CMSC 455/655 Homework 2

Please label your homework as your class number, last name, first name, and homework number. Ex: CMSC_455_Last_First_Homework2.docx. Please label your code files in a similar way.

Please remember to package your code into a single file and upload it with your homework document.

Please make answers clear and legible by highlighting your final solution if it is a number. Provide code segments in your solution where applicable. For instance, if you use the L-U decomp function in python to decompose a matrix, show the line where you are using it and the output.

Use figure captions, labels, and titles for all figures. Make your figure print ready.

Cite all source code, whether you use scipy, numerical recipes, or LAPACK.

You can delete these instructions when you turn in your homework, but please retain the questions below.

1. **Systems of Linear Equations:**

    a. Write a program that takes a matrix, A, and uses Gaussian elimination to decompose the matrix into lower and upper triangular matrices, L and U. You may check this against the built-in function, but need to use your own code to get L and U.

    b. A system of equations is shown below, where A is the 3x3 coefficient matrix. Using your code, solve for x. The solution is x = $[1,1,1]^T$. Is this what you get using your code? Compare using a built-in function.

$$\begin{bmatrix} 3 & 2 & 4 \\ 8 & -6 & -8 \\ -1 & 2 & 3 \end{bmatrix} x = \begin{bmatrix} 9 \\ -6 \\ 4 \end{bmatrix}$$

c. What is the condition number for the matrix, A? What is the determinant? You may use built-in functions.

d. Adjust the coefficient matrix slightly. Do you get the same solution? You may use your code or built-in functions.

2. **Systems of Linear Equations:** The equation to create the Hilbert matrix is defined below.

$$H_n = \left(\frac{1}{i+j+1}\right) i,j = 0,1,\ldots,n-1$$

a. (3 points) Using this matrix, display a Hilbert matrix of the order of 5.

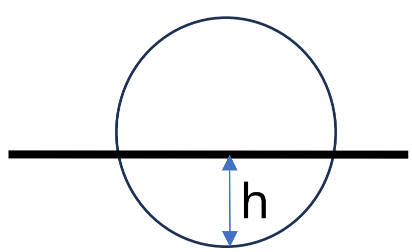
b. (4 points) What is the condition number of the 9x9 Hilbert matrix, H_9.

c. (3 points) Solve $H_9 x = [1,1,1,1,1,1,1,1,1]^T$. Then change the first component of the right-hand side to 1.01 and solve again. Which component of x is most changed?

3. **Roots:** The weight of a sphere of density, d, and radius, r, is $\frac{4}{3}\pi r^3 d$. When put in water, buoyancy acts to balance the weight of the sphere. The volume of a sphere segment is $\frac{1}{3}\pi\left(3rh^2 - h^3\right)$. Find the depth to which a sphere of density 0.5kg/m^3 sinks in water as a fraction of its radius.
The formula for buoyancy is

$$F_b = -\rho g V$$

Use 1000kg/m^3 as the density of water.



4. **Interpolation:** Use the data and divided differences to answer the following questions:

X = [-1,3,2,-2,4]
Y = [8,0,-1,15,3]

   a. Is this data from a polynomial?
   b. If so, what degree is the polynomial?
   c. If the point (5, -7) is added, what will be the degree of the new approximating polynomial?

5. **Least Squares:** When using least squares to fit a polynomial, we need to create a matrix such that we can solve for our coefficients. To do this, we can use what is called a design matrix, where n is the degree of our polynomial and N is the number of datapoints.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & \cdots & x_N \\ \vdots & \ddots & \cdots & \vdots \\ x_1^n & x_2^n & \cdots & x_N^n \end{bmatrix}$$

To find the least squares fit, we must solve the matrix equation $Ax=b$ for the coefficient vector x. Re-arranging the matrix equation gives:

$$A^{\top} A x = A^{\top}$$

$$x = \left(A^{\top} A\right)^{-1} A^{\top} b$$

Solve for x to find the coefficients of the interpolating polynomial. You may use built-in matrix operations.

   a. Solve the least squares fit for the points (1, 2), (3, 7), (4, 15). What is the polynomial?

b. Does the polynomial perfectly fit the points?

c. Given 5 points, what degree polynomial would be required to perfectly fit the data?