AG-II-Notes

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Contents

Chapter 1

Catgeory Theory Part-0

We begin by recalling some basic notions from category theory which should take some way into the course. This is far from an exhaustive account and focuses on introducing the bare minimum needed for the purposes of these lectures.

1.1 Categories: Definitions and Examples

Recall that a category $\mathscr{C}$ consists of a collection of objects $\mathrm{Ob}(\mathscr{C})$ and a collection of morphisms between these objects. The morphisms are required to satisfy certain properties:

1. For every object A in the category, there is an identity morphism 1_A from A to A.
2. For every pair of morphisms $f : A \to B$ and $g : B \to C$, there is a composite morphism $g \circ f : A \to C$.
3. Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.
4. Composition is unital: $1_B \circ f = f = f \circ 1_A$.

Example 1.1.0.1. The category **Set** has sets as objects and functions as morphisms. The identity morphism on a set A is the identity function $\mathrm{id}_A : A \to A$. The composite of two functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$. The associativity and unitality of composition follow from the corresponding properties of functions.

Example 1.1.0.2. The category **Top** has topological spaces as objects and continuous functions as morphisms. The identity morphism on a topological space X is the identity function $\mathrm{id}_X : X \to X$. The composite of two continuous functions $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f : X \to Z$. The associativity and unitality of composition follow from the corresponding properties of continuous functions.

Example 1.1.0.3. The category $\mathbf{Vect}_k$ has vector spaces over a field k as objects and linear transformations as morphisms. The identity morphism on a vector space V is the identity transformation $\mathrm{id}_V : V \to V$. The composite of two linear transformations $f : V \to W$ and $g : W \to Z$ is the transformation $g \circ f : V \to Z$. The associativity and unitality of composition follow from the corresponding properties of linear transformations.

Example 1.1.0.4. Let S be a scheme. Let $\mathbf{Sch}_S$ be the category whose objects are a pair (X, f), where X is a scheme and $f : X \to S$ a morphism. Morphisms ϕ in this category are commutative diagrams of the form

$$\begin{tikzcd} (X,f) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & (Y,g) \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

An important special case for us is the category $\mathbf{Sch}_k$ of schemes over a field $\mathrm{Spec}(k)$.

Example 1.1.0.5. Let X be a topological space. The category $\mathbf{Op}(X)$ has open sets in X as objects and inclusions as morphisms. The identity morphism on an open set U is the inclusion $U \hookrightarrow U$. The composite of two inclusions $U \hookrightarrow V$ and $V \hookrightarrow W$ is the inclusion $U \hookrightarrow W$. The associativity and unitality of composition follow from the corresponding properties of inclusions. In partiular for any two objects U and V either $\mathrm{Hom}_{\mathbf{Op}(X)}(U, V)$ is either empty or contains a unique morphism.

Example 1.1.0.6. Let $\mathscr{C}$ be a category. The opposite category $\mathscr{C}^{\mathrm{op}}$ has the same objects as $\mathscr{C}$ and morphisms reversed. That is, for every pair of objects A and B in $\mathscr{C}$, we have $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$. The identity morphism on an object A in $\mathscr{C}^{\mathrm{op}}$ is the identity morphism on A in $\mathscr{C}$. The composite of two morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}^{\mathrm{op}}$ is the composite $g \circ f : A \to C$ in $\mathscr{C}$. The associativity and unitality of composition follow from the corresponding properties of composition in $\mathscr{C}$.

1.2 Functors

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $F : \mathscr{C} \to \mathscr{D}$ assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$ and to each morphism $f : A \to B$ in $\mathscr{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathscr{D}$. Functors are required to satisfy the following properties:

1. For every object A in $\mathscr{C}$, we have $F(1_A) = 1_{F(A)}$.

2. For every pair of morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$, we have $F(g \circ f) = F(g) \circ F(f)$.

One can also have what are called as contravariant functors. A contravariant functor $F : \mathscr{C} \to \mathscr{D}$ assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$ and to each morphism $f : A \to B$ in $\mathscr{C}$ a morphism $F(f) : F(B) \to F(A)$ in $\mathscr{D}$. Contravariant functors are required to satisfy properties analogous to those for covariant functors.

Example 1.2.0.1. The forgetful functor F : **Top** → **Set** assigns to each topological space its underlying set and to each continuous function its underlying function. The identity function on a set is continuous, so the identity morphism on an object in **Top** is sent to the identity morphism on the corresponding object in **Set**. The composite of two continuous functions is continuous, so the composite of two morphisms in **Top** is sent to the composite of the corresponding morphisms in **Set**.

