HW 14 Solution

Group 1 **Grade:**

Suppose that the number of eggs laid by a certain insect has a Poisson distribution with mean λ. The probability that any egg hatches is p. Assume that the eggs hatch independently of one another. Find the

1. expected value of Y, the total number of eggs that hatch.
2. variance of Y.

Hint:

Law of Total Expectation and Law of Total Variance.

Solution

Let N be the number of eggs laid by the insect and Y be the number of eggs that hatch. Given $N = n$, Y has a binomial distribution with n trials and success probability p. Thus, $\mathbb{E}[Y \mid N = n] = np$. Since N follows as Poisson with parameter λ, $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid N]] = \mathbb{E}[Np] = p\lambda$.

$$\begin{aligned}
\operatorname{Var}[Y] &= \mathbb{E}[\operatorname{Var}[Y \mid N]] + \operatorname{Var}[\mathbb{E}[Y \mid N]] \\
&= \mathbb{E}[Np(1-p)] + \operatorname{Var}[Np] \\
&= \mathbb{E}[N]p(1-p) + p^2 \operatorname{Var}[N] \\
&= \lambda p(1-p) + p^2\lambda \\
&= \lambda p
\end{aligned}$$

Group 2 **Grade:**

Let the random variables X and Y have a joint PDF which is uniform over the triangle with vertices at $(0, 0)$, $(0, 1)$, and $(1, 0)$.

1. Find the joint PDF of X and Y.
2. Find the marginal PDF of Y.
3. Find the conditional PDF of X given Y.
4. Find $\mathbb{E}[X \mid Y = y]$, and use the total expectation theorem to find $\mathbb{E}[X]$.

Solution

$$
\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} 2 & 0 \le x \le 1-y \le 1 \\ 0 & \text{otherwise.} \end{cases} \\
f_Y(y) &= \int_0^{1-y} 2 \,\mathrm{d}x \\
&= 2(1-y) \\
f_{X|Y}(x \mid y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{2}{2(1-y)} \\
&= \frac{1}{1-y} && (\mathsf{Unif}(0, \tfrac{1}{1-y})) \\
\mathbb{E}[X \mid Y = y] &= \frac{1-y}{2} \\
\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid Y]] \\
&= \int_0^1 \frac{1-y}{2} f_Y(y) \,\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 f_Y(y) \,\mathrm{d}y - \frac{1}{2}\int_0^1 y f_Y(y) \,\mathrm{d}y \\
&= \frac{1}{2} - \frac{1}{2}\mathbb{E}[Y]
\end{aligned}
$$

Group 3 **Grade:**

Let X be a random variable with PDF

$$
f_X(x) = \begin{cases} x/4 & 1 \le x \le 3, \\ 0 & \textit{elsewhere.} \end{cases}
$$

let A be the event $X \ge 2$.

1. Find $\mathbb{E}[X]$, $\mathbb{P}(A)$, $f_{X|A}(x)$, and $\mathbb{E}[X \mid A]$.
2. Let $Y = X^2$. Find $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$.

Solution

$$
\begin{aligned}
\mathbb{E}[X] &= \int_1^3 x \frac{x}{4} \,\mathrm{d}x \\
&= \frac{13}{6} \\
\mathbb{P}(A) &= \int_2^3 \frac{x}{4} \,\mathrm{d}x \\
&= \frac{5}{8}
\end{aligned}
$$

$$
\begin{aligned}
f_{X|A}(x) &= \frac{f_X(x)}{\mathbb{P}(A)} \\
&= \frac{2}{5}x \\
\mathbb{E}[X \mid A] &= \int_2^3 x\frac{2}{5}x \, \mathrm{d}x \\
&= \frac{38}{15} \\
\mathbb{E}[Y] &= \mathbb{E}\left[X^2\right] \\
&= \int_1^3 x^2 \frac{x}{4} \, \mathrm{d}x \\
&= 5 \\
\mathrm{Var}[Y] &= \mathbb{E}\left[Y^2\right] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}\left[X^4\right] - \mathbb{E}\left[X^2\right]^2 \\
&= \int_1^3 x^4 \frac{x}{4} \, \mathrm{d}x - \mathbb{E}\left[X^2\right]^2 \\
&= \frac{91}{3} - 25 \\
&= \frac{16}{3}
\end{aligned}
$$

Group 4 **Grade:**

The random variable X has the PDF

$$
f_X(x) = \begin{cases} cx^{-2} & 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}
$$

1. Determine the value of c.
2. Let A be the event $X > 1.5$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

Solution

$$
\begin{aligned}
\int_1^2 cx^{-2} \, \mathrm{d}x &= 1 \\
c &= 2 \\
\mathbb{P}(A) &= \int_{1.5}^2 \frac{2}{x^2} \, \mathrm{d}x \\
&= \frac{1}{3} \\
f_{X|A}(x) &= \frac{f_X(x)}{\mathbb{P}(A)} \\
&= 6x^{-2}
\end{aligned}
$$

Group 5 Grade:

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?

Hint:

Law of Total Expectation.

Solution

Let X be the time until the miner reaches safety. Let D be the door that the miner chooses. Then

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid D]] \\
&= \mathbb{E}[X \mid D=1]\mathbb{P}(D=1) + \mathbb{E}[X \mid D=2]\mathbb{P}(D=2) + \mathbb{E}[X \mid D=3]\mathbb{P}(D=3) \\
&= 3 \cdot \frac{1}{3} + [5 + \mathbb{E}[X]] \cdot \frac{1}{3} + [7 + \mathbb{E}[X]] \cdot \frac{1}{3} \\
\Rightarrow 3\mathbb{E}[X] &= 15 + 2\mathbb{E}[X] \\
\Rightarrow \mathbb{E}[X] &= 15
\end{aligned}$$