

Homework 4

ISyE 6420

Spring 2020



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Due March 1, 2020, 11:55pm. HW4 is not time limited except the due date. Late submissions will not be accepted.

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1. Metropolis for Correlation Coefficient. Pairs $(X_i, Y_i), i = 1, \dots, n$ consist of correlated standard normal random variables (mean 0, variance 1) forming a sample from a bivariate normal $\mathcal{MVN}_2(\mathbf{0}, \Sigma)$ distribution, with covariance matrix

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

The density of $(X, Y) \sim \mathcal{MVN}_2(\mathbf{0}, \Sigma)$ is¹

$$f(x, y|\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2) \right\},$$

with ρ as the only parameter. Take prior on ρ by assuming Jeffreys' prior on Σ as $\pi(\Sigma) = \frac{1}{|\Sigma|^{3/2}} = \frac{1}{(1-\rho^2)^{3/2}}$, since the determinant of Σ is $1 - \rho^2$. Thus

$$\pi(\rho) = \frac{1}{(1-\rho^2)^{3/2}} \mathbf{1}(-1 \leq \rho \leq 1).$$

(a) If $(X_i, Y_i), i = 1, \dots, n$ are observed, write down the likelihood for ρ . Write down the expression for the posterior, up to the proportionality constant (that is, un-normalized posterior as the product of likelihood and prior).

(b) Since the posterior for ρ is complicated, develop a Metropolis-Hastings algorithm to sample from the posterior. Assume that $n = 100$ observed pairs (X_i, Y_i) gave the following summaries:

$$\sum_{i=1}^{100} x_i^2 = 114.9707, \quad \sum_{i=1}^{100} y_i^2 = 105.9196, \quad \text{and} \quad \sum_{i=1}^{100} x_i y_i = 82.5247.$$

¹See (6.1) on page 243 in <http://statbook.gatech.edu>.

In forming a Metropolis-Hastings chain take the following proposal distribution for ρ : At step i generate a candidate ρ' from the uniform $\mathcal{U}(\rho_{i-1} - 0.1, \rho_{i-1} + 0.1)$ distribution. Why the proposal distribution cancels in the acceptance ratio expression?

(c) Simulate 51000 samples from the posterior of ρ and discard the first 1000 samples (burn in). Plot two figures: the histogram of ρ s and the realizations of the chain for the last 1000 simulations. What is the Bayes estimator of ρ based on the simulated chain?

(d) Replace the proposal distribution from (b) by the uniform $\mathcal{U}(-1, 1)$ (independence proposal). Comment on the results of MCMC.

2. Gibbs Sampler and Lifetimes with Multiplicative Frailty. Exponentially distributed lifetimes have constant hazard rate equal to the rate parameter λ . When λ is a constant hazard rate, a simple way to model heterogeneity of hazards is to introduce a multiplicative frailty parameter μ , so that lifetimes T_i have distribution²

$$T_i \sim f(t_i|\lambda, \mu) = \lambda\mu \exp\{-\lambda\mu t_i\}, \quad t_i > 0, \lambda, \mu > 0.$$

The prior on (λ, μ) is

$$\pi(\lambda, \mu) \propto \lambda^{c-1} \mu^{d-1} \exp\{-\alpha\lambda - \beta\mu\},$$

that is, λ and μ are apriori independent with distributions $\mathcal{Ga}(c, \alpha)$ and $\mathcal{Ga}(d, \beta)$, respectively. The hyperparameters c, d, α and β are known (elicited) and positive.

Assume that lifetimes t_1, t_2, \dots, t_n are observed.

(a) Show that full conditionals for λ and μ are gamma,

$$[\lambda|\mu, t_1, \dots, t_n] \sim \mathcal{Ga}\left(n + c, \mu \sum_{i=1}^n t_i + \alpha\right),$$

and by symmetry,

$$[\mu|\lambda, t_1, \dots, t_n] \sim \mathcal{Ga}\left(n + d, \lambda \sum_{i=1}^n t_i + \beta\right).$$

(b) Using the result from (a) develop Gibbs Sampler algorithm that will sample 51000 pairs (λ, μ) from the posterior and burn-in the first 1000 simulations. Assume that $n = 20$ and that the sum of observed lifetimes is $\sum_{i=1}^{20} t_i = 512$.

Assume further that the priors are specified by hyperparameters $c = 3, d = 1, \alpha = 100$, and $\beta = 5$. Start the chain with $\mu = 0.1$.

(c) From the produced chain, plot the scatterplot of (λ, μ) as well as histograms of individual components, λ , and μ . Estimate posterior means and variances for λ and μ . Find 95% equitailed credible sets for λ and μ .

(d) A frequentist statistician estimates the product $\lambda\mu$ as $\frac{n}{\sum_{i=1}^n t_i} = 20/512 = 0.0391$. What is the Bayes estimator of this product? (Hint: It is not the product of averages, it is the average of products, so you will need to save products in the MCMC loop).

²This is a toy example. In realistic applications the multiplicative frailty depends on the i , or on a subclass of population as $\mu_{j(i)}$.