A more non-trivial functor from **Top** to **Set** is the functor Π_0.

Example 1.2.0.2. The functor Π_0 : **Top** → **Set** assigns to each topological space X the set of connected components $\Pi_0(X)$ of X and to each continuous function $f : X \to Y$ the function $\Pi_0(f) : \Pi_0(X) \to \Pi_0(Y)$ induced by f.

We now state a few properties of functors.

Definition 1.2.0.3. A functor $F : \mathscr{C} \to \mathscr{D}$ is faithful if for every pair of objects A and B in $\mathscr{C}$, the map $F : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ is injective. We say that F is fully faithful if this map is bijective.

Definition 1.2.0.4. A functor $F : \mathscr{C} \to \mathscr{D}$ is essentially surjective if for every object B in $\mathscr{D}$, there is an object A in $\mathscr{C}$ such that $F(A)$ is isomorphic to B.

The examples 1.2.0.1 and 1.2.0.2 are faithful and essentially surjective functors. Next we will discuss an important class of functors called representable functors.

Example 1.2.0.5. Let $\mathscr{C}$ be a category and A an object in $\mathscr{C}$. The representable functor $\mathrm{Hom}_{\mathscr{C}}(A, -) : \mathscr{C} \to$ **Set** assigns to each object B in $\mathscr{C}$ the set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of morphisms from A to B and to each morphism $f : B \to C$ in $\mathscr{C}$ the function $\mathrm{Hom}_{\mathscr{C}}(A, f) : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$ induced by f. The identity morphism on an object B in $\mathscr{C}$ is sent to the identity morphism on $\mathrm{Hom}_{\mathscr{C}}(A, B)$, and the composite of two morphisms $f : B \to C$ and $g : C \to D$ in $\mathscr{C}$ is sent to the composite of the corresponding morphisms $\mathrm{Hom}_{\mathscr{C}}(A, f)$ and $\mathrm{Hom}_{\mathscr{C}}(A, g)$.

Next we discuss natural transformations of functors.

Definition 1.2.0.6. Let F and G be two functors between categories $\mathscr{C}$ and $\mathscr{D}$. A natural transformation $\eta : F \to G$ assigns to each object A in $\mathscr{C}$ a morphism $\eta_A : F(A) \to G(A)$ in $\mathscr{D}$ such that for every morphism $f : A \to B$ in $\mathscr{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes. That is, we have $G(f) \circ \eta_A = \eta_B \circ F(f)$ for every morphism $f : A \to B$ in $\mathscr{C}$.

Example 1.2.0.7. Let $\mathbf{Vect}_k$ be the category of vector spaces over a field k. The double dual functor $\mathbf{Vect}_k \to \mathbf{Vect}_k$ assigns to each vector space V its double dual $V^{\vee\vee}$ and to each linear transformation $f : V \to W$ the linear transformation $f^{\vee\vee} : V^{\vee\vee} \to W^{\vee\vee}$ induced by f. The natural transformation $\eta : \mathrm{id} \to (-)^{\vee}$ assigns to each vector space V the canonical map $\eta_V : V \to V^{\vee\vee}$ and to each linear transformation $f : V \to W$ the commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"] \\ V^{\vee\vee} \arrow[r, "f^{\vee\vee}"'] & W^{\vee\vee} \end{tikzcd} .$$

Note that the vertical arrows are isomorphisms if and only if the vector spaces are finite-dimensional.

Definition 1.2.0.8. A natural transformation $\eta : F \to G$ of functors is a natural equivalence if for every object A in $\mathscr{C}$, the morphism $\eta_A : F(A) \to G(A)$ is an isomorphism in $\mathscr{D}$.

Now we are ready to state the Yoneda Lemma.

