O-minimal geometry and Hodge Theory

Shuddhodan Kadattur Vasudevan

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Chapter 1

Introduction

Consider a smooth projective variety $X/\mathbb{C}$ and let $X^{\text{an}} := X(\mathbb{C})$ be the associated compact complex manifold. Let $H_B^k(X, \mathbb{Z})$ be the k^{th}-Betti cohomology (with $\mathbb{Z}$-coefficients) of X^{an}. Then classical Hodge theory can be summarized by saying that $H_B^k(X, \mathbb{Z})$ has a *polarized Hodge structure of weight* k. Here is what it means:

(a) There exists a *Hodge decomposition*

$$H_B^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{p+q=k} H^{p,q},$$

here $H^{p,q}$ are complex vector space satisfying $(1 \otimes \sigma)H^{p,q} = H^{q,p}$[1], where $\sigma \in \text{Aut}(\mathbb{C})$ is the complex conjugation.

(b) There exists a $(-1)^k$-symmetric bilinear form

$$q : H_B^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H_B^k(X, \mathbb{Z}) \to \mathbb{Z},$$

such that $q(\alpha, \beta) =$[2] for $\alpha \in H^{p,q}$ and $\beta \in H^{p',q'}$ unless $p = q'$ and $q = p'$. Moreover for any $\alpha \in H^{p,q} \backslash \{0\}$,

$$(-1)^{q-p} q(\alpha, \bar{\alpha}) > 0.$$

Here is a nice corollary.

Corollary 1.0.0.1. *Let $X/\mathbb{C}$ be a smooth projective variety. Then the odd Betti numbers are even dimensional.*

Proof. Follows from the Hodge decomposition above. □

[1]Henceforth we shall denote $(1 \otimes \sigma)$ by the bar operation.

[2]We continue to denote the base extension of q to complex numbers by q.

Example 1.0.0.2. Every one dimensional compact complex manifold is algebraic. However in dimension 2 this fails.
For example let $\tilde{X} = \mathbb{C}^2 \backslash \{0\}$. Let λ be a non-zero complex number which is *not* a root of unity. Let $\Gamma = \mathbb{Z}$ act on $\tilde{X}$ via scaling by λ. Then the action of Γ on $\tilde{X}$ is free, properly discontinuous and via holomorphic automorphisms. Thus $X := \Gamma \backslash \tilde{X}$ is also a complex manifold of dimension 2. By choosing an appropriate closed shell in $\mathbb{C}^2 \backslash \{0\}$, we may realize X as the continuous image of a compact set. Thus X is a compact complex manifold. Evidently $\tilde{X}$ is the universal covering space of X and $\pi_1(X) \simeq \mathbb{Z}$. In particular $H^1_B(X, \mathbb{Z}) = \mathbb{Z}$. Thus X cannot have an algebraic structure!

1.0.1 Hodge theory over a point

Before we proceed here is a quick summary of how to equip $H^k_B(X, \mathbb{Z})$ with a polarized Hodge structure of weight k.

Equipping $H^k_B(X, \mathbb{Z})$ with a Hodge structure

We equip $H^k_B(X, \mathbb{Z})$ with a Hodge structure of weight k using the de Rham isomorphism and Hodge theory. The former gives us an isomorphism

$$H^k_B(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^k_{dR}(X^{\mathrm{an}}, \mathbb{C}).$$

Here $H^k_{dR}(X^{\mathrm{an}}, \mathbb{C})$ is the de Rham cohomology defined to be the hyercohomology of the de Rham complex

$$0 \to \mathscr{A}^0_X \to \mathscr{A}^1_X \cdots \to \mathscr{A}^{2n}_X \to 0,$$

where $\mathscr{A}^i_X$ is the sheaf of (complex valued $\mathscr{C}^\infty$) i-forms on X^{an} (which we assume has dimension n). Hodge theory allows us to further decompose $H^k_{dR}(X^{\mathrm{an}}, \mathbb{C})$ by *type*, the upshot of which is an isomorphism

$$H^k_{dR}(X^{\mathrm{an}}, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}}).[3]$$

It follows from Hodge theory and the de Rham isomorphism that if we set $H^{p,q} := H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}})$, then $\overline{H^{p,q}} = H^{q,p}$. In this course we will take this for granted[4].

