

PROBLEM SHEET 1

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1. Let I be a filtered category. Show that For any functor $F : I \rightarrow \mathbf{R-mod}$, the colimit exists. Then show that colimit of short exact sequences is exact.

2. Let \mathcal{C} be the category of Affine Schemes over a field k . An affine group scheme is an object G in \mathcal{C} together with the following data:

- (1) A group structure on G , i.e. a morphism of schemes $\mu : G \times_k G \rightarrow G$, an identity element $e : \text{Spec} k \rightarrow G$, and an inverse morphism $i : G \rightarrow G$.
- (2) The group structure morphisms satisfy the usual group axioms.

Now let $\text{GL}_n := \text{Spec}(k[x_{ij}]_{\det(x_{ij})})$. Here $k[x_{ij}]$ is the polynomial ring in n^2 variables and $\det(x_{ij})$ is the determinant of the $n \times n$ matrix (x_{ij}) . Show that GL_n can be endowed with the struture of affine group scheme.

HINT: By Yoneda's lemma it suffices to give functorial in A maps $m_A : \text{GL}_n(A) \times \text{GL}_n(A) \rightarrow \text{GL}_n(A)$, $i_A : \text{GL}_n(A) \rightarrow \text{GL}_n(A)$, and $e_A : \text{Spec} A \rightarrow \text{GL}_n(A)$.

3. Let X be an Noetherian integral scheme. Let \mathcal{F} be a coherent sheaf on X . Show that there exists a non-empty open subset U of X such that $\mathcal{F}|_U$ is locally free (and hence flat). Later in the course we shall generalize this and obtain Grothendieck's Theorem on generic flatness.

4. Let X be an integral Noetherian scheme. Let \mathcal{F} be a coherent sheaf on X . Show that \mathcal{F} is flat iff the rank function

$$x \rightarrow \dim_{k(x)}(\mathcal{F}_x \otimes k(x)),$$

is locally constant on X . Here $k(x)$ is the residue field at x .

5. Let X be a one dimensional connected and normal scheme. Let \mathcal{F} be a quasi-coherent sheaf on X . Show that \mathcal{F} is locally free iff \mathcal{F} is torsion free i.e the multiplication map

$$\mathcal{O}_X \rightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

is injective.

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