Wavelet Theory & Application

Author: Zhao Kau-Wen
Advisor : Chang Ren-Jung
NCKU-ME, Stochastic Data Processing
Contact : kevinzhaio@gmail.com

Contents

1 Introduction

The report can be divided into two major parts. First part is my notes of wavelet theory. It serves the quick review for myself. The second part is dealing with some practical problem.

Part I
Theory of Wavelet

In the report, I would spend more pages on the first part. Consider the integrity and completeness, stating the whole idea between Fourier transform and Wavelet transform is necessary. The whole part also cover the final project and give some insight.

2 Fourier Kingdom

Without any doubt, we have to go back to the idea of Fourier transformation. However, we still focus on the application-oriented statement, the section starts from **discrete** transformation.

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{-i2\pi kn/N} \tag{1}$$

$$X_k = \sum_{k=0}^{N-1} x_k e^{i2\pi kn/N} \tag{2}$$

where X_k is k-th Fourier coefficient and complex exponential indicates sinusoidal signals with different phase and frequency.
Intuitively, we use N-th harmonics to approximate the original signal(Figure 1). In discrete form, the signal is divided with small time step which depends on sampling rate(Figure 2).

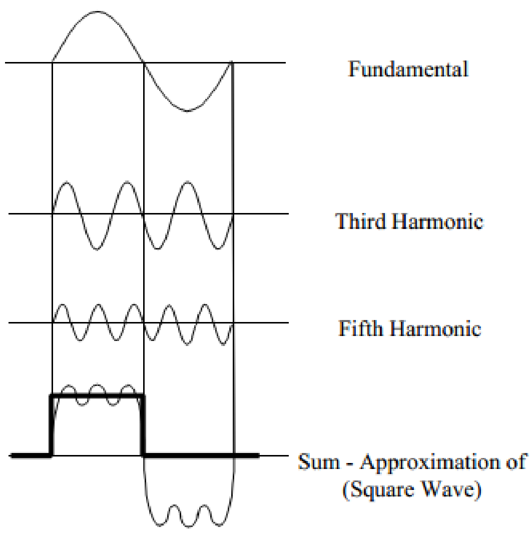

Figure 1: Harmonics

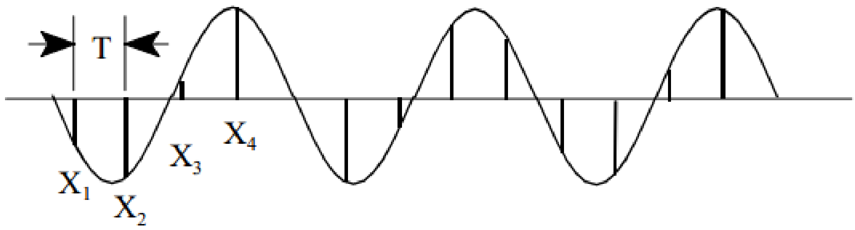

Figure 2: Waveform Sampling

In order to process a signal digitally, we need to sample the signal frequently enough to create a complete "picture" of the signal. The discrete Fourier transform (DFT) may be used in this regard. Samples are taken at uniform time intervals as shown in Figure 2 and processed. If the digital information is multiplied by the

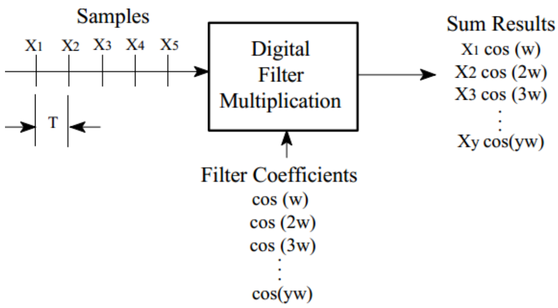

Figure 3: Digital Filtering

Fourier coefficients, a digital filter is created as shown Figure 3. If the sum of the resultant components is zero, the filter has ignored (notched out) that frequency sample. If the sum is a relatively large number, the filter has passed the signal.
To consider practical and real-life problem, we need a new method to analyze aperiodic signals. Windowed Fourier Transform was introduced. The WFT confine the signal in the specific window and constraint the signal finite in the time domain.

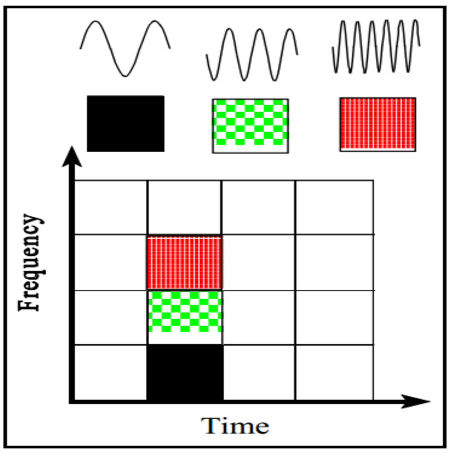

Figure 4: Windowed Fourier Transform

If the signal has sharp transitions, the input data is windowed so that the sections converge to zero at the endpoints. Because a single window is used for all frequencies in the WFT, the resolution of the analysis is the same (equally spaced) at all locations in the time-frequency domain.
Fourier analysis works well with stationary, continuous, periodic, differentiable signals, but other methods are needed to deal with non-periodic or non-stationary signals. For example the Figure 5 shows a signal and its frequency abruptly change with time.

