Take-Away Triangles

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1 Introduction

As described in Matt Parker's Math Puzzle #9, a Take-Away Triangle's initial state T_0 is an equilateral triangle with numerical values assigned to its vertices.

For a triangle

$$T_n = \triangle ABC \ ; \ val(A) = a,\ val(B) = b,\ val(C) = c$$

$$T_{n+1} = \triangle DEF;\ \angle D = \angle ADB,\ \angle E = \angle BEC,\ \angle F = \angle CFA$$
$$d = |a - b|,\ e = |b - c|,\ f = |c - a|$$

I will explore what properties are necessary for $\text{sum}(T_n) = \text{sum}(T_{n+1})$, and more general cases.

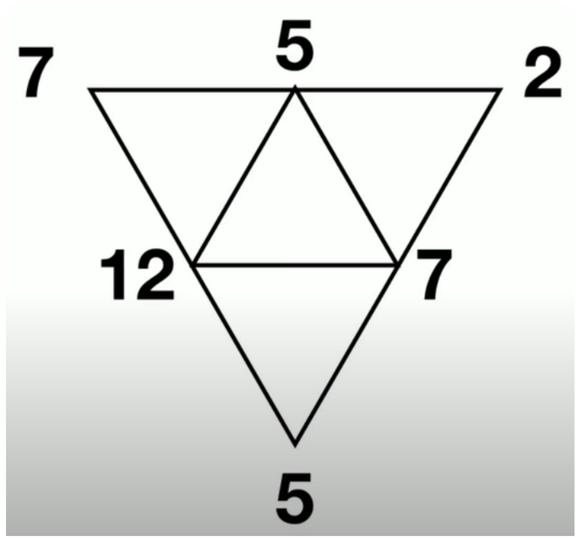

Figure 1: Example From Matt Parker's video

2 Fixed-sum Case

For a triangle $T_n = \triangle ABC$, we can assume $a \geq b \geq c$.
With vertices in no particular order, $T_{n+1} = \triangle A`B`C`$ with values $a` \geq b` \geq c`$.

$$a` = a - c,\ \begin{cases} a`{=}a{-}b,\ c`{=}b{-}c \\ b`{=}b{-}c,\ c`{=}a{-}b \end{cases}$$

If we wish to preserve our sum, we can now solve

$$a + b + c = a` + b` + c` = sum \tag{1}$$

$$a + b + c = (a - c) + (a - b) + (b - c) \tag{2}$$

$$a + b + c = 2a - 2c \tag{3}$$

$$a = b + 3c \tag{4}$$

$$sum = 2a - 2c \tag{5}$$

$$a = \frac{sum}{2} + c \tag{6}$$

$$b = \frac{sum}{2} - 2c \tag{7}$$

$$c <= \frac{sum}{6} \tag{8}$$

$$a` = \frac{sum}{2},\ \begin{cases} c{>}\frac{sum}{12}:\ b`{=}3c,\ c`{=}\frac{sum}{2}{-}3c \\ c{\leq}\frac{sum}{12}:\ b`{=}\frac{sum}{2}{-}3c,\ c`{=}3c \end{cases} \tag{9}$$

$$a` = \frac{sum}{2} + c` = \frac{sum}{2} \tag{10}$$

$$c` = 0 \tag{11}$$

$$b` = \frac{sum}{2} \tag{12}$$

$$\begin{cases} c{=}\frac{sum}{6} \\ c{=}0 \end{cases} \tag{13}$$

$$\begin{cases} c{=}\frac{sum}{6}:\ a{=}\frac{2*sum}{3} \quad b{=}\frac{sum}{6} \\ c{=}0 \quad :\ a{=}\frac{sum}{2} \quad b{=}\frac{sum}{2} \end{cases} \tag{14}$$

We have now determined that there are two cases in which T_{n+1} will have the same sum as the T_n. In either case, T_{n+1} will have values of $\frac{sum}{2}, \frac{sum}{2}$, 0; this set of values will then generate itself for all future iterations.

3 Reverse Iteration

How can we reach a state T_{n+1} with values x, y, z, from state T_n with values a, b, c. Given x, y, and z in no particular order and $a \geq b \geq c$ we can assign $x = a - b,\ y = a - c,\ z = b - c$.

$$b = a - x,\ c = a - x - z,\ c = a - y \tag{15}$$
$$y = x + z \tag{16}$$

This indicates that for any $T_n,\ n > 0$, the value of one vertex must equal the sum of the other two.

$$x = a - b,\ z = b - c \tag{17}$$
$$b = a - x \tag{18}$$
$$c = a - x - z \tag{19}$$

Given x and z, we may choose any a; our b and c are determined. If we wish for our new a, b, and c to have a prior state, then they must satisfy our equation of $a = b + c$.

$$b = a - x,\ c = a - x - z,\ a = b + c \tag{20}$$
$$b = b + c - x \tag{21}$$
$$c = x \tag{22}$$
$$a = 2x + z = y + x \tag{23}$$
$$b = x + z = y \tag{24}$$

Given that x and z are in no order, We now have two possible backwards-iterable solutions for any $T_n = \triangle XYZ$ where $y \geq x \geq z$:

$T_{n-1} = \triangle ABC \begin{cases} a{=}y{+}x,\ b{=}y,\ c{=}x \\ a{=}y{+}z,\ b{=}y,\ c{=}z. \end{cases}$

For the example $x = 3,\ z = 5,\ y = 8$ our solutions are: $a = 13,\ b = 8,\ c = 5$ and $a = 11,\ b = 8,\ c = 3$. Matching with our earlier fixed-sum results, for $a + b + c = x + y + z$ it must hold that $2y + 2(x\ or\ z) = x + y + z \implies y = (z\ or\ x) - (x\ or\ z) = x + z \implies (x\ or\ z) = 0$.

4 Conclusion

The puzzle stated by Matt Parker is as follows: find three starting numbers, not summing to 14, which eventually reaches a state T_n such that the sum of any T_{n+m} equals 14. Given our findings on fixed-sum iteration, our values at T_{n+m} must be $7,\ 7,\ 0$.
Any starting values of the form $a, a-7, a-7$ or $a, a, a-7$ will produce $7,\ 7,\ 0$ at the next step.
We can also reach $7,\ 7,\ 0$ arbitrarily many steps from our starting values by making use of the principles derived above (e.g. $T_0 = 7 * (n+1),\ 7 * n,\ 7$).