

Communication Through Category Theory

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1 Introduction

1.1 Abstract

In the Typescript world we currently don't have a clear source for how Typescript and its ecosystem surrounding functional programming relate to the categorical concepts. The current efforts in the community have been largely been comprised of limited documentation surrounding functional programming libraries and in partial blog post. This leads to a large barrier of entry for developers coming into the ecosystem either new or from another functional language. By Using a subset of Category Theory concepts as an abstract model we can create a generalized model that can be implemented into typescript.

For an abstract model to be useful it would have to be communicable across programming languages of different types meaning that concepts from the model would need to be recognizable in multiple languages modern functional languages. For the Typescript implementation to be practical inside typescripts type system it would need to be able to use existing typescript ecosystem constructs where possible, be low boilerplate, and performant in both the front and back-end ecosystems where it is used.

This document assumes the reader is a beginner with Haskell and Typescript. And presumes no mathematical knowledge outside of basic logic, algebra, and functions.

1.2 Goals

This document is broken into several parts ... **Part 1** seeks to first define an abstract model that a practical implementation can be based on. We choose to avoid typescript entirely in this section and turn to Haskell as the language of choice due to the existing documentation and libraries that define category theory concepts already. **Part 2** seeks to define an implementation and identify the caveats between the typescript representation and the Haskell representation of the model. **Part 3** is to be determined but its focus will be on communication between typescript and Haskell, translating domain types to the model, and migration schemes for existing representations of concepts. **Part 4** seeks to define existing functional architectures in terms of the abstract model as well as provide basic example implementations and possible improvements.

2 Part 1

2.1 Defining a Category

A Category consist of two things ...

- Objects
- Arrows known as morphisms

Category Theory serves as an interpretation for the foundation of mathematics much like Set Theory and Type Theory. The key focus is on the composition of mathematical structures. These structures are called objects. Again the focus is on the composition of the objects which is captured with the notion of an arrow (formally a morphism) that points to objects.

A Category C is made up of objects and arrows called morphism for which the follow properties hold true.

- **Identity** - Objects must have a an arrow originating from itself to itself as the Identity morphism. This arrow serves as a unit of composition such that when composed with any Arrow that either starts at A or ends at A it gives back the same arrow. So if f is an arrow then $f \circ id_A = f$
- **Composition** - Given the objects A, B, C and two arrows $f = A \rightarrow B$ and $g = B \rightarrow C$, there must exist a third arrow from A to C represented as $g \circ f = A \rightarrow C$.
- **Associative** - Given 3 arrows f, g, h , composition must be associative the they must be associative $h \circ (g \circ f) = (h \circ g) \circ f = h \circ g \circ f$

2.2 Haskell Types and the Category of Set

A type can be thought of as a set of values. As an example the Haskell type `Bool` is a two element set of value `True` and `False`. Sets can be finite or infinite. In Haskell `x :: Integer` is saying `x` is an element of `Integer`. The category of sets is called `Set`. Its special because we can peak in at its objects.

In the category of set **objects are sets** and **morphisms are functions between sets**

- empty set has no elements
- there are special one element sets
- functions map elements of one set to elements of another set
- functions can map two elements to one but not one element to two
- there exist an identity function that maps each element of a set to itself

In the Haskell implementation the category of Haskell types and functions is referred to as Hask. By forgetting the bottom in non terminating functions we can treat Hask as the Category Set

Functions with multiple type arguments have two interpretations as well. $a \rightarrow a \rightarrow a$ can be interpreted as a function that takes multiple arguments and the last value is the return type or it can be interpreted as $a \rightarrow (a \rightarrow a)$ function that takes an argument and returns a function requiring another argument.

Figure 1: Translation of Category of Set to Haskell ...

Set	Haskell Type	Description
Empty Set	Void	Type not inhabited by any values
Singleton Set	() "Unit"	Type has one value that always exist
Two Element Set	Bool	true or false, functions to this type are called predicates

2.3 Examples of Categories

Free Construction is the process of completing a given structure by extending it with the minimum number of items to satisfy its laws. Graphs are the free constructor for the Free category. chains of length zero serve as the identity morphism in the free category.