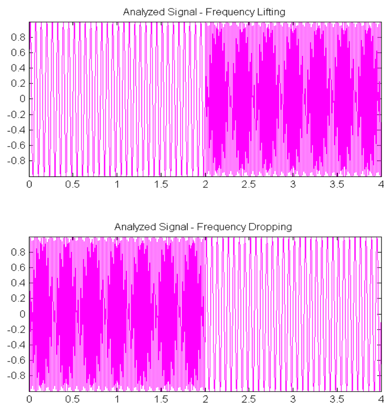

Figure 5: Top: Frequency lift from 16 Hz to 64Hz, Down: Frequency drop from 64 Hz to 16Hz

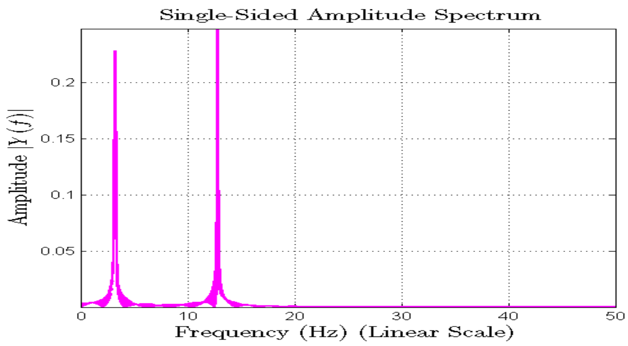

Figure 6: Amplitude spectrum can't show the information from time domain

3 Wavelet Transform

To improve and fix the problem we discussed in previous section, we need a better idea to deal with time and frequency domain concurrently.

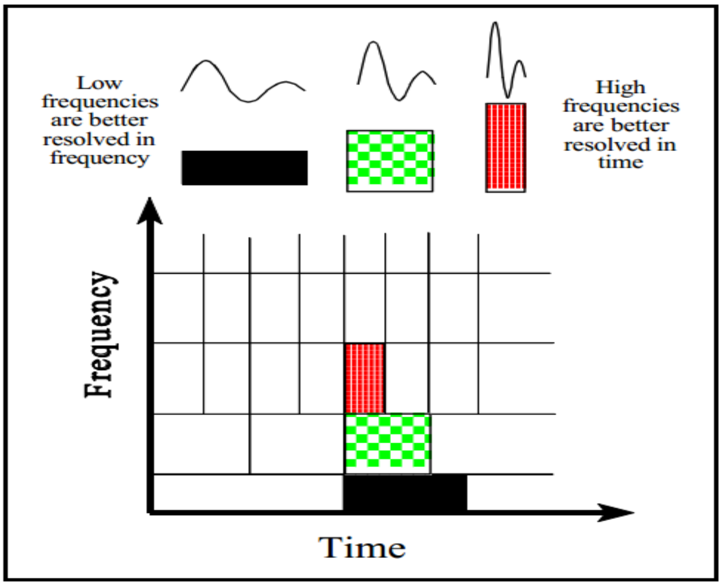

Figure 7: Wavelet Transform

The powerful tool was invented eventually. We cite the continuous wavelet transform formula to explain.

$$W(f;a,b) = \frac{1}{a}\int_{-\infty}^{\infty} f(t)\psi^*(\frac{t-b}{a})dt \tag{3}$$

where ψ is the Wavelet function, such as Daubechies, Morlet and Coiflet. Parameters a and b are important

- a : Scale Parameter
- b : Shift Parameter

Scale parameter controls the frequency of each wavelet. We can adjust them to fit the signal. Shift parameter is namely to shift in the time domain and find

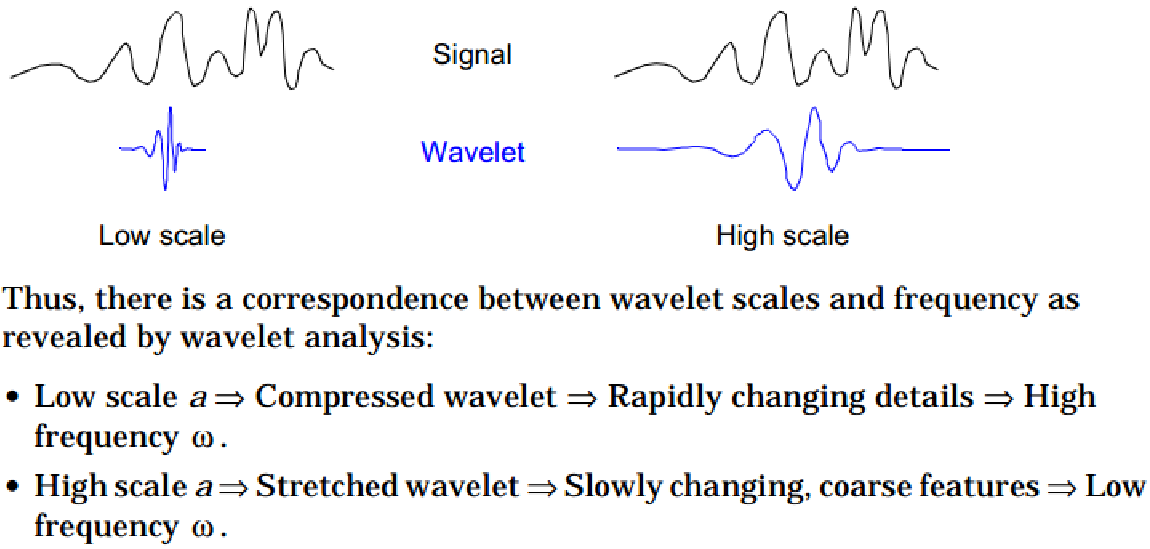

Figure 8: Scale Parameter

the better position that it can approximate the signal well.

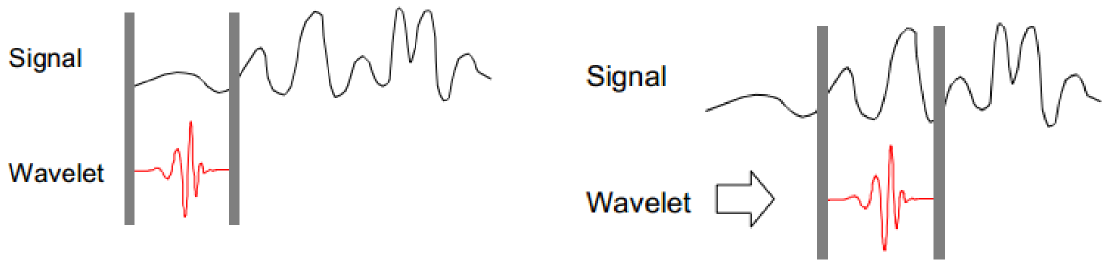

Figure 9: Shift Parameter

By the same idea with Fourier transform, we could have the same idea to analyze the signal.

