Unit - 2

Eigen Values, Eigen vectors & Quadratic forms

let A is a square matrix of order n a non-zero vector x is said to be Eigen vector (or) latent vector (or) characteristic vector.

If $Ax = \lambda x$ For some scalar λ. λ is called Eigen value or latent root (or) characteristic root.

process of finding Eigen vector and Eigen values:-

let λ is Eigen value of A and x is corresponding eigen vector then $Ax = \lambda x$.

$$Ax - \lambda x = 0.$$

$$(A - \lambda I)x = 0.$$

This is A homogeneous system of equations since x is non-zero we have determinant of $[A-\lambda I]$ $= 0$. This equation is called characteristic equation then roots are Eigens values.

using $(A-\lambda I)x = 0$ we can find the Eigen vector corresponding to each Eigen value.

The set of Eigen Values of A is called "Spectrum of A"

problems:-

1. Find the Eigen vectors and Eigen values of following matrices.

1. $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Given, $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

characteristic eq^n of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0.$$

$$(3-\lambda)\left[(2-\lambda)(5-\lambda) - 0\right] = 0.$$

$$(3-\lambda)(2-\lambda)(5-\lambda) = 0.$$

$$\lambda = 3, 2, 5$$

Eigen values are $\lambda = 2, 3, 5$.

Case (i) :- $\lambda = 2$

$$(A - \lambda I)X = 0.$$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} X = 0.$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} X = 0.$$

$R_3 \to 2R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$6x_3 = 0 \Rightarrow x_3 = 0.$

$x_1 + x_2 + 4x_3 = 0.$

$x_1 + x_2 = 0$

$x_2 = +k_1$

$x_1 = -k_1$

Eigen vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Case (iii):- $\lambda = 3$

$(A - \lambda I)X = 0.$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} X = 0$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} X = 0.$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 2 \end{bmatrix} X = 0 \quad R_3 \to 5R_3 + -R_2$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$10x_3 = 0$, $x_2 + 4x_3 = 0$

$x_3 = 0$ $\quad$ $x_2 + 0 = 0$

$x_2 = 0.$

$x_1 = k_2$

Eigen vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

case (iii):- $\lambda = 5$

$(A-\lambda I)X = 0$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} X = 0.$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-2x_1 + x_2 + 4x_3 = 0, \quad -3x_2 + 6x_3 = 0.$

$x_2 = +2x_3$

let $x_3 = k_3$.

$x_2 = 2k_3, \quad -2x_1 + 2k_3 + 4k_3 = 0.$

$-2x_1 = -6k_3$

$x_1 = 3k_3$

$x_1 = 3k_3, \; x_2 = 2k_3, \; x_3 = k_3.$

Eigens vector,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k_3 \\ 2k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

2. $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$

Given,, $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 4 \\ -1 & -1 & -2-\lambda \end{bmatrix}$$

Characteristics eqn of $|A-\lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 4 \\ -1 & -1 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(3-\lambda)(-2-\lambda)+4\right] - 1\left[(2)(-2-\lambda)+4\right] + 1\left[-2+3-\lambda\right] = 0$$

$$(2-\lambda)\left[\lambda^2-\lambda-2\right] - (-2\lambda) + 1+\lambda = 0.$$

$$(2-\lambda)(\lambda-2)(\lambda+1) + (\lambda+1) = 0.$$

$$(\lambda+1)\left[(2-\lambda)(\lambda-2)+1\right] = 0.$$

$$(\lambda+1)\left[2\lambda - 4 - \lambda^2 + 2\lambda + 1\right] = 0.$$

$$(\lambda+1)\left[-\lambda^2 + 4\lambda - 3\right] = 0.$$

$$(\lambda+1)\left[\lambda^2 - 4\lambda + 3\right] = 0$$

$$(\lambda+1)(\lambda-1)(\lambda-3) = 0.$$

$$\lambda = 1, -1, 3.$$

Case (i): $\lambda = -1$.

$$(A-\lambda I)X = 0.$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 4 \\ -1 & -1 & -1 \end{bmatrix} X = 0.$$

$R_2 \to 3R_2 - 2R_1$, $R_3 \to 3R_3 + R_1$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 10 & 10 \\ 0 & -2 & -2 \end{bmatrix} X = 0$$

$R_3 \to 5R_3 + R_2$

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 10 & 10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$3x_1 + x_2 + x_3 = 0$, $10x_2 + 10x_3 = 0$.

$x_2 = -x_3$, let $x_3 = k$,

$x_2 = -k_1$

$3x_1 + k_1 - k_1 = 0.$

$3x_1 = 0 \Rightarrow x_1 = 0.$

$\therefore x_1 = 0, \; x_2 = -k_1, \; x_3 = k_1$

Eigen vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -k_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$

case (ii):- $\lambda = 1.$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ -1 & -1 & -3 \end{bmatrix} X = 0.$$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} X = 0$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$x_1 + x_2 + x_3 = 0$, $2x_3 = 0 \Rightarrow x_3 = 0.$

$x_1 + x_2 = 0.$ let, $x_2 = k_2$

$x_1 = -k_2$

$\therefore x_1 = -k_2, \; x_2 = k_2, \; x_3 = 0.$

Eigen vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_2 \\ k_2 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

case (iii):- $\lambda = 3$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 4 \\ -1 & -1 & -5 \end{bmatrix} X = 0$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & -2 & -6 \end{bmatrix} X = 0$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} X = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-x_1 + x_2 + x_3 = 0$, $2x_2 + 6x_3 = 0$.

$x_2 = -3x_3$ let $x_3 = k_3$

$-x_1 - 3k_3 + k_3 = 0$.

$x_2 = -3k_3$

$-x_1 - 2k_3 = 0$

$x_1 = -2k_3$

$\therefore$ Eigens vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k_3 \\ -3k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

Given, $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} = A$

$$A - \lambda I = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix}$$

Charcteristics eq^n $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \end{vmatrix} = 0$$

$$(3-\lambda)\left[(-3-\lambda)(7-\lambda)+20\right]-10\left[\cancel{(3+\lambda)}(7-\lambda)(-2)+12\right]$$
$$+5\left[-10+(3+\lambda)3\right]=0.$$

$(8-\lambda)\left[(-3-\lambda)\right.$

$$(3-\lambda)\left[-21+3\lambda-7\lambda+\lambda^2+20\right]-10\left[-14+2\lambda+12\right]+$$
$$5\left[-10+9+3\lambda\right]=0.$$

$$(3-\lambda)\left(\lambda^2-4\lambda-1\right)-10\left(2\lambda-2\right)+5\left(3\lambda-1\right)=0$$

$$3\lambda^2-12\lambda-3-\lambda^3+4\lambda^2+\lambda-20\lambda+20+15\lambda-5=0$$

$$-\lambda^3+7\lambda^2-16\lambda+12=0$$

3	-1	7	-16	12	
	0	-3	12	-12	
2	-1	4	-4	0	
	0	-2	4		
2	-1	2	0		
	0	-2			
	-1	0			

$\lambda=3,2,2$

Case (i):-

$\lambda=2$

$$\begin{bmatrix}1 & 10 & 5\\ -2 & -5 & -4\\ 3 & 5 & 3\end{bmatrix}X=0$$

$R_2\to R_2+2R_1,\ R_3\to R_3-3R_1$

$$\begin{bmatrix}+1 & 10 & 5\\ 0 & 15 & 6\\ 0 & -25 & -10\end{bmatrix}X=0$$

$R_3\to\dfrac{R_3}{-5},\ R_2\to\dfrac{R_2}{3}$

$$\begin{bmatrix}1 & 10 & 5\\ 0 & 5 & 2\\ 0 & +5 & +2\end{bmatrix}X=0$$

$R_3\to R_3-R_2$

$$\begin{bmatrix}1 & 10 & 5\\ 0 & 5 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$x_1 + 10x_2 + 5x_3 = 0$, $5x_2 + 2x_3 = 0$

$x_3 = k_1$

$5x_2 + 2k_1 = 0 \Rightarrow 5x_2 = -2k_1 \Rightarrow x_2 = \frac{-2k_1}{5}$.

$x_1 + \overset{2}{10}\left(\frac{-2k_1}{5}\right) + 5k_1 = 0$.

$x_1 = +4k_1 - 5k_1$, $x_1 = -k_1$

Eigen vector, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ \frac{-2k_1}{5} \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ -2/5 \\ 1 \end{bmatrix}$.

$= -5k \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$

case (ii) :- $\lambda = 3$.

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -6 \\ 3 & 5 & 4 \end{bmatrix} X = 0.$$

$R_2 \to R_2/-2$, $R_1 \to R_1/5$

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 5 & 4 \end{bmatrix} X = 0$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 4 \end{bmatrix} X = 0 \Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & -4 & -2 \end{bmatrix} X = 0.$$

$R_3 \to R_3 - 3R_1$ $R_3 \to \frac{R_3}{-2}$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} X = 0.$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

let $x_3 = k_1 \Rightarrow 2x_2 + x_3 = 0$ $x_1 + 3x_2 + 2x_3 = 0$.

