HW 6

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Q1 **Grade:**

For each of the following choices of $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, compute the convex conjugate function f^*:

(a) $f(x) = \frac{1}{2}x^2$.

(b) For $a, b \in \mathbb{R}$, $a < b$, $f(x) = \delta_{[a,b]} = \begin{cases} 0 & x \in [a,b], \\ +\infty & \text{otherwise.} \end{cases}$

(c) $f(x) = e^x$.

Solution

We want to find

$$f^* = g(\lambda) = \sup_{x \in \mathrm{dom}(f)} \{ \langle \lambda, x \rangle - f(x) \}.$$

(a)

$$\begin{aligned} g(\lambda) &= \sup_{x \in \mathrm{dom}(f)} \left\{ \langle \lambda, x \rangle - \frac{1}{2}x^2 \right\} \\ \frac{\mathrm{d}g}{\mathrm{d}x} &= \lambda - x && \text{(Let } x = \lambda \text{ to maximize } g(\lambda)\text{)} \\ g(\lambda) &= \frac{1}{2}\lambda^2 \end{aligned}$$

(b)

$$\begin{aligned} g(\lambda) &= \sup_{x \in \mathrm{dom}(f)} \{ \langle \lambda, x \rangle - \delta_{[a,b]}(x) \} \\ &= \begin{cases} \sup_{x \in \mathrm{dom}(f)} \{ \langle \lambda, x \rangle \} & x \in [a,b] \\ \sup_{x \in \mathrm{dom}(f)} \{ -\infty \} & \text{otherwise} \end{cases} \\ \frac{dg}{dx} &= \begin{cases} \lambda & x \in [a,b] \\ 0 & \text{otherwise} \end{cases} \\ g(\lambda) &= \begin{cases} b\lambda & \lambda \geq 0 \\ a\lambda & \lambda \leq 0 \\ 0 & \lambda = 0 \end{cases} \end{aligned}$$

(c)

$$g(\lambda) = \sup_{x \in \mathrm{dom}(f)} \{ \langle \lambda, x \rangle - e^x \}$$

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \lambda - e^x \qquad \text{(Let } x = \ln\lambda \text{ to maximize } g(\lambda)\text{)}$$
$$g(\lambda) = \lambda \ln \lambda - \lambda$$

Q2 Grade:

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be positively homogeneous if

$$f(0) = 0$$
$$f(\alpha x) = \alpha f(x) \quad \forall \alpha > 0, x \in \mathbb{R}^n .$$

(Note that some definitions omit the condition $f(0) = 0$, which we include here to accord with our notion of a cone as always containing the point 0.)

(a) For any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, show that epi f is a cone in $\mathbb{R}^{n+1}$ *if and only if* f is positively homogeneous.

(b) Consider any nonempty set $X \subseteq \mathbb{R}^n$. The *support function* of X is the convex conjugate (δ_X^*) of the indicator function

$$\delta_X = \begin{cases} 0 & x \in X, \\ +\infty & \text{otherwise.} \end{cases}$$

Show that

$$\delta_X^*(y) = \sup_{x \in X} \{ \langle x, y \rangle \},$$

and this function is positively homogeneous.

(c) Show conversely that, given any positively homogeneous function f, its convex conjugate f^* is the indicator function of some closed convex set C.

(d) Given a cone K, show that $\delta_K^* = \delta_{K^*}$. That is, the conjugate of the indicator function of a K is the indicator function of its polar.

Solution

(a) *Proof.* We finish the proof by showing sufficiency and necessity.

(i) If epi f is a cone in $\mathbb{R}^{n+1}$, then $\forall (x, f(x)) \in \text{epi} f$, we have $(\alpha x, \alpha f(x)) \in \text{epi} f$ for all $\alpha > 0$.
With the definition of epi f, we have $f(\alpha x) \leq \alpha f(x)$.
Since f is proper, we have $f(\alpha x) \geq \alpha f(x)$.
Therefore, $f(\alpha x) = \alpha f(x)$, which means epi f is positively homogeneous.

(ii) If f is positively homogeneous, we have $f(\alpha x) = \alpha f(x) \leq \alpha f(x)$ for all $\alpha > 0$, which means $(\alpha x, \alpha f(x)) \in \text{epi} f$.
Since α is arbitrary, epi f is a cone by definition.

