the left derived functors of $\mathrm{Hom}(M, -)$ are denoted by $\mathrm{Ext}^i(M, -)$.

2. For any continuous map of topological spaces $f : X \to Y$, we denote by $R^i f_* \mathscr{F}$ the derived functors of the left exact functor $f_* \mathscr{F}$. These are also called the higher direct images.

3. Let $f : X \to Y$ be a morphism of Noetherian schemes. In particular f is a map of ringed spaces and hence it make sense to talk about the derived functors $R^i f_* \mathscr{F}$ for any $\mathscr{O}_X$-module $\mathscr{F}$. It is apriori not clear (or even true) that $R^i f_* \mathscr{F}$ have any additional structure that $\mathscr{F}$ may have like being coherent or quasi-coherent. These are important results and we shall discuss them later in this course

5.1 Injective Resolutions: Turning the crank

Thanks to Theorem 5.0.1.6 we know that it makes sense to talk about the derived functors of a left (or right) exact functor. But then how do we actually compute these? For simplicity we work with only left exact functors unless otherwise stated. Everything we say works well in $\mathscr{A}^{\mathrm{op}}$, and hence with projectives replaced by injectives.
What is amazing is that, once an abelian category has enough injectives, there is an *uniform* way to compute these for all possible left exact functors.
Here is an easy lemma.

Lemma 5.1.0.1. *Let $\mathscr{A}$ be an abelian category with enough injectives. Then every object X in $\mathscr{A}$ can be resolved using injectives, i.e there exists an exact complex*[2]

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots,$$

where each I^j is an injective object.

[2] Our complexes have cohomological indexing

But one can do better. Before we state the result we need a few definitions.

Definition 5.1.0.2. We denote by $C(\mathscr{A})$, the category of chain complexes with values in $\mathscr{A}$, that is objects in $\mathscr{A}$ are complexes $\mathbf{X}^{\cdot}$

$$\cdots X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \cdots ,$$

here X^i's are objects in $\mathscr{A}$ and $d^{i+1} \circ d^i = 0$ for all i[3]. Morphisms of complexes $f : \mathbf{X}^{\cdot} \to \mathbf{Y}^{\cdot}$ are given by maps $f^i : X^i \to Y^i$ for all i commuting with the differentials.

Definition 5.1.0.3. For any complex $\mathbf{X}^{\cdot}$ and any integer n, we have an exact functor called **shift by n**

$$[n] : C(\mathscr{A}) \to C(\mathscr{A}),$$

which sends $\mathbf{X}^{\cdot}$ to a complex $\mathbf{X}^{\cdot}[n]$ whose i^{th}-term is X^{i+n} and the differentials are one induced from $X^{\cdot}$.

For any i there exists a functor

$$H^i : C(\mathscr{A}) \to \mathscr{A},$$

obtained by sending

$$\mathbf{X}^{\cdot} \to \frac{\ker(d^i)}{\operatorname{Im}(d^{i-1})}.$$

We note the following easy lemma.

Lemma 5.1.0.4. *The category $C(\mathscr{A})$ can be given the structure of an abelian category with termwise kernels and cokernels. A sequence of complexes is exact iff it is termwise exact. Moreover there is an exact functor from $\mathscr{A}$ to $c(\mathscr{A})$ which sends any object X to the complex with only one non-zero object X at degree 0.*
Finally any short exact sequence of complexes induces a long exact sequence of cohomologies.

Proof. The first part is an easy exercise. For the last claim use snake lemma (see in Tag 0117). ☐

Definition 5.1.0.5. Two maps $f, g : \mathbf{X}^{\cdot} \to \mathbf{Y}^{\cdot}$ are said to be chain homotopic[4] if there exists maps $\partial^i : X^i \to Y^{i-1}$ for all i such that

$$f - g = d \circ \delta + \delta \circ d.$$

Moreover a map $f : \mathbf{X}^{\cdot} \to \mathbf{Y}^{\cdot}$ is said to be *homotopy equivalence* if there exists $g : \mathbf{Y}^{\cdot} \to \mathbf{X}^{\cdot}$ such that $g \circ f \sim 1_{\mathbf{X}^{\cdot}}$. Finally f is said to be *quasi-isomorphism* if it induces an isomorphism on the cohomology groups.

[3] In what follows we will simply denote these maps called *differentials* by d, if at all.
[4] denoted by $f \sim g$.

Here is a standard fact about chain homotopic maps.

Lemma 5.1.0.6. *Let $f, g : \mathbf{X}^{\cdot} \to \mathbf{Y}^{\cdot}$ be chain homotpic maps. Then they induce identical maps on cohomology. Moreoever if f is an homotopy equivalence then f is a quasi-isomorphism.*

Proof. Easy exercise. □

Thanks to Lemma 5.1.0.6 it makes sense to talk about the *homotopy category* of $C(\mathscr{A})$.

Definition 5.1.0.7. The homotopy category $K(\mathscr{A})$ is the category whose objects are the same as $C(\mathscr{A})$ but morphisms are given by

$$\mathrm{Hom}_{K(\mathscr{A})}(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot}) := \mathrm{Hom}_{c(\mathscr{A})}(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})/\sim,$$

where $\sim$ is the equivalence relation coming from homotopy equivalence.

Question: So why do we care about $K(\mathscr{A})$?

The easy answer is because the functors $H^i : C(\mathscr{A}) \to \mathscr{A}$ factor via $K(\mathscr{A})$ thanks to Lemma 5.1.0.6. But why care about H^i's at all?

Proposition 5.1.0.8. *Let $\mathcal{A}$ be an abelian category, and consider the following solid diagram in $\mathcal{A}$:*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "d"] \arrow[d, "f"] & I^0 \arrow[r, "d^0"] \arrow[d, dotted, "f_0"] & I^1 \arrow[r, "d^1"] \arrow[d, dotted, "f_1"] & \cdots \\
0 \arrow[r] & Y \arrow[r, "e"] & J^0 \arrow[r, "e^0"] & J^1 \arrow[r, "e^1"] & \cdots
\end{tikzcd}$$

where $I^\bullet$ and $J^\bullet$ are injective resolutions for X and Y, respectively.
Then the dotted arrows f_i exist such that the whole diagram commutes, and between any two choices of f_i and f_i', there exists a chain homotopy between them.

This allows us define for any left exact functor $F : \mathscr{A} \to \mathscr{B}$, the derived functor F^i as follows:

1. For any object X in $\mathscr{A}$ take an injective resolution $\mathbf{I}^\bullet$ as above. This amounts to replacing X by a complex of injectives which is *quasi-isomorphic* to X.

2. Define $F^i(X) := H^i(F(\mathbf{I}^\bullet))$. The latter is well defined and functorial (in X) thanks to Proposition 5.1.0.8.

We claim that these $F^i(X)$ are δ-functors and that they are universal. The universality follows from Tag 010T. For the former we need the following proposition (together with Lemma 5.1.0.4).