$x_2 = \frac{-k_1}{2}$ $x_1 = \frac{-k_1}{2}$

Eigen vectors are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1/2 \\ -k_1/2 \\ k_1 \end{bmatrix} = -2k_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

4. $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

Given, $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3-\lambda & -7 & -5 \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

char eq^n is, $|A - \lambda I| = 0$.

$$\begin{vmatrix} -3-\lambda & -7 & -5 \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)\left[(4-\lambda)(2-\lambda)-6\right] + 7\left[2(2-\lambda)-3\right] - 5\left[4-(4-\lambda)\right] = 0$$

$$(-3-\lambda)\left[8-4\lambda-2\lambda+\lambda^2-6\right] + 7\left[4-2\lambda-3\right] - 5\left[4-4+\lambda\right] = 0$$

$$(-3-\lambda)\left[\lambda^2-6\lambda+2\right] + 7\left[-2\lambda+1\right] - 5(\lambda) = 0.$$

$$-3\lambda^2 + 18\lambda - 6 + \lambda^3 + 6\lambda^2 - 2\lambda - 14\lambda + 7 - 5\lambda = 0.$$

$$\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0$$

$$(\lambda-1)^3 = 0$$

$$\lambda = 1, 1, 1.$$

case(i):- $\lambda = 1$

$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} X = 0 \qquad R_2 \to 2R_2 + R_1$$

$$R_3 \to 4R_3 + R_1$$

$$\begin{bmatrix} -4 & -7 & -5 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} X = 0$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -4 & -7 & -5 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} X = 0$$

$$\begin{bmatrix} -4 & -7 & -5 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-4x_1 - 7x_2 - 5x_3 = 0$, $-x_2 + x_3 = 0$

let, $x_3 = k_1$

$x_3 = x_2 \Rightarrow x_2 = k_1$

$-4x_1 - 7k_1 - 5k_1 = 0$.

$-4x_1 - 12k_1 = 0$.

$4x_1 = -\overset{3}{12}k_1$

$x_1 = -3k_1$

Eigen vector, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

Digonalisation of a matrix.

A matrix "A" is said to be diagonlisable if there exist a matrix 'p' such that $P^{-1}AP = D$ where p is the diagonal matrix. The process of finding p is called diagonalisation. The matrix p which reduces A in to diagonal matrix is called model matrix and D is called spectral matrix of A.

Process:- let $\lambda_1, \lambda_2, \lambda_3$ are Eigen values and X_1, X_2, X_3 are corresponding Eigen vectors of A.

define $P = [X_1 \ X_2 \ X_3]$ then $|A| \neq 0$.

$\therefore P^{-1} = \frac{\text{adj} P}{|P|}$ then $P^{-1}AP = D = \text{diag}[\lambda_1 \ \lambda_2 \ \lambda_3]$.

Suppose A is a real symmetric matrix with distinct Eigen values $\lambda_1, \lambda_2, \lambda_3$, and the corresponding eigen vectors X_1, X_2, X_3. These Eigen vectors are pair wise orthogonal. (X, Y orthogonal if and only if $X^T Y = 0$). we define $P = \left[\frac{X_1}{\|X_1\|} \ \frac{X_2}{\|X_2\|} \ \frac{X_3}{\|X_3\|}\right]$

then P is orthogonal

$\therefore P^{-1} = P^T$, $D = P^{-1}AP = P^TAP$

Powers of A:-

$$D = P^{-1}AP$$

$$PAP^{-1} = PP^{-1}APP^{-1}$$

$$A = PDP^{-1}$$

$$A^2 = AA$$
$$= PDP^{-1}PDP^{-1}$$
$$= PDIDP^{-1} = PDDP^{-1} = PD^2P^{-1},$$

$$A^3 = PD^3P^{-1}$$

In general, $A^n = PD^nP^{-1}$

1. Diagonalise the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Given, $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}$$

char equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)\left[(2-\lambda)(3-\lambda)-2\right]-1\left[2-2(2-\lambda)\right]=0.$$

$$(1-\lambda)(\lambda^2-5\lambda+4)+2-2\lambda=0.$$

$$(1-\lambda)\left[\lambda^2-5\lambda+4+2\right]=0.$$

$$(1-\lambda)(\lambda^2-5\lambda+6)=0$$

$$(1-\lambda)(\lambda-2)(\lambda-3)=0$$

$$\lambda=1,2,3.$$

case(i):- $\lambda=1.$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} X=0.$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} X=0$$

$R_2 \to 2R_2-R_1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} X=0$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0.$$

$2x_1+2x_2+2x_3=0$, $-x_3=0$

$x_3=0.$

$2x_1+2x_2+0=0$

$x_2=k_1$

$2x_1=-2x_2$

$x_1=-k_1$

Eigens vector = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Case (ii):- $\lambda = 2$.

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} X = 0.$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = 0.$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0.$$

$-X_1 - X_3 = 0$, | $2X_2 - X_3 = 0$. | $\boxed{X_3 = k_2}$

$X_1 = -X_3$ | $2X_2 = X_3$

$\boxed{X_1 = -k_2}$ | $\boxed{X_2 = \frac{k_2}{2}}$

Eigens vector,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_2 \\ k_2/2 \\ k_2 \end{bmatrix} = +\frac{k_2}{2} \begin{bmatrix} -2 \\ 1 \\ +2 \end{bmatrix}.$$

Case (iii):- $\lambda = 3$.

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} X = 0.$$

$R_2 \to 2R_2 + R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} X = 0$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-2x_1 - x_3 = 0$ $\qquad$ $-2x_2 + x_3 = 0$.

$x_1 = -\frac{x_3}{2}$ $\qquad$ $x_3 = k_3$

$x_1 = -\frac{k_3}{2}$ $\qquad$ $+2x_2 = +k_3$

$x_2 = \frac{k_3}{2}$

Eigens vector,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_3/2 \\ k_3/2 \\ k_3 \end{bmatrix} = \frac{k_3}{2}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Eigens vectors are,

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Model matrix of A is $P = [X_1 \; X_2 \; X_3]$

$$P = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

$$|P| = \begin{vmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{vmatrix}$$

$$= -1(2-2) + 2(2-0) - 1(2-0) = 4-2 = 2 \neq 0.$$

P^{-1} exist.

Adj P = cofactor P $= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ -2 & -1 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 2 & -2 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

$$\text{Adj } P = \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix} \quad [\because \text{Adj } P = (\text{cofactor } P)^T].$$

$$P^{-1} = \frac{Adj\,P}{|P|} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

Spectral matrix A is $D = P^{-1}AP$.

$$D = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -4 & -4 & 0 \\ 6 & 6 & 3 \end{bmatrix}\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

2. Diagonalised matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ Hence find A^4

→ let $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

Char equation of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$$(3-\lambda)\left[(5-\lambda)(3-\lambda)-1\right] + 1\left[-3+\lambda+1\right] + 1\left[1-5+\lambda\right] = 0$$

$$(3-\lambda)\left[15-5\lambda-3\lambda+\lambda^2-1\right] + 1\left[\lambda-2\right] + (\lambda-4) = 0$$

$$(3-\lambda)(\lambda^2-8\lambda+14) + (\lambda-2) + (\lambda-4) = 0$$

$$3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda + \lambda - 2 + \lambda - 4 = 0$$

$-\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\lambda = 2, 3, 6$

<u>case (i):-</u> $\lambda = 2$,

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} X = 0$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$2x_2 = 0$, $x_1 - x_2 + x_3 = 0$, let $x_3 = k_1$

$x_2 = 0$, $x_1 - 0 + k_1 = 0$

$x_1 = -k_1$

Eigen vector,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

<u>case (ii):-</u> $\lambda = 3$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} X = 0$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} X = 0$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} X = 0$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 .$$

$x_2 - x_3 = 0$; $x_1 - x_2 = 0$; let $x_2 = k_2$

$k_2 - x_3 = 0$; $x_1 - k_2 = 0$

$x_3 = k_2$; $x_1 = k_2$

Eigens vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

case (iii) :- $\lambda = 6$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} x = 0$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -1 \\ -3 & -1 & 1 \end{bmatrix} x = 0 .$$

$R_2 \rightarrow R_2 + R_1$, $R_3 \rightarrow R_3 + 3R_1$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix} x = 0$$

$R_2 \rightarrow R_2/-2$, $R_3 \rightarrow R_3/-4$.

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} x = 0 .$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_3 = k_3$, $x_2 + 2x_3 = 0$, $x_1 - x_2 - 3x_3 = 0$

$x = -2k_3$ $\quad$ $x_1 + 2k_3 - 3k_3 = 0$

Eigen vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_3 \\ -2k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Eigen vectors are,

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Model matrix of A is $P = [x_1 \ x_2 \ x_3]$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|P| = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = -1(1+2) - 1(0+2) + 1(0-1)$$

$$= -1(3) - 1(2) + 1(-1) = -3-2-1$$

$$= -6 \neq 0.$$

$$\text{Adj } A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & -2 & -1 \\ 0 & -2 & 2 \\ -3 & -2 & -1 \end{bmatrix}^T = \begin{bmatrix} 3 & 0 & -3 \\ -2 & -2 & -2 \\ -1 & 2 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{\text{Adj } A}{|P|} = \frac{-1}{6} \begin{bmatrix} 3 & 0 & -3 \\ -2 & -2 & -2 \\ -1 & 2 & -1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -3 & 0 & 3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

Spectral matrix of A is $D = P^{-1}AP$.

$$D = \frac{1}{6} \begin{bmatrix} -3 & 0 & 3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

we have $A^4 = PD^4P^{-1}$

$$A^4 = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -3 & 0 & 3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1506 & -2430 & 1410 \\ -2430 & 5346 & -2430 \\ 1410 & -2430 & 1506 \end{bmatrix} = \begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}$$

Second method :-

Given, $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Eigen valoes are $\lambda = 2, 3, 6$.

