Question 2:

$$p(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

log-likelihood:

$$\ell(\alpha, \beta) = \log\left(\prod_{i=1}^{n} p(x_i \mid \alpha, \beta)\right)$$

$$\ell(\alpha, \beta) = \log\left(\left[\frac{\beta^{\alpha}}{\Gamma(\alpha)}\right]^{n} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\beta \sum_{i=1}^{n} x_i}\right)$$

$$\ell(\alpha, \beta) = n\left(\alpha \ln \beta - \ln \Gamma(\alpha)\right) + (\alpha-1) \sum_{i=1}^{n} \ln x_i - \beta \sum_{i=1}^{n} x_i$$

Gradient:

$$\frac{\partial \ell}{\partial \alpha} = n\left(\ln \beta - \psi(\alpha)\right) + \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} x_i$$

It is obvious that $\hat{\beta} = \dfrac{n\alpha}{\sum_{i=1}^{n} x_i}$

Hessian:

thorme: gradient $f(x)$ = $n\left(\ln \frac{n\alpha}{\sum x_i} - \psi(\alpha)\right) + \sum_{i=1}^{n} \ln x_i$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = n\psi'(\alpha)$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n\alpha}{\beta^2}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = -\frac{n}{\beta} \qquad \cancel{\partial} = 0.$$

As a consequence:

$$f''(\alpha) = \frac{n \cancel{\alpha}}{\alpha} - n\psi'(\alpha)$$