Cycle class map

Before we can discuss about polarizations, let me briefly recall the cycle class map. Let $Z^i(X)$ denote the free abelian group generated by codimension i sub varieties of X. The cycle class

[3]Here $\Omega^*_{X^{\mathrm{an}}}$ is the sheaf of Kähler differentials on the comapct complex manifold X^{an}.
[4]Though we will not need it, I strongly encourage you to read a proof of this decomposition. A possible reference is [5, Chapter 6].

map is a morphism of abelian groups

$$\mathrm{cl}_X : Z^i(X) \to H^{2i}_B(X, \mathbb{Z}).$$

Intuitively this is easy to define, start with any codimension i sub variety Z. Hence $Z \subseteq X$ is a subvariety of dimension $n - i$ and thus defines a homology class in $H_{2n-2i}(X^{\mathrm{an}}, \mathbb{Z})$ which by Poincare duality gives us a class in $H^{2i}_B(X, \mathbb{Z})$. It is not hard to see the following.

Lemma 1.0.1.1. *For any subvariety $Z \subseteq X$ of codimension i*

$$\mathrm{cl}_X(Z) \in H^{i,i} \bigcap \mathrm{Im}(H^{2i}_B(X, \mathbb{Z}) \to H^{2i}_B(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}).$$

Proof. For a proof see [5, Proposition 11.20]. □

The Hodge conjecture predicts that the converse must be true *rationally*. More precisely.

Conjecture 1.0.1.2 (Hodge Conjecture). *The inclusion $\mathrm{Im}(\mathrm{cl}_X)_{\mathbb{Q}} \subseteq H^{i,i} \cap H^{2i}_B(X, \mathbb{Q})$ is an isomorphism.*

Remark 1.0.1.3. Here are a few remarks about the Hodge conjecture.

(i) Unlike Lemma 1.0.1.1, we need not write Im since $H^{2i}_B(X, \mathbb{Q})$ is a subspace of $H^{2i}_B(X, \mathbb{Z}) \otimes \mathbb{C}$.

(ii) The Hodge conjecture is false integrally. The earliest known counterexamples were due to Atiyah-Hirzebruch. A reference (with some history) is [4].

Let us end this section with a positive evidence for Hodge conjecture.

Proposition 1.0.1.4 (Lefschetz (1,1) theorem). *The Hodge conjecture is true for divisors (even integrally).*

Proof. Consider the exponential sequence

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}^*_X \longrightarrow 0.$$

Here $\mathbb{Z}_X$ is the constant sheaf with values in $\mathbb{Z}$. The sheaves $\mathscr{O}_X$ and $\mathscr{O}^*_X$ are the sheaves of holomorphic functions with values in $\mathbb{C}$ and $\mathbb{C}^*$ respectively. Taking the long exact sequence on cohomology, the maps of interest are the boundary map

$$c_1 : \mathrm{Pic}(X^{\mathrm{an}}) = H^1(X, \mathscr{O}^*_X) \to H^2_B(X, \mathbb{Z})$$

and

$$\tau_* : H^2_B(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X).$$

Using a standard argument, we can show that the cycle class map factors through this boundary map c_1 and that $Z^1(X) \twoheadrightarrow \mathrm{Pic}(X^{\mathrm{an}})$. Hence it suffices to show that $\ker(\tau_*)$ is precisely $H^{1,1} \bigcap \mathrm{Im}(H^2_B(X,\mathbb{Z}) \to H^2_B(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})$. This follows from the fact that τ_* can be identified with the composition

$$H^2_B(X,\mathbb{Z}) \to H^2_B(X,\mathbb{C}) \twoheadrightarrow H^{0,2} = H^2(X,\mathscr{O}_X).$$

