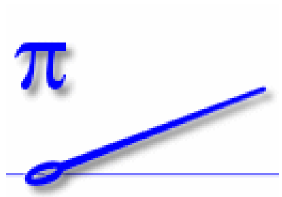
National Cheng Kung University

Mechanical Engineering

Stochastic Dynamic Data - Analysis and Processing

Buffon's needle – π estimation



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Contents

1 Introduction

1.1 Buffon's needle problem

Buffon's needle is a famous problem provided in the 18th century. Interestingly, we discovered that the deterministic value π can be estimated by stochastic process.

Problem description Suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?

1.2 Monte Carlo Method

Monte Carlo Method is a numerical method widely used in engineering and scientific field. It is a useful method for simulating systems with many coupled degrees of freedom, such as fluids and disordered materials.

We would implement this way in the MATLAB©program.

2 Problem Description

We have three random variables(uniform distribution) to determine the position of needles.

- $x \sim U(0, a)$
- $y \sim U(0, a)$
- $\theta \sim U(-\pi, \pi)$

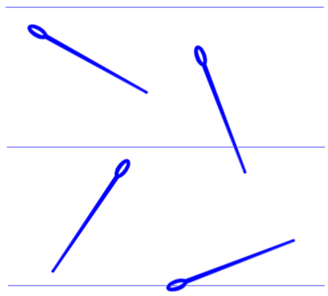

Figure 1: Scattering of Needles

2.1 Assumption

We do several assumption to simplify the derivation and use these condition in the program.

- Needle length l is always **less than or equal to the gap width** d, in other words, we do not consider the long needle situation
- I suppose the area is square and **5 times** of length d, just for computational efficiency
- Grid is parallel and aligned in the y direction and the center of needle to the nearest grid is called s
- The position and angle are **independent**

2.2 Probability distribution function

Based on the assumption, we can simplify and model the question. Reduce three variables(x, y and θ) to shortest distance and angle(s, θ).

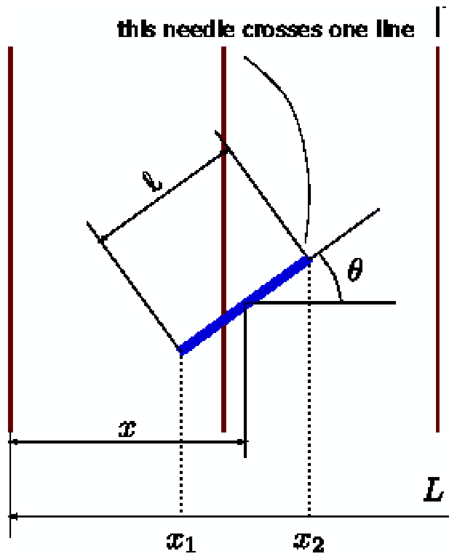

Figure 2: FBD of needle

The distribution of these two random variables.

$$Prob(s) = \begin{cases} \frac{2}{d} & when\ s \in [0, \frac{d}{2}] \\ 0 & otherwise \end{cases}$$

$$Prob(\theta) = \begin{cases} \frac{2}{\pi} & when\ \theta \in [0, \frac{\pi}{2}] \\ 0 & otherwise \end{cases}$$

Due to independent of two variables, we can find joint probability density function by multiplying them.

$$p = \begin{cases} \frac{4}{\pi d} & when\ s \in [0, \frac{d}{2}], \theta \in [0, \frac{\pi}{2}] \\ 0 & otherwise \end{cases}$$

Integrating the joint probability density function gives the probability that the needle will cross a line.

$$P = \int_{\theta=0}^{\pi/2} \int_{s=0}^{l/2 \sin\theta} p \; ds d\theta \tag{1}$$

$$P = \int_{\theta=0}^{\pi/2} \int_{s=0}^{l/2 \sin\theta} \frac{4}{\pi s} \; ds d\theta = \frac{2l}{\pi d} \tag{2}$$

We use this formula estimate pi by the following relationship

$$\hat{\pi} = \frac{2l}{Pd} = 2(\frac{l}{d}) \times \frac{number\ of\ needles}{needles\ touched\ line} \tag{3}$$

Here $\frac{l}{d} \in (0, 1]$ ratio is an important parameter ,we reassign it as L/D ratio , when we simulate. It relates to the rate of convergence. Intuitively, the ratio goes larger means needles become longer. In that case, needles tend to touch the lines.

