HW 12 Solution

Group 1 **Grade:**

A coin that has probability of heads equal to p is tossed successively and independently until a head comes twice in a row or a tail comes twice in a row. Find the expected value of the number of tosses in terms of p and $q = 1 - p$.

Hint:

The Law of Total Expectation is useful.

Solution

With the law of total probability, we can partition the event X based on the first (starting from 0) toss,

$$\mathbb{P}(X) = \mathbb{P}(X \mid H_0)\mathbb{P}(H_0) + \mathbb{P}(X \mid T_0)\mathbb{P}(T_0).$$

Similarly, each partition of the first toss can be partitioned based on the second toss,

$$\begin{aligned}\mathbb{P}(X \mid H_0) &= \mathbb{P}(X \mid H_0 \cap H_1)\mathbb{P}(H_1) + \mathbb{P}(X \mid H_0 \cap T_1)\mathbb{P}(T_1),\\ \mathbb{P}(X \mid T_0) &= \mathbb{P}(X \mid T_0 \cap H_1)\mathbb{P}(H_1) + \mathbb{P}(X \mid T_0 \cap T_1)\mathbb{P}(T_1).\end{aligned}$$

Apply expectation to both sides,

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X \mid H_0]\mathbb{P}(H_0) + \mathbb{E}[X \mid T_0]\mathbb{P}(T_0),\\ \mathbb{E}[X \mid H_0] &= \mathbb{E}[X \mid H_0 \cap H_1]\mathbb{P}(H_1) + \mathbb{E}[X \mid H_0 \cap T_1]\mathbb{P}(T_1),\\ \mathbb{E}[X \mid T_0] &= \mathbb{E}[X \mid T_0 \cap H_1]\mathbb{P}(H_1) + \mathbb{E}[X \mid T_0 \cap T_1]\mathbb{P}(T_1).\end{aligned}$$

Because the game will end with two heads/tails in a row or start counting over, we have

$$\mathbb{E}[X \mid H_0H_1] = \mathbb{E}[X \mid T_0T_1] = 2, \qquad \mathbb{E}[X \mid H_0T_1] = 1 + \mathbb{E}[X \mid T_0], \qquad \mathbb{E}[X \mid T_0H_1] = 1 + \mathbb{E}[X \mid H_0].$$

$$\begin{aligned}\mathbb{E}[X \mid H_0] &= 2p + (1 + \mathbb{E}[X \mid T_0])q,\\ \mathbb{E}[X \mid T_0] &= (1 + \mathbb{E}[X \mid H_0])p + 2q,\\ \Rightarrow \mathbb{E}[X \mid H_0] &= \frac{2 + q^2}{1 - pq},\\ \mathbb{E}[X \mid T_0] &= \frac{2 + p^2}{1 - pq},\\ \Rightarrow \mathbb{E}[X] &= \frac{2 + q^2}{1 - pq}p + \frac{2 + p^2}{1 - pq}q\\ &= \frac{2p + q^2p + 2q + p^2q}{1 - pq}\\ &= \frac{2(p + q) + (q + p)pq}{1 - pq}\end{aligned}$$

$$= \frac{2+pq}{1-pq}.$$

Group 2 Grade:

X and Y are independent identical discrete uniform $(1, 10)$ random variables. Find the conditional PMF $P_{X,Y|A}(x, y)$. Let A denote the event that

1. $\min(X, Y) > 5$.
2. $\max(X, Y) \leq 5$.

Hint:

For discrete random variables, you can always list all the possible values and their corresponding probabilities.

Solution

Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 3
Roy D. Yates and David J. Goodman

Yates and Goodman 3e Solution Set: 7.3.1

Problem 7.3.1 Solution

X and Y each have the discrete uniform PMF

$$P_X(x) = P_Y(x) = \begin{cases} 0.1 & x = 1, 2, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The joint PMF of X and Y is

$$\begin{aligned} P_{X,Y}(x, y) &= P_X(x) P_Y(y) \\ &= \begin{cases} 0.01 & x = 1, 2, \ldots, 10; y = 1, 2, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2}$$

The event A occurs iff $X > 5$ and $Y > 5$ and has probability

$$\mathrm{P}[A] = \mathrm{P}[X > 5, Y > 5] = \sum_{x=6}^{10} \sum_{y=6}^{10} 0.01 = 0.25. \tag{3}$$

Alternatively, we could have used independence of X and Y to write $\mathrm{P}[A] = \mathrm{P}[X > 5] \mathrm{P}[Y > 5] = 1/4$. From Theorem 7.6,

$$\begin{aligned} P_{X,Y|A}(x, y) &= \begin{cases} \frac{P_{X,Y}(x,y)}{\mathrm{P}[A]} & (x, y) \in A, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 0.04 & x = 6, \ldots, 10; y = 6, \ldots, 20, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

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And the same for the second case.

Group 3 **Grade:**

A supermarket has two customers waiting to pay for their purchases at counter I and one customer waiting to pay at counter II. Let Y_1 and Y_2 denote the numbers of customers who spend more than $50 on groceries at the respective counters. Suppose that Y_1 and Y_2 are independent binomial random variables, with the probability that a customer at counter I will spend more than $50 equal to .2 and the probability that a customer at counter II will spend more than $50 equal to .3. Find the

1. joint probability distribution for Y_1 and Y_2.
2. probability that not more than one of the three customers will spend more than $50.

