Homework 3

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Q1: Polar Cone Operations. Problems 3.4(a) - (c) Grade:

Show the following:

(a) For any nonempty cones $C_i \subset \mathbb{R}^{n_i}$, $i = 1, 2, \ldots, m$, we have

$$(C_1 \times C_2 \times \cdots \times C_m)^* = C_1^* \times C_2^* \times \cdots \times C_m^*.$$

(b) For any collection of nonempty cones $\{ C_i \mid i \in I \}$, we have

$$(\cup_{i \in I} C_i)^* = \cap_{i \in I} C_i^*.$$

(c) For any two nonempty cones C_1 and C_2, we have

$$(C_1 + C_2)^* = C_1^* \cap C_2^*.$$

Hint: to show $C^* = K$, the simplest general strategy is usually to show that $\langle x, y \rangle \leq 0$ for all $x \in C$ and $y \in K$, establishing $K \subseteq C^*$, and then show that if $z \notin K$, then there exists some $x \in C$ with $\langle x, z \rangle > 0$, implying that $z \notin C^*$, and thus $C^* = K$ since $z \notin K$ was arbitrary.

Solution

(a) *Proof.* Let $C = (C_1 \times C_2 \times \cdots \times C_m)^*$ and $C' = C_1^* \times C_2^* \times \cdots \times C_m^*$. We will show that $C = C'$ by showing that $C \subseteq C'$ and $C' \subseteq C$.

(i) Let $x \in C$. Then for all $y \in C_1 \times C_2 \times \cdots \times C_m$, we have $\langle x, y \rangle \leq 0$. Equivalently, that is $\sum_{i=1}^{m} x_i y_i \leq 0$ where $y_i \in C_i$ for all $i \in \{ 1, 2, \ldots, m \}$. Since C_i are cones and 0 belongs to their closure, then $\langle x_i, y_i \rangle \leq 0$ for all $i \in \{ 1, 2, \ldots, m \}$ by letting all $y_k \to 0$, $i \neq k$. Thus $x_i \in C_i^*$ for all $i \in \{ 1, 2, \ldots, m \}$. Therefore, $x \in C'$ and then $C \subseteq C'$.

(ii) Let $x \in C'$. Then $x = (x_1, x_2, \ldots, x_m)$ where $x_i \in C_i^*$ for all $i \in \{ 1, 2, \ldots, m \}$. Let $y \in C_1 \times C_2 \times \cdots \times C_m$. Then $y = (y_1, y_2, \ldots, y_m)$ where $y_i \in C_i$ for all $i \in \{ 1, 2, \ldots, m \}$. Then $\langle x_i, y_i \rangle \leq 0$ for all $i \in \{ 1, 2, \ldots, m \}$. Thus $\langle x, y \rangle \leq 0$. Therefore, $x \in C$ and then $C' \subseteq C$.

□

(b) *Proof.* Let $C = (\cup_{i \in I} C_i)^*$ and $C' = \cap_{i \in I} C_i^*$. We will show that $C = C'$ by showing that $C \subseteq C'$ and $C' \subseteq C$.

(i) Let $x \in C$. Then for all $y \in \cup_{i \in I} C_i$, we have $\langle x, y \rangle \leq 0$. Equivalently, that is $\langle x, y_i \rangle \leq 0$ where $y \in C_i$ for all $i \in I$. Thus, $x \in C_i^*$ for all $i \in I$. Therefore, $x \in C'$ and then $C \subseteq C'$.

(ii) Let $x \in C'$. Then $x \in C_i^*$ for all $i \in I$. Let $y \in \cup_{i \in I} C_i$. Then $y_i \in C_i$ for $i \in I$. Then $\langle x, y_i \rangle \leq 0$. Thus $\langle x, y \rangle \leq 0$. Therefore, $x \in C$ and then $C' \subseteq C$.

