HW 13 Solution

Group 1 **Grade:**

Two telephone calls uniformly come into a switchboard at random times in a fixed one-hour period. Assume that the calls are made independently of one another. What is the probability that the calls are made

1. in the first half hour?
2. within five minutes of each other?

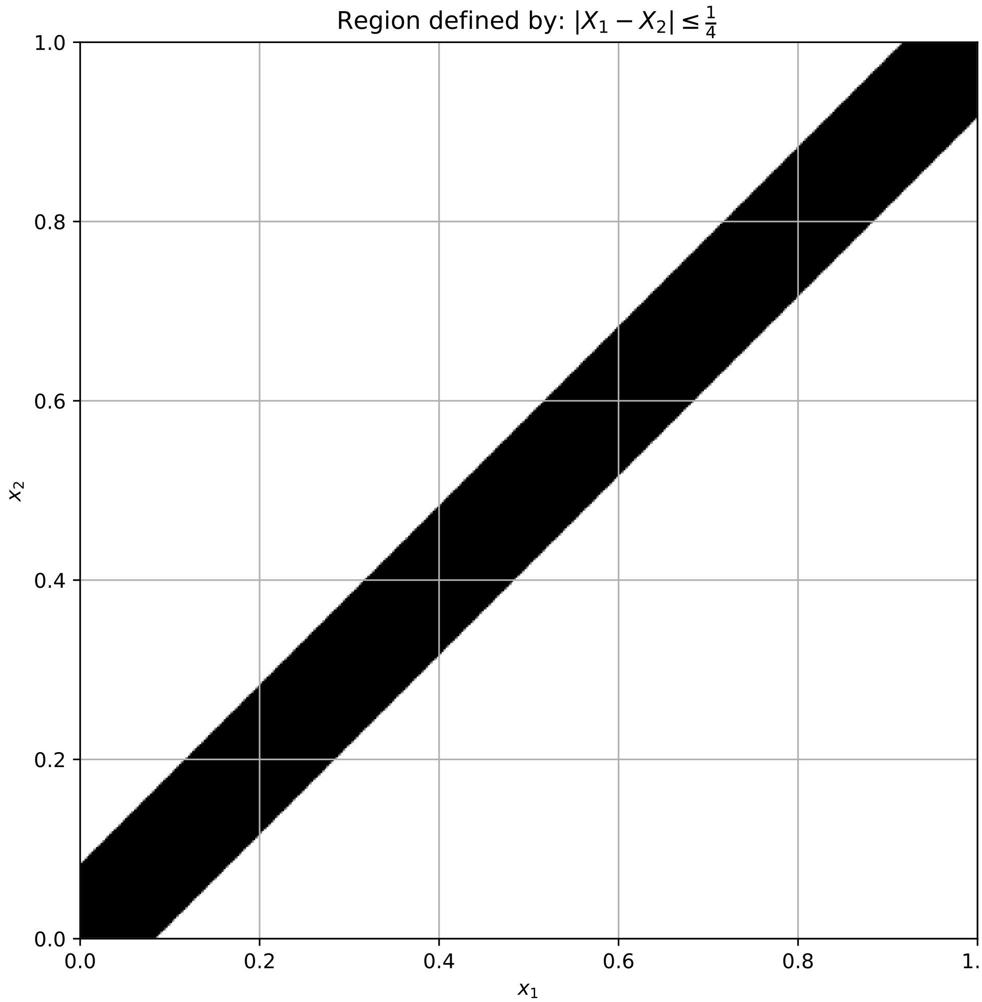
Solution

Let X_1 and X_2 be the arrival times of the two calls. Then X_1 and X_2 are independent and uniformly distributed over the interval $[0, 1]$. The joint density of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 1 & 0 \le x_1 \le 1, 0 \le x_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \mathbb{P}(X_1, X_2 \le 0.5) &= \int_0^{0.5} \int_0^{0.5} 1 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &= \frac{1}{4} \end{aligned}$$

The area of the region where $|X_1 - X_2| \le 1/12$ is



and the $\mathbb{P}(|X_1 - X_2| \leq 1/12) = 1 - \frac{1}{2}(\frac{11}{12})^2 \times 2 = \frac{23}{144}$.

Group 2 **Grade:**

A process for producing an industrial chemical yields a product containing two types of impurities. For a specified sample from this process, let Y_1 denote the proportion of impurities in the sample and let Y_2 denote the proportion of type I impurities among all impurities found. Suppose that the joint distribution of Y_1 and Y_2 can be modeled by the following probability density function:

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 2(1 - y_1) & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of the proportion of type I impurities in the sample.

Solution

$$
\begin{aligned}
\mathbb{E}[Y_1 Y_2] &= \int_0^1 \int_0^1 y_1 y_2 \cdot 2(1-y_1)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
&= \int_0^1 \int_0^1 2y_1 y_2 - 2y_1^2 y_2\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
&= \int_0^1 y_2 - \frac{2}{3} y_2\,\mathrm{d}y_2 \\
&= \frac{1}{2} - \frac{1}{3} \\
&= \frac{1}{6}
\end{aligned}
$$

Group 3 **Grade:**

Let the discrete random variables Y_1 and Y_2 have the joint probability function $P_{Y_1,Y_2}(y_1, y_2) = 1/3$, for $(y_1, y_2) = (-1, 0), (0, 1), (1, 0)$. Find $\mathrm{Cov}(Y_1, Y_2)$. Notice that Y_1 and Y_2 are dependent. (Why?) This is another example of uncorrelated random variables that are not independent.