Proposition 5.1.0.9. *For any short exact sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

one can choose injective resolutions $\boldsymbol{I}^\bullet$, $\boldsymbol{J}^\bullet$ and $\boldsymbol{K}^\bullet$ of X, Y and Z respectively such that

1. *There exists an exact sequence*

$$0 \longrightarrow \boldsymbol{I}^\bullet \longrightarrow \boldsymbol{J}^\bullet \longrightarrow \boldsymbol{K}^\bullet \longrightarrow 0,$$

   such that the obvious diagrams commute.

2. *Moreover the above exact sequence of complexes is termwise split, and hence remains exact after applying any additive functor F.*

Proof. For the former see Tag 013T. The latter follows from the fact that terms of these resolutions by choice are injective. □

Definition 5.1.0.10. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor with an universal δ-functor extending it. We say an object M is F-acyclic if $F^i(M) = 0$ for any $i > 0$.

Clearly if $\mathscr{A}$ has enough injectives then every injective is F-acyclic. Though you can have F-acyclics which are not injective, they serve the same purpose as injectives as the following lemma shows.

Lemma 5.1.0.11. *Consider an exact complex*

$$0 \longrightarrow X \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \cdots,$$

where M^i's are F-acyclics. Then $H^i(F(\boldsymbol{M}^\bullet)) \simeq F^i(X)$.

Proof. □

Sheaf Cohomology

Let us discuss a very important case of derived functors: The case of sheaf cohomology. Here $\mathscr{A}$ is the abelian category of the sheaf of $\mathscr{O}_X$-modules on a ringed space $(X, \mathscr{O}_X)$. Note that we may simply choose $\mathscr{O}_X$ to be $\mathbb{Z}_X$, so this covers the case of sheaves of abelian groups too. As we have noted earlier $\mathscr{A}$ has enough injectives hence for any $\mathscr{O}_X$-module $\mathscr{F}$ we define by

$$H^i(X, \mathscr{F}),$$

the i^{th}-derived functor of the left exact functor $\mathscr{F} \to \Gamma(X, \mathscr{F})$. With this notation $H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

A very important class of sheaves for us would be flasque sheaves.

Definition 5.1.0.12. A sheaf $\mathscr{F}$ is said to be flasque if for any $U \subseteq V$ open subsets in X, the natural map

$$\mathscr{F}(V) \to \mathscr{F}(U),$$

is surjective.

The reason they are interesting is the following:

Proposition 5.1.0.13. *There are plenty of flasque sheaves. In fact*

1. *Injective sheaves are flasque.*
2. *Flasque sheaves are F-acyclic for the global sections functor.*
3. *For any sheaf $\mathscr{F}$, the pre-sheaf God($\mathscr{F}$) defined as follows*

$$\text{God}(\mathscr{F})(U) := \prod_{x \in U} \mathscr{F}_x,$$

is a flasque sheaf. Moreover the natural map $\mathscr{F} \to$ God($\mathscr{F}$) is an injection. Thus every sheaf can be embedded canonically inside a flasque sheaf and the associated resolution is called the Godement resolution of $\mathscr{F}$.

Proof. For a proof see Tag 01EA and Tag 09SY. □

Here is a corollary to Proposition5.1.0.13.

Corollary 5.1.0.14. *Let X be a topological space with a sheaf of rings $\mathscr{O}_X$. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. Then we can also look at $\mathscr{F}$ as a sheaf of abelian groups (forgetting the $\mathscr{O}_X$-module) structure, we call this sheaf $\mathscr{F}^{ab}$. Then there exists a canonical isomorphism*

$$H^i(X, \mathscr{F}) \simeq H^i(X, \mathscr{F}^{ab}), \forall i.$$

Proof. Thanks to Proposition 5.1.0.13 and Lemma 5.1.0.11, either side of the isomorphism can be computed using Godement resolutions. □

Higher direct images

One should view sheaf cohomology as an absolute theory. There is relative version which as you will see will turn out to be equally if not more important. Given any map $f : X \to Y$ of ringed space, there is an induced left exact functor $f_* : \mathrm{Mod}_{\mathscr{O}_X} \to \mathrm{Mod}_{\mathscr{O}_Y}$.
Note that we can take Y to be a point and $\mathscr{O}_Y$ to be the unique sheaf whose global sections on the point are $H^0(X, \mathscr{O}_X)$. In this case f_* is simply the global sections functor. As before thanks to existence of enough injectives in $\mathrm{Mod}_{\mathscr{O}_X}$, we can take the right derived functors $R^i f_*$ of f_*, these are often called the *higher direct images*. Here is a simple lemma.

Lemma 5.1.0.15. *Let $f : X \to Y$ be morphism of ringed spaces which induces a homeomorphism onto a closed subset. Then $R^i f_*$ vanishes for $i > 0$.*

Proof. First note that the stalk of $f_*\mathscr{F}$ at any point $y \in Y$ is 0 if $y \notin f(X)$ and is equal to $\mathscr{F}_y$ for $y \in f(X)$. Thus $f_*\mathscr{F}$ is an exact functor. Hence the result. $\square$

5.2 Derived Categories and Derived Functors

We can do better. Recall that $\mathscr{A}$ is realized as an abelian sub category of $C(\mathscr{A})$ (Lemma 5.1.0.4). Even when begin with an object X in $\mathscr{A}$, it's injective resolution lives not in $\mathscr{A}$ but rather in $C(\mathscr{A})$. One can try to address this asymmetry.

Lemma 5.2.0.1. *Let $\mathbf{X}^\bullet$ be a complex which is exact[5] for all $i \leqslant n$ for some $n \in \mathbb{Z}$. Then there exists a complex $\mathbf{I}^\bullet$ of injectives and a quasi-isomoprphism (qis henceforth) $\alpha : \mathbf{X}^\bullet \to \mathbf{I}^\bullet$.*

Proof. For a proof see Tag 013K. $\square$

Given the importance of complexes whose cohomology vanishes in sufficiently small degrees, we introduce a notation

Notations 5.2.0.2. We denote by $C^+(\mathscr{A})$ the full abelian sub category of bounded below chain complexes i.e. complexes $\mathbf{X}^\bullet$ such that $H^i(\mathbf{X}^\bullet) = 0$ for all $i \leqslant n$ and some $n \in \mathbb{Z}$.
In a similar vein we denote by $C^b(\mathscr{A})$ the full abelian sub category of bounded chain complexes i.e. complexes $\mathbf{X}^\bullet$ such that $H^i(\mathbf{X}^\bullet) = 0$ for all $|i| \geqslant n$ and some $n \in \mathbb{N}$.

One can now ask for analogues of Proposition 5.1.0.8 and this turns out to be true verbatim.

