Unit - 5

Vector differentiation

Scalar point function:- let $p(x,y,z)$ is a point in the region E in that space such that their exist a definite function $\phi(x,y,z)$ then $\phi(x,y,z)$ is called scalar point function and E is called a scalar field

Ex:- $x^2y + y^2z + z^2x$

Vector point function:- corresponding to very point $p(x,y,z)$ in the region E in space their exist a definite vector function $\bar{f}(x,y,z)$, then $\bar{f}(x,y,z)$ is called a vector point function and E is called vector field.

Ex:- $y\bar{i} + z\bar{j} + x\hat{k}$.

Level surface:-

let $\phi(x,y,z)$ is a scalar point function and c is a constant then the set of all points (x,y,z) such that $\phi(x,y,z) = c$ is called a level surface.

Derivative of a vector:-

let $\bar{f}(t)$ is a continuous single valued function in the scalar "t". If the limit $\lim_{t\to c} \frac{\bar{f}(t)-\bar{f}(c)}{t-c}$ is exist then it is called derivate of $\bar{f}(t)$ at $t=c$

it is denoted by $\left(\frac{d\bar{f}}{dt}\right)_{t=c}$

Ex:- If $\bar{f}(t) = t\bar{i} - 3t^2\bar{j} + \sin t\,\bar{k}$, then find $\left(\frac{d\bar{f}}{dt}\right)_{t=0}$

* $\bar{f}(t) = t\bar{i} - 3t^2\bar{j} + \sin t\,\bar{k}$

$\frac{d\bar{f}}{dt} = \bar{i} - 6t\,\bar{j} + \cos t\,\bar{k}$

at, $t=0 \rightarrow \frac{d\bar{f}}{dt} = \bar{i} - 0 + \bar{k} = \bar{i} + \bar{k}$

<u>Tangent vector</u>:- let $\bar{r}(t)$ is the position vector of any point on a surface then $\frac{d\bar{r}}{dt}$ is called tangent vector to the surface at that point.

<u>Vector differential operator</u>:- The vector differential operator is denoted by "∇" and defined by

$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$.

<u>Gradient</u>:- let $\phi(x,y,z)$ is a scalar point function then the gradient of ϕ is denoted by $\nabla\phi$ (or) grad ϕ and defined as

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \Sigma\bar{i}\frac{\partial\phi}{\partial x}.$$

The gradient of a scalar function is a vector function.

<u>properties</u>:-

1. let f and g are two scalar point function then,

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla(cf) = c \cdot \nabla f$, c is constant.

(iii) $\nabla(fg) = f\nabla g + g\nabla f$

(iv) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$.

2. The necessary & sufficient condition that the scalar

function ϕ to be a constant then is $\nabla\phi = \bar{0}$

Physical interpretation of Gradient :-

The gradient of a scalar function ϕ at a point P is normal vector to the surface $\phi(x,y,z)=c$ at that point P.

problems:-

1. Find $\nabla\phi$ & $|\nabla\phi|$ if $\phi = 2xz^4 - x^2y$ at the point $(2,-2,-1)$

Given $\phi = 2xz^4 - x^2y$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \bar{i}(2z^4 - y\cdot 2x) + \bar{j}(0-x^2) + \bar{k}(2x\cdot 4z^3 - 0)$$

$$\nabla\phi = (2z^4 - 2xy)\bar{i} - x^2\bar{j} + 8xz^3\bar{k}.$$

at $(2,-2,-1)$,

$$\nabla\phi = [2(1) - 2(-4)]\bar{i} - 4\bar{j} + 8(2)(-1)^3\bar{k}$$

$$= (2+8)\bar{i} - 4\bar{j} - 8(2)\bar{k} = 10\bar{i} - 4\bar{j} - 16\bar{k}$$

$$|\nabla\phi| = \sqrt{(10)^2 + (-4)^2 + (-16)^2} = \sqrt{100+16+256} = \sqrt{372}$$

$$= 2\sqrt{93}$$

2. Find the unit normal vector to the surface $x^2+y^2+2z^2=6$ at the point $(2,2,3)$

Given, surface, $x^2+y^2+2z^2-6=0$.

let $\phi = x^2+y^2+2z^2-6$.

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= 2x\bar{i} + 2y\bar{j} + 4z\bar{k}$$

at $(2,2,3) \Rightarrow \nabla\phi = 4\bar{i} + 4\bar{j} + 12\bar{k}$

$$|\nabla\phi| = \sqrt{(4)^2+(4)^2+(12)^2} = \sqrt{176} = 4\sqrt{11}$$ next

unit normal vector to the surface.

$$= \frac{\nabla \phi}{|\nabla \phi|} = \frac{4(\bar{i}+\bar{j}+3\bar{k})}{4\sqrt{11}}$$

$$= \frac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}$$

3. Find the angle b/w the surfaces $x^2+y^2+z^2=9$, $z=x^2+y^2-3$. at the point $(2,-1,2)$

given Surfaces,

$$x^2+y^2+z^2-9=0 \quad , \quad x^2+y^2-z-3=0.$$

let, $f=x^2+y^2+z^2-9$, $g=x^2+y^2-z-3$.

$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$ $\qquad$ $\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$

$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$ $\qquad$ $= 2x\bar{i} + 2y\bar{j} - \bar{k}$.

at $p(2,-1,2)$ = $\qquad$ at $p(2,-1,2)$,

$\nabla f = 4\bar{i} - 2\bar{j} + 4\bar{k}$ $\qquad$ $\nabla g = 4\bar{i} - 2\bar{j} - \bar{k}$

Normal vector to the surface f at p

$$\bar{n}_1 = 4\bar{i} - 2\bar{j} + 4\bar{k}$$

Normal vector to the surface g is

$$\bar{n}_2 = 4\bar{i} - 2\bar{j} - 2\bar{k}$$

Angl

$$|\bar{n}_1| = \sqrt{(4)^2+(-2)^2+(4)^2} = \sqrt{16+4+16} = \sqrt{36} = 6.$$

$$|\bar{n}_2| = \sqrt{(4)^2+(2)^2+(\cancel{2})^2} = \sqrt{16+4+\cancel{4}} = \sqrt{\cancel{24}} = \cancel{2\sqrt{6}} . \sqrt{21}$$

If θ is acute angle b/w the two surfaces then θ is angle b/w the normals to the surface

$$\cos\theta = \frac{|\bar{n}_1 \cdot \bar{n}_2|}{|\bar{n}_1||\bar{n}_2|}$$

$$= \frac{|16+4-4|}{6\cdot\sqrt{21}} = \frac{8}{3\sqrt{21}}.$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

4. Find the values of a, and b so that the surfaces $ax^2 - byz = (a+2)x$, $4x^2y + z^3 = 4$ make intersect orthogonally at the point $(1,-1,2)$.

given surfaces,

$ax^2 - byz = (a+2)x$. , $4x^2y + z^3 - 4 = 0$.

$ax^2 - byz - (a+2)x = 0$

let, $f = ax^2 - byz - (a+2)x$, $g = 4x^2y + z^3 - 4$

$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$ $\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$

$= [2ax - (a+2)]\bar{i} + (-bz)\bar{j} + \bar{k}(-by)$ $= 8xy\bar{i} + 4x^2\bar{j} + 3z^2\bar{k}$

at $p(1,-1,2)$ at $p(1,-1,2)$

$\nabla f = [2a - (a+2)]\bar{i} - 2b\bar{j} + b\bar{k}$ $\nabla g = 8(-1)\bar{i} + 4\bar{j} + 12\bar{k}$

$= (a-2)\bar{i} - 2b\bar{j} + b\bar{k}$ $= -8\bar{i} + 4\bar{j} + 12\bar{k}$.

Normal vector to the surface f at p.

$$\bar{n}_1 = (a-2)\bar{i} - 2b\bar{j} + b\bar{k}$$

Normal vector to the surface g at p.

$$\bar{n}_2 = -8\bar{i} + 4\bar{j} + 12\bar{k}.$$

If θ is acute angle

given, the two surfaces are orthogonal to each other.

