Homework 4

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Q1 **Grade:**

Recall That N_C denotes the normal cone map of the set C. Show that if U is a linear subspace of $\mathbb{R}^n$, then $N_U(x) = U^\perp$ for all $x \in U$, where $U^\perp$ denotes the subspace orthogonal to U (by definition, $N_U(x) = \emptyset$ if $x \notin U$).

Solution

Proof. To show that $N_U(x) = U^\perp$, we need to show that $N_U(x) \subseteq U^\perp$ and $U^\perp \subseteq N_U(x)$.

- $N_U(x) \subseteq U^\perp$: Let $y \in N_U(x)$, then we have $y^\top(x-u) \le 0$ for all $u \in U$. Since U is a linear subspace, we have $0 \in U$. Thus, $y^\top(x-0) \le 0$, which implies $y^\top x \le 0$. Since $y^\top x \le 0$ for all $y \in N_U(x)$, we have $x \in U^\perp$. Thus, $N_U(x) \subseteq U^\perp$.
- $U^\perp \subseteq N_U(x)$: Let $y \in U^\perp$, then we have $y^\top u = 0$ for all $u \in U$. Since U is a linear subspace, we have $x - u \in U$. Thus, $y^\top(x-u) \le 0$ for all $u \in U$. Thus, $y \in N_U(x)$. Thus, $U^\perp \subseteq N_U(x)$.

□

Q2 **Grade:**

In the proof of the existence of subgradients and of the Rockafellar-Moreau theorem, we used portions of the following result: for a proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, one has

$$\operatorname{ri}\operatorname{epi} f = \{\, (x,z) \mid x \in \operatorname{ri}\operatorname{dom} f, z > f(x) \,\}.$$

In this problem, we will prove this result, using the prolongation principle. Let R denote the set on the right-hand side of the above equation. Note that you can use some form of the prolongation principle in each of the three parts of this question.

(a) Show that for any $x \in \operatorname{dom} f$, then $(x, f(x))$ cannot be in $\operatorname{ri}\operatorname{epi} f$.

(b) Show that a point $(x,z) \in \operatorname{epi} f$ that has $x \notin \operatorname{ri}\operatorname{dom} f$ cannot be in $\operatorname{ri}\operatorname{epi} f \subseteq \mathbb{R}$. Together with the previous result, this allows us to conclude that $\operatorname{ri}\operatorname{epi} f \subseteq \mathbb{R}$.

(c) Show that any $(x,z) \in \mathbb{R}$ is also in $\operatorname{ri}\operatorname{epi} f$, and hence, in view of the previous results, that $\operatorname{ri}\operatorname{epi} f = \mathbb{R}$. This may be done by showing that for any $(x',z') \in \operatorname{epi} f$, there exists $\delta > 0$ such that $(x,z) + \delta((x,z) - (x',z')) \in \operatorname{epi} f$. Hint: you should need to use another fact we proved earlier, that a convex function is continuous relative to $\operatorname{dom} f$ at all points of $\operatorname{ri}\operatorname{dom} f$, that is, if $x \in \operatorname{ri}\operatorname{dom} f$, then for any $\tau > 0$, there exists an $\epsilon > 0$ such that $x' \in \operatorname{dom} f$ and $\|x' - x\| < \epsilon$ together imply $|f(x') - f(x)| < \tau$. For example, it should be possible to show that for small enough δ, one has $z + \delta(z - z') > (z + f(x))/2$ but $f(x + \delta(x - x')) < (z + f(x))/2$.

Solution

(a) *Proof.* Assume $(x, f(x)) \in \operatorname{ri}\operatorname{epi} f$, then there exists $\epsilon > 0$ such that $B((x, f(x)), \epsilon) \subseteq \operatorname{epi} f$. Since f is a proper convex function, we have $f(x) \neq \infty$. Consider the point $(x, f(x) - \frac{\epsilon}{2})$, though it is within the ball, it is clearly not in epi f since the second component is strictly less than $f(x)$. This is contradicted to the original assumption. Thus, $(x, f(x)) \notin \operatorname{epi} f$. □

(b) *Proof.* If $x \notin \operatorname{ri}\operatorname{dom} f$, then by the prolongation principle, there is a direction $d \in \mathbb{R}^n$ such that $x + \lambda d \notin \operatorname{dom} f$ for all $\lambda > 0$. Therefore, for any z' and arbitrary small $\lambda > 0$, $(x + \lambda d, z') \notin \operatorname{epi} f$, implying (x, z) is not in the relative interior of epi f. □

(c) *Proof.* Let $(x, z) \in R$, by definition, we know $x \in \operatorname{ri}\operatorname{dom} f$ and $z > f(x)$.

