Wavelet Theory & Application

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Contents

1 Introduction

The report can be distinguished with two major parts. First part is my review notes of wavelet theory, based on a test book called **WAVELET THEORY** written by David K. Ruch and Patrick J. Van Fleet. The whole book emphasized on basic theory, inevitably, not match with the simulation of second part.

Part I

Theory of Wavelet

The wavelet is a powerful tool in signal processing, especially in image processing.

2 Haar Space

2.1 The space $L^2(\mathcal{R})$

It is a famous space, many transforms need to satisfy the space could exist. Like Fourier transformation. The space makes sure that function square-integratable. It means the rate of the kernel function decay should fast enough to resure the energy of signal is finite. By the way, functions in the space are not necessarily be continuous.

2.2 The Haar space V_0

A digital image is comprised of pixels. Graylevel ranges from 0 to 255. In particular, we define a function as this form

$$\phi(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

We need to allow summation variable to range over $\mathcal{Z}$, but we need to put conditions on the space so the signals decay. We need that any element of out space is also an element of $L^2(\mathcal{R})$.

We define the space

$$V_0 = span\{\phi(t-k)\}_{k\in\mathcal{Z}} \cap L^2(\mathcal{R})$$

We call V_0 the Haar space, generated by Haar function $\phi(t)$.

3 General Haar Space V_j

We could define a more general case

$$V_j = span\{\phi(2^j t-k)\}_{k\in\mathcal{Z}} \cap L^2(\mathcal{R})$$

And its bases are

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j t-k)$$

The space has great properties like Orthogonality, Nested property and Separation property. Otherwise, we have an important proposition formula

$$P_{g,j}(t) = \sum_{k\in\mathcal{Z}} < \phi_{j,k}(t), g(t) > \phi_{j,k}(t)$$

The projection formula has lots of to do with approximation and detail functions.

4 General Haar Wavelet Space W_j

Wavelet function is defined

$$\psi(t) = \phi(2t) - \phi(2t-1) = \begin{cases} 1, & 0 \le t < 1/2 \\ -1, & 1/2 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

We define the space

$$W_0 = span\{\psi(t-k)\}_{k\in\mathcal{Z}} \cap L^2(\mathcal{R})$$

The Haar wavelet function is given, then we can define

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t-k)$$

5 Decomposition and Reconstruction

We can decompose $f_{j+1}(t) \in V_{j+1}$ into approximation function $f_j(t) \in V_j$ and a detail function $g_j(t) \in W_j$. Due to tight time, I have to stop here.

Part II

Wavelet Application & Simulation

In simulation, I choose the kinds of wavelets. Haar, Daubechies and Coiflet. I discussed and compare them in the "standard" process and the following process I use only one wavelet to analyze.

6 Wavelets Family

6.1 Haar Wavelet

The most simple and basic one of wavelets. It is orthogonal and supported (some terminology in set theory).

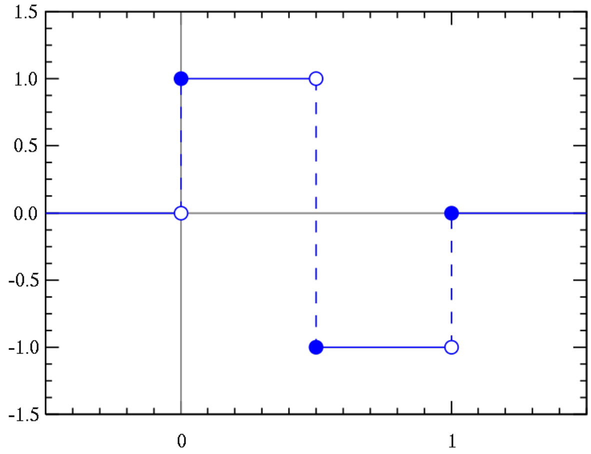

Figure 1: Haar Wavelet

6.2 Daubechies Wavelet

Named after an important scientist, Ingrid Daubechies. Daubechies wavelets are widely used in solving a broad range of problems. According to Wiki pedia's examples, e.g. self-similarity properties of a signal or fractal problems, signal discontinuities, etc. We mainly use Daubechies Wavelet in the whole report.

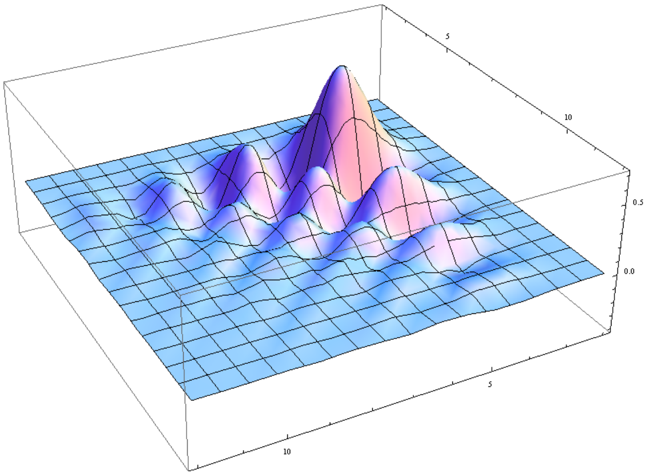

Figure 2: Daubechies Wavelet in 2D

6.3 Coiflet

Coiflet is still designed by Ingrid Daubechies, at the request of Ronald Coifman, a professor in Yale University. The coiflet is near symmetric.

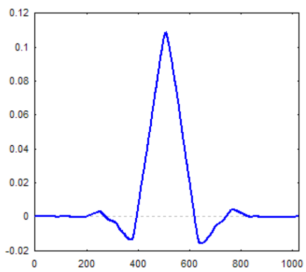

Figure 3: Coiflet

7 Sinusoidal Wave with Gaussian White Process

I prefer using this case as "standard" and check the simulation due to its clarity and predictability.

7.1 Process Specifications

In the report, we use thoroughly

- Sampling Rate = 1 Hz
- Data Length = 1000

Deterministic Signal	Sine	f = 0.003 Hz	Amplitude = 10
Stochastic Signal	Gaussian Pseudo Random	Mean = 0	Deviation = 1

Table 1: Specs of the Data

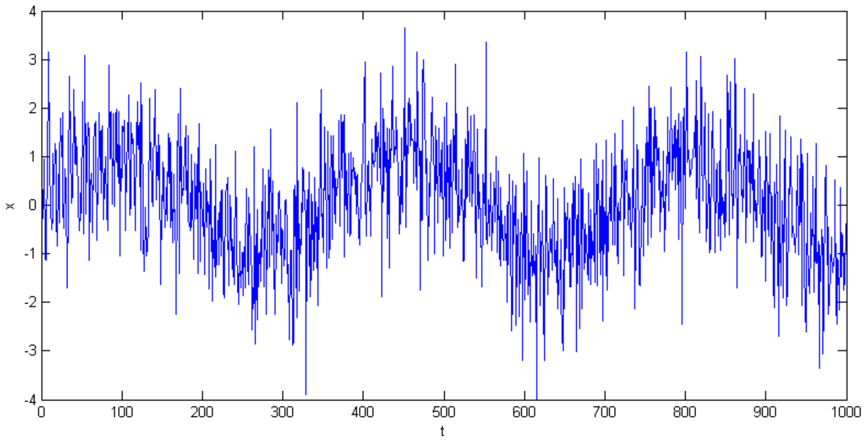

Figure 4: Sinusoidal Wave with Gaussian White Process

7.2 Wavelet Analysis

7.2.1 Decomposition

We first use Matlab toolbox to decompose signal. We all use level 5 decompose and find in this case, Daubechies looks similar to Coiflets, however Haar decomposition generate many discontinuities. The left hand-side and right hand-side individually show approximations and details. The lower level indicates higher frequency components. We can foresee that when we need to handle those coefficients when reconstructing.