Eigen vectors are, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\|X_1\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$

$\|X_2\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\|X_3\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

Model matrix of A is $P = \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix}$

$= \begin{bmatrix} \frac{-1}{\sqrt{2}} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

$P^{-1} = P^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$

Algebraic multiplicity, Geometric multiplicity of an Eigen Value :-

Let A is a matrix of order n, if λ is an Eigen value of order t, then t is called algebraic multiplicity of λ.

If s is the no. of linearly independent Eigen vectors, corresponding to the Eigen value of λ then s is called geometric multiplicity of λ. For any Eigen value λ in general $s \leq t$

Ex:- For the matrix $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ the Eigen values are 1, 1, 1.

$\therefore$ A.M of $\lambda = 1$ is 3.

The Eigen vector corresponding to $\lambda = 1$ is

$$k\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}.$$

$\therefore$ The G.M of $\lambda = 1$ is 1.

Note:-

→ The necessary and sufficient condition that the matrix A is diagonalisable is that the algebric multiplicity of any Eigen value is equal to its Geometric multiplicity $t = s$ for any λ.

1. Is the matrix $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ is diagonalisable

Given, $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix}.$$

char eq^n is $|A-\lambda I| = 0$.

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0.$$

$(3-\lambda)\left[(-3-\lambda)(7-\lambda)+20\right] - 10\left[-2(7-\lambda)+12\right] + 5\left(-10-3(-3-\lambda)\right) = 0.$

$(3-\lambda)\left[-21+3\lambda-7\lambda+\lambda^2+20\right] - 10\left[-14+2\lambda+12\right] + 5(-10+9+3\lambda) = 0$

$(3-\lambda)\left[\lambda^2-4\lambda-1\right] - 10(2\lambda-2) + 5(3\lambda-1) = 0.$

$(3-\lambda)(\lambda^2-4\lambda-1) - 20(\lambda-1) + 5(3\lambda-1) = 0.$

$(3-\lambda)(\lambda^2-4\lambda-1) + (3\lambda-1)(-20+5) = 0.$

$(3-\lambda)(\lambda^2-4\lambda-1) - 15(\lambda-1) = 0.$

$(3-\lambda)(\lambda^2-4\lambda-1) - 20\lambda + 20 + 15\lambda - 5 = 0.$

$(3-\lambda)(\lambda^2-4\lambda-1) - 5\lambda + 15 = 0.$

$(3-\lambda)(\lambda^2-4\lambda-1) + 3(3-\lambda) = 0.$

$(3-\lambda)(\lambda^2-4\lambda-1+3) = 0 \Rightarrow (3-\lambda)(\lambda^2-4\lambda+2) = 0$

$\lambda = 3, 2, 2.$

A.M of $\lambda = 2$ is 2.

A.M of $\lambda = 3$ is 1.

Case (i):- $\lambda = 2$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} X = 0$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix} X = 0$$

$R_2 \to R_2/3, \quad R_3 \to R_3/-5$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 5 & 2 \end{bmatrix} x = 0.$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$x_1 + 10x_2 + 5x_3 = 0, \quad 5x_2 + 2x_3 = 0$

$x_1 - \cancel{10}^{2} \times \frac{2}{\cancel{5}} k_1 + 5k_1 = 0. \qquad x_3 = k_1$

$5x_2 = -2k_1$

$x_1 - 4k_1 + 5k_1 = 0 \qquad x_2 = \frac{-2k_1}{5}$

$x_1 = -k_1$

$\therefore x_1 = -k_1, \quad x_2 = -\frac{2}{5} k_1, \quad x_3 = k_1$

Eigen vector,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ -2/5\, k_1 \\ k_1 \end{bmatrix} = -\frac{k_1}{5} \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$$

$\therefore$ G.M of $\lambda = 2$ is $s = 1$

for $\lambda = 2$, $AM \neq G.M$.

$\therefore$ A is not diagonalisable

Cayley Hamilton theorem:-

Every square matrix A satisfies its own characteristic equation.

let A is a square matrix of order n, then characterstic eqn of A is $|A - \lambda I| = 0$

$$\Rightarrow \quad \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0.$$

By. C.H Theorem, we have

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I = 0.$$

using C·H Theorem, we can find A^{-1} and certain powers of A.

problems:-

1. verify C·H Theorem for A. Hence find A^{-1}, where A :

i) $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

Given, $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 \\ 1 & 5-\lambda \end{bmatrix}$

char eq^n of A is $|A - \lambda I| = 0$.

$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix} = 0$

$(3-\lambda)(5-\lambda) - 2 = 0 \Rightarrow \lambda^2 - 8\lambda + 13 = 0$.

By C-H Theorem,

$A^2 - 8A + 13 = 0$.

Verification:-

$A^2 = A \cdot A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 11 & 16 \\ 8 & 27 \end{bmatrix}$

$A^2 - 8A + 13I$

$= \begin{bmatrix} 11 & 16 \\ 8 & 27 \end{bmatrix} - 8\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} + 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 11-24+13 & 16-16+0 \\ 8-8+0 & 27-40+13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

C-H. Theorem is verified.

We have, $A^2 - 8A + 13I = 0$.

$$A^{-1}(A^2 - 8A + 13I) = 0.$$

$$A - 8I + 13A^{-1} = 0.$$

$$13A^{-1} = 8I - A \quad = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$13A^{-1} = \begin{bmatrix} 5 & \cancel{8}-2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}.$$

2. $\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

let, $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

Char eq^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$$(3-\lambda)\left[(5-\lambda)(3-\lambda) - 1\right] - 1\left[-(3-\lambda) + 1\right] + 1\left[1 - (5-\lambda)\right] = 0$$

$$(3-\lambda)\left[15 - 5\lambda - 3\lambda + \lambda^2 - 1\right] - 1\left[-3 + \lambda + 1\right] + \left[1 - 5 + \lambda\right] = 0$$

$$(3-\lambda)(\lambda^2 - 8\lambda + 14) - \lambda + 2 + \lambda - 4 = 0$$

$3\lambda^2 - 2 + \lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda - 2 = 0$

$-\lambda^3 + 11\lambda^2 - 38\lambda + 40 = 0$.

$\lambda^3 - 11\lambda^2 + 38\lambda - 40 = 0$.

By C.H Theorem ;

$A^3 - 11A^2 + 38A - 40I = 0$.

Verification ;-

$A^3 = A^2 \cdot A$

$$A^2 = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{bmatrix}.$$

$$A^3 - 11A^2 + 38A - 40I = \begin{bmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{bmatrix} - 11\begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix} + 38\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 - 99 + 114 - 40 & 39 - 77 + 38 & 17 - 55 + 38 \\ -61 - 99 - 38 & 125 - 275 + 190 - 40 & -61 + 99 - 38 \\ 39 - 77 - 38 & -39 + 77 - 38 & 47 - 121 + 114 - 40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

$\therefore$ C.H theorem is verified.

We have $A^3 - 11A^2 + 38A - 40I = 0$.

$A^{-1}(A^3 - 11A^2 + 38A - 40I) = 0$

$A^2 - 11A + 38I - 40A^{-1} = 0$.

$$40A^{-1} = A^2 - 11A + 38I$$

$$40A^{-1} = \begin{bmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{bmatrix} - 11\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} + 38\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$40A^{-1} = \begin{bmatrix} 14 & -4 & -6 \\ 2 & 8 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} 14 & -4 & -6 \\ 2 & 8 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ also find A^4.

Given, $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{bmatrix}$$

Char eq^n of A is $|A-\lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)\left[(1-\lambda)^2 - 1\right] - 0 + 3\left[-2 - (1-\lambda)\right] = 0$$

$$\lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 + 3\lambda - 9 = 0.$$

$$-\lambda^3 + 3\lambda^2 + \lambda - 9 = 0.$$

$$\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

By C-H theorem, $A^3-3A^2-A+9I=0$.

$$A^3=\begin{bmatrix}4 & -9 & 21\\ 11 & -2 & 11\\ 1 & -7 & 7\end{bmatrix}$$

$$A^2=\begin{bmatrix}4 & -3 & 6\\ 3 & 2 & 4\\ 0 & -2 & 5\end{bmatrix}$$

$$A^3-3A^2-A+9I=\begin{bmatrix}4 & -9 & 21\\ 11 & -2 & 11\\ 1 & -7 & 7\end{bmatrix}-3\begin{bmatrix}4 & -3 & 6\\ 3 & 2 & 4\\ 0 & -2 & 5\end{bmatrix}-\begin{bmatrix}1 & 0 & 3\\ 2 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}+\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0.$$

$\therefore$ C-H theorem is verified.

we have, $A^3-3A^2-A+9I=0$.

$$A^{-1}(A^3-3A^2-A+9)=0.$$

$$A^2-3A-I+9A^{-1}=0.$$

$$9A^{-1}=-A^2+3A+I$$

$$9A^{-1}=-\begin{bmatrix}4 & -3 & 6\\ 3 & 2 & 4\\ 0 & -2 & 5\end{bmatrix}+3\begin{bmatrix}1 & 0 & 3\\ 2 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A^{-1}=\frac{1}{9}\begin{bmatrix}0 & 3 & 3\\ 3 & 2 & -7\\ 3 & -1 & -1\end{bmatrix}$$

Finding A^4:-

we have $A^3-3A^2-A+9I=0$.

$$A(A^3-3A^2-A+9I)=0$$

$$A^4-3A^3-A^2+9A=0.$$

$A^4 = 3A^3 + A^2 - 9A.$

$$A^4 = 3\begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12+4-9 & -27-3-0 & 63+6-27 \\ 33+3-18 & -6+2-9 & 33+4+9 \\ 3+0-9 & -21-2+9 & 21+5-9 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ hence find $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4$
$+ 8A^2 - 2A + I$.

