

# Chapter 1

## Set Theory, Probability, and Single Experiment

### 1. From Set to Probability (of the single experiment)

(a)

Set Theory	Probability
Element	Outcome
Subset	Event
Universal Set	Sample Space

(b) Outcome and Event:

- Outcomes are always **Mutually Exclusive** since there are the smallest units (i.e., Elements) in the Set.
- Event constitutes by different combinations of outcomes (through Union ( $\cup$ ) Operation).

(c)  $\mathbb{P}(\text{Event})$  is the possibility that the event appears in the sample space.

(d)  $\mathbb{P}(\emptyset) = 0$  since there is no element in *null set*, and  $\mathbb{P}(\text{Sample Space}) = 1$ .

### 2. From Set Operation to Probability Operation

(a) There are three Set Operations:  $A \cup B$ ,  $A \cap B$ ,  $A^c$ .

(b)  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ .

(c) **Union Bound:**  $\mathbb{P}(\cup_{i=1}^N A_i) \leq \sum_{i=1}^N \mathbb{P}(A_i)$ .

(d) **Mutually Exclusive:**  $\mathbb{P}(A \cap B) = 0$  so that  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ .

(e) **Collectively Exhaustive:**  $\mathbb{P}(A \cup B) = 1$ .

(f) **Partitions (i.e., Mutually Exclusive & Collectively Exhaustive):**  $\mathbb{P}(\cup_{i=1}^N A_i) = \sum_{i=1}^N \mathbb{P}(A_i) = 1$ .

### 3. Conditional Probability

(a)  $\mathbb{P}(A | B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$ .

(b) **If  $A_i$  are Mutually Exclusive:**  $\mathbb{P}(A | B) = \mathbb{P}(\cup_{i=1}^N A_i | B) = \sum_{i=1}^N \mathbb{P}(A_i | B)$ .

(c) **If  $B_i$  are Partitions** (Law of Total Number),

$$\begin{aligned}
 \mathbb{P}(A | B) &= \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} && \text{(Definition of Conditional Probability)} \\
 &= \mathbb{P}(AB) && (B \text{ is Collectively Exhaustive so } \mathbb{P}(B) = 1) \\
 &= \mathbb{P}(A \cdot \cup_{i=1}^N B_i) && (B \text{ is Mutually Exclusive}) \\
 &= \sum_{i=1}^N \mathbb{P}(AB_i) && (B \text{ is Partition}) \\
 &= \sum_{i=1}^N \mathbb{P}(A | B_i) \mathbb{P}(B_i). && \text{(Definition of Conditional Probability)}
 \end{aligned}$$

4. Bayes' Theorem:  $\mathbb{P}(A | B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}.$

5. Independent:

(a)  $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$

(b)  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B).$

(c)  $\mathbb{P}(A | B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$

## Chapter 2

# Sequential Experiments

1. Tree Diagrams
2. Counting Methods (**Essentially the outcomes in each experiment (i.e., sample space) are equiprobable**)

(a) Multiplication:  $n \times k_1 \times k_2 \times \dots$

(b) Sampling without Replacement

i. Permutation:  $\frac{n!}{(n-k)!}$ .

ii. Combination:  $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$ .

iii. Combination is Permutation without order. Combination is also called n choose k.

(c) Sampling with Replacement:  $n^k$

(d) Multiple Combination:

i.  $\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \dots k_m!}$  where  $n = \sum_{i=1}^m k_i$ .

ii. For the two cases situation,  $n = k_1 + k_2 \Rightarrow \binom{n}{k_1 k_2} = \frac{n!}{k_1! k_2!} \iff \binom{n}{k_1} \iff \binom{n}{k_2}$ .