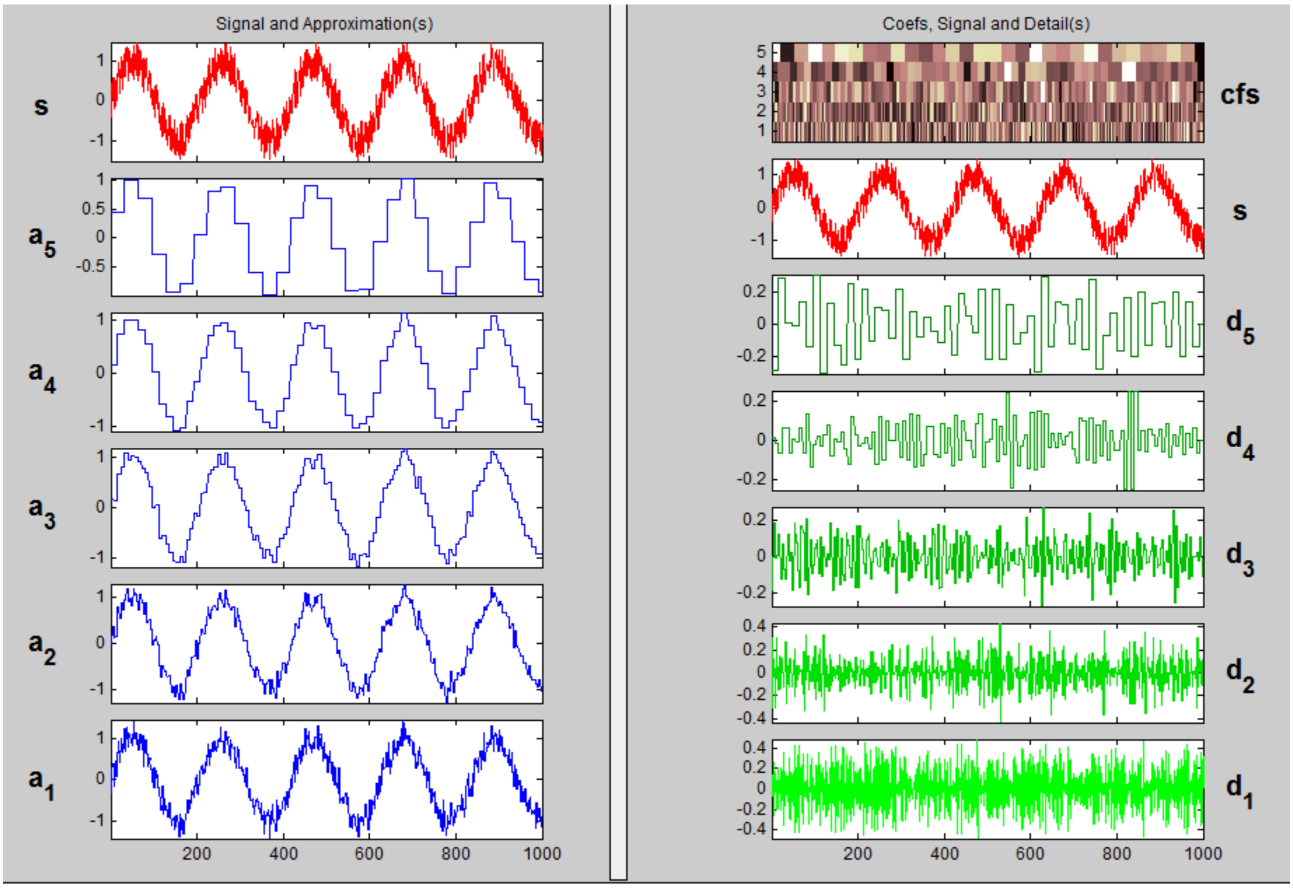

Figure 5: Level 5 decomposition of Haar Wavelets

7.2.2 Reconstruction

We first check the reconstruction is feasible, fit the theory and estimate how much information we loss.

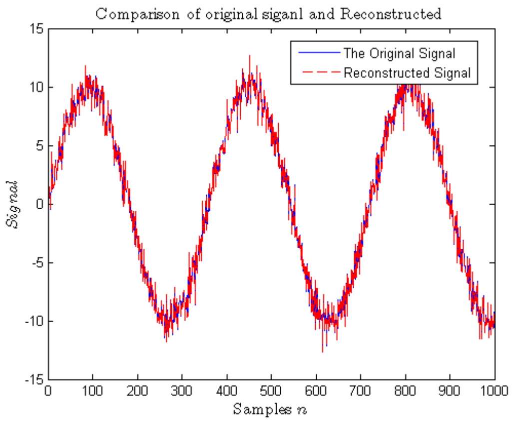

Figure 6: Compare Original and Reconstructed

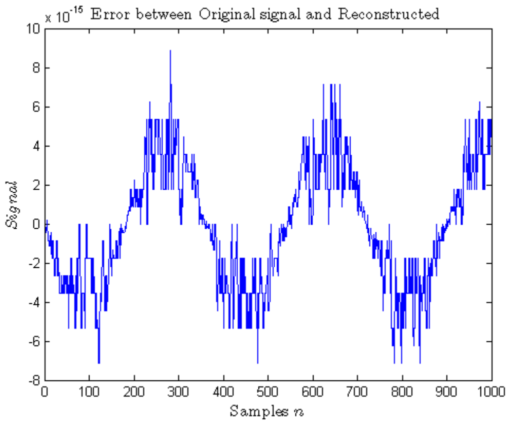

Figure 7: Error of the Reconstruction

The left one tells us that two signal are almost overlaped. And finding the simple error of two signal, it shows error in the interval of $[-7 \times 10^{-15} - 9 \times 10^{-15}]$. Such small scale gives us confidence of wavelet transform.

7.2.3 Signal Denoising

Signal denoising can be treated as one of the most significant application provided by wavelet. Here I discuss several benefits of wavelet transform.