$\therefore \theta = 90^\circ$.

$$\cos\theta = \frac{|\bar{n}_1 \cdot \bar{n}_2|}{|\bar{n}_1||\bar{n}_2|}$$

$$|\bar{n}_1 \cdot \bar{n}_2| = 0.$$

$$|(a-2)(-8) + (-2b)(4) + b \times (12)| = 0.$$

$$-8a + 16 - 8b + 12b = 0.$$

$$-8a + 4b + 16 = 0.$$

$$2a - b - 4 = 0 \quad — ①.$$

$ax^2 - byz = (a+2)x$, $p(1,-1,2)$ is poI of the surface

$\therefore$ p lies on ~~ps~~ $ax^2 - byz = (a+2)x$

$$a - b(-1)(2) = (a+2)1$$

$$\cancel{a} + 2b - \cancel{a} - 2 = 0$$

$$b = +1.$$

$$2a - (+1) - 4 = 0$$

$$2a - 1 - 4 = 0.$$

$$2a = 5$$

$$a = \frac{5}{2}.$$

$$\therefore a = \frac{5}{2},\ b = 1.$$

5. Show that $\nabla f(r) = \frac{f'(r)}{r}\bar{r}$ where, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Given, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2.$$

p. diff r w.r.to x pr

$$2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

lly. $\frac{\partial r}{\partial y} = \frac{y}{r}$. $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\nabla f(r) = \Sigma \bar{i} \frac{\partial}{\partial x}[f(r)] \qquad \nabla \phi = \Sigma \bar{i} \frac{\partial \phi}{\partial x}.$$

$$= \Sigma \bar{i} f'(r) \frac{\partial r}{\partial x}$$

$$= \Sigma \bar{i} f'(r) \frac{x}{r}$$

$$= \frac{f'(r)}{r} \Sigma x\bar{i} = \frac{f'(r)}{r}[x\bar{i}+y\bar{j}+z\bar{k}] = \frac{f'(r)}{r} \bar{r}$$

5. Find the angle btw the normals to the surface $x^2 = yz$ at the points $P(1,1,1)$ & $Q(2,4,1)$

given surface, $x^2 = yz$

let $\phi = x^2 - yz$

$$\nabla \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$= 2x\bar{i} - z\bar{j} - y\bar{k}$$

at $P(1,1,1) \Rightarrow$

$$\nabla \phi = 2\bar{i} - \bar{j} - \bar{k}$$

at $Q(2,4,1)$; $\nabla \phi = 4\bar{i} - 4\bar{j} - \bar{k}$

Normal vector to the surface ϕ at P is $\bar{n}_1 = 2\bar{i} - \bar{j} - \bar{k}$

Normal vector to the surface ϕ at Q is $\bar{n}_2 = 4\bar{i} - \bar{j} - 4\bar{k}$

$$\therefore \cos\theta = \frac{|\bar{n}_1 \cdot \bar{n}_2|}{|\bar{n}_1||\bar{n}_2|}$$

$$= \frac{|8+1+4|}{\sqrt{4+1+1}\sqrt{16+1+16}} = \frac{13}{3\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$$

. Find $\nabla(\log r)$

let, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2 + y^2 + z^2$

p.diff 'r' w.r.to x,

$2r \frac{\partial r}{\partial x} = 2x$.

$\frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$\nabla(\log r) = \Sigma \bar{i} \frac{\partial}{\partial x}(\log r)$ $\qquad \nabla\phi = \Sigma \bar{i} \frac{\partial \phi}{\partial x}$.

$= \Sigma \bar{i} \frac{1}{r} \frac{\partial r}{\partial x}$.

$= \Sigma \bar{i} \frac{1}{r} \frac{x}{r}$

$= \frac{1}{r^2} \Sigma \bar{i} x$.

$= \frac{1}{r^2} [x\bar{i} + y\bar{j} + z\bar{k}]$

$= \frac{1}{r^2} \bar{r}$

8. find $\nabla(r^n)$

→ let, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

$r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

$r^2 = x^2 + y^2 + z^2$

p. diff r w.r.to x.

$\cancel{2}r \frac{\partial r}{\partial x} = \cancel{2}x \implies \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$\nabla(r^n) = \Sigma \bar{i} \frac{\partial}{\partial r}(r^n)$

$= \Sigma \bar{i}\, n \cdot r^{n-1} \frac{\partial r}{\partial x}$.

$$= n r^{n-1} \sum \bar{i} \frac{\partial r}{\partial x}.$$

$$= n r^{n-1} \sum \bar{i} \frac{x}{r}.$$

$$= \frac{n r^{n-1}}{r} \sum \bar{i}\, x = n r^{n-2} [x\bar{i} + y\bar{j} + z\bar{k}]$$

$$= n r^{n-2} \bar{r}.$$

9. If $\bar{a}$ is constant vector, then show that $\text{grad}(\bar{a}\cdot\bar{r}) = \bar{a}$.

let, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

$$\bar{a}\cdot\bar{r} = a_1 x + a_2 y + a_3 z = \phi \text{ , say}$$

$$\text{grad}(\bar{a}\cdot\bar{r}) = \nabla\phi$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}a_1 + \bar{j}a_2 + \bar{k}a_3$$

$$= \bar{a}.$$

10. If ~~$(\nabla\phi = \bar{i}a_1 + \bar{j}a_2 + \bar{k}a_3)$~~ $\nabla\phi = 2xyz\bar{i} + x^2z\bar{j} + x^2y\bar{k}$ then find ϕ.

given, $\nabla\phi = 2xyz\,\bar{i} + x^2z\,\bar{j} + x^2y\,\bar{k}$

$$\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2xyz\,\bar{i} + x^2z\,\bar{j} + x^2y\,\bar{k}$$

$$\frac{\partial\phi}{\partial x} = 2xyz, \quad \frac{\partial\phi}{\partial y} = x^2z, \quad \frac{\partial\phi}{\partial z} = x^2y.$$

Integrating,

$\phi = 2yz\frac{x^2}{2}$ + constant independent of x $f_1(y,z)$

$\phi = x^2zy$ + constant independent of y $f_2(x,z)$

$\phi = x^2yz$ + constant independent of z $f_3(x,y)$

From the above three forms

$$\phi = x^2yz + c.$$

Directional derivative :- The directional derivative is (the derivative of a scalar point function)* the rate at which the function changes at a point in the direction of a vector

The directional derivative of a scalar point function $\phi(x,y,z)$ at a point p in the direction of the unit vector $\bar{e}$ is $\nabla\phi \cdot \bar{e}$

The maximum value of directional value is $|\nabla\phi|$ and it is along the direction of $\nabla\phi$

1. Find the directional derivative of $\phi(x,y,z) = xy^2+yz^2$ at the point $(2,-1,1)$ in the direction of the vector $\bar{i}+2\bar{j}+2\bar{k}$

→ given, $\phi = xy^2+yz^2$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\,y^2 + \bar{j}(2xy+z^2) + \bar{k}(2zy)$$

$$= y^2\bar{i} + (2xy+z^2)\bar{j} + 2zy\bar{k}$$

at $p(2,-1,1)$,

$$\nabla\phi = \bar{i} + (2\times 2(-1)+1^2)\bar{j} + 2(1)(-1)\bar{k}$$

$$\nabla\phi = \bar{i} - 3\bar{j} - 2\bar{k}$$

given vector, $\bar{a} = \bar{i}+2\bar{j}+2\bar{k}$

unit vector along $\bar{a}$ is $\bar{e} = \frac{\bar{a}}{|\bar{a}|}$

$$= \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1+4+4}}$$

$$= \bar{i}+2\bar{j}+2\bar{k}$$

D.D of ϕ at the point p along $\bar{e}$ is

$= \nabla\phi \cdot \bar{e}$

$= (\bar{i}-3\bar{j}-2\bar{k})\left(\frac{\bar{i}+2\bar{j}+2\bar{k}}{3}\right)$

$= \frac{1}{3}(1-6-4) = \frac{1}{3}(-9) = -3.$

Find the directional derivative of $f = x^2-y^2+2z^2$ at $p(1,2,3)$ in the direction of the line PQ. where Q is $(5,0,4)$