For details we refer the reader to [2, Pg 163] □

Polarization on $H^k_B(X^{\mathrm{an}},\mathbb{Z})$

The polarization on $H^k_B(X,\mathbb{Z})$ comes from the class of a hyperplane section corresponding to any embedding of $X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$. Indeed let $\omega \in H^2_B(X^{\mathrm{an}},\mathbb{Z})$ the class corresponding to an hyperplane section. Then the bilinear form

$$q : H^k_B(X,\mathbb{Z}) \otimes H^k_B(X,\mathbb{Z}) \to \mathbb{Z}$$

is defined to be

$$q(\alpha,\beta) := \mathrm{Tr}(\alpha \cup \beta \cup \omega^k)$$

induces a polarization on $H^k_B(X,\mathbb{Z})$. For a proof we refer the reader to [5, Theorem 6.32]

What about non-smooth/non-proper varieties?

As we shall see later in the course, just starting with these structures on the cohomology of a smooth projective variety we can extrapolate and obtain more complex structures on the cohomology of *any* complex algebraic variety. Moreover any morphism between such varieties induces a morphism on their cohomology respecting these additional structures. If you play this game well there is much you can say by linearizing the problem of understanding varieties and morphisms between them.

1.0.2 Hodge Theory over a base

A basic lesson one learns over time when using Algebraic geometry is that its best to set things up over a base. Often we need to deform or lift varieties or move in families even though we might be interested in proving results about a particular variety. Hodge theory in its modern avatar handles this with ease. To begin with lets recall Ehresmann fibration theorem (see [5, Theorem 9.3]).

Theorem 1.0.2.1. *Let $f : M \to N$ be a smooth and proper map between smooth manifolds, then f is locally on N a fiber bundle.*

Here is a nice corollary.

Corollary 1.0.2.2. *Let $f : M \to N$ be a proper and smooth morphism of smooth manifolds. Then $R^i f_* \mathbb{Z}_M$ is a local system on N i.e. is locally a constant sheaf. Moreover if N is connected then all the fibers have the same Betti rank.*

The upshot is that if you have a smooth and proper family then topologically not much changes when you move from one fiber to another. However this is false holomorphically/algebraically. Here is a standard example.

Example 1.0.2.3. Consider the Legendre family of elliptic curve

$$\pi : \mathscr{E} \to \Delta \backslash \{0, 1\}.$$

Here $\mathscr{E} \subseteq \mathbb{P}^2 \times \Delta^*$ is given by the vanishing of

$$Y^2 Z = X(X - Z)(X - tZ).$$

Then π is a smooth and proper family. But the j-invariant of these elliptic curves is a function of t and is in particular not constant. Thus the Hodge structure (the only relevant one here is $H^1)_B(\mathscr{E}_t, \mathbb{Z}))$ varies along the base though the underlying $\mathbb{Z}$-module remains the same. This is a typical example of a **variation of Hodge structures**.

One way to think about Example 1.0.2.3 is to choose a point say t_0 and fix the underlying $\mathbb{Z}$-module for the variation of Hodge structure say $V = H^1(\mathscr{E}_{t_0}, \mathbb{Z})$. As t varies the Hodge structure on V varies. But how do we capture this?
One way to do this would be to look at the space of all Hodge decompositions on V and then assign to t the corresponding Hodge decomposition. Of course we would want this map to be at least holomorphic. But this fails for the simple reason that $H^{1,0}$ and $H^{0,1}$ are complex conjugates, so any attempt at making one of the holomorphic would force the other to be anti holomorphic and hence the resulting map would at best be real analytic. The remedy? Well we can ignore the anti holomorphic part since it is essentially determined by the holomorphic part.