Proposition 5.2.0.3. *Given complexes*

1. *$\mathbf{X}^\bullet$, $\mathbf{Y}^\bullet$ in $C^+(\mathscr{A})$*
2. *a morphism $f : \mathbf{X}^\bullet \to \mathbf{Y}^\bullet$ and,*
3. *quasi-isomorphisms $\alpha : \mathbf{X}^\bullet \to \mathbf{I}^\bullet$, $\beta : \mathbf{Y}^\bullet \to \mathbf{J}^\bullet$ to a complex of injectives.*

There exists a morphism $\tilde{f} : \mathbf{I}^\bullet \to \mathbf{J}^\bullet$ making the obvious diagram commute. Moreover $\tilde{f}$ is unique upto homotopy.

Thanks to Proposition 5.2.0.3 we can not define derived functors not just for objects in $\mathscr{A}$ but also for bounded below complexes in $C(\mathscr{A})$. As before start with a left exact functor $F : \mathscr{A} \to \mathscr{B}$. Then we define functors $F^i : C^+(\mathscr{A}) \to \mathscr{B}$ which restrict to the usual derived functors on $\mathscr{A}$ as follows:

[5] A complex is said to be exact at an index i if it's cohomology vanishes at that index.

1. Resolve this complex $\mathbf{X}^\bullet$ by a complex of injectives (say $\mathbf{I}^\bullet$).
2. Define $F^i(\mathbf{X}^\bullet) := H^i(F(\mathbf{I}^\bullet))$. Thanks to Proposition 5.2.0.3 and Lemma 5.1.0.6 this is a functor.
3. As before using Tag 013T together with Lemma 5.1.0.4 short exact sequence of complexes give long exact sequences of their derived functors.
4. We also have an analogue of Lemma 5.1.0.11.

The upshot of all this being we have a diagram,

$$\begin{tikzcd} \mathscr{A} \arrow[r] \arrow[drrr, dashed, bend right, "F^i"'] & C^+(\mathscr{A}) \arrow[r] & K^+(\mathrm{Inj}_{\mathscr{A}}) \arrow[d] & \\ & & K^+(\mathscr{B}) \arrow[r, "H^i"] & \mathscr{B}. \end{tikzcd} \tag{5.1}$$

Here $K^+(\mathrm{Inj}_{\mathscr{A}})$ is the full subcategory of $K^+(\mathscr{C})$ consisting of complexes all of whose terms are injectives in $\mathscr{A}$. Moreover note that injective resolutions of complexes are only defined *upto quasi-isomorphisms*.This motivates the following definition. Let $\mathscr{A}$ be any abelian category.

Definition 5.2.0.4 (Derived Category)**.** Define by $D(\mathscr{A})$, the derived category of $\mathscr{A}$ as the localisation of $K(\mathscr{A})$ at quasi-isomorphisms i.e. , the objects of $D(\mathscr{A})$ are the same as those of $K(\mathscr{A})$ (or equivalently $C(\mathscr{A})$). A morphism $f : \mathbf{X}^\bullet \to \mathbf{Y}^\bullet$ in $D(\mathscr{A})$ is a triple $(\mathbf{X'}^\bullet, t, f')$ where

1. $\mathbf{X'}^\bullet$ is a complex in $K(\mathscr{A})$.
2. $t : \mathbf{X'}^\bullet \to \mathbf{X}^\bullet$ and $f' : \mathbf{X'}^\bullet \to \mathbf{Y}$ are morphisms in $K(\mathscr{A})$ (One should think of $f = \frac{f'}{t}$)
3. t is a quasi-isomorphism.

On can analogously define $D^+(\mathscr{A})$ as a localisation of $K^+(\mathscr{A})$. Of course one needs to check a few things here which we state as a theorem.

Theorem 5.2.0.5. *$D(\mathscr{A})$ is a well defined category i.e. composition is well defined. Moreover $D(\mathscr{A})$ is an additive category and the natural functor from $K(\mathscr{A}) \to D(\mathscr{A})$ is universal among those where quasi-isomorphisms are sent to isomorphisms. Finally when $\mathscr{A}$ has enough injectives then*

$$K^+(\mathit{Inj}_{\mathscr{A}}) \to D^+(\mathscr{A}),$$

is an equivalence of categories.

Proof. For a proof see Tag 05RT. For the second part it suffices to show that quasi-isomorphisms of complexes all of whose terms are injective is in fact an isomorphism in the homotopy category. This follows from Tag 013P. ☐

Corollary 5.2.0.6. *For any i the cohomology functor $H^i : C(\mathscr{A}) \to \mathscr{A}$ factors via $D(\mathscr{A})$.*

Proof. We have already seen that H^i factors via $K(\mathscr{A})$. Since H^i maps quasi-isomorphisms to isomorphisms, we are done by the universal property. ☐

Finally we have the following result.

Theorem 5.2.0.7. *The natural functor $\mathscr{D}^+(\mathscr{A}) \to D(\mathscr{A})$ is fully faithful and essentially surjective on the subcategory consisting of objects $X \in D(\mathscr{A})$[6] with $H^i(X) = 0, \forall i << 0$. Moreover, the functor $\mathscr{A} \to K(\mathscr{A}) \to D(\mathscr{A})$ sending an object X to a complex with exactly one non-zero term in degree 0, is also fully faithful and essentially surjective onto to the subcategory whose objects are $X \in D(\mathscr{A})$ with $H^i(X) = 0, \forall i \neq 0$.*

Proof. For a proof see [3, Proposition 1.7.2]. ☐

Now suppose $\mathscr{A}$ is an abelian category with enough injectives. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Then we can define an additive functor called the derived functor of F,

$$RF : D^+(\mathscr{A}) \to D^+(\mathscr{B}), \tag{5.2}$$

as the composition of the following functors,

$$D^+(\mathscr{A}) \xrightarrow{\simeq} K^+(\mathrm{Inj}_{\mathscr{A}}) \xrightarrow{F} K^+(\mathscr{B}) \longrightarrow D^+(\mathscr{B}).$$

Here is a simple consequence of the definition.

Lemma 5.2.0.8. *There exists an equivalence of functors*

$$H^i \circ RF \simeq R^i F.$$

Proof. The is an immediate consequence of the definition of RF and Corollary 5.2.0.6. ☐

We end with the following characterization of an isomorphism in $D(\mathscr{A})$.

Lemma 5.2.0.9. *A morphism $f : X \to Y$ in $D(\mathscr{A})$ is an isomorphism iff $H^i(f)$ is an isomorphism for all i, i.e. f is an isomorphism iff it is an quasi-isomorphism.*

Proof. Clearly if f is an isomorphism, then $H^i(f)$ is an isomorphism for all i, since H^i is a functor from $D(\mathscr{A})$ to $\mathscr{A}$. Conversely suppose f corresponds to a triple (X', t, f'). Then f is an quasi-isomorphism iff f' is an quasi-isomorphism. Thus the inverse of f is given by the morphism $g := (X', f', t)$ from Y to X[7]. ☐

[6] Henceforth we shall not use boldfont for complexes i.e. we shall treat objects and complexes in the same footing.

