Homework 2

ISyE 6420 Spring 2020



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Due February 9, 2020, 11:55pm. HW2 is not time limited except the due date. Late submissions will not be accepted.

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1. k-out-of-n and Weibull Lifetime. Engineering system of type k-out-of-n is operational if at least k out of n components are operational. Otherwise, the system fails. Suppose that a k-out-of-n system consists of n identical and independent elements for which the lifetime has Weibull distribution with parameters r and λ . More precisely, if T is a lifetime of a component,

$$P(T \ge t) = e^{-\lambda t^r}, t \ge 0.$$

Time t is in units of months, and consequently, rate parameter λ is in units (month)⁻¹. Parameter r is dimensionless.

Assume that n = 8, k = 4, r = 3/2 and $\lambda = 1/10$.

- (a) Find the probability that a k-out-of-n system is still operational when checked at time t=3.
- (b) At the check up at time t=3 the system was found operational. What is the probability that at that time exactly 5 components were operational?

Hint: For each component the probability of the system working at time t is $p = e^{-0.1t^{3/2}}$. The probability that a k-out-of-n system is operational corresponds to the tail probability of binomial distribution: $\mathbb{P}(X \ge k)$, where X is the number of components working. You can do exact binomial calculations or use binocdf in Octave/MATLAB (or dbinom in R, or scipy.stats.binom.cdf in Python when scipy is imported). Be careful with \le and <, because of the discrete nature of binomial distribution. Part (b) is straightforward Bayes formula.

2. Precision of Lab Measurements. The error X in measuring the weight of a chemical sample is a random variable with PDF

$$f(x) = \begin{cases} \frac{3x^2}{16}, & -2 < x < 2\\ 0, & \text{otherwise} \end{cases}$$

- (a) A measurement is considered to be accurate if |X| < 0.5. Find the probability that a randomly chosen measurement can be classified as accurate.
 - (b) Find the cumulative distribution function F(x) and sketch its graph.
- (c) The loss in thousands of dollars, which is caused by measurement error, is $Y = X^2$. Find the mean of Y (expected loss).
 - (d) Compute the probability that the loss is less than \$3.

3. 2-D Density Tasks. If

$$f(x,y) = \begin{cases} x+y, & 0 \le x \le 1; \ 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

Find:

- (a) marginal distribution $f_X(x)$.
- (b) conditional distribution f(y|x).

4. From the first page of Rand's book A Million Random Digits with 100,000 Normal Deviates.

10097	32533	76520	13586	34673	54876	80959	09117	39292	74945
37542	04805	64894	74296	24805	24037	20636	10402	00822	91665
08422	68953	19645	09303	23209	02560	15953	34764	35080	33606
99019	02529	09376	70715	38311	31165	88676	74397	04436	27659
12807	99970	80157	36147	64032	36653	98951	16877	12171	76833

The first 50 five-digit numbers form the Rand's "A Million Random Digits with 100,000 Normal Deviates" book (shown above) are rescaled to [0,1] (by dividing by 100,000) and then all numbers < 0.6 are retained. We can consider the n = 34 retained numbers as a random sample from uniform $\mathcal{U}(0,0.6)$ distribution.

0.10097	0.32533	0.13586	0.34673	0.54876	0.09117
0.39292	0.37542	0.04805	0.24805	0.24037	0.20636
0.10402	0.00822	0.08422	0.19645	0.09303	0.23209
0.02560	0.15953	0.34764	0.35080	0.33606	0.02529
0.09376	0.38311	0.31165	0.04436	0.27659	0.12807
0.36147	0.36653	0.16877	0.12171		

Pretend now that the threshold 0.6 is not known to us, that is, we are told that the sample is from uniform $\mathcal{U}(0,\theta)$ distribution, with θ to be estimated.

Let M be the maximum of the retained sample u_1, \ldots, u_{34} , in our case M = 0.54876. The likelihood is

$$f(u_1, \dots, u_{34} | \theta) = \prod_{i=1}^{34} \frac{1}{\theta} \mathbf{1}(\theta > u_i) = \theta^{-34} \mathbf{1}(\theta > M),$$

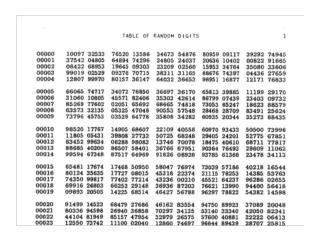


Figure 1: First page of RAND's book.

where $\mathbf{1}(A)$ is 1 if A is true, and 0 if A is false.

Assume noninformative (Jeffreys') prior on θ ,

$$\pi(\theta) = \frac{1}{\theta} \mathbf{1}(\theta > 0).$$

Posterior depends on data via the maximum M and belongs to the Pareto family, $\mathcal{P}a(c,\alpha)$, with a density

$$\frac{\alpha c^{\alpha}}{\theta^{\alpha+1}} \mathbf{1}(\theta > c).$$

- (a) What are α and c?
- (b) Estimate θ and calculate 95% equitailed credible set. Is the true value of parameter (0.6) in the credible set?

Hint: Expectation of the Pareto $\mathcal{P}a(c,\alpha)$ is $\frac{\alpha c}{\alpha-1}$ and CDF is $F(\theta) = [1 - (c/\theta)^{\alpha}] \mathbf{1}(\theta > c)$.