

Homework 1

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Q1: Affine images and preimages of convex sets.

Grade:

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \subset \mathbb{R}^n$, $D \subset \mathbb{R}^m$ be convex sets. Show that following sets are convex.

- (a) The image of C under the affine map $x \mapsto Ax + b$. That is

$$\{Ax + b \mid x \in C\} \subset \mathbb{R}^m.$$

- (b) The preimage of D under the affine map $x \mapsto Ax + b$. That is

$$\{x \mid Ax + b \in D\} \subset \mathbb{R}^n.$$

Solution

- (a) *Proof.* Let $x_1, x_2 \in C$ and $\lambda \in [0, 1]$, then we have

$$\begin{aligned} \lambda(Ax_1 + b) + (1 - \lambda)(Ax_2 + b) &= A(\lambda x_1 + (1 - \lambda)x_2) + b \\ &\in A(C) + b. \end{aligned}$$

Thus, the image of C , $A(C) + b$ is convex. □

- (b) *Proof.* Let $y_1, y_2 \in A^{-1}(D - b)$ so that $Ay_1 + b \in D$, $Ay_2 + b \in D$ and $\lambda \in [0, 1]$, then we have

$$\begin{aligned} A(\lambda y_1 + (1 - \lambda)y_2) + b &= \lambda(Ay_1 + b) + (1 - \lambda)(Ay_2 + b) \\ &\in \lambda D + (1 - \lambda)D \\ &= D. \end{aligned}$$

Thus, The preimages of D , $A^{-1}(D - b)$ is convex. □

Q2: Affine functions.

Grade:

Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R} \setminus \{-\infty, \infty\}$ always obey the convex function relation at equality, that is,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall x, y \in \mathbb{R}^n, \lambda \in [0, 1]. \quad (1)$$

Show that

- (a) If eq. (1) holds as stated for all $\lambda \in [0, 1]$, it in fact holds for all $\lambda \in \mathbb{R}$.
- (b) Any f for which eq. (1) holds must be of the form $f(x) = \langle a, x \rangle + b$ for $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ (that is, f is an inner product with some fixed vector plus a constant).
- (c) Any function of this form has the property eq. (1).

Hint: given f satisfying the condition above, show that $g : x \mapsto f(x) - f(0)$ is linear. You may then use (without proof, although the proof is very easy) that a linear function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ must be of the form $x \mapsto \langle a, x \rangle$ for some $a \in \mathbb{R}^n$.

Solution

(a) *Proof.* To extend eq. (1) to $\lambda \in \mathbb{R}^n$, we need to show that eq. (1) holds for $\lambda \in (-\infty, 0) \cup (1, \infty)$.

First, let $x, y \in \mathbb{R}^n$, and for $\lambda \in (-1, 0)$ let $\alpha = -\lambda \in (0, 1)$. Then we have

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y),$$

which shows convexity for $\lambda \in (-1, 0)$. Similarly, for $\lambda \in (1, \infty)$ let $\alpha = \frac{1}{\lambda} \in (0, 1)$, and for $\lambda \in (-\infty, -1)$ let $\alpha = -\frac{1}{\lambda} \in (0, 1)$, we can prove item (a) holds for $\lambda \in (1, \infty)$ and $\lambda \in (-\infty, -1)$ respectively. \square

(b) *Proof.* Let's define $g : \mathbb{R}^n \rightarrow \mathbb{R}$ as $g(x) = f(x) - f(0)$, then we have $g(0) = f(0) - f(0) = 0$. For any $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ (as proved above), we have

$$\begin{aligned} g(\lambda x + (1 - \lambda)y) &= f(\lambda x + (1 - \lambda)y) - f(0) \\ &\leq \lambda f(x) + (1 - \lambda)f(y) - f(0) \\ &= \lambda(f(x) - f(0)) + (1 - \lambda)(f(y) - f(0)) \\ &= \lambda g(x) + (1 - \lambda)g(y). \end{aligned}$$

This shows g is a linear function. From the hint, we can represent g as $g(x) = \langle a, x \rangle$ for some $a \in \mathbb{R}^n$. Thus, $f(x) = \langle a, x \rangle + b$ where $b = f(0)$. \square

(c) *Proof.* If $f(x) = \langle a, x \rangle + b$, then for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &= \langle a, \lambda x + (1 - \lambda)y \rangle + b \\ &= \lambda \langle a, x \rangle + (1 - \lambda) \langle a, y \rangle + b \\ &= \lambda(\langle a, x \rangle + b) + (1 - \lambda)(\langle a, y \rangle + b) \\ &\leq \lambda f(x) + (1 - \lambda)f(y). \end{aligned}$$

\square

Q3: Convex hulls.