3. Independent Trails (**Essentially the outcomes in each sample space are not necessarily equiprobable**)

(a) *Theorem 2.8:* The Probability of  $k_0$  failures and  $k_1$  successes in  $n = k_0 + k_1$  Independent Trails with success rate  $p$  is

$$\mathbb{P}(k_0, k_1) = \binom{n}{k_0} (1-p)^{k_0} p^{k_1} = \binom{n}{k_1} (1-p)^{k_0} p^{k_1}.$$

(b) *Theorem 2.9:*  $n = k_1 + k_2 + \dots + k_m$  and success rates are  $p_1, p_2, \dots, p_m$ , where  $\sum_{i=1}^m p_i = 1$  has

$$\mathbb{P}(k_1, k_2, \dots, k_m) = \binom{n}{k_1, k_2, \dots, k_m} p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}.$$

## Chapter 3

# Discrete Random Variables

1. Discrete Random Variables: Assign numerical value to discrete outcomes
2. Probability Mass Function (PMF):

$$\sum_{x \in X} P_X(x) = 1.$$

3. Families of Discrete Random Variables and their PMF

- (a) Bernoulli ( $p$ ): **E.g., Flip a coin**

$$P_X(x) = \begin{cases} 1-p & x=0, \\ p & x=1, \\ 0 & \text{otherwise.} \end{cases}$$

- (b) Binomial ( $n, p$ ): Get  $\mathbf{x}$  successes in  $\mathbf{n}$  Bernoulli ( $p$ ) experiments  $\iff$  independent trials

$$P_X(\mathbf{x}) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=0, 1, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

**Note:** Bernoulli ( $p$ )  $\iff$  Binomial ( $1, p$ ).

- (c) Poisson ( $\alpha$ ): Binomial ( $n, p$ ) with small  $p$ , large  $n$ , and  $\alpha = np$

$$P_X(x) = \begin{cases} \frac{\alpha^x e^{-\alpha}}{x!} & x=0, 1, \dots \\ 0 & \text{otherwise.} \end{cases}$$

- (d) Geometric ( $p$ ): Get the **1st** success at the  $\mathbf{x}$ -th Bernoulli ( $p$ ) experiment

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x=1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

- (e) Pascal ( $k, p$ ): Get the  $\mathbf{k}$ -th success at the  $\mathbf{x}$ -th Bernoulli ( $p$ ) experiment

$$P_X(\mathbf{x}) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & x=k, k+1, k+2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

**Note:** Geometric ( $p$ )  $\iff$  Pascal ( $1, p$ ).

- (f) Discrete Uniform ( $k, l$ ): outcomes are uniformly distributed on range ( $k, l$ ) **E.g., Roll a Die**

$$P_X(x) = \begin{cases} 1/(l-k+1) & x=k, k+1, k+2, \dots, l \\ 0 & \text{otherwise.} \end{cases}$$

## 4. Cumulative Distribution Function (CDF):

$$F_X(x) = P_X[X \leq x] = \sum_{k=0}^x P_X(k).$$

$$F_X(b) - F_X(a) = \sum_{k=0}^b P_X(k) - \sum_{k=0}^a P_X(k) = \sum_{k=a+1}^b P_X(k) = P_X(a < X \leq b).$$

The CDF of Geometric ( $p$ ) is worth to remember

$$\begin{aligned} F_X(x) &= P_X[X \leq x] \\ &= 1 - P_X[X > x] \\ &= 1 - \sum_{i=x+1}^{\infty} p(1-p)^{i-1} \\ &= 1 - (1-p)^x \sum_{i=1}^{\infty} p(1-p)^{i-1} \\ &= 1 - (1-p)^x. \end{aligned}$$

## 5. Average and Expectations

(a) In ordinary language, an **Average** is a single number taken as representative of a list of numbers.

i. Mode: The outcome appears the most often in the sample space

$$P_X(x_{\text{mode}}) \geq P_X(x).$$

ii. Median: The outcome which separates the higher half from the lower half of a sample space

$$P_X[X \leq x_{\text{med}}] \geq 1/2, \quad P_X[X \geq x_{\text{med}}] \geq 1/2.$$

iii. (Arithmetic) mean: The sum of all the outcomes divided by the number of outcomes