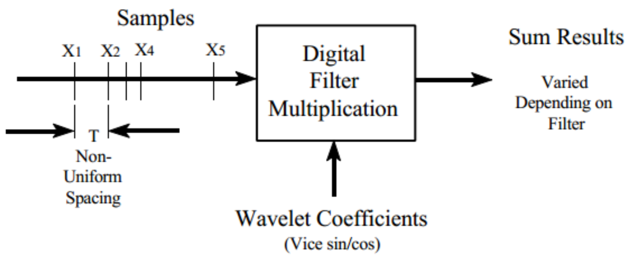

Figure 10: Wavelet Filtering

We still deal the same problem as before. We now analyze them with wavelet spectrum which can easily distinguish the difference.

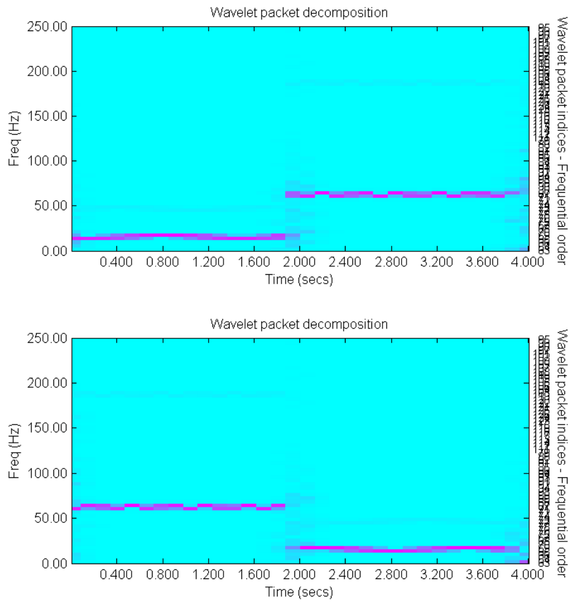

Figure 11: Wavelet Spectrum

4 Linking Together

The section is base on a paper **Wavelet Spectra compared to Fourier Spectra**. I read the content and cite few lines of deviation to make sure the relationship between Fourier and Wavelet **Power Spectrum**.

Consider s a real signal of real variable x. One may define its Fourier Transform $\hat{s}(k)$ and its power spectrum $E(k)$.

$$\hat{s}(k) = \int_{-\infty}^{\infty} s(x)e^{-ikx}dx \tag{4}$$

$$E(k) = \frac{1}{2\pi}|\hat{s}(k)|^2 \text{ for } k > 0 \tag{5}$$

And the total energy E of the signal is that

$$E = \frac{1}{2}\int_{-\infty}^{\infty} |s(k)|^2 dx = \int_{0}^{\infty} E(k)dk \tag{6}$$

Now consider ψ a function called analyzing wavelet. If it satisfies the admissibility condition and in the $L^2(R)$ space (I skip those deviation), we could go further on these bases. Within suitable hypothesis, one may define the wavelet transform $\tilde{s}$ of the signal s.

$$\tilde{s}(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} s(x)\overline{\psi(\frac{x-b}{a})}dx \tag{7}$$

Alternatively, the wavelet transform can be computed from Fourier transform of the signal

$$\tilde{s}(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} \hat{s}(\omega)\overline{\psi(a\omega)}e^{i\omega b}dx \tag{8}$$

From the energy conservation property of the wavelet transform (Skip the formula), we may define the local wavelet spectrum

$$\tilde{E}(k,x) = \frac{1}{2c_\psi k_0}|\tilde{s}(\frac{k_0}{k},k)|^2 \text{ for } k > 0 \tag{9}$$

where the k_0 is the wavenumber of the analyzing wavelet.

From this local spectrum one may in turn define a mean wavelet spectrum $\tilde{E}(k)$

$$\tilde{E}(k) = \int_{-\infty}^{\infty} \tilde{E}(k, x) dx \tag{10}$$

which is linked to the total energy

$$E = \int_{0}^{\infty} E(k) dk \tag{11}$$

From the formula (5,8,9,10) one may derive the relationship between Fourier spectrum $E(\omega)$ and the mean wavelet spectrum $\tilde{E}(k)$, namely

$$\boxed{\tilde{E}(k) = \frac{1}{c_\psi k} \int_{0}^{\infty} E(\omega) |\hat{\psi}(\frac{k_0 \omega}{k})|^2 d\omega} \tag{12}$$

The wavelet spectrum appears then as an average of the Fourier spectrum weighted by the square of the Fourier transformation of the analyzing wavelet shifted at the wavenumber k. Note that the larger$k$ is, the wider the averaging interval.

From this formula we may infer the results that the behavior of the wavelet spectrum at the large wavenumber depends strongly on the behavior of the analyzing wavelet at the small wavenumber.

Part II

Wavelet Application

We wan to compare the direct FFT amplitude spectrum with wavelet decomposed spectrum. Based on the knowledge of the wavelet decomposition, we can predict the decomposed signal characterize with its approximation or detail.

5 Random Process and Data Specification

Signal Type	Characteristic
Low frequency Sine	Amplitude = 12, frequency = 1 Hz
High frequency Sine	Amplitude = 5, frequency = 10 Hz
Gaussian White Process	$\mu_x = 0$, $\sigma_x = 3$

Table 1: Specs of the Signal

As the Table 1 states that two different sine components could be easily distinguished. We expect, then, in the simulation, these two frequencies could make

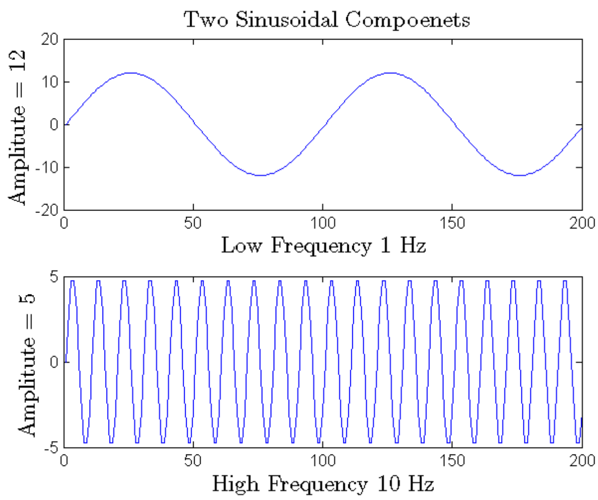

Figure 12: High and low frequencies Sine Wave

different in the spectrum.