Given, $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

Char eq^n of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(2-\lambda) - 1\right] = 0.$$

$$(1-\lambda)(4 + \lambda^2 - 4\lambda - 1) = 0.$$

$$\lambda^2 - 4\lambda + 3 - 3\lambda - \lambda^3 + 4\lambda^2 = 0.$$

$$\lambda^3 - \lambda^2 + \dots = 0 \quad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$$

By C-H theorem, $A^3 - 5A^2 + 7A - 3I = 0$ — ①.

$$A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$A^3 - 5A^2 + 7A - 3I =$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2) + 8A^2 - 2A + I$

$= A^5(0) + A(-7A + 3I) + 8A^2 - 2A + I$ [from ①].

$= -7A^2 + 3A + 8A^2 - 2A + I$

$= A^2 + A + I$ — ②

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

5. $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ find A^8 by using C-H Theorem.

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix}$$

char equation, of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0.$$

$$-(1-\lambda)(1+\lambda) - 4 = 0$$

$$-(1-\lambda^2) - 4 = 0$$

$$1 - \lambda^2 + 4 = 0.$$

$$\lambda^2 - 5 = 0.$$

By C.H. Theorem, $A^2 - 5I = 0$.

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 0.$$

we have,

$$A^8 = (A^2)^4 = (5I)^4 = 625 I^4$$

properties of Eigen Values & Eigen vector:-

→ If λ is an Eigen value of A corresponding to Eigen vector x, then λ^n is Eigen value of A^n corresponding to same Eigen vector x.

proof :- Given, Eigen value of A, with Eigen vector x

$$Ax = \lambda x \quad \text{—— (1)}.$$

This proof is given by mathematical Induction

pre multiply eq^n (1) with A.

$A(Ax) = A(\lambda x)$.

$(AA)x = \lambda(Ax)$ [applying associative property]

$A^2x = \lambda(\lambda x)$. by ①

$A^2x = \lambda^2 x$.

λ^2 is Eigen value of A^2 corresponding to Eigen vector x.

let λ^k is Eigen value of A^k with Eigen vector x

$\therefore \lambda^k$ is

$$A^k x = \lambda^k x$$

$$A(A^k x) = A(\lambda^k x)$$

$$(AA^k)x = \lambda^k (Ax)$$

$$A^{k+1} x = \lambda^k \lambda x \quad [\because \text{from } ①]$$

$$A^{k+1} x = \lambda^{k+1} x$$

λ^{k+1} is Eigen value of A^{k+1} with Eigen vector x

$\therefore$ for $n = k+1$ statement is true.

$\therefore$ By Mathematical Induction, λ^n is Eigen value of A^n with Eigen vector x.

Ex:- If $-1, 2, -3$ are Eigen values of A, then Eigen values of A^3 are $(-1)^3, (2)^3, (-3)^3 = -1, 8, -27$

iii) If λ is an Eigen value of A with Eigen vector x, then $\lambda + k$ is Eigen value of $A + kI$ with same Eigen vector x.

proof:- Given λ is Eigen value of A with Eigen vector x

$\therefore Ax = \lambda x$ — ①

$(A+kI)x = Ax + kIx = \lambda x + kx$ — from ①

$(A+kI)x = (\lambda + k)x$

$\lambda + k$ is Eigen value of $A + kI$ with Eigen vector x.

Note:- If $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are Eigen values of A, then Eigen values are of $A+kI$ are $(\lambda_1+k), (\lambda_2+k) \ldots (\lambda_n+k)$

Ex:- If 1, -1, 2 are Eigen values of A, then Eigen

Eigen values of $(A+3I)$ are 4, 2, 5.

iii) If λ is Eigen value of A with Eigen vector x, then $\lambda - k$ is Eigen value of $A - Ik$ with same Eigen vector x.

Proof:- Same proof (ii).

iv) If λ is Eigen value of A, then $k\lambda$ is Eigen value of kA with same Eigen vector x.

v) If λ is Eigen value of non-singular matrix A, then $\frac{1}{\lambda}$ Eigen value of A^{-1} with same Eigen vector x

Proof:- Given λ is Eigen value of A with Eigen vector

$$AX = \lambda X \quad — ① \quad (\lambda \neq 0)\ (A^{-1} \text{ exist})$$

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$\frac{1}{\lambda} IX = A^{-1}X$$

$$\frac{1}{\lambda} X = A^{-1}X \Rightarrow A^{-1}X = \frac{1}{\lambda} X$$

$\therefore 1/\lambda$ is Eigen value of A^{-1} with Eigen vector x.

Ex:- If 1, 3, 6, are Eigen values of non-singular matrix A, then Eigen values of A^{-1} are 1, 1/3, 1/6

vi) If λ is Eigen value of non-singular matrix A, then $\frac{|A|}{\lambda}$ is Eigen value of adj A with same Eigen vector x.

Proof:- Given Eigen value of A with eigen vector x

$$AX = \lambda X — ①$$

$$\text{adj} A(AX) = \text{adj} A(\lambda X)$$

$$[(\text{adj} A)A]X = \lambda(\text{adj} A)X$$

$$|A| I x = \lambda (\text{adj} A) x$$

$$\frac{|A|}{\lambda} x = (\text{adj} A) x.$$

$$(\text{adj} A) x = \frac{|A|}{\lambda} x.$$

$\therefore \frac{|A|}{\lambda}$ is Eigen value of adj A.

(vii)

A square matrix and its transpose have same Eigen values.

proof:- Given Eigen value of A with Eigen vector x

$$\therefore A x = \lambda x \quad \text{--- } ①.$$

$$(A - \lambda I)^T = A^T - (\lambda I)^T$$

$$= A^T - \lambda I^T = A^T - \lambda I$$

$$(A - \lambda I)^T = A^T - \lambda I.$$

$$|(A - \lambda I)^T| = |A^T - \lambda I| \qquad (kA)^T = k A^T$$

$$|A - \lambda I| = |A^T - \lambda I|. \qquad |B^T| = |B|.$$

$$|A - \lambda I| = 0 \text{ iff } |A^T - \lambda I| = 0.$$

$\therefore \lambda$ is Eigen value of A iff λ is Eigen value of A^T.

$\therefore$ A and A^T have same Eigen values.

(viii)

λ is Eigen value of an orthogonal matrix A then, $1/\lambda$ is also Eigen value of A.

proof:- Given A is an orthogonal matrix then,

$$A^{-1} = A^T \quad \text{--- } ①$$

λ is Eigen value of A

$\Rightarrow \frac{1}{\lambda}$ is Eigen value of A^{-1} (prop - 5)

$\frac{1}{\lambda}$ is Eigen value of A^T by ①

$\frac{1}{\lambda}$ is Eigen value of A (prop - 7).

Ex:- If A is an orthogonal matrix of order 3 and 1, 2 are two Eigen values of A then, the third Eigen value is $\frac{1}{2}$.

(ix) If A and P are square matrices of same order such that P is non-singular then A & $P^{-1}AP$ have same Eigen values

proof:- Given, A & P are square matrices of same order and P is non-singular.

$$|P^{-1}AP - \lambda I| = |P^{-1}AP - \lambda P^{-1}IP|$$

$$= |P^{-1}(A-\lambda I)P| \qquad |AB| = |A||B|$$

$$= |P^{-1}||A-\lambda I||P| \qquad |P^{-1}| = \frac{1}{|P|}$$

$$= \frac{1}{\cancel{|P|}}|A-\lambda I|\cancel{|P|}$$

$$= |A-\lambda I|$$

$$|P^{-1}AP - \lambda I| = |A-\lambda I|$$

$$\Rightarrow |A-\lambda I| = 0 \text{ iff } |P^{-1}AP - \lambda I| = 0$$

λ is eigen value of A iff λ is Eigen value of $P^{-1}AP$

A & P are square matrices of same order such that P is non-singular, A & $P^{-1}AP$ have same eigen values.

Results:-

1. If A and B are square matrices of same order then AB and BA have same Eigen values.

Replacing P with B & A with BA in above property.

2. If A and B are two square matrices of same order and A is invertable, then $A^{-1}B$ & BA^{-1} have same Eigen values.

(x) If λ is Eigen value of A then Eigen value of $f(A) = a_0A^2 + a_1A + a_2I$ is $f(\lambda) = a_0\lambda^2 + a_1\lambda + a_2$

(xi) The sum of the Eigen values of a square matrix is equal to its trace and product of the Eigen values is equal to its determinants.

proof:-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Char eq^n of A is $|A-\lambda I| = 0$.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) \cdots (a_{nn}-\lambda) +$ a polynomial of degree $(n-2) = 0$.

$(-\lambda)^n + (a_{11}+a_{22}+\cdots+a_{nn})(-\lambda)^{n-1} + a$ poly of degree $(n-2)$

+ a poly of degree of $(n-2)$...

