Special Topics in Management Science 26:711:685

Convex Analysis and Optimization

Fall 2013 Rutgers University Prof. Eckstein

Solutions to Homework 2

1. Suppose $0 < \alpha_1 < \alpha_2$. Then note that we can express $x + \alpha_1 d$ as a convex combination of x and $x + \alpha_2 d$ as follows:

$$\left(1 - \frac{\alpha_1}{\alpha_2}\right) x + \frac{\alpha_1}{\alpha_2}(x + \alpha_2 d) = x - \frac{\alpha_1}{\alpha_2} x + \frac{\alpha_1}{\alpha_2} x + \alpha_1 d = x + \alpha_1 d.$$

   Therefore, the convexity of f yields

$$f(x + \alpha_1 d) \le \left(1 - \frac{\alpha_1}{\alpha_2}\right) f(x) + \frac{\alpha_1}{\alpha_2} f(x + \alpha_2 d).$$

   Subtracting $f(x)$ from both sides, we have

$$f(x + \alpha_1 d) - f(x) \le \frac{\alpha_1}{\alpha_2}\left(f(x + \alpha_2 d) - f(x)\right).$$

   Dividing through by $\alpha_1 > 0$, we have

$$\frac{f(x + \alpha_1 d) - f(x)}{\alpha_1} \le \frac{f(x + \alpha_2 d) - f(x)}{\alpha_2}.$$

   Since we chose $0 < \alpha_1 < \alpha_2$ arbitrarily, we must have a nondecreasing function.

2. (a) The simplest approach here is to use coercivity. Define

$$f(w) = \begin{cases} \|w - x\|, & w \in C \\ +\infty, & \text{otherwise,} \end{cases}$$

   Clearly f is proper, because C is nonempty and $f(w) = 0 < \infty$ for $w \in C$. The function $g : w \mapsto \|w - x\|$ is continuous, so $\operatorname{epi} g$ is closed. It is easily seen that $\operatorname{epi} f = \operatorname{epi} g \cap (C \times \mathbb{R})$. Since C is closed, $C \times \mathbb{R}$ is closed, and then $\operatorname{epi} f$ is closed since it is the intersection of two closed sets. To summarize, f is proper and closed. Furthermore, it is also coercive because clearly $f(w) \to \infty$ if $\|w\| \to \infty$ (either the distance from x eventually increases without bound or $f(w) = +\infty$ because $w \notin C$). Thus, it has a nonempty compact set of minima as given by Proposition 2.1.1(3). Note that f need not be convex, and convexity is not required in Proposition 2.1.1. Alternatively, one could just follow the proof given in part (a) of the projection theorem, noting that convexity is not used until part (b).

   Note that it is not necessary to square the norm $\|w - x\|$ in this analysis. That was done in the convex projection theorem in order to derive some additional results that don't follow in the nonconvex case.

(b) Fix any $\hat{w} \in C$. For any $x \in \mathbb{R}^n$, it is clear that $0 \le \inf_{w \in C} \|w - x\| \le \|\hat{w} - x\|$, so we have $0 \le \mathrm{dist}_C(x) \le \|\hat{w} - x\|$ and dist_C must be finite-valued.

Now take any two points $x, y \in \mathbb{R}^n$ and let $w_x \in P_C(x)$ and $w_y \in P_C(y)$. Then

$$\begin{aligned} \mathrm{dist}_C(x) &= \|w_x - x\| \\ &\le \|w_y - x\| && \text{[Since } w_y \in C \text{ and } w_x \text{ is closest in } C \text{ to } x\text{]} \\ &\le \|w_y - y\| + \|y - x\| && \text{[Triangle inequality]} \\ &= \mathrm{dist}_C(y) + \|y - x\| . \end{aligned}$$

In summary, $\mathrm{dist}_C(x) \le \mathrm{dist}_C(y) + \|x - y\|$. Reversing the roles of x and y, however, we also conclude that $\mathrm{dist}_C(y) \le \mathrm{dist}_C(x) + \|x - y\|$. Combining these two inequalities, we conclude that

$$\left| \mathrm{dist}_C(x) - \mathrm{dist}_C(y) \right| \le \|x - y\| ,$$

that is, dist_C is Lipschitz continuous with modulus 1, and hence continuous.

(c) In $\mathbb{R}^1$, consider $C = \{0, 1\}$. Then

$$\mathrm{dist}_C(x) = \begin{cases} -x, & x \le 0, \\ x, & 0 \le x \le 1/2 \\ 1 - x, & 1/2 \le x \le 1 \\ x - x, & x \ge 1. \end{cases}$$

The graph of this function looks like a "W" and is clearly not convex. In particular, $\mathrm{dist}_C(0) = 0$, $\mathrm{dist}_C(1/2) = 1/2$, and $\mathrm{dist}_C(1) = 0$, so we have $\mathrm{dist}_C(1/2) > (1/2)\,\mathrm{dist}_C(0) + (1/2)\,\mathrm{dist}_C(1) = 0$, violating the convexity inequality.

3. (a) The epigraph of δ_X is $X \times [0, \infty)$. If X is closed, this is a Cartesian product of closed sets, and hence closed.

   (b) Again, $\mathrm{epi}\, f = X \times [0, \infty)$. If X is convex, $\mathrm{epi}\, f$ is the Cartesian product of convex sets and hence convex.

4. Since K is a closed convex set, and $y \notin K$, the separating hyperplane theorem guarantees that there exists some $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, $\epsilon > 0$, such that $\langle a, y \rangle = b$ and $\langle a, x \rangle \le b - \epsilon$ for all $x \in K$ and $\langle a, y \rangle = b$. Since $0 \in K$, can immediately deduce that $\langle a, 0 \rangle = 0 \le b - \epsilon$, and hence that $b \ge \epsilon > 0$. So $\langle a, y \rangle = b > 0$. On the other hand, suppose that $\langle a, \bar{x} \rangle > 0$ for some $\bar{x} \in K$. Since K is a cone, $\alpha \bar{x} \in K$ for any $\alpha \ge 0$, but if one chooses α $\alpha > (b - \epsilon)/\langle a, \bar{x} \rangle$, we have $\alpha \bar{x} \in K$, while at the same time

$$\langle a, \alpha \bar{x} \rangle = \alpha \langle a, \bar{x} \rangle > \frac{b - \epsilon}{\langle a, \bar{x} \rangle} \langle a, \bar{x} \rangle = b - \epsilon$$

which contradicts the construction of a as having $\langle a, x \rangle \le b - \epsilon$ for all $x \in K$. Therefore it is not possible to have $\langle a, \bar{x} \rangle > 0$ for any $\bar{x} \in K$. In other words, one must have $\langle a, x \rangle \le 0$ for all $x \in K$.

Quite a few students tried to use the supporting hyperplane theorem here, rather than the separating hyperplane theorem. That will not work, because you need strict separation to argue that $\langle a, y \rangle > 0$.

I also saw a group of multiple students try to use a set-based separation theorem here, separating K from $\{y\}$, but that's unnecessary because we are trying to separate a point and a set. The simpler forms of the separation theorems all start with a point and a set, and that's all you need here.

I saw obvious signs of collaboration on these assignments. Collaboration is OK for regular homework, but will not be accepted on the take-home exams.