Wavelet Theory & Application

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Contents

1 Introduction

The report can be distinguished with two major parts. First part is my notes of wavelet theory. It serves the quick review for myself. The second part is dealing with some practical problem.

Part I

Theory of Wavelet

2 Denoising Problem Formulation

A common posing of the denoising problem is as follows. Suppose that there are n noisy samples of a function f:

$$y_i = f(t_i) + \sigma\epsilon_i, \quad i = 1...n$$

where ϵ_i are $N(0,1)$ and the noise level σ may be known or unknown. The goal of denosing is find $\hat{f}$ which satisfies:

$$\hat{f} = min||\hat{f} - f||_2$$

3 Wavelet Thresholding

The orthogonality of DWT (assuming that orthogonal wavelets are used with periodic bound- ary conditions) leads to the feature that white noise is transformed into white noise.

$$y_{jk} = w_{jk} + \sigma\hat{\epsilon_{jk}}$$

where w_{jk} are the clean wavelet coefficients of $f(t_i)$, consisting of the approximation and detail and coefficients of $\hat{\epsilon_{jk}}$ and $N(0,1)$. Unfortunately, due to limited time, I have to stop here.

Part II

Wavelet Application

4 Wavelet Packet

Wavelet Packets is a wavelet transform where the discrete-time (sampled) signal is passed through more filters than the discrete wavelet transform. We could compare the difference between two of them. Each level we only decompose the

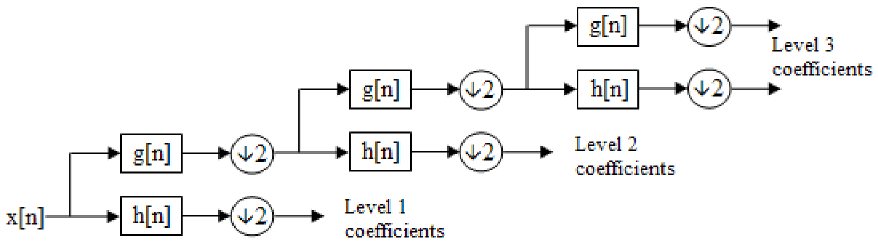


Figure 1: Wavelet Decomposition

low frequency part. The high frequency components are abandoned and energy could loss.

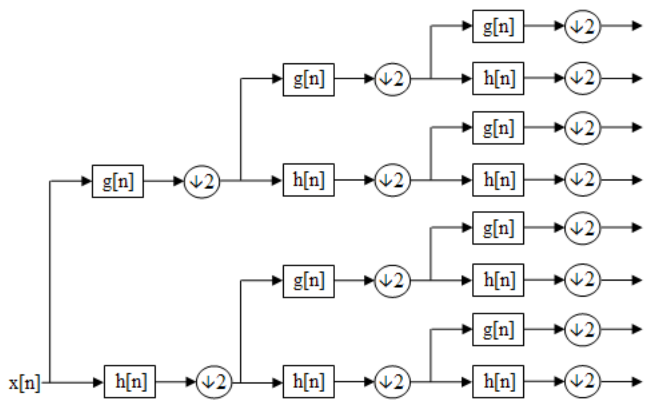


Figure 2: Wavelet Packet Decomposition

To save this, Wavelet packet executes whole decompositions. It completely conserve the energy and uncover the whole information from signal.

4.1 Best Tree Algorithm

Wavelet packet indeed serve as a powerful tool that we observe the signal without reservation. However, in practical usage, we hope to reconstruct the signal like music and image efficiently. On 1992, Coifman and Wickerhauser provide a method to find the best basis. It finds a set of bases that provide the most desirable representation of the data relative to a particular cost function.
They define , mostly attractive, the cost function as the Shannon Entropy.

$$H(A) = -\sum p_i \log_2 p_i$$

The method is based on tree algorithm which can pick the expansion of lessor entropy and continue up to some maximum interval size. And the time complexity of algorithm is

$$O(N \log N)$$

It is really an excellent method to reconstruct wavelet packet.