2.3 Define Monte Carlo algorithm

We defined the domain of the problem and now providing possible inputs. On

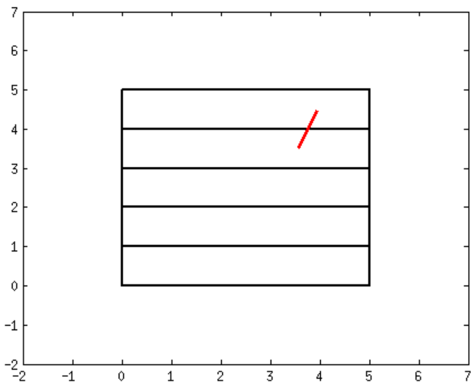

Figure 3: Count (Red)

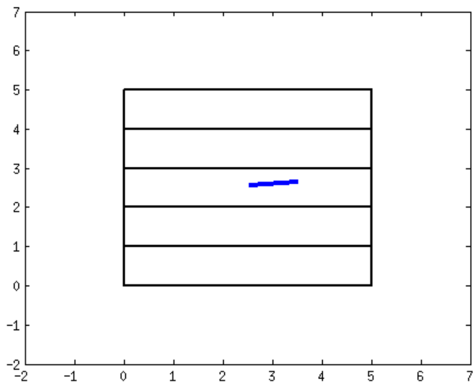

Figure 4: Not count (Blue)

the left hand side, the needle crossed the line, that is an available count. On the opposite side, three needles are in the interval of lines, we define those needles are unavailable in Monte Carlo algorithm.

3 Estimation of π

Goal Plot and discuss your results of the estimated π with respect to different numbers of test, N, L and D.

Simulation Steps

1. Visualize the problem

2. Use $L/D = 1$ to find the number of needles can converge to π with tolerance under ± 0.01

3. Change the L/D ratio and find the relation between ratio and number of needles

3.1 function BuffonsNeedle()

Before analyzing the problem, I write a program followed the rule mentioned above to estimate π and plot the result.

Buffon's needle

```
 %% Buffon's needle
% N : numbers of needles
% a : length of space edge, but not a variable
% we suppose the edge equals to 5 times of d
% d : line spacing
% l : length of needle
function est_pi = BuffonsNeedle(N,d,l)
if l > d
    error('In␣the␣program,l␣must␣less␣or␣equal␣to␣d') ;
   % random variables
end
     x = rand(N,1)*5*d ;
     y = rand(N,1)*5*d ;
     theta = rand(N,1) ;

     dx = abs( .5*l*cos( 2*pi*theta ) );
     dy = abs( .5*l*sin( 2*pi*theta ) );
     xn = [ x+dx , x-dx] ;
     yn = [ y+dy , y-dy ] ;
```

```
    yd = floor(yn/d) ;
    isTouch = yd(:,1) - yd(:,2) ;
    r = sum( isTouch ) ;
    p = r / N ;
    % return the estimation value of pi
    est_pi = 2*l/p/d ;
    %% plot
    if 1
        figure() ;
        % grid
        fill([0 5*d 5*d 0],[0 0 5*d 5*d],'k',...
               'FaceColor','none','EdgeColor','k','linewidth',2)
        axis([-2 ,5*d+2, -2, 5*d+2])
        hold on
       for i = 1 : 5
           plot([0 5*d],i*d*ones(1,2),'k','linewidth',2)
       end
       % needles
       for i = 1:N
           if isTouch(i)
               plot(xn(i,:),yn(i,:),'r','linewidth',1.5)
           else
                plot(xn(i,:),yn(i,:),'b','linewidth',1.5)
           end
       end
    end

  end
}
```

The result of tossing 100 needles is showed below.

