Dynamic portfolio choice under nonlinear dynamics*

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Abstract

I study a nonlinear dynamic portfolio choice of an investor endowed with recursive preferences. I estimate a multivariate quadratic autoregressive (QAR) process for the evolution of the investment opportunity set, including the aggregate vacancy rate as a return predictor. Jointly incorporating multivariate nonlinear dynamics in conditional means and stochastic volatility substantially improves standard measures of portfolio performance relative to using a linear time series model: the cumulative wealth path between 1972 and 2014 by 93%, Sharpe ratio by 20%, and utility-based economic welfare by 48%. Methodologically, I derive an analytical approximation of the optimal dynamic portfolio in a multivariate nonlinear environment embedded in the QAR model.

Keywords: Portfolio choice, Volatility, Market timing, Nonlinear dynamics, Perturbation solution, Quadratic autoregressions

JEL Codes: G11

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1 Introduction

A large literature in empirical asset pricing has documented nonlinearities of aggregate asset returns, while recent studies in macroeconomics emphasize nonlinearities of macroeconomic variables that serve as return predictors. The goal of this paper is to develop a tractable framework that incorporates predictable nonlinear components into portfolio choice problems. Long-term investors seek to detect these predictable nonlinear components in asset returns and return predictors, and use this information to optimally hedge state-dependent risks.

For this purpose, I specify and estimate a flexible multivariate nonlinear time series model for the joint dynamics of the aggregate U.S. stock excess return, the risk-free rate, and a predictor variable. Then I quantify the importance of incorporating the estimated nonlinearities into the optimal portfolio selection problem by documenting improved portfolio performance using various standard measures including cumulative wealth path, Sharpe ratio, and economic welfare.

To flexibly capture the nonlinearities of asset returns and their predictors, I adopt a multivariate quadratic autoregressive (QAR) model. This is a multivariate extension of the univariate model proposed and estimated for U.S. macroeconomic series data in Aruoba et al. (2017). This time series model allows for nonlinear dynamics in conditional means and stochastic volatilities, while preserving the stationarity of the process. It can be derived as the second-order approximation of general smooth nonlinear dynamics. It nests linear vector autoregression (VAR) models and a range of nonlinear time series models in the existing portfolio choice literature.

Furthermore, I develop a perturbation method for the dynamic portfolio choice problem under recursive preferences in the case of unitary intertemporal elasticity of substitution when the underlying state space is described by the QAR model. The approximate solution has a closed form represented in the form of the QAR model, which allows for a transparent interpretation of the effects of alternative sources of nonlinearity in the QAR model on intertemporal hedging portfolios. The unitary-IES assumption fits the description of investment problems solved by individuals and institutions who require a constant expenditure share of their wealth including university endowments, sovereign wealth funds, or pension funds.

I apply this tractable empirical framework to the U.S. aggregate stock and bond investment problem in the sample between 1972 and 2014. Motivated by the recent macro-labor literature (Hall (2017), Liu (2007), which hypothesizes that discount rates are important determinants of firms' vacancy posting behavior and validates this prediction empirically, I employ the aggregate vacancy rate as the predictor of the aggregate stock excess return. In fact, I corroborate existing findings by showing that the aggregate vacancy rates dominate predictive power of the price-dividend ratios in forecasting excess stock return.

I find significant predictability of nonlinear parts of excess returns, risk-free rates, and vacancy rates that cannot be captured by VAR models or univariate QAR models. The aggregate vacancy rate exhibits significant nonlinear dynamics in its conditional mean, reflecting rapid declines rapidly during recessions and following gradual recoveries. This asymmetry is consistent with that of other labor market quantities like unemployment rates as observed by Dupraz et al. (2021). These nonlinear conditional mean dynamics produce additional predictability of future risk-free rates beyond the linear conditional mean.

Furthermore, the multivariate model also predicts the stochastic volatilities of excess stock returns and risk-free rates. Excess stock returns are predictable not only by the lagged transient excess stock return itself but also by the persistent aggregate vacancy rate. The stochastic volatility of risk-free rates is predictable by all three predictors in the model. The estimated volatilities of excess stock returns and riskfree rates are positively correlated with financial uncertainty and macroeconomic uncertainty measures constructed in Jurado et al. (2015) and Ludvigson et al. (2021), highlighting the importance of incorporating multiple types of stochastic volatility in investment opportunities and using the multivariate framework for their forecasting.

By incorporating the estimated nonlinear state space dynamics into the optimal portfolio rule, I find that the nonlinear conditional means have a quantitatively small impact on the model-implied portfolio allocation in the historical data. By contrast, stochastic volatility significantly changes the portfolio dynamics. The presence of the stochastic volatility reduces stock demand on average and amplifies the fluctuations of optimal portfolio weights. The countercyclical volatility of excess stock returns makes the aggregate stock market a poorer hedging instrument against changes in future riskfree rates during recessions. Moreover, stochastic volatilities of excess stock returns and riskfree rates expose future portfolio returns to larger fluctuations and reduce stock demand to unload risk during market downtowns. Finally, excess stock returns are lower when the volatility of future portfolio returns goes up so that this volatility hedging motive further reduces stock demand.

These model recommendations for reducing the stock portfolio share during crises fit the observation by Nagel et al. (2021) that investors in the top 1% and top 0.1 % of the overall income distribution, retired individuals and individuals at the very top of the dividend income distribution, heavily engaged in stock selling during the recent finanical crisis in 2008 and 2009. Wealth of these investors heavily overweight in financial assets rather than claims on labor income so they are highly vulnerable to the large volatility movements in the financial market. As a result, though the risk premium spikes, they unload the exposure to the stock market risks during these market turmoils.

I finally assess the performance of the optimal portfolio that fully exploits hedging against predictable changes in investment opportunities induced by nonlinear components. I use three measures: the cumulative wealth path and the realized Sharpe ratio are standard measures of portfolio performance that do not depend on the choice of the utility function, while a welfare-based measure takes stand on the utility function but incorporates the full welfare impact of nonlinearities. The qualitative results are consistent across those measures. Incorporating nonlinear dynamics improves the realized cumulative wealth path relative to using a linear VAR model by 93%, Sharpe ratio by 20%, and utility-based economic welfare by 48%. Stochastic volatilities account for all of these differences, while the nonlinear conditional mean is the least important. These results suggest the practical importance of jointly incorporating the nonlinearities in the multivariate dynamics into portfolio selection.

Related literature: I contribute to the optimal portfolio selection literature by providing an empirical framework with a more comprehensive set of nonlinearities and applying it to real financial data. Following the convention of this literature, I postulate an exogenous data generating process for asset return dynamics in a partial equilibrium setup and focus on investors' individual decision problem.

Methodologically, I adopt the series expansion method by Holmes (1995), Lombardo (2010), and especially Borovicka and Hansen (014b) and apply it to the portfolio selection problem when the underlying state space model is a multivariate QAR model. As emphasized by Aruoba et al. (2017), this nonlinear time-series model allows for both nonlinearity in conditional mean and stationarity simultaneously, which is crucial for obtaining a tractable solution to dynamic portfolio choice problem. Moreover, the multivariate QAR model incorporates the multiple stochastic volatilities of all the state variables jointly, not just excess returns of risky assets as seen in Bansal et al. (2014); Campbell et al. (2018); Moreira and Tyler (2019).

By contrast, the past literatures derive the approximate solutions or compute global solutions under linear VAR models without allowing for nonlinearity in conditional mean (for approximate solutions, Campbell and Viceira (1999); Barberis (2000); Watcher (2002); Campbell et al. (2003); Chacko and Viceira (2005); Laborda and Olmo (2022); for global solutions, Buraschi et al. (2010); Zhou and Zhu (2012); Moreira and Tyler (2019)). Moreover, they abstract from the stochastic volatilities of risk-free rates and return predictors.

In terms of the application to real financial data, this paper is the first to estimate the multivariate QAR model, extending the estimation of the univariate model by a MCMC method by Aruoba et al. (2017) to a multivariate model, which was left for future research in the previous literature. This extension allows for utilizing the multivariate information to improve the forecasts for a conditional mean and time-varying volatility. With the model with more comprehensive nonlinearity than the previous literature, I detect which aspects of nonlinearity are important for the optimal portfolio choice. This question has not been studied in the previous literature.

Other related papers that study the relationship between volatility and expected returns include Shanken and Tamayo (2012) and Johannes et al. (2014). Johannes et al. (2014) solve a Bayesian learning problem that accounts for time-varying volatility when forming out of sample expected return forecasts. These features are abstracted from in this paper and left for future research.

Structure of the paper: This paper is structured as follows. Section 2 describes the portfolio choice and consumption-saving problem in a general setting, and then Section 3 describes a newly developed approximation method to solve this problem. Section 4 estimates a multivariate QAR model for U.S. aggregate stock market returns and risk-free rates. Section 5 explores the implications of estimated nonlinearities for the characterization and investment performance of the optimal portfolio. Finally, Section 6 concludes. The Appendix A contains details of the theoretical derivations.

2 Theoretical framework

The model is set in discrete time. I assume an infinitely long-lived investor with Epstein-Zin recursive preferences defined over a stream of consumption. I allow an arbitrary set of traded assets and state variables and do not assume a market completeness.

There are $n+1$ assets available for investment, n risky assets with returns r^i_{t+1} $(i = 1, ..., n)$, and a real risk-free asset with return r^f_t. The investor allocates after-consumption wealth among these assets. The real portfolio return r^p_{t+1} is given by

$$r^p_{t+1} = \sum_{i=1}^{n} \alpha^i_t (r^i_{t+1} - r^f_t) + r^f_t,$$

where α_t^i is the portfolio weight on risky asset $i = 1, ..., n$.

The dynamics of relevant exogenous state variables are formulated as a general stationary Markov process whose law of motion is denoted by a smooth function ψ[1]:.

$$X_{t+1} = \psi(X_t, W_{t+1}),$$

where $\{W_{t+1}\}$ is a white noise process such that $W_{t+1} \sim N(0, I)$. Exogenous state variables contain not only asset returns but any other exogenous variables that are relevant as predictors of the dynamics of asset returns.

Epstein-Zin recursive preferences has the desirable property that the notion of risk aversion is separated from that of the intertemporal elasticity of substitution(IES). I assume that the intertemporal elasticity of substitution is equal to one, which turns out to greatly simplify the derivation of the approximately analytical solution for the portfolio choice. The unitary IES has been adopted by the literature including **?** and Moreira and Tyler (2019), mimicking the decisions of individuals and institutions such as university endowments, sovereign wealth funds, or pension funds.

The value function V_t is defined as:

$$\log V_t = \max_{C_t, \alpha_t} [1 - \exp(-\delta)] \log C_t + \frac{\exp(-\delta)}{1-\gamma} \log \mathbb{E}_t[(V_{t+1})^{1-\gamma}], \tag{1}$$

where C_t is consumption at time t, α_t is the vector of portfolio weights on risky assets, $\gamma > 0$ is the relative risk aversion coefficient, $\delta > 0$ is the time discount rate, and $\mathbb{E}_t$ is the conditional expectation operator under the data generating process.