We, first, have a signal with two main frequency. It is so clean that we even can determine the properties with naked eye.

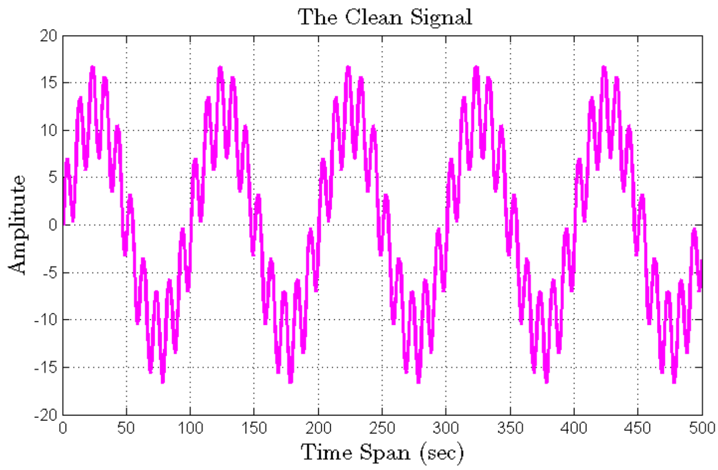

Figure 13: High and low frequencies Sine Wave

Then, Gaussian white process is added to them.

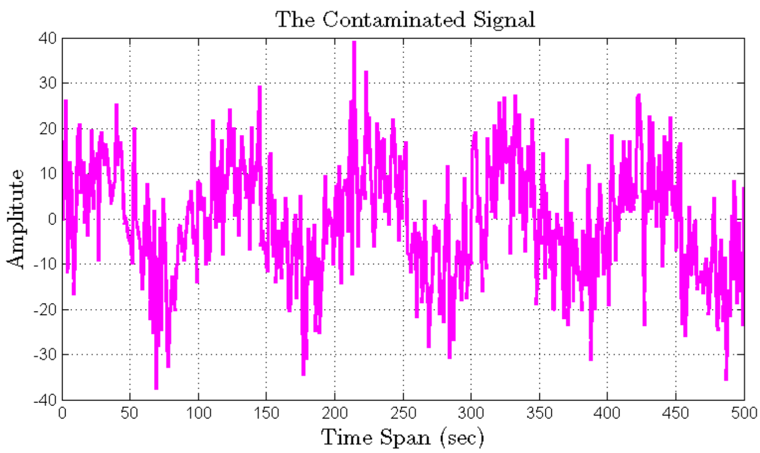

Figure 14: High and low frequencies Sine Wave

Now, the signal is polluted that we can broadly find its periodicity but can not declare its property.

We list the specification of random process here.

Sampling Rate	Time Span	Data Length	SNR
100 Hz	20 sec	2×10^3	3.28 dB

Table 2: Specs of the Process

6 Direct Analysis with FFT

6.1 Amplitude Spectrum

The Fourier transformation is powerful. We first try the amplitude spectrum with FFT algorithm gives the result of Figure 15. Amazingly, we could easily capture

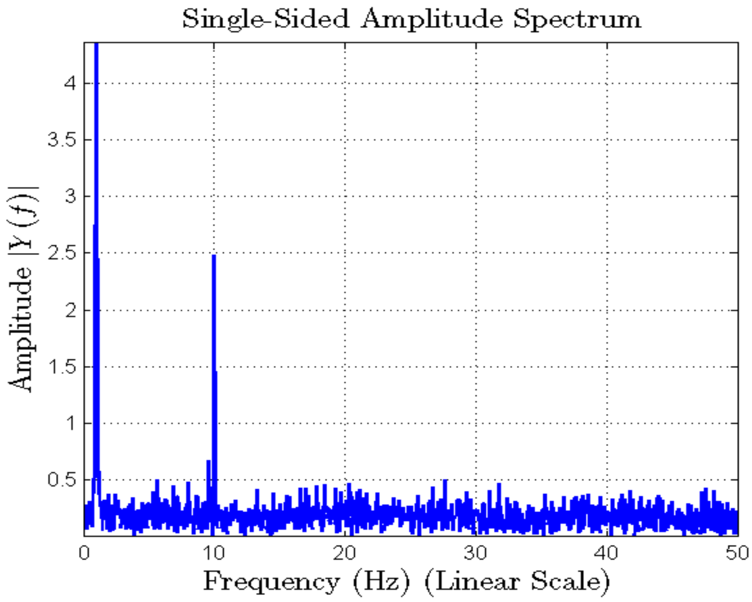

Figure 15: Amplitude Spectrum of the Polluted Signal

the main frequencies, 1 Hz and 10 Hz and the rest frequency components are provided by white process. The main frequencies and residual are facile to tell apart and not confusing.

6.2 Power Spectrum

Then, we take the power spectrum. According to the experience we used before, power spectrum might introduce energy leak and decay. It dominates by the Parsavel theory. We still can capture these two main frequencies but the white

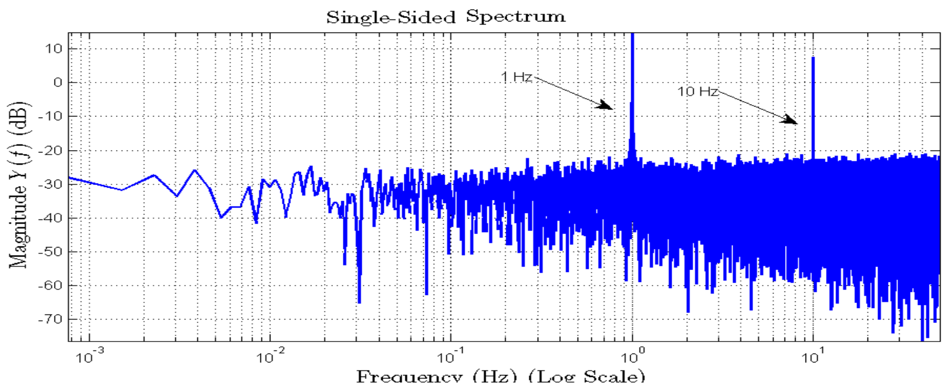

Figure 16: Power Spectrum of the Polluted Signal

process causes significantly influence on the spectrum.