- "Flexible" bases : We can select "best" wavelets function as our solution base, different with Fourier Transformation, only one kernel $e^{-j2\pi\omega t}$
- Multiresolution : Its property can capture precise characteristics of signal
- Low entropy : It means wavelets' coefficients relatively sparse that the information entropy is lower than traditional FIR filter
- De-autocorrelation : Wavelet transformation "whiten" the noise signal and make it easier to denoise

There are bunch of Matlab functions help us denoising. In Matlab, criteria of naming function follows that

- wv : choose Wavelet
- wp : choose Wavelet packet
- den : Denoise
- cmp : Compress

So, there functions like ddencmp(), wbmpen(),wpdencmp() and etc. I choose the simplest one as

$$wden()$$

which means wavelet denoising function. I want to discuss one of the parameters in the function. It is definite a threshold selection rule. The basic one is called **Stein's unbiased risk estimate (SURE)**. It is an unbiased estimator of the mean-squared error of "a nearly arbitrary, nonlinear biased estimator." In other word, it provides an indication of the accuracy of a given estimator. Another is heuristic SURE. It is kind of adaptive case from first option.

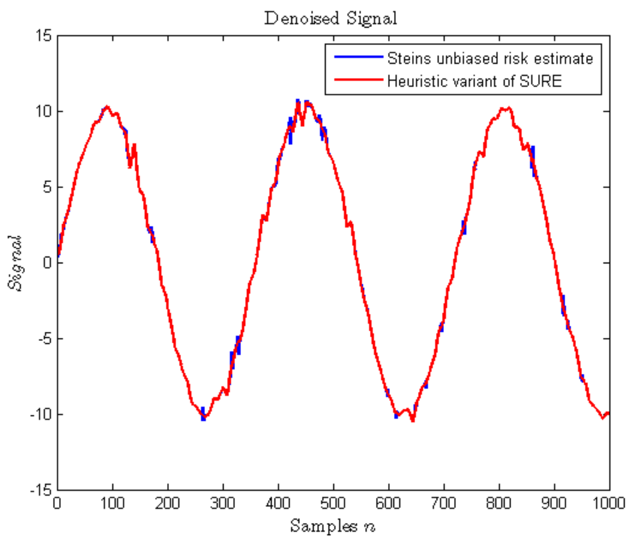

Figure 8: Comparison of SURE and heuristic SURE

We can find that heuristic SURE can preform better in most cases.

7.3 Statistical Estimation

In the last stage, we justify the four statistical estimations.

7.3.1 Mean value and Standard Deviation

- - -	Original Signal	Reconstructed Signal	Denoised Signal	Noise
Average $\hat{\mu_x}$	0.5068	0.5068	0.5137	-0.0069
Deviation $\hat{\sigma_x}$	7.1544	7.1544	7.0774	0.9578

Table 2: Specs of the Data

The last column indicates that we extract the noise with zero mean and unity deviation which perfectly fit the Gaussian noise we add at the first.

7.3.2 Autocorrelation Function

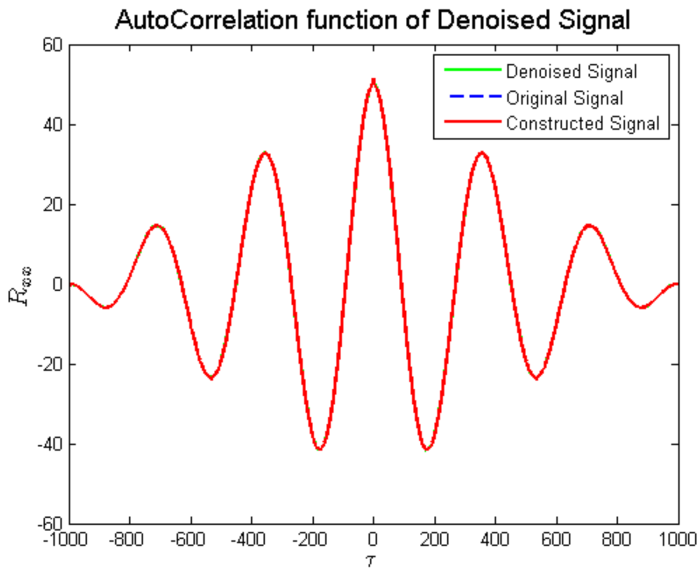

Figure 9: Autocorrelation of Original, Reconstructed and Denoised Signal

Surprisingly, these three signals' autocorrelation function coincidently overlap. It is too beautiful to believe. I use Matlab function biased mode and check the $R_{xx}(0)$ is correct.

I want to verify one of the benefits from wavelet transformation listed above. Which is wavelet transformation can whiten the noise and reduce the autocorrelation function. The verification is simple, I compare the autocorrelation between original noise and the noise by estimated. I suppose the noise model is

$$y(t) = s(t) + w(t)$$

where $s(t)$ is deterministic signal and $w(t)$ is Gaussian white and $y(t)$ is what we measured.

The estimated noise is

$$\hat{w(t)} = y(t) - s_d(t)$$

where $s_d(t)$ is denoised signal The result indicates that the noise is actually be

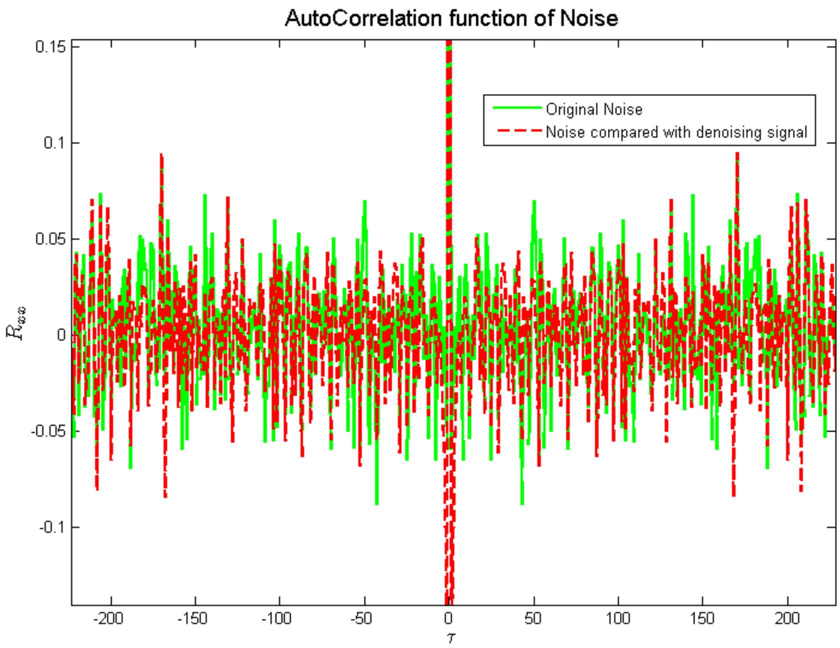

Figure 10: Autocorrelation of Original Noise and Whitened Noise

whitened! The profile of red line becomes smaller which means correlation decreases. The noise could be easier to remove because it is unrelated.