Solution

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\begin{aligned}
P_{Y_1}(y_1) &= \binom{2}{y_1} 0.2^{y_1} 0.8^{2-y_1}, \qquad y_1 = 0,1,2 \\
P_{Y_2}(y_2) &= \binom{1}{y_2} 0.3^{y_2} 0.7^{1-y_2}, \qquad y_2 = 0,1 \\
P_{Y_1,Y_2}(y_1,y_2) &= P_{Y_1}(y_1) P_{Y_2}(y_2) \\
&= \binom{2}{y_1} 0.2^{y_1} 0.8^{2-y_1} \binom{1}{y_2} 0.3^{y_2} 0.7^{1-y_2}, \qquad y_1 = 0,1,2 \quad y_2 = 0,1 \\
\mathbb{P}(Y_1 + Y_2 \le 1) &= \mathbb{P}(0,0) + \mathbb{P}(1,0) + \mathbb{P}(0,1) \\
&= 0.8^2 \cdot 0.7 + 2 \cdot 0.2 \cdot 0.8 \cdot 0.7 + 0.8^2 \cdot 0.3 \\
&= 0.864
\end{aligned}
$$

Group 4 **Grade:**

The length of life Y for fuses of a certain type is modeled by the exponential distribution, with rate $\lambda = 1/3$ (The measurements are in hundreds of hours.).

(a) If two such fuses have independent lengths of life Y_1 and Y_2, find the joint probability density function for Y_1 and Y_2.

(b) One fuse in item (a) is in a primary system, and the other is in a backup system that comes into use only if the primary system fails. The total effective length of life of the two fuses is then $Y_1 + Y_2$. Find $\mathbb{P}(Y_1 + Y2 \le 1)$.

Solution

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\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{Y_1}(y_1) f_{Y_2}(y_2) \\
&= \frac{1}{3} e^{-\frac{1}{3} y_1} \frac{1}{3} e^{-\frac{1}{3} y_2} \\
&= \frac{1}{9} e^{-\frac{1}{3}(y_1+y_2)} \\
\mathbb{P}(Y_1 + Y_2 \le 1) &= \int_0^1 \int_0^{1-y_1} \frac{1}{9} e^{-\frac{1}{3}(y_1+y_2)} \, \mathrm{d}y_2 \, \mathrm{d}y_1 \\
&= 1 - \frac{4}{3} e^{-\frac{1}{3}}
\end{aligned}
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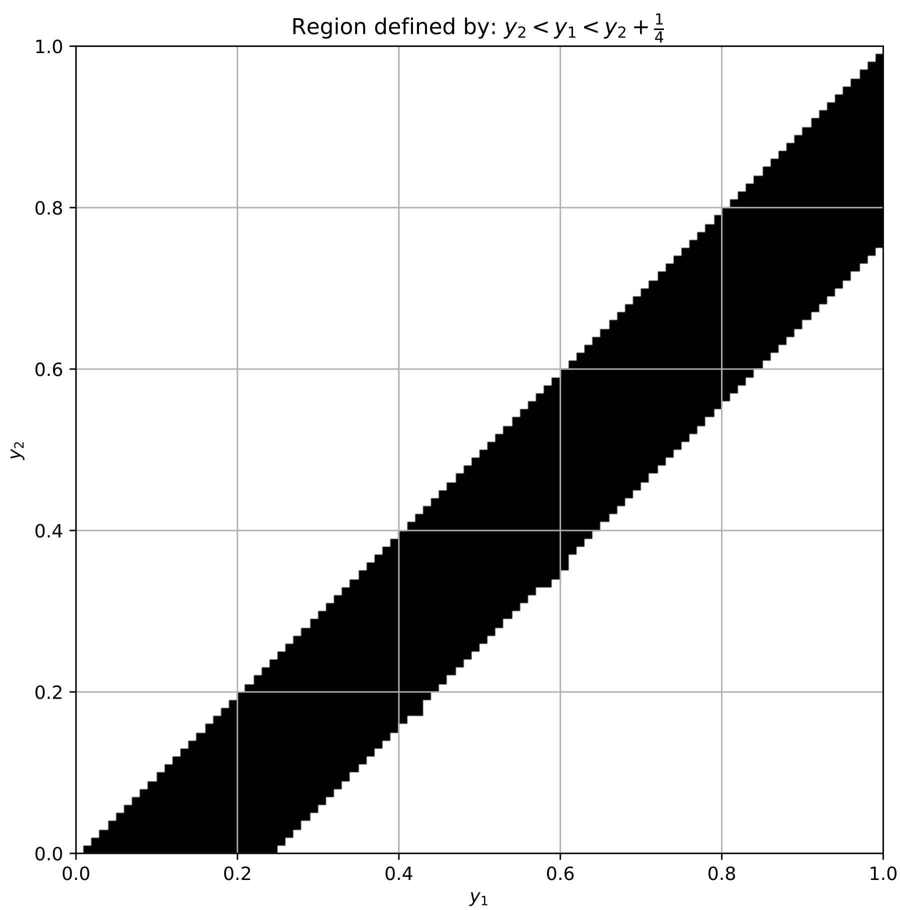
Group 5 **Grade:**

A bus arrives at a bus stop at a uniformly distributed time over the interval 0 to 1 hour. A passenger also arrives at the bus stop at a uniformly distributed time over the interval 0 to 1 hour. Assume that the arrival times of the bus and passenger are independent of one another and that the passenger will wait for up to 1/4 hour for the bus to arrive. What is the probability that the passenger will catch the bus?

Hint:

1. Let Y_1 denote the bus arrival time and Y_2 the passenger arrival time.
2. Determine the joint density of Y_1 and Y_2.
3. Find $\mathbb{P}(Y_2 \leq Y_1 \leq Y_2 + 1/4)$.

Solution



The black area is

$$1 \cdot 1 \cdot \frac{1}{2} - \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{32}.$$