HW

Group 1 **Grade:**

Given number $\{0, 1, 2, 3, 4, 5\}$, how many odd numbers can be formed with 5 unrepeated digits?

Group 2 **Grade:**

Six couples sitting in a table. How many ways can they be seated such that all couples sits together?

Group 3 **Grade:**

We want to order 4 green balls, 3 red balls, and 2 blue balls. The green balls cannot be next to each other. How many ways can we do that?

Group 4 **Grade:**

We want to place 10 balls in 7 boxes. How many ways can we do that?

Group 5 **Grade:**

There are 8 books on a shelf, of 2 are paperbacks and 6 are hardbacks. How many possible selections of 4 books from this shelf Include at least one paperback?

Group 1 **Grade:**

A committee of 7 people have to be chosen among 11 women and 8 men. How many ways can be chosen if there must be more women than men.

Group 2 **Grade:**

let A and B be events with probability $\mathbb{P}(A) = 3/4$ and $\mathbb{P}(B) = 1/3$, show that $1/12 \leq \mathbb{P}(A \cap B) \leq 1/3$.

Group 3 **Grade:**

Prove that $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$

Group 4 **Grade:**

A box contains 30 red balls, 30 white balls, and 30 blue balls. If 10 balls are selected at random, without replacement, what is the probability that at least one color will be missing from the selection?

Group 5 **Grade:**

If the probability that student A will fail a certain examination is 0.5, the probability that student B will fail the examination is 0.2, and the probability that both student A and student B will fail the examination is 0.1, what is the probability that at least one of these two students will fail the examination?

Group 1 **Grade:**

When utilizing theorems of probability theory (and other theorems), it's crucial to carefully distinguish between the generalized form and the special case. For instance, the generalized version of the combination formula, $\binom{n}{k_1, k_2, \ldots k_r}$, mirrors the pattern of $\binom{n}{k}$. However, the generalized form of De Morgan's Law is not as straightforward as its special case. In this problem, we will delve into the concept of the **independence of three events**.

Definition 0.1. A_1, A_2, and A_3 are mutually independent if and only if

1. A_1 and A_2 are independent.
2. A_1 and A_3 are independent.
3. A_2 and A_3 are independent.
4. $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$.

Question: Rolling two dices independently. Let X and Y be the scores on two fair dice taking values in the set $\{1, 2, \ldots, 6\}$. Let $A_1 = \{X + Y = 9\}$, $A_2 = \{X \in \{1, 2, 3\}\}$, and $A_3 = \{X \in \{3, 4, 5\}\}$. Are A_1, A_2, and A_3 are mutually independent.
Note: In upcoming lectures, we will learn that the assumption of independence is both strong and commonly employed in many probability modeling theorems. Nevertheless, in real-world problems, it can be quite challenging to meet this assumption. There is a Ph.D. level course, High Dimensional Probability, that tackles this topic in real-life scenarios. For the moment, let's proceed under the assumption that it holds true.

Group 2 **Grade:**

A diagnostic test for a disease is such that it (correctly) detects the disease in 90% of the individuals who actually have the disease. Also, if a person does not have the disease, the test will report that he or she does not have it with probability .9. Only 1% of the population has the disease in question. If a person is chosen at random from the population and the diagnostic test indicates that she has the disease, what is the conditional probability that she does, in fact, have the disease? Are you surprised by the answer? Would you call this diagnostic test reliable?
Note: Always keep *Law of Total Number* in mind.

Group 3 **Grade:**

The *symmetric difference* between two events A and B is the set of all sample points that are in exactly one of the sets and is often denoted $A \triangle B$. Note that $A \triangle B = (A \cap B^{\mathsf{c}}) \cup (A^{\mathsf{c}} \cap B)$. Prove that $\mathbb{P}(A \triangle B) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$.

Group 4 **Grade:**

Of the travelers arriving at a small airport, 60% fly on major airlines, 30% fly on privately owned planes, and the remainder fly on commercially owned planes not belonging to a major airline. Of those traveling on major airlines, 50% are traveling for business reasons, whereas 60% of those arriving at private planes and 90% of those arriving at other commercially owned planes are traveling for business reasons. Suppose that we randomly select one person arriving at this airport. What is the probability that

1. The person is traveling on business?
2. The person is traveling for business on a privately owned plane?
3. The person arrived at a privately owned plane, given that the person is traveling for business reasons?
4. The person is traveling on business, given that the person is flying on a commercially owned plane?

Group 5 **Grade:**

If A_1, A_2, and A_3 are three events and $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_3) \neq 0$ but $\mathbb{P}(A_2 \cap A_3) = 0$, show that

$$\mathbb{P}(\text{at least one } A_i) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - 2\mathbb{P}(A_1 \cap A_2).$$

Note: Venn Diagram is helpful, but we correctly expect something beyond that.

Group 1 **Grade:**

Two gamblers bet \$1 each on the successive tosses of a coin. Each has a bank of \$6. What is the probability that

1. they break even after six tosses of the coin?
2. One player—say, Jones—wins all the money on the tenth toss of the coin?

Group 2 **Grade:**

Alice searches for her term paper in her filing cabinet, which has several drawers. Each drawer would be selected with probability $p_i > 0 (i = 1, 2, \ldots, j-1, j, j+1, \ldots)$, and the term paper is in drawer j, but Alice doesn't know which one is. The drawers are so messy that even if she correctly guesses that the term paper is in drawer j (*i.e.* with probability p_j), the probability that she finds it is only d. Alice searches in a particular drawer, but the search is unsuccessful. Condition on this, what is the probability that she chose the correct drawer? And what is the probability that she chose the wrong one?

Note: The calculation of this problem is not difficult, but the statement can be misleading. Try to describe the problem in your own words.
Note: You probably have experienced this situation in your life. But you may not notice that a failure search still provides information and increase the chance of success. In designing a strategy to maximize the long term success rate of a sequential experiment, this is a very important concept to start with.

Group 3 **Grade:**

A hunter has two hunting dogs. One day, on the trail of some animal, the hunter comes to a place where the road diverges into two paths. He knows that each dog, independent of the other, will choose the correct path with probability p. The hunter decides to let each dog choose a path, and if they agree, take that one, and if they disagree, to randomly pick a path. Is his strategy better than just letting one of the two dogs decide on a path?

Group 4 **Grade:**

You are a member of a class of 18 students. A bowl contains 18 chips: 1 blue and 17 red. Each student is to take one chip from the bowl without replacement. The student who draws the blue chip is guaranteed an 'A' for the course.

1. If you have a choice of drawing first, fifth or last, which would you choose?
2. Suppose the bowl contains 2 blue chips and 16 red chips. What position would you now choose (Not limited to first, fifth or last)?

Group 5 **Grade:**

A hospital receives 40% of its flu vaccine from Company A and the remainder from Company B. Each shipment contains many vials of vaccine. From Company A, 3% of the vials are ineffective, from Company B, 2% are ineffective. A hospital tests $n = 25$ randomly selected vials from one shipment and finds that two are ineffective. What is the conditional probability that the shipment came from Company B?

Group 1 **Grade:**

You are a contestant on a TV game show; there are four identical-looking suitcases containing \$200, \$400, \$800, and \$1600. You start the game by randomly choosing a suitcase. Among the three unchosen suitcases, the game show host opens the suitcase that holds the least money. The host then asks you if you want to keep your suitcase or switch one of the other remaining suitcases. For the following analysis, use the following notation for events:

- C_i is the event that you choose a suitcase with i dollars.
- O_i denotes the event that the host opens a suitcase with i dollars.
- R is the reward in dollars that you keep.

Sketch a tree diagram for this experiment, and answer the following questions.

1. You refuse the host's offer and open the suitcase you first chose. Find the PMF of R and the expected value $\mathbb{E}[R]$.

2. You always switch and randomly choose one of the two remaining suitcases. You receive the R dollars in this chosen suitcase. Find the PMF and expected value of R.
3. Can you do better than either always switching or always staying with your original choice? Explain.

Group 2 **Grade:**

Show that the variance of $\mathsf{Bin}(n, p)$ is

$$\mathrm{Var}(X) = np(1-p).$$

Group 3 **Grade:**

In the New Jersey state lottery, each \$ 1 ticket has six randomly marked numbers out of $1, 2, \ldots, 46$. A ticket is a winner if the six marked numbers match six numbers drawn at random at the end of a week. For each ticket sold, 50 cents are added to the pot for the winners. If there are k winning tickets, the pot is divided equally among the k winners. Suppose you bought a winning ticket in a week in which $2n$ tickets are sold, and the pot is n dollars. Given that your ticket is a winner, what is the probability that there are "$\mathbb{E}[k]$" winners?
Note: For this problem, first explain that the number of winners is a binomial random variable, determine the corresponding parameters and then find the solution. Assume that $2nq$ is an integer, where $q = \frac{1}{\binom{46}{6}}$.
Note: In this statement, there are a lot of redundant information. Try to extract the essential information and express the problem in your own words.

Group 4 **Grade:**

An internet service provider uses 50 modems to serve the needs of 1000 customers. It is estimated that at a given time, each customer will need a connection with probability 0.01, independent of the other customers.

(a) What is the PMF of the number of customers that need a connection at a given time?

(b) Repeat item (a) by approximating the PMF of the number of customers that need a connection with a Poisson PMF

(c) What is the probability that there are more customers needing a connection than there are modems? Provide an exact (item (a)), as well as an approximate formula based on the Poisson approximation of item (b).