7.3.3 Probability density function

The PDF is estimated by matlab kernel smoother function. PDF of green comes

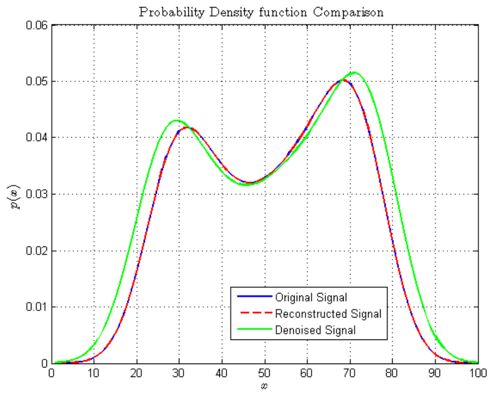

Figure 11: PDF of Original, Reconstructed and Denoised Signal

wider and higher. Maybe it's an interesting topic of nonstationary estimation.

7.3.4 Power Spectrum Density

The spectrum highly indicate the peak frequency of the sinusoidal wave. In the view point of energy conservation, it's easy to understand the green decays sooner.

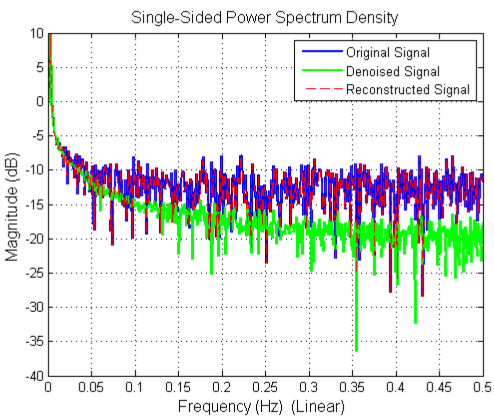

Figure 12: PSD of Original, Reconstructed and Denoised Signal

We want to see another case. We know that Haar wavelets are discontinuous. When the level is low, it becomes significant flunctuation. When low level of Daubechies would cause the same problem.

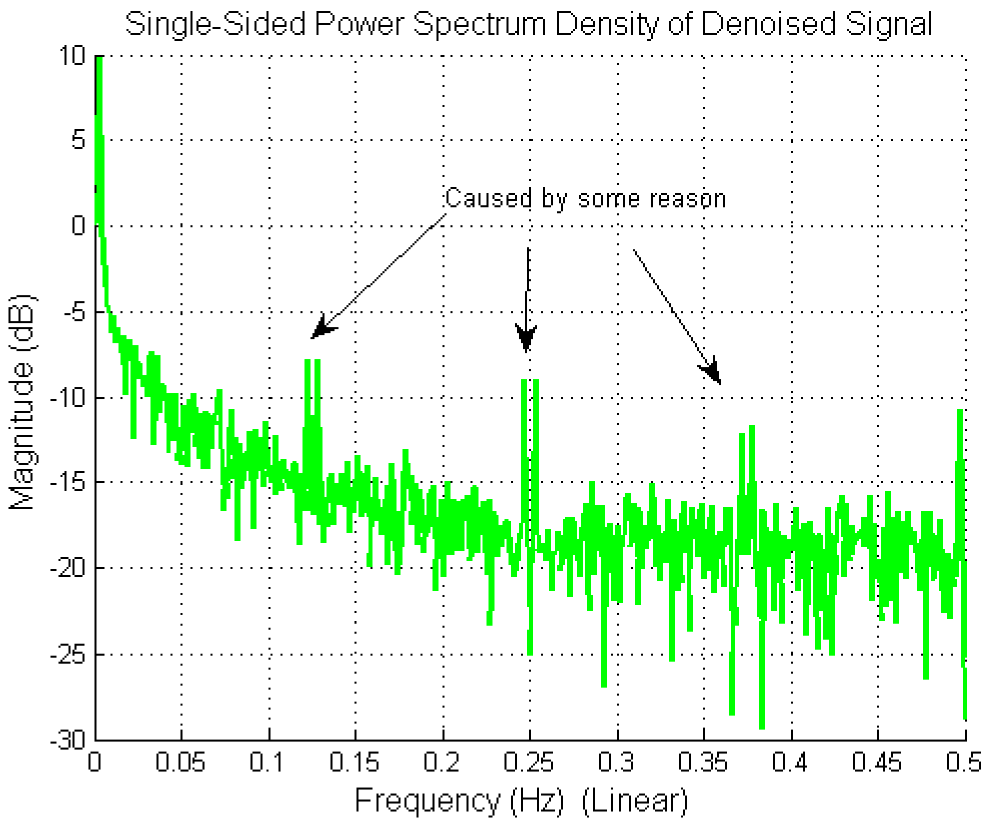

Figure 13: low level Haar Wavelets caused some confusing peaks

7.4 Trend Detection and SNR

In the end, we do extract the trend from contaminated signal. It is a sinusoidal wave with the frequency we design at first. The Signal-Noise Ratio

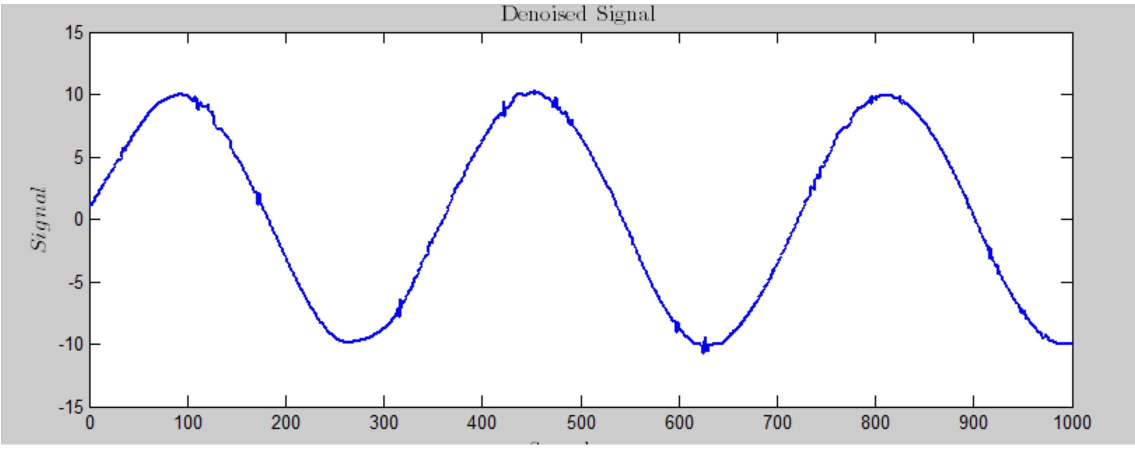

Figure 14: Trend Detection : Sinusoidal Wave

$$SNR = 10\log_{10}\frac{\sum s(t)^2}{\sum w(t)^2} = 17.4878\ dB$$

And with different wavelets, the SNRs are almost the same.