Lemma 1.2.0.9 (Yoneda Lemma). *Let $\mathscr{C}$ be a category and A an object in $\mathscr{C}$. Let $F : \mathscr{C} \to$ **Set** be a functor. Then the natural transformations Hom$_{\mathscr{C}}(A, -) \to F$ are in bijection with the elements of $F(A)$.*

Proof. Let $\eta : \mathrm{Hom}_{\mathscr{C}}(A, -) \to F$ be a natural transformation. In particular, $\eta_A : \mathrm{Hom}_{\mathscr{C}}(A, A) \to F(A)$ is a morphism in **Set**. The desired element in $F(A)$ is simply the image of the identity morphism on A under η_A. Conversely, given an element x in $F(A)$, we can define a natural transformation $\eta : \mathrm{Hom}_{\mathscr{C}}(A, -) \to F$ by setting $\eta_B(f) = F(f)(x)$ for every object B in $\mathscr{C}$ and morphism $f : A \to B$ in $\mathscr{C}$. The naturality of η follows from the properties of functors. □

In particular we note the following corollary.

Corollary 1.2.0.10. *Let C be a category. Then the functor Hom$_{\mathscr{C}}(A, -) : \mathscr{C}^{op} \to$ Func($\mathscr{C}$, **Set**) is fully faithful, where Func($\mathscr{C}$, **Set**) is the category whose objects are functors from $\mathscr{C}$ to **Set** and morphisms are natural transformations.*

Chapter 2

Flatness

Consider the following three maps:

1. $f : \mathrm{Bl}_{(0,0)}\mathbb{A}^2 \to \mathbb{A}^2$,where $\mathrm{Bl}_{(0,0)}\mathbb{A}^2$ is the blow-up of $\mathbb{A}^2$ at the origin and f is the projection map.

2. $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ with $f(z) = z^2$.

3. $f : G_m \to G_m$ with $f(z) = z^2$. Here G_m is $\mathbb{A}^1 \backslash \{0\}$.

The map (1) here is an isomorphism on the complement of the origin, but over the origin the fiber is $\mathbb{P}^1$. The map (2) is nice outside the origin, with the inverse image of any $z \neq 0$ consisting of two points. But at the origin the fiber consists of exactly one point. The map (3) is simply the base change of (1) along the open immersion $G_m \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$, and hence all points have as inverse image exactly two distinct points.

Question 2.0.0.1. How do we capture the discontinuous jump in the fiber dimension at the origin in Example 1? Note that even though Example (2) has a *bad* fiber over the origin, it is still of dimension 0 like every other fiber.

The answer lies in the notion of flatness, a purely algebraic construct!

2.1 Flatness: Definition and Properties

We begin by defining flatness and faithful flatness.

Definition 2.1.0.1. Let A be a ring and M be an A-module. We say that M is **flat** over A if the right-exact functor $- \otimes_A M$ is exact. A map of rings $A \to B$ is said to be **flat** if B is flat as an A-module.

Definition 2.1.0.2. A flat A-module M is said to be **faithfully flat** if the functor $- \otimes_A M$ is faithful.

Let us see some examples of flat and faithfully flat modules.

Example 2.1.0.3.
1. The ring A is flat over itself.
2. Since tensor products are right adjoint, they commute with arbitrary colimits. moreover *filtered* colimits of exact sequences is exact. Combining these two, we get that filtered colimits of flat modules are flat.
3. Combining (1) and (2) we get that filtered colimits of the form $\mathrm{colim}_i M_i$, where each M_i is abstractly isomorphic to A is flat. Note that we dont care what the maps are as long as the indexing category is filtered.

Example 2.1.0.3, (3) has the following corollary.

Corollary 2.1.0.4. *The ring A_f is flat over A. More generally for any multiplicative subset S of A, the ring $A[S^{-1}]$ is flat.*

Proof. The first claim follows from the isomorphism

$$A_f \simeq \mathrm{colim}\{A \to A \to A \cdots\},$$

where the transition maps are multiplication by f. The second part of the claim follows from the isomorphism

$$A[S^{-1}] = \mathrm{colim}_{f \in S} A_f,$$

where the colimit is over the directed set indexed by elements of S, with $f \leqslant g$ if $g = ff'$ for some $f' \in A$. This is directed because S is multiplicative and further the first part of the Corollary implies each of the A_f's are flat. Hence the result. ☐

Corollary 2.1.0.5. *For any ring A, arbitrary direct sums of A is a flat A-module. In particular when A is a field, all A-modules are flat.*

Corollary 2.1.0.6. *For any ring R the map $R \to R[x]$ is flat.*

Proof. Direct sums are colimits over an directed set with no non-identity arrows, hence the result. ☐

Next we list some properties of flatness.