□

(c) *Proof.* Let $C = (C_1 + C_2)^*$ and $C' = C_1^* \cap C_2^*$. We will show that $C = C'$ by showing that $C \subseteq C'$ and $C' \subseteq C$.

(i) Let $x \in C$. Then for all $y \in C_1 + C_2$, we have $\langle x, y\rangle \leq 0$. Equivalently, that is $\langle x, y_1 + y_2\rangle \leq 0$ where $y_1 \in C_1$ and $y_2 \in C_2$. Thus, $\langle x, y_1\rangle + \langle x, y_2\rangle \leq 0$. Since C_1 and C_2 are cones and 0 belongs to their closure, following the same logic in (a), $\langle x, y_1\rangle \leq 0$ and $\langle x, y_2\rangle \leq 0$. Thus $x \in C_1^*$ and $x \in C_2^*$. Therefore, $x \in C'$ and then $C \subseteq C'$.

(ii) Let $x \in C'$. Then $x \in C_1^*$ and $x \in C_2^*$. Let $y \in C_1 + C_2$. Then $y = y_1 + y_2$ where $y_1 \in C_1$ and $y_2 \in C_2$. Then $\langle x, y_1\rangle \leq 0$ and $\langle x, y_2\rangle \leq 0$. Thus $\langle x, y_1 + y_2\rangle \leq 0$. Therefore, $\langle x, y\rangle \leq 0$. Therefore, $x \in C$ and then $C' \subseteq C$.

□

I didn't follow the hint. Please let me know my mistakes if this proof doesn't work. Thanks!

Q2: Cone Separation — Grade:

Suppose $K \in \mathbb{R}^n$ is a nonempty closed convex cone. Show that if $z \in \mathbb{R}^n$ and $z \notin K$, then there exists $a \in K^*$ with $\langle a, z\rangle > 0$.

Solution

Proof. Using **Separating Hyperplane Theorem**, if K is a nonempty closed convex cone and $z \notin K$, then by the theorem, there exists a hyperplane that can separate z from K. This means there exists $a \neq 0$ such that $\langle a, x\rangle \leq 0$ for all $x \in K$ and $\langle a, z\rangle > 0$. By definition of polar cone, $a \in K^*$. Thus, there exists $a \in K^*$ with $\langle a, z\rangle > 0$. □

Q3: Sums of Convex Cones — Grade:

Show that if $C_1, C_2 \subseteq \mathbb{R}^n$ are convex cones, then $C_1 + C_2$ is a convex cone.

Solution

Proof. We will show that $C_1 + C_2$ is a convex cone by showing that $C_1 + C_2$ is a cone and $C_1 + C_2$ is convex.

(a) Convexity. Let $x, y \in C_1 + C_2$ and $\theta \in [0, 1]$. Then $x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1, y_1 \in C_1$ and $x_2, y_2 \in C_2$. Then $\theta x + (1-\theta)y = \theta x_1 + (1-\theta)y_1 + \theta x_2 + (1-\theta)y_2$. Since C_1 and C_2 are convex, $\theta x_1 + (1-\theta)y_1 \in C_1$ and $\theta x_2 + (1-\theta)y_2 \in C_2$, which implies $\theta x + (1-\theta)y \in C_1 + C_2$. Therefore, $C_1 + C_2$ is convex.

(b) Coneness. Let $x \in C_1 + C_2$ and $\theta \geq 0$. Then $x = x_1 + x_2$ where $x_1 \in C_1$ and $x_2 \in C_2$. Then $\theta x = \theta x_1 + \theta x_2$. Since C_1 and C_2 are cones, $\theta x_1 \in C_1$ and $\theta x_2 \in C_2$, which implies $\theta x \in C_1 + C_2$. Therefore, $C_1 + C_2$ is a cone.