Given, $f = x^2-y^2+2z^2$

$\nabla\phi = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$= \bar{i}\,2x - 2y\bar{j} + 4z\bar{k}.$

at $p(1,2,3)$,

$\nabla f = 2\bar{i} - 4\bar{j} + 12\bar{k}$

Given, point $Q(5,0,4)$

vector along the line $\overline{PQ}$ is $\bar{a} = \overline{PQ}$

$= \overline{OQ} - \overline{OP}$

$= 5\bar{i} + 4\bar{k} - \bar{i} - 2\bar{j} - 3\bar{k}$

$\bar{a} = 4\bar{i} - 2\bar{j} + \bar{k}$

unit vector along $\bar{a}$ is $\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{4\bar{i}-2\bar{j}+\bar{k}}{\sqrt{16+4+1}}$

$= \frac{4\bar{i}-2\bar{j}+\bar{k}}{\sqrt{21}}$

D.D of f at the point p along $\bar{e}$ is

$= \nabla f \cdot \bar{e}$

$= (2\bar{i} - 4\bar{j} + 12\bar{k}) \cdot \left(\frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}\right)$

$= \frac{1}{\sqrt{21}}(8+8-12) = \frac{28}{\sqrt{21}}.$

3. Find the directional derivative of xyz^2+xz at the point $(1,1,1)$ in the direction of the normal vector to the surface $3xy^2+y=z$ at $(0,1,1)$.

given, $f = xyz^2+xz$.

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$= \bar{i}(yz^2+z) + \bar{j}(xz^2) + \bar{k}(2xyz+x).$$

at $p(1,1,1) \Rightarrow$

$$\nabla f = 2\bar{i} + \bar{j} + 3\bar{k}$$

let, $g = 3xy^2+y-z$.

$$\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$$

$$= \bar{i}(3y^2) + \bar{j}(6xy+1) + \bar{k}(-1).$$

at $Q(0,1,1)$ $\nabla g = 3\bar{i} + \bar{j} - \bar{k}$

Normal vector to the surface g at Q is

$$\bar{a} = 3\bar{i}+\bar{j}-\bar{k}.$$

unit vector along $\bar{a}$ is $\bar{e}$ $= \dfrac{|3\bar{i}+\bar{j}-\bar{k}|}{\sqrt{9+1+1}}$

$$= \frac{(3\bar{i}+\bar{j}-\bar{k})}{\sqrt{11}}$$

D.D of f at the point p along $\bar{e}$ is.

$$= \nabla f \cdot \bar{e}$$

$$= (2\bar{i}+\bar{j}+3\bar{k}) \cdot \frac{1}{\sqrt{11}}(3\bar{i}+\bar{j}-\bar{k})$$

$$= \frac{1}{\sqrt{11}}(6+1-3) = \frac{4}{\sqrt{11}}$$

1. Find the directional derivative of $xy^2+yz^2+zx^2$ along the tangent to the curve $x=t, y=t^2, z=t^3$ at the point $(1,1,1)$.

given, $\phi = xy^2+yz^2+zx^2$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(y^2+2zx) + \bar{j}(2xy+z^2) + \bar{k}(2yz+x^2)$$

At $p(1,1,1) \Rightarrow$

$$\nabla\phi = 3\bar{i} + 3\bar{j} + 3\bar{k}$$

given, curve, $x=t, y=t^2, z=t^3$.

$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} \quad = t\bar{i}+t^2\bar{j}+t^3\bar{k}$

tangent vector to the curve is,

$$\frac{d\bar{r}}{dt} = \bar{i}+2t\bar{j}+3t^2\bar{k}$$

at $p(1,1,1) \Rightarrow \frac{d\bar{r}}{dt} = \bar{i}+2\bar{j}+3\bar{k}$

$x=1, y=1, z=1$

$t=1, t^2=1, t^3=1$

$\therefore t=1.$

unit vector along $\frac{d\bar{r}}{dt}$ is $\bar{e} = \frac{|\bar{i}+2\bar{j}+3\bar{k}|}{\sqrt{1+4+9}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$.

D.D of ϕ at the point p along $\bar{e}$ is.

$$= \nabla\phi\cdot\bar{e}$$

$$= (3\bar{i}+3\bar{j}+3\bar{k})\cdot\frac{1}{\sqrt{14}}(\bar{i}+2\bar{j}+3\bar{k})$$

$$= \frac{1}{\sqrt{14}}(3+6+9) = \frac{18}{\sqrt{14}}.$$

5. Find the directional derivative of $\frac{1}{r}$ in the direction of $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ at $(1,1,2)$

$\rightarrow$ given, $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

p. diff 'r' w.r.to x.

$2r \cdot \frac{\partial r}{\partial x} = 2x \cdot \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

let, $\phi = \frac{1}{r}$.

$$\nabla\phi = \Sigma\,\bar{i}\frac{\partial\phi}{\partial x} = \Sigma\,\bar{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \Sigma\,\bar{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x}.$$

$$= \Sigma\,\bar{i}\left(\frac{-1}{r^2}\right)\frac{x}{r} = \frac{-1}{r^3}\Sigma\bar{i}\,x.$$

$$= \frac{-1}{r^3}(x\bar{i}+y\bar{j}+z\bar{k})$$

$$= \frac{-1}{r^3}\cdot\bar{r}.$$

$$\nabla\phi = \frac{-(x\bar{i}+y\bar{j}+z\bar{k})}{(x^2+y^2+z^2)^{3/2}} = \frac{-(\bar{i}+\bar{j}+2\bar{k})}{(1+1+4)^{3/2}}$$

$$= \frac{-(\bar{i}+\bar{j}+2\bar{k})}{6\sqrt{6}}.$$

given, $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

at $(1,1,2)$; $\bar{r} = \bar{i}+\bar{j}+2\bar{k} = \bar{a}$, say.

unit vector along $\bar{a}$ is.

$$\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{i+j+2\bar{k}}{\sqrt{1+1+4}} = \frac{i+j+2k}{\sqrt{6}}.$$

D.D of ϕ at $(1,1,2)$ along $\bar{e}$ is.

$$= \nabla\phi \cdot \bar{e}$$

$$= \frac{-1}{6\sqrt{6}}(\bar{i}+\bar{j}+2\bar{k}) \cdot \frac{1}{\sqrt{6}}(\bar{i}+\bar{j}+2\bar{k})$$

$$= \frac{-1}{36}(1+1+4) = \frac{-6}{36_6} = \frac{-1}{6}.$$

6. In what direction from the point $(-1,1,2)$ the D.D of $\phi = xy^2z^3$ is maximum and what is its magnitude.

given, $\phi = xy^2z^3$.

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\, y^2z^3 + 2xyz^3\,\bar{j} + 3xy^2z^2\,\bar{k}.$$

at $(-1,1,2)$,

$$\nabla\phi = \bar{i}(1)^2(2)^3 + 2(-1)(1)(2)^3\bar{j} + 3(-1)(1)(2)^2\bar{k}.$$

$$= 8\bar{i} - 16\bar{j} - 12\bar{k}$$

$\therefore$ The directional derivative is maximum along the direction of $\nabla\phi = 8\bar{i} - 16\bar{j} - 12\bar{k}$.

Maximum Value of D.D $= |\nabla\phi|$

$$= \sqrt{64+256+144}$$

$$= \sqrt{464} = \sqrt{16\times 29} = 4\sqrt{29}$$

Divergence:- let $\bar{f}$ is a vector point function the divergence of $\bar{f}$ is denoted by $\nabla\cdot\bar{f}$ (or) $\operatorname{div}\bar{f}$ and defined by $\nabla\cdot\bar{f} = \bar{i}\cdot\frac{\partial\bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial\bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial\bar{f}}{\partial z} = \Sigma\,\bar{i}\cdot\frac{\partial\bar{f}}{\partial x}$.

