HW 6

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Q1 **Grade:**

For each of the following choices of $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, compute the convex conjugate function f^*:

(a) $f(x) = \frac{1}{2}x^2$.

(b) For $a, b \in \mathbb{R}$, $a < b$, $f(x) = \delta_{[a,b]} = \begin{cases} 0 & x \in [a, b], \\ +\infty & \text{otherwise.} \end{cases}$

(c) $f(x) = e^x$.

Q2 **Grade:**

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be positively homogeneous if

$$\begin{aligned} f(0) &= 0 \\ f(\alpha x) &= \alpha f(x) \quad \forall \alpha > 0, x \in \mathbb{R}^n . \end{aligned}$$

(Note that some definitions omit the condition $f(0) = 0$, which we include here to accord with our notion of a cone as always containing the point 0.)

(a) For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, show that $\operatorname{epi} f$ is a cone in $\mathbb{R}^{n+1}$ *if and only if* f is positively homogeneous.

(b) Consider any nonempty set $X \subseteq \mathbb{R}^n$. The *support function* of X is the convex conjugate (δ_X^*) of the indicator function

$$\delta_X = \begin{cases} 0 & x \in X, \\ +\infty & \text{otherwise.} \end{cases}$$

Show that

$$\delta_X^*(y) = \sup_{x \in X} \{ \langle x, y \rangle \},$$

and this function is positively homogeneous.

(c) Show conversely that, given any positively homogeneous function f, its convex conjugate f^* is the indicator function of some closed convex set C.

(d) Given a cone K, show that $\delta_K^* = \delta_{K^*}$. That is, the conjugate of the indicator function of a K is the indicator function of its polar.

Q3 **Grade:**

Consider the standard primal linear programming problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{S.T.} \quad & Ax = b \\ & x \geqslant 0. \end{aligned}$$

Model this problem as $\min f(x) + g(Mx)$, where

$$f(x) \doteq \begin{cases} c^T x & x \geqslant 0 \\ +\infty & \text{otherwise} \end{cases} \qquad M \doteq A \qquad g(z) \doteq \begin{cases} 0 & z = b \\ +\infty & \text{otherwise,} \end{cases}$$

where A is any $m \times n$ matrix and $b \in \mathbb{R}^m$. Show that the corresponding Fenchel dual is equivalent to the standard dual programming problem

$$\begin{aligned} \max_{u \in \mathbb{R}^m} \quad & b^T u \\ \text{S.T.} \quad & A^T u \leqslant c \end{aligned}$$

in the sense that any solution y^* of the Fenchel dual is equal to $-u^*$ is some optimal solution to the standard dual linear programming problem.