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i.$$

(b) Expectation: Weighted (Arithmetic) mean

i. Definition:

$$\mu_x = \mathbb{E}[X] = \sum_{x \in S_X} x P_X(x). \quad (\text{First Moment of } X)$$

$$\mathbb{E}[X^2] = \sum_{x \in S_X} x^2 P_X(x). \quad (\text{Second Moment of } X)$$

ii. Important Expectations

A. Bernoulli ( $p$ ):

$$\mathbb{E}[X] = 0 \cdot P_X(0) + 1 \cdot P_X(1) = 0(1-p) + 1(p) = p.$$

B. Binomial ( $n, p$ ):

$$\mathbb{E}[X] = np.$$

C. Poisson ( $\alpha$ ):

$$\mathbb{E}[X] = \alpha.$$

D. Geometric ( $p$ ):

$$\mathbb{E}[X] = 1/p.$$

E. Pascal ( $k, p$ ):

$$\mathbb{E}[X] = k/p.$$

F. Discrete Uniform (k, l):

$$\mathbb{E}[X] = (k + l)/2.$$

- (c) From an engineering perspective, **Mean (including Expectations, etc.)** is numerically easier to calculate, either using human brain or computers, than Mode and Median, when the sample space is humongous.
- (d) In most cases, average, mean and expectation refer to the same concept.

6. Derived Random Variable:  $Y = g(X)$

- (a)  $P_Y(y) = P[Y = y] = P[Y = g(x)] = P[g^{-1}(Y) = g^{-1}(g(x))] = P[X = x] = P_X(x)$
- (b)  $\mathbb{E}[Y] = \sum y P_Y(y) = \sum g(x) P_X(x)$
- (c)  $\mathbb{E}[X - \mu_x] = \sum_{x \in S_X} (x - \mu_x) P_X(x) = \sum_{x \in S_X} x P_X(x) - \mu_x \sum_{x \in S_X} P_X(x) = \mathbb{E}[X] - \mu_x \cdot 1 = \mu_x - \mu_x = 0$
- (d)  $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b \Rightarrow \mathbb{E}[b] = \mathbb{E}[0 \cdot X + b] = b$

7. Variance ( $\sigma_x^2$ ) and Standard Deviation ( $\sigma_x$ )

(a)

$$\begin{aligned} \sigma_x^2 &= \text{Var}(X) \\ &= \mathbb{E}[(X - \mu_x)^2] \\ &= \mathbb{E}[X^2 - 2\mu_x X + \mu_x^2] \\ &= \mathbb{E}[X^2] - 2\mu_x \mathbb{E}[X] + \mathbb{E}[\mu_x^2] \\ &= \mathbb{E}[X^2] - 2\mu_x^2 + \mu_x^2 \\ &= \mathbb{E}[X^2] - \mu_x^2 \end{aligned}$$

(b)  $\text{Var}(X) \geq 0$

(c)  $\text{Var}(aX + b) = a^2 \text{Var}(X)$

(d) Important Variance:

i. Bernoulli (p):

$$\text{Var}(X) = p(1 - p).$$

ii. Binomial (n, p):

$$\text{Var}(X) = np(1 - p).$$

iii. Poisson ( $\alpha$ ):

$$\text{Var}(X) = \alpha.$$

iv. Geometric (p):

$$\text{Var}(X) = (1 - p)/p^2.$$

v. Pascal (k, p):

$$\text{Var}(X) = k(1 - p)/p^2.$$

vi. Discrete Uniform (k, l):

$$\text{Var}(X) = (l - k)(l - k + 2)/12.$$

# Chapter 4

## Continuous Random Variables

### 4.1 Continuous sample space

**Axiom.** A random variable  $X$  is continuous if the range  $S_X$  consists of one or more intervals. For  $x \in S_X$ ,  $\mathbb{P}(X = x) = 0$ .