To prove $(x, z) \in \operatorname{ri}\operatorname{epi} f$, we must show that for every $(x', z') \in \operatorname{epi} f$, there exists a $\delta > 0$ such that

$$(x, z) + \delta((x, z) - (x', z')) \in \operatorname{epi} f.$$

The point halfway between (x, z) and (x', z') is

$$\left(\frac{x + x'}{2}, \frac{z + z'}{2}\right).$$

Due to the convexity of f, this lies strictly above the graph of f at x.

For $x \in \operatorname{ri}\operatorname{dom} f$, by continuity of convex functions, for any $\tau > 0$, there exists $\epsilon > 0$ such that if $\|x' - x\| < \epsilon$ and $x' \in \operatorname{dom} f$, then $|f(x') - f(x)| < \tau$.

Choose $\tau = \frac{z - f(x)}{2}$. By continuity, there exists $\epsilon > 0$ ensuring that

$$f(x') < f(x) + \tau$$

whenever $\|x' - x\| < \epsilon$. Given our choice of τ, this means

$$f(x') < \frac{z + f(x)}{2}$$

for $\|x' - x\| < \epsilon$.

Choose δ small enough that the point

$$(x, z) + \delta((x, z) - (x', z'))$$

is within an ϵ-distance from x in its first coordinate, and lies below the midway point of (x, z) and (x', z') in its second coordinate. This ensures that this point lies strictly above the graph of f.

Thus, for any $(x', z') \in \operatorname{epi} f$, there exists a $\delta > 0$ such that $(x, z) + \delta((x, z) - (x', z')) \in \operatorname{epi} f$, proving that any $(x, z) \in R$ is also in $\operatorname{ri}\operatorname{epi} f$. □ □

Q3 Grade:

In this problem, we will prove the following "almost industrial strength" generalization of Proposition 4.2.5(a): let $\mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function and let A be an $m \times n$ matrix. Define $g(x) = f(Ax)$, which is also a convex function. Then, for all $x \in \mathbb{R}^n$,

$$\partial g(x) \supseteq A^\mathsf{T} \partial f(Ax). \tag{1}$$

Furthermore, if $\operatorname{ri}\operatorname{dom} f \cap \operatorname{im} A \neq \emptyset$, that is, there exists some point in $\bar{z} \in \operatorname{ri}\operatorname{dom} f$ that may be expressed as $\bar{z} = A\bar{x}$ for some $\bar{x} \in \mathbb{R}^n$, then for any $x \in \mathbb{R}^n$,

$$\partial g(x) = A^\mathsf{T} \partial f(Ax). \tag{2}$$

(a) Prove eq. (1).

(b) Define $U = \{ (x,z) \in \mathbb{R}^n \times \mathbb{R}^m \mid z = Ax \}$, which is a linear subspace of $\mathbb{R}^n \times \mathbb{R}^m$, along with the following functions $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{ +\infty \}$:

$$F_1(x,z) = f(z)$$
$$F_2(x,z) = \begin{cases} 0, & z = Ax \\ +\infty, & z \neq Ax \end{cases}$$
$$F(x,z) = F_1(x,z) + F_2(x,z) = \begin{cases} f(z), & z = Ax \\ +\infty, & z \neq Ax \end{cases}$$

Show that F_1, F_2 and F defined in this manner are convex and that $d \in \partial g(x)$ implies $(d,0) \in \partial F(x, Ax)$.

(c) Show that

$$\partial F_1(x,z) = \{ 0 \} \times \partial f(z)$$
$$\partial F_2(x,z) = \begin{cases} \{ (A^\mathsf{T} w, -w) \mid w \in \mathbb{R}^m \}, & z = Ax \\ \emptyset, & z \neq Ax \end{cases}$$

You may use the elementary linear-algebra fact that for any $p \times q$ matrix M, the subspace orthogonal to the subspace $\{ y \in \mathbb{R}^q \mid My = 0 \}$ is $\left\{ M^\mathsf{T} w \mid w \in \mathbb{R}^q \right\}$.