Group 5 **Grade:**

Let $X \sim \mathsf{Bin}(n, p)$ and $Y \sim \mathsf{Bin}(n, 1-p)$ represent binomial random variables with parameters n and p, show that

$$\mathbb{P}(X \leq i) + \mathbb{P}(Y \leq n - i - 1) = 1.$$

Group 1 **Grade:**

The number of eggs laid on a tree leaf by an insect of a certain type is a Poisson random variable with parameter α. However, such a random variable can be observed only if it is positive, since if it is 0, then we cannot know that such an insect was on the leaf. If we let Y denote the observed number of eggs, then

$$\mathbb{P}(Y = i) = \mathbb{P}(X = i \mid X > 0)$$

where X is Poisson with parameter α. Find $\mathbb{E}[Y]$.
Hint: Regular Poisson X has values starting from 0 while Y starts from 1. We can eliminate $P_X(0)$ from X and re-normalize it to get the PMF of Y.

Group 2 **Grade:**

A jar contains n chips. Suppose that a boy successively draws a chip from the jar, each time replacing the one drawn before drawing another. The process continues until the boy draws a chip that he has previously drawn. Let X denote the number of draws, and compute its probability mass function.

Group 3 **Grade:**

If X is a geometric random variable, show that

$$\mathbb{P}(X = n + k \mid X > n) = \mathbb{P}(X = k).$$

This is called memoryless property. Using the interpretation of a geometric random variable, give a verbal argument as to why the preceding equation is true.

Group 4 **Grade:**

True or False: For any random variable X, $\mathbb{E}[1/X] = 1/\mathbb{E}[X]$.

Group 5 **Grade:**

Find $\mathbb{P}(K < \mathbb{E}[K])$ when

1. K is $\mathsf{Geo}(1/3)$
2. K is $\mathsf{Bin}(6, 1/2)$
3. K is $\mathsf{Poisson}(3)$

Group 1 **Grade:**

Calamity Jane goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customers ahead of her. The service time of the customer ahead, if present, is exponentially distributed with parameter λ. What is the CDF of Jane's waiting time?

Group 2 **Grade:**

A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over $(0, 100)$. There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?

Group 3 **Grade:**

The time X (in minutes) between customer arrivals at a bank is exponentially distributed with mean 1.5 minutes.

1. If a customer has just arrived, what is the probability that no customer will arrive in the next 2 minutes?
2. What is the probability that no customer will arrive within the next minute, given that no customer had arrived in the past minute?

Group 4 **Grade:**

The random variable X has the probability density function

$$f(x) = \begin{cases} ax + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $\mathbb{E}[X] = 0.6$, find

1. $\mathbb{P}\big(X < \frac{1}{2}\big)$.
2. $\mathrm{Var}(X)$.

Group 5 **Grade:**

The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take more than 10 years before a year occurs having a rainfall of more than 50 inches? Suppose the rainfall in each year is independent of the rainfall in other years and the distribution of rainfall in each year is the same as in the present year.

Group 1 **Grade:**

If X is uniformly distributed over (a, b), $a < b$, what is the probability density function of $Y = cX + d$ for any constants c and d.

Group 2 **Grade:**

A die is biased in such a way that even numbers are three times as likely to be rolled as odd numbers. Approximate the probability that the number 5 will appear at most 15 times in 100 throws.

Group 3 **Grade:**

If Y has an exponential distribution with mean $\frac{1}{\lambda}$, find (as a function of λ) the median of Y.

Group 4 **Grade:**

The random variable X has probability density function

$$f_X(x) = \begin{cases} cx & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

1. What is the constant c to make the f_X a valid PDF? The result will be used in the following questions.
2. Find the CDF of X.
3. Find $\mathbb{E}[X]$ and $\text{Var}[X]$.

The CDF of random variable Y is

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ (y+1)/2 & -1 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

1. Find the PDF of Y.
2. Find $\mathbb{E}[Y]$ and $\text{Var}[Y]$.

Group 5 **Grade:**

X is a uniform random variable with expected value $\mu_X = 7$ and variance $\text{Var}[X] = 3$. What is the PDF of X?

Group 1 **Grade:**

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability function of

- X_1, X_2.
- X_1, X_2, X_3.

Solution

$$P_{X_1X_2}(x_1x_2) = \begin{cases} \frac{5}{13}\frac{4}{12} & \text{if } x_1 = x_2 = 1 \\ \frac{5}{13}\frac{8}{12} & \text{if } x_1 = 1, x_2 = 0 \\ \frac{8}{13}\frac{5}{12} & \text{if } x_1 = 0, x_2 = 1 \\ \frac{8}{13}\frac{7}{12} & \text{if } x_1 = x_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{X_1X_2X_3}(x_1x_2x_3) = \begin{cases} \frac{5}{13}\frac{4}{12}\frac{3}{11} & \text{if } x_1 = x_2 = x_3 = 1 \\ \frac{5}{13}\frac{4}{12}\frac{8}{11} & \text{if } x_1 = x_2 = 1, x_3 = 0 \\ \frac{5}{13}\frac{8}{12}\frac{4}{11} & \text{if } x_1 = 1, x_2 = 0, x_3 = 1 \\ \frac{5}{13}\frac{8}{12}\frac{7}{11} & \text{if } x_1 = 1, x_2 = x_3 = 0 \\ \frac{8}{13}\frac{5}{12}\frac{4}{11} & \text{if } x_1 = 0, x_2 = x_3 = 1 \\ \frac{8}{13}\frac{5}{12}\frac{7}{11} & \text{if } x_1 = 0, x_2 = 1, x_3 = 0 \\ \frac{8}{13}\frac{7}{12}\frac{5}{11} & \text{if } x_1 = x_2 = 0, x_3 = 1 \\ \frac{8}{13}\frac{7}{12}\frac{6}{11} & \text{if } x_1 = x_2 = x_3 = 0 \\ 0 & \text{otherwise} \end{cases}$$

Group 2 **Grade:**

Consider a sequence of independent Bernoulli trails each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first and second successes. Find the joint probability mass function of X_1 and X_2.

Solution

$$P_{X_1X_2}(x_1x_2) = \begin{cases} p^2(1-p)^{x_1+x_2} & \text{if } x_1, x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Group 3 **Grade:**

The time that it takes to service a car is an exponential random variable with rate 1.

1. If A brings his car in at time 0 and B brings his car in at time t, what is the probability that B's car is ready before A's? Assume that the service times are independent, and the service begins upon arrival.
2. If both cars are brought in at time 0, with work starting on B's car only when A's car has been completely serviced, what is the probability that B's car is ready before 2?

Hint:

1. Find the Joint PDF of X_A and X_B.
2. $\mathbb{P}(X_A > X_B + t)$ can be written in a form of $\mathbb{P}(X_A \in (a, b), X_B \in (c, d))$.
3. Then $\mathbb{P}(X_A > X_B + t) = \int_a^b \int_c^d P_{X_A,Y_B}(x, y)\,\mathrm{d}y\,\mathrm{d}x$.
4. With the same logic, we can solve $\mathbb{P}(X_A + X_B < 2)$.

Solution

Let X_A be the service time of A's car and X_B be the service time of B's car. Due to independent, the joint PDF is

$$f_{X_A,X_B}(x_A, x_B) = \begin{cases} e^{-(x_A+x_B)} & x_A, x_B \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
\mathbb{P}(X_A > X_B + t) &= \mathbb{P}(0 < X_B + t < X_A < \infty) \\
&= \int_0^\infty \int_{x_B+t}^\infty e^{-(x_A+x_B)}\,\mathrm{d}x_A\,\mathrm{d}x_B \\
&= \int_0^\infty e^{-x_B} \int_{x_B+t}^\infty e^{-x_A}\,\mathrm{d}x_A\,\mathrm{d}x_B \\
&= \int_0^\infty e^{-x_B} e^{-(x_B+t)}\,\mathrm{d}x_B \\
&= \int_0^\infty e^{-2x_B-t}\,\mathrm{d}x_B \\
&= \frac{1}{2}e^{-t}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(X_A + X_B < 2) &= \mathbb{P}(0 < X_A < 2 - X_B < 2) \\
&= \int_0^2 \int_0^{2-x_B} e^{-(x_A+x_B)}\,\mathrm{d}x_A\,\mathrm{d}x_B \\
&= \int_0^2 e^{-x_B} \int_0^{2-x_B} e^{-x_A}\,\mathrm{d}x_A\,\mathrm{d}x_B \\
&= \int_0^2 e^{-x_B} - e^{-2}\,\mathrm{d}x_B \\
&= 1 - 3e^{-2}
\end{aligned}
$$

Group 4 Grade:

Suppose X and Y are independent normal random variables with parameters (μ_X, σ_X) and (μ_Y, σ_Y), respectively. Find x such that $\mathbb{P}(X - Y > x) = \mathbb{P}(X + Y > a)$ for some constant a.

Hint:

1. $X - Y$ and $X + Y$ are normal random variables as well. Find their parameters.
2. Convert $X - Y$ and $X + Y$ to standard normal random variables.
3. Solve the problem with Φ function, the CDF of standard normal random variable.