4.2 Threshold Selection

1994, Donoho published a method called VisuShrink, which also called universal threshold.

$$\lambda^{univ} = \sigma\sqrt{2ln(N)}$$

And define risk function as

$$R(t) = \frac{1}{N}||\hat{f} - f||^2$$

After transforming into wavelet domain, we find the optimal solution

$$arg\ min\ ER(t)$$

When N approaches infinity, the median absolute deviation of the highpass highpass values values converges converges to $.6745\sigma$. So the estimate for σ is $MAD(d)/.6745$. (Where σ is deviation of white noise)

We will apply the theorem mentioned above. In Matlab, there is an function helps us select the threshold which is the **Birge-Massart Strategy.**
The Birge-Massart strategy is based on results on adaptive functional estimation in regression. It uses level-dependent thresholds obtained by the following wavelet coefficients selection rule.
Let j_0 as decomposition level, m be the length of approximation and α is an integer larger than 1. These parameters define the strategy :

- at level $j_0 + 1$, everything is kept.
- for level j from 1to j_0, the k_j larger coefficients in absolute value, are kept with: $k_j = m/(j_0 + 1 - j)^\alpha$

Typically the parameter α is equal to 1.5 for compression and α is equal to 3 for de-noising.

5 Signal Denoising

We decompose the signal and denoise them with specific threshold. Then reconstruct and compare with the original signal.

5.1 Signal Specifications

In the whole report, we use the data with

- Sampling Rate : 1 Hz
- Data Length : 1,000
- Noise : Gaussian white noise with zero mean and unity deviation

Signal Type	Characteristic	SNR
Sine with noise	Amplitude = 10, frequency = 0.003 Hz	17.68 dB
Polynomial function with noise	ramp with slope 0.01 $y(t) = 0.01t$	9.62 dB

Table 1: Specs of the Signal

5.2 Wavelet Packet

We pick two kinds of WPs

- Haar Wavelet : level 7
- Daubechies Wavelet 5 : level 5

For some simple reasons that Haar wavelet is the most simple one and Daubechies wavelets perform well in many situations.

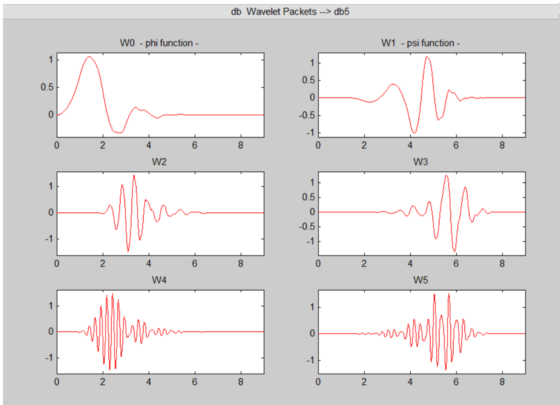


Figure 3: Daubechies Wavelet level 5

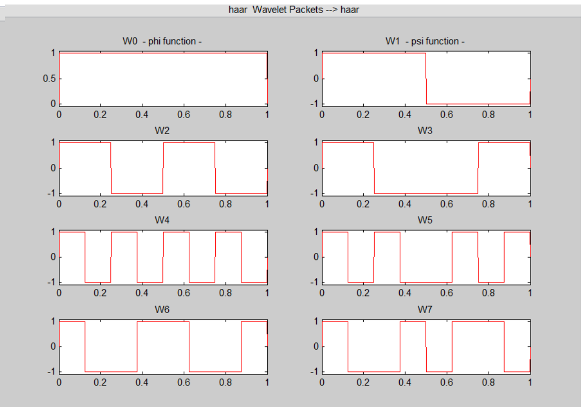


Figure 4: Haar Wavelet level 7

5.3 Best Tree

We observe that the decomposition tree of contaminated sine wave.