At each time t, the investor uses all relevant information to make optimal consumption

[1] I abstract from state dynamics represented by non-differentiable rules. An example are regime-switching models studied in the context of portfolio choices in Ang and Bakaert (2002), and Guidolin and Timmermann (2007)

and portfolio choice decisions. The intertemporal budget constraint is

$$A_{t+1} = r^p_{t+1}(A_t - C_t),$$

where A_t is wealth at time t. In the absence of labor income, this problem also fits the description of the investment problem faced by wealthy and /or retired households.

This problem yields the consumption and portfolio choice policies as functions of the endogenous state A_t and exogenous states X_t denoted by $C(A_t, X_t)$ and $\alpha_t(A_t, X_t)$ respectively. They satisfy the following optimality conditions: the FONC for the consumption-saving problem:

$$\mathbb{E}_t(M_{t+1} r^f_t) = 1;$$

the FONC for the portfolio choice problem: for $i = 1, ...n$

$$\mathbb{E}_t(M_{t+1} exr^i_{t+1}) = 0,$$

where M_{t+1} is the stochastic discount factor (SDF, thereafter):

$$M_{t+1} = \exp(-\delta - \Delta \log C_{t+1}) \frac{\exp[(1-\gamma)(\log V_{t+1} + \log C_t)]}{\mathbb{E}_t \exp[(1-\gamma)(\log V_{t+1} + \log C_t)]}$$

and exr^i_{t+1} is the excess return of asset i over the riskfree rate, $r^i_{t+1} - r^f_t$. I define the change of measure associated with this SDF

$$\tilde{M}_{t+1} = \frac{\exp[(1-\gamma)(\log V_{t+1} + \log C_t)]}{\mathbb{E}_t \exp[(1-\gamma)(\log V_{t+1} + \log C_t)]}. \quad (2)$$

The time-differential operator is denoted by Δ such that $\Delta Y_{t+1} = Y_{t+1} - Y_t$.

3 Solution method

Since a general global solution is not available in this environment, I utilize the series expansion method to obtain an approximate solution that characterizes the optimal portfolio for an arbitrary number of state variables that follow a quadratic approximation of the non-linear asset return dynamics. The solution method in this paper builds on the early work by Holmes (1995) and Lombardo (2010) and especially on the expansion method for robust and recursive preferences developed by Borovicka and Hansen (014b). Detailed derivations of the approximate solution are discussed in the Appendix.

3.1 An approximate framework for exogenous process

I start with expanding the general asset return dynamics ψ up to the second order. As proposed in Borovicka and Hansen (014a), the following set of dynamics nests the approximated dynamics:

$$X_{1,t+1} = \Theta_{10} + \Theta_{11}X_{1,t} + \Lambda_{10}W_{t+1}$$

and

$$\begin{aligned} X_{2,t+1} =& \Theta_{21}X_{1,t} + \Theta_{11}X_{2,t} + \Theta_{23}(X_{1,t} \otimes X_{1,t}) + \\ &+ \Lambda_{21}(X_{1,t} \otimes W_{t+1}) + \Lambda_{22}(W_{t+1} \otimes W_{t+1}). \end{aligned}$$

$\{X_{1,t}\}$ can be regarded as the first-order approximation of the original series $\{X_t\}$, while $\{X_{2,t}\}$ as the second-order approximation. The series expansion method imposes the restriction that the square matrix Θ_{11} is the lag coefficient matrix in both processes. Hence, if Θ_{11} has spectral radius strictly less than one in absolute value, both first- and second-order processes are jointly stationary. This restriction gives more tractability of the system than that obtained by Taylor expansion method that often leads to approximate second-order dynamics that has explosive paths.

The approximate dynamics is the multivariate extension of the univariate QAR model proposed and estimated for macroeconomic variables by Aruoba et al. (2017). The first-order process is a linear homoskedastic VAR model often employed in the portfolio choice literature (Campbell and Viceira (1999); Campbell et al. (2003);Laborda and Olmo (2022)).

The second-order process allows for nonlinear conditional means Θ_{23} and flexible stochastic volatility terms Λ_{21}. Unlike the nonlinear time series model employed in the existing portfolio choice literature (Chacko and Viceira (2005); Buraschi et al. (2010); Moreira and Tyler (2019)), the multivariate QAR model allows every variable to be exposed to all the interactions of the lagged state variable and innovations. I will show in the later sections 4 and 5 that the estimated asset return dynamics and their predictors are in fact exposed to various interactions of stochastic volatilities and predictable components that are crucial for portfolio performance in practice.

3.2 Approximate solution to portfolio choice

Because of the unitary IES assumption, the consumption-saving decision is straightforward with consumption being proportional to wealth. Combined with the budget constraint, the consumption growth rate is equalized with the portfolio return $\Delta \log C_{t+1} = r^p_{t+1}$.

Turning to the portfolio choice problem determining the share α_t, I derive an approximate solution that is fully characterized by the asset return dynamics up to second order derived as the QAR model in the previous section. The approximation method builds on the insight of Borovicka and Hansen (014b), who construct a perturbation by scaling variances by the perturbation parameter q while simultaneously scaling the risk aversion parameter by the reciprocal of the perturbation parameter $1/q$. In the limit ($q \to 0$), the impact of relative risk aversion does not vanish even in the first-order dynamics since the scaled relative risk aversion goes to infinity, offsetting the second-order nature of the risk aversion.

The first-order condition from the first-order approximation is then expressed as

$$\mathbb{E}_t(exr_{1,t+1}) = (\gamma - 1)\mathbb{E}_t\big[\Delta\mathbb{E}_{t+1}(exr_{1,t+1}) \cdot \Delta\mathbb{E}_{t+1}\big(\sum_{j=1}^{\infty} \exp(-\delta j) r^p_{1,t+j}\big)\big]$$

This illustrates the usual risk-return tradeoff in financial investment. The increase in risk exposure arising from holding an additional unit of a risky asset on left hand side is compensated by the expected excess return (risk premium) on the right hand side. Since the first-order process is linear and homoskedastic, the right hand side of this equation is constant over time. Therefore, the risk premium on the left hand side must be constant under the first-order dynamics. This restriction will be imposed later in section 4 when I estimate the asset return dynamics.

The resulting constant portfolio share α_0 takes the Merton (1971)'s form, consisting of a myopic portfolio and an intertemporal hedging portfolio. The myopic portfolio shows up even in the static portfolio choice. The myopic portfolio in the first order dynamics is given by the relative risk aversion times a constant risk premium divided by the variance of the first-order excess return. The intertemporal hedging portfolio arises in the dynamic setting since investors would like to hedge the change in future investment opportunities. The intertemporal hedging portfolio in first order hedges the changes in expected future portfolio returns. Since the risk premium does not have any predictable components in the first order dynamics, investors structure the first-order hedging portfolio only to hedge changes in expected future risk-free rates.

The optimality condition in the second order dynamics is also characterized as a risk-return tradeoff relationship:

$$\tilde{\mathbb{E}}_t(exr_{2,t+1}) = \frac{3}{2}(\gamma - 1)\tilde{\mathbb{E}}_t\big[\Delta\tilde{\mathbb{E}}_{t+1}(exr_{1,t+1}) \cdot \Delta\tilde{\mathbb{E}}_{t+1}\big(\sum_{j=1}^{\infty}\exp(-\delta j)r^p_{2,t+j}\big)\big],$$

where $\tilde{\mathbb{E}}_t$ is the conditional expectation under the probability measure implied by the first-order dynamics of the change of measure $\tilde{M}_{t+1}$ in (2). The second-order dynamics are heteroskedastic with nonlinear conditional means but the right-hand side is still solely described by the linear combination of first-order variables. This implies the second-order risk premium in the left hand side will be a linear function of these variables as well. Again, this restriction will be imposed on the asset return dynamics when I estimate the QAR model in section 4.

The approximate solution is similar to Merton (1971)'s form with a linear combination of first-order variables. Since the second-order risk premium is time-varying, so is the myopic portfolio. Regarding the intertemporal hedging portfolio, investors demand risk compensation not only for the additional exposure to the change in future expected portfolio returns but also to changes in the volatility of future portfolio returns. Both nonlinear conditional means (Θ_{23}) and stochastic volatilities (Λ_{21}) affect the hedging demand. In the empirical application in later sections, I will disentangle the various roles of these nonlinearities into different components of intertemporal hedging portfolios, exploiting the novel analytical structure of the approximate solution.

The solution method developed in this paper adopts a series expansion method in contrast to Taylor expansions applied in the previous portfolio selection literature (Devereux and Sutherland (2010) and Devereux and Sutherland (2011)). The series expansion method ensures stationarity of higher-order dynamics if the original dynamics are locally stationary around the approximation point, while the Taylor expansion induces explosive dynamics as documented by Fernandez-Villaverde et al. (2016). Stationarity of higher-order dynamics yields empirical tractability of the present framework.

Moreover, the current methodology accommodates nonlinearities in asset return dynamics in the form of nonlinear conditional means and stochastic volatilities in a flexible way and an arbitrary number of state variables unlike linear time series models in Campbell and Viceira (1999), Campbell et al. (2003), Laborda and Olmo (2022) or nonlinear time series model that accommodates only some of the features studies in this paper like Chacko and Viceira (2005), Buraschi et al. (2010), Moreira and Tyler (2019).

On the other hand, the approximate solution in this paper abstracts from economic forces present in fully global solutions as in Liu (2007) and Moreira and Tyler (2019) to provide the tractability mentioned above. For example, although the global solution yields time-varying volatility-hedging portfolios, the approximate solution features only a constant volatility hedging portfolio. A higher-order approximation than second-order would overcome this problem and is left for future work.

4 An empirical application: stocks and real riskfree bond

I apply the framework to the environment where investors make the portfolio decision between a risky aggregate stock and the risk-free rate, whose dynamics are described by the QAR model. I include aggregate vacancy rate in the state vector as a return predictor. Liu (2019) argues that firms' discount rates are important determinants of vacancy posting decisions in the model. He also empirically documents that this variable is a relevant predictor for the aggregate excess stock returns, comparable to other well-known predictors such as a price-dividend ratio. Moreover, in a labor search model, Petrosky-Nadeau and Zhang (2021) shows the importance of nonlinearities associated with the labor market variables in shaping the aggregate labor market behavior during crises. These empirical and theoretical studies motivate the importance of incorporating the labor market variable and its nonlinear parts into the state vector as a return predictor.

In this section, I describe the data, estimate the econometric specification of the time-series model, and investigate the estimated nonlinearities.

4.1 Data description and econometric specification

The sample of aggregate vacancy rates, risk-free rates, and excess stock returns spans the period between 1959M2 and 2014M12 at monthly frequency. I use the aggregate vacancy rates constructed by Petrosky-Nadeau and Zhang (2021)[2].

[2]For a detailied description of the construction of this series, see Petrosky-Nadeau and Zhang (2021), Section 2.1.2.