Solution

$X - Y \sim \mathsf{N}(\mu_X - \mu_Y, \sqrt{\sigma_X^2 + \sigma_Y^2})$, $X + Y \sim \mathsf{N}(\mu_X + \mu_Y, \sqrt{\sigma_X^2 + \sigma_Y^2})$.

$$
\begin{aligned}
&\mathbb{P}(X - Y > x) = \mathbb{P}(X + Y > a) \\
\Rightarrow &\mathbb{P}(X - Y \le x) = \mathbb{P}(X + Y \le a) \\
\Rightarrow &\mathbb{P}\left(\frac{X - Y - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}} \le \frac{x - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) = \mathbb{P}\left(\frac{X + Y - (\mu_X + \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}} \le \frac{a - (\mu_X + \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) \\
\Rightarrow &\Phi\left(\frac{x - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) = \Phi\left(\frac{a - (\mu_X + \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) \\
\Rightarrow &\frac{x - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}} = \frac{a - (\mu_X + \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}
\end{aligned}
$$

$\Rightarrow x = a - 2\mu_Y$

Group 5 **Grade:**

Let Y_1 and Y_2 be uncorrelated random variables and consider $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$.

1. Find the $\text{Cov}(U_1, U_2)$ in terms of the variances of Y_1 and Y_2.
2. Find an expression for the coefficient of correlation between U_1 and U_2.
3. Is it possible that $\text{Cov}(U_1, U_2) = 0$? When does this occur?

Hint:

1. $\text{Cov}(U_1, U_2) = \mathbb{E}[U_1U_2] - \mathbb{E}[U_1]\mathbb{E}[U_2] = \ldots$.
2. $\rho = \frac{\sigma_{U_1U_2}}{\sigma_{U_1}\sigma_{U_2}}$

Solution

$$
\begin{aligned}
\mathbb{E}[U_1] &= \mathbb{E}[Y_1 + Y_2] = \mathbb{E}[Y_1] + \mathbb{E}[Y_2] \\
\mathbb{E}[U_2] &= \mathbb{E}[Y_1 - Y_2] = \mathbb{E}[Y_1] - \mathbb{E}[Y_2] \\
\mathbb{E}[U_1U_2] &= \mathbb{E}[(Y_1 + Y_2)(Y_1 - Y_2)] = \mathbb{E}\left[Y_1^2\right] - \mathbb{E}\left[Y_2^2\right] \\
\text{Cov}(U_1U_2) &= \mathbb{E}[U_1U_2] - \mathbb{E}[U_1]\mathbb{E}[U_2] \\
&= \mathbb{E}\left[Y_1^2\right] - \mathbb{E}\left[Y_2^2\right] - (\mathbb{E}[Y_1] + \mathbb{E}[Y_2])(\mathbb{E}[Y_1] - \mathbb{E}[Y_2]) \\
&= \mathbb{E}\left[Y_1^2\right] - \mathbb{E}\left[Y_2^2\right] - \left(\mathbb{E}[Y_1]^2 - \mathbb{E}[Y_2]^2\right) \\
&= \text{Var}(Y_1) - \text{Var}(Y_2) \\
\rho &= \frac{\text{Cov}(U_1, U_2)}{\sqrt{\text{Var}(U_1)\,\text{Var}(U_2)}} \\
&= \frac{\text{Var}(Y_1) - \text{Var}(Y_2)}{\sqrt{\text{Var}(Y_1)\,\text{Var}(Y_2)}}
\end{aligned}
$$

When $\text{Var}(Y_1) = \text{Var}(Y_2)$, $\rho = 0$.

Group 1 **Grade:**

Suppose your grade in a probability course depends on your exam scores X_1 and X_2. The professor, a fan of probability, releases exam scores in a normalized fashion such that X_1 and X_2 are independent Gaussian ($\mu = 0, \sigma = \sqrt{2}$) random variables. Your semester average is $X = 0.5(X_1 + X_2)$.

1. You earn an A grade if $X > 1$. What is $\mathbb{P}(A)$?
2. To improve his SIRS (Student Instructional Rating Service) score, the professor decides he should award more A's. Now you get an A if $\max(X_1, X_2) > 1$. What is $\mathbb{P}(A)$ now?
3. The professor found out he is unpopular at ratemyprofessor.com and decides to award an A if either $X > 1$ or $\max(X_1, X_2) > 1$. Now what is $\mathbb{P}(A)$?
4. Under criticism of grade inflation from the department chair, the professor adopts a new policy.

An A is awarded if $\max(X_1, X_2) > 1$ and $\min(X_1, X_2) > 0$. Now what is $\mathbb{P}(A)$?

Hint:

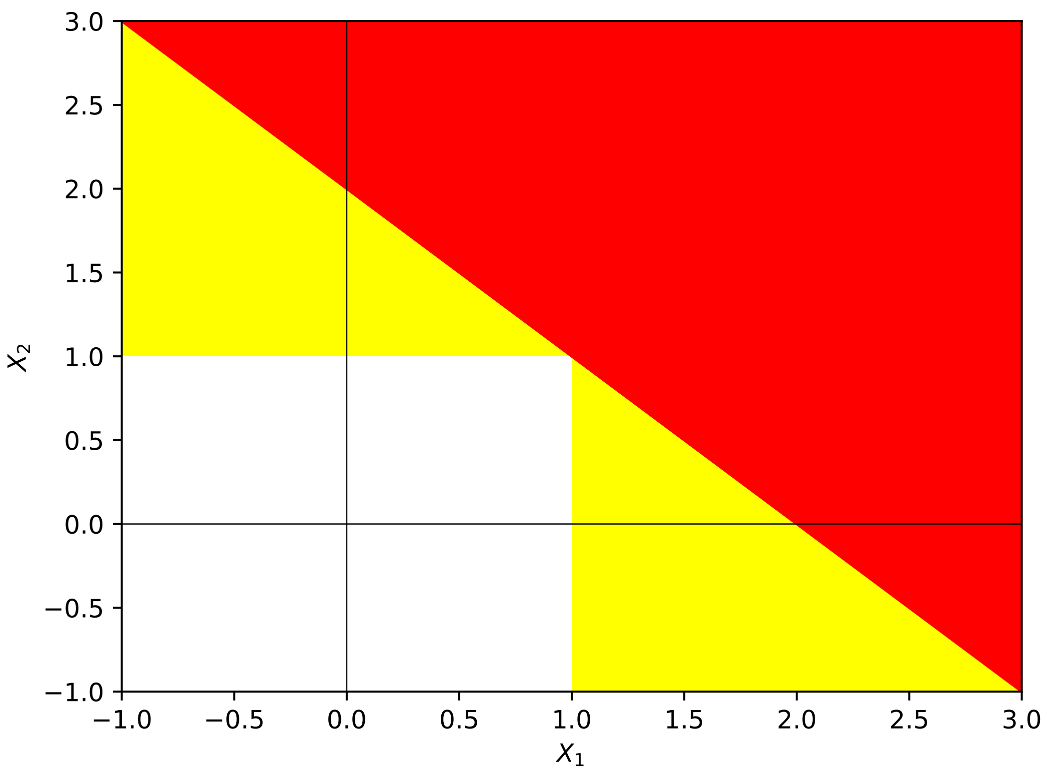
1. Sum of independent Gaussian and Φ function.
2. $\mathbb{P}(\max(X_1, X_2) > 1) = 1 - \mathbb{P}(\max(X_1, X_2) \le 1)$.
3. Draw the Venn Diagram to show the relationship between $X > 1$ and $\max(X_1, X_2) > 1$.
4. Draw the Venn Diagram to show the relationship between $\max(X_1, X_2) > 1$ and $\min(X_1, X_2) > 0$.

Solution

It's easy to show that $X \sim \mathsf{N}(0, 1)$ which is a standard normal random variable.

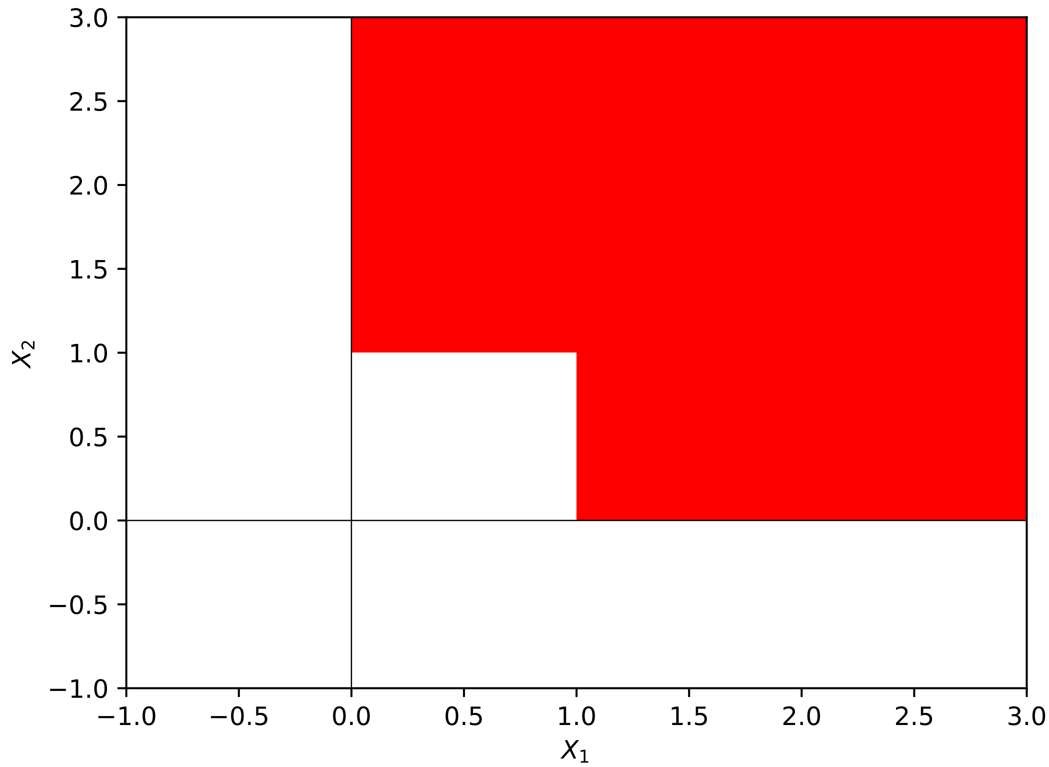
$$\begin{aligned}
\mathbb{P}(X > 1) &= 1 - \mathbb{P}(X \le 1) = 1 - \Phi(1) \\
\mathbb{P}(\max(X_1, X_2) > 1) &= 1 - \mathbb{P}(\max(X_1, X_2) \le 1) \\
&= 1 - \mathbb{P}(X_1 \le 1, X_2 \le 1) \\
&= 1 - \mathbb{P}(X_1 \le 1)\mathbb{P}(X_2 \le 1) \\
&= 1 - \Phi(1/\sqrt{2})^2
\end{aligned}$$