### 4.2 The Cumulative Distribution Function

**Definition 4.1** (Cumulative Distribution Function (CDF)). The CDF of random variable  $X$  is

$$F_X(x) = \mathbb{P}(X \leq x).$$

**Theorem 4.2.** For any random variable  $X$ ,

1.  $F_X(-\infty) = 0$
2.  $F_X(\infty) = 1$
3.  $\mathbb{P}(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

### 4.3 Probability Density Function

Start with a continuous random variable  $X$  with CDF  $F_X(x)$ . The probability of “ $X$  with volume  $\Delta$ ” is defined as:

$$\begin{aligned} \mathbb{P}(x < X \leq x + \Delta) &= F_X(x + \Delta) - F_X(x) \\ &= \frac{F_X(x + \Delta) - F_X(x)}{(x + \Delta) - x} \cdot \Delta. \end{aligned}$$

**Definition 4.3** (Probability Density Function (PDF)).

$$\begin{aligned} f_X(x) &= \lim_{\Delta \rightarrow 0} \frac{F_X(x + \Delta) - F_X(x)}{\Delta} \\ &= \frac{d F_X(x)}{d x}. \end{aligned}$$

**Theorem 4.4.** For a continuous random variable  $X$  with PDF  $f_X(x)$ ,

1.  $f_X(x) \geq 0$  for all  $x$
2.  $F_X(x) = \int_{-\infty}^x f_X(u) du$
3.  $\int_{-\infty}^{\infty} f_X(x) dx = 1$

**Theorem 4.5.**

$$\mathbb{P}(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x) dx.$$

## 4.4 Expected Value

**Definition 4.6** (Expected value).

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx.$$

**Theorem 4.7** (Derived Random Variable).

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx.$$

**Theorem 4.8.** For any random variable  $X$ ,

1.  $\mathbb{E}[X - \mu_x] = 0$ ,
2.  $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ ,
3.  $\text{Var}[X] = \mathbb{E}[X^2] - \mu_x^2$ ,
4.  $\text{Var}[aX + b] = a^2 \text{Var}[X]$ .

## 4.5 Families of Continuous Random Variables

1. Continuous Uniform  $\text{Unif}(k, l)$ : A continuous counterpart of Discrete Uniform

$$f_X(x) = \begin{cases} \frac{1}{l-k} & k \leq x \leq l \\ 0 & \text{otherwise.} \end{cases}$$

$$F_X(x) = \frac{x-k}{l-k}, \quad x \in (k, l)$$

$$\mathbb{E}[X] = (l+k)/2.$$

$$\text{Var}[X] = (l-k)^2/12.$$

2. Exponential  $\text{Exp}(\lambda)$ : A continuous counterpart of  $\text{Geom}(1 - e^{-\lambda})$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$F_X(x) = 1 - e^{-\lambda x}.$$

$$\mathbb{E}[X] = 1/\lambda.$$

$$\text{Var}[X] = 1/\lambda^2.$$

3. Erlang  $\text{Erlang}(n, \lambda)$ : A continuous counterpart of  $\text{Pascal}(n, 1 - e^{-\lambda})$

$$f_X(x) = \begin{cases} \frac{\lambda(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$F_X(x) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!} = \mathbb{P}_{\text{poisson}}(k \geq n).$$

$$\mathbb{E}[X] = n/\lambda.$$

$$\text{Var}[X] = n/\lambda^2.$$



## 4.6 Gaussian Random Variables

**Theorem 4.9** (Gaussian Integral).

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

**Definition 4.10** (Gaussian Random Variable).  $X$  is a Gaussian( $\mu, \sigma$ ) random variable if the PDF of  $X$  is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

$X$  is also called Normal( $\mu, \sigma$ ) random variable. We will use  $N(\mu, \sigma)$  in the following content.

**Theorem 4.11** (The Expectation and Variance of  $X \sim N(\mu, \sigma)$ ).

$$\mathbb{E}[X] = \mu, \quad \text{Var}(X) = \sigma^2.$$

**Theorem 4.12.** If  $X$  is  $N(\mu, \sigma)$ ,  $Y = aX + b$  is  $N(a\mu + b, a\sigma)$ .

**Theorem 4.13** (Standard Normal Random Variable). The  $N(\mu, \sigma)$  with  $\mu = 0, \sigma = 1$  is called standard normal random variable  $Z \sim N(0, 1)$ . The PDF is,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

