FLATNESS-I

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Consider the following three maps:

(1) $f : \mathrm{Bl}_{(0,0)}\mathbb{A}^2 \to \mathbb{A}^2$,where $\mathrm{Bl}_{(0,0)}\mathbb{A}^2$ is the blow-up of $\mathbb{A}^2$ at the origin and f is the projection map.
(2) $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ with $f(z) = z^2$.
(3) $f : G_m \to G_m$ with $f(z) = z^2$. Here G_m is $\mathbb{A}^1 \backslash \{0\}$.

The map (1) here is an isomorphism on the complement of the origin, but over the origin the fiber is $\mathbb{P}^1$. The map (2) is nice outside the origin, with the inverse image of any $z \neq 0$ consisting of two points. But at the origin the fiber consists of exactly one point. The map (3) is simply the base change of (1) along the open immersion $G_m \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$, and hence all points have as inverse image exactly two distinct points.

Question 0.0.1. How do we capture the discontinuous jump in the fiber dimension at the origin in Example 1? Note that even though Example (2) has a *bad* fiber over the origin, it is still of dimension 0 like every other fiber.

The answer lies in the notion of flatness, a purely algebraic construct!

1. Flatness: Definition and Properties

We begin by defining flatness and faithful flatness.

Definition 1.0.1. Let A be a ring and M be an A-module. We say that M is **flat** over A if the right-exact functor $- \otimes_A M$ is exact. A map of rings $A \to B$ is said to be **flat** if B is flat as an A-module.

Definition 1.0.2. A flat A-module M is said to be **faithfully flat** if the functor $- \otimes_A M$ is faithful.

Let us see some examples of flat and faithfully flat modules.

Example 1.0.3. (1) The ring A is flat over itself.
(2) Since tensor products are right adjoint, they commute with arbitrary colimits. moreover *filtered* colimits of exact sequences is exact. Combining these two, we get that filtered colimits of flat modules are flat.
(3) Combining (1) and (2) we get that filtered colimits of the form $\mathrm{colim}_i\, M_i$, where each M_i is abstractly isomorphic to A is flat. Note that we dont care what the maps are as long as the indexing category is filtered.

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Example 1.0.3, (3) has the following corollary.

Corollary 1.0.4. *The ring A_f is flat over A. More generally for any multiplicative subset S of A, the ring $A[S^{-1}]$ is flat.*

Proof. The first claim follows from the isomorphism

$$A_f \simeq \operatorname{colim}\{A \to A \to A \cdots\},$$

where the transition maps are multiplication by f. The second part of the claim follows from the isomorphism

$$A[S^{-1}] = \operatorname{colim}_{f \in S} A_f,$$

where the colimit is over the directed set indexed by elements of S, with $f \leqslant g$ if $g = ff'$ for some $f' \in A$. This is directed because S is multiplicative and further the first part of the Corollary implies each of the A_f's are flat. Hence the result. □

Corollary 1.0.5. *For any ring A, arbitrary direct sums of A is a flat A-module. In particular when A is a field, all A-modules are flat.*

Proof. Direct sums are colimits over an directed set with no non-identity arrows, hence the result. □

Next we list some properties of flatness.

Proposition 1.0.6. *We will need the following facts about flatness. Let $\phi : A \to B$ be a map of rings, M be an A-module and N a B-module. Then the following hold*

(1) M is flat over A iff for all finitely generated ideals $\mathfrak{a}$ of A the induced map

$$\mathfrak{a} \otimes_A M \to M,$$

is injective.

(2) (Base-Change) M is flat over A implies $M \otimes_A B$ is flat over B.

(3) (Transitivity) B flat over A and N flat over B implies N is flat over A.

(4) (Local Nature) M is flat over A iff $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of A.

(5) N is flat over A iff $N_\mathfrak{q}$ is flat over $A_\mathfrak{p}$ for all prime ideals $\mathfrak{q}$ of B, here $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$.

(6) For a short exact sequence of A-modules

$$0 \to M' \to M \to M'' \to 0,$$

M is flat if M' and M'' are flat. Also if M and M'' are flat, so is M'.