$\{\max(X_1, X_2) > 1 \cup X > 1\} = \{\max(X_1, X_2) > 1\}$ can be visualized by the following plot.



$$\mathbb{P}(\max(X_1, X_2) > 1 \cup X > 1) = \mathbb{P}(\max(X_1, X_2) > 1) = 1 - \Phi(1/\sqrt{2})^2$$

$\{\max(X_1, X_2) > 1 \cup \min(X_1, X_2) > 0\}$ can be visualized by the following plot.



$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(\max(X_1, X_2) > 1 \cup \min(X_1, X_2) > 0) \\
&= \mathbb{P}(X_1 > 0, X_2 > 0) - \mathbb{P}(0 < X_1 < 1, 0 < X_2 < 1) \\
&= \mathbb{P}(X_1 > 0)\mathbb{P}(X_2 > 0) - \mathbb{P}(0 < X_1 < 1)\mathbb{P}(0 < X_2 < 1) \\
&= (1 - \mathbb{P}(X_1 \leq 0))(1 - \mathbb{P}(X_2 \leq 0)) - (\mathbb{P}(X_1 \leq 1) - \mathbb{P}(X_1 \leq 0))(\mathbb{P}(X_2 \leq 1) - \mathbb{P}(X_2 \leq 0)) \\
&= (1 - \Phi(0))^2 - (\Phi(1/\sqrt{2}) - \Phi(0))^2
\end{aligned}$$

Group 2 **Grade:**

Let X and Y be continues random variables with joint PDF

$$f(x, y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

1. What are $\mathbb{E}[X]$ and $\mathrm{Var}(X)$?
2. What are $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$?
3. What are $\mathrm{Cov}(X, Y)$ and $\mathrm{Corr}(X, Y)$?
4. What are $\mathbb{E}[X + Y]$ and $\mathrm{Var}(X + Y)$?

Solution

$$
\begin{aligned}
\mathbb{E}[X] &= \int_0^1 \int_y^1 x \cdot 2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^1 1 - y^2 \, \mathrm{d}y \\
&= \frac{2}{3} \\
\mathbb{E}\left[X^2\right] &= \int_0^1 \int_y^1 x^2 \cdot 2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^1 \frac{2}{3}(1 - y^3) \, \mathrm{d}y \\
&= \frac{1}{2} \\
\mathrm{Var}(X) &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\
&= \frac{1}{2} - \frac{4}{9} \\
&= \frac{1}{18} \\
\mathbb{E}[Y] &= \int_0^1 \int_0^x y \cdot 2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 x^2 \, \mathrm{d}x \\
&= \frac{1}{3} \\
\mathbb{E}\left[Y^2\right] &= \int_0^1 \int_0^x y^2 \cdot 2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \frac{2}{3} x^3 \, \mathrm{d}x \\
&= \frac{1}{6} \\
\mathrm{Var}(Y) &= \mathbb{E}\left[Y^2\right] - \mathbb{E}[Y]^2 \\
&= \frac{1}{6} - \frac{1}{9} \\
&= \frac{1}{18} \\
\mathbb{E}[XY] &= \int_0^1 \int_0^x xy \cdot 2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 x^3 \, \mathrm{d}x \\
&= \frac{1}{4} \\
\mathrm{Cov}(X, Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \frac{1}{4} - \frac{2}{9} \\
&= \frac{1}{36}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Corr}(X,Y) &= \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} \\
&= \frac{1}{2} \\
\mathbb{E}[X+Y] &= \mathbb{E}[X] + \mathbb{E}[Y] \\
&= 1 \\
\operatorname{Var}(X+Y) &= \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X,Y) \\
&= \frac{1}{18} + \frac{1}{18} + 2\frac{1}{36} \\
&= \frac{1}{6}
\end{aligned}
$$

Group 3 **Grade:**

A company receives shipments from two factories. Depending on the size of the order, a shipment can be in

1. 1 box for a small order,
2. 2 boxes for a medium order,
3. 3 boxes for a large order.

The company has two different suppliers. Factory Q is 60 miles from the company. Factory R is 180 miles from the company. An experiment consists of monitoring a shipment and observing B, the number of boxes, and M, the number of miles the shipment travels. The following probability model describes the experiment:

	Factory Q	Factory R
small order	0.3	0.2
medium order	0.1	0.2
large order	0.1	0.1

1. Find $P_{B,M}(b,m)$, the joint PMF of the number of boxes and the distance.
2. What is $\mathbb{E}[B]$, the expected number of boxes?
3. Are B and M independent?

Solution

$$
P_{B,M}(b,m) = \begin{cases}
0.3 & b=1, m=60 \\
0.2 & b=1, m=180 \\
0.1 & b=2, m=60 \\
0.2 & b=2, m=180 \\
0.1 & b=3, m=60 \\
0.1 & b=3, m=180 \\
0 & \text{otherwise}
\end{cases}
$$

$$\mathbb{E}[B] = 1\cdot 0.3 + 1\cdot 0.2 + 2\cdot 0.1 + 2\cdot 0.2 + 3\cdot 0.1 + 3\cdot 0.1 = 1.7$$

No, B and M are not independent. Because $P_{B,M}(1,60) = 0.3 \neq 0.25 = 0.5\cdot 0.5 = P_B(1)P_M(60)$.

Group 4 **Grade:**

Random Variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} \frac{(1-p)^{n-1}p}{n} & k = 1, 2, \ldots, n; n = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Find $P_N(n)$, $\mathbb{E}[N]$, $\text{Var}(N)$, $\mathbb{E}\left[N^2\right]$, $\mathbb{E}[K]$, $\text{Var}(K)$, $\mathbb{E}[N+K]$, $r_{N,K}$, $\text{Cov}(N,K)$.

Hint:

Please solve the first four and leave the rest for me.

Solution

$$P_N(n) = \begin{cases} \sum_{k=1}^{n} \frac{(1-p)^{n-1}p}{n} = n \cdot \frac{(1-p)^{n-1}p}{n} = (1-p)^{n-1}p & n = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[N] = \frac{1}{p} \qquad \text{Var}[N] = \frac{1-p}{p^2} \qquad \mathbb{E}\left[N^2\right] = \text{Var}[N] + \mathbb{E}[N]^2 = \frac{2-p}{p^2}$$

$$\mathbb{E}[K] = \sum_{n=1}^{\infty}\sum_{k=1}^{n} k \cdot \frac{(1-p)^{n-1}p}{n} = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}p}{n} \sum_{k=1}^{n} k = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}p}{n} \frac{n(n+1)}{2}$$

Since $\sum_{n=1}^{\infty}(1-p)^{n-1}p = 1$

$$\mathbb{E}[K] = \mathbb{E}\left[\frac{N+1}{2}\right] = \frac{\mathbb{E}[N]+1}{2} = \frac{1}{2p} + \frac{1}{2}$$

Following the same logic,

$$\mathbb{E}\left[K^2\right] = \mathbb{E}\left[\frac{(N+1)(2N+1)}{6}\right] = \frac{2\mathbb{E}\left[N^2\right] + 3\mathbb{E}[N] + 1}{6} = \frac{2-p}{3p^2} + \frac{1}{2p} + \frac{1}{6}$$

$$\text{Var}[K] = \mathbb{E}\left[K^2\right] - \mathbb{E}[K]^2 = \frac{2-p}{3p^2} + \frac{1}{2p} + \frac{1}{6} - \left(\frac{1}{2p} + \frac{1}{2}\right)^2$$

$$\begin{aligned}
r_{N,K} &= \mathbb{E}[NK] \\
&= \mathbb{E}\left[\sum_{n=1}^{\infty}\sum_{k=1}^{n} nk \cdot \frac{(1-p)^{n-1}p}{n}\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{\infty}(1-p)^{n-1}p \sum_{k=1}^{n} k\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{\infty}(1-p)^{n-1}p \frac{n(n+1)}{2}\right] \\
&= \mathbb{E}\left[\frac{N(N+1)}{2}\right] \\
&= \frac{1}{2}\mathbb{E}\left[N^2\right] + \frac{1}{2}\mathbb{E}[N] \\
&= \frac{2-p}{2p^2} + \frac{1}{2p} = \frac{1}{p^2}
\end{aligned}$$

$$\text{Cov}(N,K) = r_{N,K} - \mathbb{E}[N]\mathbb{E}[K] = \frac{1}{p^2} - \frac{1}{p}\frac{1}{2p} - \frac{1}{2p} = \frac{1}{2p^2} - \frac{1}{2p}$$

Group 5 **Grade:**

X and Y are random variables such that X has expected value $\mu_X = 0$ and standard deviation $\sigma_X = 3$ while Y has expected value $\mu_Y = 1$ and standard deviation $\sigma_Y = 4$. In addition, X and Y have covariance $\operatorname{Cov}[X, Y] = -3$. Find the expected value and variance of $W = 2X + 2Y$.

