

## HW 6

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## Q1

Grade:

For each of the following choices of  $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ , compute the convex conjugate function  $f^*$ :

(a)  $f(x) = \frac{1}{2}x^2$ .

(b) For  $a, b \in \mathbb{R}$ ,  $a < b$ ,  $f(x) = \delta_{[a,b]} = \begin{cases} 0 & x \in [a, b], \\ +\infty & \text{otherwise.} \end{cases}$

(c)  $f(x) = e^x$ .

## Q2

Grade:

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is said to be positively homogeneous if

$$\begin{aligned} f(0) &= 0 \\ f(\alpha x) &= \alpha f(x) \quad \forall \alpha > 0, x \in \mathbb{R}^n. \end{aligned}$$

(Note that some definitions omit the condition  $f(0) = 0$ , which we include here to accord with our notion of a cone as always containing the point 0.)

- (a) For any proper function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ , show that  $\text{epi } f$  is a cone in  $\mathbb{R}^{n+1}$  if and only if  $f$  is positively homogeneous.
- (b) Consider any nonempty set  $X \subseteq \mathbb{R}^n$ . The *support function* of  $X$  is the convex conjugate ( $\delta_X^*$ ) of the indicator function

$$\delta_X = \begin{cases} 0 & x \in X, \\ +\infty & \text{otherwise.} \end{cases}$$

Show that

$$\delta_X^*(y) = \sup_{x \in X} \langle x, y \rangle,$$

and this function is positively homogeneous.

- (c) Show conversely that, given any positively homogeneous function  $f$ , its convex conjugate  $f^*$  is the indicator function of some closed convex set  $C$ .
- (d) Given a cone  $K$ , show that  $\delta_K^* = \delta_{K^*}$ . That is, the conjugate of the indicator function of a  $K$  is the indicator function of its polar.

**Q3****Grade:**

Consider the standard primal linear programming problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{S.T.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

Model this problem as  $\min f(x) + g(Mx)$ , where

$$f(x) \doteq \begin{cases} c^\top x & x \geq 0 \\ +\infty & \text{otherwise} \end{cases} \quad M \doteq A \quad g(z) \doteq \begin{cases} 0 & z = b \\ +\infty & \text{otherwise,} \end{cases}$$

where  $A$  is any  $m \times n$  matrix and  $b \in \mathbb{R}^m$ . Show that the corresponding Fenchel dual is equivalent to the standard dual programming problem

$$\begin{aligned} \max_{u \in \mathbb{R}^m} \quad & b^\top u \\ \text{S.T.} \quad & A^\top u \leq c \end{aligned}$$

in the sense that any solution  $y^*$  of the Fenchel dual is equal to  $-u^*$  is some optimal solution to the standard dual linear programming problem.