CATEGORY THEORY: PART 0

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1. Categories: Definitions and Examples

Recall that a category $\mathscr{C}$ consists of a collection of objects Ob$(\mathscr{C})$ and a collection of morphisms between these objects. The morphisms are required to satisfy certain properties:

(1) For every object A in the category, there is an identity morphism 1_A from A to A.
(2) For every pair of morphisms $f : A \to B$ and $g : B \to C$, there is a composite morphism $g \circ f : A \to C$.
(3) Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.
(4) Composition is unital: $1_B \circ f = f = f \circ 1_A$.

Example 1.0.1. The category **Set** has sets as objects and functions as morphisms. The identity morphism on a set A is the identity function $\mathrm{id}_A : A \to A$. The composite of two functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$. The associativity and unitality of composition follow from the corresponding properties of functions.

Example 1.0.2. The category **Top** has topological spaces as objects and continuous functions as morphisms. The identity morphism on a topological space X is the identity function $\mathrm{id}_X : X \to X$. The composite of two continuous functions $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f : X \to Z$. The associativity and unitality of composition follow from the corresponding properties of continuous functions.

Example 1.0.3. The category **Vect**$_k$ has vector spaces over a field k as objects and linear transformations as morphisms. The identity morphism on a vector space V is the identity transformation $\mathrm{id}_V : V \to V$. The composite of two linear transformations $f : V \to W$ and $g : W \to Z$ is the transformation $g \circ f : V \to Z$. The associativity and unitality of composition follow from the corresponding properties of linear transformations.

Example 1.0.4. Let S be a scheme. Let **Sch**$_S$ be the category whose objects are a pair (X, f), where X is a scheme and $f : X \to S$ a morphism. Morphisms ϕ in this category are commutative diagrams of the form

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$$\begin{tikzcd} (X,f) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & (Y,g) \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

An important special case for us is the category **Sch**$_k$ of schemes over a field $\mathrm{Spec}(k)$.

Example 1.0.5. Let X be a topological space. The category **Op**(X) has open sets in X as objects and inclusions as morphisms. The identity morphism on an open set U is the inclusion $U \hookrightarrow U$. The composite of two inclusions $U \hookrightarrow V$ and $V \hookrightarrow W$ is the inclusion $U \hookrightarrow W$. The associativity and unitality of composition follow from the corresponding properties of inclusions. In partiular for any two objects U and V either $\mathrm{Hom}_{\mathbf{Op}(X)}(U, V)$ is either empty or contains a unique morphism.

Example 1.0.6. Let $\mathscr{C}$ be a category. The opposite category $\mathscr{C}^{\mathrm{op}}$ has the same objects as $\mathscr{C}$ and morphisms reversed. That is, for every pair of objects A and B in $\mathscr{C}$, we have $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$. The identity morphism on an object A in $\mathscr{C}^{\mathrm{op}}$ is the identity morphism on A in $\mathscr{C}$. The composite of two morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}^{\mathrm{op}}$ is the composite $g \circ f : A \to C$ in $\mathscr{C}$. The associativity and unitality of composition follow from the corresponding properties of composition in $\mathscr{C}$.

2. FUNCTORS

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $F : \mathscr{C} \to \mathscr{D}$ assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$ and to each morphism $f : A \to B$ in $\mathscr{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathscr{D}$. Functors are required to satisfy the following properties:

(1) For every object A in $\mathscr{C}$, we have $F(1_A) = 1_{F(A)}$.
(2) For every pair of morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$, we have $F(g \circ f) = F(g) \circ F(f)$.

One can also have what are called as contravariant functors. A contravariant functor $F : \mathscr{C} \to \mathscr{D}$ assigns to each object A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$ and to each morphism $f : A \to B$ in $\mathscr{C}$ a morphism $F(f) : F(B) \to F(A)$ in $\mathscr{D}$. Contravariant functors are required to satisfy properties analogous to those for covariant functors.

Example 2.0.1. The forgetful functor F : **Top** $\to$ **Set** assigns to each topological space its underlying set and to each continuous function its underlying function. The identity function on a set is continuous, so the identity morphism on an object in **Top** is sent to the identity morphism on the corresponding object in **Set**. The composite of two continuous functions is continuous, so the composite of two morphisms in **Top** is sent to the composite of the corresponding morphisms in **Set**.