Proposition 2.1.0.7. *We will need the following facts about flatness. Let $\phi : A \to B$ be a map of rings, M be an A-module and N a B-module. Then the following hold*

1. *M is flat over A iff for all finitely generated ideals $\mathfrak{a}$ of A the induced map*
$$\mathfrak{a} \otimes_A M \to M,$$
is injective.
2. *(Base-Change) M is flat over A implies $M \otimes_A B$ is flat over B.*
3. *(Transitivity) B flat over A and N flat over B implies N is flat over A.*
4. *(Local Nature) M is flat over A iff $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of A.*
5. *N is flat over A iff $N_\mathfrak{q}$ is flat over $A_\mathfrak{p}$ for all prime ideals $\mathfrak{q}$ of B, here $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$.*
6. *For a short exact sequence of A-modules*
$$0 \to M' \to M \to M'' \to 0,$$
M is flat if M' and M'' are flat. Also if M and M'' are flat, so is M'.
7. *For a Noetherian local ring A, a finitely generated module M is flat over A iff M is free over A.*

Proof. (1) is proved in Tag 00HD, (2) in Tag 051D, (3) in Tag 051D, (4) and (5) in Tag 051D, (6) in Tag 00HM and finally (7) in Tag 00NZ[1] □

We can now globalize the definition of flatness to schemes.

Definition 2.1.0.8 (Flatness)**.** Let $f : X \to Y$ be a morphism of schemes. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ (resp. f) is flat over Y at a point $x \in X$ if the stalk $\mathscr{F}_x$ (resp. $\mathscr{O}_{X,x}$) is flat as a $\mathscr{O}_{Y,f(y)}$-module. If this holds for all points x in X we say $\mathscr{F}$ is flat over Y (resp. f is a flat morphism).

Remark 2.1.0.9. Note that flatness is local on both the source and the base. Meaning to check a sheaf $\mathscr{F}$ is flat (over Y) it suffices to check this on an open cover of either X or Y or both.

Now we translate Proposition 2.1.0.7 into the language of scheme.

Proposition 2.1.0.10. *Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ a $\mathscr{O}_X$-module of X. Then the following hold.*

[1] If you assume A is Noetherian, the proof can be simplified. As in the proof by Nakayama's Lemma we can pick a surjection $A^n \to M$ where n is the dimension of $\frac{M}{\mathfrak{m}M}$. Here $\mathfrak{m}$ is the unique maximal ideal of A. Suppose K is the kernel of this surjection. Then tensoring this exact sequence with $\frac{A}{\mathfrak{m}}$, we get that $\frac{K}{\mathfrak{m}K}$ is trivial by flatness of M, which by Nakayama implies K is trivial. (Question: Where did we use A is Noetherian?)

1. *If f is an open immersion then it is flat.*

2. *Suppose both X and Y are affine schemes, say $X = \mathrm{Spec} B$ and $Y = \mathrm{Spec} A$. Then $\mathscr{F}$ is flat over Y iff M is flat over A where M is the A-module corresponding to $\mathscr{F}$.*

3. *A base change of a flat quasi-coherent sheaf[2] is flat. That is if we have a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

and assume that $\mathscr{F}$ is flat and quasi-coherent, then the pullback $\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'}$ is flat over Y'.

4. *Suppose f was morphism over a base scheme S. If $\mathscr{F}$ is flat over Y and Y is flat over S, then $\mathscr{F}$ is flat over S. In particular composition of flat morphisms is flat.*

5. *Suppose we have a short exact sequence of quasi-coherent sheaves*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

Then $\mathscr{F}$ is flat if $\mathscr{F}'$ and $\mathscr{F}''$ are flat. Also if $\mathscr{F}$ and $\mathscr{F}''$ are flat, so is $\mathscr{F}'$.

6. *Let X be a Noetherian scheme and $\mathscr{F}$ a coherent sheaf. Then $\mathscr{F}$ is flat iff $\mathscr{F}$ is locally free aka a vector bundle.*

Proof. (1) is immediate from the definition since the induced map on local rings is an isomorphism. (2) follows from Proposition 2.1.0.7, (5). The claims (3)-(6) are now a consequence of Remark 2.1.0.9 and Proposition 2.1.0.7. □

Remark 2.1.0.11. 1. Fix a base scheme S. Consider the subcategory of $\mathbf{Sch}_S$ where we only allow morphisms which are flat between the objects. This is a subcategory of $\mathbf{Sch}_S$, and is closed under composition and base change.