$(-1)^n \lambda^n + (-1)^{n-1} Tr(A) \lambda^{n-1} + k_2 \lambda^{n-2} + k_3 \lambda^{n-3} + \cdots + k_{n-1}\lambda + k_n$

$= 0 \quad - (1)$

The roots of the equation are Eigen values.

$\therefore$ sum of the roots $= -\dfrac{\text{coeff of } \lambda^{n-1}}{\text{Coeff of } \lambda^n}$

Sum of the Eigen Values $= \dfrac{-(-1)^{n-1} Tr(A)}{(-1)^n} = Tr(A)$

from (1),

$|A - \lambda I| = (-1)^n \lambda^n + (-1)^n Tr A \lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_{n-1}\lambda + k_n$

$\lambda = 0 \Rightarrow |A| = k_n$

product of the root $= \dfrac{(-1)^n \text{ const}}{\text{Coeff of } \lambda^n}$

product of the Eigen values $= \dfrac{(-1)^n k_n}{(-1)^n} = |A|$.

(xii)

For a real symmetric matrix, the Eigen vectors corresponding to distinct Eigen values are orthogonal

proof:-

Two Eigen vectors x and y are said to be orthogonal, if $x^T y = 0$.

let A is real symmetrical matrix

$\therefore A^T = A$

let, λ_1, λ_2 are distinct Eigen values of A, x_1, x_2 are corresponding Eigen vectors.

$Ax_1 = \lambda_1 x_1, \quad Ax_2 = \lambda_2 x_2.$

$Ax_1 = \lambda_1 x_1 \Rightarrow x_2^T A x_1 = x_2^T \lambda_1 x_1$

$(x_2^T A x_1)^T = (x_2^T \lambda_1 x_1)^T$ $\left[\because (AB)^T = B^T A^T, \; (KA)^T = KA^T\right]$

$x_1^T A^T (x_2^T)^T = x_1^T \lambda_1 (x_2^T)^T$

$x_1^T A x_2 = \lambda_1 x_1^T x_2$ — ①.

~~$Ax_2 = \lambda_2 x_2 \Rightarrow x_1^T A x_2 = x_1^T \lambda_2 x_2$~~

~~$(x_1^T A x_2)^T = (x_1^T \lambda_2 x_2)^T$~~

~~$x_2^T A^T (x_1^T)^T = \lambda_2 x_2^T (x_1^T)^T$~~

~~$x_2^T A x_1 = \lambda_2 x_2^T x_1.$~~

$Ax_2 = \lambda_2 x_2 \Rightarrow x_1^T A x_2 = x_1^T \lambda_2 x_2$ — ②.

From ① & ②, we get

$\lambda_1 x_1^T x_2 = x_1^T \lambda_2 x_2$

$\lambda_1 x_1^T x_2 - \lambda_2 x_1^T x_2 = 0.$

$(\lambda_1 - \lambda_2) x_1^T x_2 = 0.$

$x_1^T x_2 = 0.$

$(\because \lambda_1 \neq \lambda_2)$

$\therefore x_1, x_2$ are orthogonal.

problems:-

1\. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find Eigen values of $3A^3 + 5A^2 - 6A + 3I$.

→ let, $f(A) = 3A^3 + 5A^2 - 6A + 3I$

Since, the given matrix A is triangular matrix.

$\therefore$ The Eigen values of A are, 1, 3, -2.

$\therefore$ Eigen values of A&A F(A) are f(1), f(3), f(-2)

$f(1) = 3(1)^3 + 5(1)^2 - 6(1) + 2 = 3 + 5 - 6 + 2 = 4$

$f(3) = 3(3)^3 + 5(3)^2 - 6(3) + 2 = 81 + 45 - 18 + 2 = 110$

$f(-2) = 3(-2)^3 + 5(-2)^2 - 6(-2) + 2 = -24 + 20 + 12 + 2 = 10$

Eigen values of f(A) are 4, 110, 10.

2. Find the Eigen values and Eigen vectors of $B = 2A^2 - \frac{1}{2}A + 3I$ where $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$.

$\rightarrow$ Given, $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$.

let, $f(A) = 2A^2 - 1/2\,A + 3I$

$A - \lambda I = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$

Char eqn is $|A - \lambda I| = 0$.

$\begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$.

$(8-\lambda)(2-\lambda) + 8 = 0$.

$16 - 8\lambda - 2\lambda + \lambda^2 + 8 = 0$.

$\lambda^2 - 10\lambda + 24 = 0$.

$\lambda = 6, 4$

The Eigen values of B are 4

$B = B(6) = 2(6)^2 - \frac{1}{2}(\overset{3}{\cancel{6}}) + 3 = 72$

$B(4) = 2(4)^2 - \frac{1}{2}(4^2) + 3 = 32 - 2 + 3 = 33.$

conjugate matrix:- let A is the given matrix the matrix obtained by replacing the complex numbers of A with their corresponding conjugate complex numbers is called conjugate matrix of A. It is denoted by $\bar{A}$.

Ex:- $A = \begin{bmatrix} 1+i & -2i & 3 \\ 2-3i & 4 & i \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 1-i & 2i & 3 \\ 2+3i & 4 & -i \end{bmatrix}$

Transpose of conjugate matrix:- let A is the given matrix then the transpose of conjugate matrix of A is called transposed conjugate matrix of A. It is denoted by $(\bar{A})^T$ (or) A^{θ}.

Ex: $A = \begin{bmatrix} 4i & 8 \\ -2-i & 3+2i \end{bmatrix}$ $\bar{A} = \begin{bmatrix} -4i & 8 \\ -2+i & 3-2i \end{bmatrix}$

$A^\theta = \begin{bmatrix} -4i & -2+i \\ 8 & 3-2i \end{bmatrix}$.

properties:-

1. $(A \pm B)^\theta = A^\theta \pm B^\theta$

2. $(kA)^\theta = \bar{k} A^\theta$

3. $(AB)^\theta = B^\theta A^\theta$

4. $(A^\theta)^\theta = A$

Hermition matrix:- A square matrix A is said to be Hermition if $A^\theta = A$ (or) $(\bar{A})^T = A$ (or) $\bar{A} = A^T$

In a Hermition matrix the diagonal elements are real numbers ~~then~~ and ij^{th} element is conjugate of ji^{th} element

Ex:- $\begin{bmatrix} 2 & 2-i \\ 2+i & 3 \end{bmatrix}$

Skew-Hermition matrix:- A square matrix A is said to be skew-Hermition, if transpose $A^\theta = -A$ (or) $(\bar{A})^T = -A$ (or) $\bar{A} = -A^T$.

In a skew-Hermition matrix the diagonal elements are zero (or) purely imaginary and ij^{th} element is negative of conjugate of ji^{th} element.

Ex:- $\begin{bmatrix} i & 1+2i & 3-4i \\ -1+2i & -2i & -i \\ -3-4i & -i & 0 \end{bmatrix}$

Unitary matrix:- A square matrix of A is said to be unitary if $AA^{\theta} = A^{\theta}A = I$ (or) $A^{\theta} = A^{-1}$

Ex:- $\begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$

Quadratic forms:- A homogeneous expression of second degree of any no. of ~~no~~ variables is called quadratic form.

Ex:- 1. $x^2+3xy-y^2$ is a quadratic from in two variables.

2. $x^2+y^2+yz+5zx$ is a quadratic form in three variable.

In general an expression of the form

$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ is a quadratic form in n variables $x_1, x_2, \ldots x_n$.

Here a_{ij} are constants. If these are real numbers then Q is called a real quadratic form. Every real quadratic form can be expressed as $X^T A X$. where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. A is called matrix of the quadrant form which is a symmetric matrix. If A

* If A is singular then the quadratic form is also singular otherwise non-singular.

problems:-

I. Find the matrix of following quadratic form.

1. $x^2-2y^2+3z^2+xy-4yz+3zx$.

→ Matrix of the quadratic form,

$$A=\begin{bmatrix} 1 & 1/2 & 3/2 \\ 1/2 & -2 & -2 \\ 3/2 & -2 & 3 \end{bmatrix}$$

2. $2x_1x_2+6x_1x_3-4x_2x_3$.

Matrix of the quadratic form,

$$A=\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -2 \\ 3 & -2 & 0 \end{bmatrix}.$$

3. $x_1^2+2x_3^2+4x_1x_2-6x_3x_4$.

matrix of the Quadratic form

$$A=\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

II. Find the Quadratic form of the following matrices.

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$,

Quadratic form of the matrix.

$x^2+z^2+4xy+6yz+6xz$.

(or)

$Q = x^T A x$.

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x+2y+3z & 2x+3z & 3x+3y+z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left[x^2+2xy+3xz+2xy+3zy+3xz+3yz+z^2\right]$$

$$= x^2+z^2+4xy+6xz+6zx$$

2. $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$

$Q = x^T A x$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2x+y+5z & x+3y-2z & 5x-2y+4z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left[2x^2+xy+5zx+xy+3y^2-2zy+5xz-2yz+4z^2\right]$$

$$= 2x^2+3y^2+4z^2+2xy+10xz-4zy.$$

Linear transformation:-

Consider the linear equations $x' = ax+by$, $y' = cx+dy$ These equations can be written as the matrix form $X = PY$ where

$X = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $Y = \begin{bmatrix} x \\ y \end{bmatrix}$

The transformation $X = PY$ is called linear transformation if P is singular then $X = PY$ is singular transformation otherwise non-singular transformation. A non-singular transformation is also called regular transformation. In the case of regualar transformation inverse transformation is exist and it is $Y = P^{-1}X$.

