Problem Set 4
Due date 26th March 2025

Shuddhodan Kadattur Vasudevan

March 10, 2025

1. In this problem you will prove the following theorem due to Grothendieck:

Theorem 0.0.1 (Grothendieck)**.** *Let X be a Noetherian topological space. Then $H^p(X, \mathscr{F}) = 0$ for any $p > \dim(X)$ and any abelian sheaf $\mathscr{F}$ on X*[1]*.*

We shall do this in steps.

(a) Show that we may assume X is finite dimensional and prove the theorem for $\dim(X) = 0$. We now assume $\dim(X) = d > 0$ and the result known for all X with $\dim(X) \leqslant d - 1$.

(b) Use Lemma 6.1.0.2 from the class notes and induction on the number of irreducible components of X, to reduce to the case where X is irreducible.

(c) Since X is irreducible, show that $\mathbb{Z}_X$ is flasque and hence $H^p(X, \mathbb{Z}_X) = 0$[2] for $p > 0$. Use this and induction hypothesis to conclude that $H^p(X, j_!\mathbb{Z}_U) = 0$ for $p > \dim(X)$ where $j : U \hookrightarrow X$ is an open immersion.

(d) Next recall that for any sheaf $\mathscr{F}$, there exists a surjection

$$\oplus_{(j:U\hookrightarrow X, s\in\mathscr{F}(U))} j_!\mathbb{Z}_U \to \mathscr{F}.$$

Let A denote the indexing subset of the above direct sum i.e elements of A consists of a pair $U \hookrightarrow X$ and an element $s \in \mathscr{F}(U)$. For any finite subset $S \subset A$, denote by $\mathscr{F}_S$, the image of

$$\oplus_{(U,s)\in S} j_!\mathbb{Z}_U \to \mathscr{F}.$$

Now let I be the collection of all finite subsets of A ordered by inclusion. Show that I is filtered and that

$$\varinjlim_{i\in I} \mathscr{F}_i \simeq \mathscr{F}.$$

[1] Observe that this is *half* of what you see in topology. For example when $X = \mathbb{P}^1_{\mathbb{C}}$, then the closed points of X, $X(\mathbb{C}) \simeq S^2$, hence $H^2(X(\mathbb{C}), \mathbb{Z}) \neq 0$.

[2] This is the only place where we "compute" cohomology!

(e) Using the fact that cohomology commutes with filtered limits of sheaves on a Noetherian topological space (see Tag 01FF) to reduce to the case where $\mathscr{F} = \mathscr{F}_S$, where $S \subset A$ is a finite set. Next induct on the number of elements in S to reduce to the case where $\#S = 1$. Hence we may assume there exists a short exact sequence

$$0 \to \mathscr{G} \to j_! \mathbb{Z}_U \to \mathscr{F} \to 0,$$

where $j : U \hookrightarrow X$ is an open immersion.

(f) Let $\eta \in X$ be the unique generic point. Show that $\mathscr{G} = 0$ iff $\mathscr{G}_\eta = 0$. Conclude that is $\mathscr{G}_\eta = 0$, then we are done by step (c) above.

(g) Now suppose $\mathscr{G}_\eta = d\mathbb{Z} \subseteq \mathbb{Z}$ with $d > 0$. Show that the map $\mathscr{G} \to j_! \mathbb{Z}_U$, factors through the sub sheaf $j_! d\mathbb{Z}_U \subseteq j_! \mathbb{Z}_U$. Thus we have an exact sequence

$$0 \to \mathscr{G} \to j_! d\mathbb{Z}_U \to \mathscr{F}' \to 0.$$

(h) Next show that $\mathscr{F}'_\eta = 0$ and hence[3] conclude that $\mathscr{F}' = i'_* \mathscr{F}''$ for some proper closed subset $i' : Z' \subset U$. Combined with step (c) this show that $H^p(X, \mathscr{G}) = 0$ for $p > \dim(X)$.

(i) Conclude the proof by combining steps (c), (e) and (h).

2. Let X be a Noetherian scheme. Prove that

(a) X is affine iff X_{red} is affine.

(b) X is affine iff its irreducible components (with any scheme structure) are affine.

Finally show that if $f : Y \to X$ is a finite surjective morphism. Then Y is affine iff X is affine[4].

3. Let X be a proper scheme over a Noetherian ring A and $\mathscr{L} \in \text{Pic}(X)$. Prove that

(a) $\mathscr{L}$ is ample iff $\mathscr{L}|_{X_{\text{red}}}$ is ample.

(b) $\mathscr{L}$ is ample iff its restriction to each irreducible component (with any scheme structure) is ample.

Finally show that if $f : Y \to X$ be a finite surjective morphism. Then $\mathscr{L}$ is ample iff $f^*\mathscr{L}$ is.

4. Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective i.e has an ample line bundle. You may assume that the natural map $\text{Pic}(X) \to \text{Pic}(X_{\text{red}})$ is a surjection. For hints on how to proceed see [1, Chapter III, Exercise 5.8]

[3] This step uses the fact that $\mathscr{F}'$ is "generated" by a single element.
[4] For hints see [1, Chapter III, Exercise 4.2]

References

[1] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, 1977. 2, 3