6.3 Power Spectrum Density

Welch's Spectrum density gets the better result. All in all, the amplitude spectrum reach my satisfaction, so the following analysis takes advantage of it.

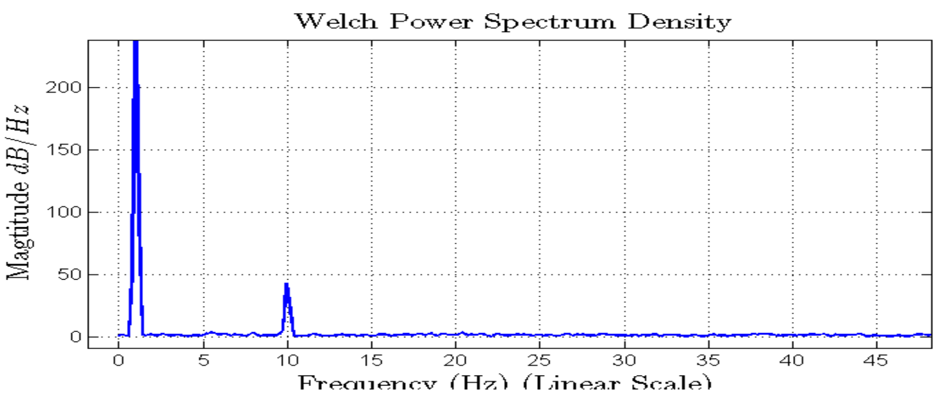

Figure 17: Power Spectrum Density of the Polluted Signal

7 Wavelet Packet Decomposition and FFT

In the section, we do the wavelet packet decomposition first. We extract each elements in the signal and analyze them with amplitude spectrum.

7.1 Wavelet Packet

We choose two kinds of wavelet packets and the following results are implemented mainly be **Daubechies Wavelet**. If the result is done by Haar wavelet, it would be noticed.

- Haar Wavelet : level
- Daubechies Wavelet 5 : level 3

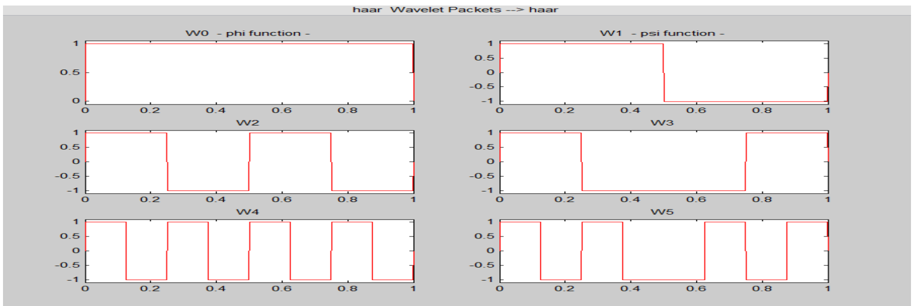

Figure 18: Haar Wavelet, level 5

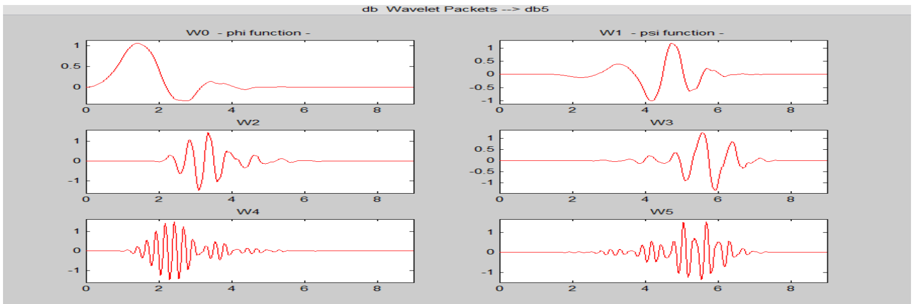

Figure 19: Daubechies Wavelet 5, level 3

7.2 Best Tree and Decomposition

We apply the Matlab function *wpdec*() and decompose the signal. With Shannon entropy and entropy-based algorithm to find the best tree (Figure 20).

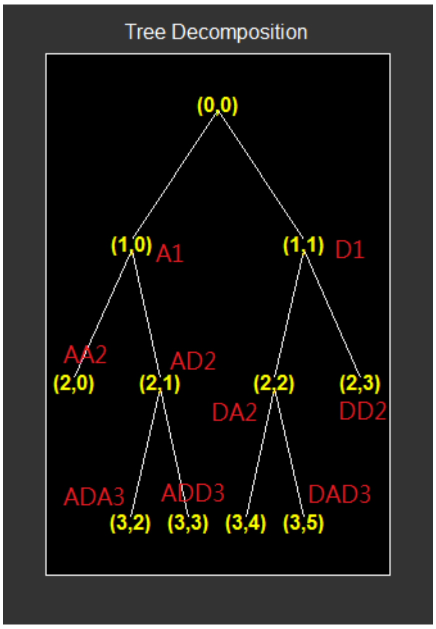

Figure 20: Best Tree of Daubechies Wavelet Packet Decomposition, Level 3

Here we follow the widely-used notation that indicates each decomposed signal. A means Approximation, D indicates detail, the order illustrates their relation in the tree and the sub-number tell the level in the tree.