If P is orthogonal then $X = PY$ is called orthogonal transformation.

consider a quadratic form $Q = X^TAX$ and a linear transformation $X = PY$

$$\therefore Q = (PY)^T A (PY)$$

$$Q = Y^T P^T A P Y$$

$$Q = Y^T D Y \quad [\because D = P^T A P]$$

Here D is symmetric matrix

$\therefore Y^TDY$ represents a quadratic form.

$\therefore$ A quadratic form reduces to another quadratic form by using a linear transformation.

Normal Form (or) Sum of Squares form (or) canonical form:-

let $Q = X^T A X$ is the given Quadratic form having n variables then there exist a linear transformation $X = PY$ such that Q will be reduced into the form $Y^T D Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$, where $D = \text{diag}[\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n]$ the above form is called normal form of Q.

Rank of the Quadratic form:- The no of terms in the normal form of the given quadratic form Q is called Rank of Quadr-atic form Q. It is denoted by "R". "r".

r R = Rank of A

Index of the Quadratic form:- The no. of positive terms in the normal form of the Quadratic form is called index of the Quadratic form. It is denoted by "S"

Signature of the Quadratic form:- The excess no. of positive terms over the no. of negative terms.

Normal form of the Quadratic form is called signature.

* If R r is rank and s is index then the signature,

Signature = no. of positive terms − no. of negative terms

$= s - (r - s) = 2s - r$

Nature of the Quadratic form:- let $Q = X^T A X$ is a quadratic form having 'n' variables 'r' is the rank, 's' is the index of the quadratic form.

positive definite:- If $r = n, s = r$ (or) all Eigen values of A are positive then nature of Q is positive definite.

positive semi definite:- If $r < n$, $s = r$ or atleast one Eigen value of A is zero and remaining are +ve then the nature of Q is +ve semi definite.

3. Negative definite:- If $r=n$, $s=0$ (or) all Eigen values of A are -ve then the quadratic form then Q is +ve definite.

4. Negative Semi definite:- If $r<n$, $s=0$ (or) atleast one Eigen value is zero and remaining -ve then Q is negative semi definite.

In definite:- In all the remaining cases Nature of Q is indefinite.

problems:-

1. Find the nature of Quadratic form $x^2+y^2+2z^2-2xy+4xz+4zy$.

→ The matrix of quadratic form is

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$(A-\lambda I) = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

char eqn of A is $|A-\lambda I|=0$.

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)\left[(1-\lambda)(2-\lambda)-4\right]+1\left[-(2-\lambda)-4\right]+2\left(-2-(1-\lambda)\right)=0$$

$$(1-\lambda)(\lambda^2-3\lambda-2)+\lambda-6+2(2\lambda-4)=0$$

$\lambda^2 - 3\lambda - 2 - \lambda^3 + 3\lambda^2 + 2\lambda + \lambda - 6 + 4\lambda - 8 = 0.$

$-\lambda^3 + 4\lambda + 4\lambda^2 - 16 = 0.$

$\lambda^3 - 4\lambda - 4\lambda^2 + 16 = 0.$

$\lambda^2 - 4 = 0.$

$\lambda^2 = 4.$

$\lambda = \pm 2$

$\therefore$ Eigen values are $-2, 2, 4$.

$\therefore$ The nature of Quadratic form is indefinite.

Reduction to normal form by linear transformation:-

let $Q = X^T A X$ is the given quadratic form, the symmetric matrix A can be written as $A = IAI$ using row and column operations, reduce the above matrix into the form $D = P^T A P$ where D is the diagonal matrix. Apply every row transformation to LHS and pre factor to RHS. Apply similar to column transformation to LHS and post factor of RHS.

The normal form of Q is given by $Y^T D Y$ and the linear transformation is $X = PY$.

1. Reduce the following Quadratic form into normal form by a linear transformation. Hence find the rank & Index & signature, also find the nature & the linear transformation.

1. $x_1^2+2x_2^2+3x_3^2+4x_1x_2-2x_2x_3+6x_3x_1$.

Symmetric matrix of the given quadratic form

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 2 & -1\\ 3 & -1 & 3\end{bmatrix}$$

A can be written as $A = IAI$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\ 2 & 2 & -1\\ 3 & -1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$R_2 \to R_2-2R_1$, $R_3 \to R_3-3R_1$

$$\begin{bmatrix}1 & 2 & 3\\ 0 & -2 & -7\\ 0 & -7 & -6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$C_2 \to C_2-2C_1$, $C_3 \to C_3-3C_1$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -2 & -7\\ 0 & -7 & -6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$R_3 \to 2R_3-7R_2$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -2 & -7\\ 0 & 0 & 37\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ +8 & 0 & 2\end{bmatrix} A \begin{bmatrix}1 & -2 & -3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$C_3 \to 2C_3-7C_2$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 874 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ +8 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & +8 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This is in the form $D = P^T A P$.

where, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 874 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -2 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Normal form is $Y^T D Y = y_1^2 - 2y_2^2 + 74y_3^2$.

$\therefore$ rank r = no. of terms $= 3$.

index $= s =$ no. of positive terms $= 2$.

signature $= 2s - r = 2(2) - 3 = 1$.

Nature of quadratic form is indefinite,

the linear transformation is $X = PY$.

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

2. $10x^2 + 2y^2 + 5z^2 - 4xy - 10xz + 6yz$.

Symmetric matrix of the given quadratic form.

$$A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

A can be written as $A = IAI$

$$\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 5R_2 + R_1$, $R_3 \to 2R_3 + R_1$

$$\begin{bmatrix} 10 & -2 & -5 \\ 0 & 8 & 10 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 5C_2 + C_1$, $C_3 \to 2C_3 + C_1$

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$C_3 \to 2C_3 - C_2$

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

This is in the form of $D = P^T A P$

where, $D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$

Normal form is $Y^T D Y = 10x^2 + 40y^2$.

rank r = no. of terms = 2

index = s = no. of positive terms = 2

Signature $= 2s - r = 2(2) - 2 = 2$

Nature of the quadratic form is semi positive definite

the linear transformation is $X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Reduction to Normal form by orthogonal transformation:-

let $Q = X^T A X$ is the given quadratic form let the Eigen values of A are $\lambda_1, \lambda_2, \lambda_3$ and corresp-onding Eigen vectors x_1, x_2, x_3 since A is symme-tric the Eigen vectors are pair wise orthogonal.

Define, $P = \left[\frac{x_1}{\|x_1\|} \quad \frac{x_2}{\|x_2\|} \quad \frac{x_3}{\|x_3\|}\right]$ then p is orthogonal

$\therefore P^{-1} = P^T$

$$\therefore D = P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Normal form is $y^T D y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$ and orthogonal transformation is $x = Py$.

1. Reduce the following quadratic form in to normal form by orthogonal transformation. Hence find the rank, Index, and signature.

1. $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$.

Symmetric matrix of the normal form.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

characteristics of the equation,

$|A-\lambda I|=0.$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$(3-\lambda)\{(3-\lambda)^2-1\} - 1(3-\lambda+1) + 1(-1-3+\lambda) = 0$

$(3-\lambda)(9+\lambda^2-6\lambda-1) - 1(4-\lambda) + 1(-4+\lambda) = 0$

$(3-\lambda)(\lambda^2-6\lambda+8) - 1(4-\lambda) - 1(4-\lambda) = 0$

$(3-\lambda)(\lambda^2-4\lambda-2\lambda+8) - 2(4-\lambda) = 0.$

$(3-\lambda)[\lambda(\lambda-4)-2(\lambda-4)] - 2(4-\lambda) = 0.$

$(3-\lambda)(\lambda-2)(\lambda-4) - 2(\lambda-4) = 0.$

$((3-\lambda)(\lambda-2)+2)(\lambda-4) = 0.$

$\lambda-4=0, \quad 3\lambda-6-\lambda^2+2\lambda+2=0.$

$-\lambda^2+5\lambda-4=0.$

$\lambda = 1, 4, 4.$

Eigen values are 1, 4, 4.

Case (i):- $\lambda=1$

$(A-\lambda I)X=0.$

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} X = 0.$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} X = 0$$

$R_2 \to 2R_2 - R_1$, $R_3 \to 2R_3 - R_1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} x = 0.$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$3x_2 - 3x_3 = 0$, $2x_1 + x_2 + x_3 = 0$

$x_2 - x_3 = 0$, $2x_1 = -x_2 - x_3$

$x_2 = x_3$, $2x_1 = -2k_1$

$x_1 = -k_1$

$\therefore$ let, $x_3 = k_1$.

$x_2 = k_1$.

Eigen vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_2 \\ k_3 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

case (iii):-

$\lambda = 4$.

$(A - \lambda I)x = 0$.

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} x = 0.$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$-x_1 + x_2 + x_3 = 0$ let, $x_2 = k_2$, $x_3 = k_3$.

