Homework 4

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Q1 Grade:

Recall That N_C denotes the normal cone map of the set C. Show that if U is a linear subspace of $\mathbb{R}^n$, then $N_U(x) = U^\perp$ for all $x \in U$, where $U^\perp$ denotes the subspace orthogonal to U (by definition, $N_U(x) = \emptyset$ if $x \notin I$).

Q2 Grade:

In the proof of the existence of subgradients and of the Rockafellar-Moreau theorem, we used portions of the following result: for a proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{ \infty \}$, one has

$$\operatorname{ri} \operatorname{epi} f = \{ (x, z) \mid x \in \operatorname{ri} \operatorname{dom} f, z > f(x) \}.$$

In this problem, we will prove this result, using the prolongation principle. Let R denote the set on the right-hand side of the above equation. Note that you can use some form of the prolongation principle in each of the three parts of this question.

(a) Show that for any $x \in \operatorname{dom} f$, then $(x, f(x))$ cannot be in $\operatorname{ri} \operatorname{epi} f$.

(b) Show that a point $(x, z) \in \operatorname{epi} f$ that has $x \notin \operatorname{ri} \operatorname{dom} f$ cannot be in $\operatorname{ri} \operatorname{epi} f \subseteq R$.

(c) Show that any $(x, z) \in R$ is also in $\operatorname{ri} \operatorname{epi} f$, and hence, in view of the previous results, that $\operatorname{ri} \operatorname{epi} f = R$. This may be done by showing that for any $(x', z') \in \operatorname{epi} f$, there exists $\delta > 0$ such that $(x, z) + \delta((x, z) - (x', z')) \in \operatorname{epi} f$. Hint: you should need to use another fact we proved earlier, that a convex function is continuous relative to $\operatorname{dom} f$ at all points of $\operatorname{ri} \operatorname{dom} f$, that is, if $x \in \operatorname{ri} \operatorname{dom} f$, then for any $\tau > 0$, there exists an $\epsilon > 0$ such that $x' \in \operatorname{dom} f$ and $\|x' - x\| < \epsilon$ together imply $|f(x') - f(x)| < \tau$. For example, it should be possible to show that for small enough τ, one has $z + \tau(z - z') > (z + f(x))/2$ but $f(x + \tau(x - x')) < (z + f(x))/2$.

Q3 Grade:

In this problem, we will prove the following "almost industrial strength" generalization of Proposition 4.2.5(a): let $\mathbb{R}^m \to (-\infty, +\infty]$ be a proper convex function and let A be an $m \times n$ matrix. Define $g(x) = f(Ax)$, which is also a convex function. Then, for all $x \in \mathbb{R}^n$,

$$\partial g(x) \supseteq A^\top \partial f(Ax). \tag{1}$$

Furthermore, if $\operatorname{ri} \operatorname{dom} f \cap \operatorname{im} A \neq \emptyset$, that is, there exists some point in $\bar{z} \in \operatorname{ri} \operatorname{dom} f$ that may be expressed as $\bar{z} = A\bar{x}$ for some $\bar(x) \in \mathbb{R}^n$, then for any $x \in \mathbb{R}^n$,

$$\partial g(x) = A^\top \partial f(Ax). \tag{2}$$

(a) Prove eq. (1).

(b) Define $U = \{ (x, z) \in \mathbb{R}^n \times \mathbb{R}^m \mid z = Ax \}$, which is a linear subspace of $\mathbb{R}^n \times \mathbb{R}^m$, along with the following functions $\mathbb{R}^n \times \mathbb{R}^m \to (-\infty, +\infty]$:

$$F_1(x, z) = f(z)$$

$$F_2(x,z) = \begin{cases} 0, & z = Ax \\ +\infty, & z \neq Ax \end{cases}$$

$$F(x,z) = F_1(x,z) + F_2(x,z) = \begin{cases} f(z), & z = Ax \\ +\infty, & z \neq Ax \end{cases}$$

Show that F_1, F_2 and F defined in this manner are convex and that $d \in \partial g(x)$ implies $(d, 0) \in \partial F(x, Ax)$.

(c) Show that

$$\partial F_1(x,z) = \{\, 0 \,\} \times \partial f(z)$$

$$\partial F_2(x,z) = \begin{cases} \left\{\, (A^{\mathsf{T}} w, -w) \mid w \in \mathbb{R}^m \,\right\}, & z = Ax \\ \emptyset, & z \neq Ax \end{cases}$$

You may use the elementary linear-algebra fact that for any $p \times q$ matrix M, the subspace orthogonal to the subspace $\{\, y \in \mathbb{R}^q \mid My = 0 \,\}$ is $\left\{\, M^{\mathsf{T}} w \mid w \in \mathbb{R}^q \,\right\}$.

(d) For the reminder of this problem, assume $\operatorname{ri}\operatorname{dom} f \cap \operatorname{im} A \neq \emptyset$. Show that, in this case, $\operatorname{ri}\operatorname{dom} F_1$ and $\operatorname{ri}\operatorname{dom} F_2$ must intersect.

(e) Find an expression for $\partial F(x,z) = \partial(F_1 + F_2)(x,z)$. You may use version of the Moreau-Rockafellar theorem, which asserts that if $\operatorname{ri}\operatorname{dom} f_1 \cap \operatorname{ri}\operatorname{dom} f_2 \neq \emptyset$, then $\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$ for all $x \in \mathbb{R}^n$.

(f) Combine the above results to show that $\partial g(x) = A^{\mathsf{T}} \partial f(Ax)$.