HW 10 Solution

Group 1 **Grade:**

Suppose your grade in a probability course depends on your exam scores X_1 and X_2. The professor, a fan of probability, releases exam scores in a normalized fashion such that X_1 and X_2 are independent Gaussian ($\mu = 0, \sigma = \sqrt{2}$) random variables. Your semester average is $X = 0.5(X_1 + X_2)$.

1. You earn an A grade if $X > 1$. What is $\mathbb{P}(A)$?
2. To improve his SIRS (Student Instructional Rating Service) score, the professor decides he should award more A's. Now you get an A if $\max(X_1, X_2) > 1$. What is $\mathbb{P}(A)$ now?
3. The professor found out he is unpopular at ratemyprofessor.com and decides to award an A if either $X > 1$ or $\max(X_1, X_2) > 1$. Now what is $\mathbb{P}(A)$?
4. Under criticism of grade inflation from the department chair, the professor adopts a new policy. An A is awarded if $\max(X_1, X_2) > 1$ and $\min(X_1, X_2) > 0$. Now what is $\mathbb{P}(A)$?

Hint:

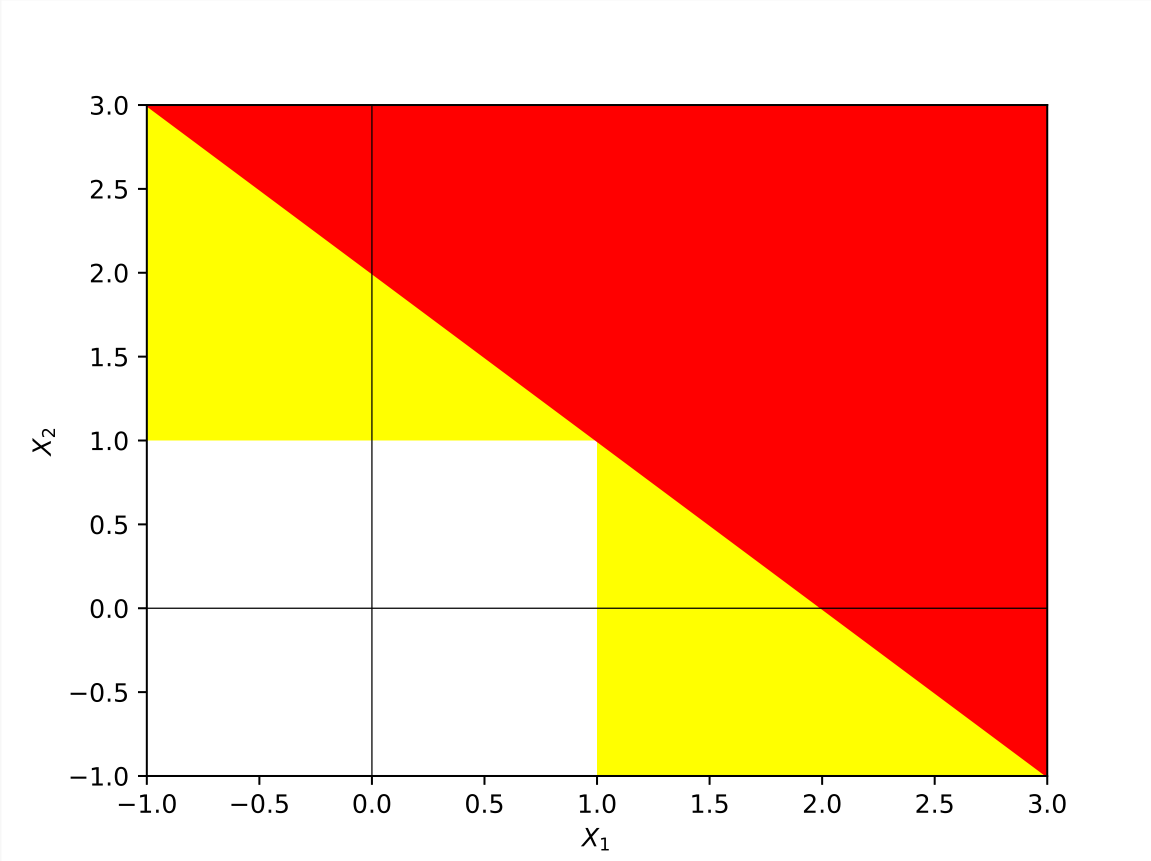
1. Sum of independent Gaussian and Φ function.
2. $\mathbb{P}(\max(X_1, X_2) > 1) = 1 - \mathbb{P}(\max(X_1, X_2) \leq 1)$.
3. Draw the Venn Diagram to show the relationship between $X > 1$ and $\max(X_1, X_2) > 1$.
4. Draw the Venn Diagram to show the relationship between $\max(X_1, X_2) > 1$ and $\min(X_1, X_2) > 0$.