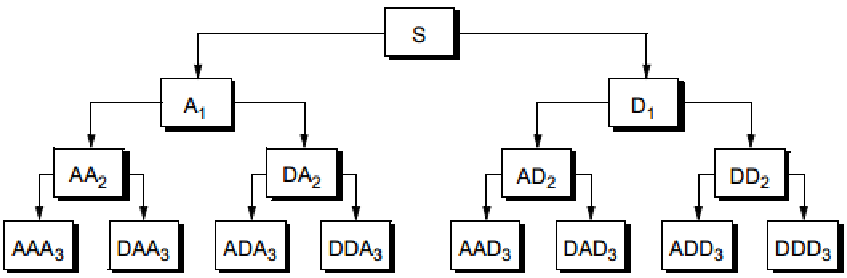

Figure 21: Notation of each Decomposed Signal

7.3 Amplitude Spectrum of each Bank

We decompose them and exam each concurrently. The right-hand side are the approximations and left-hand are details. We also analyze them with FFT. Roughly

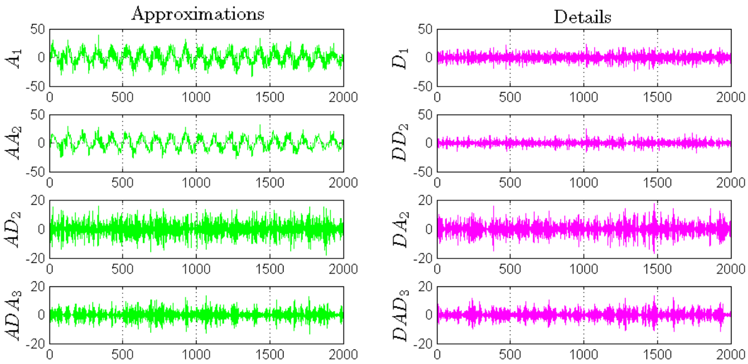

Figure 22: Approximations and Details

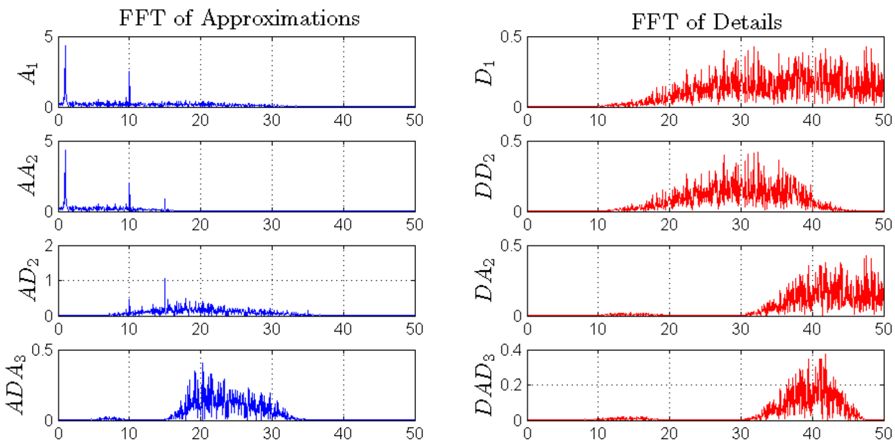

Figure 23: Approximations and Details

we could observe that each bank possesses its own character and their spectrum shows different properties. We iscuss them individually in next section.

7.3.1 Approximation A_1

The first level of approximation gives the best broad sight of the whole signal.

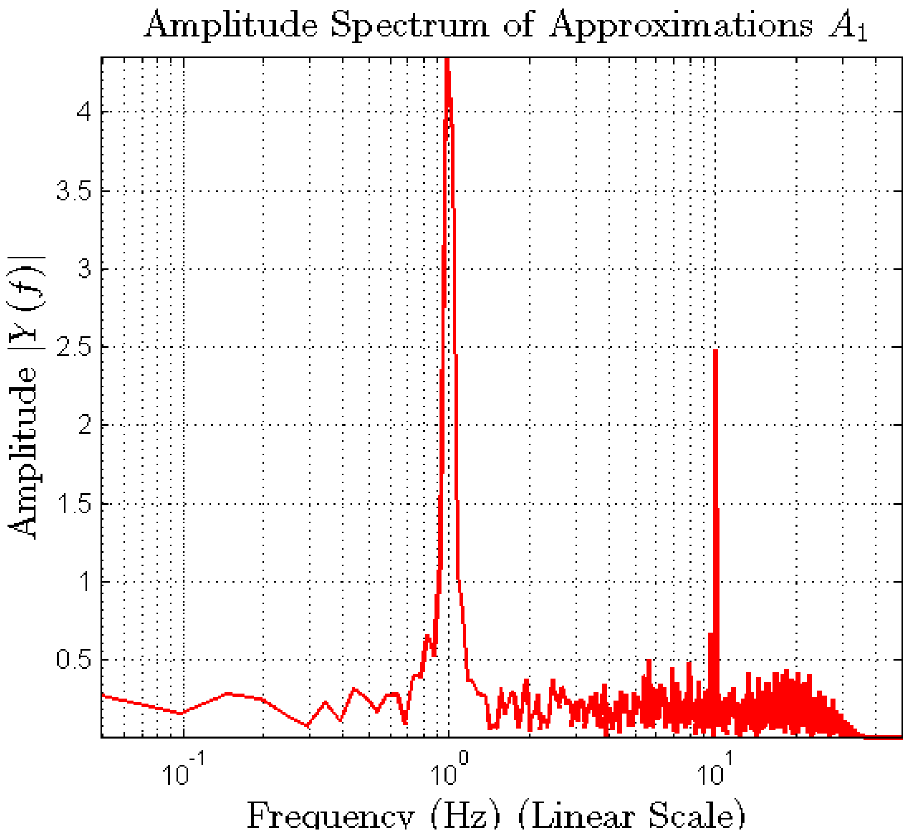

Figure 24: Amplitude Spectrum of first Approximation

It captures the main property well and clearly indicate frequencies. Besides, the white process do not cause too much damage to the spectrum.

7.3.2 Approximation AA_2

However, the second level of approximation gives some bad news.