A more non-trivial functor from **Top** to **Set** is the functor π_0.

Example 2.0.2. the functor π_0 : **top** → **set** assigns to each topological space x the set of connected components $\pi_0(x)$ of x and to each continuous function $f : x \to y$ the function $\pi_0(f) : \pi_0(x) \to \pi_0(y)$ induced by f.

we now state a few properties of functors.

Definition 2.0.3 (faithful and fully faithful functors)**.** A functor $f : \mathscr{C} \to \mathscr{D}$ is faithful if for every pair of objects a and b in $\mathscr{C}$, the map $f : \mathrm{Hom}_{\mathscr{C}}(a, b) \to \mathrm{Hom}_{\mathscr{D}}(f(a), f(b))$ is injective. we say that f is fully faithful if this map is bijective.

Definition 2.0.4 (essentially surjective functor)**.** A functor $f : \mathscr{C} \to \mathscr{D}$ is essentially surjective if for every object b in $\mathscr{D}$, there is an object a in $\mathscr{C}$ such that $f(a)$ is isomorphic to b.

Definition 2.0.5 (equivalence of categories)**.** A functor $f : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories if it is both fully faithful and essentially surjective.

the examples **??** and 2.0.2 are faithful and essentially surjective functors. neither is an equivalence of categories. an example of such a functor would be the one from the category of affine schemes to commutative rings that sends a scheme to its ring of functions.
next we will discuss an important class of functors called representable functors.

Example 2.0.6. let $\mathscr{C}$ be a category and a an object in $\mathscr{C}$. the representable functor $\mathrm{hom}_{\mathscr{C}}(-, a) : \mathscr{C}^{op} \to$ **set** assigns to each object b in $\mathscr{C}$ the set $\mathrm{hom}_{\mathscr{C}}(b, a)$ of morphisms from b to a and to each morphism $f : b \to c$ in $\mathscr{C}$ the morphism $\mathrm{hom}_{\mathscr{C}}(a, f) : \mathrm{hom}_{\mathscr{C}}(c, a) \to \mathrm{hom}_{\mathscr{C}}(b, a)$ induced by f. the identity morphism on an object b in $\mathscr{C}$ is sent to the identity morphism on $\mathrm{hom}_{\mathscr{C}}(b, a)$, and the composite of two morphisms $f : b \to c$ and $g : c \to d$ in $\mathscr{C}$ is sent to the composite of the corresponding morphisms $\mathrm{hom}_{\mathscr{C}}(a, g)$ and $\mathrm{hom}_{\mathscr{C}}(a, f)$.

next we discuss natural transformations of functors.

Definition 2.0.7 (Natural Transformation)**.** Let F and G be two functors between categories $\mathscr{C}$ and $\mathscr{D}$. A natural transformation $\eta : F \to G$ assigns to each object A in $\mathscr{C}$ a morphism $\eta_A : F(A) \to G(A)$ in $\mathscr{D}$ such that for every morphism $f : A \to B$ in $\mathscr{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes. That is, we have $G(f) \circ \eta_A = \eta_B \circ F(f)$ for every morphism $f : A \to B$ in $\mathscr{C}$.

Example 2.0.8. Let $\mathbf{Vect}_k$ be the category of vector spaces over a field k. The double dual functor $\mathbf{Vect}_k \to \mathbf{Vect}_k$ assigns to each vector space V its double dual $V^{\vee\vee}$ and to each linear transformation $f : V \to W$ the linear transformation $f^{\vee\vee} : V^{\vee\vee} \to W^{\vee\vee}$ induced by f. The natural transformation $\eta : \mathrm{id} \to (-)^{\vee}$ assigns to each vector space V the canonical map $\eta_V : V \to V^{\vee\vee}$ and to each linear transformation $f : V \to W$ the commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"] \\ V^{\vee\vee} \arrow[r, "f^{\vee\vee}"'] & W^{\vee\vee} \end{tikzcd} .$$

Note that the vertical arrows are isomorphisms if and only if the vector spaces are finite-dimensional.