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then

$$\nabla\cdot\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

The divergence of a vector function is a scalar function.

properties:-

1. If $\bar{f}$ is a constant vector then div $\bar{f} = 0$.

2. Divergence of $c\bar{f}$, $div(c\bar{f}) = c\, div \bar{f}$, c is constant

3. $div(\bar{f} \pm \bar{g}) = div\,\bar{f} \pm div\,\bar{g}$.

4. $(\bar{a}\cdot\nabla)\phi = \Sigma(\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x}$

proof:-

$$(\bar{a}\cdot\nabla)\phi = \left[\bar{a}\cdot\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\right]\phi.$$

$$= \left[(\bar{a}\cdot\bar{i})\frac{\partial}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial}{\partial z}\right]\phi.$$

$$= (\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial\phi}{\partial z}$$

$$= \Sigma(\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x}.$$

5. $(\bar{a}\cdot\nabla)\bar{f} = \Sigma(\bar{a}\cdot\bar{i})\frac{\partial\bar{f}}{\partial x}$

Solenodial vector:- A vector $\bar{f}$ is said to be solenoidal if divergence $\bar{f} = 0$

Every constant vector is solenodial.

physical significance of divergence:-

let $\bar{f}$ is velocity of a fluid in a fluid flow then divergence $\bar{f}$ represents the rate of fluid flow through unit volume.

1. If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find divergence $\bar{f}$ at $(1,-1,1)$.

given, $\bar{f} = \underset{f_1}{xy^2}\bar{i} + \underset{f_2}{2x^2yz}\bar{j} - \underset{f_3}{3yz^2}\bar{k}$

$$div\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$\operatorname{div}\bar{f} = y^2 + 2x^2 z - 6yz$.

at $p(1,-1,1)$, $\operatorname{div}\bar{f} = (-1)^2 + 2(1)(1) - 6(-1)(1)$

$= 1 + 2 + 6 = 9$.

2. Find divergence $\bar{f}$ ($\operatorname{div}\bar{f}$) where $\bar{f} = \operatorname{grad}(x^3+y^3+z^3-3xyz)$

→ given, $\bar{f} = \operatorname{grad}(x^3+y^3+z^3-3xyz)$.

$\phi = x^3+y^3+z^3-3xyz$

$\bar{f} = \operatorname{grad}\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\bar{f} = \bar{i}(\underbrace{3x^2-3yz}_{f_1}) + \bar{j}(\underbrace{3y^2-3xz}_{f_2}) + \bar{k}(\underbrace{3z^2-3xy}_{f_3})$

$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$\operatorname{div}\bar{f} = 6x + 6y + 6z$

3. If $\bar{f} = (x+2y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal, Find p

given, $\bar{f} = (x+2y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$.

$\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$\operatorname{div}\bar{f} = 1 + 1 + p$

So given $\bar{f}$ is solenoidal

$\therefore \operatorname{div}\bar{f} = 0$

$p + 2 = 0$

$p = -2$.

4. Find divergence of $r^n\bar{r}$.

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

P. diff r w.r.to x.

$$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial \bar{r}}{\partial x} = \bar{i}$$

$$\text{div } r^n\bar{r} = \nabla \cdot (r^n \bar{r})$$

$$= \Sigma \bar{i} \cdot \frac{\partial}{\partial x}(r^n \bar{r})$$

$$= \Sigma \bar{i} \cdot \left[n r^{n-1} \frac{\partial r}{\partial x} \cdot \bar{r} + r^n \frac{\partial \bar{r}}{\partial x} \right]$$

$$= \Sigma \bar{i} \cdot \left[n r^{n-1} \frac{x}{r} \bar{r} + r^n \bar{i} \right]$$

$$= \Sigma \bar{i} \cdot \left(n r^{n-1} \frac{x}{r} \bar{r} \right) + \Sigma \bar{i} \cdot (r^n \bar{i})$$

$$= \Sigma n r^{n-2} \cdot x (\bar{i} \cdot \bar{r}) + \Sigma r^n (\bar{i} \cdot \bar{i})$$

$$= n r^{n-2} \Sigma x(x) + r^n \Sigma 1$$

$$= n r^{n-2} (x^2 + y^2 + z^2) + r^n (3)$$

$$= n r^{n-2} r^2 + 3r^n = n r^n + 3r^n = (3+n) r^n.$$

Note:-

1. Find n such that $r^n \bar{r}$ is solenoidal.
2. Show that $\bar{r}/r^3$ is solenoidal.

Find divergence $\bar{r}$.

let, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

p. diff w.r.to x

$$2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial \bar{r}}{\partial x} = \bar{i}$$

$$\text{div } \bar{r} = \nabla \cdot (\bar{r}) = \Sigma \bar{i} \cdot \frac{\partial}{\partial x} \bar{r} = \Sigma \bar{i} \cdot \bar{i} = \Sigma 1 = 3.$$

Show that $\nabla\left[\nabla\cdot\frac{\bar{r}}{r}\right] = -\frac{2}{r^3}\bar{r}$.

let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$.

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$.

p. diff 'r' w.r.to x

$2r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$. , $\frac{\partial \bar{r}}{\partial x} = \bar{i}$

$\operatorname{div}\frac{\bar{r}}{r} = \nabla\cdot\frac{\bar{r}}{r}$

$= \Sigma\,\bar{i}\cdot\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r}\right)$.

$\cancel{= \Sigma\frac{\bar{i}}{r}\cdot\frac{\partial\bar{r}}{\partial x} = \frac{1}{r}\Sigma\bar{i}\cdot\bar{i} = \frac{1}{r}3}$

$= \Sigma\,\bar{i}\cdot\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\bar{r}\right)\right)$.

$= \Sigma\,\bar{i}\cdot\left[\frac{-1}{r^2}\frac{\partial r}{\partial x}\bar{r}+\frac{1}{r}\frac{\partial\bar{r}}{\partial x}\right]$.

$= \Sigma\,\bar{i}\cdot\left[\frac{-\bar{r}}{r^2}\cdot\frac{x}{r}+\frac{1}{r}\bar{i}\right]$.

$= \Sigma\,\bar{i}\cdot\left[\frac{-x\bar{r}}{r^3}+\frac{\bar{i}}{r}\right]$.

$= -\Sigma\,\bar{i}\cdot\frac{x\bar{r}}{r^3}+\Sigma\,\bar{i}\cdot\frac{\bar{i}}{r}$

$\cancel{= -\frac{x}{r^3}\Sigma\bar{i}\,x + \frac{1}{r}\Sigma\bar{i}}$

$\cancel{= -x^3 = -\frac{x}{r^3}}$

$= \Sigma-\frac{x}{r^3}(\bar{i}\cdot\bar{r})+\frac{1}{r}\Sigma\,\bar{i}\cdot\bar{i}$.

$= \frac{-1}{r^3}\Sigma\,x\cdot x+\frac{1}{r}\Sigma 1$.

$= \frac{-1}{r^3}(x^2+y^2+z^2)+\frac{3}{r}$.

$= -\frac{r^2}{r^3}+\frac{3}{r} = \frac{-1}{r}+\frac{3}{r} = \frac{2}{r}$.

$$\nabla\left[\nabla\cdot\frac{\bar{r}}{r}\right] = \nabla\cdot\frac{2}{r}$$

$$= \Sigma\,\bar{i}\cdot\frac{\partial}{\partial x}\frac{2}{r}$$

$$= 2\Sigma\,\bar{i}\cdot\frac{\partial}{\partial x}\left(\frac{1}{r}\right).$$

$$= 2\,\Sigma\,\bar{i}\cdot\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x}.$$

$$= 2\,\Sigma\,\bar{i}\cdot\left(\frac{-1}{r^2}\right)\frac{x}{r}.$$

$$= -\frac{2}{r^3}\,\Sigma\,\bar{i}\cdot x = \frac{-2}{r^3}\,\Sigma\,x\cdot\bar{i}$$

$$= \frac{-2}{r^3}\left(x\bar{i}+y\bar{j}+z\bar{k}\right)$$

$$= \frac{-2}{r^3}\,\bar{r}.$$

Show that, $\nabla\cdot\left[r\,\nabla\frac{1}{r^3}\right] = \frac{3}{r^4}$.

