HW 11 Solution

Group 1 **Grade:**

The expectation of $\mathsf{Geo}(p)$ can be proved by conditional expectation as follows:

Proof. For $X \sim \mathsf{Geo}(p)$,

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X \mid X=1]\mathbb{P}(X=1) + \mathbb{E}[X \mid X>1]\mathbb{P}(X>1) \\ &= \mathbb{E}[X=1]\cdot p + \mathbb{E}[X+1](1-p) \\ \Rightarrow \mathbb{E}[X] &= \frac{1}{p}.\end{aligned}$$

□

Following the same logic, derive the Variance of $\mathsf{Geo}(p)$.

Solution

$$\begin{aligned}\mathbb{E}\left[X^2\right] &= \mathbb{E}\left[X^2 \mid X=1\right]\mathbb{P}(X=1) + \mathbb{E}\left[X^2 \mid X>1\right]\mathbb{P}(X>1) \\ &= \mathbb{E}\left[X=1^2\right]\cdot p + \mathbb{E}\left[(X+1)^2\right](1-p) \\ &= 1\cdot p + \mathbb{E}\left[X^2+2X+1\right](1-p) \\ &= p + \mathbb{E}\left[X^2\right](1-p) + 2\mathbb{E}[X](1-p) + 1(1-p) \\ \Rightarrow \mathbb{E}\left[X^2\right] &= \frac{2}{p^2} - \frac{1}{p} \\ \mathrm{Var}[X] &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\ &= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} \\ &= \frac{1-p}{p^2}.\end{aligned}$$

Group 2 **Grade:**

Let Y_1 denote the weight (in tons) of a bulk item stocked by a supplier at the beginning of a week and suppose that Y_1 has a uniform distribution over the interval $0 \le y_1 \le 1$. Let Y_2 denote the amount (by weight) of this item sold by the supplier during the week and suppose that Y_2 has a uniform distribution over the interval $0 \le y_2 \le y_1$, where y_1 is a specific value of Y_1.

1. Find the joint density function for Y_1 and Y_2.
2. If the supplier stocks a half-ton of the item, what is the probability that she sells more than a quarter-ton?
3. If it is known that the supplier sold a quarter-ton of the item, what is the probability that she

had stocked more than a half-ton?

Solution

1.

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{Y_2|Y_1}(y_2 \mid y_1) f_{Y_1}(y_1) \\ &= \frac{1}{y_1} \cdot 1 \\ f_{Y_1,Y_2}(y_1,y_2) &= \begin{cases} \frac{1}{y_1} & 0 \le y_2 \le y_1 \le 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

2.

$$\begin{aligned} f_{Y_2|Y_1}(y_2 \mid y_1) &= \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_1}(y_1)} \\ &= \frac{\frac{1}{y_1}}{\int_0^{y_1} \frac{1}{y_1}\,\mathrm{d}y_2} \\ &= \frac{\frac{1}{y_1}}{1} \\ &= \frac{1}{y_1} \\ \mathbb{P}(Y_2 > 0.25 \mid Y_1 = 0.5) &= \int_{0.25}^{0.5} \frac{1}{0.5}\,\mathrm{d}y_2 \\ &= 0.5 \end{aligned}$$

3.

$$\begin{aligned} f_{Y_1|Y_2}(y_1 \mid y_2) &= \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_2}(y_2)} \\ &= \frac{\frac{1}{y_1}}{\int_{y_2}^{1} \frac{1}{y_1}\,\mathrm{d}y_1} \\ &= \frac{\frac{1}{y_1}}{-\ln y_2} \\ &= \frac{1}{y_1(-\ln y_2)} \\ \mathbb{P}(Y_1 > 0.5 \mid Y_2 = 0.25) &= \int_{0.5}^{1} \frac{1}{y_1(-\ln 0.25)}\,\mathrm{d}y_1 \\ &= \frac{\ln 2}{2\ln 2} \\ &= 0.5 \end{aligned}$$

Group 3 **Grade:**

A quality control plan calls for randomly selecting three items from the daily production (assumed large) of a certain machine and observing the number of defectives. However, the proportion p of defectives produced by the machine varies from day to day and is assumed to have a uniform distribution on the interval (0, 1). For a randomly chosen day, find the unconditional probability that exactly two defectives are observed in the sample.