$-x_1 + k_2 + k_3 = 0 \Rightarrow x_1 = k_2 + k_3$

Eigen vector is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2+k_3 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} k_2 \\ k_2 \\ k_3 \end{bmatrix} + \begin{bmatrix} k_3 \\ k_2 \\ k_3 \end{bmatrix}$$

$$= k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

But, the Eigen vectors, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are not orthogonal

let, $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ consider,

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} k_2+k_3 \\ k_2 \\ k_3 \end{bmatrix}$

$$1(k_2+k_3)+1(k_2)+0=0.$$

$$2k_2+k_3=0.$$

$$k_3=-2k_2.$$

$$\begin{bmatrix} k_2+k_3 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} k_2-2k_2 \\ k_2 \\ -2k_2 \end{bmatrix} = \begin{bmatrix} -k_2 \\ k_2 \\ -2k_2 \end{bmatrix}$$

$$= -k_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

let $X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

$\|x_1\| = \sqrt{(-1)^2+(1)^2+(1)^2} = \sqrt{3}$

$\|x_2\| = \sqrt{1^2+1^2+0} = \sqrt{2}$

$\|x_3\| = \sqrt{1+1+4} = \sqrt{6}$

$$P = \begin{bmatrix} \dfrac{x_1}{\|x_1\|} & \dfrac{x_2}{\|x_2\|} & \dfrac{x_3}{\|x_3\|} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

$D = P^{-1}AP = P^TAP.$

$$= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Normal form $y^T D y = y_1^2 + 4y_2^2 + 4y_3^2.$

Rank $r = 3$

Index = no. of positive terms = $s = 3$.

Signature = ~~no.~~ $2s - r = 3$.

Nature is +ve definite.

orthogonal transformation $x = Py$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

2. $2x^2+2y^2+2z^2-2xy+2zx-2yz$.

Symmetric matrix of the Normal form

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

Char eqn $= |A-\lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0.$$

$$(2-\lambda)\left[(2-\lambda)(2-\lambda)-1\right] + 1\left[-(2-\lambda)+1\right] + 1\left(1-(2-\lambda)\right) = 0$$

$$(2-\lambda)\left[4+\lambda^2-4\lambda-1\right] + 1(-2+\lambda+1) + (1-2+\lambda) = 0$$

$$(2-\lambda)(\lambda^2-4\lambda+3) + 1(\lambda-1) + (\lambda-1) = 0$$

$$(2-\lambda)(\lambda^2-3\lambda-\lambda+3) + 2(\lambda-1) = 0$$

$$(2-\lambda)\left[\lambda(\lambda-3)-1(\lambda-3)\right] + 2(\lambda-1) = 0$$

$$(2-\lambda)(\lambda-1)(\lambda-3) + 2(\lambda-1) = 0$$

$$(\lambda-1)\left[(2-\lambda)(\lambda-3)+2\right] = 0$$

$$(\lambda-1)(2\lambda-6-\lambda^2+3\lambda+2) = 0$$

$$(\lambda-1)(-\lambda^2+5\lambda-4) = 0$$

$$(\lambda-1)(\lambda^2-5\lambda+4) = 0 \Rightarrow (\lambda-1)(\lambda-1)(\lambda-4) = 0$$

$$\lambda = 1, 1, 4.$$

Case (i): $\lambda = 1$

$$(A-\lambda I)X = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} X = 0.$$

$R_2 \to R_2 + R_1, \; R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X = 0.$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 - x_2 + x_3 = 0$

let, $x_2 = \alpha, \; x_3 = \beta$

$x_1 - \alpha + \beta = 0.$

$x_1 = \alpha - \beta$

Eigen vector,

$$X_1 = \begin{bmatrix} \alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \beta \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) :- $\lambda = 4$.

$(A - \lambda I) X = 0.$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} X = 0.$$

$R_2 \to 2R_2 - R_1, \; R_3 \to 2R_3 + R_1$

$$\begin{bmatrix} -2 & -1 & 1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} X = 0$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} -2 & -1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-2x_1 - x_2 + x_3 = 0$ $\qquad$ $-3x_2 - 3x_3 = 0.$ $\qquad$ let, $x_3 = k_1$

$-2x_1 + k_1 + k_1 = 0.$ $\qquad$ $-3x_2 = +3k_1$

$-2x_1 = -2k_1$ $\qquad$ $x_2 = -k_1$

$x_1 = k_1.$

Eigen vector,

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

But, the Eigen vectors, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are not orthogonal

let, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

consider,

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} \alpha - \beta \\ \alpha \\ \beta \end{bmatrix}$

$$(\alpha - \beta) + \alpha + \beta(0) = 0.$$

$$\alpha - \beta + \alpha = 0.$$

$$2\alpha - \beta = 0$$

$$2\alpha = \beta$$

then,

$$\begin{bmatrix} \alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha - 2\alpha \\ \alpha \\ 2\alpha \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \\ 2\alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$\|X_1\| = \sqrt{(+1)^2 + (1)^2 + (0)^2} = \sqrt{2}$

$\|X_2\| = \sqrt{(1)^2 + (-1)^2 + (1)^2} = \sqrt{3}$

$\|X_3\| = \sqrt{(-1)^2 + (1)^2 + (2)^2} = \sqrt{6}.$

$$P = \left[\frac{X_1}{\|X_1\|} \quad \frac{X_2}{\|X_2\|} \quad \frac{X_3}{\|X_3\|}\right].$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$$

$$P^{-1} = P^T$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

$$D = P^T A P = P^{-1} A P$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Normal form $Y^T D Y = y_1^2 + 4y_2^2 + y_3^2$

rank $r = 3$

Index = no. of positive terms $= 3 = s$

Signature $= 2s - r = 3$

Nature is +ve definite

orthogonal transformation

$$X = PY$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

3. $3x^2+5y^2+3z^2-2yz+2zx-2xy=0$.

Symmetric matric of the Normal form

$$A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$$

$$[A-\lambda I]=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3-\lambda & -1 & 1\\ -1 & 5-\lambda & -1\\ 1 & -1 & 3-\lambda\end{bmatrix}.$$

Char eq^n is $|A-\lambda I|=0$

$$\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 5-\lambda & -1\\ 1 & -1 & 3-\lambda\end{vmatrix}=0.$$

$$(3-\lambda)\left[(5-\lambda)(3-\lambda)-1\right]+1\left[-(3-\lambda)+1\right]+1\left(1-5+\lambda\right)=0.$$

$$(3-\lambda)\left(15-5\lambda-3\lambda+\lambda^2-1\right)+1(-3+\lambda+1)+1(\lambda-4)=0.$$

$$(3-\lambda)\left[\lambda^2-8\lambda+14\right]+(\lambda-2)+(\lambda-4)=0.$$

$$(3-\lambda)(\lambda^2-8\lambda+14)+2\lambda-6=0$$

$$(3-\lambda)(\lambda^2-8\lambda+14)-2(3-\lambda)=0.$$

$$(3-\lambda)(\lambda^2-8\lambda+14-2)=0.$$

$$(3-\lambda)(\lambda^2-8\lambda+12)=0.$$

$$(3-\lambda)(\lambda-6)(\lambda-2)=0.$$

$$\lambda=3,6,2.$$

case(i): $\lambda=2$

$$(A-\lambda I)X=0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} X = 0$$

$R_2 \to R_2 + R_1, \quad R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$x_1 - x_2 + x_3 = 0, \quad 2x_2 = 0. \quad , x_3 = k_1,$

$x_1 - 0 + k_1 = 0. \quad x_2 = 0.$

$x_1 = -k_1.$

Eigen vector,

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Case (ii) :- $\lambda = 3.$

$(A - \lambda I)X = 0.$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} X = 0$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} X = 0$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} X = 0.$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$x_1 - x_2 = 0, \quad x_2 - x_3 = 0 \quad x_3 = k_2$

$x_1 = x_2, \quad x_2 = x_3$

$x_1 = k_2 \quad x_2 = k_2$

Eigen vector,

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 \\ k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

case (iii) :- $\lambda = 6$

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} X = 0$$

$R_2 \to 3R_2 - R_1, \; R_3 \to 3R_3 + R_1$

$$\sim \begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix} X = 0$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$-3x_1 - x_2 + x_3 = 0, \quad -2x_2 - 4x_3 = 0$

$-3x_1 + 2k_3 + k_3 = 0. \quad -2x_2 = +\cancel{4}^{2} k_3 \quad x_3 = k_3$

$-3x_1 = -3k_3 \quad x_2 = -2k_3$

$x_1 = k_3$

Eigen vector,

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_3 \\ -2k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\|X_1\| = \sqrt{(-1)^2 + 0 + (1)^2} = \sqrt{2}$$

$$\|X_2\| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$\|X_3\| = \sqrt{(1)^2 + (-2)^2 + (1)^2} = \sqrt{6}$$

$$P = \left[\frac{X_1}{\|X_1\|} \quad \frac{X_2}{\|X_2\|} \quad \frac{X_3}{\|X_3\|} \right]$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$P^{-1} = P^T$$

$$= \begin{bmatrix} -1/2 & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$D = P^{-1}AP = P^TAP$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & +1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Normal form, $Y^T D Y = 2y_1^2 + 3y_2^2 + 6y_3^2$

rank $r = 3$

Index $= s = 3$

Signature $= 2s - r = 3$

Nature is +ve definite

orthogonal transformation, $X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Lagrange's Reduction :- Reduce the given quadratic form by rearranging the terms into normal form.

problems :-

1. $x^2+2y^2-7z^2-4xy+8xz$

$\rightarrow$ Given, $x^2+2y^2-7z^2-4xy+8xz$

$= x^2-2x(2y-4z)+2y^2-7z^2$

$= x^2-2x(2y-4z)+(2y-4z)^2-(2y-4z)^2+2y^2-7z^2$

$= [x-(2y-4z)]^2-(4y^2+16z^2-16yz)+2y^2-7z^2$

$= (x-2y+4z)^2-2y^2+16yz-23z^2$

$= (x-2y+4z)^2-2(y^2-8yz)-23z^2$

$= (x-2y+4z)^2-2(y^2-2y(4z)+16z^2-16z^2)-23z^2$

$= (x-2y+4z)^2-2(y-4z)^2+9z^2$

$= y_1^2-2y_2^2+9y_3^2$

where, $y_1 = x-2y+4z$

$y_2 = y-4z$

$y_3 = z$.