Variation of Hodge structures

More generally given a family $\pi : X \to S$ of smooth projective varieties, we obtain for any k a variation of Hodge structures on the lattice $V = H^k_B(X_s, \mathbb{Z})$. One possible way to capture the change (or variation) in the Hodge structure as one moves along S is by looking at the *period map*[5].

$$\Phi : S \to D^{6},$$

here D is the space of all possible Hodge structures on V, also called a *period domain*. Thus Φ captures the Hodge theoretic complexity of f (or its fibers).

[5]The name is justified by the Legendre family example, where the variation is literally captured by the period.

[6]This is not correct as written, but is a very good approximation. We shall make this precise eventually.

To see why this is useful, suppose the Hodge conjecture were true. Then the fibers of f which have extra algebraic cycles of codimension k, are those for which $H^{k,k}$ is particularly large. This in turn is captured by looking at the period map for $2k$ and looking at special points in its period domain i.e. those for which the corresponding Hodge structure has a bigger $H^{k,k}$.

What can we say about the period map?

Even though we started with algebraic objects like π, X and S, the period domain itself is rarely algebraic. The best one can do is put an analytic structure on D and consequentially ϕ is a morphism of analytic spaces. This is where o-minimal geometry enters the picture.
One of the aims of this course is to show that period domains are semi-algebraic or equivalently definable in $\mathbb{R}_{\mathrm{alg}}$, (the simplest o-minimal structure) in a functorial way which we shall make precise later. However the period map Φ itself is more complicated. For example the period map for the Legendre family involves hypergeometric functions. So one cannot expect any algebraicity for Φ. In fact one of the main results of [1] is the following:

Theorem 1.0.2.4 (Bakker-Klingler-Tsimerman). *The period maps Φ are definable in $\mathbb{R}_{\mathrm{an,exp}}$.*

This result is amplified in strength by the following o-minimal GAGA by Peterzil-Starchenko.

Theorem 1.0.2.5 (Peterzil-Starchenko). *Let S be a quasi-projective complex variety and let $Z \subset S$ be a closed analytic subset. If there exists an o-minimal structure expanding $\mathbb{R}_{\mathrm{an}}$ in which Z is definable, then Z is algebraic.*

Combining Theorems 1.0.2.4, 1.0.2.5 and a key algebraization result for definable images allows the authors to prove Griffiths conjecture on algebraicity of the images of period maps.

Chapter 2

Hodge Structures

In this chapter we shall discuss the necessary background from Hodge theory. As we mentioned in class, we are going to take Hodge decomposition on compact Kähler manifolds (or smooth projective varieties) for granted. That this is not a buzzkill is to me a feature of the theory than a bug.

2.1 Pure Hodge structures

Let $H_{\mathbb{Z}}$ be a finitely generated abelian group. We denote by $H_{\mathbb{R}}$ and $H_{\mathbb{C}}$ the $\mathbb{R}$ and $\mathbb{C}$ vector spaces obtained by extension of scalars from $H_{\mathbb{Z}}$.

Definition 2.1.0.1. A pure Hodge structure of weight k on $H_{\mathbb{Z}}$ is a decomposition[1]

$$H_{\mathbb{C}} \simeq \bigoplus_{p+q=k} H^{p,q},$$

such that $\overline{H^{p,q}} = H^{q,p}$, where the bar operation is the complex conjugation on $H_{\mathbb{C}}$.

In fact we can make a category out of pure Hodge Structures.

Definition 2.1.0.2. Let $H_{\mathbb{Z}}$ and $H'_{\mathbb{Z}}$ be finitely generated abelian groups with pure Hodge structures. Let $f : H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ be a morphism of abelian groups. We say that f is a morphism of pure Hodge structures if $f_{\mathbb{C}} : H_{\mathbb{C}} \to H'_{\mathbb{C}}$ respects the Hodge decomposition.

In an obvious way we can form a category whose objects are finitely generated abelian groups with pure Hodge structures and morphisms are those which respect the Hodge structure.

Example 2.1.0.3. For any integer k there is functor from the category of smooth projective varieties to the category of pure Hodge structures given by sending a variety $X \to H^k(X, \mathbb{Z})$[2].