And the CDF is

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du.$$

**Theorem 4.14.** If  $X$  is  $N(\mu, \sigma)$ , the CDF of  $X$  is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that  $X$  is in the interval  $(a, b)$  is

$$\mathbb{P}(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

**Theorem 4.15.**  $\Phi(-z) = 1 - \Phi(z)$ .

## 4.7 Delta Function, Mixed (Being Discrete and Continuous at the same time) Random Variable

**Definition 4.16** (Unit Impulse (Delta) Function). Let

$$d_\epsilon(x) = \begin{cases} 1/\epsilon & -\epsilon/2 \leq x \leq \epsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$

The **unit impulse function** is

$$\delta(x) = \lim_{\epsilon \rightarrow 0} d_\epsilon(x).$$

Since

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$

The  $\delta(x)$  is indeed a PDF given it is also non-negative.

**Theorem 4.17.** For any continuous function  $g(x)$ ,

$$\int_{-\infty}^{\infty} g(x) \delta(x - x_0) dx = g(x_0).$$

**Definition 4.18** (Unit Step Function). The **unit step function** is

$$u(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

**Theorem 4.19** (CDF of  $\delta(x)$  and connection to the unit step function).

$$\int_{-\infty}^x \delta(v) \, dv = u(x).$$

And thus

$$\delta(x) = \frac{du(x)}{dx}.$$

**Corollary 4.20.** The theorem 4.19 allows us to define a generalized PDF that applies to discrete random variables as well as to continuous random variables. Consider the CDF of a discrete random variable,  $X$ . It is constant (let's say 0 for now) everywhere except at point  $x_i \in S_X$ , where it has jumps of height  $P_X(x_i)$ . Using the **unit step function**, the CDF of  $X$  is

$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i) u(x - x_i).$$

And the PDF can be defined with  $\delta(x)$  as

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i) \delta(x - x_i).$$

Then the Expectation will be

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x \sum_{x_i \in S_X} P_X(x_i) \delta(x - x_i) \, dx \\ \mathbb{E}[X] &= \sum_{x_i \in S_X} \int_{-\infty}^{\infty} x P_X(x_i) \delta(x - x_i) \, dx \\ &= \sum_{x_i \in S_X} x_i P_X(x_i) \end{aligned}$$

**Theorem 4.21.** For a random variable  $X$  (not specified whether it is discrete or continuous), we have

$$\begin{aligned} q &= \mathbb{P}(X = x_0) && \text{(General expression)} \\ &= P_X(x_0) && \text{(PMF)} \\ &= F_X(x_0^+) - F_X(x_0^-) && \text{(CDF)} \\ &= f_X(x_0) = q\delta(0). && \text{(PDF \& delta function)} \end{aligned}$$

**Theorem 4.22.**  $X$  is a **mixed** random variable if and only if  $f_X(x)$  contains both impulses and nonzero, finite values.

## Chapter 5

# Multiple Random Variables

### 5.1 Joint CDF

**Definition 5.1** (Joint CDF). The joint CDF of random variables  $X$  and  $Y$  is

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y).$$

The joint CDF is a **complete** probability model for any pair of random variables  $X$  and  $Y$ .

**Theorem 5.2.** For any pair of random variables,  $X$  and  $Y$ , the following properties hold:

- (a)  $0 \leq F_{X,Y}(x, y) \leq 1$ ,
- (b)  $F_{X,Y}(\infty, \infty) = 1$ ,
- (c)  $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$ ,
- (d)  $F_X(x) = F_{X,Y}(x, \infty)$  and  $F_Y(y) = F_{X,Y}(\infty, y)$ ,
- (e)  $F_{X,Y}(x, y)$  is non-decreasing in  $x$  and  $y$ .

### 5.2 Joint PMF

**Definition 5.3** (Joint PMF). The joint PMF of random variables  $X$  and  $Y$  is

$$P_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y).$$

The joint PMF is a **complete** probability model for any pair of discrete random variables  $X$  and  $Y$ .