Thin categories are those where Hom-set is the set of morphisms from object a to object b in the category of C written as $C(a,b)$ or $\text{Hom}_C(a,b)$

A hom-set is the set of morphisms from object a to object b in the category of C written as $C(a,b)$ or $\text{Hom}_C(a,b)$ Every hom-set is either empty, a singleton, or a preorder. Preorders can have cycles where as partial orders cannot

Sorting algorithms like quick-sort, bubble sort, merge sort, only work on total orders. Partial orders use topological sorts there is at most one morphism going from any object to any other object.

Figure 2: Examples of Categories ...

Category	Objects	Morphisms	Description
Empty	None	None	A category without objects or morphisms
Set	Sets	Functions	A category in which we can peek into the objects
Graph (Free Category)	Nodes that represent objects	all possible chains of composable edges	A category for Free construction
Preorder (Thin Category)	general non specific objects	relationship between objects where objects \leq	only one morphism between two objects
Partial Order	general non specific objects	relationship between objects where objects are \leq and that if $a \leq b$ and $b \leq a$ then a must be the same as b	only one morphism between two objects
Linear/Total Order	general non specific objects	relationship between objects where objects are \leq and that if $a \leq b$ and $b \leq a$ then a must be the same as b . Finally it states any two objects are in relation to one another	only one morphism between two objects

2.4 Subcategories and Haskell Typeclasses

In the category of C a subcategory of C is a sub-collection of the objects of C and a sub-collection of the morphism of C which map only between objects in the sub-collection of the subcategory

Subcategories can be represented as typeclasses in Haskell and have two interpretations depending on the amount of type parameters

- A single parameter typeclass can be interpreted as describing the members of a set of types
- Multi-parameter typeclasses is interpreted as a relation on types

Type class can be used to defined a collection of types that are members of a particular class. A type signature with a universally quantified type variable constrained by the typeclass

Example type signatures and their interpretation

- `Eq a => a`
the collections of types that are members of the class `Eq`. Members of `Eq` are a sub-collection of objects in the Category of `Hask`
- `Eq a, Eq b => (a -> b)`
is a sub-collection of the morphisms in `Hask` mapping between objects in the sub-collection of objects defined in `Eq`

2.5 Monoids

Monoids are a mathematical concept found across different branches of mathematics.

In set theory a monoid is a set equipped with a binary function that is associative and a unit element. An example is addition on the set of integers is equipped with the pseudo function $(+) :: a \rightarrow a \rightarrow a$.

Monoid M is a set with a unit element e and binary operation. Such that if $a, b, c \in M$ then ...

$$\begin{aligned}a \circ b &\in M \\(a \circ b) \circ c &= a \circ (b \circ c) \\e \circ a &= a \circ e = a\end{aligned}$$

In category theory a monoid is a one object category with a set of morphisms that follow the rules of composition.

Given a single object category C . C 's object is c , C 's morphism $c \rightarrow c$. C 's unit is $1_c : c \rightarrow c$

Operations on M are exactly the same as operations on $Morph(C)$ allowing us to regard the category C as the monoid M . The correspondence is reversible

$$\begin{aligned}
f \circ g &\in \text{Morph}(C) \\
(f \circ g) \circ h &= f \circ (g \circ h) \\
1_c \circ f &= f \circ 1_c = f
\end{aligned}$$

given the category of $C = c$ we obtain a monoid M whose elements are arrows of C . This allows us to extract a set from a single object category.

because $c \in C$ and the hom-set of c is $\text{Hom}_C(c, c)$ resulting in a Set monoid M whose elements are morphisms of C .

In the Hask we can define a subcategory definition for the Monoid using a typeclass.

```

class Monoid m where
  mempty  :: m
  mappend :: m -> m -> m

```