Parameters/Dependent	Pre-sample	Vacancy v_{t+1}	Riskfree rates r^f_{t+1}	Excess returns exr_{t+1}
Θ_{10}	1959.M2-1971.M12	$\mathcal{N}(0.0082, 6.7e-5)$	$\mathcal{N}(9.3e-5, 8.7e-9)$	$\mathcal{N}(0.032, 0.001)$
Θ_{11} on:	1959.M2-1971.M12			
v_t		$\mathcal{N}(0.98, 0.97)$	$\mathcal{N}(6.6e-6, 4.4e-11)$	zero by restriction
r^f_t		$\mathcal{N}(39, 1.5e3)$	$\mathcal{N}(0.88, 0.78)$	zero by restriction
exr_t		$\mathcal{N}(0.3, 0.09)$	$\mathcal{N}(3.5e-4, 1.3e-7)$	zero by restriction
Θ_{21} on:	1959.M2-1971.M12			
v_t		zero by restriction	zero by restriction	$\mathcal{N}(-0.006, 3.8e-5)$
r^f_t		zero by restriction	zero by restriction	$\mathcal{N}(-7.23, 52.34)$
exr_{t+1}		zero by restriction	zero by restriction	$\mathcal{N}(0.107, 0.01)$
Θ_{23} on:	—			
$v_t \times v_t$		$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	zero by restriction
$v_t \times r^f_t$		$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	zero by restriction
$v_t \times exr_t$		$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	zero by restriction
$r^f_t \times r^f_t$		$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	zero by restriction
$r^f_t \times exr_t$		$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	zero by restriction
$exr_t \times exr_t$		$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	zero by restriction
Λ_{21} on:	—			
$X_{1,t}(i) \times W_{t+1}(j)$		$\mathcal{N}(0,2)$	$\mathcal{N}(0,2)$	$\mathcal{N}(0,2)$
$\Lambda_{10}\Lambda'_{10}$	1959.M2-1971.M12		$\mathcal{IW}(\nu, \Phi)$	
			$\nu = 5.6$	
			Φ: mean VCV matrix:	
		v_{t+1}	r^f_{t+1}	exr_{t+1}
	v_{t+1}	0.089	1.2e-6	4.8e-4
	r^f_{t+1}	1.2e-6	5.4e-8	-9.7e-7
	exr_{t+1}	4.8e-4	-9.7e-7	0.0014

Table 1: Prior distribution for multivariate QAR model. The prior for Θ_{11} is truncated to ensure stationarity. The Inverse-Wishart distribution is parameterized as: $p(\Lambda_{10}\Lambda'_{10}; \nu, \Phi) = |\Phi|^{\nu/2}|\Lambda_{10}\Lambda'_{10}|^{-(\nu+n+1)/2}e^{-0.5tr(\Phi(\Lambda_{10}\Lambda'_{10})^{-1})}/(2^{\nu \times n/2}\Gamma_n(\nu/2))$, where $n = 3$ is the number of the exogenous series.

I use the data series for the risk-free rates and aggregate stock excess stock returns constructed by Schorfheide et al. (2018). They obtain the risk-free rate series as a fitted value from a projection of the ex-post real rate on the current nominal yield and inflation over the previous year. The aggregate stock return is the CRSP value-weighted portfolio return of all stocks traded on the NYSE, AMEX, and NASDAQ deflated by CPI[3].

I cast the QAR model into a state space representation to connect the model to observed data. Let X^O_t denotes the observed state vector and I assume the state vector in the model coincides with the observable without measurement errors, following Aruoba et al. (2017): $X_t = X^O_t$. With the definition that $X_t = X_{1,t} + X_{2,t}$, the first-order and second-order

[3]See Schorfheide et al. (2018), Section 5.1. for more details

dynamics produced by the series expansion yield the following econometric specification:

State equation:

$$X_{1,t+1} = \Theta_{10} + \Theta_{11} X_{1,t} + \Lambda_{10} W_{t+1},$$

where the first-order variables $X_{1,t}$ serve as latent variables in the econometric setting.

Observation equation:

$$\begin{aligned} X^o_{t+1} = & \Theta_{10} + \Theta_{11} X^o_t + \Theta_{21} X_{1,t} + \Theta_{23}(X_{1,t} \otimes X_{1,t}) + \Lambda_{10} W_{t+1} + \\ & + \Lambda_{21}(X_{1,t} \otimes W_{t+1}) + \Lambda_{22}(W_{t+1} \otimes W_{t+1}). \end{aligned}$$

Note that the observables have nonlinear conditional means (Θ_{23}) and stochastic volatilities terms (Λ_{21}).

I impose several zero restrictions on the parameters in the observation equation. The first restriction is $\Lambda_{22} = 0$ for two reasons. First, the portfolio formula up to the second approximation does not contain any terms involving Λ_{22} so that this term is not important for investors up to the order of approximation studied in this paper. Moreover, this assumption implies the conditional normality of the predictive density, which saves me from adopting a simulation-based estimation method to approximate the non-analytical likelihood function. This assumption for the practical convenience is also made in the univariate setting of Aruoba et al. (2017).

The second set of restrictions comes from the approximate Euler equations. They imply that the first-order dynamics contain only a constant risk premium and the second-order dynamics contain only the expected excess return predicted by the first-order variables but not second-order variables. These restrictions imply that the row in Θ_{11} and Λ_{23} corresponding to the excess stock return must be zero. Since the shock vector W_{t+1}, is normally distributed, this assumption further implies that the first-order excess stock return is i.i.d. over time.

The restricted model still provides a flexible multivariate framework. It does not restrict any stochastic volatility exposures Λ_{21}, which means that a single variable can be exposed to various sources of stochastic volatility. By contrast, the empirical models used in the literature often postulate a single stochastic volatility (Chacko and Viceira (2005); Moreira and Tyler (2019)) or assume that each innovation loads only on one time-varying volatility for each variable (Buraschi et al. (2010)). I will later show that the multivariate QAR model in this paper capture volatilities of risk-free rates and stock excess returns based on multivariate information beyond the flexibility of the time-series models employed by these past studies.

Moreover, the aggregate vacancy rate and risk-free rate are allowed to have flexible nonlinear conditional means, which is a feature often neglected in the empirical setting in the portfolio literature. However, the univariate study of the QAR model by Aruoba et al. (2017) documents the significance of the nonlinear conditional mean in the Federal Funds rate after 1980, illustrating the asymmetric behavior of this series across recessions and recovery phases of the economy. In the subsequent section, I provide empirical evidence for the presence of nonlinear conditional means in the estimated model.

Following the same estimation methodology as in the univariate case of Aruoba et al. (2017), I estimate the model by a Bayesian method, specifically Random-Walk Metropolis-Hastings algorithm. I adopt a similar specification for the prior as in Aruoba et al. (2017), reported in Table 1. I calibrate the prior distribution so that the posterior distribution has an adequate chance to update the prior information by observable data. I pin down the mean of parameters in the first-order process and Θ_{21} by estimating a linear VAR model using the presample data from 1959M2 to 1971M12. The prior standard deviations of these parameters are larger than the point estimate from the presample VAR estimation. For parameters governing the nonlinear conditional means and stochastic volatilities (Θ_{23} and Λ_{21}), their prior distribution is highly diffuse centered around zero.

Parameters/Dependent	Vacancy v_{t+1}	Riskfree rates r^f_{t+1}	Excess returns exr_{t+1}
Θ_{21} on:			
$v_{1,t}$	zero by restriction	zero by restriction	-0.0025**
$r^f_{1,t}$	zero by restriction	zero by restriction	-1.58
$exr_{1,t}$	zero by restriction	zero by restriction	0.0475
Predictive/Horizon	One month	One year	Five year
Model	0.01	0.046	0.27
Log P-D ratio	0.002	0.015	0.13

Table 2: Loading of excess stock return on lagged state variable and their predictabilities. The the last column in the upper panel reports the posterior means of loadings of the future excess stock return on current first-order state variables. The double asterisk indicates the posterior mean is away from zero by two standard deviations. The lower panel reports the R-squared from the predictability of excess stock returns in the QAR model and predictability regressions onto log price-dividend ratio at different horizons. The QAR model is evaluated at the posterior mean.

4.2 QAR estimation results

Table 8 reports all the estimated posterior means with double asterisks indicating that they are different from zero by two standard deviations of the posterior distribution. I document salient predictability of both the mean and volatility of excess stock returns and other return predictors below.

I start with describing the predictability of excess stock returns in the estimated model. The upper panel in Table 2 reports the coefficients for the linear conditional mean of the excess stock return. The vacancy rate significantly predicts the future excess stock returns and the risk premium is countercyclical with respect to the vacancy rate. Other variables such as the riskfree rate and excess stock return do not have significant predictive power for the future excess stock returns. Figure 1 plots the model-implied forecasts and the realized excess stock returns at one-month, one-year, and five-year horizons. The model-implied risk premium tracks the slow-moving components of the excess returns with R-squared increasing in the predictive horizon.

The lower panel in Table 2 compares the predictability of excess stock returns in the estimated model with that from a predictive regression on a standard return predictor, log price-dividend ratio. The model produces R-squared's at both short and medium frequencies

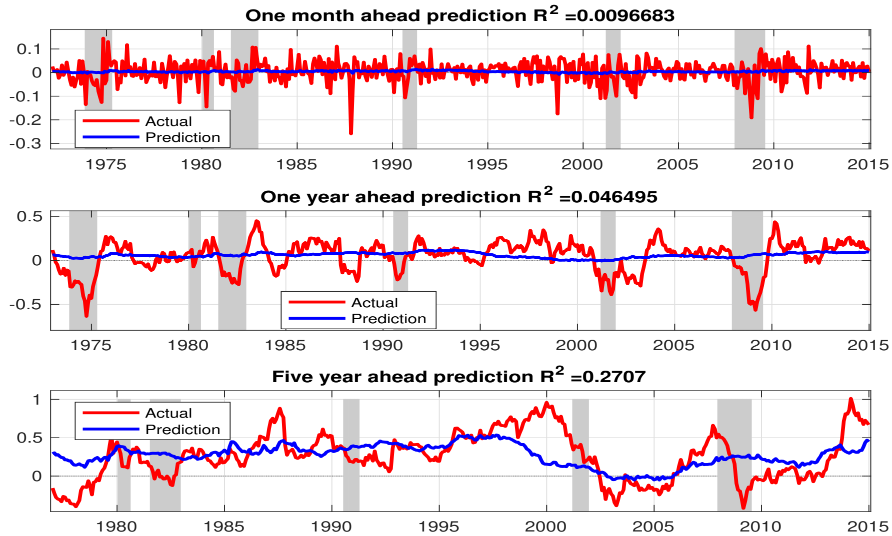

Figure 1: Predicted excess stock returns in the QAR model in blue and actual returns in red at various horizons. Each title also reports the R-squared implied in the model.

comparable to those generated by a standard predictive regression. This validates the estimated model has a reasonable magnitude of return predictability in line with past empirical asset pricing literature.

The upper panel in Table 3 reports the predictability of the linear components of aggregate vacancy rates and risk-free rates. The riskfree rates are predicted not only by its own lag but also by the vacancy rate. This highlights the importance of employing a multivariate model to exploit the cross-predictability across variables.