2. Thanks to Corollary 2.1.0.6 and Remark 2.1.0.9, for any scheme X, the morphism $\mathbb{A}^n_X \to X$ is flat. More generally for an locally free sheaf $\mathscr{E}$ on a scheme X, the map $\mathbb{A}(\mathscr{E}) \to X$ is flat. Again using Remark 2.1.0.9, we can conclude that $\mathbb{P}(\mathscr{E}) \to X$ is flat.

Recall for any topological space X and a pair of points x and y in X, we have the following:

(a) x is a specialization of y if $x \in \overline{\{y\}}$.

(b) x is a generalisation of y if $y \in \overline{\{x\}}$.

[2] Hartshorne forgets writing quasi-coherent in Chapter III.9, Proposition 9.2 (b).

In particular when $X = \mathrm{Spec}(A)$, the constructible subsets of X which are stable under generalisation are open and those stable under specialization are closed (see [1, Chapter II, Exercise 3.18]

Proposition 2.1.0.12. *Let $f : X \to Y$ be a flat morphism of schemes. Then the image*[3] *of f is stable under generalization.*

Proof. Let y be a point in the image of f. We need to show that any point $y' \in Y$ such that $y \in \overline{\{y'\}}$, also belongs to the image of f. Choose an affine open $V \ni y$ and an affine open $U \ni x$ such that $f(x) = y$ and $f(U) \subseteq V$. It suffices to show that there is a point $x' \in U$ such that $f(x') = y'$. But this is precisely the going down theorem from local algebra (see Tag 00HS). □

Corollary 2.1.0.13 (Openness of flat morphisms). *Let $f : X \to Y$ be a flat morphism, locally of finite presentation*[4]*. Then f is universally open i.e the image of any base change of f is open.*

Proof. Since both flat morphisms and morphisms of finite presentation satisfy BC, we are reduced to showing the openness of f. We have already shown that the image of f is stable under generalizations (without any finite presentation assumptions). As before we can assume that both $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine with the map $A \to B$ bein of finite presentations. By Chevalley's theorem (see Tag 00FE), Im(f) is constructible and by Prop 2.1.0.12 it is stable under generalizations and hence is open. □

Corollary 2.1.0.14. *Let $f : A \to B$ be a local and flat morphism of local rings. Then the induced maps on Spec is surjective.*

Proof. This is essentially the content of going down theorem. Every point of $\mathrm{Spec}(A)$ is a generalisation of the unique closed point. □

Corollary 2.1.0.15. *Let $f : X \to Y$ be flat and proper morphism of finite presentation such that Y is irreducible. The f is surjective.*

2.2 Flatness and dimension of fibers

The following Proposition tells us that flat morphisms have well behaved fibers. This is mysterious (at least to me) given that flatness itself had a very algebraic definition.

[3]the set theoretic image

[4]For those who want to remain in the Noetherian world, anytime I say finite presentation you may assume that the schemes are Noetherian and that the morphism is of finite type.

Proposition 2.2.0.1. *Let $f : X \to Y$ be a flat morphism of locally Noetherian*[5] *schemes. Then for any point $x \in X$ we have,*

$$\dim(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{Y,f(x)}) + \dim(\mathscr{O}_{X_y,x}).$$

Proof. Since everything is local in x and y we may assume everything is sight is the Spectrum of a Noetherian ring. In which case the result follows from Tag 00ON. □

Example 2.2.0.2. This shows that the morphism (1) in the beginning of the chapter is not flat! The fiber over the origin is of dimension 1, while the fibers over other points are of dimension 0.

We derive one more corollary from Proposition 2.2.0.1.

Corollary 2.2.0.3. *Let $f : X \to Y$ be a flat morphism of schemes finite type over a field k with Y equidimensional*[6]*. Then TFAE*

1. *X is equidimensional of dimension equal to dim$(Y) + n$.*

2. *All fibers (not necessarily over closed points) of f are equidimensional of dimension n.*

In particular if both X and Y are irreducible then dim$(X) \geqslant$ dim(Y) and all the fibers are equidimensional of dimension dim(X) – dim(Y).

