SM Exercise 11.8

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A forensic laboratory assesses if the DNA profile from a specimen found at a crime scene matches the DNA profile of a suspect. The technology is not perfect, as there is a (small) probability ρ that a match occurs by chance even if the suspect was not present at the scene, and a (larger) probability γ that a match is reported even if the profiles are different; this can arise due to laboratory error such as cross-contamination or accidental switching of profiles.

1. Let R, S, and M denote the events that a match is reported, that the specimen does indeed come from the suspect, and that there is a match between the profiles, and suppose that

$$\Pr(R \mid M \cap S) = \Pr(R \mid M \cap \bar{S}) = \Pr(R \mid M) = 1,$$
$$\Pr(\bar{M} \mid S) = 0, \quad \Pr(R \mid S) = 1.$$

Show that the posterior odds of the profiles matching, given that a match has been reported, depend on

$$\frac{\Pr(R \mid S)}{\Pr(R \mid \bar{S})} = \frac{\Pr(R \mid M \cap S)\Pr(M \mid S) + \Pr(R \mid \bar{M} \cap S)\Pr(\bar{M} \mid S)}{\Pr(R \mid M \cap \bar{S})\Pr(M \mid \bar{S}) + \Pr(R \mid \bar{M} \cap \bar{S})\Pr(\bar{M} \mid \bar{S})},$$

and establish that this is $\{\rho + \gamma(1-\rho)\}^{-1}$.

From the information given in the question, let us suppose $\rho = \Pr(M|\bar{S})$ and $\gamma = \Pr(R|\bar{M})$.

Assuming $\Pr(R \mid M \cap S) = (R \mid M \cap \bar{S}) = \Pr(R|M) = 1$, we then have that R is conditionally independent of S, conditional on M.

Note that the complement events are also independent, and so R is also conditionally independent of $\bar{S}$, conditional on $\bar{M}$.

Thus, we can show the desired result as follows:

$$
\begin{aligned}
\frac{\Pr(R \mid S)}{\Pr(R \mid \bar{S})} &= \frac{\Pr(R \mid M \cap S)\Pr(M \mid S) + \Pr(R \mid \bar{M} \cap S)\Pr(\bar{M} \mid S)}{\Pr(R \mid M \cap \bar{S})\Pr(M \mid \bar{S}) + \Pr(R \mid \bar{M} \cap \bar{S})\Pr(\bar{M} \mid \bar{S})} \\
&= \frac{(1)\Pr(M \mid S) + \Pr(R \mid \bar{M} \cap S)(0)}{(1)\Pr(M \mid \bar{S}) + \Pr(R \mid \bar{M} \cap \bar{S})\Pr(\bar{M} \mid \bar{S})} \quad \text{based on assumptions} \\
&= \frac{\Pr(M \mid S)}{\Pr(M \mid \bar{S}) + \Pr(R \mid \bar{M} \cap \bar{S})\Pr(\bar{M} \mid \bar{S})} \\
&= \frac{\Pr(M \mid S) + \Pr(\bar{M} \mid S)}{\Pr(M \mid \bar{S}) + \Pr(R \mid \bar{M} \cap \bar{S})\Pr(\bar{M} \mid \bar{S})} \quad \text{since } \Pr(\bar{M} \mid S) = 0 \\
&= \frac{1}{\Pr(M \mid \bar{S}) + \Pr(R \mid \bar{M} \cap \bar{S})\Pr(\bar{M} \mid \bar{S})} \\
&= \frac{1}{\Pr(M \mid \bar{S}) + \Pr(R \mid \bar{M} \cap \bar{S})(1 - \Pr(M \mid \bar{S}))} \\
&= \frac{1}{\Pr(M \mid \bar{S}) + \Pr(R \mid \bar{M})(1 - \Pr(M \mid \bar{S}))} \quad \text{since } R \perp \bar{S} | \bar{M} \\
&= \frac{1}{\rho + \gamma(1-\rho)} \\
&= \{\rho + \gamma(1-\rho)\}^{-1}
\end{aligned}
$$

Next, we can show how the posterior odds of the profiles matching, given that a match has been reported, depend on the expression above.

The posterior odds that we are interested in computing is:

$$\frac{\Pr(S \mid R)}{\Pr(\bar{S} \mid R)}$$

Then,

$$
\begin{aligned}
\frac{\Pr(S \mid R)}{\Pr(\bar{S} \mid R)} &= \frac{\frac{\Pr(S \cap R)}{\Pr(R)}}{\frac{\Pr(\bar{S} \cap R)}{\Pr(R)}} \\
&= \frac{\Pr(S \cap R)}{\Pr(\bar{S} \cap R)} \\
&= \frac{\Pr(R \mid S)\Pr(S)}{\Pr(R \mid \bar{S})\Pr(\bar{S})} \\
&= \frac{\Pr(R \mid S)}{\Pr(R \mid \bar{S})}\frac{\Pr(S)}{\Pr(\bar{S})}
\end{aligned}
$$

Therefore, the posterior odds of the profile matching, given that a match has been report, depends on $\frac{\Pr(R|S)}{\Pr(R|\bar{S})}$.

2. Tabulate $\Pr(R \mid S)/\Pr(R \mid \bar{S})$ when $\rho = 0, 10^{-9}, 10^{-6}, 10^{-3}$ and $\gamma = 0, 10^{-4}, 10^{-3}, 10^{-2}$.

```
##     rhos gammas    post_odds
## 1  0e+00  0e+00          Inf
## 5  0e+00  1e-04 1.000000e+04
## 9  0e+00  1e-03 1.000000e+03
## 13 0e+00  1e-02 1.000000e+02
## 2  1e-09  0e+00 1.000000e+09
```

```
## 6  1e-09  1e-04 9.999900e+03
## 10 1e-09  1e-03 9.999990e+02
## 14 1e-09  1e-02 9.999999e+01
## 3  1e-06  0e+00 1.000000e+06
## 7  1e-06  1e-04 9.901000e+03
## 11 1e-06  1e-03 9.990020e+02
## 15 1e-06  1e-02 9.999010e+01
## 4  1e-03  0e+00 1.000000e+03
## 8  1e-03  1e-04 9.091736e+02
## 12 1e-03  1e-03 5.002501e+02
## 16 1e-03  1e-02 9.099181e+01
```