The lower panel in Table 3 tabulates the coefficients on the nonlinear conditional means of the same variables. The aggregate vacancy rate loads positively on the lagged quadratic vacancy rate. The aggregate vacancy rate drops rapidly but recovers gradually in line with other labor market variables such as unemployment rates observed by Dupraz et al. (2021). This nonlinearity is useful for predicting future risk-free rates but as we will see later, it has a quantitatively small impact on portfolio choice.

The upper panel in Table 4 shows the loadings on the stochastic volatilities and the lower panel tabulates the conditional mean parameters of stochastic volatilities. Recall that

Parameters/Dependent	Vacancy v_{t+1}	Riskfree rates r^f_{t+1}	Excess returns exr_{t+1}
Θ_{11} on:			
$v_{1,t}$	0.995^{**}	$-1.11e-5^{**}$	zero by restriction
$r^f_{1,t}$	8.58^{**}	0.9865^{**}	zero by restriction
$exr_{1,t}$	0.228	$2.66e-4$	zero by restriction
Θ_{23} on:			
$v_{1,t} \times v_{1,t}$	-0.0029^{**}	$6.72e-6$	zero by restriction
$v_{1,t} \times r^f_{1,t}$	-0.9679	$-3.37e-4$	zero by restriction
$v_{1,t} \times exr_{1,t}$	0.0856	$4.11e-4$	zero by restriction
$r^f_{1,t} \times r^f_{1,t}$	0.089	-0.1307	zero by restriction
$r^f_{1,t} \times exr_{1,t}$	-0.1723	-0.2177	zero by restriction
$exr_{1,t} \times exr_{1,t}$	-0.3629	0.0062	zero by restriction

Table 3: Posterior mean estimates of conditional mean parameters for the linear (upper panel) and nonlinear components (lower panel).

Parameters/Dependent	Vacancy rates v_{t+1}	Riskfree rates r^f_{t+1}	Excess returns exr_{t+1}
Λ_{21} on:			
$v_{1,t}W_{t+1}(1)$	-0.042	$2.071e-5$	-0.0018
$v_{1,t}W_{t+1}(2)$	-0.0022	$5.52e-5^{**}$	-0.0017
$v_{1,t}W_{t+1}(3)$	0.0096	$2.46e-5^{**}$	0.0029^{**}
$r^f_{1,t}W_{t+1}(1)$	2.0954	-0.011	1.62
$r^f_{1,t}W_{t+1}(2)$	-0.0832	-0.0267^{**}	-0.1926
$r^f_{1,t}W_{t+1}(3)$	-0.4389	-0.0122	1.416
$exr_{1,t}W_{t+1}(1)$	-0.0063	$2.64e-4$	0.2645
$exr_{1,t}W_{t+1}(2)$	-0.1108	-0.0014^{**}	9.13e-4
$exr_{1,t}W_{t+1}(3)$	-0.4708	$-2.66e-4$	-0.1652^{**}
Θ_{11} on:			
$v_{1,t}$	0.989^{**}	$1.5e-5^{**}$	zero by restriction
$r^f_{1,t}$	8.19^{**}	0.9715^{**}	zero by restriction
$exr_{1,t}$	0.1574	$-2.4e-4$	zero by restriction

Table 4: Posterior mean estimates of loadings on stochastic volatilities (upper panel) and the loadings of stochastic volatilities on lagged variables (lower panel). Recall the stochastic volatilities evolve as $X_{1,t+1} + \Theta_{10} + \Theta_{11}X_{1,t} + \Lambda_{10}W_{t+1}$.

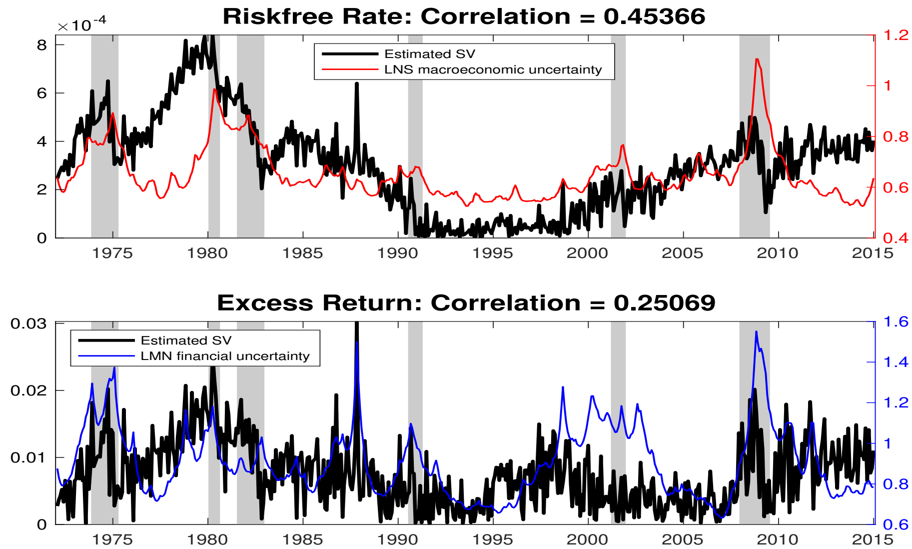

Figure 2: Comparisons of stochastic volatilities. The upper panel plots the estimated stochastic volatility of risk-free rates in black together with the macroeconomic uncertainty measure constructed by Ludvigson et al. (2021) The lower panel plots the estimated stochastic volatility of excess stock returns with the financial uncertainty measure constructed by Jurado et al. (2015) Each title also reports the correlation between the two series in the figure.

the stochastic volatilities are governed by the first-order process. The stochastic volatility of the riskfree rate is predicted by multiple variables including the lagged vacancy rate, risk-free rate, and excess stock return. The excess stock return is exposed to the stochastic volatilities loading on the lagged vacancy rate and excess stock return. On the one hand, since the aggregate vacancy rate is highly persistent, the stochastic volatility associated with this variable describes the low-frequency movement of the stock excess return volatility as in Chacko and Viceira (2005), Bansal et al. (2014), Campbell et al. (2018). On the other hand, since the first-order excess return process is transient, the stochastic volatility loading on this lagged variable corresponds to the high-frequency movement of the excess stock return volatility as in Moreira and Tyler (2019). The empirical result in this paper shows the multivariate nonlinear framework jointly captures the various sources of volatility of each variable.

To provide a comparison with existing literature, I plot the time series of the estimated stochastic volatilities of risk-free rates and excess stock returns with the macroeconomic and financial uncertainty measures constructed by Jurado et al. (2015) and Ludvigson et al. (2021), respectively. Those series are constructed from large sets of macroeconomic and financial variables, respectively, and represent common components of the uncertainty of those types of variables. The volatility of the risk-free rate is correlated with macroeconomic uncertainty while that of the excess stock return co-moves with the financial uncertainty. This highlights that investors are confronted with different types of uncertainty and the importance of incorporating multiple volatilities in investment opportunities.

5 Portfolio selection under estimated nonlinear dynamics

This section explores the implications of the estimated nonlinearities for the optimal portfolio choice with parameters of the underlying dynamics, evaluated at the posterior mean. Section 5.1 explains how estimated nonlinearities affect the distinct components of the optimal portfolio in the historical data. Section 5.2 evaluates the investment performance of the optimal portfolio relative to suboptimal portfolios similar to those studied in the literature in terms of cumulative wealth, Sharpe ratio, and utility-based welfare.

5.1 Dynamic portfolio choices in historical data

I set the discount rate δ to 0.0017 in line with the literature, Campbell et al. (2003). The relative risk aversion is calibrated to 4.27, which implies that the average stock portfolio share is unity in the sample under the homoskedastic VAR model ($\Theta_{23} = \Lambda_{23} = 0$).

Figure 3 plots the various components of the optimal portfolio in black and the portfolio assuming the underlying asset return dynamics are described by the VAR model $\Theta_{23} = \Lambda_{23} = 0$ in red. To smooth out high frequency fluctuations, I plot the 12-month backward moving average of the original portfolio shares. The upper left panel compares the overall optimal

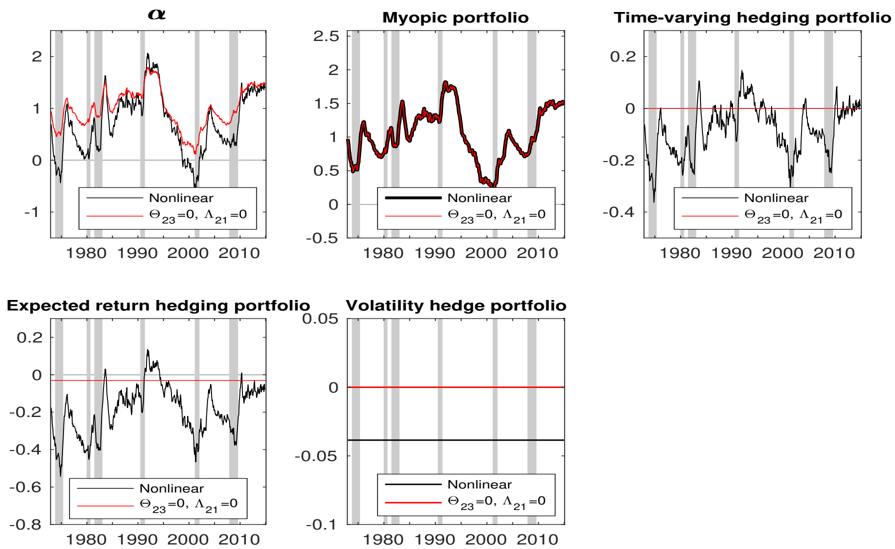

Figure 3: History of model-implied risky portfolio share under the linear model in red and under the QAR model describing the investment opportunities (1972.M12-2014.M12). This figure plots time series of the 12-month backward-moving-average total optimal portfolio weights on stocks for investors with relative risk aversion γ 4.27 and discount rate $\delta = 0.0017$. The upper left panel plots time series of the stock portfolio weight. The rest of the panels decompose the contributions of different components to the total portfolio weight. The upper right panel plots the myopic portfolio demand. The upper left panel plots the portfolio demand associated with the time-varying hedging properties of excess stock returns. The bottom left panel plots the intertemporal hedging portfolio that hedges future expected returns. The bottom middle panel plots the constant hedging portfolio on the volatilities of the future returns.

portfolio with the suboptimal portfolio. Without any nonlinearities, the stock portfolio share would be higher on average and less volatile.

To understand the source of the discrepancy between the two portfolios in Figure 3, Figure 4 compares the optimal portfolio with the portfolio that ignores the stochastic volatility by setting $\Lambda_{21} = 0$. The latter portfolio behaves similarly to the linear portfolio in Figure 3, implying that the stochastic volatility, not the nonlinear conditional mean, is the main source of the nonlinearity that affects the optimal portfolio.