Proof. Suppose X if equidimensional of dimension dim$(Y) + n$. Let y be a *closed* point in Y with residue field $k(y)$. We would like to show that $X_y := X \times_{k(y)} Y$ is equidimensional of dimension n. Choose any irreducible component of X_y and in that component choose a closed point x in X_y. Note that x is closed in X (Why?). Then the dimension of X, X_y and Y can be computed using the dimension of the local rings at the points x and y. Thus we are done by Proposition 2.2.0.1.
Reduction the case y a closed point: Now suppose y is a possibly non closed point of Y. Then note that the map $\mathrm{Spec}(k(y)) \to Y$ factors via $Y \times_k k(y)$ and X_y can be considered as a fiber of the map induced between $X \times_k k(y) \to Y \times_k k(y)$ over the closed point $k(y)$ of $Y \times_k k(y)$. Note that both $X \times_k k(y)$ and $Y \times_k k(y)$ continue being equidimensional of dimension dim(X) and dim(Y) respectively (see Tag 00P4).
For the converse, choose a closed point $x \in X$, then $f(x) \in Y$ is a closed point (why?). Then again we are done by Proposition 2.2.0.1. □

But more is true! We have the following *miraculous* result, known colloquially as the *Miracle Flatness Theorem* due to Hironaka.

Theorem 2.2.0.4 (Miracle Flatness Theorem)**.** *Let $R \to S$ be a local morphism of Noetherian local rings. Assume that*

[5]We really need this to ensure dimensions are finite.
[6]Each irreducible component of Y has the same dimension.

1. *R is a regular local ring.*

2. *S is Cohen-Macaulay.*

3. *The dimension formula holds i.e,*

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S),$$

   where $\mathfrak{m}$ is the maximal ideal of R.

Then $R \to S$ is flat!

This has the following very useful corollary.

Corollary 2.2.0.5 (Miracle Flatness Theorem for schemes). *Let $f : X \to Y$ be a morphism of locally Noetherian schemes such that X is Cohen-Macaulay and Y is regular. Then f is flat iff the dimension formula holds.*

Example 2.2.0.6. This immediately implies that the examples (2) and (3) in the beginning of the chapter are flat. The fibers are of constant dimension 0.

Chapter 3

Faithful Flatness

3.1 Faithfully flat morphisms

Let $\phi : A \to B$ be a flat morphism of rings. We say ϕ is *faithfully* flat if B is a faithfully flat A-module. Surprisingly faithful flatness can be captured set theoretically!

Lemma 3.1.0.1. *ϕ is faithfully flat iff it is flat and the induced map $\phi^{\#} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.*

Proof. Let $\mathfrak{p}$ be a prime in A, then the induced map $A \to k(\mathfrak{p})$ is non-zero iff $A \otimes_A B \to k(\mathfrak{p}) \otimes_A B$ is non-zero.The latter necessarily implies the fiber over $\mathfrak{p}$ is non-empty. Conversely suppose $\phi^{\#}$ is surjective. We shall prove that for any A-module M, $M \otimes_A B = 0$ iff $M = 0$, a well known criterion for $m \in M$ different from zero inducing an injection

$$0 \longrightarrow \frac{A}{I} \longrightarrow M,$$

here I is the annhilator of $m \in M$. Tensoring the above exact sequence with the flat ring B and knowing that $B \otimes_A \frac{A}{I}$ is non-zero implies the required result. $\square$

Combining Corollary 2.1.0.14 and Lemma 3.1.0.1 we obtain the following result.

Corollary 3.1.0.2. *Flat and local maps of local rings are faithfully flat.*

Motivated by Lemma 3.1.0.1 we have the following definition.

Definition 3.1.0.3. A morphism of schemes $f : X \to Y$ is said to be faithfully flat if it is flat and surjective.

Example 3.1.0.4. Now we give some examples of faithfully flat morphisms

1. Any extension of fields $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is faithfully flat.

2. Any proper and flat morphism whose target is an irreducible scheme is faithfully flat.

3. Let X be an affine scheme and let $X_{f_i}, 1 \leqslant i \leqslant n$ be a finite cover by basic affines, then

$$\sqcup_i X_{f_i} \to X,$$

is faithfully flat.

4. Let X be the projective space $\mathbb{P}^n$ and let $D(x_i), 0 \leqslant i \leqslant n$ be the standard affine covering corresponding to a choice of homogeneous coordinates. Then

$$\sqcup_i D(x_i) \to \mathbb{P}^n,$$

is faithfully flat.

We note the following obvious lemma.