Definition 2.0.9 (Natural Equivalence)**.** A natural transformation $\eta : F \to G$ of functors is a natural equivalence if for every object A in $\mathscr{C}$, the morphism $\eta_A : F(A) \to G(A)$ is an isomorphism in $\mathscr{D}$.

The last thing I would like to discuss in this section is the notion of adjoint functors.

Definition 2.0.10 (Adjoint Functors)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A pair of functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are adjoint if for every pair of objects A in $\mathscr{C}$ and B in $\mathscr{D}$, there is a natural (in A and B) bijection

$$\mathrm{Hom}_{\mathscr{D}}(F(A), B) \cong \mathrm{Hom}_{\mathscr{C}}(A, G(B)).$$

Example 2.0.11. The forgetful functor F : **Top** → **Set** has a left adjoint G : **Set** → **Top** that assigns to each set X the discrete topological space $G(X)$ with underlying set X.

Example 2.0.12. Let V be a vector space over a field k. The tensor product functor $- \otimes V : \mathbf{Vect}_k \to \mathbf{Vect}_k$ has a right adjoint $\mathrm{Hom}(V, -) : \mathbf{Vect}_k \to \mathbf{Vect}_k$ that assigns to each vector space W the vector space $\mathrm{Hom}(V, W)$ of linear transformations from V to W.

Adjoint functors show up in plenty in Algebraic Geometry and topology. For example, Poincare duality is a manifestation of an adjointness result.

2.1. **Yoneda's Lemma.** Now we are ready to state the Yoneda's Lemma.

Lemma 2.1.1 (Yoneda's Lemma)**.** *Let $\mathscr{C}$ be a category and A an object in $\mathscr{C}$. Then the set of natural transformations from the functor $\mathrm{Hom}_{\mathscr{C}}(-, A)$ to any other contravariant functor $F : \mathscr{C} \to$ **Set** is naturally isomorphic to the set $F(A)$.*

Proof. Let $\eta : \mathrm{Hom}_{\mathscr{C}}(-, A) \to F$ be a natural transformation. In particular, $\eta_A : \mathrm{Hom}_{\mathscr{C}}(A, A) \to F(A)$ is a morphism in **Set**. The desired element in $F(A)$ is simply the image of the identity morphism on A under η_A. Conversely, given an element x in $F(A)$, we can define a natural transformation $\eta : \mathrm{Hom}_{\mathscr{C}}(-, A) \to F$ by setting $\eta_B(f) = F(f)(x)$ for every object B in $\mathscr{C}$ and morphism $f : B \to A$ in $\mathscr{C}$. The naturality of η follows from the properties of functors. $\square$

In particular we note the following corollary.

Corollary 2.1.2. *Let C be a category. Then the functor Hom$_{\mathscr{C}}(-, A) : \mathscr{C} \to$ Func$(\mathscr{C}^{op}, $ **Set**$)$ is fully faithful, where Func$(\mathscr{C}^{op}, $ **Set**$)$ is the category whose objects are functors from $\mathscr{C}^{op}$ to **Set** and morphisms are natural transformations.*

Example 2.1.3. Let us unpack this when C is $\mathbf{Sch}_k$. For any two schemes X and Y over k, giving a morphism $f : X \to Y$ is equivalent to giving a natural transformation of functors $\mathrm{Hom}_{\mathbf{Sch}_k}(-, X) \to \mathrm{Hom}_{\mathbf{Sch}_k}(-, Y)$, which is the same as giving a functorial map of sets

$$\mathrm{Hom}_{\mathbf{Sch}_k}(Z, X) \to \mathrm{Hom}_{\mathbf{Sch}_k}(Z, Y)$$

as Z varies over all schemes over k. The sets $\mathrm{Hom}(Z, X)$ and $\mathrm{Hom}(Z, Y)$ are the sets of morphisms from Z to X and Y respectively. You should think of these Z-valued points on X and Y. For example if k is algebraically closed and $Z = \mathrm{Spec}(k)$, then $\mathrm{Hom}(\mathrm{Spec}(k), X)$ is simply the set of closed points of X.