Myopic portfolio, plotted in the upper middle panel, tends to keep rising during the NBER recessions since the aggregate vacancy rate tends to keep declining during these recessions and negatively predicts the future excess stock return. The upper right panel shows the time-varying hedging portfolio, which shows how the stochastic volatility changes

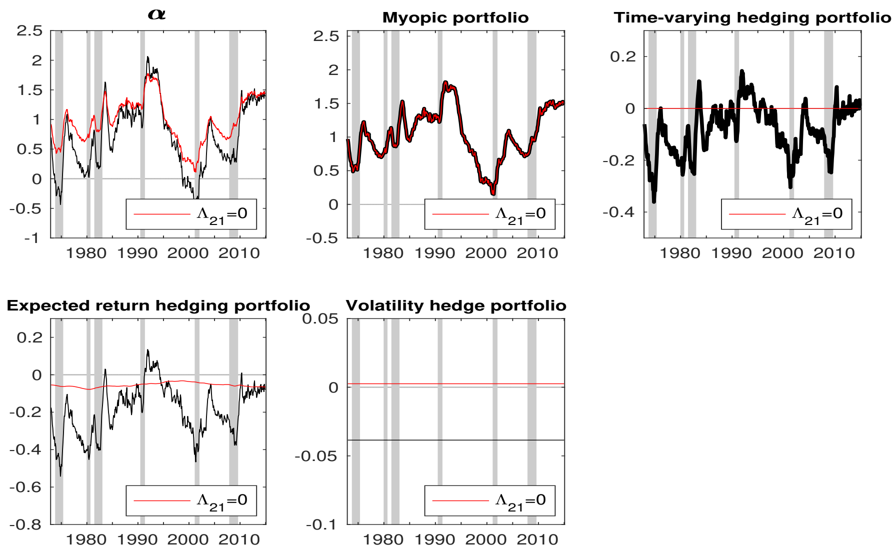

Figure 4: History of model-implied stock portfolio shares by risk averse investors under the QAR model without stochastic volatilities $\Lambda_{21} = 0$ and under the QAR model describing the investment opportunities (1972.M12-2014. M12). For the description of each panel and construction of the series, see the label of Figure 3.

the hedging property of the excess stock return against the one-period-ahead unexpected change in excess stock return and persistent changes in risk-free rates. Since this portfolio tends to decline during the NBER recessions, the stochastic volatility induces the excess stock return to be a poor hedge against these risks.

The lower left panel plots the expected return hedging portfolio, which shows how the stochastic volatility changes the hedging demand for risky asset agains unexpected changes in future portfolio returns. This portfolio also drops during the recessions. On average, the excess stock return is a poor hedging against the portfolio return risk. In addition, this poor hedging property becomes more severe during the NBER recessions.

The lower middle panel plots the volatility hedging portfolio, which hedges the future rise in the volatility of the portfolio return. This portfolio is negative since when the stock delivers lower return, it increases the volatility of the future excess stock return, increasing the volatility of the future portfolio return.

Overall, the intertemporal hedging portfolio recommends reducing the risky portfolio

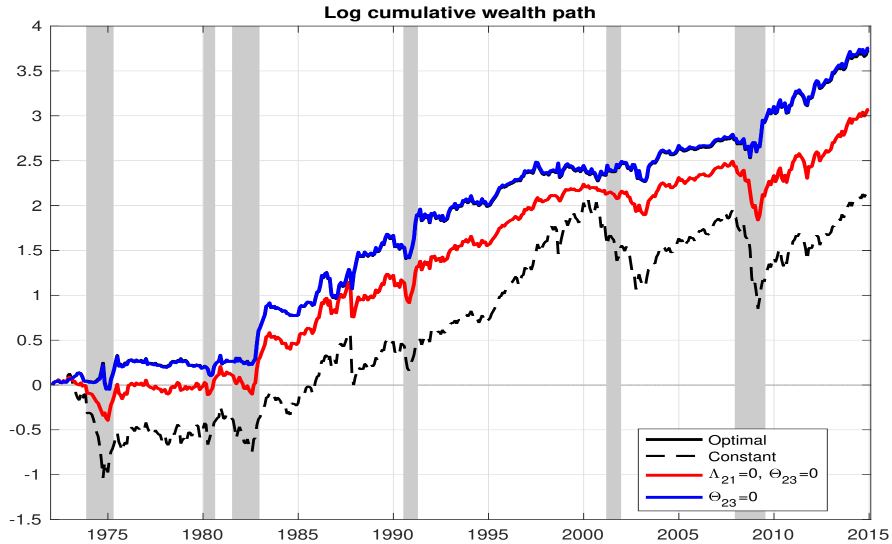

Figure 5: Log cumulative paths of various model-implied portfolios between 1972.M12-2014.M12. At the initial month, every investor starts with \$1 and continuously reinvests all the cumulative returns on a stock and riskfree bond, following a portfolio decision rules. The optimal portfolio is in black, the portfolio ignoring nonlinear conditional means $\Theta_{23} = 0$ in blue, linear portfolio ignoring all the nonlinearities $\Theta_{23} = \Lambda_{21} = 0$ in red, and constant portfolio $\Theta_{23} = \Lambda_{21} = \Theta_{21} = 0$ in dashed black line.

share during the recessions due to the heightened stochastic volatility. This model recommendation is consistent with the observation by Nagel et al. (2021) that the stock selling during the recent financial crisis in 2008 and 2009 is largely due to stock investors in the top 1% and top 0.1 % of the overall income distribution, retired individuals and individuals at the very top of the dividend income distribution. Their wealth is more heavily tilted toward financial assets rather than labor income so they are highly vulnerable to the large volatility movements in the financial market. As a result, in spite of the hike in the risk premium, they unload the exposure to the stock market risks during these market turmoils.

5.2 Portfolio performance

Now I investigate the performance of the optimal portfolio relative to suboptimal portfolios that do not or only partially incorporate the nonlinearities into the intertemporal hedging portfolios. For this purpose, I compute various standard measures of portfolio performances

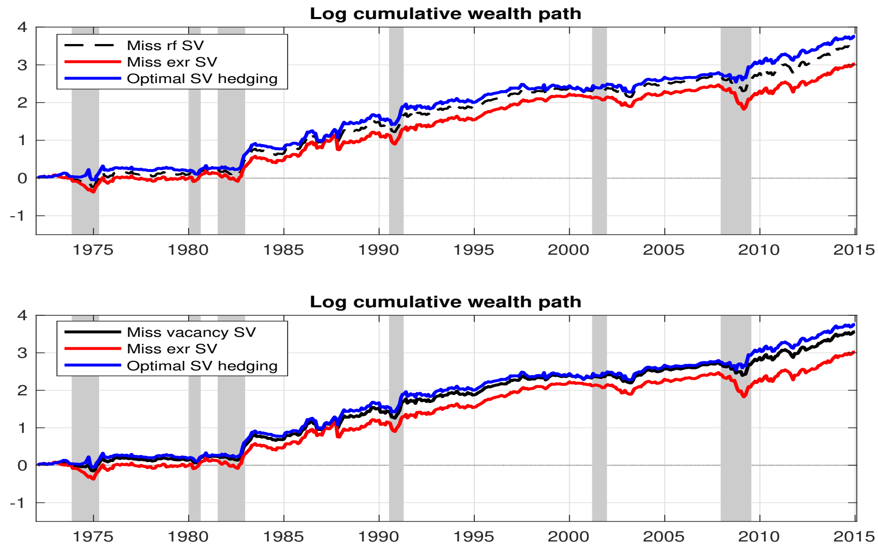

Figure 6: Log cumulative paths of various model-implied portfolios between 1972.M12-2014.M12. In the initial month, every investor starts with \$1 and continuously reinvests all the cumulative returns, following the given portfolio decision rules. The portfolio with optimal hedging of stochastic volatility but ignoring nonlinear conditional means $\Theta_{23} = 0$ in blue, In the upper panel, the portfolio ignoring nonlinear conditional means and stochastic volatilities of excess stock returns is in red, and the portfolio ignoring nonlinear conditional means and stochastic volatilities of riskfree rates is in dashed black line. In the lower panel, the portfolio ignoring nonlinear conditional means and stochastic volatilities of excess stock returns is in red, and the portfolio ignoring nonlinear conditional means and stochastic volatilities of aggregate vacancy rates is in dashed black line.

Optimal (5)	constant (1)	$\Theta_{23} = 0,\ \Lambda_{21} = 0$ (2)	
\$41.71	\$8.69	\$21.58	
$\Theta_{23} = 0,\ \Lambda_{21}(v) = 0$ (3)	$\Theta_{23} = 0,\ \Lambda_{21}(r_f) = 0$ (3)	$\Theta_{23} = 0,\ \Lambda_{21}(exr) = 0$ (3)	$\Theta_{23} = 0$ (4)
\$35.24	\$34.13	\$20.57	\$42.83

Table 5: Final wealth level after the portfolio investments between 1972.M12-2014.M12. At the initial month, every investor starts with \$1 and continuously reinvests all the cumulative returns on a stock and riskfree bond, following various portfolio decision rules.

including the cumulative wealth path, Sharpe ratio, and the utility-based economic welfare, using these portfolios. I consider the following types of portfolios: (1) constant portfolio weight determined by constant myopic portfolio and constant risk-free rate hedging portfolio ($\Theta_{21} = 0$, $\Theta_{23} = 0$, $\Lambda_{21} = 0$), (2) VAR portfolio which is the sum of time-varying myopic portfolio and constant risk-free rate hedging portfolio ($\Theta_{23} = 0$, $\Lambda_{21} = 0$), (3) portfolios hedging only stochastic volatilities of a single variable ($\Theta_{23} = 0$, $\Lambda_{21} = 0$ for one row), (4) fully volatility hedging portfolio ($\Theta_{23} = 0$), and (5) optimal portfolio.

Figure 5 plots the log cumulative wealth paths of the investors who start with one dollar as the initial wealth at the start of the sample and follow various portfolio rules (1), (2), (4), and (5). The fully volatility hedging portfolio (4) in blue the optimal portfolio (5) in black, meaning that it performs almost equally well to the optimal one and implying the little quantitative importance of the nonlinear conditional means during the sample. These two portfolios consistently dominate other portfolios (1) and (2), highlighting the importance of the stochastic volatilities in portfolio determinations. In particular, during recessions or market crashes, these portfolios lose less money due to the intertemporal hedging portfolio arising from the stochastic volatilities. As a result, the wealth path is more stable under the management of these two portfolio without reducing the portfolio returns outside of these adverse events.

In order to further uncover the role of each stochastic volatility, I plot the log cumulative wealth paths generated by portfolios (3) and (4) in Figure 7. The red lines in the figure show that failure to incorporate the stochastic volatilities of excess stock returns yield significant wealth losses especially during adverse events of the market. Moreover, the black dashed and black solid line describes the paths generated by the portfolios that fail to incorporate the stochastic volatilities of riskfree rates and aggregate vacancy rates, respectively. These portfolios perform more poorly than the optimal portfolio and fully volatility hedging portfolio. Table 5 shows the wealth level at the end of the sample.