Grade:

Show that for any set $X \subseteq \mathbb{R}^n$, the convex hull $\text{conv}(X)$ of X (the intersection of all convex sets containing X) is equal to the set of all convex combinations of points in X .

Hint: Define Y to be the set of all convex combinations of points from X , that is,

$$Y = \left\{ \sum_{i=1}^m \lambda_i x_i \mid m \geq 1, \lambda_i > 0, \sum_{i=1}^m \lambda_i = 1 \right\},$$

and then prove that both $Y \subseteq \text{conv}(X)$ (which may be accomplished by showing that it is convex and contains X), and $\text{conv}(X) \subseteq Y$ (which may be accomplished by showing that every convex set containing X also contains Y).

Solution

1. $Y \subseteq \text{conv}(X)$

Proof. Let $y_1, y_2 \in Y$. By definition of Y ,

$$\begin{aligned} y_1 &= \sum_{i=1}^{m_1} \alpha_i x_i, & \sum_{i=1}^{m_1} \alpha_i &= 1, \\ y_2 &= \sum_{j=1}^{m_2} \beta_j x_j, & \sum_{j=1}^{m_2} \beta_j &= 1. \end{aligned}$$

For any $\lambda \in [0, 1]$, consider the point $y = \lambda y_1 + (1 - \lambda)y_2$. Then

$$\begin{aligned} y &= \lambda \sum_{i=1}^{m_1} \alpha_i x_i + (1 - \lambda) \sum_{j=1}^{m_2} \beta_j x_j \\ &= \sum_{i=1}^{m_1} (\lambda \alpha_i) x_i + \sum_{j=1}^{m_2} ((1 - \lambda) \beta_j) x_j \end{aligned}$$

where

$$\begin{aligned} &\sum_{i=1}^{m_1} \lambda \alpha_i + \sum_{j=1}^{m_2} (1 - \lambda) \beta_j \\ &= \lambda \sum_{i=1}^{m_1} \alpha_i + (1 - \lambda) \sum_{j=1}^{m_2} \beta_j \\ &= \lambda + (1 - \lambda) \\ &= 1. \end{aligned}$$

Clearly, $y \in Y$, which shows Y is convex. Also, every point $x_i \in X$ is in Y with $\lambda_i = 1$, which shows $X \subseteq Y$. Since Y is convex and contains X , then it must contain $\text{conv}(X)$ as $\text{conv}(X)$ is the intersection of all convex sets containing X . \square

2. $\text{conv}(X) \subseteq Y$

Proof. Let Z be any convex set containing X . We want to show that Z also contains Y . Take any $y \in Y$, since Z is convex and contains X , Z must contain y , the convex combination of points in X . Thus, Z contains Y . Since arbitrary Z contains Y , $\text{conv}(X)$ must contain Y as $\text{conv}(X)$ is the intersection of all convex sets containing X . Therefore, $\text{conv}(X) \subseteq Y$. \square

Q4: Affine sets and hulls.

Grade:

The scalars λ_i in this problem may take negative values.

- (a) The textbook defines a set $X \subseteq \mathbb{R}^n$ as being affine if it is of the form $S + x = \{s + x \mid s \in S\}$ for some $x \in \mathbb{R}^n$ and linear subspace S of \mathbb{R}^n . Show that X is affine according to this definition if and only if X is

$$\left. \begin{aligned} x_1, x_2, \dots, x_m &\in X \\ \lambda_1, \lambda_2, \dots, \lambda_m &\in \mathbb{R} \\ \sum_{i=1}^m \lambda_i &= 1 \end{aligned} \right\} \Rightarrow \sum_{i=1}^m \lambda_i x_i \in X$$

Hint: For the “if”, take any $x \in X$ and show that the set $S = X - x = \{x' - x \mid x' \in X\}$ is a linear subspace of \mathbb{R}^n .