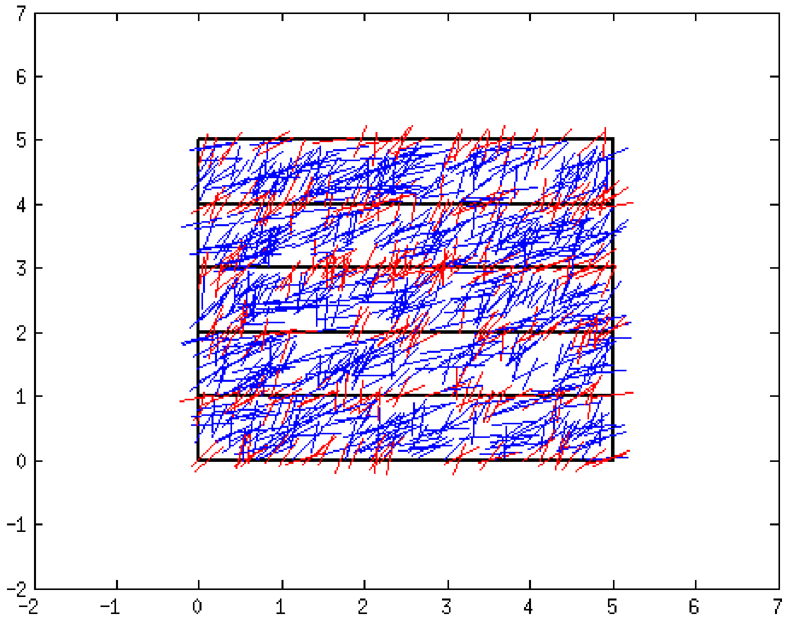

Figure 5: Scattered Needles,$\hat{\pi} = 3.318$

In fact, it really hard to acquire information from the graph. So I do another graph to indicate the number of needles versus the estimation of π value.

I use semi-log axis as different scattered needles number. The Red line is the π assigned by MATLAB©and the Blue line is $\hat{\pi}$ generated by the function.

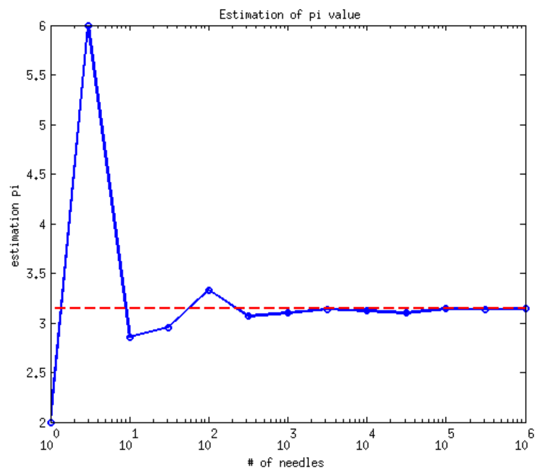

Figure 6:

The both graphs illustrate that the error can significantly reduced when needles number exceeds 10^3. I want to find the precise N that $\hat{\pi} \to \pi$ with tolerance ± 0.01

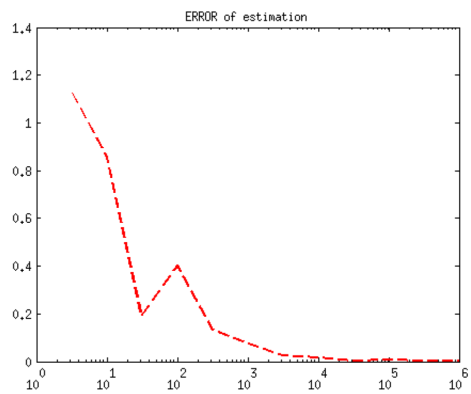

Figure 7:

3.2 Relation between L/D and N

We can define these two parameters with physical meaning.

- L/D ratio : An index of effective tossing. Larger of the ration, more plausible needles would touched the lines.
- N number of needles : An index of convergence rate. Less needles can reach the tolerance indicates high rate of convergence.

I use the function called $nConvergence()$ to find N when the error is less than ± 0.01. The reason I choose the tolerance equals to 0.01 is that we usually use π as 3.14 in application. So I pick the precision to second digit after the decimal point. The following simulation I would run the program a hundred times and average the numbers of N.

nConverge()

```
%% this program start from n ~ 1e3 to find pi estimation
% set tolerance is 1%, as to say, pi_hat ~ 3.14 +- error
% with input L/D, tolerance ratio
    n = 5e2; step = 1e2;
    tol = abs( pi - fastBuffonsNeedle(1,1,r) ) ;
    while tol > error
        tol = abs( pi - fastBuffonsNeedle(n,1,r) ) ;
        n = n + step ;
    end
end
}
```