Wavelets	SNR
Haar	17.6453 dB
Coiflet	17.4054 dB
Daubechies	17.4878 dB

Table 3: SNR with different Wavelets function

7.5 "Old Ways"

If I found this problem just 1 or 2 weeks ago, what would I do? First, of course, find the spectrum and extract the main peak. Then, I would apply the narrow band-

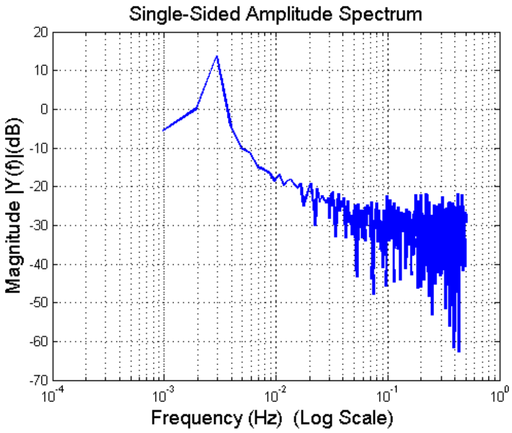

Figure 15: FFT finding Main Freq

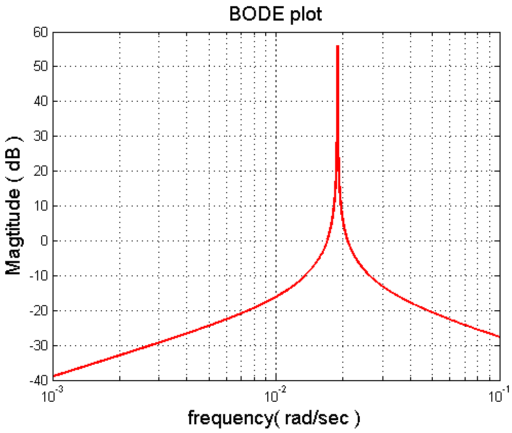

Figure 16: Narrow band pass filter

passed filter because I found the spectrum indicate the single peak. Unfortunately, I stuck. Because I do not know how to design a NBP filter with both performance and stability.

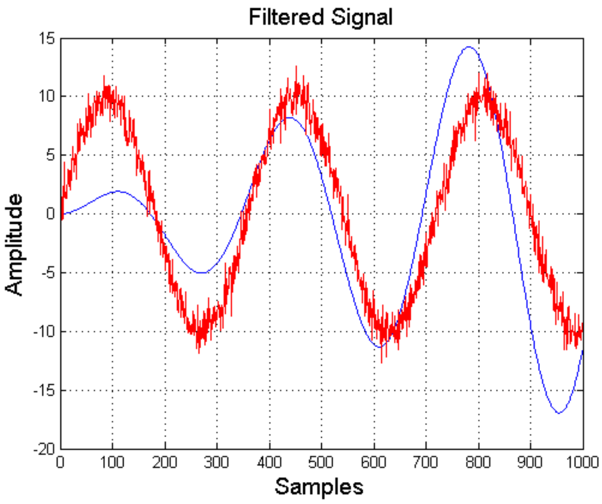

Figure 17: Filter the noise and get the trend

Finally, I design a horrible filter which can diverse any moment.

8 Gaussian White Process

In the report, we use thoroughly

- Sampling Rate = 1 Hz
- Data Length = 1000

8.1 Process Specifications

Stochastic Signal	Gaussian Pseudo Random	$\mu_x = 0, \sigma_x = 1$

Table 4: Specs of the Data

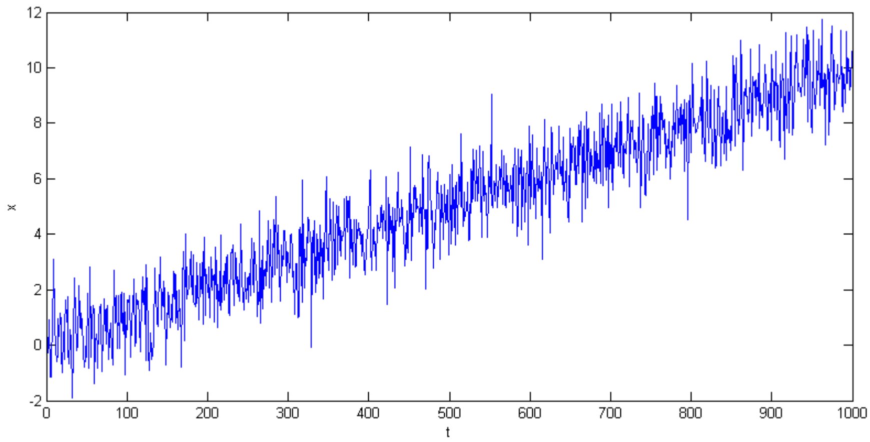

Figure 18: Gaussian White Process

8.2 Wavelet Analysis

8.2.1 Decomposition

We all use level 5 Daubechies decompose in this case. The left hand-side and right