[1] called the Hodge decomposition

[2] We shall henceforth ignore the subscript B, cohomology will always refer to Betti/singular cohomology.

Here is a simple lemma.

Lemma 2.1.0.4. *Let $H_\mathbb{Z}$ and $H'_\mathbb{Z}$ be pure Hodge structures of weight k and k' respectively. Then there exists a non-zero morphism of Hodge structures between $H_\mathbb{Z}$ and $H'_\mathbb{Z}$ iff $k = k'$.*

Proof. Immediate from the definition. □

We have already noted in the introduction that the Hodge decomposition is not the *correct* object if you want to move in families. In fact every Hodge decomposition gives rise to a Hodge filtration as follows

$$F^i H_\mathbb{C} := \bigoplus_{p \geqslant i} H^{p,k-p}.$$

Clearly $F^\bullet H_\mathbb{C}$ is a decreasing filtration[3] and let us denote by $\overline{F}^\bullet$ the filtration obtain by applying complex conjugation (and hence also called as the *conjugate filtration*). We have the following

$$H_\mathbb{C} := F^p \oplus \overline{F}^{k-p+1} \tag{2.1}$$

and

$$H^{p,q} = F^p \cap \overline{F}^q. \tag{2.2}$$

Definition 2.1.0.5. Let $H_\mathbb{Z}$ be a finitely generated abelian group. Let $F^\bullet$ be a decreasing filtration on $H_\mathbb{C}$. We say $F^\bullet$ is a Hodge filtration of weight k if equations (2.1) and (2.2) hold.

We have already seen that a pure Hodge structure of weight k induces a Hodge filtration of weight k on $H_\mathbb{C}$. What is amazing is that this is a reversible process!

Lemma 2.1.0.6. *Let $H_\mathbb{Z}$ be a finitely generated abelian group such that there exists a Hodge filtration of weight k on $H_\mathbb{C}$. Then there exists a Hodge structure of weight k on $H_\mathbb{Z}$ such that the associated Hodge filtration is the one given to us.*

Proof. Well clearly we need to define

$$H^{p,q} := F^p \cap \overline{F}^q.$$

By definition each $H^{p,q}$ is a complex subspace of $H_\mathbb{C}$ and essentially by definition $\overline{H}^{p,q} = H^{q,p}$. Moreover for $p, q >> 0$ the subspaces $H^{p,q} = 0$. Now it suffices to show that the natural map

$$\bigoplus_{p+q=k} H^{p,q} \to H_\mathbb{C}$$

[3]Meaning $F^i H_\mathbb{C} \subseteq F^{i-1} H_\mathbb{C}$.

is an isomorphism. In fact it suffices to show that

$$\bigoplus_{p \geqslant i} H^{p,k-p} \to F^i H_{\mathbb{C}},$$

is an isomorphism for all i. This is because for i sufficiently small $F^i H_{\mathbb{C}} = H_{\mathbb{C}}$.
We prove the result by an increasing induction on i. For $i >> 0$, both sides are 0 and we are done. Note that by induction hypothesis

$$\bigoplus_{p \geqslant i} H^{p,k-p} \simeq H^{i,k-1} \bigoplus F^{i+1}.$$

Combining this with (2.2) gives the required result. □

Deligne torus

There is yet one more way of defining pure Hodge structures which is more group theoretic in nature. This is also well suited for the purposes of this course. Before we can state the definition, we recall some results from representation theory of tori.
Let K be field and we denote by $\overline{K}$ an algebraic closure of K. We denote by $\mathbb{G}_{m,K}$ (we ignore K if the underlying field is clear) the algebraic group whose underlying scheme is $\operatorname{Spec}(K[t, t^{-1}])$.

Definition 2.1.0.7. Let G/K be an algebraic group. We say G/K is a tori if $G \times_K \overline{K} \simeq \mathbb{G}^r_{m,\overline{K}}$ for some integer r. Here the isomorphism is that of algebraic groups over $\overline{K}$.