3.3 Varying L/D ratio testing

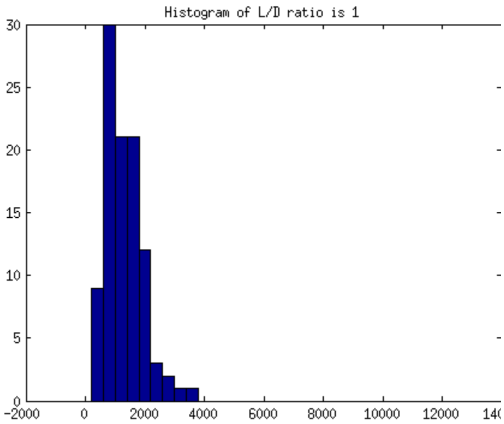

Figure 8: $D/L = 1, \bar{N} = 1353$

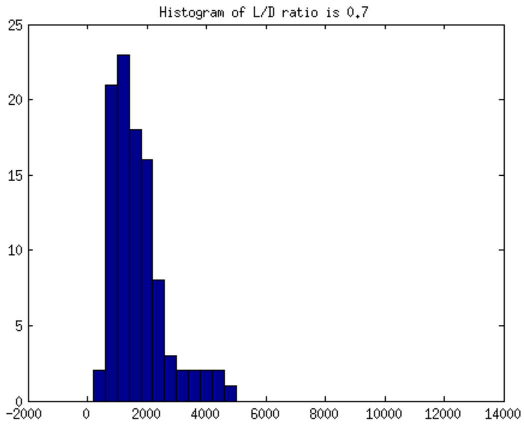

Figure 9: $D/L = 0.7, \bar{N} = 1750$

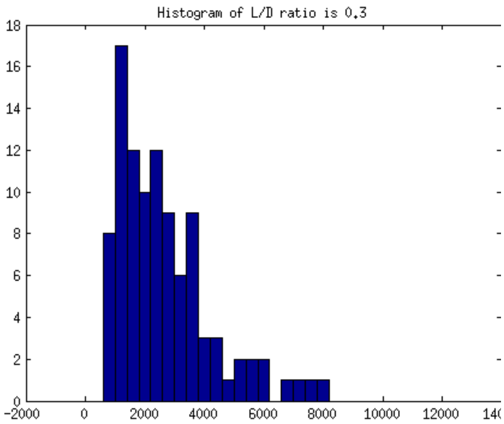

Figure 10: $D/L = 0.3, \bar{N} = 2714$

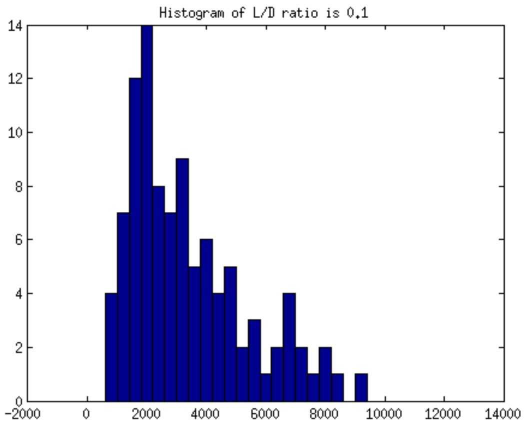

Figure 11: $D/L = 0.1, \bar{N} = 3447$

These histograms show us that as L/D goes higher, the N goes smaller. Just fit our intuition. It makes sense that more often effective tossing occurs, more rapid the estimation converge.

# graph	L/D	$\bar{N}$
$1st$	1.0	1353
$2nd$	0.7	1750
$3rd$	0.3	2714
$4th$	0.1	3447

4 Discussion - Cost analysis

Objective Can we find the quickest way to get $\hat{\pi}$?
I think the Buffon's Needles is a fabulous and insightful problem. It is amazing that the deterministic value π can be estimated by stochastic process. I also want to know from the formula below.

$$\hat{\pi} = \frac{2l}{Pd} = 2(\frac{l}{d}) \times \frac{number\ of\ needles}{needles\ touched\ line} \tag{4}$$

Are there "the best parameters" that can lead to the fastest convergence and find reasonable π?
To analyze this question, I write another program called $fastBuffonsNeedle()$ which is neat and more efficient to execute.

fastBuffonsNeedle()

```
function est_pi = fastBuffonsNeedle(N,rr)
     y = rand(N,1)*5;
     theta = rand(N,1) ;
     dy = abs( .5*rr*sin( 2*pi*theta ) );
     yn = [ y+dy , y-dy ] ; yn= floor(yn) ;
     yn(:,1) = yn(:,1) - yn(:,2) ; r = sum( yn(:,1) ) ;
     % return the estimation value of pi
      est_pi = 2*rr*N/r ;
   end
}
```

Environment Setting Changing the ratio L/D with step 0.025 from $0 \sim 1$, and each L/D ratio runs 100 times and find its average as time consumption.