→ let, $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$$r = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.\quad , \frac{\partial\bar{r}}{\partial x} = \bar{i}$$

$$\nabla\left(\frac{1}{r^3}\right) = \Sigma\,\bar{i}\,\frac{\partial}{\partial x}\frac{1}{r^3}$$

$$= \frac{-3}{r^4}\,\Sigma\,\bar{i}\,\frac{\partial r}{\partial x}$$

$$= \frac{-3}{r^4}\,\Sigma\,\bar{i}\,\frac{x}{r}. = \frac{-3}{r^4}\cdot\frac{1}{r}\Sigma\,x\bar{i}.$$

$$= \frac{-3}{r^4}\frac{1}{r}\left(x\bar{i}+y\bar{j}+z\bar{k}\right)$$

$$= \frac{-3}{r^4}\frac{\bar{r}}{r}.$$

$$r\nabla\frac{1}{r^3} = r\times\frac{-3}{r^4}\frac{\bar{r}}{r} = \frac{-3\bar{r}}{r^4}$$

$$\nabla \cdot \left[r \nabla \frac{1}{r^3} \right] = \nabla \cdot \frac{\partial}{\partial x} \left(\frac{-3 \bar{r}}{r^4} \right).$$

$$= -3 \sum \bar{i} \cdot \frac{\partial}{\partial x} \left(\frac{\bar{r}}{r^4} \right).$$

$$= -3 \sum \bar{i} \cdot \left(\frac{-4}{r^5} \frac{\partial r}{\partial x} \bar{r} + \frac{1}{r^4} \frac{\partial \bar{r}}{\partial x} \right).$$

$$= -3 \sum \bar{i} \cdot \left(-\frac{4}{r^5} \cdot \frac{x}{r} \bar{r} + \frac{1}{r^4} \bar{i} \right).$$

$$= -3 \left[\sum -\frac{4x}{r^6} \bar{i} \cdot \bar{r} + \sum \frac{\bar{i}}{r^4} \bar{i} \cdot \bar{i} \right]$$

$$= -3 \left[\frac{-4\bar{r}}{r^6} \sum \hat{x} \cdot x + \frac{\bar{r}}{r^4} \sum 1 \right].$$

$$= -3 \left[\frac{-4\bar{r}}{r^6} \sum x \cdot x + \frac{\bar{r}}{r^4} (3) \right].$$

$$= -3 \left[\frac{-4}{r^6} r^2 + \frac{3}{r^4} \right]$$

$$= \frac{12}{r^4} - \frac{9}{r^4} = \frac{3}{r^4}$$

8. If $\bar{a}$ is constant vector then $\operatorname{div}(\bar{a} \times \bar{r}) = 0$.

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

$$r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2.$$

given, $\bar{a}$ is constant vector.

$$\operatorname{div}(\bar{a} \times \bar{r}) = \sum \bar{i} \cdot \frac{\partial}{\partial x} (\bar{a} \times \bar{r})$$

$$= \sum \bar{i} \cdot \left[\bar{a} \times \frac{\partial \bar{r}}{\partial x} + \frac{\partial \bar{a}}{\partial x} \times \bar{r} \right]$$

$$= \sum \bar{i} \cdot \left[\bar{a} \times \bar{i} + \bar{0} \times \bar{r} \right]$$

$$= \sum \bar{i} \cdot (\bar{a} \times \bar{i}) = \sum 0 = 0.$$

Curl of a vector: let $\bar{f}$ is a vector point function then the curl of $\bar{f}$ is denoted by curl $\bar{f}$ (or) $\nabla \times \bar{f}$ and defined by $\nabla \times \bar{f} = \bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$

(or) $\nabla \times \bar{F} = \Sigma \bar{i} \times \frac{\partial \bar{F}}{\partial x}$

If $\bar{F} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$ then

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

properties:-

1. If $\bar{f}$ is constant vector then, curl $\bar{f} = \bar{0}$.
2. curl $(c\bar{f}) = c$ curl $\bar{f}$; c is constant.
3. curl $(\bar{f} \pm \bar{g}) =$ curl $\bar{f} \pm$ curl $\bar{g}$
4. The curl of a vector function is a vector

Irrotational vector:- The vector $\bar{f}$ is said to be irrotational if curl $\bar{f} = \bar{0}$

If $\bar{f}$ is an irrotational vector then there Exists a scalar function ϕ such that $\bar{f} = \nabla\phi$. Here ϕ is called scalar potential of $\bar{f}$. $\bar{f}$ is a conservative vector field.

problems:-

1. If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 2yz^2\bar{k}$ find curl $\bar{f}$ at (1,-1,1)

given, $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 2yz^2\bar{k}$.

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -2yz^2 \end{vmatrix}$$

$$= \bar{i}\left[-2z^2 \cdot 1 - 2x^2y \cdot 1\right] - \bar{j}\left[0 - 0\right] + \bar{k}\left[4xyz - 2xy\right]$$

$$= \bar{i}\left[-2z^2 - 2x^2y\right] + \bar{k}\left[4xyz - 2xy\right].$$

at $(1,-1,1) \Rightarrow \operatorname{curl}\bar{f} = (-2+2)\bar{i} + \bar{k}(-4+2)$

$$= -2\bar{k}.$$

. If $\bar{f} = e^{x+y+z}(\bar{i}+\bar{j}+\bar{k})$. find $\operatorname{curl}\bar{f}$.

given, $\bar{f} = e^{x+y+z}(\bar{i}+\bar{j}+\bar{k})$.

$$\operatorname{curl}\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+y+z} & e^{x+y+z} & e^{x+y+z} \end{vmatrix}$$

$\left[\mp i\left[e^{x+y+z}\right.\right.$

$$= e^{x+y+z} \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+y+z} & e^{x+y+z} & e^{x+y+z} \end{vmatrix}$$

$$= e^{x+y+z}\left[\bar{i}(0-0) + \bar{j}(0-0) + \bar{k}(0-0)\right]$$

$= \bar{0}$. $\Big]$ ×

$$= \bar{i}\left(e^{x+y+z} - e^{x+y+z}\right) - \bar{j}\left(e^{x+y+z} - e^{x+y+z}\right) + \bar{k}\left(e^{x+y+z} - e^{x+y+z}\right)$$

$= \bar{0}$.

. Find the constants a, b, c such that,

$\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\bar{j} + (2x+cy+3z)\bar{k}$ is irrotational.

given, ~~and~~ $\bar{f}$ is irrotational, $\operatorname{curl}\bar{f} = 0$

$$\operatorname{curl}\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = \bar{0}$$

$$\bar{i}\left[c-3\right] - \bar{j}\left[2-a\right] + \bar{k}\left[b-3\right] = \bar{0}$$

$c-3=0$, $2-a=0$, $b=3$

$c=3$, $a=2$, $b=3$.

4. Show that $\text{curl}(\bar{a}\times\bar{r}) = 2\bar{a}$ if $\bar{a}$ is constant vect

let, $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$.

$$\text{curl}(\bar{a}\times\bar{r}) = \Sigma\, \bar{i}\times\frac{\partial}{\partial x}(\bar{a}\times\bar{r})$$

$$= \Sigma\, \bar{i}\times\left[\bar{a}\times\frac{\partial\bar{r}}{\partial x}\right]$$

$$= \Sigma\, \bar{i}\times[\bar{a}\times\bar{i}]$$

$$= \Sigma[(\bar{i}\cdot\bar{i})\bar{a} - (\bar{i}\cdot\bar{a})\bar{i}]$$

$$= \Sigma(\bar{a} - a_1\bar{i})$$

$$= \Sigma\bar{a} - \Sigma a_1\bar{i} = 3\bar{a} - (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})$$

$$= 3\bar{a} - \bar{a} = 2\bar{a}.$$

5. Show that $\text{curl}(r^n\bar{r}) = \bar{0}$

let, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

$$r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2.$$

p.diff r w.r.to x.