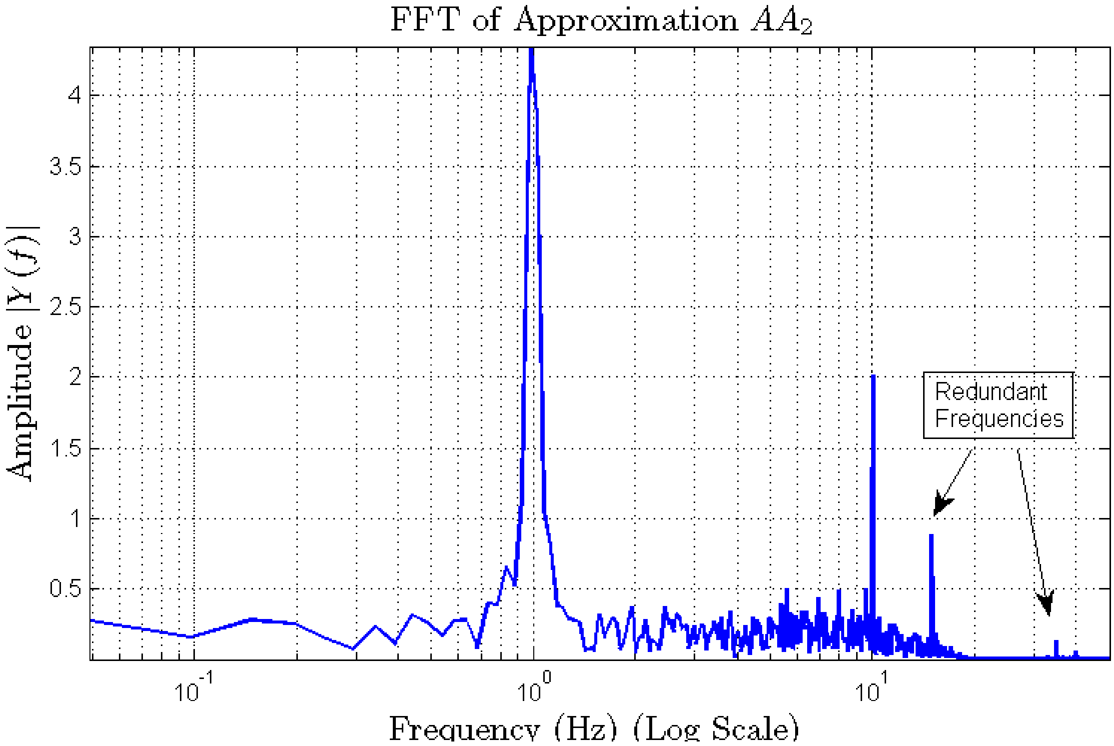

Figure 25: Amplitude Spectrum of first Approximation

The spectrum not only provides us desired 1 Hz and 10 Hz, but also introduce new and redundant frequencies. It is a fatal phenomena because the introduced-frequency is not negligible and might cause misleading.
I try to figure out what includes this frequency-component. First, I directly compare the approximation A_1 and AA_2 (Figure 15) but find A_1 fluctuates severely than AA_2. It is intuitively think that spectrum of AA_2 is more simple. But it is not. Second, I take spectrum with pure sinusoidal signal without Gaussian process(Figure16). The phenomena still exists. Finally I suppose this is caused by the wavelet function since that I change the wavelet to *haar* or $coif1$, the phenomena would be more serious.

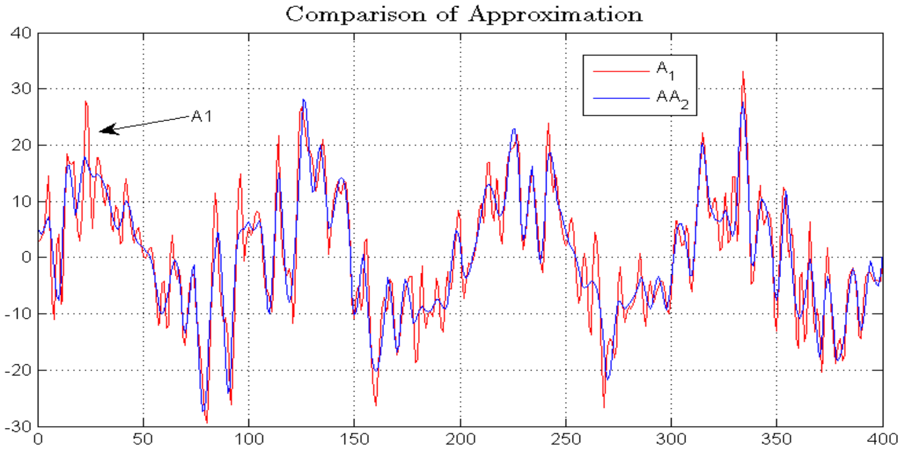

Figure 26: Comparison of Approximation A_1 and AA_2

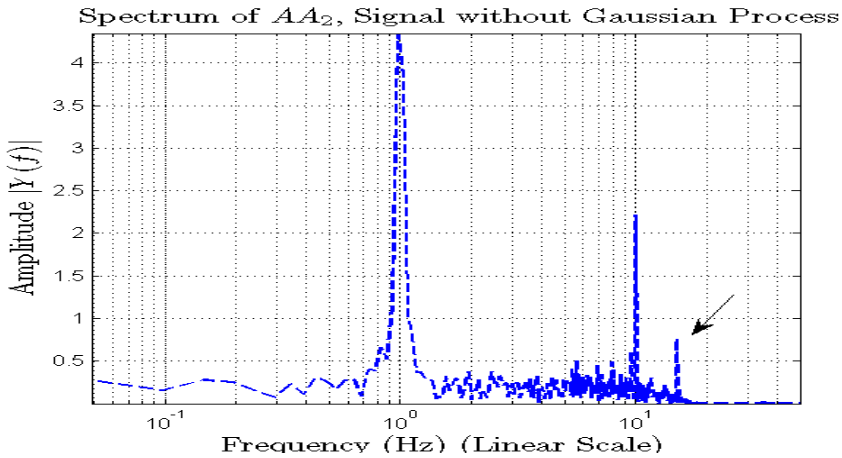

Figure 27: Amplitude Spectrum of Pure Signal without Noise

7.3.3 Approximation ADD_3

Let take a look at the ADD_3. It is interesting and reasonable. ADD_3 contains approximation that the profile shows two peaks and also possesses details that frequencies are rich. The detail parts provide high frequencies(Figure 29,30).

