Homework 2

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Q1 Grade:

Suppose that $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex function and $x \in \operatorname{dom} f$. Show that for any $d \in \mathbb{R}^n$ the function $g_d : (0, \infty) \to (-\infty, +\infty]$ defined by

$$g_d(\alpha) = \frac{f(x + \alpha d) - f(x)}{\alpha}$$

is nondecreasing.

Q2: Nonconvex Projections (similar to exercise 2.11 in the text). Grade:

Let $C \subset \mathbb{R}^n$ be a nonempty closed set (but possibly not convex), and consider any point $x \in \mathbb{R}^n$.

(a) Show that the function $g(w) \doteq \|w - x\|$ must have a nonempty, compact set of minima over C. Denote this set by $P_C(x)$.

(b) Show that $\operatorname{dist}_C(x) \doteq \inf_{w \in C} \|w - x\|$ is an everywhere finite-valued and continuous function of $x \in \mathbb{R}^n$. (If you like, you can show that it is Lipschitz continuous with modulus 1, which implies continuity.)

(c) Give an example showing that if C is not convex, dist_C need not be convex.

Q3 Grade:

Given a set $X \subseteq \mathbb{R}^n$, its *indicator function* is the function $\delta_X : \mathbb{R}^n \to (-\infty, +\infty]$ given by

$$\delta_X(x) = \begin{cases} 0 & \text{if } x \in X \\ +\infty & \text{if } x \notin X \end{cases}$$

1. Show that if X is a closed set, δ_X is a closed function.
2. Show that if X is a convex set, δ_X is a convex function.

Q4 Grade:

Suppose $K \subset \mathbb{R}^n$ is a nonempty closed convex cone and $y \notin K$. Using the separating hyperplane theorem, show that there exists a vector $a \in \mathbb{R}^n$ such that $\langle a, x \rangle \leq 0$ for all $x \in K$ and $\langle a, y \rangle > 0$ (this is equivalent to showing that there is a hyperplane separating y from K that passes through the origin).