[7] Of course hidden in all this is the well definedness of composition of morphisms in $D(\mathscr{A})$.

5.2.1 Spectral Sequences

Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be two left exact functors between abelian categories. Assume that $\mathscr{A}$ and $\mathscr{B}$ have enoug injectives. Thus we can talk about

$$RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$$

and

$$RG : D^+(\mathscr{B}) \to D^+(\mathscr{C}).$$

Under what conditions do we have

$$R(G \circ F) = RG \circ RF?$$

Let us first work out a necessary condition. Suppose X is an injective object in $\mathscr{A}$. Then both $RF(X)$ and $R(G \circ F)(X)$ are complexes with no cohomology outside degree 0. So for an equality as above to hold, we must have that $RG(F(X))$, is also a complex with no cohomology outside degree 0. Put differently F maps injective objects to G-acyclic objects. Turns out this is all that we need.

Theorem 5.2.1.1 (Grothendieck). *Let F and G be abelian functors as above. If F maps injective objects to G-acyclic objects then $RG \circ RF$ and $R(G \circ F)$ are naturally equivalent.*

Proof. Tag 015M ☐

Before we move onto applications, here is a simple criterion which ensures that injectives are mapped to injectives (and hence to acyclics, provided there are enough injectives!).

Lemma 5.2.1.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Suppose F has a left adjoint which is exact, then F maps injectives to injectives.*

Proof. Let I be an injectives object in $\mathscr{A}$. We need to show that the functors $Y \to \mathrm{Hom}_{\mathscr{B}}(Y, F(I))$ is exact. Let (G, F) be an adjoint pair. Then there exists a natural (in Y) isomorphism $\mathrm{Hom}(G(Y), I) \simeq \mathrm{Hom}(Y, F(I))$. By assumption G is exact and $\mathscr{I}$ is injective, hence the functor $Y \to \mathrm{Hom}_{\mathscr{B}}(Y, F(I))$ is exact and thus $F(I)$ is injective. ☐

Here are some useful applications.

Proposition 5.2.1.3. *Let $f : X \to Y$ be a map of ringed spaces and let $\mathscr{F}$ be an $\mathscr{O}_X$ module. Then the sheaf $R^i f_* \mathscr{F}$ is canonically isomorphic to the sheafification of the presheaf $U \mapsto H^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$ for U open in Y.*

Proof. Let $\mathscr{A}$ be the abelian category sheaves of $\mathscr{O}_X$-modules, let $\mathscr{B}$ be the pre-sheaves of $\mathscr{O}_Y$-modules and $\mathscr{C}$ be the sheaves of $\mathscr{O}_Y$-modules. Let F be the composition of the forgetful functor from $\mathscr{A}$ to presheaves of $\mathscr{O}_X$-modules on X followed by f_*. Note that f_* is an exact functor on the category of presheaves, while the forgetful functor is only left exact. Thus F is left exact and G being the sheafification functor is exact. Hence by Theorem 5.2.1.1 $R^i f_* \mathscr{F} = G \circ F^i \mathscr{F}$.
We claim $F^i \mathscr{F}$ is the required pre sheaf. To see this note that to compute $\mathscr{F}$, we replace $\mathscr{F}$ by an injective resolution $\mathscr{I}^\bullet$and then apply the forgetful functor to presheaves followed by the exact functor f_*. Thus on any open set U of Y, $F^i\mathscr{F}(U)$ is the same as $H^i(\mathscr{I}(f^{-1}(U)))$ (Why?). The latter computes $H^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$ because injective sheaves are flasque (see Proposition 5.1.0.13). □

Corollary 5.2.1.4. *Let $f : X \to Y$ and $g : Y \to Z$ be morphism of ringed spaces. Then f_* takes injectives to g_*-acyclic sheaves. Hence $R(g \circ f)_* = Rg_* \circ Rf_*$*

Proof. Let $\mathscr{F}$ be an injective $\mathscr{O}_X$-module. Then Proposition 5.1.0.13 implies that $\mathscr{F}$ is flasque. Thus $f_*\mathscr{F}$ is flasque and hence for any open $U \hookrightarrow Y$, $f_*\mathscr{F}|_U$ is flasque. Thus by Proposition 5.2.1.3 we are done. □

So how do we apply Corollary 5.2.1.4?

1. Suppose f_* is exact (for example a closed immersion see Lemma 5.1.0.15). Then $R^i(g \circ f)_*\mathscr{F} = R^i g_*(f_*\mathscr{F})$ for any sheaf $\mathscr{F}$. Put differently we can compute the cohomology of $\mathscr{F}$ after pushing it to the ambient space. Similarly if g_* is exact then $R^i(g \circ f)_*\mathscr{F} = g_* R^i f_*$.

2. Suppose $\mathscr{F}$ is a sheaf on X with no higher direct images. Then we claim $R^i(g \circ f)_*\mathscr{F} = R^i g_*(f_*\mathscr{F})$.

In general though it is not so easy to relate $R^i g_*(R^j f_*\mathscr{F})$ and $R^{i+j}(g \circ f)_*\mathscr{F}$. They are related by what are called spectral sequences which can be thought of as a collection of many long exact sequences whose cohomologies compute what we want. Our spectral sequence has an E_2-page indexed by two non-negative integers p and q, which looks like

$$E_2^{p,q} = R^p f_*\Big(R^q g_* M\Big).$$

The E_2-page of our spectral sequence has maps

$$d_2 \colon E_2^{p,q} \longrightarrow E_2^{p+2,q-1},$$

which are called differentials, since any composition of two of these maps is zero. Since we have differentials, we can take cohomology, and the E_3-page is defined exactly as that,

$$E_3^{p,q} = \frac{\ker\Big(d_2 \colon E_2^{p,q} \to E_2^{p+2,q-1}\Big)}{\operatorname{im}\Big(d_2 \colon E_2^{p-2,q+1} \to E_2^{p,q}\Big)}.$$

Again we have differentials, called

$$d_3 \colon E_3^{p,q} \longrightarrow E_3^{p+3,q-2},$$

and this process continues to the E_4-page. In general, we have an E_r-page of our spectral sequence defined in the obvious way. This is a first quadrant spectral sequence, so we can see that since these differentials grow larger as r increases, eventually an element $E_r^{p,q}$ with $r > p, q$ cannot be hit or receive nonzero differentials, and hence

$$E_r^{p,q} \cong E_k^{p,q} \quad \text{for all } k \geqslant r.$$

In this situation we define

$$E_r^{p,q} = E_\infty^{p,q},$$

where the position has stabilised.
All of this information leads us to the following theorem, which is also true in the generality of Proposition 11.11.