Solution

$$
\begin{aligned}
P_{Y_1}(y_1) &= \begin{cases} 1/3 & y_1 = -1, 0, 1 \\ 0 & \text{otherwise.} \end{cases} \\
P_{Y_2}(y_2) &= \begin{cases} 2/3 & y_2 = 0 \\ 1/3 & y_2 = 1 \\ 0 & \text{otherwise.} \end{cases} \\
\mathbb{E}[Y_1] &= -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \\
&= 0 \\
\mathbb{E}[Y_2] &= 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} \\
&= \frac{1}{3} \\
\mathbb{E}[Y_1 Y_2] &= -1 \cdot 0 \cdot \frac{1}{3} + 0 \cdot 1 \cdot \frac{1}{3} + 1 \cdot 0 \cdot \frac{1}{3} \\
&= 0 \\
\mathrm{Cov}(Y_1, Y_2) &= \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_1]\mathbb{E}[Y_2] \\
&= 0 - 0 \cdot \frac{1}{3} \\
&= 0
\end{aligned}
$$

Group 4 **Grade:**

Suppose that Y_1 and Y_2 have correlation coefficient $\rho = .2$. What is the value of the correlation coefficient between

1. $1 + 2Y_1$ and $3 + 4Y_2$?
2. $1 + 2Y_1$ and $3 - 4Y_2$?
3. $1 - 2Y_1$ and $3 - 4Y_2$?

Hint:

Use the variance of derived random variable and the definition of correlation coefficient.

Solution

$$\begin{aligned}
\rho_{1+2Y_1,3+4Y_2} &= \frac{\mathrm{Cov}(1+2Y_1, 3+4Y_2)}{\sqrt{\mathrm{Var}(1+2Y_1)}\sqrt{\mathrm{Var}(3+4Y_2)}} \\
&= \frac{\mathrm{Cov}(2Y_1, 4Y_2)}{\sqrt{\mathrm{Var}(2Y_1)}\sqrt{\mathrm{Var}(4Y_2)}} \\
&= \frac{2 \cdot 4\,\mathrm{Cov}(Y_1, Y_2)}{2 \cdot 4\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= \rho = 0.2 \\
\rho_{1+2Y_1,3-4Y_2} &= \frac{\mathrm{Cov}(1+2Y_1, 3-4Y_2)}{\sqrt{\mathrm{Var}(1+2Y_1)}\sqrt{\mathrm{Var}(3-4Y_2)}} \\
&= \frac{\mathrm{Cov}(2Y_1, -4Y_2)}{\sqrt{\mathrm{Var}(2Y_1)}\sqrt{\mathrm{Var}(-4Y_2)}} \\
&= \frac{2 \cdot (-4)\,\mathrm{Cov}(Y_1, Y_2)}{2 \cdot 4\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= -\frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= -\rho = -0.2 \\
\rho_{1-2Y_1,3-4Y_2} &= \frac{\mathrm{Cov}(1-2Y_1, 3-4Y_2)}{\sqrt{\mathrm{Var}(1-2Y_1)}\sqrt{\mathrm{Var}(3-4Y_2)}} \\
&= \frac{\mathrm{Cov}(-2Y_1, -4Y_2)}{\sqrt{\mathrm{Var}(-2Y_1)}\sqrt{\mathrm{Var}(-4Y_2)}} \\
&= \frac{(-2) \cdot (-4)\,\mathrm{Cov}(Y_1, Y_2)}{2 \cdot 4\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= \rho = 0.2
\end{aligned}$$

Group 5 **Grade:**

Let Y_1 and Y_2 have a joint density function given by

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 3y_1 & 0 \le y_2 \le y_1 \le 1, \\ 0 & elsewhere. \end{cases}$$

1. Find the marginal density functions of Y_1 and Y_2.
2. Find $\mathbb{P}(Y_1 \le 3/4 \mid Y_2 \le 1/2)$.
3. Find the conditional density function of Y_1 given $Y_2 = y_2$.
4. Find $\mathbb{P}(Y_1 \le 3/4 \mid Y_2 = 1/2)$.

Solution

$$\begin{aligned}
P_{Y_1}(y_1) &= \int_0^{y_1} 3y_1 \, \mathrm{d}y_2 \\
&= 3y_1^2 \\
P_{Y_2}(y_2) &= \int_{y_2}^{1} 3y_1 \, \mathrm{d}y_1 \\
&= \frac{3}{2}(1-y_2^2) \\
\mathbb{P}(Y_1 \le 3/4 \mid Y_2 \le 1/2) &= \frac{\mathbb{P}(Y_1 \le 3/4, Y_2 \le 1/2)}{\mathbb{P}(Y_2 \le 1/2)} \\
&= \frac{\int_0^{1/2} \int_{y_2}^{3/4} 3y_1 \, \mathrm{d}y_1 \, \mathrm{d}y_2}{\int_0^{1/2} \frac{3}{2}(1-y_2^2) \, \mathrm{d}y_2} \\
&= \frac{\frac{23}{64}}{\frac{11}{16}} \\
&= \frac{23}{44} \\
f_{Y_1|Y_2}(y_1 \mid y_2) &= \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_2}(y_2)} \\
&= \frac{3y_1}{\frac{3}{2}(1-y_2^2)} \\
&= \frac{2y_1}{1-y_2^2} \\
\mathbb{P}(Y_1 \le 3/4 \mid Y_2 = 1/2) &= \int_0^{3/4} \frac{2y_1}{1-(1/2)^2} \, \mathrm{d}y_1 \\
&= \frac{3}{4}
\end{aligned}$$