(7) For a Noetherian local ring A, a finitely generated module M is flat over A iff M is free over A.

Proof. (1) is proved in Tag 00HD, (2) in Tag 051D, (3) in Tag 051D, (4) and (5) in Tag 051D, (6) in Tag 00HM and finally (7) in Tag 00NZ[1] □

We can now globalize the definition of flatness to schemes.

Definition 1.0.7 (Flatness)**.** Let $f : X \to Y$ be a morphism of schemes. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ (resp. f) is flat over Y at a point $x \in X$ if the stalk $\mathscr{F}_x$ (resp. $\mathscr{O}_{X,x}$) is flat as a $\mathscr{O}_{Y,f(y)}$-module. If this holds for all points x in X we say $\mathscr{F}$ is flat over Y (resp. f is a flat morphism).

Remark 1.0.8. Note that flatness is local on both the source and the base. Meaning to check a sheaf $\mathscr{F}$ is flat (over Y) it suffices to check this on an open cover of either X or Y or both.

Now we translate Proposition 1.0.6 into the language of scheme.

Proposition 1.0.9. *Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ a $\mathscr{O}_X$-module of X. Then the following hold.*

1. *If f is an open immersion then it is flat.*
2. *Suppose both X and Y are affine schemes, say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Then $\mathscr{F}$ is flat over Y iff M is flat over A where M is the A-module corresponding to $\mathscr{F}$.*
3. *A base change of a flat quasi-coherent sheaf[2] is flat. That is if we have a cartesian diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

   and assume that $\mathscr{F}$ is flat and quasi-coherent, then the pullback $\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'}$ is flat over Y'.
4. *Suppose f was morphism over a base scheme S. If $\mathscr{F}$ is flat over Y and Y is flat over S, then $\mathscr{F}$ is flat over S. In particular composition of flat morphisms is flat.*
5. *Suppose we have a short exact sequence of quasi-coherent sheaves*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

   Then $\mathscr{F}$ is flat if $\mathscr{F}'$ and $\mathscr{F}''$ are flat. Also if $\mathscr{F}$ and $\mathscr{F}''$ are flat, so is $\mathscr{F}'$.

[1] If you assume A is Noetherian, the proof can be simplified. As in the proof by Nakayama's Lemma we can pick a surjection $A^n \to M$ where n is the dimension of $\frac{M}{\mathfrak{m}M}$. Here $\mathfrak{m}$ is the unique maximal ideal of A. Suppose K is the kernel of this surjection. Then tensoring this exact sequence with $\frac{A}{\mathfrak{m}}$, we get that $\frac{K}{\mathfrak{m}K}$ is trivial by flatness of M, which by Nakayama implies K is trivial. (Question: Where did we use A is Noetherian?)

[2] Hartshorne forgets writing quasi-coherent in Chapter III.9, Proposition 9.2 (b).

(6) *Let X be a Noetherian scheme and $\mathscr{F}$ a coherent sheaf. Then $\mathscr{F}$ is flat iff $\mathscr{F}$ is locally free aka a vector bundle.*

Proof. (1) is immediate from the definition since the induced map on local rings is an isomorphism. (2) follows from Proposition 1.0.6, (5). The claims (3)-(6) are now a consequence of Remark 1.0.8 and Proposition 1.0.6. $\square$

Remark 1.0.10. Fix a base scheme S. Consider the subcategory of **Sch**$_S$ where we only allow morphisms which are flat between the objects. This is a subcategory of **Sch**$_S$, and is closed under composition and base change.

Recall for any topological space X and a pair of points x and y in X, we have the following:

(a) x is a specialization of y if $x \in \overline{\{y\}}$.
(b) x is a generalisation of y if $y \in \overline{\{x\}}$.

In particular when $X = \mathrm{Spec}(A)$, the constructible subsets of X which are stable under generalisation are open and those stable under specialization are closed (see [1, Chapter II, Exercise 3.18]).