The other performance measures further corroborates the above finding from the realized

Optimal (5)	constant (1)	$\Theta_{23} = 0,\ \Lambda_{21} = 0$ (2)	
18.8%	9.9%	15.6%	
$\Theta_{23} = 0,\ \Lambda_{21}(v) = 0$ (3)	$\Theta_{23} = 0,\ \Lambda_{21}(r_f) = 0$ (3)	$\Theta_{23} = 0,\ \Lambda_{21}(exr) = 0$ (3)	$\Theta_{23} = 0$ (4)
18.8%	17.9%	15.7%	18.7%

Table 6: Average Sharpe ratios of various portfolio investment strategies between 1972.M12-2014.M12. At the initial month, every investor starts with $1 and continuously reinvests all the cumulative returns on a stock and riskfree bond, following portfolio decision rules.

constant	$\Theta_{23} = 0,\ \Lambda_{21} = 0$	$\Theta_{23} = 0,\ \Lambda_{21}(v) = 0$
91%	48.3%	10.4%
$\Theta_{23} = 0,\ \Lambda_{21}(r_f) = 0$	$\Theta_{23} = 0,\ \Lambda_{21}(exr) = 0$	$\Theta_{23} = 0$
18.8%	48.6%	0.13%

Table 7: Equivalent lifetime utility losses in terms of initial wealth from following suboptimal portfolios. Each entry shows the percentage of the initial wealth loss investors are willing to lose if they give up a suboptimal portfolio and adopt the optimal portfolio.

wealth path that the multiple sources of stochastic volatility is crucial in investment outcomes. Table 6 displays the average Sharpe ratios generated in the sample by the portfolios from (1) to (5) and Table 7 tabulates the equivalent initial wealth loss that investors are willing to give up in order to switch from following the suboptimal portfolios to the optimal portfolios evaluated at unconditional moments of the QAR model. For the latter measure, since the elasticity of intertemporal substitutions is unity, this initial wealth loss is equivalent to the loss of consumption by the same amount every period.

6 Conclusion

In this paper, I explored the implications of predictable nonlinear components for portfolio decision making in a new flexible nonlinear time series model. Given a reasonable degree of excess stock return predictability by aggregate vacancy rates in the estimated multivariate QAR model, I found evidence for nonlinear conditional means and stochastic volatilities that are predictable based on multivariate information, not just univariate one. The newly developed optimal portfolio solution suggests the practical importance of comprehensively incorporating these nonlinear predictabilities into the intertemporal hedging portfolios in terms of various measures of portfolio performance.

The framework used in this paper can be used for other settings. For example, the approximate solution to the optimal portfolio can be applied in the international portfolio allocation to investigate the benefit of international diversification under nonlinear dynamics as in Ang and Bakaert (2002), Benigno and Nistico (2012), and Coeurdacier and Gourinchas (2016). Moreover, the framework can be applied to household consumption-saving problems in the presence of nontradable labor incomes as in Viceira (2001). These extensions would further shed light on observed behaviors of various investors and practical investment advice under nonlinear dynamics. These potential extensions are left for future research.

A. Appendix

This Appendix provides additional details of the results described in the main text. Section A.1 reviews the preliminary algebra, while Section A.2 provides formula to compute various objects in the second-order dynamics. Section A.3 approximates the utility recursion and A.4 derives the exogenous dynamics under a risk-neutral probability. Section A.5 and A.6 give the full approximate solution for the portfolio choice problem.

A.1 Preliminary algebra

I provide some details on some algebraic operations used in the paper, especially rules of Kronecker products. Materials are heavily borrowed from Borovicka and Hansen (014b).

1. For an $m \times n$ matrix H, $vec(H)$ produces a column vector of length mn created by stacking the columns of H:

$$h_{(j-1)m+i} = [vec(h)]_{(j-1)m+i} = H_{ij}.$$

2. For a vector (column or row) h of length mn, $mat_{m,n}(h)$ produces an $m \times n$ matrix H created by 'columnizing' the vector:

$$H_{ij} = [mat_{m,n}(h)]_{ij} = h_{(j-1)m+i}.$$

3. For $X_{n\times 1}$,

$$vec(XX') = X \otimes X. \tag{3}$$

As an application,

$$\begin{aligned}\mathbb{E}_t(W_{t+1} \otimes W_{t+1}) &= \mathbb{E}_t(vecW_{t+1}W'_{t+1}) = \\ &= vec\mathbb{E}_t(W_{t+1}W'_{t+1}) = \\ &= vecI_{k\times k}.\end{aligned}$$

4. For a row vector $H_{1\times nk}$ and column vectors $X_{n\times 1}$ and $W_{n\times 1}$,

$$H(X\otimes W)=X'[mat_{k,n}(H)]'W. \tag{4}$$

5. For $A_{n\times k}$, $X_{n\times 1}$, and $W_{k\times 1}$,

$$X'AW=(vecA')'(X\otimes W). \tag{5}$$

As an application, for a $n\times 1$ vector a,

$$\begin{aligned} a'XX'a &= (\Theta_{11}'\iota_{\Delta c})'(XX')(\Theta_{11}'\iota_{\Delta c}) = \\ &= (vec(XX')')'(a'\otimes a) = \\ &= (a\otimes a)'vec(XX') = \\ &= (a\otimes a)'(X\otimes X). \end{aligned}$$

6. For $A_{n\times n}$, $X_{n\times 1}$, $W_{k\times 1}$,

$$(AX)\otimes W=(A\otimes W)X, \tag{6}$$

$$W\otimes (AX)=(W\otimes A)X. \tag{7}$$

7. For matrices A, B, C, and D such that AC and BD are well-defined,

$$(A\otimes B)(C\otimes D)=(AC)\otimes (BD). \tag{8}$$

A.2 Useful Algebra

1. The infinite expected discounted sum of the first-order variables:

$$\sum_{i=0}^{\infty} \exp(-\delta i)\mathbb{E}_t(X_{1,t+1+i}) = \big\{[1-\exp(-\delta)]^{-1} I - \Theta_{11}[I-\exp(-\delta)\Theta_{11}]^{-1}\big\}[I-\Theta_{11}]^{-1}\Theta_{10} + \\ + \Theta_{11}[I-\exp(-\delta)\Theta_{11}]^{-1}X_{1,t}.$$

2. The innovation on the expected discounted sum of future exogenous second-order state variables:

$$\begin{aligned}
&\sum_{i=0}^{\infty} \exp(-\delta i)\Delta\tilde{\mathbb{E}}_{t+1}(X_{2,t+1+i}) = \tilde{\Sigma}_{2,t,\tilde{W}}\tilde{W}_{t+1} + \tilde{\Sigma}_{2,\tilde{W}\tilde{W}}(\tilde{W}_{t+1}\otimes\tilde{W}_{t+1}) \\
&\quad \Big\{[I-\exp(-\delta)\Theta_{11}]^{-1}[\Lambda_{21}(X_{1,t}\otimes I_{k\times k})] + \\
&\quad + \exp(-\delta)[I-\exp(-\delta)\Theta_{11}]^{-1}\tilde{\Theta}_{21}[I-\exp(-\delta)\Theta_{11}]^{-1}\Lambda_{10} + \\
&\quad + \exp(-\delta)[I-\exp(-\delta)\Theta_{11}]^{-1}\Theta_{23}\big\{([I-\Theta_{11}]^{-1}\tilde{\Theta}_{10})\otimes([I-\exp(-\delta)\Theta_{11}]^{-1})\Lambda_{10} - \\
&\quad - [I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(\Theta_{11}[I-\Theta_{11}]^{-1}\tilde{\Theta}_{10}\otimes I_{n\times n})\Lambda_{10} + \\
&\quad + ([I-\exp(-\delta)\Theta_{11}]^{-1}\otimes[I-\Theta_{11}]^{-1}\tilde{\Theta}_{10})\Lambda_{10} - \\
&\quad - [I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(I_{n\times n}\otimes\Theta_{11}[I-\Theta_{11}]^{-1}\tilde{\Theta}_{10})\Lambda_{10} + \\
&\quad + [I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(\tilde{\Lambda}_{10}\otimes\Theta_{11})(I_{k\times k}\otimes X_{1,t}) + \\
&\quad + [I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(\Theta_{11}\otimes\tilde{\Lambda}_{10})(X_{1,t}\otimes I_{k\times k})\big\}\Big\}\tilde{W}_{t+1} + \\
&\quad + \Big\{[I-\exp(\delta)\Theta_{11}]^{-1}\Lambda_{22} + \\
&\quad + \exp(-\delta)[I-\exp(-\delta)\Theta_{11}]^{-1}\Theta_{23}[I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(\Lambda_{10}\otimes\Lambda_{10})\Big\}(\tilde{W}_{t+1}\otimes\tilde{W}_{t+1}).
\end{aligned} \tag{9}$$

3. The linear combination of the time-varying volatility matrix can be expressed as a

linear function of $X_{1,t}$:

$$x'\tilde{\Sigma}_{2,t,\tilde{W}}y = x'\tilde{\Sigma}_{2,\tilde{W}}y + \Big\{ y' mat_{n\times n}(x'[I-\exp(-\delta)\Theta_{11}]^{-1}\Lambda_{21}) +$$

$$+ \exp(-\delta)y'\big[mat_{n\times n}\big(x'[I-\exp(-\delta)\Theta_{11}]^{-1}\Theta_{23}[I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(\Lambda_{10}\otimes\Theta_{11})\big)\big]'+$$

$$+ \exp(-\delta)y' mat_{n\times n}\big(x'[I-\exp(-\delta)\Theta_{11}]^{-1}\Theta_{23}[I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(\Theta_{11}\otimes\Lambda_{10})\big)\Big\} X_{1,t}$$

where

$$\tilde{\Sigma}_{2,\tilde{W}} = \Big\{ \exp(-\delta)[I-\exp(-\delta)\Theta_{11}]^{-1}\tilde{\Theta}_{21}[I-\exp(-\delta)\Theta_{11}]^{-1}\Lambda_{10}+$$

$$+ \exp(-\delta)[I-\exp(-\delta)\Theta_{11}]^{-1}\Theta_{23}\big\{[([I-\Theta_{11}]^{-1}\tilde{\Theta}_{10})\otimes([I-\exp(-\delta)\Theta_{11}]^{-1})]\Lambda_{10}-$$

$$- [I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(\Theta_{11}[I-\Theta_{11}]^{-1}\tilde{\Theta}_{10}\otimes I_{n\times n})\Lambda_{10}+$$

$$+ ([I-\exp(-\delta)\Theta_{11}]^{-1}\otimes[I-\Theta_{11}]^{-1}\tilde{\Theta}_{10})\Lambda_{10}-$$

$$- [I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(I_{n\times n}\otimes\Theta_{11}[I-\Theta_{11}]^{-1}\tilde{\Theta}_{10})\Lambda_{10}\big\}\Big\}.$$

A.3 Approximating the utility recursion

The following sections describe the steps to derive the semi-analytical solution for portfolio choice. I characterize up to first order, $\alpha_t = \alpha_0 + \alpha_{1,t}$, where α_0 is a constant zeroth-order portfolio derived from the first-order dynamics. $\alpha_{1,t}$ is a time-varying portfolio as a linear combination of the first-order variables $X_{1,t}$, which is derived from the second-order dynamics so it incorporates the implications of the nonlinear conditional mean and heteroskedasticity in the asset return dynamics.