Theorem 5.2.1.5. *The sheaf $R^i(f \circ g)_*\mathscr{F}$ has a decreasing filtration*

$$F^p R^i(f \circ g)_*\mathscr{F} \subset R^i(f \circ g)_*\mathscr{F},$$

with

$$F^{-1} = R^i(f \circ g)_*\mathscr{F} \quad \textit{and} \quad F^i = 0,$$

such that the associated graded object is

$$\mathrm{gr}^p\, R^i(f \circ g)_*\mathscr{F} = E_\infty^{p,i-p}$$

in the spectral sequence defined above. In the usual language of spectral sequences we may write

$$E_2^{p,q} = R^p f_*\Big(R^q g_*\mathscr{F}\Big) \quad \Longrightarrow \quad R^{p+q}(f \circ g)_*\mathscr{F}.$$

A specific case of the above spectral sequence is the so called *Leray* spectral sequence which is obtained by taking $Z = \{pt\}$. Then we get a spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*\mathscr{F}) \implies H^{p+q}(X, \mathscr{F})$$

These spectral sequences are special cases of filtered complex spectral sequence, which we shall state now. Let $\mathcal{A}$ be an abelian category. A *filtered complex* in $\mathcal{A}$ is a complex

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots,$$

together with a *decreasing* filtration $\{F^pC^\bullet\}_{p\in\mathbb{Z}}$ by subcomplexes, that is, for each $p \in \mathbb{Z}$ and for all n,

$$F^pC^n \subseteq C^n,$$

and

$$d^n\big(F^pC^n\big) \subseteq F^pC^{n+1}.$$

These filtrations satisfy

$$\cdots \supseteq F^pC^\bullet \supseteq F^{p+1}C^\bullet \supseteq \cdots .$$

The Filtered Complex Spectral Sequence

Given the filtered complex $(C^\bullet, F^\bullet)$, one forms the *associated graded complex* by setting

$$\mathrm{Gr}^p\, C^n := \frac{F^pC^n}{F^{p+1}C^n}.$$

Since the differential d is compatible with the filtration, it induces a differential

$$d_0^{p,q}\colon\ \mathrm{Gr}^p\, C^{p+q} \longrightarrow \mathrm{Gr}^p\, C^{p+q+1}.$$

Thus, we define the E_0-page of the spectral sequence by

$$E_0^{p,q} := \mathrm{Gr}^p\, C^{p+q}.$$

Taking cohomology with respect to d_0 gives the E_1-page:

$$E_1^{p,q} := H^{p+q}\Big(\mathrm{Gr}^p\, C^\bullet\Big).$$

In general, one obtains a spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ with differentials

$$d_r^{p,q}\colon E_r^{p,q} \longrightarrow E_r^{p+r,\,q-r+1},$$

and, under appropriate boundedness conditions, the spectral sequence converges (or *abuts*) to $H^{p+q}(C^\bullet)$ and thus one has

$$E_\infty^{p,q} \cong \mathrm{Gr}^p\, H^{p+q}(C^\bullet) = \frac{F^pH^{p+q}(C^\bullet)}{F^{p+1}H^{p+q}(C^\bullet)}.$$

Example: Two Spectral Sequences Associated to a Total Complex

Suppose we are given a first quadrant *double complex* $C^{\bullet,\bullet}$ in $\mathcal{A}$, with horizontal differentials d_h and vertical differentials d_v satisfying

$$d_h^2 = d_v^2 = d_hd_v + d_vd_h = 0.$$

The *total complex* $\mathrm{Tot}(C)$ is defined by

$$\mathrm{Tot}^n(C) = \bigoplus_{p+q=n} C^{p,q},$$

with differential

$$d = d_h + d_v.$$

There are two natural filtrations on $\mathrm{Tot}(C)$:

(a) The Column Filtration

Define the filtration by columns:

$$F^p_{\mathrm{col}}\ \mathrm{Tot}^n(C) := \bigoplus_{\substack{p' \geqslant p \\ p'+q=n}} C^{p',q}.$$

The associated spectral sequence has:

$$E_0^{p,q} = C^{p,q},$$

with the vertical differential d_v yielding

$$E_1^{p,q} = H^q\Big(C^{p,\bullet}, d_v\Big).$$

The differential $d_1^{p,q}$ on the E_1-page is induced by the horizontal differential d_h:

$$d_1^{p,q}\colon H^q\Big(C^{p,\bullet}\Big) \longrightarrow H^q\Big(C^{p+1,\bullet}\Big).$$

This spectral sequence converges to the cohomology $H^{p+q}(\mathrm{Tot}(C))$.

(b) The Row Filtration

Alternatively, define the filtration by rows:

$$F^q_{\mathrm{row}}\ \mathrm{Tot}^n(C) := \bigoplus_{\substack{q' \geqslant q \\ p+q'=n}} C^{p,q'}.$$

Then the associated spectral sequence has:

$$E_0^{p,q} = C^{p,q},$$

and the horizontal differential d_h gives

$$E_1^{p,q} = H^p\Big(C^{\bullet,q}, d_h\Big).$$

The differential on the E_1-page, $d_1^{p,q}$, is induced by the vertical differential d_v:

$$d_1^{p,q}\colon H^p\Big(C^{\bullet,q}\Big) \longrightarrow H^p\Big(C^{\bullet,q+1}\Big).$$

This spectral sequence also converges to $H^{p+q}(\mathrm{Tot}(C))$.

Both of these spectral sequences provide computational tools for analyzing the cohomology of the total complex, by first computing cohomology along one of the two directions (columns or rows) and then taking the cohomology of the resulting complex.

Chapter 6

Cohomology: Basic Computations

In this chapter we will compute the cohomology of *all* affine schemes and the projective space. The former is particularly simple and the latter (unsurprisingly) uses the former.

6.1 A fundamental exact sequence

We fix a base ring R (for example $\mathbb{Z}$). By a sheaf we shall me sheaves of R-modules[1]. Let X be any topological space. Let $j : U \hookrightarrow X$ be an open subset and let $i : Z \hookrightarrow X$ be the closed complement.

Definition 6.1.0.1. Let $\mathscr{F}$ be a pre sheaf on U. The extension by zero $j_!\mathcal{F}$ is defined as follows

$$j_!(\mathscr{F})(V) = \begin{cases} \mathscr{F}(V) & \text{if } V \subseteq U, \\ 0 & \text{otherwise.} \end{cases}$$

As before we let $j : U \hookrightarrow X$ be an open immersion and we denote by $i : Z \hookrightarrow X$ the corresponding closed immersion.