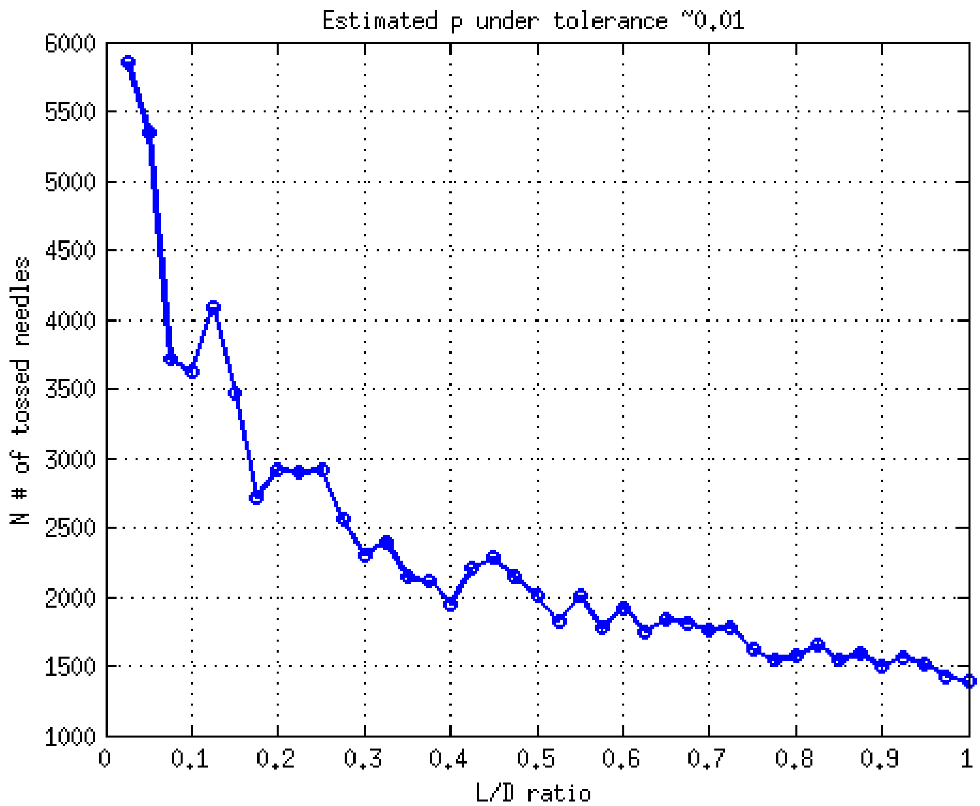

Figure 12: Material cost and Time consumption

Conclusion Obviously, the nonlinear result can be predicted as the formula. In **engineering viewpoint(if it is a real problem),** L/D **ratio means material cost and** N **stands for time cost.** If we are constrained by both factors, we can just take L/D ratio around 0.3. That we can get dramatically decrease of time consumption.(it resembles to the picture of time constant)

The problem owns a beautiful pattern.

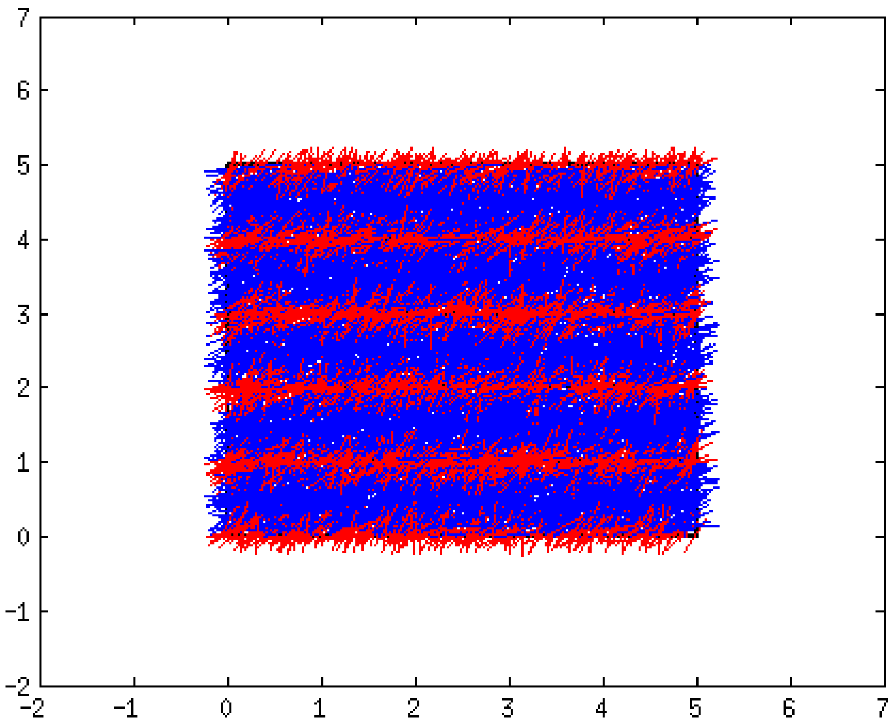

Figure 13: When $\hat{\pi} = 3.142024$

References

[1] MATLAB©: Users guide. Massachusetts: The Math Works Inc., 2009.

[2] Wikipedia : Monte Carlo method, 2012

[3] Wikipedia : Buffon's needle, 2012