Solution

$$\begin{aligned}
\mathbb{P}(Y = y \mid p) &= \binom{3}{y} p^y (1-p)^{3-y} \qquad y = 0, 1, 2, 3 \\
\mathbb{P}(Y = 2) &= \int_0^1 \mathbb{P}(Y = 2, p)\, \mathrm{d}p \\
&= \int_0^1 \mathbb{P}(Y = 2 \mid p) f_P(p)\, \mathrm{d}p \\
&= \int_0^1 \binom{3}{2} p^2 (1-p)^{3-2} \cdot 1\, \mathrm{d}p \\
&= \int_0^1 3p^2(1-p)\, \mathrm{d}p \\
&= \int_0^1 (3p^2 - 3p^3)\, \mathrm{d}p \\
&= \left[p^3 - \frac{3}{4} p^4 \right]_0^1 \\
&= 1 - \frac{3}{4} \\
&= \frac{1}{4}
\end{aligned}$$

Group 4 Grade:

Suppose that the probability that a head appears when a coin is tossed is p and the probability that a tail occurs is $q = 1 - p$. Person A tosses the coin until the first head appears and stops. Person B does likewise. The results obtained by persons A and B are assumed to be independent. What is the probability that A and B stop on exactly the same number toss? Explain how you think. The calculation is not mandatory.

Solution

$$\begin{aligned}
&\mathbb{P}(A = 1, B = 1) + \mathbb{P}(A = 2, B = 2) + \mathbb{P}(A = 3, B = 3) + \dots \\
=& \mathbb{P}(A = 1)\mathbb{P}(B = 1) + \mathbb{P}(A = 2)\mathbb{P}(B = 2) + \mathbb{P}(A = 3)\mathbb{P}(B = 3) + \dots \\
=& \sum_{i=1}^{\infty} \mathbb{P}(A = i)\mathbb{P}(B = i) \\
=& \sum_{i=1}^{\infty} pq^{i-1} pq^{i-1} \\
=& p^2 \sum_{i=1}^{\infty} (q^2)^{i-1} \\
=& \frac{p^2}{1 - q^2}
\end{aligned}$$

Group 5 **Grade:**

Let Y_1 and Y_2 be independent exponentially distributed random variables, each with mean 1. Find $P(Y_1 > Y_2 \mid Y_1 < 2Y_2)$ and $P(Y_1 < 2Y_2 \mid Y_1 > Y_2)$.

Hint:

1. Find the joint probability of Y_1 and Y_2.
2. Determine the range of Y_1 and Y_2 that satisfies the condition.
3. Use the definition of conditional probability.
4. Surprisingly, $P(Y_1 < 2Y_2 \mid Y_1 > Y_2)$ is much easier to calculate than $P(Y_1 = 2Y_2 \mid Y_1 > Y_2)$.

Solution

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\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{Y_1}(y_1)f_{Y_2}(y_2) \\
&= e^{-y_1}e^{-y_2} \\
&= e^{-(y_1+y_2)} \\
\mathbb{P}(Y_1 > Y_2 \mid Y_1 < 2Y_2) &= \frac{\mathbb{P}(Y_1 > Y_2, Y_1 < 2Y_2)}{\mathbb{P}(Y_1 < 2Y_2)} \\
&= \frac{\int_0^\infty \int_{y_2}^{2y_2} e^{-(y_1+y_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2}{\int_0^\infty \int_0^{2y_2} e^{-(y_1+y_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2} \\
&= \frac{\frac{1}{6}}{\frac{2}{3}} \\
&= \frac{1}{4} \\
\mathbb{P}(Y_1 < 2Y_2 \mid Y_1 > Y_2) &= \frac{\mathbb{P}(Y_1 < 2Y_2, Y_1 > Y_2)}{\mathbb{P}(Y_1 > Y_2)} \\
&= \frac{\int_0^\infty \int_{y_2}^{2y_2} e^{-(y_1+y_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2}{\int_0^\infty \int_{y_2}^{\infty} e^{-(y_1+y_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2} \\
&= \frac{\frac{1}{6}}{\frac{1}{2}} \\
&= \frac{1}{3}
\end{aligned}
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