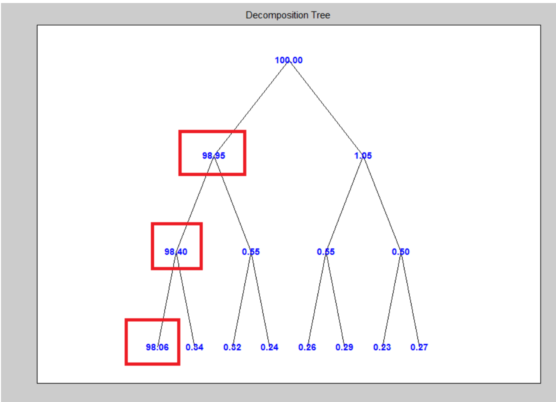


Figure 5: Energy of 5 level Tree

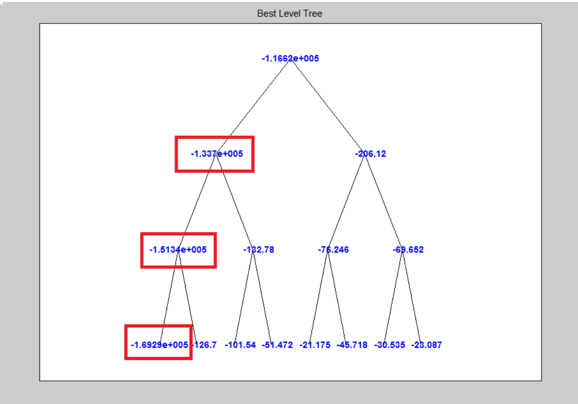


Figure 6: Entropy of 5 level Tree

We find that the right-hand side branch contains higher energy and lower entropy.

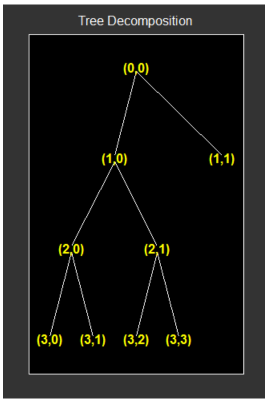


Figure 7: Best Tree

It means they have more information and we could reconstruct the signal with them. And the best tree algorithm gives the predictable results.

5.4 Sine Wave Denoising

5.4.1 The Original Signal

The signal before processed is contaminated with Gaussian white noise.

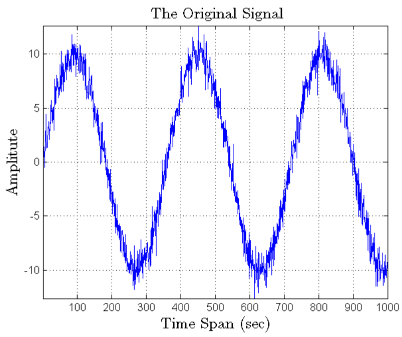


Figure 8: Sine with AGWN

5.4.2 Denoised Signal and Extracted Noise

I denoise the signal with Matlab function *wpdencmp*().

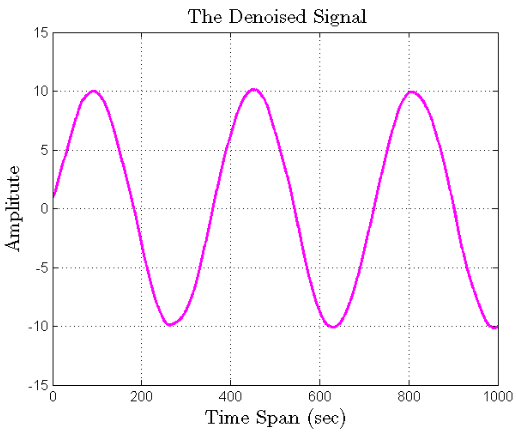


Figure 9: With Daubechies WP level 5

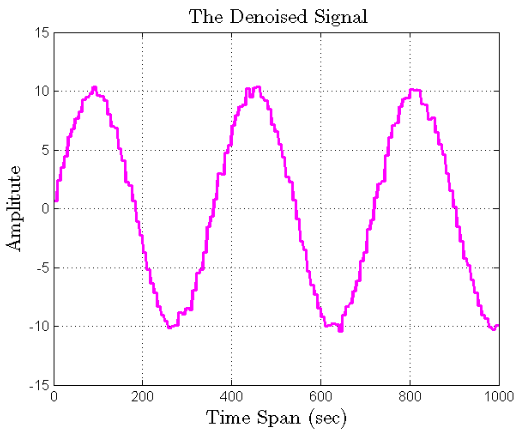


Figure 10: With Haar WP level 7

If we compare with the original signal and estimate the SNR.