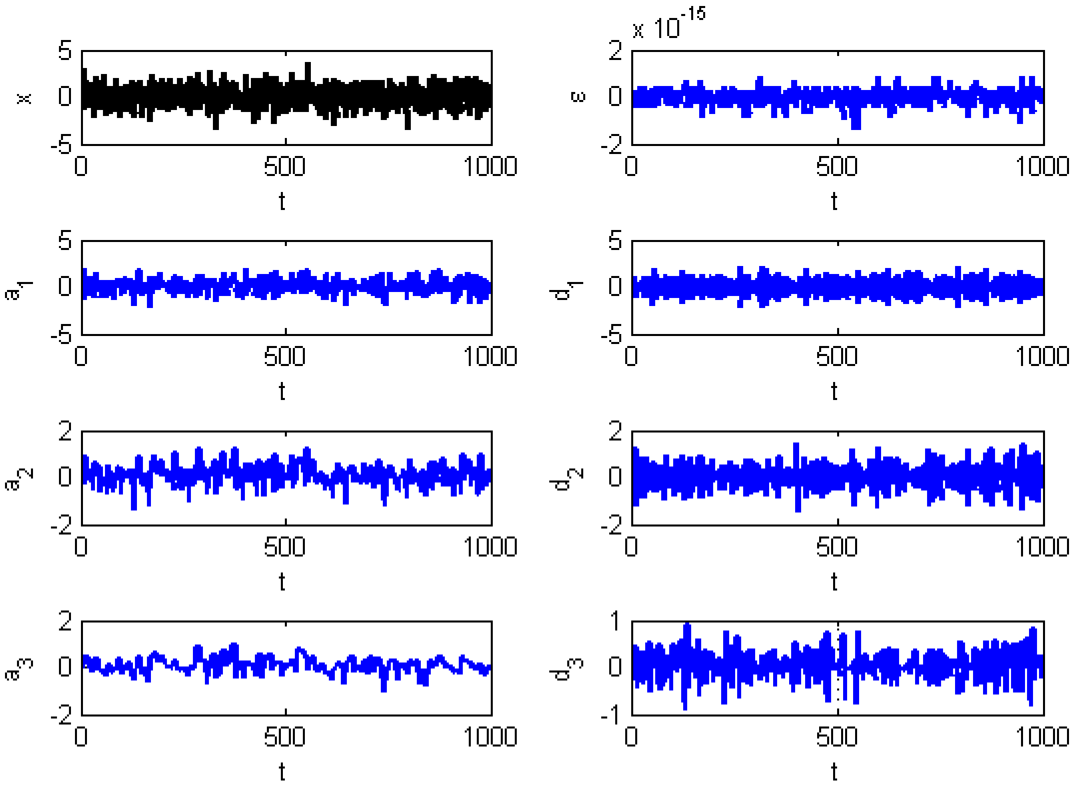

Figure 19: Level 3 decomposition of Haar Wavelets

hand-side individually show approximations and details. The lower level indicates higher frequency components.

8.2.2 Reconstruction

We first check the reconstruction is feasible, fit the theory and estimate how much information we loss.

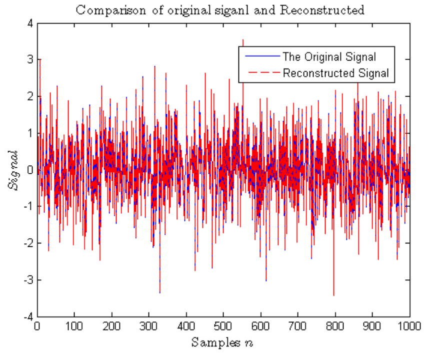

Figure 20: Compare Original and Reconstructed

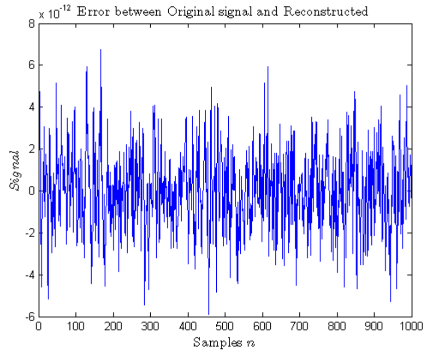

Figure 21: Error of the Reconstruction

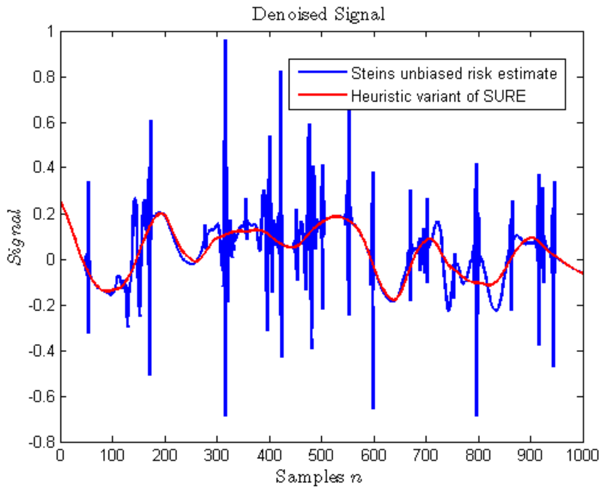

Figure 22: Comparison of SURE and heuristic SURE

We can find that heuristic SURE can preform better in most cases.

8.3 Statistical Estimation

In the last stage, we justify the four statistical estimations.

8.3.1 Mean value and Standard Deviation

- - -	Original Signal	Reconstructed Signal
Average $\hat{\mu_x}$	0.0284	0.0284
Deviation $\hat{\sigma_x}$	1.0116	1.0116

Table 5: Specs of the Data

The last column indicates that we extract the noise with zero mean and unity deviation which perfectly fit the Gaussian noise we add at the first.

8.3.2 Autocorrelation Function

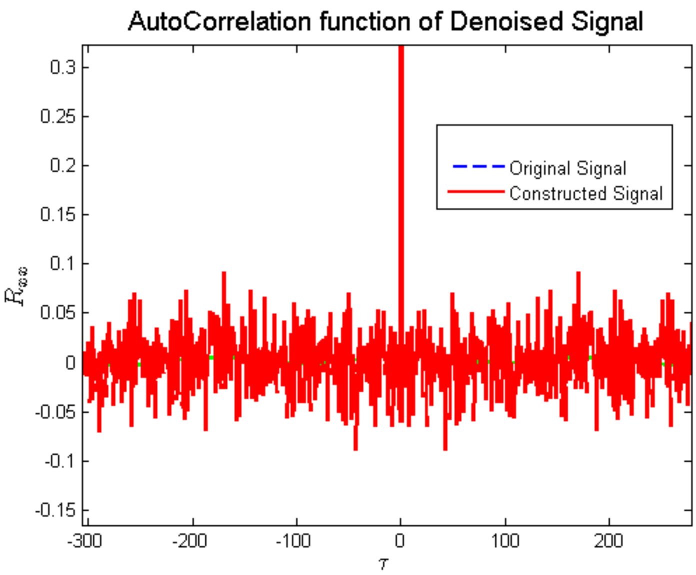

Figure 23: Autocorrelation of Original, Reconstructed

Again, these three signals' autocorrelation function coincidently overlap.

8.3.3 Probability density function

The PDF is estimated by matlab kernel smoother function. PDF of green comes

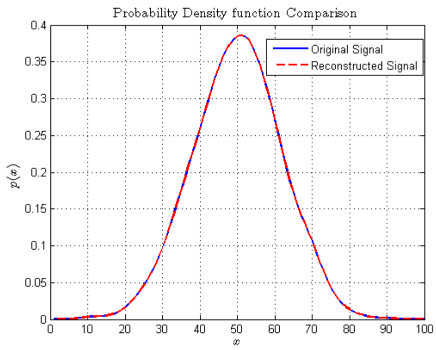

Figure 24: PDF of Original, Reconstructed

wider and higher. Maybe it's an interesting topic of nonstationary estimation.

8.3.4 Power Spectrum Density

The spectrum highly indicate the peak frequency of the sinusoidal wave. In the view point of energy conservation, it's easy to understand the green decays sooner.