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial\bar{r}}{\partial x} = \bar{i}.$$

$$r^n\bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}$$

$$\text{curl}(r^n\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \bar{i}\left[z n r^{n-1}\frac{\partial r}{\partial y} - y n r^{n-1}\frac{\partial r}{\partial z}\right] + \bar{j} -$$

$$\bar{j}\left[z n r^{n-1}\frac{\partial r}{\partial x} - x n r^{n-1}\frac{\partial r}{\partial z}\right] + \bar{k}\left[y n r^{n-1}\frac{\partial r}{\partial x} - x n r^{n-1}\frac{\partial r}{\partial y}\right]$$

$$= n r^{n-1}\left[z\frac{y}{r} - y\frac{z}{r}\right]\bar{i} - n r^{n-1}\left[z\frac{x}{r} - x\frac{z}{r}\right]\bar{j} + n r^{n-1}\left[y\frac{x}{r} - x\frac{y}{r}\right]\bar{k}$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}.$$

6\. Show that the vector $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ is irrotational. Find its scalar potential.

Given, $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$.

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \bar{i}\left[-x + x\right] - \bar{j}\left[-y + y\right] + \bar{k}\left[-z + z\right]$$

$$= \bar{0}.$$

$\therefore \bar{f}$ is irrotation.

Let ϕ is scalar potential of $\bar{f}$

$$\therefore \bar{f} = \nabla\phi$$

$$(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = x^2 - yz, \quad \frac{\partial\phi}{\partial y} = y^2 - zx, \quad \frac{\partial\phi}{\partial z} = z^2 - xy$$

Integrating

$\phi = \frac{x^3}{3} - xyz +$ constant not containing x.

$\phi = \frac{y^3}{3} - xyz +$ constant not containing y

$\phi = \frac{z^3}{3} - xyz +$ constant not containing z.

$\therefore$ from above 3 forms $\frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c.$

7. S.T. $\bar{f} = 2xyz^2\bar{i} + (x^2z^2 + z\cos yz)\bar{j} + (2x^2yz + y\cos yz)\bar{k}$ is irrotational. Find its scalar potential.

given, $\bar{f} = 2xyz^2\bar{i} + (x^2z^2 + z\cos yz)\bar{j} + (2x^2yz + y\cos yz)\bar{k}$

$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \bar{i}\left[\left[2x^2z + (-yz\sin yz) + \cos yz\right] - \left[2x^2z + \cos yz - yz\sin yz\right]\right]$$

$$- \bar{j}\left[4xyz - 4xyz\right] + \bar{k}\left[2xz^2 - 2xz^2\right]$$

$$= \bar{0}$$

$\therefore \bar{f}$ is irrotational.

let ϕ is scalar potential of $\bar{f}$

$\therefore \bar{f} = \nabla\phi$

$$2xyz^2\bar{i} + (x^2z^2 + z\cos yz)\bar{j} + (2x^2yz + y\cos yz)\bar{k} =$$

$$\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}.$$

$$\frac{\partial\phi}{\partial x} = 2xyz^2 \; ; \; \frac{\partial\phi}{\partial y} = x^2z^2 + z\cos yz \; , \; \frac{\partial\phi}{\partial z} = 2x^2yz + y\cos yz$$

Integrating

$\phi = 2yz^2 \cdot \frac{x^2}{2} = x^2yz^2 +$ constant not containg x.

$\phi = x^2z^2y + \frac{z\sin yz}{z} +$ constant not containing y

$\phi = x^2z^2y + \sin yz +$ constant not containing y

$\phi = x^2z^2y + \sin yz +$ constant not containg z

from above three form

$$\phi = x^2yz^2 + \sin yz + c$$

Laplacian operator :- The Laplacian operator is denoted by ∇^2 and defined by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. If ϕ is a scalar function then $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$.

The equation $\nabla^2\phi = 0$ is called Laplace eqn. Here ϕ is called Harmonic function.

Note that $\nabla^2\phi = \nabla\cdot(\nabla\phi)$

1. show that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

let, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$.

p.diff r w.r.to x.

$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial \bar{r}}{\partial x} = \bar{i}$.

we have, $\nabla^2 f(r) = \nabla\cdot[\nabla f(r)]$

$\nabla f(r) = \Sigma \bar{i} \frac{\partial}{\partial x}[f(r)]$

$= \Sigma \bar{i} f'(r) \frac{\partial r}{\partial x}$

$= \Sigma \bar{i} f'(r) \frac{x}{r} = \frac{f'(r)}{r} \Sigma x\bar{i}$

$= \frac{f'(r)}{r}[x\bar{i} + y\bar{j} + z\bar{k}] = \frac{f'(r)}{r}\cdot\bar{r}$

$\nabla\cdot[\nabla f(r)] = \nabla\cdot\left[\frac{f'(r)}{r}\bar{r}\right]$

$= \Sigma \bar{i}\cdot\frac{\partial}{\partial x}\left[f'(r)\cdot\frac{1}{r}\cdot\bar{r}\right]$

$= \Sigma \bar{i}\cdot\left[\frac{f'(r)}{r}\frac{\partial \bar{r}}{\partial x} + f'(r)\bar{r}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \frac{\bar{r}}{r} f''(r)\frac{\partial r}{\partial x}\right]$

$$= \sum \bar{i} \cdot \left[f'(r) \frac{1}{r} \bar{i} + f'(r) \cdot \bar{r} \left(\frac{-1}{r^2}\right) \frac{x}{r} + \frac{1}{r} \bar{r} f''(r) \frac{x}{r} \right]$$

$$= \sum \bar{i} \cdot \frac{f'(r)}{r} \bar{i} + \sum \bar{i} \cdot f'(r) \bar{r} \left(\frac{-1}{r^2}\right) \frac{x}{r} + \sum \bar{i} \cdot \frac{\bar{r}}{r} f''(r) \frac{x}{r}$$

$$= \frac{f'(r)}{r} \sum \bar{i} \cdot \bar{i} - \frac{f'(r)}{r^3} \sum \bar{i} \cdot \bar{r} x + f''(r) \frac{1}{r^2} \sum \bar{i} \cdot \bar{r} x$$

$$= \frac{f'(r)}{r} (3) - \frac{f'(r)}{r^3} r^2 + f''(r) \frac{1}{r^2} r^2$$

$$= \frac{3 f'(r)}{r} - \frac{f'(r)}{r} + f''(r)$$

$$= f''(r) + \frac{2 f'(r)}{r}.$$

2. Find $\nabla^2 \log r$

let, $f(r) = \log r$.

we know that,

$$\nabla^2 f(r) = f''(r) + \frac{2 f'(r)}{r}$$

$$f'(r) = \frac{1}{r}$$

$$f''(r) = \frac{-1}{r^2}$$

$$\nabla^2 \log r = \frac{-1}{r^2} + 2 \frac{1}{r} \cdot \frac{1}{r} = -\frac{1}{r^2} + \frac{2}{r^2} = +\frac{1}{r^2}.$$

Vector identities :-

1. $\mathrm{Div}(\phi \bar{f}) = (\mathrm{grad}\, \phi) \cdot \bar{f} + \phi \,\mathrm{div} \bar{f}$. (or)

$$\nabla \cdot (\phi \bar{f}) = \nabla \phi \cdot \bar{f} + \phi (\nabla \cdot \bar{f}).$$

where, $\phi, \bar{f}$ are differentiable scalar, vector functions

Proof :- L.H.S $= \nabla \cdot (\phi \bar{f})$

$$= \sum \bar{i} \cdot \frac{\partial}{\partial x} (\phi \bar{f})$$

$$= \Sigma \bar{i} \cdot \left[\frac{\partial \phi}{\partial x} \bar{f} + \phi \frac{\partial \bar{f}}{\partial x}\right].$$

$$= \Sigma \bar{i} \cdot \frac{\partial \phi}{\partial x} \bar{f} + \Sigma \bar{i} \cdot \phi \frac{\partial \bar{f}}{\partial x}.$$

$$\left(= \frac{\partial \phi}{\partial x} \Sigma \bar{i} \cdot \bar{f} + \phi \Sigma \bar{i} \cdot \frac{\partial \bar{f}}{\partial x}.\right) \times$$