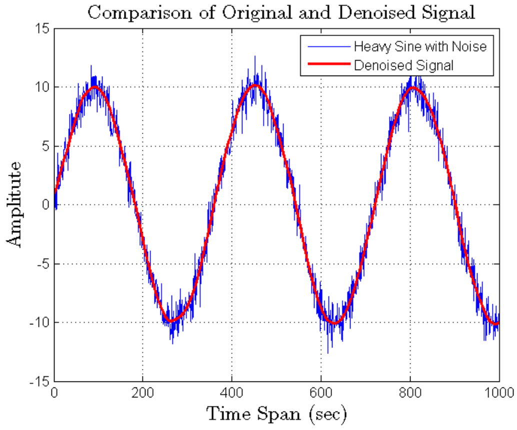


Figure 11: With Daubechies WP level 5

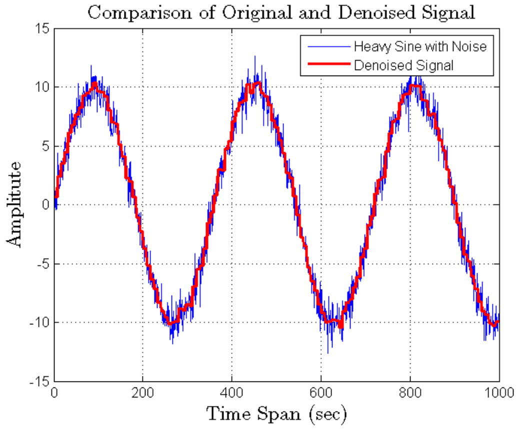


Figure 12: With Haar WP level 7

WP type	Before Denoising	After Denoising
Haar	17.68 dB	17.02 dB
Daubechies	17.68 dB	16.98 dB

Table 2: Comparison of Signal-Noise Ratio

5.4.3 Statistical Properties of Extracted Noise

We extract the noise by substracting the denoised signal and could get.

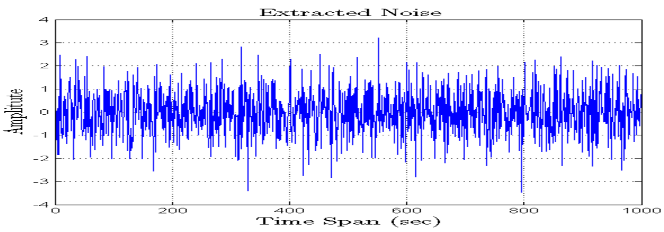
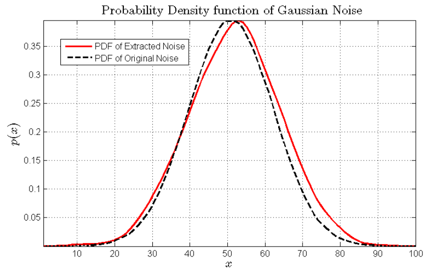


Figure 13: Extracted Noise

We examine its statistics properties with PDF



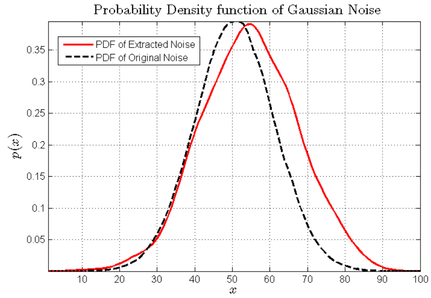
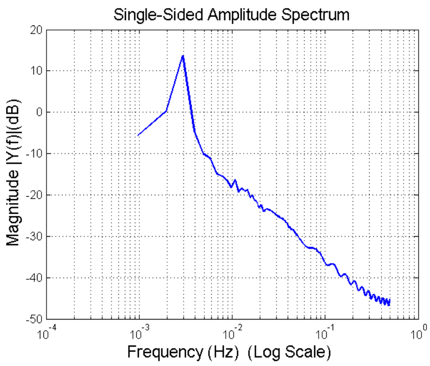


Figure 14: PDF With Daubechies level 5 Figure 15: PDF With Haar WP level 7

WP type	Mean Value	Standard Deviation
Haar	0	0.9989
Daubechies	-0.0039	1.0029

Table 3: Specifications of Extracted White Noise



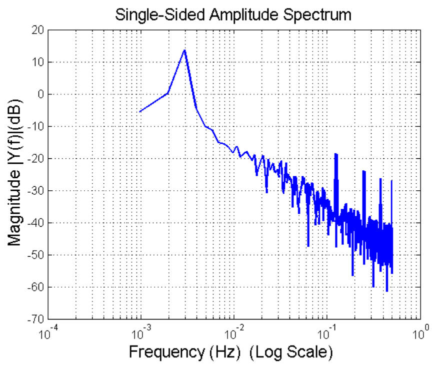


Figure 16: Spectrum With Daubechies Figure 17: Spectrum With Haar

5.4.4 Comparison of Denoised and Original Signal

We could find that the performance of wavelet packet is terrific. The sine wave is almost the same as uncontaminated.