The starting point of the approximation is to expand the Epstein- Zin utility recursion. Unlike the standard perturbation method, I scale the relative risk aversion parameter by the perturbation parameter $1/q$ jointly scales the volatilities of the shocks W_{t+1} by q, following Borovička and Hansen (2014b)[4] The third-order expansion of the scaled continuation value

[4]Alternatively, the adjusted parameter can be regarded as the penalty parameter associated with the relative entropy in the robust control problem of Hansen and Sargent (2008) or the risk sensitive parameter of

is given by

$$\log U_{t+1} \log V_t - \log C_t \approx \log U_{0,t+1} + q \cdot \log U_{1,t+1} + \frac{q^2}{2} \log U_{2,t+1} + \frac{q^3}{6} \log U_{3,t+1}.$$

Similarly, the stochastic discount factor is now parameterized by q and approximated up to second order

$$\begin{aligned} M_{t+1}(q) =& \exp(-\delta - \Delta \log C_{t+1}) \frac{\exp[\frac{1-\gamma}{q}(\log U_{t+1}(q) + \Delta \log C_{t+1}(q))]}{\mathbb{E}_t \exp[\frac{1-\gamma}{q}(\log U_{t+1}(q) + \Delta \log C_{t+1}(q))]} \\ \approx & M_{0,t+1} + q \cdot M_{1,t+1} + \frac{q^2}{2} M_{2,t+1}. \end{aligned}$$

I will repeatedly utilize the zeroth-order stochastic discount factors to describe the risk-neutral dynamics of the exogenous process.

The zeroth-order term is characterized by taking the derivative with respect to q and evaluating both sides at $q = 0$.

$$\begin{aligned} M_{0,t+1} =& \exp(-\delta - \Delta \log C_{0,t+1}) \frac{\exp[-(\gamma-1)(\log U_{1,t+1} + \Delta \log C_{1,t+1})]}{\mathbb{E}_t \exp[-(\gamma-1)(\log U_{1,t+1}(q) + \Delta \log C_{1,t+1})]} \\ =& \exp(-\delta - \Delta \log C_0) \tilde{M}_{0,t+1}, \end{aligned}$$

where $\tilde{M}_{0,t+1}$ is the zeroth-order change of measure, satisfying $\mathbb{E}_t \tilde{M}_{0,t+1} = 1$

$$\tilde{M}_{0,t+1} \frac{\exp[-(\gamma-1)(\log U_{1,t+1} + \Delta \log C_{1,t+1})]}{\mathbb{E}_t \exp[-(\gamma-1)(\log U_{1,t+1} + \Delta \log C_{1,t+1})]}. \tag{10}$$

This change of measure captures the risk-adjustment of the objective probability measure. I denote this conditional risk neutral measure by $\tilde{\mathbb{P}}_t$ and the associated expectational operator by $\tilde{\mathbb{E}}_t$.

Similarly, the first-order term is obtained similarly by taking the second-order derivative

Tallarini (2000). The results here can be regarded as the empirically tractable framework of robust portfolio selection model of Maenhout (2004). The current framework can incorporate a flexible time-series evolution of investment opportunities without restricting them to a set of i.i.d. stochastic processes.

with respect to q:

$$M_{1,t+1} = -M_{0,t+1}\Big\{\Delta\log C_{1,t+1} + \frac{\gamma-1}{2}\big[\log U_{2,t+1} + \Delta\log C_{2,t+1} - \\ - \mathbb{E}_t(\tilde{M}_{0,t+1}[\log U_{2,t+1} + \Delta\log C_{2,t+1}])\big]\Big\} \tag{11}$$

Now I express these approximated continuation values and SDF in terms of consumption growth moments. The utility recursion is parameterized by the perturbation parameter q and I scale the relative risk aversion parameter in the same manner as I did for the stochastic discount factor:

$$\log U_t(q) = -\frac{\exp(-\delta)}{\gamma-1} q \cdot \log\mathbb{E}_t\Bigg(\exp\Big[-\frac{\gamma-1}{q}(\log U_{t+1}(q) + \Delta\log C_{t+1}(q))\Big]\Bigg).$$

The first-order continuation value is obtained by taking derivative of the utility recursion with respect to q and evaluating at $q=0$: using the log-normality,

$$\begin{aligned}\log U_{1,t} =& -\frac{\exp(-\delta)}{\gamma-1}\log\mathbb{E}_t\Bigg(\exp\Big[-(\gamma-1)(\log U_{1,t+1} + \Delta\log C_{1,t+1})\Big]\Bigg) \\ &= \exp(-\delta)\mathbb{E}_t[\log U_{1,t+1} + \Delta C_{1,t+1}] - \exp(-\delta)\frac{\gamma-1}{2}\mathbb{V}_t(\Delta\mathbb{E}_{t+1}[\log U_{1,t+1} + \Delta C_{1,t+1}]).\end{aligned}$$

This expansion highlights the mean-variance tradeoff of the evolution of consumption growth rates in first order. Note that the variance term is constant over time due to the homoskedastic structure of the first-order exogenous processes[5]. Solved forward, the first-order continuation value is the discounted sum of future consumption growth rates up to an additive constant.

$$\log U_{1,t} = \mathbb{E}_t\Bigg[\sum_{i=1}^{\infty}\exp(-\delta i)\Delta\log C_{1,t+i}\Bigg] - \frac{\gamma-1}{2}\sum_{i=1}^{\infty}\mathbb{E}_t\mathbb{V}_{t+i-1}(\Delta\mathbb{E}_{t+i}[\log U_{1,t+i} + \Delta C_{1,t+i}]). \tag{12}$$

I characterize the implication of the risk adjustment in terms of 1st-order consumption

[5]See Hansen et.al. (2007) for a similar argument in log-normal framework with IES = 1.

growth moments by obtaining the change of measure induced by the risk adjustment in terms of these moments:

$$\tilde{M}_{0,t+1} = \frac{\exp[-(\gamma-1)\Delta\mathbb{E}_{t+1}\sum_{i=0}^{\infty}\exp(-\delta i)\Delta\log C_{1,t+1+i}]}{\mathbb{E}_t\exp[-(\gamma-1)\Delta\mathbb{E}_{t+1}\sum_{i=0}^{\infty}\exp(-\delta i)\Delta\log C_{1,t+1+i}]}, \tag{13}$$

where $\Delta\mathbb{E}_{t+1}Y_{t+1}$ is the innovation on Y_{t+1} arriving at period $t+1$ defined as $\mathbb{E}_{t+1}Y - \mathbb{E}_t Y$. The distribution of the shock W_{t+1} under the zeroth-order risk neutral measure $\tilde{\mathbb{P}}_t$ has a constant distorted mean denoted by $\tilde{\mu}(W)$.

Similarly, the second-order continuation value is expressed as the discounted sum of expected consumption growth rates under the risk neutral measure $\tilde{\mathbb{P}}_t$:

$$\log U_{2,t} = \tilde{\mathbb{E}}_t \sum_{i=1}^{\infty} \exp(-\delta i)\Delta\log C_{2,t+1}. \tag{14}$$

Finally, I derive the third-order utility recursions:

$$\log U_{3,t} = -\frac{3}{4}(\gamma-1)\exp(-\delta)\tilde{\mathbb{V}}_t(\log U_{2,t+1}+\Delta\log C_{2,t+1})+\exp(-\delta)\tilde{\mathbb{E}}_t(\log U_{3,t+1}+\Delta\log C_{3,t+1}).$$

The third-order utility recursion features the mean-variance tradeoff of the evolution of consumption growth rates in second order. Solving forward for the third-order utility recursion:

$$\log U_{3,t} = -\frac{3}{2}(\gamma-1)\sum_{i=1}^{\infty}\exp(-\delta i)\tilde{\mathbb{E}}_t\tilde{\mathbb{V}}_{t+i-1}(\log U_{2,t+i}+\Delta\log C_{2,t+i})+\sum_{i=1}^{\infty}\exp(-\delta i)\tilde{\mathbb{E}}_t(\Delta\log C_{3,t+i}).$$

Now I express the zeroth-order stochastic discount factor in terms of consumption growth moments. Due to the log-normality of the first-order processes, the zeroth-order stochastic discount factor is given by

$$\begin{aligned}
M_{0,t+1} =& \exp(-\delta - \Delta \log C_0) \exp\Big[-(\gamma-1)\Delta\mathbb{E}_t\big(\sum_{i=1}^{\infty} \exp(-\delta i)\Delta \log C_{1,t+i+1} + \Delta \log C_{1,t+1}\big) \\
& - \frac{(\gamma-1)^2}{2}\mathbb{E}_t\mathbb{V}_{t+i-1}\big(\Delta\mathbb{E}_{t+i}\big(\sum_{i=0}^{\infty} \exp(-\delta i)\Delta \log C_{1,t+i+1}\big)\big)\Big] \\
=& \exp(-\delta - \Delta \log C_0) \exp\Big[-(\gamma-1)\Delta\mathbb{E}_t\big(\sum_{i=0}^{\infty} \exp(-\delta i)\Delta \log C_{1,t+i+1}\big) + \mathit{constant}\Big].
\end{aligned} \tag{15}$$

The zeroth-order SDF has a constant exposure to the underlying innovations in first order dynamics. Similarly, the first-order stochastic discount factor is expressed in terms of the expected consumption growth rates under the risk-neutral measure.

A.4 Exogenous dynamics under risk-neutral measures

Given the zeroth-order risk-neutral measures derived in the previous section, I construct the evolutions of the exogenous processes under those measures as in Hansen et.al. (2007) and Borovicka and Hansen (2014b).

$$X_{1,t+1} = \tilde{\Theta}_{10} + \Theta_{11}X_{1,t} + \Lambda_{10}\tilde{W}_{t+1} \tag{16}$$

$$\begin{aligned}
X_{2,t+1} =& \tilde{\Theta}_{21}X_{1,t} + \Theta_{11}X_{2,t} + \Theta_{23}(X_{1,t} \otimes X_{1,t}) \\
& + \tilde{\Lambda}_{20}\tilde{W}_{t+1} + \Lambda_{21}(X_{1,t} \otimes \tilde{W}_{t+1}) + \Lambda_{22}(\tilde{W}_{t+1} \otimes \tilde{W}_{t+1}),
\end{aligned} \tag{17}$$

where $\tilde{W}_{t+1}$ is a normal random vector with unit standard deviation, orthogonal each other contemporaneously and serially under the zeroth-order risk-neutral measure. The coefficients are defined as:

$$\tilde{\Theta}_{10} = \Theta_{10} + \Lambda_{10}\tilde{\mu}(W)$$

where

$$\tilde{\Theta}_{21} = \Theta_{21} + \Lambda_{21}[I_{n\times n} \otimes \tilde{\mu}(W)]$$

$$\tilde{\Lambda}_{20} = \Lambda_{22}\big\{[\tilde{\mu}(W) \otimes I_{k\times k}] + [I_{k\times k} \otimes \tilde{\mu}(W)]\big\}.$$

The adjustment for the constant term Θ_{10} comes from the risk adjustment from the shock exposure in the first order process. The coefficient matrix for the first order lagged terms Θ_{21} arises from the risk adjustment from the exposure to the stochastic volatility Λ_{21}.

