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problem 1:

$$h_1(x) = 1 \; ; \; h_2(x) = x \; ; \; h_3(x) = x^2$$

$$h_4(x) = x^3 \; ; \; h_5(x) = (x - \xi)^3_+$$

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x - \xi)^3_+$$

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^3 - 3\beta_4 x^2 \xi + 3\beta_4 x \xi^2 -$$

$$f(x) = (\beta_0 - \xi^3) + x(\beta_1 + 3\beta_4 \xi^2) + x^2(\beta_2 - 3\beta_4$$
$$+ x^3(\beta_3 + \beta_4)$$

a). Find cubic polynomial

$$f_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$$

$$\forall x \le \xi \qquad f(x) = f_1(x)$$

$$\Longleftrightarrow \begin{cases} a_1 = \beta_0 - \xi^3 \\ b_1 = \beta_1 + 3\beta_4 \xi^2 \\ c_1 = \beta_2 - 3\beta_4 \xi \\ d_1 = \beta_3 + \beta_4 \end{cases}$$

b). Similarly:

$$f_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3$$

$$f(x) = f_2(x) \quad \forall x \geq \xi$$

$$\Leftrightarrow \begin{cases} a_2 = \beta_0 - \xi^3 \\ b_2 = \beta_1 + 3\beta_4 \xi^2 \\ c_2 = \beta_2 - 3\beta_4 \xi \\ d_2 = \beta_3 + \beta_4 \end{cases}$$

c). Therefore $f_1(\xi) = f_2(\xi)$

$\rightarrow f(x)$ is continuous at ξ.

$$\lim_{x \to \xi_-} f_1(x) = (\beta_0 - \xi^3) + (\xi)(\beta_1 + 3\beta_4 \xi^2) + \xi^2(\beta_2 - 3\beta_4 \xi) + \xi^3(\beta_3 + \beta_4)$$

$$\lim_{x \to \xi_+} f_2(x) = (\beta_0 - \xi^3) + \xi(\beta_1 + 3\beta_4 \xi^2) + \xi^2(\beta_2 - 3\beta_4 \xi) + \xi^3(\beta_3 + \beta_4)$$

$$\lim_{x \to \xi_-} f_1(x) = f_1(\xi) = f_2(\xi) = \lim_{x \to \xi_+} f_2(x)$$

d) $$f_1'(x) = \beta_1 + 3\beta_4 \xi^2 + 2x(\beta_2 - 3\beta_4 \xi) + 3x^2(\beta_3 + \beta_4)$$

$$f_2'(x) = \beta_1 + 3\beta_4 \xi^2 + 2x(\beta_2 - 3\beta_4 \xi) + 3x^2(\beta_3 + \beta_4)$$

$$\therefore f_1'(\xi) = f_2'(\xi)$$

e.

$$f_1''(x) = 2(\beta_2 - 3\beta_4 \xi) + 6x(\beta_3 + \beta_4)$$

$$f_2''(x) = 2(\beta_2 - 3\beta_4 \xi) + 6x(\beta_3 + \beta_4)$$

$$\therefore \quad f_1''(\xi) = f_2''(\xi)$$

problem 2:

a^{th} sample:

$$\hat{f}_{(a)}(x_a) = \frac{\sum_{i=1}^{n} K(x_a; x_i)\, y_i - K(x_a; x_a)\, y_a}{\sum_{i=1}^{n} K(x_a; x_i) - K(x_a; x_a)}$$

$$CV_a := y_a - \hat{f}_{(a)}(x_a) =$$

$$= \frac{y_a \sum_{i=1}^{n} K(x_a; x_i) - y_a K(x_a; x_a) - \sum_{i=1}^{n} K(x_a; x_i)\, y_i + K(x_a; x_a)\, y_a}{\sum_{i=1}^{n} K(x_a; x_i) - K(x_a; x_a)}$$

$$CV_a = \frac{\sum_{i=1}^{n} \left[(y_a - y_i)\left(K(x_a; x_i)\right)\right]}{\sum_{i=1}^{n} K(x_a; x_i) - K(x_a; x_a)} \qquad (1)$$

$$\frac{y_a - \hat{f}(x_a)}{1 - \frac{K(x_a;x_a)}{\sum_{i=1}^{n} K(x_a;x_i)}} = \frac{y_a \sum_{i=1}^{n} K(x_a;x_i) - \hat{f}(x_a)\sum_{i=1}^{n} K(x_a;x_i)}{\sum_{i=1}^{n} K(x_a;x_i) - K(x_a;x_a)}$$

$$= \frac{y_a \sum_{i=1}^{n} K(x_a;x_i) - \sum_{i=1}^{n} K(x_a;x_i)\, y_i}{\sum_{i=1}^{n} K(x_a;x_i) - K(x_a;x_a)}$$

$$= \frac{\sum_{i=1}^{n} \left[(y_a - y_i)\left(K(x_a;x_i)\right)\right]}{\sum_{i=1}^{n} K(x_a;x_i) - K(x_a;x_a)} \qquad (2)$$

From (1) and (2)

$$CV_a = \frac{y_a - \hat{f}(x_a)}{1 - \frac{K(x_a;x_a)}{\sum_{i=1}^{n} K(x_a;x_a)}}$$

Given this fact $\rightarrow$ we only need to fit model $(n-1)$ times to calculate mean square cross validation error.