2.2. **Limits and Colimits.** Let $\mathscr{C}$ be a category and I a category. A functor $F : I \to \mathscr{C}$ is called a diagram in $\mathscr{C}$ indexed by I. A cone over F is an object A in $\mathscr{C}$ together with morphisms $A \to F(i)$ for every object i of I compatible with the functor F. A limit of F is a terminal object[1] in the category of cones over F. Dually, a colimit of F is an initial object[2] in the category of co-cones[3] over F. Note that limits and colimits **maynot** exist in general but when they do they are unique upto unique isomorphism.
Limits are denoted by

$$\lim_{i \in I} F(i)$$

and colimits are denoted by

$$\mathrm{colim}_{i \in I} F(i).$$

Example 2.2.1. Let V and W be vector spaces over a field k, and let $f : V \to W$ be a linear transformation. The kernel of f is the limit of the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

Here the indexing category is the category with two objects and one non-identity morphism between them.

We have the dual example.

[1] Meaning it maps uniquely to any other cone
[2] Meaning it gets an unique map from every cone
[3] Guess its definition!

Example 2.2.2. Let V and W be vector spaces over a field k, and let $f : V \to W$ be a linear transformation. The cokernel of f is the colimit of the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow & & \\ 0 & & . \end{array}$$

As before the indexing category is the category with two objects and one non-identity morphism between them.

You have seen these before!

Example 2.2.3. Let X and Y be schemes over S. The fibre product of X and Y (over S) is the limit of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & S. \end{array}$$

Here the indexing category is the category with three objects and two non-identity morphisms between them.

Here is a basic and important example.

Example 2.2.4. Let I be a set. We say a category $\mathscr{C}$ has products (resp. coproducts) indexed by I if every functor indexed by I has limits (resp. colimits). Here I is considered as a category with objects indexed by elements of I and no non-identity morphisms.

Example 2.2.5. We say that a category $\mathscr{C}$ has finite limits (resp. colimits) if it has limits (resp. colimits) indexed by any category with finitely many objects and morphisms

Another important class of indexing category for us are the *flitered* ones. Let me give an example first.

Example 2.2.6. Let $\mathbb{N}$ be the set of natural numbers. We can consider $\mathbb{N}$ as a category with objects indexed by natural numbers and a unique morphism between any two objects. This is a filtered category.

Here is a formal definition.

Definition 2.2.7 (Filtered Category)**.** A category I is called filtered if for every pair of objects i and j in I, there is an object k in I and morphisms $f : i \to k$ and $g : j \to k$. Moreover for a pair of morphisms $f, g : i \to j$ in I, there is an object k in I and a morphism $h : j \to k$ such that $h \circ f = h \circ g$.

Example 2.2.8. Let R be a ring and M an R-module. Then M is a filtered colimit of its finitely generated submodules. This is often used to reduce statements about arbitrary modules to statements about finitely generated modules.

We have the following *very useful* but formal result.

Lemma 2.2.9. *Limits commute with right adjoints and colimits commute with left adjoints.*

3. Abelian Categories

We begin with the definition of an additive category.

Definition 3.0.1 (Additive Category)**.** An additive category is a category $\mathscr{A}$ with the following properties:

(1) For every pair of objects A and B in $\mathscr{A}$, the morphism set $\text{Hom}_{\mathscr{C}}(A, B)$ is an abelian group[4].
(2) Composition of morphisms is bilinear.
(3) $\mathscr{A}$ has a zero object, that is, an object that is both initial and terminal[5].
(4) $\mathscr{A}$ has finite products and coproducts i.e. the indexing set is finite.

Clearly the opposites of an additive category can also be naturally given a structure of an additive category. We will see additive categories later too, when we discuss cohomology. For now you may think of them as categories where you can add morphisms and have a zero object.

Definition 3.0.2 (Additive Functor)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be additive categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is additive if for every pair of objects A and B in sC, the map $F : \text{Hom}_{\mathscr{C}}(A, B) \to \text{Hom}_{\mathscr{D}}(F(A), F(B))$ is a group homomorphism. Moreover, F is required to preserve finite products and coproducts[6].