Lemma 6.1.0.2. *The functor $j_!$ takes sheaves to sheaves. Moreover*

1. *For any $x \in X$ and sheaf $\mathscr{F}$ on X,*

$$(j_!\mathscr{F})_x = \begin{cases} \mathscr{F}_x & \textit{if } x \in U, \\ 0 & \textit{otherwise.} \end{cases}$$

2. *$j_!$ is an exact functor.*

[1]Alternatively you can think of every topological space as a ringed space with the sheaf of rings given by the constant sheaf with values in R

3. *For any sheaf $\mathscr{F}$ there exists an exact sequence of sheaves*

$$0 \longrightarrow j_! j^* \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow i_* i^* \mathscr{F} \longrightarrow 0.$$

Proof. That $j_!$ takes sheaves to sheaves is clear from the definition. So it the claim about its stalk. This implies (2) and (3). □

Here is simple observation about $j_!$ which will be pretty handy.

Lemma 6.1.0.3. *There exists an adjoint triple $(j_!, j^*, j_*)$.*

Proof. We only need to show that $(j_!, j^*)$ form an adjoint pair or equivalently that there exists a natural isomorphism,

$$\mathrm{Hom}(j_! \mathscr{F}, G) \simeq \mathrm{Hom}(\mathscr{F}, j^* \mathscr{G}).$$

This is clear from the definition of $j_!$. □

Here is a simple corollary to the adjointness.

Corollary 6.1.0.4. *For any pre sheaf $\mathscr{F}$ there exists a natural isomorphism*[2]

$$\mathit{Hom}(j_! R_U, \mathscr{F}) \simeq \mathscr{F}(U).$$

In particular there exists a natural surjection (of pre sheaves)

$$\bigoplus_{(j:U \hookrightarrow X, s \in \mathscr{F}(U))} j_! R_U \to \mathscr{F},$$

where the sum is over all open subsets $U \hookrightarrow X$ and sections $s \in \mathscr{F}(U)$.

Proof. By adjunction $\mathrm{Hom}(j_! R_U, \mathscr{F})$ is isomorphic to $\mathrm{Hom}(R_U, j^* \mathscr{F})$ which is isomorphic to $\mathscr{F}(U)$. □

We end this section with an useful proposition. To state it we need a few notations. Let $\mathscr{U} := \{U_i\}_{i \in I}$ be a finite open cover of X indexed by $I := \{0, 1 \cdots n\}$. For any subset $J \subseteq I$, we denote by U_J the intersection of open subsets indexed elements in J and by j_J the corresponding open immersion. Moreover for any two subsets $K \subseteq J$, we have a natural map

$$r_{JK} : j_{K!} R \to j_{J!} R.$$

With these notations we have the following, inclusion-exclusion principle.

[2]For any topological space X, by R_X we mean the constant sheaf with values in R. We shall drop the subscript X, when there is no scope of confusion.

Proposition 6.1.0.5. *The following sequence of sheaves is exact*

$$0 \longrightarrow j_{I!}R \longrightarrow \bigoplus_{J\subseteq I,|J|=|I|-1} j_{J!}R \longrightarrow \cdots \qquad \bigoplus_{J\subseteq I,|J|=1} j_{J!}R \longrightarrow R_X \longrightarrow 0, \tag{6.1}$$

where the map

$$j_{K!}R \to \bigoplus_{J\subseteq K\subseteq I,|J|=|K|-1} j_{J!}R,$$

is given by $\sum(-1)^{K\setminus J} r_{JK!}$. *Moreover if* $X \in \mathscr{U}$, *then the above sequence of sheaves in null homotopic to* 0.

Proof. Exactness of a sequence of sheaves can be checked on an open cover. Hence by replacing X by open sets in $\mathscr{U}$ we may assume $X \in \mathscr{U}$, say $X = U_i$ for some $0 \leqslant i \leqslant n$. In particular it suffices to prove the second part of the proposition.
Denote the complex of sheaves by $\mathscr{C}_{\mathscr{U}}$. To prove null-homotopy we need to give a map

$$\partial : \mathscr{C}_{\mathscr{U}} \to \mathscr{C}_{\mathscr{U}}[-1],$$

inducing an homotopy equivalence between the zero map and identity. The key point is to note that for any subset $J \subseteq I$ not containing i, $J' := \{i\} \cup J$ satisfies

1. $|J'| = |J| + 1$.

2. $U_{J'} = U_J$.

We consider the sheaf R_X to be in degree 0. Thus we define

$$\partial_0 : R_X \to \bigoplus_{J,|J|=1} r_{J!}R,$$

via the isomorphism $R_X \simeq r_{\{i\}!}R$. For the lower ∂'s (upto a sign) they are given by identifying $r_{J!}R \simeq r_{J'!}R$ for $i \notin J$. We leave it to the reader to work out the signs. ☐

6.2 Čech Cohomology

In this section we will define and study some properties of a very useful tool called Čech Cohomology, this can be thought of as a sheaf theoretic inclusion-exclusion principle. We begin by defining the Čech complex.
As before we fix an open covering $\mathscr{U} = \{U_i\}_{i\in I}$ of our topological space X indexed by a finite set $I = \{0, 1 \cdots n\}$. We continue using the notations from Section 6.1. The Čech complex $(_{\mathscr{U}})$ is a functor from pre sheaves of abelian groups on X to complex of abelian groups.

Definition 6.2.0.1. The Čech complex of $\mathscr{F}$ with respect to $\mathscr{U}$ is the following complex of abelian groups

$$\prod_{J\subseteq I,|J|=1}\mathscr{F}(U_J) \longrightarrow \prod_{J\subseteq I,|J|=2}\mathscr{F}(U_J) \longrightarrow \cdots \qquad \mathscr{F}(\textstyle\bigcap_{i\in J}U_i),$$

where the map

$$\mathscr{F}(U_J) \to \prod_{K\subseteq I,|K|=|J|+1}\mathscr{F}(U_K),$$

is given by sending a section $s \to \prod_{J\subseteq K\subseteq I,|K|=|J|+1}((-1)^{K\setminus J}s|_{U_K})$.

We need one more notation before we can state an important corollary to Proposition 6.1.0.5.

Notations 6.2.0.2. Denote by $\check{H}^p(\mathscr{U},\mathscr{F})$ the p^{th} cohomology of the Čech complex of $\mathscr{F}$ with respect to $\mathscr{U}$.