Example 2.1.0.8. Clearly split-toris i.e. algebraic groups $G = \mathbb{G}^r_{m,K}$ are examples of tori.

Here is an example of a non-split tori.

Example 2.1.0.9. Let $\mathbb{S}^1 := \operatorname{Spec}(R)$, where $R = \frac{\mathbb{R}[x,y]}{(x^2+y^2-1)}$. Note that $\mathbb{S}^1(\mathbb{R})$ can be identified with the circle S^1 and hence has a group law on it. Using this one can give a compatible group law on $\mathbb{S}^1(A)$ for any $\mathbb{R}$-algebra A, thus making $\mathbb{S}^1$ a group scheme over $\mathbb{R}$. Moreover $\mathbb{S}^1_{\mathbb{C}} := \mathbb{S}^1 \times_{\mathbb{R}} \mathbb{C} \simeq \mathbb{G}_{m,\mathbb{C}}$. However $\mathbb{S}^1$ itself is not isomorphic to $\mathbb{G}_{m,\mathbb{C}}$ since their $\mathbb{R}$-points are different (as manfiolds).

The key example for us will be the Deligne torus denote by $\mathbb{S}$.

Example 2.1.0.10. Let $\mathbb{S} \subset \mathrm{GL}_{2,\mathbb{R}}$ be defined as the subgroup of matrices of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

It is clear that $\mathbb{S}$ is a linear algebraic group and that $\mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}^2_{m,\mathbb{C}}$ under the map

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mapsto (a+ib, a-ib).$$

Moreover $\mathbb{S}$ is not split over $\mathbb{R}$, since its $\mathbb{R}$-points are isomorphic to $S^1 \times \mathbb{R}^\times$. In addition have the weight cocharacter

$$w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}^1, \tag{2.3}$$

given by the inclusion of the diagonal tori and the Norm character

$$\mathrm{Nm} : \mathbb{S}^1 \to \mathbb{G}_{m,\mathbb{R}},$$

given by restricting the determinant map to $\mathbb{S}$.

Here are a few results about tori that we shall need. We will state them without a proof. For proofs we refer the reader to [3, Chapter 12].

Proposition 2.1.0.11. *Let K be a field. Then the natural map $\mathbb{Z}^{\oplus r} \to Hom(\mathbb{G}^r_{m,K}, \mathbb{G}_{m,K})$[4] sending*

$$(n_1, n_2 \cdots n_r) \mapsto ((t_1, t_2 \cdots t_r) \to t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r})$$

is an isomorphism of abelian groups.

Combing the above proposition with complete reducibility of representations of $\mathbb{G}^r_m$, we obtain the following.

Proposition 2.1.0.12. *Let V_K/K be a finite dimensional representation of $G = \mathbb{G}^r_m$. Then there exists an isomorphism*

$$V_K \simeq \bigoplus_{\chi \in X^*(G)} V^\chi,$$

V^χ is a subspace of V on which the action of G is via the character χ.

Next we deal with non-split toris. As before let G/K be a tori. There is a natural action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathrm{Hom}(G_{\overline{K}}, \mathbb{G}_{m,\overline{K}})$ given by the formula

$$\sigma.\chi(g) = \sigma(\chi(\sigma^{-1}g)).$$

Example 2.1.0.13. Let us compute this Galois action for the Deligne torus $\mathbb{S}$. The character group $X^*(\mathbb{S}_\mathbb{C})$ is a free abelian group of rank 2 generated by characters χ_1 and χ_2 given by

$$\chi_1(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}) = a + ib,$$

$$\chi_2(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}) = a - ib.$$

It is clear from the definition of the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ that $\sigma.\chi_1 = \chi_2$ for the unique non trivial element $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Thus $X^*(\mathbb{S}_\mathbb{C})$ is isomorphic to $\mathbb{Z}^{\oplus 2}$ as a Galois module where