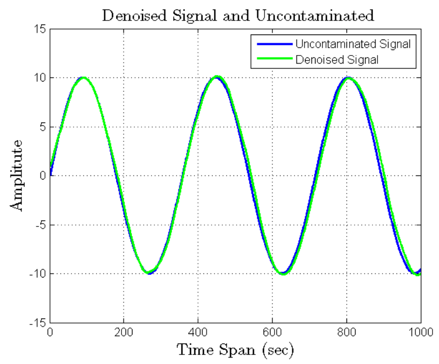


Figure 18: With Daubechies Wavelet

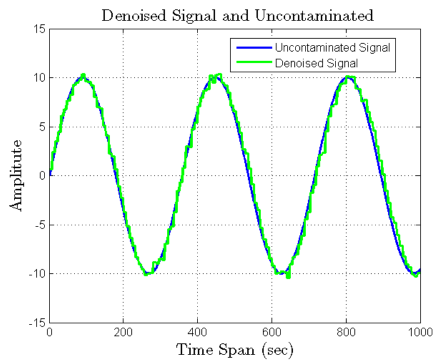


Figure 19: With Haar Wavelet

Even in the worse case, the Haar wavelet packet still provide an excellent result.

5.5 Ramp signal Denoising

The signal before processed is contaminated with Gaussian white noise.

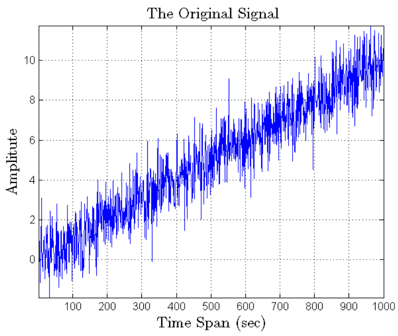


Figure 20: Ramp with AGWN

5.5.1 Denoised Signal and Extracted Noise

I denoise the signal with Matlab function *wpdencmp*().

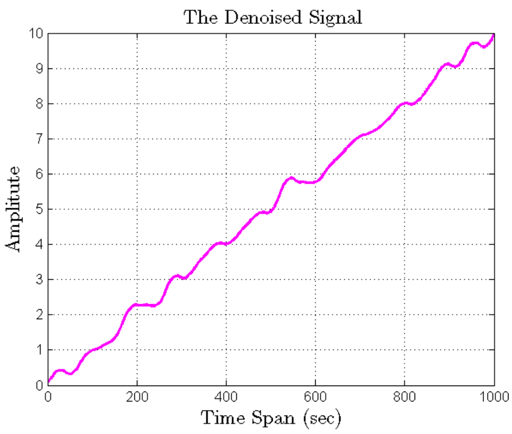


Figure 21: With Daubechies WP level 5

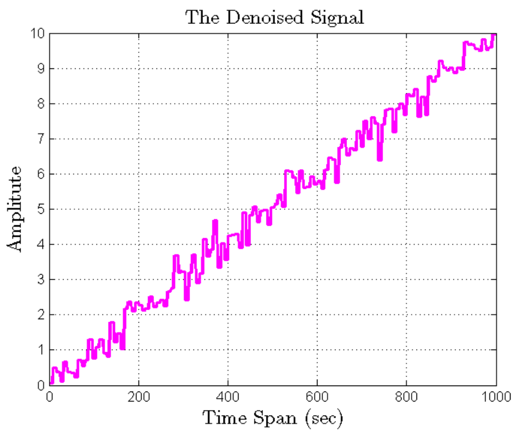


Figure 22: With Haar WP level 7

If we compare with the original signal and estimate the SNR.