**Theorem 5.4.** For discrete random variables  $X$  and  $Y$  and any set  $B$  in the  $X, Y$  plane, the probability of the event is

$$\mathbb{P}([B]) = \sum_{(x,y) \in B} P_{X,Y}(x, y).$$

Apparently, the joint PMF is non-negative and sums to one.

$$\sum_{x \in S_X} \sum_{y \in S_Y} P_{X,Y}(x, y) = 1.$$

### 5.3 Marginal PMF

**Theorem 5.5.** For discrete random variables  $X$  and  $Y$  with joint PMF  $P_{X,Y}(x, y)$ ,

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x, y), \quad P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x, y).$$

For discrete random variables, the marginal PMF  $P_X(x)$  and  $P_Y(y)$  are probability models for the individual random variables  $X$  and  $Y$ , but they only provide an **incomplete** probability model for the pair of random variables  $X$  and  $Y$ .

## 5.4 Joint PDF

**Definition 5.6** (Joint PDF). The joint CDF of continuous random variables  $X$  and  $Y$  is a function  $f_{X,Y}(x, y)$  with the property

$$F_{X,Y}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(u, v) \, du \, dv.$$

Apparently, we can then derive the joint PDF as follows,

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y).$$

The joint PDF is a **complete** probability model for any pair of continuous random variables  $X$  and  $Y$ .

**Theorem 5.7.** *The probability that the continuous random variables  $(X, Y)$  are in  $A$*

$$\mathbb{P}([A]) = \iint_A f_{X,Y}(x, y) \, dx \, dy.$$

The joint PDF is non-negative and integrates to one.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = 1.$$

## 5.5 Marginal PDF

**Theorem 5.8.** *For continuous random variables  $X$  and  $Y$  with joint PDF  $f_{X,Y}(x, y)$ ,*

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx.$$

For continuous random variables, the marginal PDFs  $f_X(x)$  and  $f_Y(y)$  are probability models for the individual random variables  $X$  and  $Y$ , but they only provide an **incomplete** probability model for the pair of random variables  $X$  and  $Y$ .

## 5.6 Independent Random Variables

**Definition 5.9** (Independent Random Variables). Random variables  $X$  and  $Y$  are independent if and only if

$$\begin{aligned} P_{X,Y}(x, y) &= P_X(x)P_Y(y); & \text{(Discrete)} \\ f_{X,Y}(x, y) &= f_X(x)f_Y(y). & \text{(Continuous)} \end{aligned}$$

It's easy to show that if  $X$  and  $Y$  are independent, then

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y) = F_X(x)F_Y(y).$$

## 5.7 Expected Value of a Function of Two Random Variables

**Theorem 5.10** (Expected Value of a Function of Two Random Variables). *The expected value of a function  $g(X, Y)$  of two random variables  $X$  and  $Y$  is*

$$\begin{aligned} \mathbb{E}[g(X, Y)] &= \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y}(x, y); & \text{(Discrete)} \\ \mathbb{E}[g(X, Y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) \, dx \, dy. & \text{(Continuous)} \end{aligned}$$

**Theorem 5.11.**

$$\mathbb{E}\left[\sum_{i=1}^n a_i g_i(X, Y)\right] = \sum_{i=1}^n a_i \mathbb{E}[g_i(X, Y)].$$

**Theorem 5.12.** *For any two random variables  $X$  and  $Y$ ,*

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y].$$

## 5.8 Covariance, Correlation and Independent

**Definition 5.13** (Covariance). The covariance of two random variables  $X$  and  $Y$  is

$$\sigma_{xy} = \text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

**Theorem 5.14.** The variance of the sum of two random variables is

$$\text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y] + 2ab \text{Cov}[X, Y].$$

**Definition 5.15** (Correlation Coefficient). The correlation coefficient of two random variables  $X$  and  $Y$  is

$$\rho_{xy} = \text{Corr}[X, Y] = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X] \text{Var}[Y]}} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}.$$

**Note:** In some definition of correlation coefficient,  $\rho_{xy}$  is defined as  $\rho_{xy} = \sigma_{xy}$  (e.g., in stochastic analysis where state space is unit free).

**Theorem 5.16.**

$$-1 \leq \rho_{xy} \leq 1.$$

**Theorem 5.17.** If  $X$  and  $Y$  are independent, then

$$(a) \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y],$$

$$(b) \text{Cov}[X, Y] = 0,$$

$$(c) \text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y].$$