□

Q4 — Grade:

Show that if $C_1, C_2\, \mathbb{R}^m$ are closed convex cones, then $(C_1 \cap C_2)^* = \mathrm{cl}\big(C_1^* + C_2^*\big)$.
Note: this is the main result of problem 3.4(d).
Hint: to show that $z \notin \mathrm{cl}(C_1^* + C_2^*)$ implies $z \notin (C_1 \cap C_2)^*$, use problem 2, problem 1(c), and the polar cone theorem.

Solution

Proof. We need to show that $\mathrm{cl}\big(C_1^* + C_2^*\big) \subseteq (C_1 \cap C_2)^*$ and $z \notin \mathrm{cl}(C_1^* + C_2^*)$ implies $z \notin (C_1 \cap C_2)^*$.

(i) For any $y \in \mathrm{cl}(C_1^* + C_2^*)$, there exists $y_1 \in C_1^*$ and $y_2 \in C_2^*$ such that for any $\epsilon > 0$, $\|y - (y_1 + y_2)\| < \epsilon$. Let

$x \in C_1 \cap C_2$, we have $\langle y, x\rangle = \langle y_1 + y_2, x\rangle + \langle y - (y_1 + y_2), x\rangle$. Because y_1 is in the polar of C_1 and y_2 is in the polar of C_2, $\langle y_1, x\rangle \le 0$ and $\langle y_2, x\rangle \le 0$, which implies $\langle y_1 + y_2, x\rangle \le 0$. Using Cauchy-Schwarz inequality, $\langle y - (y_1 + y_2), x\rangle \le \|y - (y_1 + y_2)\|\|x\| < \epsilon\|x\|$. Since ϵ is arbitrary, $\langle y - (y_1 + y_2), x\rangle \le 0$. Thus, $\langle y, x\rangle \le 0$. Therefore, $y \in (C_1 \cap C_2)^*$ and then $\mathrm{cl}(C_1^* + C_2^*) \subseteq (C_1 \cap C_2)^*$.

(ii) If $z \notin \mathrm{cl}(C_1^* + C_2^*)$, by the cone separation theorem, there exists an x such that

$$\langle x, z\rangle > 0, \qquad \langle x, p\rangle \le 0$$

for all $p \in \mathrm{cl}(C_1^* + C_2^*)$. The second condition implies that $x \in (\mathrm{cl}(C_1^* + C_2^*))^* = (C_1^* + C_2^*)^* = C_1^{**} \cap C_2^{**} = C_1 \cap C_2$. Then the first condition implies, $z \in C_1 \cap C_2$ which means $z \notin \mathrm{cl}(C_1 \cap C_2)^*$.

Thus we complete the proof. □

Q5 Grade:

Let A be an $m \times n$ real matrix, and $C \subseteq \mathbb{R}^m$ be a closed convex cone. Define

$$K = \{\, x \in \mathbb{R}^n \mid Ax \in C \,\} \qquad P = \left\{\, A^\mathsf{T} y \mid y \in C^* \,\right\}.$$

(a) Show that K is a closed convex cone.

(b) Show that $K^* = \mathrm{cl}\, P$. Hint: to show that $z \notin \mathrm{cl}\, P$ implies $z \notin K^*$, use problem 2.

(c) Show that $P^* = K$ (by using the polar cone theorem).

Solution

(a) *Proof.* To show that K is a closed convex cone, we need to prove the convexity, closedness and coneness of K.

(i) Coneness. Given any $x \in K$, $Ax \in C$, and $\lambda \ge 0$, since C is a cone, $A\lambda x = \lambda Ax \in C$. Thus, $\lambda x \in K$ and K is a cone.

(ii) Convexity. Given $x_1, x_2 \in K$, $Ax_1, Ax_2 \in C$, since C is a convex set, for any $\lambda \in [0, 1]$, we have

$$\lambda Ax_1 + (1 - \lambda)Ax_2 = A(\lambda x_1 + (1 - \lambda)x_2) = A \cdot \lambda x_1 + A \cdot (1 - \lambda)x_2.$$

Thus, $\lambda x_1 + (1 - \lambda)x_2 \in K$ and K is a convex set.