$$= \Sigma \bar{i} \frac{\partial \phi}{\partial x} \cdot \bar{f} + \Sigma\left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x}\right) \phi$$

$$= \nabla \phi \cdot \bar{f} + (\nabla \cdot \bar{f}) \phi.$$

4. $\text{curl}(\phi \bar{f}) = (\text{grad } \phi) \times \bar{f} + \phi (\text{curl } \bar{f})$. (or)

$$\nabla \times (\phi \bar{f}) = \nabla \phi \times \bar{f} + \phi (\nabla \times \bar{f}).$$

$$LHS = \nabla \times (\phi \bar{f})$$

$$= \Sigma \bar{i} \times \frac{\partial}{\partial x} (\phi \bar{f})$$

$$= \Sigma \bar{i} \times \left[\frac{\partial \phi}{\partial x} \bar{f} + \phi \frac{\partial \bar{f}}{\partial x}\right]$$

$$= \Sigma \bar{i} \times \frac{\partial \phi}{\partial x} \bar{f} + \Sigma \bar{i} \times \phi \frac{\partial \bar{f}}{\partial x}$$

$$= \Sigma \bar{i} \frac{\partial \phi}{\partial x} \times \bar{f} + \Sigma\left(\bar{i} \times \frac{\partial \bar{f}}{\partial x}\right) \phi$$

$$= \nabla \phi \times \bar{f} + (\nabla \times \bar{f}) \phi.$$

5. $\text{curl}(\text{grad } \phi) = \bar{0}$ (or)

$$\nabla \times (\nabla \phi) = \bar{0}.$$

we have, $\nabla \phi = \frac{\partial \phi}{\partial x} \bar{i} + \frac{\partial \phi}{\partial y} \bar{j} + \bar{k} \frac{\partial \phi}{\partial z}$

$$\nabla \times (\nabla \phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right] - \bar{j}\left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right] +$$

$$\bar{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right] = \bar{0}$$

$$\nabla \times (\nabla \phi) = \bar{0}$$

$\nabla \phi$ is irrotational

8. $\nabla \times (\bar{f} \times \bar{g}) = \bar{f}(\operatorname{div} \bar{g}) - \bar{g}(\operatorname{div} \bar{f}) + (\bar{g} \cdot \nabla)\bar{f} - (\bar{f} \cdot \nabla)\bar{g}$

$$\nabla \times (\bar{f} \times \bar{g}) = \Sigma \bar{i} \times \frac{\partial}{\partial x}(\bar{f} \times \bar{g})$$

$$= \Sigma \bar{i} \times \left[\frac{\partial \bar{f}}{\partial x} \times \bar{g} + \bar{f} \times \frac{\partial \bar{g}}{\partial x}\right]$$

$$= \Sigma \left[\bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) + \bar{i} \times \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right)\right]$$

$$= \Sigma \left[(\bar{i} \cdot \bar{g}) \frac{\partial \bar{f}}{\partial x} - \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x}\right)\bar{g} + \left(\bar{i} \cdot \frac{\partial \bar{g}}{\partial x}\right)\bar{f} - (\bar{i} \cdot \bar{f})\left(\frac{\partial \bar{g}}{\partial x}\right)\right]$$

$$= \Sigma (\bar{g} \cdot \bar{i}) \frac{\partial \bar{f}}{\partial x} - \Sigma \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x}\right)\bar{g} + \Sigma \left(\bar{i} \cdot \frac{\partial \bar{g}}{\partial x}\right)\bar{f} - \Sigma (\bar{f} \cdot \bar{i}) \frac{\partial \bar{g}}{\partial x}$$

$$= (\bar{g} \cdot \nabla)\bar{f} - (\operatorname{div} \bar{f})\bar{g} + (\operatorname{div} \bar{g})\bar{f} - (\bar{f} \cdot \nabla)\bar{g}$$

$$= \bar{f}(\operatorname{div} \bar{g}) - \bar{g}(\operatorname{div} \bar{f}) + (\bar{g} \cdot \nabla)\bar{f} - (\bar{f} \cdot \nabla)\bar{g}.$$

9. $\operatorname{grad}(\bar{f} \cdot \bar{g}) = (\bar{g} \cdot \nabla)\bar{f} + (\bar{f} \cdot \nabla)\bar{g} + \bar{g} \times \operatorname{curl} \bar{f} + \bar{f} \times \operatorname{curl} \bar{g}$

$$\bar{f} \times \operatorname{curl} \bar{g} = \bar{f} \times \Sigma \bar{i} \times \frac{\partial \bar{g}}{\partial x}$$

$$= \Sigma \bar{f} \times \left(\bar{i} \times \frac{\partial \bar{g}}{\partial x}\right)$$

$$= \Sigma \left[\left(\bar{f} \cdot \frac{\partial \bar{g}}{\partial x}\right)\bar{i} - (\bar{f} \cdot \bar{i})\frac{\partial \bar{g}}{\partial x}\right]$$

$$= \Sigma \left(\bar{f} \cdot \frac{\partial \bar{g}}{\partial x}\right)\bar{i} - \Sigma (\bar{f} \cdot \bar{i})\frac{\partial \bar{g}}{\partial x}.$$

$$= \Sigma \left(\bar{f} \cdot \frac{\partial \bar{g}}{\partial x}\right)\bar{i} - (\bar{f} \cdot \nabla)\bar{g} \quad — (1)$$

Inter change $\bar{f}$ & $\bar{g}$ in (1)

$$\bar{g} \times \operatorname{curl} \bar{f} = \Sigma \left(\bar{g} \cdot \frac{\partial \bar{f}}{\partial x}\right)\bar{i} - (\bar{g} \cdot \nabla)\bar{f} \quad — (2)$$

By adding (1) & (2)

$$\bar{f}\times\text{curl}\,\bar{g}+\bar{g}\times\text{curl}\,\bar{f}=\Sigma\left(\bar{f}\cdot\frac{\partial\bar{g}}{\partial x}\right)\bar{i}-(\bar{f}\cdot\nabla)\bar{g}+$$

$$\Sigma\left(\bar{g}\cdot\frac{\partial\bar{f}}{\partial x}\right)\bar{i}-(\bar{g}\cdot\nabla)\bar{f}.$$

$$\bar{f}\times\text{curl}\,\bar{g}+\bar{g}\times\text{curl}\,\bar{f}+(\bar{f}\cdot\nabla)\bar{g}+(\bar{g}\cdot\nabla)\bar{f}=\Sigma\left[\bar{f}\cdot\frac{\partial\bar{g}}{\partial x}+\bar{g}\cdot\frac{\partial\bar{f}}{\partial x}\right]\bar{i}$$

$$=\Sigma\,\bar{i}\frac{\partial}{\partial x}(\bar{f}\cdot\bar{g})$$

$$=\text{grad}(\bar{f}\cdot\bar{g}).$$

$$\therefore\ \bar{f}\times\text{curl}\,\bar{g}+\bar{g}\times\text{curl}\,\bar{f}+(\bar{f}\cdot\nabla)\bar{g}+(\bar{g}\cdot\nabla)\bar{f}=\text{grad}(\bar{f}\cdot\bar{g}).$$

4. $\text{div}(\text{curl}\,\bar{f})=0$ (or) $\nabla\cdot(\nabla\times\bar{f})=0$.

let, $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

$$\text{curl}\,\bar{f}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}$$

$$=\bar{i}\left[\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right]-\bar{j}\left[\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right]+\bar{k}\left[\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial z}\right].$$

$$=\bar{i}\left[\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right]+\bar{j}\left[\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right]+\bar{k}\left[\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial z}\right]$$

$\text{div}(\text{curl}\,\bar{f})$

$$=\frac{\partial}{\partial x}\left[\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right]+\frac{\partial}{\partial y}\left[\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right]+\frac{\partial}{\partial z}\left[\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right]$$

$$=\cancel{\frac{\partial^2 f_3}{\partial x\partial y}}-\cancel{\frac{\partial^2 f_2}{\partial x\partial z}}+\cancel{\frac{\partial^2 f_1}{\partial y\partial z}}-\cancel{\frac{\partial^2 f_3}{\partial y\partial x}}+\cancel{\frac{\partial^2 f_2}{\partial z\partial x}}-\cancel{\frac{\partial^2 f_1}{\partial z\partial y}}$$

$$=0$$

5. $\nabla\cdot(\bar{f}\times\bar{g})=\bar{g}\cdot(\nabla\times\bar{f})-\bar{f}\cdot(\nabla\times\bar{g})$.