Corollary 6.2.0.3. *Let $\mathscr{F}$ be a sheaf on X. Then we have the following*

1. *$\check{H}^p(\mathscr{U},\mathscr{F})$ vanishes for $p > |I|$.*
2. *$\check{H}^0(\mathscr{U},\mathscr{F}) \simeq \mathscr{F}(X)$.*
3. *$\check{H}^p(\mathscr{U},\mathscr{F})$ vanishes for $p > 0$ if $\mathscr{F}$ is an injective sheaf.*
4. *$\check{H}^p(\mathscr{U},\mathscr{F})$ vanishes for $p > 0$ if $X \in \mathscr{U}$.*

Proof. (1) is clear from the definition of the Čech complex, since there are no terms in the complex of degree greater than $|I|$. (2) follows from the definition of the Čech complex and the fact that $\mathscr{F}$ is a sheaf. For (3) and (4) simply note that, thanks to Corollary 6.1.0.4, Čech complex can be obtained by applying the functor $\mathrm{Hom}(-,\mathscr{F})$ to the exact sequence of sheaves in (6.1) and ignoring the degree 0 term (which as we just saw is $\mathscr{F}(X)$). If $\mathscr{F}$ is injective, then the functor $\mathrm{Hom}(-,\mathscr{F})$ is exact and hence implies (3). On the other hand if $X \in \mathscr{U}$, the complex in (6.1) is null homotopic and thus continues to be null homotopic after applying the additive functor $\mathrm{Hom}(-,\mathscr{F})$. Hence the result. □

Now suppose $\mathscr{F}$ is an arbitrary sheaf. Then what do we do? Of course as always we first resolve $\mathscr{F}$ by injectives. Thus get a complex of injectives

$$\mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots$$

quasi-isomorphic to $\mathscr{F}$. Each of those $\mathscr{I}^j$'s has its own Čech complex. Thus we obtain a double complex

$$
\begin{tikzcd}
\prod_{\substack{J\subseteq I\\|J|=1}} \mathscr{I}^0(U_J) \arrow[r, "\check{\delta}^0"] \arrow[d, "d^0"'] & \prod_{\substack{J\subseteq I\\|J|=2}} \mathscr{I}^0(U_J) \arrow[r, "\check{\delta}^1"] \arrow[d, "d^0"'] & \cdots \arrow[r] & \mathscr{I}^0\Big(\bigcap_{i\in J} U_i\Big) \arrow[d, "d^0"] \\
\prod_{\substack{J\subseteq I\\|J|=1}} \mathscr{I}^1(U_J) \arrow[r, "\check{\delta}^0"] \arrow[d, "d^1"'] & \prod_{\substack{J\subseteq I\\|J|=2}} \mathscr{I}^1(U_J) \arrow[r, "\check{\delta}^1"] \arrow[d, "d^1"'] & \cdots \arrow[r] & \mathscr{I}^1\Big(\bigcap_{i\in J} U_i\Big) \arrow[d, "d^1"] \\
\vdots \arrow[r] \arrow[d, dotted] & \vdots \arrow[r] \arrow[d, dotted] & {} & \vdots \arrow[d, dotted] \\
\prod_{\substack{J\subseteq I\\|J|=1}} \mathscr{I}^q(U_J) \arrow[r, "\check{\delta}^0"] \arrow[d, dotted] & \prod_{\substack{J\subseteq I\\|J|=2}} \mathscr{I}^q(U_J) \arrow[r, "\check{\delta}^1"] \arrow[d, dotted] & \cdots \arrow[r] & \mathscr{I}^q\Big(\bigcap_{i\in J} U_i\Big) \arrow[d, dotted] \\
{} \arrow[r, dotted] & {} \arrow[r, dotted] & \cdots \arrow[r, dotted] & {}
\end{tikzcd}
$$

Here are a few remarks about this double complex:

1. Thanks to Corollary 6.2.0.3, each row is exact outside degree 0 and at degree 0 its cohomology is the global sections of the corresponding injective sheaf.

2. On a fixed column the cohomology is simply the product of the cohomologies of $\mathscr{F}|_{U_J}$, thanks to restriction of injective sheaf to open subsets being flasque (see Proposition 5.1.0.13).

In short we have the following,

Proposition 6.2.0.4. *There exists a first quadrant E_1 spectral sequence*

$$E_1^{p,q} := \prod_{\varnothing \subsetneq J \subseteq I, |J|=p+1} H^q(U_J, \mathscr{F}|_{U_J}) \implies H^{p+q}(X, \mathscr{F}), \tag{6.2}$$

such that the differentials $E_1^{p,0}$ are the one's corresponding to the Čech complex.

Here is a quick corollary.

Corollary 6.2.0.5. *Suppose $\mathscr{U}$ is an open cover such that $H^q(U_J, \mathscr{F}|_{U_J}) = 0$ for any $J \subseteq I$ with $|J| \geqslant 1$. Then the spectral sequence in (6.2) degenerates in the E_1-page and*

$$H^p(X, \mathscr{F}) \simeq \check{H}^p(\mathscr{U}, \mathscr{F}).$$

6.3 Cohomology of affine schemes

The aim of this section is to show that affine schemes have no higher cohomology with coefficients in a quasi-coherent sheaf. As we shall see this uniquely characterizes affine schemes among quasi compact and (quasi) separated schemes.

Theorem 6.3.0.1. *Let X be an affine scheme and $\mathscr{F}$ be a quasi-coherent sheaf on X. Then $H^p(X, \mathscr{F})$ vanishes for all $p > 0$.*

Proof. Thanks to Corollary 5.1.0.14, we can work in the category of abelian sheaves and thus use results from Section 6.2.
Claim:

1. Given any topological space X, a sheaf $\mathscr{F}$ on it and any class $\alpha \in H^q(X, \mathscr{F}), q > 0$, there exists an open cover $\mathscr{U} = \{U_i\}_{i\in I}$ of X such that $\alpha|_{U_i} \in H^q(U_i, \mathscr{F}|_{U_i})$ vanishes. Moreover if X is quasi-compact we can choose this cover to be finite.

2. Let X be an affine scheme and $\mathscr{U}$ be a finite open cover of X by *basic affine opens*. Then $\check{H}^q(X, \mathscr{F})$ vanishes for $q > 0$ and any quasi-coherent sheaf $\mathscr{F}$.

Taking these for granted we complete the proof of the theorem. Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open affine cover of X indexed by $I = \{0, 1, 2 \cdots n\}$. Then we have an E_1 spectral sequence (6.2) computing $H^p(X, \mathscr{F})$. This is a first quadrant spectral sequence and we note that the complex $E_1^{*,0}$ is simply the Čech complex of the cover which by claim (2) is exact in positive degrees. Thus $E_\infty^{p,0}$ vanishes for $p > 0$.
Further the filtration of $H^1(X, \mathscr{F})$ induced by this is particularly simple, it is a two-step filtration i.e

1. $F^iH^1(X, \mathscr{F}) = H^1(X, \mathscr{F})$ for all $i \leqslant 0$.

2. $F^iH^1(X, \mathscr{F}) = 0$ for all $i \geqslant 2$.

Thus we have an exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1(X, \mathscr{F}) \longrightarrow E_\infty^{0,1} \longrightarrow 0.$$

Notice also that $E_\infty^{0,1} = \ker(\prod_{i\in I} H^1(U_i, \mathscr{F}|_{U_i}) \to \prod_{i_o<i_i} H^1(U_{i_0} \cap U_{i_i}, \mathscr{F}_{U_{i_0}\cap U_{i_1}}))$ and that $E_\infty^{0,1} = \check{H}^1(\mathscr{U}, \mathscr{F})$[3].

