Midterm 1

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October 30, 2023

Q1: Normal cones to level sets. Grade:

Suppose $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable convex function, and consider the level set $L(0, h) = \{ x \in \mathbb{R}^n \mid h(x) < 0 \}$. Assuming that there exists some point $\bar{x} \in \mathbb{R}^n$ with $h(\bar{x}) < 0$, prove that, for any $x \in L(0, h)$, the normal cone $N_{L(0,h)}(x)$ to $L(0, h)$ at x is given by the formula

$$N_{L(0,h)}(x) = \{ \alpha \nabla h(x) \mid \alpha \geq 0, \alpha h(x) = 0 \} \begin{cases} \emptyset & \text{if } h(x) > 0 \\ \{ \alpha \nabla h(x) \mid \alpha \geq 0 \} & \text{if } h(x) = 0 \\ \{ 0 \} & \text{if } h(x) < 0 \end{cases}$$

Q2: Optimality conditions for convex problems with "mixed" constraint sets. Grade:

Consider an optimization problem of the form

$$\begin{aligned} \min \quad & f(x) \\ \text{S.T.} \quad & Ax = b \\ & h_j(x) \leq 0 \qquad j = 1, 2, \ldots, r \\ & x \in X \end{aligned} \tag{1}$$

where

- $f : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \}$ is a convex function
- A is an $m \times n$ matrix and $b \in \mathbb{R}^m$
- For $j = 1, 2, \ldots, r$, $h_j : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function
- X is a convex set.

Let a_i denote row i of A, $i = 1, 2, dots, m$ represented as a column vector. Suppose that there exists a point $\bar{x} \in \mathbb{R}^n$ with the following properties:

- $\bar{x} \in \operatorname{ri} \operatorname{dom} f$
- $A\bar{x} = b$
- For $j = 1, 2, \ldots, r$, $h_j(\bar{x}) < 0$
- $\bar{x} \in \operatorname{ri} X$.

Show that for $x^* \in \mathbb{R}^n$ to be a solution of eq. (1), it is necessary and sufficient that there exist $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^r$ such that

$$\partial f(x^*) + \sum_{i=1}^{m} \lambda_i^* a_i + \sum_{j=1}^{r} \mu_j^* \nabla h_j(x^*) + N_X(x^*) \ni 0$$

$$
\begin{aligned}
\sum_{j=1}^{r} \mu_j^* h_j(x^*) &= 0 \\
Ax^* &= b \\
h_j(x^*) &\le 0 \qquad j = 1, 2, \dots, r \\
\mu_j^* &\ge 0 \qquad j = 1, 2, \dots, r.
\end{aligned}
$$

Q3: Optimality conditions for convex cone programming. Grade:

Below, suppose $K \subseteq \mathbb{R}^m$ be a nonempty closed convex cone, A is an $m \times n$ matrix, and $b \in \mathbb{R}^m$, and let $Z = \{\, x \in \mathbb{R}^n \mid Ax - b \in K \,\}$. Assume that Z is nonempty.

(a) Show that for any $x \in K$,

$$F_K(x) = \{\, z - \alpha x \mid z \in K, \alpha \ge 0 \,\}.$$

(b) Show that for any $x \in K$,

$$N_K(x) = \{\, y \in K^* \mid \langle x, y \rangle = 0 \,\}.$$

Hint: you may use the results of homework 3, problem 1(c) and 4(c).

(c) Show that, for $x \in Z$,

$$N_Z(x) = \mathrm{cl}\big\{\, A^\mathsf{T}\lambda \mid \lambda \in K^*, \langle Ax - b, \lambda \rangle = 0 \,\big\}.$$

Hint: you may use the results of homework 3, problem 4.

(d) Show that $\mathrm{ri}\, Z \supseteq \{\, x \in \mathbb{R}^n \mid Ax - b \in \mathrm{ri}\, K \,\}$.

(e) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function, suppose that the cone $A^\mathsf{T} K^* = \big\{\, A^\mathsf{T}\lambda \mid \lambda \in K^* \,\big\}$ is closed, and consider the problem

$$
\begin{aligned}
\min \quad & f(x) \\
\text{S.T.} \quad & Ax - b \in K.
\end{aligned}
\tag{2}
$$

Further suppose that there exists some point $\bar{x} \in \mathrm{ri}\,\mathrm{dom}\, f$ such that $A\bar{x} - b \in \mathrm{ri}\, K$. Show that, in order for $x^* \in \mathbb{R}^n$ to solve eq. (2), it is necessary and sufficient that there exists $\lambda^* \in \mathbb{R}^m$ such that

$$
\begin{aligned}
\partial f(x^*) + A^\mathsf{T}\lambda^* &\ni 0 \\
\lambda^* &\in K^* \\
\langle Ax^* - b, \lambda^* \rangle &= 0.
\end{aligned}
$$