Solution

It's easy to show that $X \sim \mathsf{N}(0, 1)$ which is a standard normal random variable.

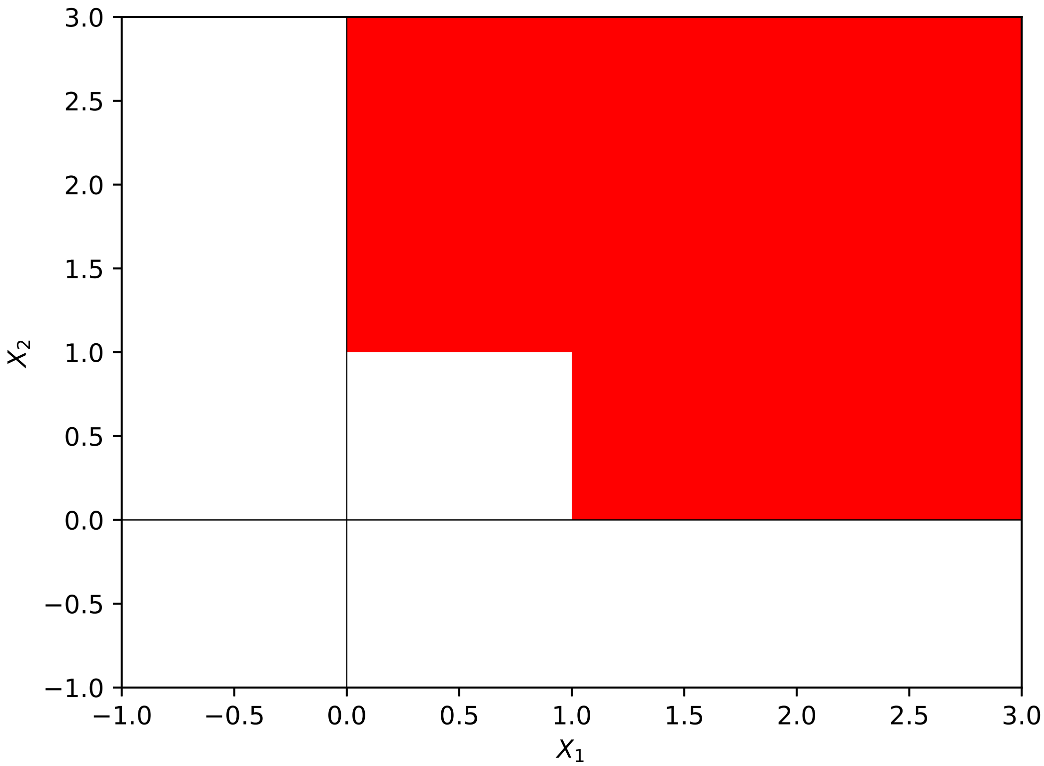
$$\begin{aligned}
\mathbb{P}(X > 1) &= 1 - \mathbb{P}(X \leq 1) = 1 - \Phi(1) \\
\mathbb{P}(\max(X_1, X_2) > 1) &= 1 - \mathbb{P}(\max(X_1, X_2) \leq 1) \\
&= 1 - \mathbb{P}(X_1 \leq 1, X_2 \leq 1) \\
&= 1 - \mathbb{P}(X_1 \leq 1)\mathbb{P}(X_2 \leq 1) \\
&= 1 - \Phi(1/\sqrt{2})^2
\end{aligned}$$

$\{\max(X_1, X_2) > 1 \cup X > 1\} = \{\max(X_1, X_2) > 1\}$ can be visualized by the following plot.



$$\mathbb{P}(\max(X_1, X_2) > 1 \cup X > 1) = \mathbb{P}(\max(X_1, X_2) > 1) = 1 - \Phi(1/\sqrt{2})^2$$

$\{\max(X_1, X_2) > 1 \cup \min(X_1, X_2) > 0\}$ can be visualized by the following plot.