□

(b) *Proof.* We have shown the $\delta_X^*(y) = \sup_{x \in X} \{ \langle x, y \rangle \}$ in Q1. We only need to prove δ_X^* is positively homogeneous.

$$\begin{aligned} \delta_X^*(\alpha y) &= \sup_{x \in X} \{ \langle x, \alpha y \rangle \} \\ &= \sup_{x \in X} \{ \alpha \langle x, y \rangle \} \end{aligned}$$

$$= \alpha \sup_{x \in X} \{ \langle x, y \rangle \}$$
$$= \alpha \delta_X^*(y)$$

□

(c) *Proof.* Based on definition, we have $f^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - f(x) \}$. Construct a set C which is $C = \{ y \in \mathbb{R}^n \mid \langle x, y \rangle \leq f(x), \forall x \in \mathbb{R}^n \}$.

  (i) For $y \in C$, we have $\langle x, y \rangle \leq f(x)$ for all $x \in \mathbb{R}^n$ and thus $\langle x, y \rangle - f(x) \leq 0$. By positively homogeneous $f(0) = 0$, and we know $\langle 0, y \rangle = 0$. So 0 is attainable in the supremum. Therefore, $f^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - f(x) \} = 0$.

  (ii) For $y \notin C$, there exists $x \in \mathbb{R}^n$ such that $\langle x, y \rangle > f(x)$. Since f is positively homogeneous, we have $\langle \alpha x, y \rangle > f(\alpha x) = \alpha f(x)$ for all $\alpha > 0$. Therefore, $\langle \alpha x, y \rangle - f(\alpha x) = \alpha(\langle x, y \rangle - f(x))$. This means for a given $y \notin C$, we can scale x by arbitrary $\alpha \geq 0$ and the supremum will be unbounded. Therefore, $f^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - f(x) \} = +\infty$.

The above two terms show the f^* is the indicator function of C. Now we need to show C is closed and convex to finish the proof.

  (iii) For any $y_1, y_2 \in C$ and $\lambda \in [0, 1]$, we have $\langle x, \lambda y_1 + (1 - \lambda) y_2 \rangle \leq \lambda f(x) + (1 - \lambda) f(x)$ for all $x \in \mathbb{R}^n$. Therefore, $\lambda y_1 + (1 - \lambda) y_2 \in C$ and C is convex. The continuity of $\langle \cdot, \cdot \rangle$ implies C is closed.

□

(d) *Proof.* We have $\delta_K^* = \sup_{x \in K} \{ \langle x, y \rangle \}$ from Q2(b).
For $y \in K^*$, it satisfies $\langle x, y \rangle \leq 0, \forall x \in K$, which matches $\delta_K^*(y) = \sup_{x \in K} \{ \langle x, y \rangle \} = 0$.
For $y \notin K^*$, there exists $x \in K$ such that $\langle x, y \rangle > 0$. Since K is a cone, we have $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $\alpha > 0$. Therefore, $\langle \alpha x, y \rangle$ is unbounded and matches $\delta_K^*(y) = \sup_{x \in K} \{ \langle x, y \rangle \} = +\infty$.

□

Q3 Grade:

Consider the standard primal linear programming problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{S.T.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Model this problem as $\min f(x) + g(Mx)$, where

$$f(x) \doteq \begin{cases} c^\top x & x \geq 0 \\ +\infty & \text{otherwise} \end{cases} \qquad M \doteq A \qquad g(z) \doteq \begin{cases} 0 & z = b \\ +\infty & \text{otherwise,} \end{cases}$$

where A is any $m \times n$ matrix and $b \in \mathbb{R}^m$. Show that the corresponding Fenchel dual is equivalent to the standard dual programming problem

$$\begin{aligned} \max_{u \in \mathbb{R}^m} \quad & b^\top u \\ \text{S.T.} \quad & A^\top u \leq c \end{aligned} \tag{1}$$

in the sense that any solution y^* of the Fenchel dual is equal to $-u^*$, where u^* is some optimal solution to the standard dual linear programming problem.

Solution

Proof. Solving $\sup_{x\geq 0}\{\langle x, y\rangle - f(x)\}$ and get

$$f^*(y) = \begin{cases} 0 & y \leq c \\ +\infty & \text{otherwise.} \end{cases}$$

Since $g(z)$ is an indicator function, using the result of Q2(b), we have

$$g^*(w) = w^\mathsf{T} b.$$

The Fenchel dual problem is

$$\begin{aligned} &\max_{w\in\mathbb{R}^m} \{ -f^*(-w) - g^*(w) \} \\ &\max_{w\in\mathbb{R}^m} \left\{ -f^*(-w) - w^\mathsf{T} b \right\} \\ \Rightarrow &\begin{cases} -w^\mathsf{T} b & -w \leq c \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Let $-u = w$, we get the standard dual problem eq. (1). Note the w here is corresponding to y of Fenchel dual problem in the question statement. $\square$