PROBLEM SET 2 (DUE 02/05/25)

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1**[1]. Prove the following: For any morphism $f : X \to Y$ of finite type between Noetherian schemes and $\mathscr{F}$ a coherent sheaf on X. The locus in X

$$\{x \in X | \mathscr{F} \text{ is flat over } Y \text{ at } x\},$$

is open.

2. Let $f : X \to Y$ be a morphism from a reduced scheme to a Noetherian normal scheme Y of dimension 1. Then f is flat iff every generic point of X maps to a generic point of Y.
HINT: Use the fact that the local rings of Y are PID's and that flatness is the same as torsion freeness over a PID. First assume X is integral and solve the problem. Then see what is that you need and do the general case.

3. Let X be a scheme and let $\{U_i\}_{1 \leqslant i \leqslant n}$ be a finite open cover. Let $Y := \sqcup_i U_i$ and let $f : Y \to X$ be the natural map. Verify faithfully flat descent of quasi-coherent sheaves for f.

4. In this exercise we will see an application of Galois descent. A rank one torus T over $\mathbb{R}$ is an affine group scheme such that $T \times_{\mathbb{R}} \mathbb{C} \simeq \mathrm{GL}_1$ as affine group schemes over $\mathbb{C}$. Assume that the endomorphism group of GL_1 (as an affine group scheme) is $\mathbb{Z}$.[2]. Classify all rank one tori over $\mathbb{R}$.
HINT: First come up with a $\mathbb{R}$-torus which is **not** isomorphic to GL_1 as $\mathbb{R}$-group schemes but becomes isomorphic to it over $\mathbb{C}$. For this you may want to use the coordinate ring $\frac{\mathbb{C}[x,y]}{xy-1}$. Then use Galois descent to give an upper bound on the possible number of such tori.

5. Let $\pi : \tilde{X} \to X$ be the normalization of a non-normal one-dimensional integral scheme of finite type over a field. Show that π is never flat.

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[1]Skip this for now. We will revisit this after we discuss Tor

[2]This is not hard. We will eventually prove this.