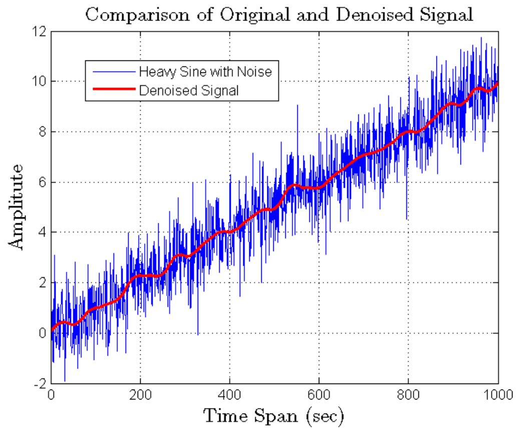


Figure 23: With Daubechies WP level 5

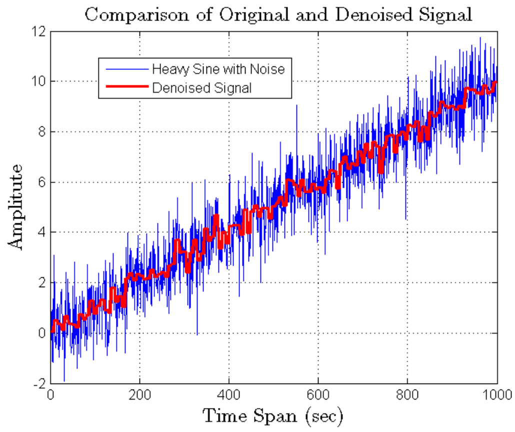


Figure 24: With Haar WP level 7

WP type	Before Denoising	After Denoising
Haar	9.62 dB	9.49 dB
Daubechies	9.62 dB	9.16 dB

Table 4: Comparison of Signal-Noise Ratio

5.5.2 Statistical Properties of Extracted Noise

We extract the noise by substracting the denoised signal and could get.

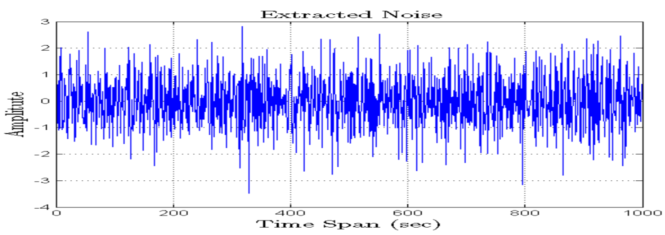
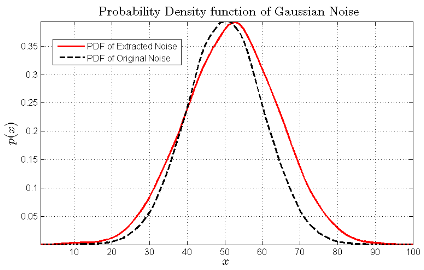


Figure 25: Extracted Noise

We examine its statistics properties with PDF



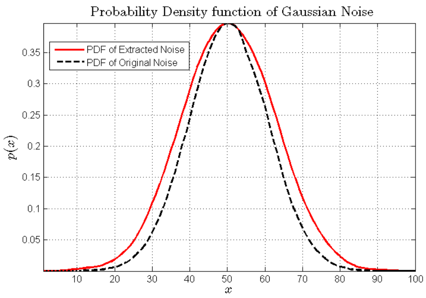


Figure 26: PDF With Daubechies level 5 Figure 27: PDF With Haar WP level 7

WP type	Mean Value	Standard Deviation
Haar	0	0.95
Daubechies	0	1.005

Table 5: Specifications of Extracted White Noise

5.5.3 Comparison of Denoised and Original Signal

The left-hand side, Haar WP gives a worse result but the trend still can easily be observed.

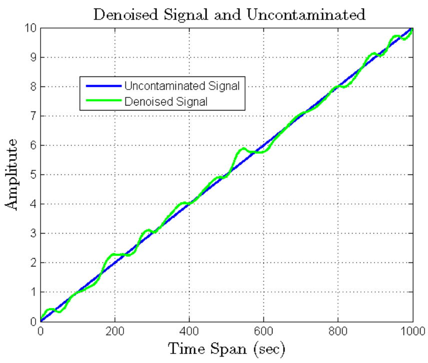


Figure 28: With Daubechies Wavelet

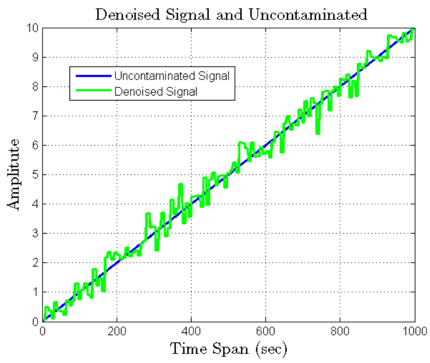


Figure 29: With Haar Wavelet

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