Proposition 1.0.11. *Let $f : X \to Y$ be a flat morphism of schemes. Then the image*[3] *of f is stable under generalization.*

Proof. Let y be a point in the image of f. We need to show that any point $y' \in Y$ such that $y \in \overline{\{y'\}}$, also belongs to the image of f. Choose an affine open $V \ni y$ and an affine open $U \ni x$ such that $f(x) = y$ and $f(U) \subseteq V$. It suffices to show that there is a point $x' \in U$ such that $f(x') = y'$. But this is precisely the going down theorem from local algebra (see Tag 00HS). $\square$

Corollary 1.0.12 (Openness of flat morphisms)**.** *Let $f : X \to Y$ be a flat morphism, locally of finite presentation*[4]*. Then f is universally open i.e the image of any base change of f is open.*

Proof. Since both flat morphisms and morphisms of finite presentation satisfy BC, we are reduced to showing the openness of f. We have already shown that the image of f is stable under generalizations (without any finite presentation assumptions). As before we can assume that both $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine with the map $A \to B$ bein of finite presentations. By Chevalley's theorem (see Tag 00FE), Im(f) is constructible and by Prop 1.0.11 it is stable under generalizations and hence is open. $\square$

Corollary 1.0.13. *Let $f : A \to B$ be a local and flat morphism of local rings. Then the induced maps on* Spec *is surjective.*

[3] the set theoretic image

[4] For those who want to remain in the Noetherian world, anytime I say finite presentation you may assume that the schemes are Noetherian and that the morphism is of finite type.

Proof. This is essentially the content of going down theorem. Every point of $\mathrm{Spec}(A)$ is a generalisation of the unique closed point. ☐

Corollary 1.0.14. *Let $f : X \to Y$ be flat and proper morphism of finite presentation such that Y is irreducible. The f is surjective.*

2. Flatness and dimension of fibers

The following Proposition tells us that flat morphisms have well behaved fibers. This is mysterious (at least to me) given that flatness itself had a very algebraic definition.

Proposition 2.0.1. *Let $f : X \to Y$ be a flat morphism of locally Noetherian[5] schemes. Then for any point $x \in X$ we have,*

$$\dim(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{Y,f(x)}) + \dim(\mathscr{O}_{X_y,x}).$$

Proof. Since everything is local in x and y we may assume everything is sight is the Spectrum of a Noetherian ring. In which case the result follows from Tag 00ON. ☐

Corollary 2.0.2. *Let $f : X \to Y$ be a flat morphism of locally Noetherian schemes such that the dimensions of X and Y are constant. Then the dimension of the fibers of f are also constant.*

Example 2.0.3. This shows that the morphism (1) in the beginning of the lecture is not flat! The fiber over the origin is of dimension 1, while the fibers over other points are of dimension 0.

We derive one more corollary from Proposition 2.0.1.

Corollary 2.0.4. *Let $f : X \to Y$ be a flat morphism of schemes finite type over a field k with Y irreducible. Then TFAE*

(1) *Every irreducible component of X has dimension equal to dim$(Y) + n$.*
(2) *All fibers of f are of dimension n.*

Proof. see [1, Corollary 9.6] ☐

But more is true! We have the following *miraculous* result, known colloquially as the *Miracle Flatness Theorem* due to Hironaka.

Theorem 2.0.5 (Miracle Flatness Theorem)**.** *Let $R \to S$ be a local morphism of Noetherian local rings. Assume that*

(1) *R is a regular local ring.*
(2) *S is Cohen-Macaulay.*
(3) *The dimension formula holds i.e,*

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S),$$

where $\mathfrak{m}$ is the maximal ideal of R.

[5] We really need this to ensure dimensions are finite.

Then $R \to S$ *is flat!*

This has the following very useful corollary.

Corollary 2.0.6 (Miracle Flatness Theorem for schemes)**.** *Let* $f : X \to Y$ *be a morphism of locally Noetherian schemes such that* X *is regular and* Y *is Cohen-Macaulay. Then* f *is flat iff the dimension formula holds.*

Example 2.0.7. This immediately implies that the examples (2) and (3) in the beginning of the lecture are flat. The fibers are of constant dimension 0.

REFERENCES

[1] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, 1977. 4, 5