Figure 25: PSD of Original, Reconstructed

8.4 Trend Detection and SNR

In the end, we do extract the trend from contaminated signal. It is a sinusoidal wave with the frequency we design at first. The Signal-Noise Ratio

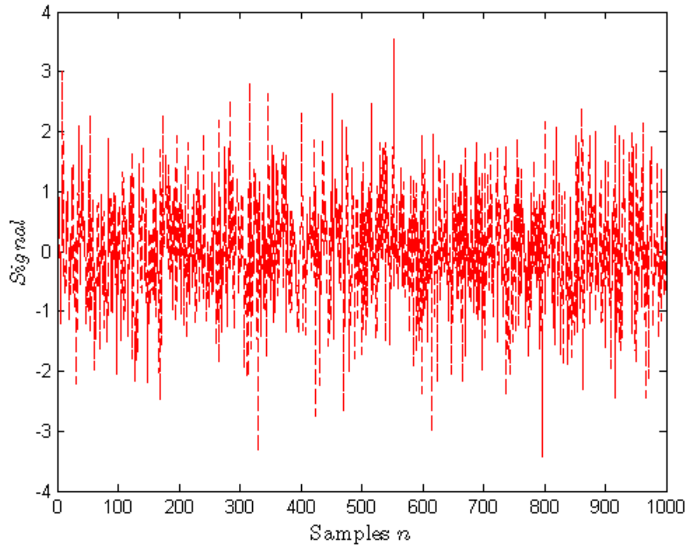

Figure 26: Trend Detection : Gaussian White

$$SNR = 10\log_{10}\frac{\sum s(t)^2}{\sum w(t)^2} = -15.7274\ dB$$

9 Polynomial with Gaussian White Process

In the report, we use thoroughly

- Sampling Rate = 1 Hz
- Data Length = 1000

9.1 Process Specifications

Deterministic Signal	y(t) = t	–
Stochastic Signal	Gaussian Pseudo Random	$\mu_x = 0, \sigma_x = 1$

Table 6: Specs of the Data

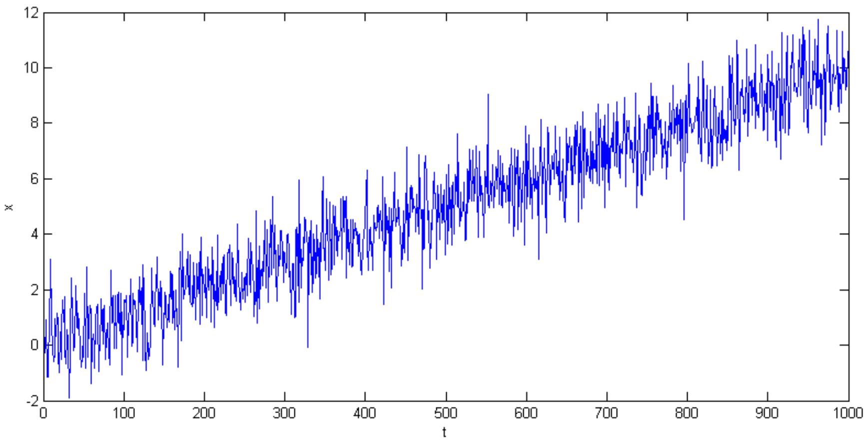

Figure 27: Polynomial with Gaussian White Process

9.2 Wavelet Analysis

9.2.1 Decomposition

We all use level 5 Daubechies decompose in this case. The left hand-side and right

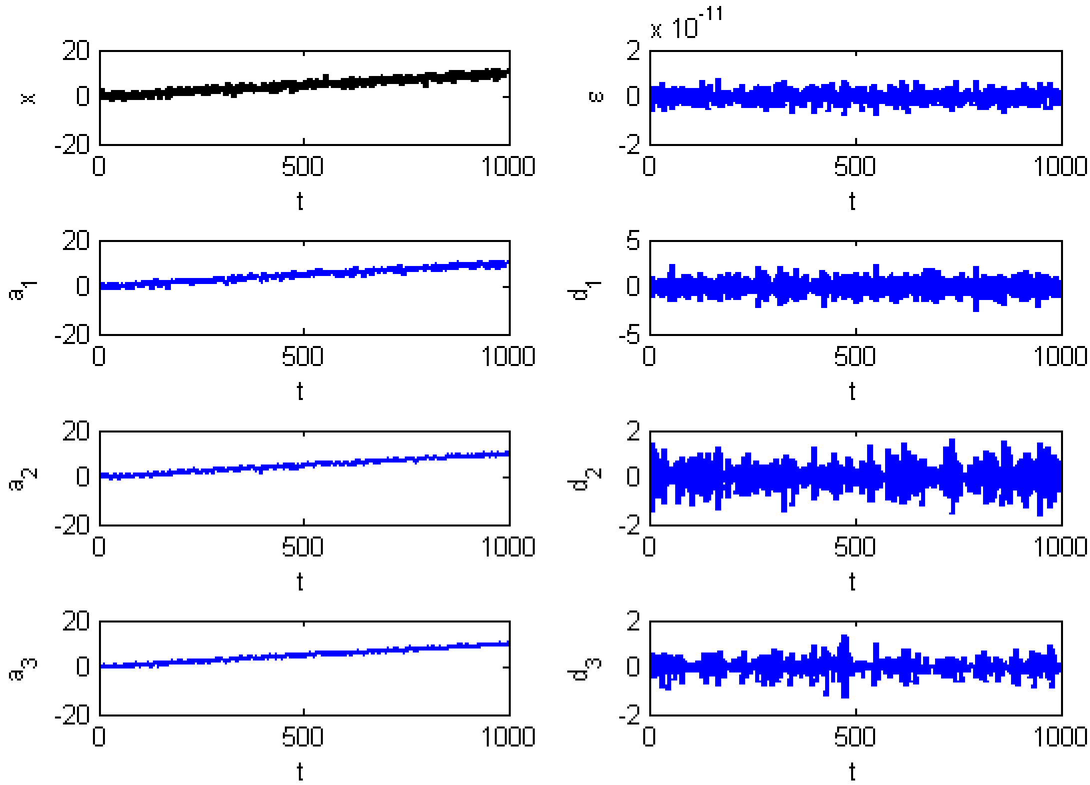

Figure 28: Level 3 decomposition of Haar Wavelets

hand-side individually show approximations and details. The lower level indicates higher frequency components.

9.2.2 Reconstruction

We first check the reconstruction is feasible, fit the theory and estimate how much information we loss.