$$
\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(\max(X_1, X_2) > 1 \cup \min(X_1, X_2) > 0) \\
&= \mathbb{P}(X_1 > 0, X_2 > 0) - \mathbb{P}(0 < X_1 < 1, 0 < X_2 < 1) \\
&= \mathbb{P}(X_1 > 0)\mathbb{P}(X_2 > 0) - \mathbb{P}(0 < X_1 < 1)\mathbb{P}(0 < X_2 < 1) \\
&= (1 - \mathbb{P}(X_1 \leq 0))(1 - \mathbb{P}(X_2 \leq 0)) - (\mathbb{P}(X_1 \leq 1) - \mathbb{P}(X_1 \leq 0))(\mathbb{P}(X_2 \leq 1) - \mathbb{P}(X_2 \leq 0)) \\
&= (1 - \Phi(0))^2 - (\Phi(1/\sqrt{2}) - \Phi(0))^2
\end{aligned}
$$

Group 2 **Grade:**

Let X and Y be continues random variables with joint PDF

$$
f(x, y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}
$$

1. What are $\mathbb{E}[X]$ and $\mathrm{Var}(X)$?
2. What are $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$?
3. What are $\mathrm{Cov}(X, Y)$ and $\mathrm{Corr}(X, Y)$?
4. What are $\mathbb{E}[X + Y]$ and $\mathrm{Var}(X + Y)$?

Solution

$$
\begin{aligned}
\mathbb{E}[X] &= \int_0^1 \int_y^1 x \cdot 2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^1 1 - y^2 \, \mathrm{d}y \\
&= \frac{2}{3} \\
\mathbb{E}\left[X^2\right] &= \int_0^1 \int_y^1 x^2 \cdot 2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^1 \frac{2}{3}(1 - y^3) \, \mathrm{d}y \\
&= \frac{1}{2} \\
\operatorname{Var}(X) &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\
&= \frac{1}{2} - \frac{4}{9} \\
&= \frac{1}{18} \\
\mathbb{E}[Y] &= \int_0^1 \int_0^x y \cdot 2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 x^2 \, \mathrm{d}x \\
&= \frac{1}{3} \\
\mathbb{E}\left[Y^2\right] &= \int_0^1 \int_0^x y^2 \cdot 2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \frac{2}{3} x^3 \, \mathrm{d}x \\
&= \frac{1}{6} \\
\operatorname{Var}(Y) &= \mathbb{E}\left[Y^2\right] - \mathbb{E}[Y]^2 \\
&= \frac{1}{6} - \frac{1}{9} \\
&= \frac{1}{18} \\
\mathbb{E}[XY] &= \int_0^1 \int_0^x xy \cdot 2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 x^3 \, \mathrm{d}x \\
&= \frac{1}{4} \\
\operatorname{Cov}(X, Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \frac{1}{4} - \frac{2}{9} \\
&= \frac{1}{36}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Corr}(X,Y) &= \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} \\
&= \frac{1}{2} \\
\mathbb{E}[X+Y] &= \mathbb{E}[X] + \mathbb{E}[Y] \\
&= 1 \\
\operatorname{Var}(X+Y) &= \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X,Y) \\
&= \frac{1}{18} + \frac{1}{18} + 2\frac{1}{36} \\
&= \frac{1}{6}
\end{aligned}
$$

Group 3 **Grade:**

A company receives shipments from two factories. Depending on the size of the order, a shipment can be in

1. 1 box for a small order,
2. 2 boxes for a medium order,
3. 3 boxes for a large order.

The company has two different suppliers. Factory Q is 60 miles from the company. Factory R is 180 miles from the company. An experiment consists of monitoring a shipment and observing B, the number of boxes, and M, the number of miles the shipment travels. The following probability model describes the experiment:

	Factory Q	Factory R
small order	0.3	0.2
medium order	0.1	0.2
large order	0.1	0.1

1. Find $P_{B,M}(b,m)$, the joint PMF of the number of boxes and the distance.
2. What is $\mathbb{E}[B]$, the expected number of boxes?
3. Are B and M independent?

Solution

$$
P_{B,M}(b,m) = \begin{cases} 0.3 & b=1, m=60 \\ 0.2 & b=1, m=180 \\ 0.1 & b=2, m=60 \\ 0.2 & b=2, m=180 \\ 0.1 & b=3, m=60 \\ 0.1 & b=3, m=180 \\ 0 & \text{otherwise} \end{cases}
$$

$$\mathbb{E}[B] = 1\cdot 0.3 + 1\cdot 0.2 + 2\cdot 0.1 + 2\cdot 0.2 + 3\cdot 0.1 + 3\cdot 0.1 = 1.7$$

No, B and M are not independent. Because $P_{B,M}(1,60) = 0.3 \neq 0.25 = 0.5\cdot 0.5 = P_B(1)P_M(60)$.