A.5 Solution to the asset allocation problem

The starting point of deriving the approximate solution to the dynamic programming problem is the observation that the unitary IES implies the constant wealth-consumption ratio. Then intertemporal budget constraint restricts the wealth growth rate to equal to the return to the entire wealth portfolio r^p. Therefore, the consumption growth rate is also proportional to this return

$$\Delta \log C_{t+1} \propto r^p_{t+1}.$$

The approximated wealth portfolio returns are

$$r^p_{1,t+1} = \alpha'_{0,t} exr_{1,t+1} + r^f_{1,t}$$

$$r^p_{2,t+1} = \alpha'_{1,t} exr_{1,t+1} + \alpha'_{0,t} exr_{2,t+1} + r^f_{2,t}.$$

Note that the zeroth-order excess returns are always zero. I assume that the variations up to second order capture the whole fluctuations of varialbes so that

$$\Delta \log C_{3,t+i} = \alpha'_{1,t+i-1} exr_{2,t+i}.$$

I derive the zeroth-order portfolio weights by replacing the consumption growth with the wealth return in the first-order utility recursion and taking the first-order condition with respect to $\alpha_{0,t}$:

$$\frac{1}{\gamma-1}\mathbb{E}_t(exr_{1,t+1}) = \mathbb{E}_t\left(\left[\Delta\mathbb{E}_{t+1}\sum_{i=1}^{\infty}\Delta \log C_{1,t+i}\right]\cdot exr_{1,t+1}\right).$$

Solving for the portfolio weight α_0,

$$\alpha_0 = \frac{1}{\gamma - 1}(\Sigma_{exr_1}\Sigma'_{exr_1})^{-1}\mathbb{E}_t(exr_{1,t+1}) - (\Sigma_{exr_1}\Sigma'_{exr_1})^{-1}\Sigma_{exr_1}\Sigma'_{r_1^f},$$

where r_0 is the zeroth-order constant asset return, Σ_{exr_1} is the exposure of the excess returns to risky assets to underlying innovations in the system

$$\Sigma_{exr_1}W_{t+1} = \Delta\mathbb{E}_{t+1}(exr_{1,t+1}),$$

and $\Sigma_{r_1^f}$ is the exposure of the future expected discounted sum of riskfree rate returns to innovations

$$\Sigma_{r_1^f}W_{t+1} = \exp(-\delta)\iota'_{r_f}\Theta_{11}[I - \exp(-\delta)\Theta_{11}]^{-1}.$$

The first term in the right hand side of the zeroth-order portfolio is a myopic portfolio. The second-term is the intertemporal hedging portfolio on the changes in the future expected risk-free rates. Note that terms other than the risk premium $\mathbb{E}_t(exr_{1,t+1})$ are constant, implying the restriction that the first-order risk premium should also be constant.

Given the zeroth-order portfolio weight, I obtain the fully structural characterizations of the zeroth-order change of measure associated with the risk-neutral probability $\tilde{M}_{0,t+1}$.The future consumption growth innovation is expressed as

$$\sum_{i=0}^{\infty}\exp(-\delta i)\Delta\mathbb{E}_{t+1}(\Delta\log C_{1,t+1}) = \sum_{i=0}^{\infty}\exp(-\delta i)\Delta\mathbb{E}_{t+1}(r^p_{1,t+1+i}).$$

The zeroth-order SDF is now expressed in terms of portfolio weights and exogenous objects:

$$\tilde{M}_{0,t+1} = r_0^{-1}\exp\left[- (\gamma - 1)(\alpha_0 \cdot \Sigma_{exr_1}W_{t+1} + \Sigma_{r_1^f}W_{t+1})\right], \tag{18}$$

Moreover, it turns out that the zeroth-order portfolio obtained above satisfies the first-order expansion of the Euler equation:

$$0 = \mathbb{E}_t[M_{0,t+1} r_{1,t+1}] + \mathbb{E}_t[M_{1,t+1} r_{0,t+1}].$$

Then the Euler equation for the excess returns is now reduced to

$$0 = \mathbb{E}_t[M_{0,t+1} exr_{1,t+1}]$$
$$\Rightarrow 0 = \tilde{\mathbb{E}}_t[exr_{1,t+1}],$$

implying that the first-order risk premia are zero under the zeroth-order risk-neutral measure.

I utilize the third-order utility recursion to derive the first-order portfolio since the second-order utility recursion cannot pin down the first-order portfolio due to the property above that the first-order risk premium is always zero. By substituting out the third-order consumption growth with the wealth portfolio return, the first-order condition with respect to $\alpha_{1,t}$ leads to:

$$0 = \tilde{\mathbb{E}}_t(exr_{2,t+1}) - \frac{3}{2}(\gamma - 1)\tilde{\mathbb{E}}_t\left[\Delta\tilde{\mathbb{E}}_{t+1}(\sum_{j=0}^{\infty} \exp(-\delta i)\Delta \log C_{2,t+1+j}) \cdot exr_{1,t+1}\right].$$

The first-order portfolio is then derived as

$$\alpha_{1,t} = \frac{1}{\gamma - 1} 2r_0 \Sigma^{-1}_{exr_1 exr_1} \tilde{\mathbb{E}}_t(exr_{2,t+1}) - r_0 \Sigma^{-1}_{exr_1 exr_1} \Sigma_{exr_1} \tilde{\Sigma}'_{\Upsilon_{2,t}}, \tag{19}$$

where $\Sigma_{\Upsilon_{2,t}} \tilde{W}_{t+1}$ is the time-varying exposure of future second-order consumption growth to innovations, whose complete expression is reported in the following section.

A.6 Additional theoretical results

1. The fully structural characterization of the first-order portfolio weight:

$$\alpha_{1,t} = \Sigma^{-1}_{exr_1 exr_1}\left[\frac{2}{\gamma-1}\tilde{\Sigma}_{exr_1}\tilde{\Theta}_{20} - \Sigma_{exr_1}\tilde{\Sigma}'_{2,\tilde{W}}\iota_{\Upsilon_2}\right]+$$
$$+\Sigma^{-1}_{exr_1 exr_1}\left[\frac{2}{\gamma-1}\iota_{exr}\tilde{\Theta}_{21} - \Sigma_{exr_1}\left\{mat_{n\times n}(\iota'_{\Upsilon_2}[I-\exp(-\delta)\Theta_{11}]^{-1}\Lambda_{21})+\right.\right.$$
$$+\exp(-\delta)mat_{n\times n}(\iota'_{\Upsilon_2}[I-\exp(-\delta)\Theta_{11}]^{-1}\Theta_{23}[I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(\Lambda_{10}\otimes\Theta_{11}))'+$$
$$\left.\left.+\exp(-\delta)mat_{n\times n}(\iota'_{\Upsilon_2}[I-\exp(-\delta)\Theta_{11}]^{-1}\Theta_{23}[I-\exp(-\delta)(\Theta_{11}\otimes\Theta_{11})]^{-1}(\Theta_{11}\otimes\Lambda_{10}))'\right\}\right]X_{1,t}.$$

where

$$\iota'_{\Upsilon_2} = \alpha_0\iota_{exr} + \exp(-\delta)\iota_{r_f}.$$

2. The innovation representation with respect to the risk neutral measure presents the second-order component of the expected discounted sum of future consumption growth as

$$\sum_{i=0}^{\infty}\exp(-\delta i)\Delta\tilde{\mathbb{E}}_{t+1}(\Delta\log C_{2,t+1+i}) = \Delta\tilde{\mathbb{E}}_{t+1}\sum_{i=0}^{\infty}\exp(-\delta i)[r_0^{-1}r^c_{2,t+1+i}] =$$
$$= \alpha_{1,t}\Sigma_{exr_1}\tilde{W}_{t+1} + \Sigma_{\Upsilon_{2,t}}\tilde{W}_{t+1},$$

where

$$\Sigma_{\Upsilon_{2,t}}\tilde{W}_{t+1} = \iota'_{\Upsilon_2}\tilde{\Sigma}_{2,t,\tilde{W}}\tilde{W}_{t+1}$$

$$\iota_{\Upsilon}(\alpha'_0\iota_{exr})' + \exp(-\delta)\iota_{r_f}.$$

and ι_{exr_1} is a selection vector choosing the excess returns from the state vector X_t, while $\iota_{r_{-1}}$ denote that choosing the lagged riskfree rate. The term $\Sigma_{\Upsilon_{2,t}}\tilde{W}_{t+1}$ captures the risk exposure of the future consumption growth to the underlying innovations. This exposure is time-varying, inheriting the nonlinear conditional mean and heteroskedasticity of the second-order process.

Parameters/Dependent	Vacancy v_{t+1}	Riskfree rates r^f_{t+1}	Excess returns exr_{t+1}
Θ_{10}	0.02^{**}	$-2.86e-6$	0.007^{**}
Θ_{11} on:			
$v_{1,t}$	0.995^{**}	$-1.11e-5^{**}$	zero by restriction
$r^f_{1,t}$	8.58^{**}	0.9865^{**}	zero by restriction
$exr_{1,t}$	0.228	$2.66e-4$	zero by restriction
Θ_{21} on:			
$v_{1,t}$	zero by restriction	zero by restriction	-0.0025**
$r^f_{1,t}$	zero by restriction	zero by restriction	-1.58
$exr_{1,t}$	zero by restriction	zero by restriction	0.0475
Θ_{23} on:			
$v_{1,t} \times v_{1,t}$	-0.0029^{**}	$6.72e-6$	zero by restriction
$v_{1,t} \times r^f_{1,t}$	-0.9679	$-3.37e-4$	zero by restriction
$v_{1,t} \times exr_{1,t}$	0.0856	$4.11e-4$	zero by restriction
$r^f_{1,t} \times r^f_{1,t}$	0.089	-0.1307	zero by restriction
$r^f_{1,t} \times exr_{1,t}$	-0.1723	-0.2177	zero by restriction
$exr_{1,t} \times exr_{1,t}$	-0.3629	0.0062	zero by restriction
Λ_{10} :	v_{t+1}	r^f_{t+1}	exr_{t+1}
v_{t+1}	0.1364^{**}	0	0
r^f_{t+1}	$-2.11e-5$	$1.83e-4^{**}$	0
exr_{t+1}	-0.0014	-0.0018	0.0396^{**}
Λ_{21} on:			
$v_{1,t}W_{t+1}(1)$	-0.042	$2.071e-5$	-0.0018
$v_{1,t}W_{t+1}(2)$	-0.0022	$5.52e-5^{**}$	-0.0017
$v_{1,t}W_{t+1}(3)$	0.0096	$2.46e-5^{**}$	0.0029^{**}
$r^f_{1,t}W_{t+1}(1)$	2.0954	-0.011	1.62
$r^f_{1,t}W_{t+1}(2)$	-0.0832	-0.0267^{**}	-0.1926
$r^f_{1,t}W_{t+1}(3)$	-0.4389	-0.0122	1.416
$exr_{1,t}W_{t+1}(1)$	-0.0063	$2.64e-4$	0.2645
$exr_{1,t}W_{t+1}(2)$	-0.1108	-0.0014^{**}	9.13e-4
$exr_{1,t}W_{t+1}(3)$	-0.4708	$-2.66e-4$	-0.1652^{**}

Table 8: QAR estimates. The double asterisk indicates the posterior mean is away from zero by two standard deviations of the posterior distribution.

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