[4] Such homomorphisms are called *characters* of G.

the action of Gal$(\mathbb{C}/\mathbb{R})$ on $\mathbb{Z}^{\oplus 2}$ is by sending $(p,q) \to (q,p)$. Finally note that there is a morphism induced by the weight homomorphims (2.3)

$$X^*(\mathbb{S}_{\mathbb{C}}) \to X^*(\mathbb{G}_{m,\mathbb{C}}),$$

which under the above identification of $X^*(\mathbb{S}_{\mathbb{C}}) \simeq \mathbb{Z}^{\oplus 2}$ is simply the addition map.

Thanks to Galois descent we have the following corollary to Proposition 2.1.0.12.

Corollary 2.1.0.14. *Let G/K be a tori and let V/K be a finite dimensional representation of G. Then there exists an isomorphism*

$$V_{\overline{K}} \simeq \bigoplus_{\chi \in X^*(G_{\overline{K}})} V^{\chi},$$

such that the Galois action of Gal$(\overline{K}/K)$ on the left sends $V^{\chi} \to V^{\sigma \cdot \chi}$.

Example 2.1.0.15. Let $V/\mathbb{R}$ be a finite dimensional $\mathbb{R}$-vector space with an action of $\mathbb{S}$. Then by Corollary 2.1.0.14 there exists an isomorphism

$$V_{\mathbb{C}} \simeq \bigoplus_{(p,q) \in \mathbb{Z}^{\oplus 2}} V^{p,q},$$

such that the complex conjugation on the left send $V^{p,q} \mapsto V^{q,p}$.

Thus we have the following alternate definition of a pure Hodge structure of weight k.

Proposition 2.1.0.16. *Let $V_{\mathbb{Z}}$ be a finitely generated abelian group. Then giving a pure Hodge structure of weight k on $V_{\mathbb{Z}}$ is equivalent to giving a representation*

$$\rho : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}}),$$

such that $\rho \circ w : \mathbb{G}_{m,\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ is via the single character $k \in X^*(\mathbb{G}_{m,\mathbb{R}})$.

The restriction of ρ to the weight co-character may seem artificial. In fact given any representation ρ of $\mathbb{S}$ on a real vector space $V_{\mathbb{R}}$, by restricting it to $\mathbb{G}_{m,\mathbb{R}}$ we can decompose $V_{\mathbb{R}}$ into its isotypic components (or weights) for this action. Since $\mathbb{S}$ is a commutative group, this decomposition is respected by the action of $\mathbb{S}$.

Definition 2.1.0.17. Let $\mathbb{Q}$-HS be the category whose objects are a pair $(V_{\mathbb{Q}}, \rho)$ where $V_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space and $\rho : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ is a representation of the Deligne torus. Morphisms in this category are morphisms of $\mathbb{Q}$-vector spaces which respect the action of Deligne torus under extension of scalars. A $\mathbb{Q}$-HS $(V_{\mathbb{Q}}, \rho)$ is said to be pure of weight k if the $\rho \circ w$ has exactly one isotypic component of weight k.

Thus the category of $\mathbb{Q}$-HS has

(a) direct sums

(b) tensor products: The tensor product of two pure Hodge structurs of weight k and k' is pure of weight $k + k'$.

(c) kernels and cokernels.

Bibliography

[1] Benjamin Bakker, Bruno Klingler, and Jacob Tsimerman. Tame topology of arithmetic quotients and algebraicity of hodge loci. *Journal of the American Mathematical Society*, 33(4):917–939, 2020. 8

[2] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. Wiley Classics Library. John Wiley & Sons, New York, 1978. 6

[3] James S. Milne. *Linear Algebraic Groups*, volume 150 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2017. 12

[4] Christophe Soulé and Claire Voisin. Torsion cohomology classes and algebraic cycles on complex projective manifolds. *Advances in Mathematics*, 198(1):107–127, 2005. 5

[5] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. 4, 5, 6