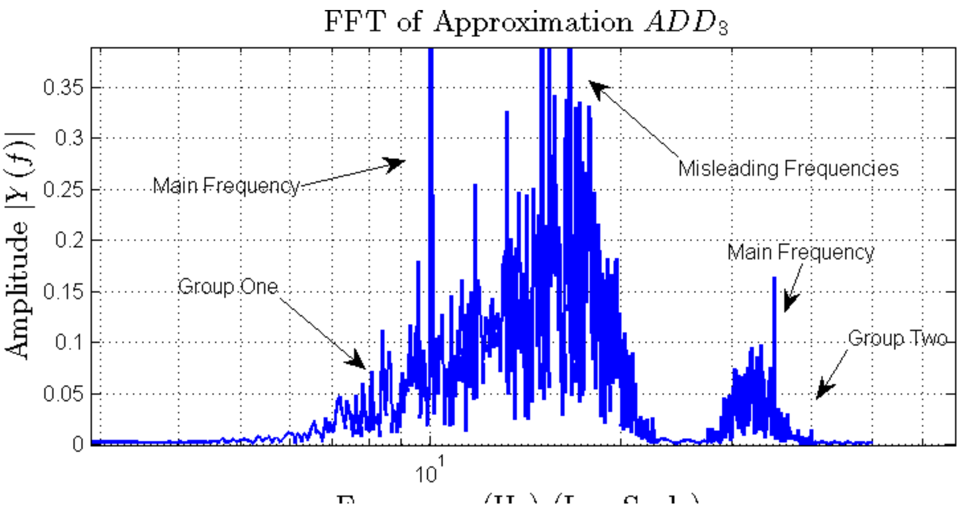

Figure 28: Spectrum of Approximation ADD_3

7.3.4 Detail D_1 and DD_2

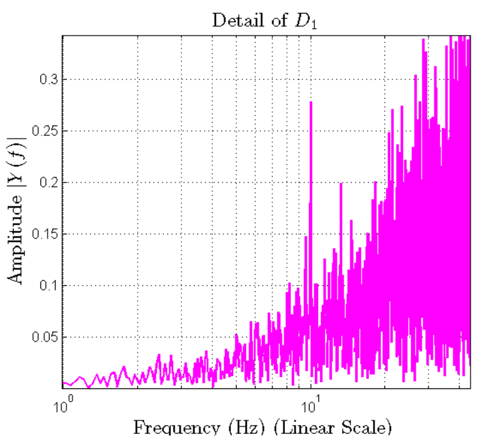

Figure 29: Spectrum of D_1

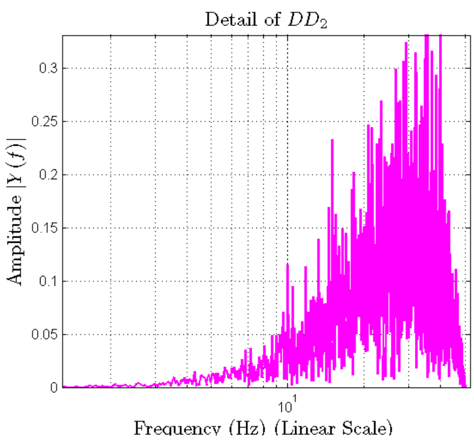

Figure 30: Spectrum of DD_2

7.4 Direct Fourier Spectrum v.s Decomposed Spectrum

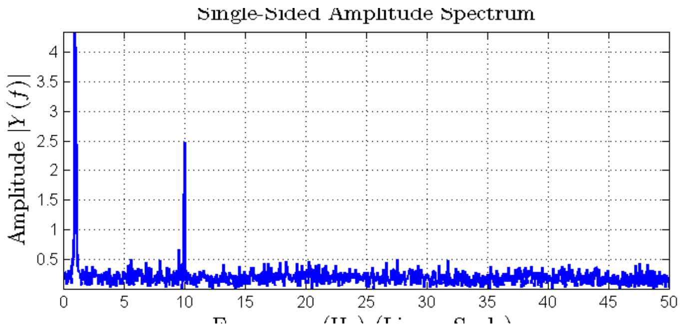

Figure 31: Spectrum of Original Signal

By comparison, the Figure 32 clearly illustrate different banks "take charge" different "frequency-span".

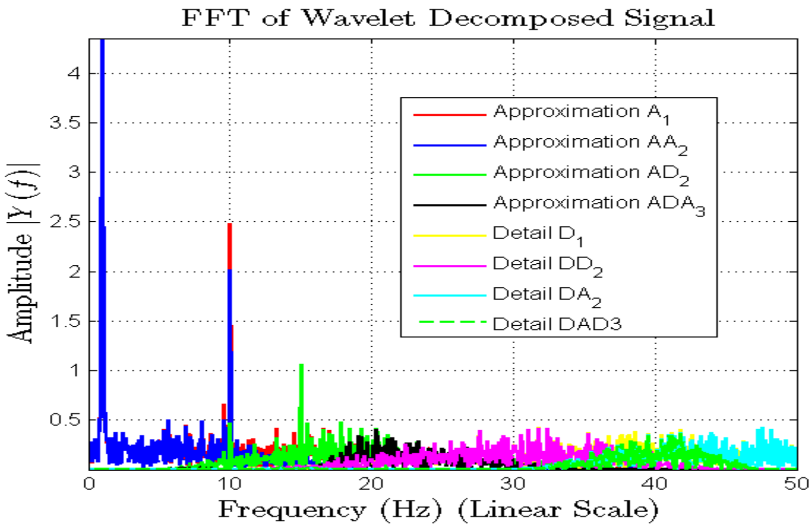

Figure 32: Spectrum of Decomposed Signal

References

[1] Wavelet Spectra compared to Fourier Spectra , by Val'erie Perrier , Thierry Philipovitch , Claude Basdevant 1995

[2] Wikipedia : Wavelet Transform, 2012

[3] Matlab User's Guide : Wavelet toolbox