Solution

$$\mathbb{E}[W] = \mathbb{E}[2X + 2Y] = 2\mathbb{E}[X] + 2\mathbb{E}[Y] = 2 \cdot 0 + 2 \cdot 1 = 2$$
$$\operatorname{Var}(W) = \operatorname{Var}(2X + 2Y) = 4\operatorname{Var}(X) + 4\operatorname{Var}(Y) + 8\operatorname{Cov}(X, Y) = 4 \cdot 3^2 + 4 \cdot 4^2 + 8 \cdot (-3) = 76$$

Group 1 **Grade:**

The expectation of $\mathsf{Geo}(p)$ can be proved by conditional expectation as follows:

Proof. For $X \sim \mathsf{Geo}(p)$,

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X \mid X = 1]\mathbb{P}(X = 1) + \mathbb{E}[X \mid X > 1]\mathbb{P}(X > 1) \\
&= \mathbb{E}[X = 1] \cdot p + \mathbb{E}[X + 1](1 - p) \\
\Rightarrow \mathbb{E}[X] &= \frac{1}{p}.
\end{aligned}$$

$\square$

Following the same logic, derive the Variance of $\mathsf{Geo}(p)$.

Solution

$$\begin{aligned}
\mathbb{E}\left[X^2\right] &= \mathbb{E}\left[X^2 \mid X = 1\right]\mathbb{P}(X = 1) + \mathbb{E}\left[X^2 \mid X > 1\right]\mathbb{P}(X > 1) \\
&= \mathbb{E}\left[X = 1^2\right] \cdot p + \mathbb{E}\left[(X + 1)^2\right](1 - p) \\
&= 1 \cdot p + \mathbb{E}\left[X^2 + 2X + 1\right](1 - p) \\
&= p + \mathbb{E}\left[X^2\right](1 - p) + 2\mathbb{E}[X](1 - p) + 1(1 - p) \\
\Rightarrow \mathbb{E}\left[X^2\right] &= \frac{2}{p^2} - \frac{1}{p} \\
\operatorname{Var}[X] &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\
&= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} \\
&= \frac{1 - p}{p^2}.
\end{aligned}$$

Group 2 **Grade:**

Let Y_1 denote the weight (in tons) of a bulk item stocked by a supplier at the beginning of a week and suppose that Y_1 has a uniform distribution over the interval $0 \le y_1 \le 1$. Let Y_2 denote the amount (by weight) of this item sold by the supplier during the week and suppose that Y_2 has a uniform distribution over the interval $0 \le y_2 \le y_1$, where y_1 is a specific value of Y_1.

1. Find the joint density function for Y_1 and Y_2.
2. If the supplier stocks a half-ton of the item, what is the probability that she sells more than a quarter-ton?
3. If it is known that the supplier sold a quarter-ton of the item, what is the probability that she had stocked more than a half-ton?

Solution

1.

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{Y_2|Y_1}(y_2 \mid y_1) f_{Y_1}(y_1) \\ &= \frac{1}{y_1} \cdot 1 \\ f_{Y_1,Y_2}(y_1,y_2) &= \begin{cases} \frac{1}{y_1} & 0 \le y_2 \le y_1 \le 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

2.

$$\begin{aligned} f_{Y_2|Y_1}(y_2 \mid y_1) &= \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_1}(y_1)} \\ &= \frac{\frac{1}{y_1}}{\int_0^{y_1} \frac{1}{y_1}\,\mathrm{d}y_2} \\ &= \frac{\frac{1}{y_1}}{1} \\ &= \frac{1}{y_1} \\ \mathbb{P}(Y_2 > 0.25 \mid Y_1 = 0.5) &= \int_{0.25}^{0.5} \frac{1}{0.5}\,\mathrm{d}y_2 \\ &= 0.5 \end{aligned}$$

3.

$$\begin{aligned} f_{Y_1|Y_2}(y_1 \mid y_2) &= \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_2}(y_2)} \\ &= \frac{\frac{1}{y_1}}{\int_{y_2}^{1} \frac{1}{y_1}\,\mathrm{d}y_1} \\ &= \frac{\frac{1}{y_1}}{-\ln y_2} \\ &= \frac{1}{y_1(-\ln y_2)} \\ \mathbb{P}(Y_1 > 0.5 \mid Y_2 = 0.25) &= \int_{0.5}^{1} \frac{1}{y_1(-\ln 0.25)}\,\mathrm{d}y_1 \\ &= \frac{\ln 2}{2\ln 2} \\ &= 0.5 \end{aligned}$$

Group 3 **Grade:**

A quality control plan calls for randomly selecting three items from the daily production (assumed large) of a certain machine and observing the number of defectives. However, the proportion p of defectives produced by the machine varies from day to day and is assumed to have a uniform distribution on the interval (0, 1). For a randomly chosen day, find the unconditional probability that exactly two defectives are observed in the sample.

Solution

$$\begin{aligned}
\mathbb{P}(Y = y \mid p) &= \binom{3}{y} p^y (1-p)^{3-y} \qquad y = 0,1,2,3 \\
\mathbb{P}(Y = 2) &= \int_0^1 \mathbb{P}(Y = 2, p)\,\mathrm{d}p \\
&= \int_0^1 \mathbb{P}(Y = 2 \mid p) f_P(p)\,\mathrm{d}p \\
&= \int_0^1 \binom{3}{2} p^2 (1-p)^{3-2} \cdot 1\,\mathrm{d}p \\
&= \int_0^1 3p^2(1-p)\,\mathrm{d}p \\
&= \int_0^1 (3p^2 - 3p^3)\,\mathrm{d}p \\
&= \left[p^3 - \frac{3}{4}p^4\right]_0^1 \\
&= 1 - \frac{3}{4} \\
&= \frac{1}{4}
\end{aligned}$$

Group 4 **Grade:**

Suppose that the probability that a head appears when a coin is tossed is p and the probability that a tail occurs is $q = 1 - p$. Person A tosses the coin until the first head appears and stops. Person B does likewise. The results obtained by persons A and B are assumed to be independent. What is the probability that A and B stop on exactly the same number toss? Explain how you think. The calculation is not mandatory.

Solution

$$\begin{aligned}
&\mathbb{P}(A = 1, B = 1) + \mathbb{P}(A = 2, B = 2) + \mathbb{P}(A = 3, B = 3) + \dots \\
=&\mathbb{P}(A = 1)\mathbb{P}(B = 1) + \mathbb{P}(A = 2)\mathbb{P}(B = 2) + \mathbb{P}(A = 3)\mathbb{P}(B = 3) + \dots \\
=&\sum_{i=1}^{\infty} \mathbb{P}(A = i)\mathbb{P}(B = i) \\
=&\sum_{i=1}^{\infty} pq^{i-1}pq^{i-1}
\end{aligned}$$

$$
\begin{aligned}
&= p^2 \sum_{i=1}^{\infty} (q^2)^{i-1} \\
&= \frac{p^2}{1-q^2}
\end{aligned}
$$

Group 5 **Grade:**

Let Y_1 and Y_2 be independent exponentially distributed random variables, each with mean 1. Find $P(Y_1 > Y_2 \mid Y_1 < 2Y_2)$ and $P(Y_1 < 2Y_2 \mid Y_1 > Y_2)$.

Hint:

1. Find the joint probability of Y_1 and Y_2.
2. Determine the range of Y_1 and Y_2 that satisfies the condition.
3. Use the definition of conditional probability.
4. Surprisingly, $P(Y_1 < 2Y_2 \mid Y_1 > Y_2)$ is much easier to calculate than $P(Y_1 = 2Y_2 \mid Y_1 > Y_2)$.

Solution

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1, y_2) &= f_{Y_1}(y_1) f_{Y_2}(y_2) \\
&= e^{-y_1} e^{-y_2} \\
&= e^{-(y_1+y_2)} \\
\mathbb{P}(Y_1 > Y_2 \mid Y_1 < 2Y_2) &= \frac{\mathbb{P}(Y_1 > Y_2, Y_1 < 2Y_2)}{\mathbb{P}(Y_1 < 2Y_2)} \\
&= \frac{\int_0^\infty \int_{y_2}^{2y_2} e^{-(y_1+y_2)} \, \mathrm{d}y_1 \, \mathrm{d}y_2}{\int_0^\infty \int_0^{2y_2} e^{-(y_1+y_2)} \, \mathrm{d}y_1 \, \mathrm{d}y_2} \\
&= \frac{\frac{1}{6}}{\frac{2}{3}} \\
&= \frac{1}{4} \\
\mathbb{P}(Y_1 < 2Y_2 \mid Y_1 > Y_2) &= \frac{\mathbb{P}(Y_1 < 2Y_2, Y_1 > Y_2)}{\mathbb{P}(Y_1 > Y_2)} \\
&= \frac{\int_0^\infty \int_{y_2}^{2y_2} e^{-(y_1+y_2)} \, \mathrm{d}y_1 \, \mathrm{d}y_2}{\int_0^\infty \int_{y_2}^{\infty} e^{-(y_1+y_2)} \, \mathrm{d}y_1 \, \mathrm{d}y_2} \\
&= \frac{\frac{1}{6}}{\frac{1}{2}} \\
&= \frac{1}{3}
\end{aligned}
$$

Group 1 **Grade:**

A coin that has probability of heads equal to p is tossed successively and independently until a head comes twice in a row or a tail comes twice in a row. Find the expected value of the number of tosses in terms of p and $q = 1 - p$.