[3] This short exact sequence is valid for any topological space and a finite cover $\mathscr{U}$ of X.

Since $E_\infty^{1,0}$ vanishes, this implies that $H^1(X,\mathscr{F}) \simeq E_\infty^{0,1} \hookrightarrow \prod_i H^1(U_i, \mathscr{F}|_{U_i})$. Note that U_i's are *any* finite open affine cover of X. Since any non-zero element of α can be (locally) killed by choosing an appropriate open affine cover, we must have $H^1(X,\mathscr{F}) = 0$ for any affine scheme X and a quasi-coherent sheaf $\mathscr{F}$.

Now suppose we have proved the vanishing of $H^p(X,\mathscr{F})$ for $1 \leqslant p \leqslant q-1$ and *any* affine X with a quasi-coherent sheaf $\mathscr{F}$. Assuming this we shall prove the vanishing at $H^q(X,\mathscr{F})$. By the induction step our E_1 spectral sequence will have vanishing $E_1^{p,q'}$ for all $1 \leqslant p \leqslant q-1$ and any q'. Thus $E_\infty^{p,q'} = 0$ for all $1 \leqslant p \leqslant q-1$ and any q'. Together with the vanishing of $E_\infty^{p,0}, p \geqslant 1$ this implies as before that

$$H^q(X,\mathscr{F}) \simeq E_\infty^{0,q} \hookrightarrow \prod_i H^q(U_i, \mathscr{F}|_{U_i}).$$

Arguing as before we show that $H^q(X,\mathscr{F})$ cannot have any non-zero classes. Thus proving the theorem modulo the claim, which we shall prove now.

Proof of Claim 1: This is simple. By definition to compute $H^q(X,\mathscr{F})$, we take an injective resolution $\mathscr{I}^\bullet$ of $\mathscr{F}$, and thus any element $\alpha \in H^q(X,\mathscr{F})$ begins life in $\mathscr{I}^q(X)$ as the kernel of the map to $\mathscr{I}^{q+1}(X)$. Since $\mathscr{I}^\bullet$ is an injective resolution, of sheaves, there exists an open cover $\mathscr{U}$ of X such that $\alpha|U_i$ is necessarily in the image of the map from $\mathscr{I}^{q-1}(U_i) \to \mathscr{I}^q(U_i)$. Since injective sheaves are flasque (see Proposition 5.1.0.13), this means that as a class in cohomology, α vanishes when restricted to the U_i's.

Proof of Claim 2:

Suppose $X = \mathrm{Spec}(A)$, $\mathscr{F} = \tilde{M}$ and $U_i = \mathrm{Spec}(A_{f_i})$ we need to show that the following complex is exact:

$$0 \to M \to \prod_i M\left[f_i^{-1}\right] \to \prod_{i_0<i_1} M\left[(f_{i_1}f_{i_2})^{-1}\right] \to \cdots$$

We may check exactness after a faithfully flat base change $\sqcup_i \mathrm{Spec}(A_{f_i}) \to \mathrm{Spec}(A)$, and since finite product of exact sequences is exact it suffices to check this after tensoring this with A_{f_i}. In which case the result follows from Corollary 6.2.0.3.

□

We now deduce some corollaries.

Corollary 6.3.0.2. *Let X be a separated (over $\mathbb{Z}$) scheme and $\mathscr{U}$ be a finite affine cover of X. Then $\check{H}^p(X,\mathscr{F}) \simeq H^p(X,\mathscr{F})$ for any quasi-coherent sheaf $\mathscr{F}$ on X. Moreover let I be the indexing set of such a cover. Then $H^p(X,\mathscr{F})$ vanishes for $p > |I|-1$.*

Proof. The first part follows from Corollary 6.2.0.5. The second part is a consequence of the fact that the Čech complex has non-zero terms only upto degree $|I|-1$. □

Corollary 6.3.0.3. *Let $f : X \to Y$ be an affine morphism and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then $R^q f_* \mathscr{F}$ vanishes for $q > 0$ and hence f_* is exact on the abelian category of quasi-coherent sheaves. In particular $H^p(X, \mathscr{F}) \simeq H^p(Y, f_*\mathscr{F})$*

Proof. Combine Theorem 6.3.0.1 and Proposition 5.2.1.3. □

We end this section with Serre's affineness criterion, which is a cohomological criterion to detect affineness.

Theorem 6.3.0.4 (Serre's criterion for affineness). *Let X be a quasi-compact scheme. Then TFAE*

1. *X is affine.*
2. *$H^p(X, \mathscr{F}) = 0$ for all $p > 0$ and any quasi-coherent sheaf $\mathscr{F}$ on X.*
3. *$H^1(X, \mathscr{I}) = 0$ for any quasi-coherent sheaf of ideals $\mathscr{I}$ of $\mathscr{O}_X$.*

Proof. Thanks to Theorem 6.3.0.1 it suffices to show (3) implies (1). To do so we shall use [2, II, Exercise 2.17 (b)]. Thus we need to find $f_i, 1 \leqslant i \leqslant n$ such that

1. the ideal generated by the f_i's is $H^0(X, \mathscr{O}_X)$.
2. The X_{f_i}'s are affine

Let $x \in X$ be a closed point of X. Choose an affine open $U \ni x$ and let Z be the complement of U with the reduced induced structure. Let $Z' = Z \cup \{x\}$, also with the reduced induced structure. Consider the short exact sequence

$$0 \to \mathscr{I}_{Z'} \to \mathscr{I}_Z \to \frac{\mathscr{I}_Z}{\mathscr{I}_{Z'}} \to 0.$$

Since $Z \cap (X \backslash x) = Z' \cap (X' \backslash x)$, the quotient $\frac{\mathscr{I}_Z}{\mathscr{I}_{Z'}}$ is supported only at x, further restricting to the affine open U we conclude that $\frac{\mathscr{I}_Z}{\mathscr{I}_{Z'}} = i_{x*}k(x)$, where $i_x : x \to X$ is the corresponding closed immersion. Taking the long exact sequence in cohomology of the above short exact sequence of sheaves together with the vanishing condition (3) we get a surjection

$$H^0(X, \mathscr{I}_Z) \twoheadrightarrow k(x).$$

Let $f \in H^0(X, \mathscr{I}_Z) \subseteq H^0(X, \mathscr{O}_X)$ be a lift of the $1 \in k(x)$. Then $X_f = U_f$ and hence is affine since U was affine by choice. We can do this around every closed point. Let $\tilde{X}$ be the union of open affines X_f obtained as above. Then their complement if non-empty is a non-empty quasi-compact scheme with no closed points. This is not possible (Why?). Thus $\tilde{X} = X$ and by quasi-compactness we can assume that X is covered by finitely many X_{f_i}'s. Now we shall show that the f_i's generate $H^0(X, \mathscr{O}_X)$.

□

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