Problem Set 3
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1. Let $X \to \mathrm{Spec}(k)$ be a morphism of finite type with $\Omega^1_{X/k} = 0$. Show that $X \simeq \sqcup_{i=1}^{n} \mathrm{Spec}(k_i)$ with k_i/k finite and separable. Conclude that étale morphisms are quasi-finite.

HINT: First base change to $\bar{k}$, then show that the local ring at any closed point of $X_{\bar{k}}$ is a field (and of course isomorphic to $\bar{k}$).

2. Let X/k be a smooth variety and let $Y \subseteq X$ be a smooth (over k) sub variety of codimension $r \geqslant 2$. Let $\tilde{X} := \mathrm{Bl}_Y(X)$ be the blow up of X along Y and let $\pi : \tilde{X} \to X$ be the blow up map.

1. Show that $\tilde{X}$ is also smooth over k. In particular we may identify CaCl$(\tilde{X})$, Cl$(\tilde{X})$ and Pic$(\tilde{X})$.
2. Show that $\pi|_E : E \to Y$ is the projective bundle $\mathbb{P}(\mathscr{I}/\mathscr{I}^2)$, here $\mathscr{I}$ is the ideal sheaf of Y in X. Thus $\pi|_E$ is a projective bundle of relative dimension $r-1$.
3. Let $E \subseteq \tilde{X}$ be the exceptional divisor. Prove that the map Pic$(X) \oplus \mathbf{Z}.[E] \to$ Pic$(\tilde{X})$ sending $([Z], r[E]) \mapsto \pi^*[Z] + r[E]$ is an isomorphism.
4. Finally show that $\omega_{\tilde{X}} = \pi^*\omega_X \otimes \mathscr{O}((r-1)E)$.

HINT: For (1) use étale local coordinates and reduce to the case of affine space using the fact that blow up commutes with flat base change Tag 0805. For (3) use the fact that Cl$(X) \simeq$ Cl$(X \backslash Y)$. For (4) use (3) to first conclude that $\omega_{\tilde{X}} = \pi^*\omega_X \otimes \mathscr{O}(qE)$ for some integer q. To determine q again use the trick from (1), to reduce to the case of $X = \mathbb{A}^n_k$ and $Y = \mathbb{A}^{n-r}_k$ and then use adjunction formula for $E \hookrightarrow \tilde{X}$ in combination with (2).

3. Do Exercise 8.4 from [1, Chapter II.8].

4. Let $f : X \to Y$ be a morphism of varieties over a field k. Show that

   1. Assuming char$(k) = 0$ and f *dominant* show that there exists an open dense $U \hookrightarrow X$ such that $f|_U : U \to Y$ is smooth. You may take generic flatness for granted.

   2. Assuming char$(k) = 0$ and X is smooth, show that there exists an open dense $U \hookrightarrow Y$ such that $f^{-1}(U) \to U$ is smooth.

   3. Give counterexamples to (1) and (2)with k perfect but of positive characteristic.

   4. Assuming f is proper and flat suppose that there exists a $y \in Y$ such that the fiber X_y is smooth over $\mathrm{Spec}(k(y))$. Show that there exists a non-empty open neighbourhood $U \hookrightarrow Y$ of y such that $f^{-1}(U) \to U$ is smooth.

References

[1] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, 1977. 1