(iii) Closedness. Since C is a closed set, Ax is a continuous function. Thus, $K = A^{-1}Ax$ is a closed set.

□

(b) *Proof.* Given $z \in K^*$, for all $x \in K$, we have

$$\langle z, x\rangle \le 0, \qquad \forall x, s.t. Ax \in C.$$

By definition of polar cone C^*, $y \in C^*$ if and only if $\langle y, Ax\rangle \le 0$. Hence, for the same x,

$$\left\langle A^\mathsf{T} y, x\right\rangle \le 0.$$

If z is an element in $A^\mathsf{T} y \in P$. Since P is not necessarily closed, we have $z \in \mathrm{cl}\, P$, which is $K^* \subseteq \mathrm{cl}\, P$. If $z \notin \mathrm{cl}\, P$, by the cone separation theorem, there exists $x \in (\mathrm{cl}\, P)^* = P^*$ such that $\langle x, z\rangle > 0$, and $\langle x, p\rangle \le 0$ for all $p \in \mathrm{cl}\, P$. Given that $P = \left\{ A^\mathsf{T} y \mid y \in C^* \right\}$, we can write the second condition as $\left\langle x, A^\mathsf{T} y\right\rangle \le 0$ for all $y \in C^*$. This is equivalent to $\langle y, Ax\rangle \le 0$. This implies $Ax \in C$ and $x \in K$. Since $\langle x, z\rangle > 0$ and $x \in K$, we have $z \in K$ and so the $z \notin K^*$. Then the proof is completed by the hint. □

(c) *Proof.* The polar cone theorem states that for a convex cone C, $C^{**} = C$ and $(\mathrm{cl}\, C)^* = C^*$. From (b), $K^* = \mathrm{cl}\, P \Rightarrow (K^*)^* = P^* \Rightarrow K = P^*$. Thus, $P^* = K$. □

Q6 Grade:

A cone K is called *self-dual* if $K^* = -K$. Show that the following cones are self-dual:

(a) The non-negative orthant $\{ x \in \mathbb{R}^n \mid x \geq 0 \}$.

(b) The *Lorentz cone* (also called the "ice cream cone") in $\mathbb{R}^{n+1}$, defined as follows:

$$K = \{ (x, w) \in \mathbb{R}^n \times \mathbb{R} \mid w \geq \|x\| \}.$$

Solution

(a) *Proof.* Let $K = \{ x \in \mathbb{R}^n \mid x \geq 0 \}$. Then $K^* = \{ y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 0, \forall x \in K \}$. Since $x \geq 0$, $\langle x, y \rangle \leq 0$ for all $y \leq 0$. Thus, $K^* = \{ y \in \mathbb{R}^n \mid y \leq 0 \}$. Therefore, $K^* = -K$ and K is self-dual. □

(b) *Proof.* Let $K = \{ (x, w) \in \mathbb{R}^n \times \mathbb{R} \mid w \geq \|x\| \}$. Then $K^* = \{ (y, z) \in \mathbb{R}^n \times \mathbb{R} \mid \langle (x, w), (y, z) \rangle \leq 0, \forall (x, w) \in K \}$. If $x = 0$, then $w \geq \|x\| = 0$ and $\langle (x, w), (y, z) \rangle = x^\intercal y + wz \leq 0$ implies $z \leq 0$. If $x \neq 0$, let $u = \frac{x}{\|x\|}$ be unit vector, $\langle (x, w), (y, z) \rangle = x^\intercal y + wz \leq 0 \Rightarrow u^\intercal y + z \leq 0$. To make the inequality hold, there must be $y = 0$ and $z \leq 0$ since u represents all directions. Thus, $K^* = \{ (0, z) \in \mathbb{R}^n \times \mathbb{R} \mid z \leq 0 \}$. Therefore, $K^* = -K$ and K is self-dual. □