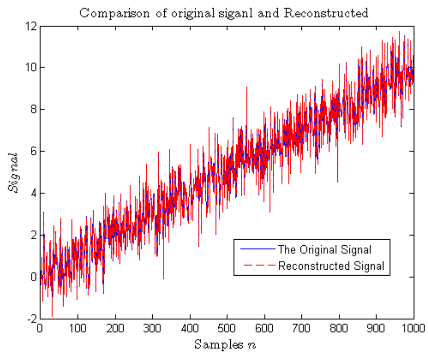

Figure 29: Compare Original and Reconstructed

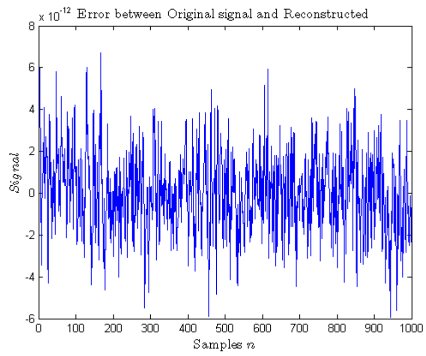

Figure 30: Error of the Reconstruction

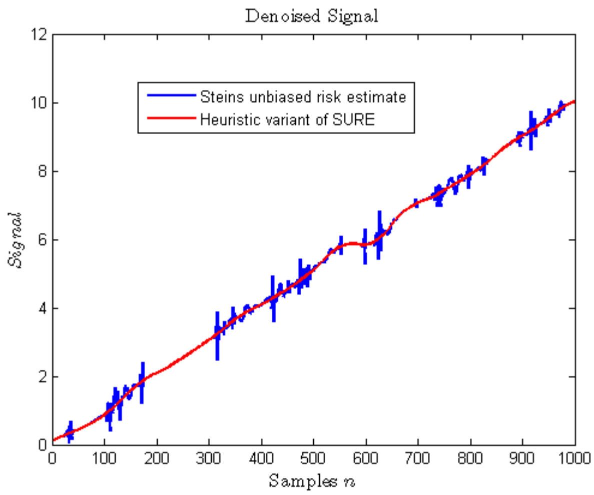

Figure 31: Comparison of SURE and heuristic SURE

We can find that heuristic SURE can preform better in most cases.

9.3 Statistical Estimation

In the last stage, we justify the four statistical estimations.

9.3.1 Mean value and Standard Deviation

- - -	Original Signal	Reconstructed Signal	Denoised Signal	Noise
Average $\hat{\mu_x}$	5.0234	5.0234	5.0261	-0.0027
Deviation $\hat{\sigma_x}$	3.0383	3.0383	2.8817	0.9586

Table 7: Specs of the Data

The last column indicates that we extract the noise with zero mean and unity deviation which perfectly fit the Gaussian noise we add at the first.

9.3.2 Autocorrelation Function

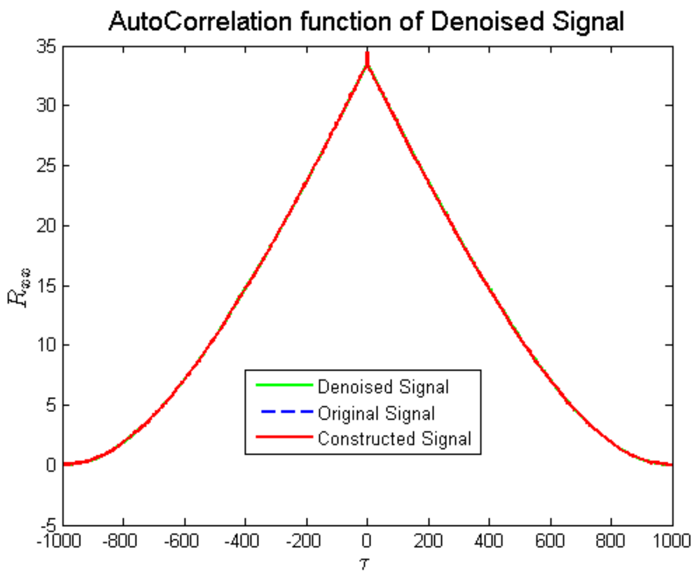

Figure 32: Autocorrelation of Original, Reconstructed and Denoised Signal

Again, these three signals' autocorrelation function coincidently overlap.

9.3.3 Probability density function

The PDF is estimated by matlab kernel smoother function. PDF of green comes

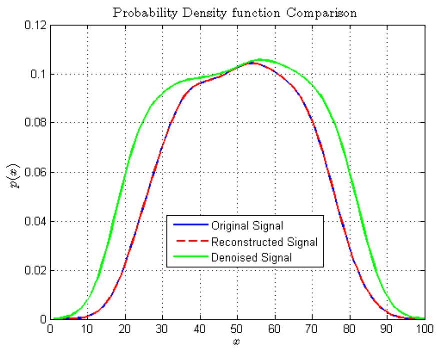

Figure 33: PDF of Original, Reconstructed and Denoised Signal

wider and higher. Maybe it's an interesting topic of nonstationary estimation.

9.3.4 Power Spectrum Density

The spectrum highly indicate the peak frequency of the sinusoidal wave. In the view point of energy conservation, it's easy to understand the green decays sooner.

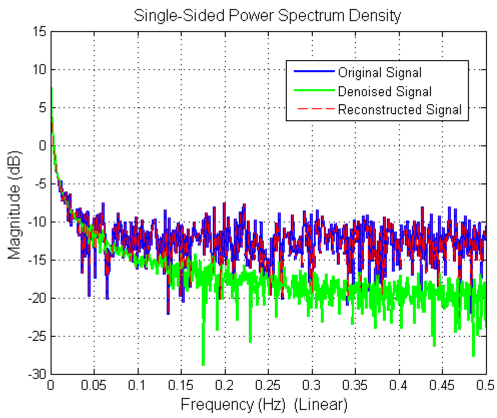

Figure 34: PSD of Original, Reconstructed and Denoised Signal

9.4 Trend Detection and SNR

In the end, we do extract the trend from contaminated signal. It is a sinusoidal wave with the frequency we design at first. The Signal-Noise Ratio

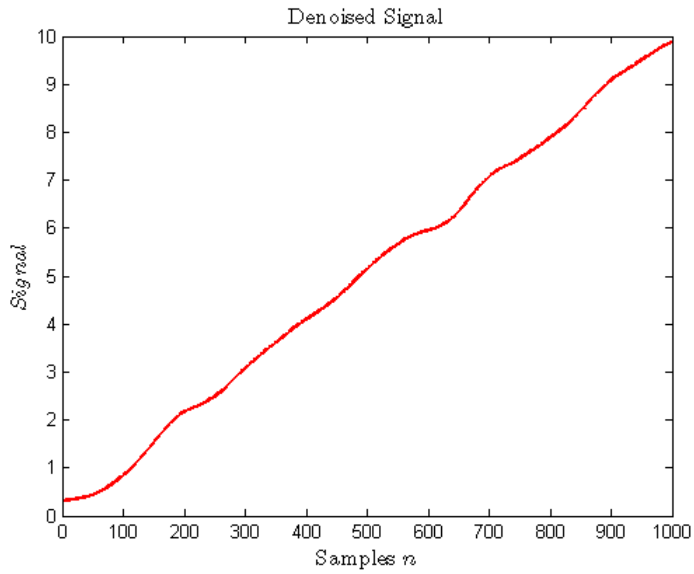

Figure 35: Trend Detection : Sinusoidal Wave

$$SNR = 10\log_{10}\frac{\sum s(t)^2}{\sum w(t)^2} = 15.1940\ dB$$

References

[1] WAVELET THEORY by David K. Ruch and Patrick J. Van Fleet

[2] Wikipedia, information entropy, Stein's unbiased risk estimate, Wavelet

[3] MIT - OpenCourseWare Wavelets, Filter Banks and Application