Lemma 3.1.0.5. *Faithfull flatness is stable under base change and composition.*

3.2 Faithfully flat descent

Let X be any scheme and let $\{U_i\}_{1 \leqslant i \leqslant n}$ be an open cover of X. We have the following cartesian diagram

$$\begin{tikzcd} \sqcup_{i,j} U_i \cap U_j \arrow[r, "p_2"] \arrow[d, "p_1"] & \sqcup_i U_i \arrow[d, "f"] \\ \sqcup_j U_j \arrow[r, "f"] & X \end{tikzcd} .$$

Moreover for any schem T giving a morphism $X \to T$ is the same as giving a collection of morphisms $U_i \to T$ which agree on the intersections $U_i \cap U_j$. Put differently the following sequence of sets is exact

$$\mathsf{Hom}(X,T) \xrightarrow{f^*} \prod_i \mathsf{Hom}(U_i, T) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{i,j} \mathsf{Hom}(U_i \cap U_j, T).$$

There is nothing special about schemes here, one could have donethe same starting with any toplogical space X and a cover $\{U_i\}_{1 \leqslant i \leqslant n}$. However doing so obscures the following important fact, the exactness of the above sequence is a consequence of faithfully flatness of f! This is the content of the following theorem.

Theorem 3.2.0.1 (Faithfully Flat descent). *Let $f : Y \to X$ be a fully faithful and quasi-compact morphism of schemes over S*[1] *Then for any scheme T over S, the sequence of sets*

$$Hom_S(X,T) \xrightarrow{f^*} Hom_S(Y,T) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} Hom_S(Y \times_X Y, T),$$

is exact. Here p_1 and p_2 are projections $Y \times_X Y \to Y$.

For a proof see Tag 023Q.
We shall concern ourselves with applications of this result. Let K/k be a finite Galois extension of field with Galois group G. Let X, Y be schemes over k. Let

$$X_K := X \times_k K, Y_K := Y \times_k K.$$

Every element $\sigma \in G$ acts on K while fixing k, thus inducing a morphism of $\mathrm{Spec}(K)$ as k-scheme. By functoriality of the fiber product we get an induced action of σ on $X_K := X \times_k K$ and $Y_K := Y \times_k K$. We denote this action by σ_X and σ_Y. Note that σ_X and σ_Y are *not* morphisms of K-schemes, rather they are only morphisms of k-schemes. Finally we get an action of G on $\mathrm{Hom}_K(X_K, Y_{K)})$ as follows:

$$f \to \sigma_Y \circ f \circ \sigma_X^{-1}. \tag{3.1}$$

Corollary 3.2.0.2 (Galois Descent). *The natural map $Hom_k(X,Y) \to Hom_K(X_K, Y_K)$ has image*

$$Hom_K(X_K, Y_K)^G,$$

i.e. precisely those morphisms that are invariant under G.

Proof. Lets start with some basic analysis. Since K/k is Galois we choose an $\alpha \in K$, such that $K = k(\alpha)$ as k-algebras. If $f(x)$ is the minimal polynomial of α, then we have

$$K \simeq \frac{k[x]}{(f(x))},$$

with $x \to \alpha$ under this isomorphism. Using the above isomorphism we identify

$$K \otimes_k K \simeq K \otimes_k \frac{k[x]}{(f(x))} \simeq \frac{K[x]}{(f(x))}.$$

Note that under the above isomorphism $\alpha \otimes 1 \to \alpha$ while $1 \otimes \alpha \to x$. Since K is the splitting field of $f(x)$, we can further identify

$$\psi : K \otimes_k K \simeq \prod_i \frac{K[x]}{(X - \alpha_i)} \simeq \prod_i K,$$

[1] Grothendieck coined the acronym *fpqc* (fidèlement plat et quasi-compact) for such morphisms.

where α_i's are the conjugates of α in K. Note that Ψ is a map of k-algebras and maps $\alpha \otimes 1 \to \alpha$ while $1 \otimes \alpha \to \alpha_i$ along the i^{th}-component. Put differently $1 \otimes \alpha \to \prod_{\sigma \in G} \sigma(\alpha)$. To summarize the diagram

$$K \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} K \otimes_k K$$

is isomorphic to the diagram

$$K \underset{\prod_{\sigma \in G} \sigma}{\overset{\Delta}{\rightrightarrows}} \prod_i K. \tag{3.2}$$

Now we can get back to proving the corollary. Consider the Cartesian diagram

$$\begin{tikzcd}
X_K \times_X X_K \arrow[r, "p_2"] \arrow[d, "p_1"] & X_K \arrow[d, "f"] \\
X_K \arrow[r, "f"] & X
\end{tikzcd}$$