(d) For the reminder of this problem, assume $\operatorname{ri}\operatorname{dom} f \cap \operatorname{im} A \neq \emptyset$. Show that, in this case, $\operatorname{ri}\operatorname{dom} F_1$ and $\operatorname{ri}\operatorname{dom} F_2$ must intersect.

(e) Find an expression for $\partial F(x,z) = \partial(F_1 + F_2)(x,z)$. You may use version of the Moreau-Rockafellar theorem, which asserts that if $\operatorname{ri}\operatorname{dom} f_1 \cap \operatorname{ri}\operatorname{dom} f_2 \neq \emptyset$, then $\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$ for all $x \in \mathbb{R}^n$.

(f) Combine the above results to show that $\partial g(x) = A^\mathsf{T} \partial f(Ax)$.

Solution

(a) *Proof.* Let's take any $v \in \partial f(Ax)$. By the definition of subgradients, we have

$$f(y) \geq f(Ax) + v^\mathsf{T}(y - Ax) \quad \forall y \in \mathbb{R}^m .$$

Let $y = Ax + Az$, then we have

$$f(Ax + Az) \geq f(Ax) + v^\mathsf{T}(Ax + Az - Ax) = f(Ax) + v^\mathsf{T}(Az).$$

Notice that the left side is $g(x+z)$, and the function on the right involves z which is the perturbation in x.

$$g(x+z) \geq g(x) + v^\mathsf{T} Az \quad \forall z \in \mathbb{R}^n .$$

This is the definition of the subgradients of g at x, thus, we have $A^\mathsf{T} v \in \partial g(x)$. Since v is arbitrary, we have $\partial g(x) \supseteq A^\mathsf{T} \partial f(Ax)$. □

(b) *Proof.* The proof is as follow:

- F_1 **is convex:** $F_1(x,z)$ is convex since f is given to be a proper convex function.
- F_2 **is convex:** Consider any two points (x_1, z_1) and (x_2, z_2) in $\mathbb{R}^n \times \mathbb{R}^m$ and any $\lambda \in (0,1)$.
  - If $z_1 = Ax_1$ and $z_2 = Ax_2$, then the line segment between (x_1, Ax_1) and (x_2, Ax_2) is entirely contained in the set $\{(x,z) \mid z = Ax\}$, and hence F_2 is zero along this segment.
  - If either $z_1 \neq Ax_1$ or $z_2 \neq Ax_2$, then F_2 takes the value $+\infty$ at one or both of these points, and it is trivially convex as $\infty \leq \infty$.

- F **is convex:** F is the sum of F_1 and F_2, and the sum of two convex functions is also convex.
- $d \in \partial g(x)$ **implies** $(d, 0) \in \partial F(x, Ax)$**:** By the definition of subgradients and function g, we have:

$$g(x + h) \geq g(x) + d^\top h \text{ for all } h \in \mathbb{R}^n.$$

  Given $g(x) = f(Ax)$, this can be rewritten as:

$$f(A(x + h)) \geq f(Ax) + d^\top h.$$

  Considering the definition of F, we can express this inequality as:

$$F(x + h, A(x + h)) \geq F(x, Ax) + d^\top h.$$

  Given the definition of the subgradients for functions of two variables, this means:

$$(d, 0) \in \partial F(x, Ax)$$

□

(c) *Proof.* The proof is as follow:

- Since F_1 is only dependent on z, its subgradients with respect to x will simply be 0. With respect to z, the subgradients will be the same as the subgradients of f at z. Thus, we have $\partial F_1(x, z) = \{0\} \times \partial f(z)$.
- – When $z = Ax$: To find the subgradients of F_2, we want to find all vectors (d, w) such that:

$$F_2(x + h, z + k) \geq F_2(x, z) + \langle d, h \rangle + \langle w, k \rangle$$

    for all (h, k). Given that $F_2(x, z) = 0$ for $z = Ax$, the inequality becomes

$$F_2(x + h, z + k) \geq \langle d, h \rangle + \langle w, k \rangle.$$

    Considering perturbing z slightly by some k such that $z + k \neq A(x + h)$. In this case $F_2(x + h, z + k) = +\infty$, thus, the inequality holds for all (d, w). Thus, only need to deal with $z + k = A(x + h)$. Now the inequality becomes