Hint:

The Law of Total Expectation is useful.

Solution

With the law of total probability, we can partition the event X based on the first (starting from 0) toss,

$$\mathbb{P}(X) = \mathbb{P}(X \mid H_0)\mathbb{P}(H_0) + \mathbb{P}(X \mid T_0)\mathbb{P}(T_0).$$

Similarly, each partition of the first toss can be partitioned based on the second toss,

$$\begin{aligned}
\mathbb{P}(X \mid H_0) &= \mathbb{P}(X \mid H_0 \cap H_1)\mathbb{P}(H_1) + \mathbb{P}(X \mid H_0 \cap T_1)\mathbb{P}(T_1), \\
\mathbb{P}(X \mid T_0) &= \mathbb{P}(X \mid T_0 \cap H_1)\mathbb{P}(H_1) + \mathbb{P}(X \mid T_0 \cap T_1)\mathbb{P}(T_1).
\end{aligned}$$

Apply expectation to both sides,

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X \mid H_0]\mathbb{P}(H_0) + \mathbb{E}[X \mid T_0]\mathbb{P}(T_0), \\
\mathbb{E}[X \mid H_0] &= \mathbb{E}[X \mid H_0 \cap H_1]\mathbb{P}(H_1) + \mathbb{E}[X \mid H_0 \cap T_1]\mathbb{P}(T_1), \\
\mathbb{E}[X \mid T_0] &= \mathbb{E}[X \mid T_0 \cap H_1]\mathbb{P}(H_1) + \mathbb{E}[X \mid T_0 \cap T_1]\mathbb{P}(T_1).
\end{aligned}$$

Because the game will end with two heads/tails in a row or start counting over, we have

$$\mathbb{E}[X \mid H_0H_1] = \mathbb{E}[X \mid T_0T_1] = 2, \qquad \mathbb{E}[X \mid H_0T_1] = 1 + \mathbb{E}[X \mid T_0], \qquad \mathbb{E}[X \mid T_0H_1] = 1 + \mathbb{E}[X \mid H_0].$$

$$\begin{aligned}
\mathbb{E}[X \mid H_0] &= 2p + (1 + \mathbb{E}[X \mid T_0])q, \\
\mathbb{E}[X \mid T_0] &= (1 + \mathbb{E}[X \mid H_0])p + 2q, \\
\Rightarrow \mathbb{E}[X \mid H_0] &= \frac{2 + q^2}{1 - pq}, \\
\mathbb{E}[X \mid T_0] &= \frac{2 + p^2}{1 - pq}, \\
\Rightarrow \mathbb{E}[X] &= \frac{2 + q^2}{1 - pq}p + \frac{2 + p^2}{1 - pq}q \\
&= \frac{2p + q^2p + 2q + p^2q}{1 - pq} \\
&= \frac{2(p + q) + (q + p)pq}{1 - pq} \\
&= \frac{2 + pq}{1 - pq}.
\end{aligned}$$

Group 2 **Grade:**

X and Y are independent identical discrete uniform $(1, 10)$ random variables. Find the conditional PMF $P_{X,Y|A}(x, y)$. Let A denote the event that

1. $\min(X, Y) > 5$.
2. $\max(X, Y) \leq 5$.

Hint:

For discrete random variables, you can always list all the possible values and their corresponding probabilities.

Solution

Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
Roy D. Yates and David J. Goodman

Yates and Goodman 3e Solution Set: 7.3.1

Problem 7.3.1 Solution

X and Y each have the discrete uniform PMF

$$P_X(x) = P_Y(x) = \begin{cases} 0.1 & x = 1, 2, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The joint PMF of X and Y is

$$\begin{aligned} P_{X,Y}(x, y) &= P_X(x) P_Y(y) \\ &= \begin{cases} 0.01 & x = 1, 2, \ldots, 10; y = 1, 2, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2}$$

The event A occurs iff $X > 5$ and $Y > 5$ and has probability

$$\mathrm{P}[A] = \mathrm{P}[X > 5, Y > 5] = \sum_{x=6}^{10} \sum_{y=6}^{10} 0.01 = 0.25. \tag{3}$$

Alternatively, we could have used independence of X and Y to write $\mathrm{P}[A] = \mathrm{P}[X > 5] \mathrm{P}[Y > 5] = 1/4$. From Theorem 7.6,

$$\begin{aligned} P_{X,Y|A}(x, y) &= \begin{cases} \frac{P_{X,Y}(x,y)}{\mathrm{P}[A]} & (x, y) \in A, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 0.04 & x = 6, \ldots, 10; y = 6, \ldots, 20, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

1

And the same for the second case.

Group 3 **Grade:**

A supermarket has two customers waiting to pay for their purchases at counter I and one customer waiting to pay at counter II. Let Y_1 and Y_2 denote the numbers of customers who spend more than \$50 on groceries at the respective counters. Suppose that Y_1 and Y_2 are independent binomial random variables, with the probability that a customer at counter I will spend more than \$50 equal to .2 and the probability that a customer at counter II will spend more than \$50 equal to .3. Find the

1. joint probability distribution for Y_1 and Y_2.
2. probability that not more than one of the three customers will spend more than \$50.

Solution

$$
\begin{aligned}
P_{Y_1}(y_1) &= \binom{2}{y_1} 0.2^{y_1} 0.8^{2-y_1}, \qquad y_1 = 0,1,2 \\
P_{Y_2}(y_2) &= \binom{1}{y_2} 0.3^{y_2} 0.7^{1-y_2}, \qquad y_2 = 0,1 \\
P_{Y_1,Y_2}(y_1,y_2) &= P_{Y_1}(y_1) P_{Y_2}(y_2) \\
&= \binom{2}{y_1} 0.2^{y_1} 0.8^{2-y_1} \binom{1}{y_2} 0.3^{y_2} 0.7^{1-y_2}, \qquad y_1 = 0,1,2 \quad y_2 = 0,1 \\
\mathbb{P}(Y_1 + Y_2 \leq 1) &= \mathbb{P}(0,0) + \mathbb{P}(1,0) + \mathbb{P}(0,1) \\
&= 0.8^2 \cdot 0.7 + 2 \cdot 0.2 \cdot 0.8 \cdot 0.7 + 0.8^2 \cdot 0.3 \\
&= 0.864
\end{aligned}
$$

Group 4 **Grade:**

The length of life Y for fuses of a certain type is modeled by the exponential distribution, with rate $\lambda = 1/3$ (The measurements are in hundreds of hours.).

(a) If two such fuses have independent lengths of life Y_1 and Y_2, find the joint probability density function for Y_1 and Y_2.

(b) One fuse in item (a) is in a primary system, and the other is in a backup system that comes into use only if the primary system fails. The total effective length of life of the two fuses is then $Y_1 + Y_2$. Find $\mathbb{P}(Y_1 + Y_2 \leq 1)$.

Solution

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{Y_1}(y_1) f_{Y_2}(y_2) \\
&= \frac{1}{3} e^{-\frac{1}{3} y_1} \frac{1}{3} e^{-\frac{1}{3} y_2} \\
&= \frac{1}{9} e^{-\frac{1}{3}(y_1 + y_2)} \\
\mathbb{P}(Y_1 + Y_2 \leq 1) &= \int_0^1 \int_0^{1-y_1} \frac{1}{9} e^{-\frac{1}{3}(y_1+y_2)} \, \mathrm{d}y_2 \, \mathrm{d}y_1 \\
&= 1 - \frac{4}{3} e^{-\frac{1}{3}}
\end{aligned}
$$