The morphism f is fpqc and hence by Theorem 3.2.0.1 we have the exact sequence

$$\mathrm{Hom}_k(X, Y) \xrightarrow{f^*} \mathrm{Hom}_k(X_K, Y) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_k(X_K \times_X X_K, Y).$$

Note that we have isomorphisms $X_K \times_X X_K \simeq X \times_k (K \otimes_K K) \simeq \sqcup_{\sigma \in G} X$, where the first one comes from properties of fiber product and the last one is the isomorphism ψ above. Further under this identification we may identify $p_1 : X_K \times_X X_K \to X$ with map which is identity on each of the factors, while $p_2 : X_K \times_X X_K \to X$ is identified with the map which sends the factor X corresponding to σ by σ_X onto X. Consider the diagram

$$\begin{tikzcd}
X_K \arrow[r, "\sigma_X"] \arrow[d] & X_K \arrow[r, "\phi"] \arrow[d] & Y \arrow[d] \\
\mathrm{Spec}(K) \arrow[r, "\sigma_{\mathrm{Spec}(K)}"] & \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd} \tag{3.3}$$

Note that by properties of fiber product any such ϕ induces a map $\tilde{\phi} : X_K \to Y_K$ and similarly $\phi \circ \sigma_X$ induces a map $\widetilde{\phi \circ \sigma_X} : X_K \times Y_K$ for all $\sigma \in G$. Thanks to the diagram 3.3 we get the desired conclusion. $\square$

Here is a simple example to see this in action.

Example 3.2.0.3. Let $X = Y = \mathrm{Spec}(\mathbb{R}[x])$. A morphism $f : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is given by $x \to p(x)$, for a complex polynomial $p(x)$. By our criterion this descends iff $\bar{p}(x) = p(x)$, here $\bar{p}(x)$ is the polynomial obtained by applying the uniqe non-trivial element of Gal($\mathbb{C}/\mathbb{R}$) on the coeffcicients of $p(x)$. In other words $p(x)$ should be a polynomial with real coefficients.

Corollary 3.2.0.4. *Let $\phi : X \to Y$ be a morphism in **Sch**$_S$. Let $f : S' \to S$ be a fpqc morphism. Then ϕ is an isomorphisms iff its base change along f is.*

Example 3.2.0.5. Suppose $f : X \to Y$ is a morphism of varieties over the rational numbers $\mathbb{Q}$. Let us say you want to prove that f is an isomorphism. Corollary 3.2.0.4 implies that we can base change to $\mathbb{C}$ to prove this. In certain situations this can be quite profitable, for example one can use analytic techniques over $\mathbb{C}$ to prove this which apriori were not accesible over $\mathbb{Q}$.

Corollary 3.2.0.4 is the tip of the fpqc descent iceberg. Colloquially Corollary 3.2.0.4 is referred to by saying that isomorphisms descent along fpqc covers. Here $f : S' \to S$ is thought of as an "cover" of S. We have the following beautiful result.

Theorem 3.2.0.6. *The following properties of morphisms descend along a fpqc cover:*

1. *separatedness,*
2. *properness,*
3. *affineness,*
4. *open immersion,*
5. *closed immersion,*
6. *finitness,*
7. *quasi-finiteness.*

For a proof see Tag 02YJ. Before we end this section I would like to state one more result which is a consequence of faithfully flat descent. Let us revisit Example 3.1.0.4 (4). This open covering was crucial in constructing quasi-coherent sheaves on projective space. Well it turns out that all we needed was that the covering was faithfully flat. This is the content of the following theorem.

Theorem 3.2.0.7. *Let $f : Y \to X$ be a fpqc morphism of schemes. Then there is an equivalence of categories between quasi-coherent sheaves on X and those quasi-coherent sheaves $\mathscr{F}$ on Y which satisfy gluing (or more appropriately descend) conditions:*

1. *There exists an isomorphism $\alpha : p_1^*\mathscr{F} \simeq p_2^*\mathscr{F}$ on $Y \times_X Y$.*
2. *α satisfies the cocycle condition on $Y \times_X Y \times_X Y$,*
$$p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha.$$
   Here p_{ij} is the projection onto the i^{th} and j^{th} factors.

Moreover the equivalence above respects coherence, local freeness etc..For a proof we refer to Tag 023R

Bibliography

[1] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, 1977. 13