We need few more definitions before we can define an abelian category. In what follows we assume A is an additive category and all functors are additive.

Definition 3.0.3 (Kernels and Cokernels)**.** Let $f : A \to B$ be a morphism in a category $\mathscr{A}$. A kernel of f is the limit of the diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}.$$

Dually a cokernel of f is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ 0 & \end{tikzcd}.$$

[4]In particular there is a 0 morphism.

[5]This makes it unique upto an unique isomorphism

[6]The product or coproduct indexed by the empty set is the 0-object. Hence F is required to take the zero object to the zero object.

Remark 3.0.4. Kernels and cokernels may not exist in general. However when they do they are unique upto unique isomorphism.

We now define monomorphisms and epimorphisms in an additive category.

Definition 3.0.5. A morphism $f : A \to B$ in an additive category is a monomorphism if for every object C and morphisms $g, h : C \to A$ such that $f \circ g = f \circ h$, we have $g = h$. A morphism $f : A \to B$ in an additive category is an epimorphism if for every object C and morphisms $g, h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

Now we can state the definition of an abelian category.

Definition 3.0.6 (Abelian Category)**.** An abelian category is an additive category $\mathscr{A}$ with the following properties:

(1) Every morphism in $\mathscr{A}$ has a kernel and a cokernel[7].
(2) Every monomorphism in $\mathscr{A}$ is the kernel of its cokernel.
(3) Every epimorphism in $\mathscr{A}$ is the cokernel of its kernel.

Example 3.0.7. The category of abelian groups can be given the structure of an abelian category. The zero object is the trivial group, the product is the direct sum, and the coproduct is the direct product. The kernel of a morphism $f : A \to B$ is the subgroup of elements a in A such that $f(a) = 0$, and the cokernel is the quotient group $B/\text{im}(f)$. The monomorphisms are the injective group homomorphisms, and the epimorphisms are the surjective group homomorphisms.

We can even restrict to the category of finitely generated abelian groups and get an abelian category. Note that finite coproducts are the same as finite products in both these cases. This is not a coincidence. In general in an abelian category finite products and coproducts are the same.
The last thing we need to get us going into geometry is exactness of functors.

Definition 3.0.8. Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is left exact if it preserves finite limits, and right exact if it preserves finite colimits. A functor $F : \mathscr{C} \to \mathscr{D}$ is exact if it is both left and right exact.

This coincides with the more usual definition as shown below[8].

Proposition 3.0.9. *Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then the following are equivalent:*

(1) F is left exact.
(2) For every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

[7]This implies that in addition to have finite products and coproducts by virtue of $\mathscr{A}$ being additive, it also has finite limits and colimits (see Tag 010D)

[8]Ignore on first reading

in $\mathscr{C}$, the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact in $\mathscr{D}$.

(3) F preserves kernels.

Proof. Clearly (2) implies (3). That (1) implies (3) follows from Definition 3.0.3. To see that (3) implies (2), it suffices to show that the image of $F(f)$ is the kernel of $F(g)$. Since F preserves monomorphisms, $F(f)$ is a monomorphism. Thus the image of $F(f)$ is naturally isomorphic to $F(A)$, which is the kernel of $F(g)$, since F preserves kernels and A is the kernel of g.
We are left to show that (3) implies (1). For this we use a general result that finite limits can be expressed in terms of kernels and finite products (see Tag 002P for a reference). Since F preserves both we are done.

$\square$

We note here a very useful corollary to Lemma 2.2.9

Corollary 3.0.10. *Any right adjoint functor is left-exact and any left adjoint functor is right-exact.*

We conclude this section with a couple of examples.

Example 3.0.11. Let R be any ring[9] and M an R-module. The tensor product functor $-\otimes M$: **Mod**$_R$ → **Mod**$_R$ is right-exact. This follows from the fact that it is a right adjoint to the Hom functor.

Example 3.0.12. Let $f : X \to Y$ be a morphism of schemes. The lower shriek functor f_* is left-exact. This follows from the fact that it is a left adjoint to the pullback functor f^* (which in turn is necessarily right-exact).

[9] All rings in this course are commutative with unity.