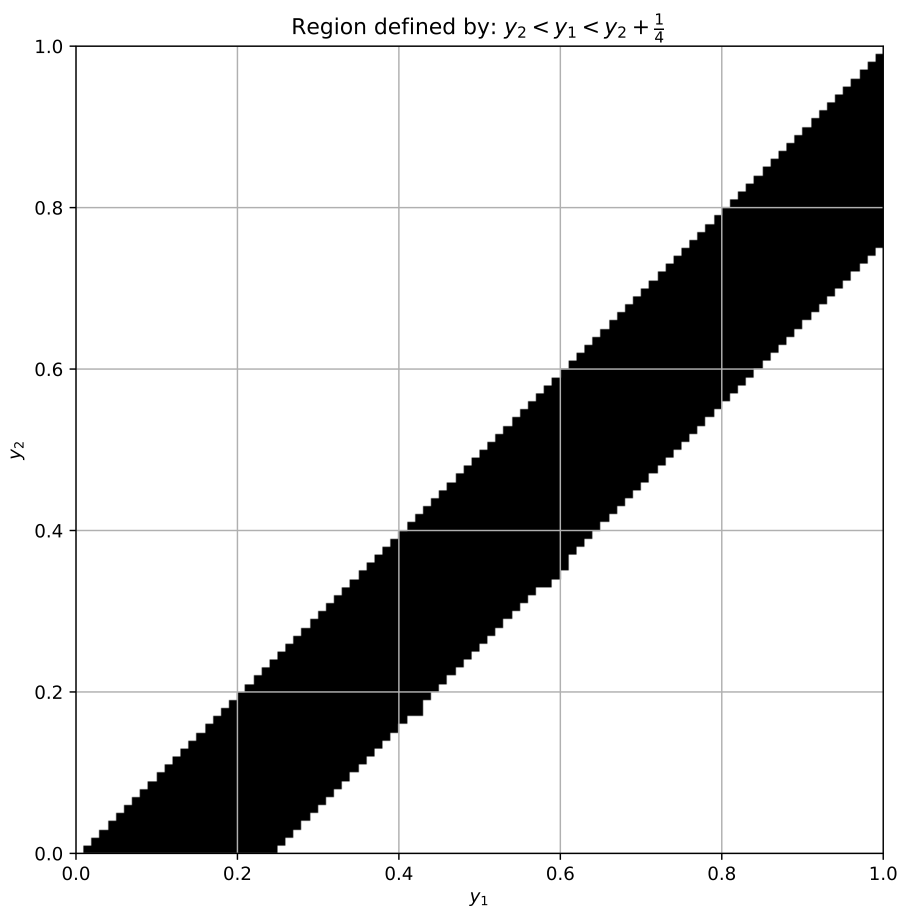
Group 5 **Grade:**

A bus arrives at a bus stop at a uniformly distributed time over the interval 0 to 1 hour. A passenger also arrives at the bus stop at a uniformly distributed time over the interval 0 to 1 hour. Assume that the arrival times of the bus and passenger are independent of one another and that the passenger will wait for up to 1/4 hour for the bus to arrive. What is the probability that the passenger will catch the bus?

Hint:

1. Let Y_1 denote the bus arrival time and Y_2 the passenger arrival time.
2. Determine the joint density of Y_1 and Y_2.
3. Find $\mathbb{P}(Y_2 \leq Y_1 \leq Y_2 + 1/4)$.

Solution



The black area is

$$1 \cdot 1 \cdot \frac{1}{2} - \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{32}.$$

Group 1 **Grade:**

Two telephone calls uniformly come into a switchboard at random times in a fixed one-hour period. Assume that the calls are made independently of one another. What is the probability that the calls are made

1. in the first half hour?
2. within five minutes of each other?

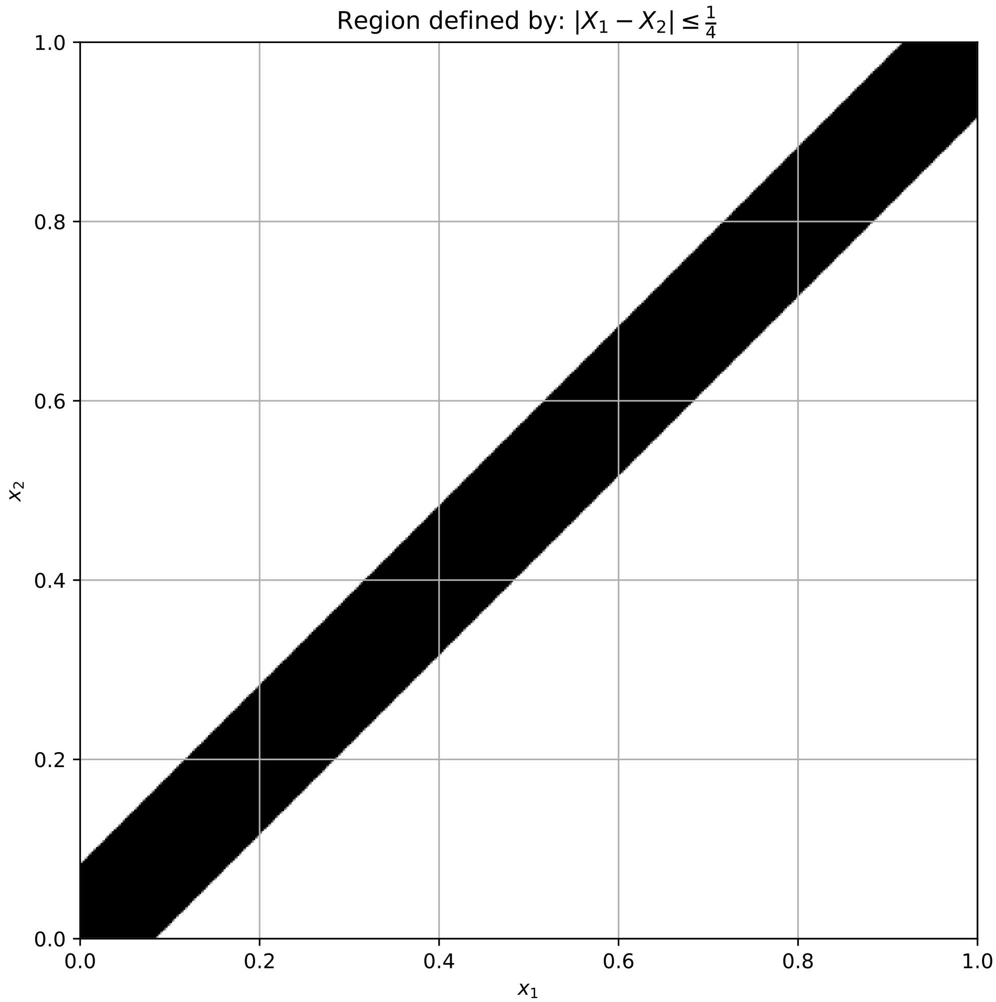
Solution

Let X_1 and X_2 be the arrival times of the two calls. Then X_1 and X_2 are independent and uniformly distributed over the interval $[0, 1]$. The joint density of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 1 & 0 \le x_1 \le 1, 0 \le x_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \mathbb{P}(X_1, X_2 \le 0.5) &= \int_0^{0.5} \int_0^{0.5} 1 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &= \frac{1}{4} \end{aligned}$$

The area of the region where $|X_1 - X_2| \le 1/12$ is



and the $\mathbb{P}(|X_1 - X_2| \leq 1/12) = 1 - \frac{1}{2}(\frac{11}{12})^2 \times 2 = \frac{23}{144}$.

Group 2 **Grade:**

A process for producing an industrial chemical yields a product containing two types of impurities. For a specified sample from this process, let Y_1 denote the proportion of impurities in the sample and let Y_2 denote the proportion of type I impurities among all impurities found. Suppose that the joint distribution of Y_1 and Y_2 can be modeled by the following probability density function:

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 2(1 - y_1) & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of the proportion of type I impurities in the sample.

Solution

$$
\begin{aligned}
\mathbb{E}[Y_1Y_2] &= \int_0^1\int_0^1 y_1y_2\cdot 2(1-y_1)\,\mathrm{d}y_1\,\mathrm{d}y_2\\
&= \int_0^1\int_0^1 2y_1y_2 - 2y_1^2y_2\,\mathrm{d}y_1\,\mathrm{d}y_2\\
&= \int_0^1 y_2 - \frac{2}{3}y_2\,\mathrm{d}y_2\\
&= \frac{1}{2}-\frac{1}{3}\\
&= \frac{1}{6}
\end{aligned}
$$

Group 3 **Grade:**

Let the discrete random variables Y_1 and Y_2 have the joint probability function $P_{Y_1,Y_2}(y_1,y_2) = 1/3$, for $(y_1,y_2) = (-1,0),(0,1),(1,0)$. Find $\mathrm{Cov}(Y_1,Y_2)$. Notice that Y_1 and Y_2 are dependent. (Why?) This is another example of uncorrelated random variables that are not independent.

Solution

$$
\begin{aligned}
P_{Y_1}(y_1) &= \begin{cases} 1/3 & y_1 = -1,0,1\\ 0 & \text{otherwise.}\end{cases}\\
P_{Y_2}(y_2) &= \begin{cases} 2/3 & y_2 = 0\\ 1/3 & y_2 = 1\\ 0 & \text{otherwise.}\end{cases}\\
\mathbb{E}[Y_1] &= -1\cdot\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}\\
&= 0\\
\mathbb{E}[Y_2] &= 0\cdot\frac{2}{3}+1\cdot\frac{1}{3}\\
&= \frac{1}{3}\\
\mathbb{E}[Y_1Y_2] &= -1\cdot 0\cdot\frac{1}{3}+0\cdot 1\cdot\frac{1}{3}+1\cdot 0\cdot\frac{1}{3}\\
&= 0\\
\mathrm{Cov}(Y_1,Y_2) &= \mathbb{E}[Y_1Y_2]-\mathbb{E}[Y_1]\mathbb{E}[Y_2]\\
&= 0-0\cdot\frac{1}{3}\\
&= 0
\end{aligned}
$$

Group 4 **Grade:**

Suppose that Y_1 and Y_2 have correlation coefficient $\rho = .2$. What is the value of the correlation coefficient between

1. $1 + 2Y_1$ and $3 + 4Y_2$?
2. $1 + 2Y_1$ and $3 - 4Y_2$?
3. $1 - 2Y_1$ and $3 - 4Y_2$?

Hint:

Use the variance of derived random variable and the definition of correlation coefficient.