$$0 \geq \langle d, h \rangle + \langle w, k \rangle.$$

    Given $k = A(x + h) - Ax = Ah$, the inequalitycan be written as:

$$0 \geq \langle d, h \rangle + \langle w, Ah \rangle.$$

    For this to hold for all h, d must be orthogonal to A and w must be orthogonal to the nullspace of A^T. Using the hint, we have

$$d = A^\mathsf{T} w$$

    for some $w \in \mathbb{R}^m$. Next for any h:

$$k = Ah \Rightarrow -k = -Ah.$$

    Thus w should be the negative of any vector in $\mathbb{R}^m$ to ensure the orthogonality condition. Thus, we have $\partial F_2(x, z) = \left\{ (A^\mathsf{T} w, -w) \mid w \in \mathbb{R}^m \right\}$.
  – When $z \neq Ax$: In this case, $F_2(x, z) = +\infty$. Thus, the subgradients are empty.

□

(d) *Proof.* $\operatorname{ri}\operatorname{dom} F_1$ is the set of all z such that $f(z) < +\infty$, which means it is the relative interior of the domain of f, *i.e.* $ri \operatorname{dom} f$. $\operatorname{ri}\operatorname{dom} F_2$ is the set of all z such that $z = Ax$ for some x, which means it is the image of A, *i.e.* imA. Given that $\operatorname{ri}\operatorname{dom} f \cap \operatorname{im} A \neq \emptyset$, there exists some $\bar{z} \in \operatorname{ri}\operatorname{dom} f$ such that $\bar{z} = A\bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. Therefore, $\bar{z}$ belongs to both $\operatorname{ri}\operatorname{dom} F_1$ and $\operatorname{ri}\operatorname{dom} F_2$, which means $\operatorname{ri}\operatorname{dom} F_1$ and $\operatorname{ri}\operatorname{dom} F_2$ must intersect. □

(e) *Proof.* The Moreau-Rockafellar theorem states that if ri dom $f_1 \cap$ri dom $f_2 \neq \emptyset$, then $\partial(f_1+f_2)(x) = \partial f_1(x)+\partial f_2(x)$ for all $x \in \mathbb{R}^n$. Since ri dom F_1 and ri dom F_2 intersect, we have $\partial F(x,z) = \partial(F_1+F_2)(x,z) = \partial F_1(x,z)+\partial F_2(x,z)$. Thus, we have

- When $z = Ax$:

$$\begin{aligned}\partial F(x,z) &= \{\,0\,\} \times \partial f(z) + \left\{\,(A^{\mathsf{T}} w, -w) \mid w \in \mathbb{R}^m\,\right\} \\ &= \left\{\,(A^{\mathsf{T}} w, v-w) \mid w \in \mathbb{R}^m, v \in \partial f(z)\,\right\}.\end{aligned}$$

- When $z \neq Ax$:

$$\begin{aligned}\partial F(x,z) &= \emptyset + \emptyset \\ &= \emptyset.\end{aligned}$$

□

(f) *Proof.* To find $\partial g(x)$, we use the property that any d in $\partial g(x)$ must satisfy $(d,0) \in \partial F(x,Ax)$. Given $F(x,z) = F_1(x,z) + F_2(x,z)$, $F(x,Ax) = F_1(x,Ax) + F_2(x,Ax) = f(Ax) = g(x)$. Thus, for any $d \in \partial g(x)$, the corresponding $(d,0) \in \partial F(x,Ax)$ must have the form $(A^{\mathsf{T}} w, v-w)$ where $w \in \mathbb{R}^m$ and $v \in \partial f(Ax)$. But the second coordinate is 0, which implies $v = w \Rightarrow (d,0) = (A^{\mathsf{T}} w, 0)$. This means $d = A^{\mathsf{T}} w$ for some $w \in \partial f(Ax)$. In other words:

$$d \in A^{\mathsf{T}} \partial f(Ax).$$

This is $\partial g(x) \subseteq A^{\mathsf{T}} \partial f(Ax)$. Combining the results from (a), we have $\partial g(x) = A^{\mathsf{T}} \partial f(Ax)$. □