Rank $= r = 3$.

index = no. of positive terms $= s = 2$

Signature $= 2s-r = 4-3 = 1$.

Nature is not definite. (or) indefinite.

$\therefore$ Normal form is $y_1^2-2y_2^2+9y_3^2$.

we have, $z = y_3$

$y_2 = y - 4z \Rightarrow y = y_2 + 4z = y_2 + 4y_3$

$y_1 = x - 2y + 4z$

$x = y_1 + 2y - 4z = y_1 + 2(y_2 + 4y_3) - 4y_3$

$= y_1 + 2y_2 + 4y_3$

Linear transformation,

$X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

2. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$.

Given, $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

$= 6x_1^2 + 3x_2^2 + 3x_3^2 - 2x_1(2x_2 - 2x_3) + 3x_3^2 + 3 - 2x_2x_3$

$= 6\left[x_1^2 - 2x_1\left(\frac{2x_2 - 2x_3}{6}\right)\right] + 3x_2^2 + 3x_3^2 - 2x_2x_3$

$= 6\left[x_1^2 - 2x_1\left(\frac{x_2}{3} - \frac{x_3}{3}\right) + \left(\frac{x_2}{3} - \frac{x_3}{3}\right)^2 - \left(\frac{x_2}{3} - \frac{x_3}{3}\right)^2\right] + 3x_2^2 + 3x_3^2 - 2x_2x_3$.

$= 6\left[\left(x_1 - 2\left(\frac{x_2}{3} - \frac{x_3}{3}\right)\right)^2\right] - 6\left(\frac{x_2}{3} - \frac{x_3}{3}\right)^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$

$= 6\left[\left(x_1 - \left(\frac{x_2}{3} - \frac{x_3}{3}\right)\right)^2\right] - 6\left(\frac{x_2^2}{9} + \frac{x_3^2}{9} - \frac{2x_2x_3}{9}\right) + 3x_2^2 + 3x_3^2 - 2x_2x_3$.

$= 6\left[\left(x_1 - \left(\frac{x_2}{3} - \frac{x_3}{3}\right)\right)^2\right] - 6\,\frac{2x_1^2}{3} - \frac{2x_3^2}{3} + \frac{4x_2x_3}{3} + 3x_2^2 + 3x_3^2 - 2x_2x_3$.

$= 6\left[\left(x_1 - \left(\frac{x_2}{3} - \frac{x_3}{3}\right)\right)^2\right] + \frac{7}{3}x_2^2 + \frac{7}{3}x_3^2 - \frac{2}{3}x_2x_3$

$$= 6\left[x_1 - \frac{x_2}{3} + \frac{x_3}{3}\right]^2 + \frac{7}{3}\left[x_2^2 - \frac{2}{7}x_2 x_3\right] + \frac{7}{3}x_3^2$$

$$= 6\left[x_1 - \frac{x_2}{3} + \frac{x_3}{3}\right]^2 + \frac{7}{3}\left[x_2^2 - 2x_2\left(\frac{x_3}{7}\right) + \left(\frac{x_3}{7}\right)^2 - \left(\frac{x_3}{7}\right)^2\right] + \frac{7}{3}x_3^2$$

$$= 6\left[x_1 - \frac{x_2}{3} + \frac{x_3}{3}\right]^2 + \frac{7}{3}\left[\left(x_2 - \frac{x_3}{7}\right)^2\right] - \frac{7}{3}\left(\frac{x_3}{(7)^2}\right)^2 + \frac{7}{3}x_3^2$$

$$= 6\left[x_1 - \frac{x_2}{3} + \frac{x_3}{3}\right]^2 + \frac{7}{3}\left[\left(x_2 - \frac{x_3}{7}\right)^2\right] + \frac{16}{7}x_3^2$$

$$= 6y_1^2 + \frac{7}{3}y_2^2 + \frac{16}{7}y_3^2$$

where, $y_1 = x_1 - \frac{x_2}{3} + \frac{x_3}{3}$

$y_2 = x_2 - \frac{x_3}{7}$

$y_3 = x_3$

$\therefore$ Normal form is $6y_1^2 + \frac{7}{3}y_2^2 + \frac{16}{7}y_3^2$

Rank $= r = 3$

index $=$ no. of positive terms $= s = 3$

Signature $= 2s - r = 3$

Nature is positive definite

Application

Free vibration of two mass system :-

In free vibration the object is not under the influence of external force

Ex:- Vibrations of a pendulum.

The vibrating system which requires two coordinates to describe its motion is called two degrees of freedom

In two degree of freedom, system, two coordinates and two equation of motion are required to represent the system.

A degree of freddom have two natural frequen-cies (Eigen values) and two normal modes (Eigen vectors). The equation of motion in the matrix form is $(K-\lambda M)X=0$.

where, K = stiffness, M = mass of the vibrating system.

problems:-

1. Find the natural frequency and normal mode of the vibrating system for

$$M=\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix},\ K=\begin{bmatrix}6 & -2\\ -2 & 9\end{bmatrix}.$$

The equation of motion of the vibrating system is $(K-\lambda M)X=0$.

Cha equation is $|K-\lambda M|=0$

$$\begin{vmatrix}6-2\lambda & -2\\ -2 & 9-4\lambda\end{vmatrix}=0.$$

$$(6-2\lambda)(9-4\lambda)-4=0.$$

$$8\lambda^2-42\lambda+50=0.$$

$$4\lambda^2-21\lambda+25=0.$$

$$\lambda=\frac{21\pm\sqrt{(-21)^2-4\cdot4\cdot25}}{2(4)}$$

$$=\frac{21\pm\sqrt{441-400}}{8}=\frac{21\pm\sqrt{41}}{8}$$

$\therefore$ Natural frequencies, $\lambda = \dfrac{21 \pm \sqrt{41}}{8}$.

Case (i):- $\lambda = \dfrac{21+\sqrt{41}}{8}$

$$(K-\lambda M)X=0.$$

$$\begin{bmatrix} 6-2\lambda & -2 \\ -2 & 9-4\lambda \end{bmatrix} X = 0$$

$$\begin{bmatrix} 6-2\left(\frac{21+\sqrt{41}}{8}\right) & -2 \\ -2 & 9-4\left(\frac{21+\sqrt{41}}{8}\right) \end{bmatrix} X = 0$$

$$\begin{bmatrix} \frac{3-\sqrt{41}}{4} & -2 \\ -2 & \frac{-3-\sqrt{41}}{2} \end{bmatrix} X = 0$$

$$R_2 \to \frac{3-\sqrt{41}}{4} R_2 + 2R_1$$

$$\begin{bmatrix} \frac{3-\sqrt{41}}{4} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\frac{3-\sqrt{41}}{4} x_1 - 2x_2 = 0.$$

let, $x_1 = k$, $2x_2 = \dfrac{3-\sqrt{41}}{4} x_1$

$$x_2 = \frac{3-\sqrt{41}}{8} k$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ \frac{3-\sqrt{41}}{8} k \end{bmatrix} = \frac{k}{8} \begin{bmatrix} 8 \\ 3-\sqrt{41} \end{bmatrix}$$

Case (ii):- $\lambda = \dfrac{21-\sqrt{41}}{8}$

$$(K-\lambda M)X=0.$$

$$\begin{bmatrix} 6-2\lambda & -2 \\ -2 & 9-4\lambda \end{bmatrix} X = 0.$$

$$\begin{bmatrix} 6-2\left(\frac{21-\sqrt{41}}{8}\right) & -2 \\ -2 & 9-4\left(\frac{21-\sqrt{41}}{8}\right) \end{bmatrix} X = 0.$$

$$\begin{bmatrix} \frac{3+\sqrt{41}}{4} & -2 \\ -2 & -\frac{3+\sqrt{41}}{2} \end{bmatrix} X = 0.$$

$$R_2 \rightarrow \frac{3+\sqrt{41}}{4} R_2 + 2R_1$$

$$\begin{bmatrix} \frac{3+\sqrt{41}}{4} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$\frac{3+\sqrt{41}}{4} x_1 - 2x_2 = 0 \qquad \text{let } x_1 = k_2$$

$$\frac{3+\sqrt{41}}{4} k_2 - 2x_2 = 0.$$

$$2x_2 = \frac{3+\sqrt{41}}{4} k_2$$

$$x_2 = \frac{3+\sqrt{41}}{8} k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_2 \\ \frac{3+\sqrt{41}}{8} k_2 \end{bmatrix} = \frac{k_2}{8} \begin{bmatrix} 8 \\ 3+\sqrt{41} \end{bmatrix}.$$