Solution

$$\begin{aligned}
\rho_{1+2Y_1,3+4Y_2} &= \frac{\mathrm{Cov}(1+2Y_1, 3+4Y_2)}{\sqrt{\mathrm{Var}(1+2Y_1)}\sqrt{\mathrm{Var}(3+4Y_2)}} \\
&= \frac{\mathrm{Cov}(2Y_1, 4Y_2)}{\sqrt{\mathrm{Var}(2Y_1)}\sqrt{\mathrm{Var}(4Y_2)}} \\
&= \frac{2 \cdot 4\,\mathrm{Cov}(Y_1, Y_2)}{2 \cdot 4\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= \rho = 0.2 \\
\rho_{1+2Y_1,3-4Y_2} &= \frac{\mathrm{Cov}(1+2Y_1, 3-4Y_2)}{\sqrt{\mathrm{Var}(1+2Y_1)}\sqrt{\mathrm{Var}(3-4Y_2)}} \\
&= \frac{\mathrm{Cov}(2Y_1, -4Y_2)}{\sqrt{\mathrm{Var}(2Y_1)}\sqrt{\mathrm{Var}(-4Y_2)}} \\
&= \frac{2 \cdot (-4)\,\mathrm{Cov}(Y_1, Y_2)}{2 \cdot 4\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= -\frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= -\rho = -0.2 \\
\rho_{1-2Y_1,3-4Y_2} &= \frac{\mathrm{Cov}(1-2Y_1, 3-4Y_2)}{\sqrt{\mathrm{Var}(1-2Y_1)}\sqrt{\mathrm{Var}(3-4Y_2)}} \\
&= \frac{\mathrm{Cov}(-2Y_1, -4Y_2)}{\sqrt{\mathrm{Var}(-2Y_1)}\sqrt{\mathrm{Var}(-4Y_2)}} \\
&= \frac{(-2) \cdot (-4)\,\mathrm{Cov}(Y_1, Y_2)}{2 \cdot 4\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)}\sqrt{\mathrm{Var}(Y_2)}} \\
&= \rho = 0.2
\end{aligned}$$

Group 5 **Grade:**

Let Y_1 and Y_2 have a joint density function given by

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 3y_1 & 0 \leq y_2 \leq y_1 \leq 1, \\ 0 & elsewhere. \end{cases}$$

1. Find the marginal density functions of Y_1 and Y_2.
2. Find $\mathbb{P}(Y_1 \leq 3/4 \mid Y_2 \leq 1/2)$.
3. Find the conditional density function of Y_1 given $Y_2 = y_2$.
4. Find $\mathbb{P}(Y_1 \leq 3/4 \mid Y_2 = 1/2)$.

Solution

$$\begin{aligned}
P_{Y_1}(y_1) &= \int_0^{y_1} 3y_1 \, \mathrm{d}y_2 \\
&= 3y_1^2 \\
P_{Y_2}(y_2) &= \int_{y_2}^{1} 3y_1 \, \mathrm{d}y_1 \\
&= \frac{3}{2}(1 - y_2^2) \\
\mathbb{P}(Y_1 \leq 3/4 \mid Y_2 \leq 1/2) &= \frac{\mathbb{P}(Y_1 \leq 3/4, Y_2 \leq 1/2)}{\mathbb{P}(Y_2 \leq 1/2)} \\
&= \frac{\int_0^{1/2} \int_{y_2}^{3/4} 3y_1 \, \mathrm{d}y_1 \, \mathrm{d}y_2}{\int_0^{1/2} \frac{3}{2}(1 - y_2^2) \, \mathrm{d}y_2} \\
&= \frac{\frac{23}{64}}{\frac{11}{16}} \\
&= \frac{23}{44} \\
f_{Y_1|Y_2}(y_1 \mid y_2) &= \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} \\
&= \frac{3y_1}{\frac{3}{2}(1 - y_2^2)} \\
&= \frac{2y_1}{1 - y_2^2} \\
\mathbb{P}(Y_1 \leq 3/4 \mid Y_2 = 1/2) &= \int_0^{3/4} \frac{2y_1}{1 - (1/2)^2} \, \mathrm{d}y_1 \\
&= \frac{3}{4}
\end{aligned}$$

Group 1 **Grade:**

Suppose that the number of eggs laid by a certain insect has a Poisson distribution with mean λ. The probability that any egg hatches is p. Assume that the eggs hatch independently of one another. Find the

1. expected value of Y, the total number of eggs that hatch.
2. variance of Y.

Hint:

Law of Total Expectation and Law of Total Variance.

Solution

Let N be the number of eggs laid by the insect and Y be the number of eggs that hatch. Given $N = n$, Y has a binomial distribution with n trials and success probability p. Thus, $\mathbb{E}[Y \mid N = n] = np$. Since N follows as Poisson with parameter λ, $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid N]] = \mathbb{E}[Np] = p\lambda$.

$$\begin{aligned}
\mathrm{Var}[Y] &= \mathbb{E}[\mathrm{Var}[Y \mid N]] + \mathrm{Var}[\mathbb{E}[Y \mid N]] \\
&= \mathbb{E}[Np(1-p)] + \mathrm{Var}[Np] \\
&= \mathbb{E}[N]p(1-p) + p^2 \mathrm{Var}[N] \\
&= \lambda p(1-p) + p^2\lambda \\
&= \lambda p
\end{aligned}$$

Group 2 Grade:

Let the random variables X and Y have a joint PDF which is uniform over the triangle with vertices at $(0,0)$, $(0,1)$, and $(1,0)$.

1. Find the joint PDF of X and Y.
2. Find the marginal PDF of Y.
3. Find the conditional PDF of X given Y.
4. Find $\mathbb{E}[X \mid Y = y]$, and use the total expectation theorem to find $\mathbb{E}[X]$.

Solution

$$\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} 2 & 0 \le x \le 1 - y \le 1 \\ 0 & \text{otherwise.} \end{cases} \\
f_Y(y) &= \int_0^{1-y} 2 \,\mathrm{d}x \\
&= 2(1-y) \\
f_{X|Y}(x \mid y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{2}{2(1-y)} \\
&= \frac{1}{1-y} && (\mathsf{Unif}(0, \tfrac{1}{1-y}))
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[X \mid Y = y] &= \frac{1-y}{2} \\
\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid Y]] \\
&= \int_0^1 \frac{1-y}{2} f_Y(y)\,\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 f_Y(y)\,\mathrm{d}y - \frac{1}{2}\int_0^1 y f_Y(y)\,\mathrm{d}y \\
&= \frac{1}{2} - \frac{1}{2}\mathbb{E}[Y]
\end{aligned}
$$

Group 3 **Grade:**

Let X be a random variable with PDF

$$
f_X(x) = \begin{cases} x/4 & 1 \le x \le 3, \\ 0 & \textit{elsewhere}. \end{cases}
$$

let A be the event $X \ge 2$.

1. Find $\mathbb{E}[X]$, $\mathbb{P}(A)$, $f_{X|A}(x)$, and $\mathbb{E}[X \mid A]$.
2. Let $Y = X^2$. Find $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$.

Solution

$$
\begin{aligned}
\mathbb{E}[X] &= \int_1^3 x \frac{x}{4}\,\mathrm{d}x \\
&= \frac{13}{6} \\
\mathbb{P}(A) &= \int_2^3 \frac{x}{4}\,\mathrm{d}x \\
&= \frac{5}{8} \\
f_{X|A}(x) &= \frac{f_X(x)}{\mathbb{P}(A)} \\
&= \frac{2}{5}x \\
\mathbb{E}[X \mid A] &= \int_2^3 x \frac{2}{5} x\,\mathrm{d}x \\
&= \frac{38}{15} \\
\mathbb{E}[Y] &= \mathbb{E}\left[X^2\right] \\
&= \int_1^3 x^2 \frac{x}{4}\,\mathrm{d}x \\
&= 5 \\
\mathrm{Var}[Y] &= \mathbb{E}\left[Y^2\right] - \mathbb{E}[Y]^2
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\big[X^4\big] - \mathbb{E}\big[X^2\big]^2 \\
&= \int_1^3 x^4 \frac{x}{4} \,\mathrm{d}x - \mathbb{E}\big[X^2\big]^2 \\
&= \frac{91}{3} - 25 \\
&= \frac{16}{3}
\end{aligned}
$$

Group 4 **Grade:**

The random variable X has the PDF

$$
f_X(x) = \begin{cases} cx^{-2} & 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}
$$

1. Determine the value of c.
2. Let A be the event $X > 1.5$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

Solution

$$
\begin{aligned}
\int_1^2 cx^{-2} \,\mathrm{d}x &= 1 \\
c &= 2 \\
\mathbb{P}(A) &= \int_{1.5}^2 \frac{2}{x^2} \,\mathrm{d}x \\
&= \frac{1}{3} \\
f_{X|A}(x) &= \frac{f_X(x)}{\mathbb{P}(A)} \\
&= 6x^{-2}
\end{aligned}
$$

Group 5 **Grade:**

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?

Hint:

Law of Total Expectation.

Solution

Let X be the time until the miner reaches safety. Let D be the door that the miner chooses. Then

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid D]] \\
&= \mathbb{E}[X \mid D = 1]\mathbb{P}(D = 1) + \mathbb{E}[X \mid D = 2]\mathbb{P}(D = 2) + \mathbb{E}[X \mid D = 3]\mathbb{P}(D = 3) \\
&= 3 \cdot \frac{1}{3} + [5 + \mathbb{E}[X]] \cdot \frac{1}{3} + [7 + \mathbb{E}[X]] \cdot \frac{1}{3} \\
\Rightarrow 3\mathbb{E}[X] &= 15 + 2\mathbb{E}[X] \\
\Rightarrow \mathbb{E}[X] &= 15
\end{aligned}$$