$$\nabla\cdot(\bar{f}\times\bar{g})=\Sigma\,\bar{i}\cdot\frac{\partial}{\partial x}(\bar{f}\times\bar{g})$$

$$=\Sigma\,\bar{i}\cdot\left[\frac{\partial\bar{f}}{\partial x}\times\bar{g}+\bar{f}\times\frac{\partial\bar{g}}{\partial x}\right]$$

$$= \sum \bar{i} \cdot \left[\frac{\partial \bar{f}}{\partial x} \times \bar{g} - \frac{\partial \bar{g}}{\partial x} \times \bar{f} \right]$$

$$= \sum \bar{i} \cdot \left[\frac{\partial \bar{f}}{\partial x} \times \bar{g} \right] - \sum \bar{i} \cdot \left[\frac{\partial \bar{g}}{\partial x} \times \bar{f} \right].$$

$$= \sum \left(\bar{i} \times \frac{\partial \bar{f}}{\partial x} \right) \cdot \bar{g} - \sum \left(\bar{i} \times \frac{\partial \bar{g}}{\partial x} \right) \bar{f}.$$

$$= (\nabla \times \bar{f}) \cdot \bar{g} - (\nabla \times \bar{g}) \cdot \bar{f}$$

$$= \bar{g} \cdot (\nabla \times \bar{f}) - \bar{f} \cdot (\nabla \times \bar{g}).$$

6. $\operatorname{div}(\phi \nabla \varphi) = \phi \nabla^2 \varphi + \nabla \phi \cdot \nabla \varphi$

we have, $\nabla \varphi = \bar{i} \frac{\partial \varphi}{\partial x} + \bar{j} \frac{\partial \varphi}{\partial y} + \bar{k} \frac{\partial \varphi}{\partial z}$

$$\phi \nabla \varphi = \bar{i} \phi \frac{\partial \varphi}{\partial x} + \bar{j} \phi \frac{\partial \varphi}{\partial y} + \bar{k} \phi \frac{\partial \varphi}{\partial z}$$

$$\operatorname{div}[\phi \nabla \varphi] = \frac{\partial}{\partial x}\left[\phi \frac{\partial \varphi}{\partial x}\right] + \frac{\partial}{\partial y}\left[\phi \frac{\partial \varphi}{\partial y}\right] + \frac{\partial}{\partial z}\left[\phi \frac{\partial \varphi}{\partial z}\right]$$

$$= \phi \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \varphi}{\partial y} + \phi \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \varphi}{\partial y} + \phi \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial \phi}{\partial z} \frac{\partial \varphi}{\partial z}$$

$$= \phi \left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \right] + \frac{\partial \phi}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \varphi}{\partial y} + \frac{\partial \phi}{\partial z} \frac{\partial \varphi}{\partial z}.$$

$$= \phi \nabla^2 \varphi + \left[\bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z} \right] \cdot \left[\bar{i} \frac{\partial \varphi}{\partial x} + \bar{j} \frac{\partial \varphi}{\partial y} + \bar{k} \frac{\partial \varphi}{\partial z} \right]$$

$$= \phi \nabla^2 \varphi + \nabla \phi \cdot \nabla \varphi$$

7. $\nabla \times (\nabla \times \bar{f}) = \nabla(\nabla \cdot \bar{f}) - \nabla^2 \bar{f}$

L.H.S $= \nabla \times (\nabla \times \bar{f}) = \sum \bar{i} \times \frac{\partial}{\partial x} (\nabla \times \bar{f})$ —— (1)

$$= \bar{i} \times \frac{\partial}{\partial x} (\nabla \times \bar{f}) = \bar{i} \times \frac{\partial}{\partial x} \left[\sum \bar{i} \times \frac{\partial \bar{f}}{\partial x} \right].$$

$$= \bar{i} \times \frac{\partial}{\partial x} \left[\bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z} \right]$$

$$= \bar{i} \times \left[\bar{i} \times \frac{\partial^2 \bar{f}}{\partial x^2} + \bar{j} \times \frac{\partial^2 \bar{f}}{\partial y \partial z} + \bar{k} \times \frac{\partial^2 \bar{f}}{\partial x \partial z} \right]$$

$$= \bar{i} \times \left[\bar{i} \times \frac{\partial^2 \bar{f}}{\partial x^2} \right] + \bar{i} \times \left[\bar{j} \times \frac{\partial^2 \bar{f}}{\partial x \partial y} \right] + \bar{i} \times \left[\bar{k} \times \frac{\partial^2 \bar{f}}{\partial x \partial z} \right]$$

$$= \left(\bar{i} \cdot \frac{\partial^2 \bar{f}}{\partial x^2} \right) \bar{i} - (\bar{i} \cdot \bar{i}) \frac{\partial^2 \bar{f}}{\partial x^2} + \left(\bar{i} \cdot \frac{\partial^2 \bar{f}}{\partial x \partial y} \right) \bar{j} - (\bar{i} \cdot \bar{j}) \frac{\partial^2 \bar{f}}{\partial x \partial y}$$
$$+ \left(\bar{i} \cdot \frac{\partial^2 \bar{f}}{\partial x \partial z} \right) \bar{k} - (\bar{i} \cdot \bar{k}) \frac{\partial^2 \bar{f}}{\partial x \partial z}$$

$$= \bar{i} \frac{\partial}{\partial x} \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x} \right) + \bar{j} \frac{\partial}{\partial y} \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x} \right) + \bar{k} \frac{\partial}{\partial z} \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x} \right) - \frac{\partial^2 \bar{f}}{\partial x^2}$$

$$= \nabla \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x} \right) - \frac{\partial^2 \bar{f}}{\partial x^2}$$

from ①,

$$\nabla \times (\nabla \times \bar{f}) = \Sigma \bar{i} \times \frac{\partial}{\partial x} (\nabla \times \bar{f})$$

$$= \Sigma \left[\nabla \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x} \right) - \frac{\partial^2 \bar{f}}{\partial x^2} \right]$$

$$= \Sigma \left(\nabla \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x} \right) \right) - \Sigma \frac{\partial^2 \bar{f}}{\partial x^2}$$

$$= \nabla \Sigma \bar{i} \cdot \frac{\partial \bar{f}}{\partial x} - \left(\frac{\partial^2 \bar{f}}{\partial x^2} + \frac{\partial^2 \bar{f}}{\partial y^2} + \frac{\partial^2 \bar{f}}{\partial z^2} \right)$$

$$= \nabla (\nabla \cdot \bar{f}) - \nabla^2 \bar{f}$$

1. If ϕ satisfies laplace equation then show that $\nabla \phi$ is both solenoidal & irrotational.

Given, ϕ satisfies laplace eq^n

$$\nabla^2 \phi = 0.$$

$$\nabla \cdot (\nabla \phi) = 0$$

$\nabla \phi$ is solenoidal

from identity-③, $\nabla \times (\nabla \phi) = \bar{0}$

$\nabla \phi$ is irrotational.

2. If f, g are scalar functions then show that $\nabla f \times \nabla g$ is solenoidal.

from Identity-5, we have

$$\nabla\cdot(\bar{a}\times\bar{b}) = \bar{b}\cdot(\nabla\times\bar{a}) - \bar{a}\cdot(\nabla\times\bar{b})$$

let, $\bar{a} = \nabla f$, $\bar{b} = \nabla g$

$$\nabla\cdot(\nabla f\times\nabla g) = \nabla g\cdot(\nabla\times\nabla f) - \nabla f\cdot(\nabla\times\nabla g)$$

$$= \nabla g\cdot\bar{0} - \nabla f\cdot\bar{0}$$

$$= 0$$

$\nabla f\times\nabla g$ is solenoidal.