PROBLEM SHEET 1

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1. Let I be a filtered category. Show that For any functor $F : I \to$ **R-mod**, the colimit exists. Then show that colimit of short exact sequences is exact.

2. Let $\mathscr{C}$ be the category of Affine Schemes over a field k. An affine group scheme is an object G in C together with the following data:

   (1) A group structure on G, i.e. a morphism of schemes $\mu : G \times_k G \to G$, an identity element $e : \mathrm{Spec}k \to G$, and an inverse morphism $i : G \to G$.
   (2) The group structure morphisms satisfy the usual group axioms.

Now let $\mathrm{GL}_n := \mathrm{Spec}(k[x_{ij}]_{\det(x_{ij})})$. Here $k[x_{ij}]$ is the polynomial ring in n^2 variables and det(x_{ij}) is the determinant of the $n \times n$ matrix (x_{ij}). Show that GL_n can be endowed with the struture of affine group scheme.
HINT: By Yoneda's lemma it suffices to give functorial in A maps $m_A : \mathrm{GL}_n(A) \times \mathrm{GL}_n(A) \to \mathrm{GL}_n(A)$, $i_A : \mathrm{GL}_n(A) \to \mathrm{GL}_n(A)$, and $e_A : \mathrm{Spec}A \to \mathrm{GL}_n(A)$.

3. Let X be an Noetherian integral scheme. Let $\mathscr{F}$ be a coherent sheaf on X. Show that there exists a non-empty open subset U of X such that $\mathscr{F}|_U$ is locally free (and hence flat). Later in the course we shall generalize this and obtain Grothendieck's Theorem on generic flatness.

4. Let X be a Noetherian scheme. Let $\mathscr{F}$ be a coherent sheaf on X. Show that $\mathscr{F}$ is flat iff the rank function

$$x \to \dim_{k(x)}(\mathscr{F}_x \otimes k(x)),$$

is locally constant on X. Here $k(x)$ is the residue field at x.

5. Let X be a one dimensional connected and normal scheme. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Show that $\mathscr{F}$ is locally free iff $\mathscr{F}$ is torsion free i.e the multiplication map

$$\mathscr{O}_X \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$$

is injective.

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