- (b) In the text, the *affine hull* $\text{aff}(Y)$ of a set Y is defined to be the intersection of all affine sets containing Y . Show

that

$$\text{aff } Y = \left\{ \sum_{i=1}^m \lambda_i y_i \mid m \geq 1, \lambda_i \in \mathbb{R}, \sum_{i=1}^m \lambda_i = 1, y_i \in Y \right\},$$

that is, the affine hull of Y is the set of all affine combinations of points in Y .

Solution

(a) • “if”

Proof. Assume that the condition holds for X . We want to show that X is affine by showing S is a linear subspace of \mathbb{R}^n . Take any $x \in X$ and let $S = X - x$, we have

(1) $0 \in S$ because $x' - x = 0$ for $x' = x$ and $x' \in X$.

(2) For $s_1, s_2, \dots, s_m \in S$, $\sum_{i=1}^m \lambda_i s_i = \sum_{i=1}^m \lambda_i (x_i - x) = (\sum_{i=1}^m \lambda_i x_i) - x \in S$ for $\sum_{i=1}^m \lambda_i = 1$.

(3) For any $s \in S$ and any scalar λ , $\lambda s = \lambda(x' - x) = [\lambda(x') + (1 - \lambda)x] - x \in S$.

Thus, S is a linear subspace of \mathbb{R}^n , which shows X is affine. \square

• “only if”

Proof. Suppose X is affine as $X = S + x$ for some $x \in \mathbb{R}^n$ and some linear subspace S of \mathbb{R}^n , we want to show that when the conditions hold, $\sum_{i=1}^m \lambda_i x_i \in X$. Since $X = S + x$, for any point $x_i = s_i + x$, we have

$$\sum_{i=1}^m \lambda_i x_i = \sum_{i=1}^m \lambda_i (s_i + x) = \sum_{i=1}^m \lambda_i s_i + \sum_{i=1}^m \lambda_i x = \sum_{i=1}^m \lambda_i s_i + x \in X.$$

\square

(b) The proof of this is similar to Q3 without condition $\sum_{i=1}^m \lambda_i = 1$.

Q5: Arithmetic-Geometric Mean Inequality.

Grade:

Show that if $\lambda_1, \lambda_2, \dots, \lambda_m$ are positive scalars with $\sum_{i=1}^m \lambda_i = 1$, then for every set of positive scalars x_1, x_2, \dots, x_m , we have

$$\prod_{i=1}^m x_i^{\lambda_i} \leq \sum_{i=1}^m \lambda_i x_i,$$

with equality if and only if $x_1 = x_2 = \dots = x_m$.

Hint: Show that $-\ln x$ is a strictly convex function on $(0, \infty)$.

Solution

Consider the function $f(x) = -\ln x$, then $f''(x) = \frac{1}{x^2} > 0$ for $x > 0$. Thus, $f(x)$ is strictly convex on $(0, \infty)$. By Jensen's inequality, we have

$$\begin{aligned} -\ln \left(\sum_{i=1}^m \lambda_i x_i \right) &= f \left(\sum_{i=1}^m \lambda_i x_i \right) \\ &\leq \sum_{i=1}^m \lambda_i f(x_i) \\ &= \sum_{i=1}^m \lambda_i (-\ln x_i) \end{aligned}$$

$$\begin{aligned} &= -\ln\left(\prod_{i=1}^m x_i^{\lambda_i}\right) \\ \Rightarrow \exp\left\{-\ln\left(\sum_{i=1}^m \lambda_i x_i\right)\right\} &\leq \exp\left\{-\ln\left(\prod_{i=1}^m x_i^{\lambda_i}\right)\right\} \\ \Rightarrow \sum_{i=1}^m \lambda_i x_i &\geq \prod_{i=1}^m x_i^{\lambda_i}. \end{aligned}$$

Since $f(x)$ is strictly convex, the equality holds if and only if $x_1 = x_2 = \cdots = x_m$.