Group 4 **Grade:**

Random Variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} \frac{(1-p)^{n-1}p}{n} & k = 1,2,\dots,n; n = 1,2,\dots \\ 0 & \text{otherwise.} \end{cases}$$

Find $P_N(n)$, $\mathbb{E}[N]$, $\text{Var}(N)$, $\mathbb{E}\left[N^2\right]$, $\mathbb{E}[K]$, $\text{Var}(K)$, $\mathbb{E}[N+K]$, $r_{N,K}$, $\text{Cov}(N,K)$.

Hint:

Please solve the first four and leave the rest for me.

Solution

$$P_N(n) = \begin{cases} \sum_{k=1}^{n} \frac{(1-p)^{n-1}p}{n} = n \cdot \frac{(1-p)^{n-1}p}{n} = (1-p)^{n-1}p & n = 1,2,\dots \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[N] = \frac{1}{p} \qquad \text{Var}[N] = \frac{1-p}{p^2} \qquad \mathbb{E}\left[N^2\right] = \text{Var}[N] + \mathbb{E}[N]^2 = \frac{2-p}{p^2}$$

$$\mathbb{E}[K] = \sum_{n=1}^{\infty}\sum_{k=1}^{n} k \cdot \frac{(1-p)^{n-1}p}{n} = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}p}{n} \sum_{k=1}^{n} k = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}p}{n} \frac{n(n+1)}{2}$$

Since $\sum_{n=1}^{\infty}(1-p)^{n-1}p = 1$

$$\mathbb{E}[K] = \mathbb{E}\left[\frac{N+1}{2}\right] = \frac{\mathbb{E}[N]+1}{2} = \frac{1}{2p} + \frac{1}{2}$$

Following the same logic,

$$\mathbb{E}\left[K^2\right] = \mathbb{E}\left[\frac{(N+1)(2N+1)}{6}\right] = \frac{2\mathbb{E}\left[N^2\right] + 3\mathbb{E}[N] + 1}{6} = \frac{2-p}{3p^2} + \frac{1}{2p} + \frac{1}{6}$$

$$\text{Var}[K] = \mathbb{E}\left[K^2\right] - \mathbb{E}[K]^2 = \frac{2-p}{3p^2} + \frac{1}{2p} + \frac{1}{6} - \left(\frac{1}{2p} + \frac{1}{2}\right)^2$$

$$\begin{aligned} r_{N,K} &= \mathbb{E}[NK] \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} nk \cdot \frac{(1-p)^{n-1}p}{n} \\ &= \sum_{n=1}^{\infty}(1-p)^{n-1}p \sum_{k=1}^{n} k \\ &= \sum_{n=1}^{\infty}(1-p)^{n-1}p \frac{n(n+1)}{2} \\ &= \mathbb{E}\left[\frac{N(N+1)}{2}\right] \\ &= \frac{1}{2}\mathbb{E}\left[N^2\right] + \frac{1}{2}\mathbb{E}[N] \\ &= \frac{2-p}{2p^2} + \frac{1}{2p} = \frac{1}{p^2} \end{aligned}$$

$$\text{Cov}(N,K) = r_{N,K} - \mathbb{E}[N]\mathbb{E}[K] = \frac{1}{p^2} - \frac{1}{p}\frac{1}{2p} - \frac{1}{2p} = \frac{1}{2p^2} - \frac{1}{2p}$$

Group 5 **Grade:**

X and Y are random variables such that X has expected value $\mu_X = 0$ and standard deviation $\sigma_X = 3$ while Y has expected value $\mu_Y = 1$ and standard deviation $\sigma_Y = 4$. In addition, X and Y have covariance $\mathrm{Cov}[X, Y] = -3$. Find the expected value and variance of $W = 2X + 2Y$.

Solution

$$\mathbb{E}[W] = \mathbb{E}[2X + 2Y] = 2\mathbb{E}[X] + 2\mathbb{E}[Y] = 2 \cdot 0 + 2 \cdot 1 = 2$$

$$\mathrm{Var}(W) = \mathrm{Var}(2X + 2Y) = 4\,\mathrm{Var}(X) + 4\,\mathrm{Var}(Y) + 8\,\mathrm{Cov}(X, Y) = 4 \cdot 3^2 + 4 \cdot 4^2 + 8 \cdot (-3) = 76$$