Homework 2

Kailong Wang

November 16, 2023

Q1 Grade:

Suppose that $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex function and $x \in \operatorname{dom} f$. Show that for any $d \in \mathbb{R}^n$ the function $g_d : (0, \infty) \to (-\infty, +\infty]$ defined by

$$g_d(\alpha) = \frac{f(x + \alpha d) - f(x)}{\alpha}$$

is non-decreasing.

Solution

Since f is convex, we have

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y). \quad \forall x, y \in \mathbb{R}^n, \lambda \in [0, 1] \tag{1}$$

To show that $g_d(\alpha)$ is non-decreasing, we need to show

$$\alpha_1 \leqslant \alpha_2 \Rightarrow g_d(\alpha_1) \leqslant g_d(\alpha_2). \quad \forall \alpha_1, \alpha_2 \in (0, \infty)$$

Let $\lambda = \frac{\alpha_1}{\alpha_2} \in [0, 1]$ (because $\alpha_1 \leqslant \alpha_2$), then

$$\begin{aligned}
f(x + \alpha_1 d) = f(\frac{\alpha_1}{\alpha_2}(x + \alpha_2 d) + (1 - \frac{\alpha_1}{\alpha_2})x) \leqslant \frac{\alpha_1}{\alpha_2} f(x + \alpha_2 d) + (1 - \frac{\alpha_1}{\alpha_2})f(x) & \quad \text{by eq. (1)} \\
\Rightarrow \alpha_2 f(x + \alpha_1 d) \leqslant \alpha_1 f(x + \alpha_2 d) + (\alpha_2 - \alpha_1)f(x) & \quad \text{multiply } \alpha_2 \text{ on both sides} \\
\Rightarrow \alpha_2 f(x + \alpha_1 d) - \alpha_2 f(x) \leqslant \alpha_1 f(x + \alpha_2 d) - \alpha_1 f(x) & \quad \text{subtract } \alpha_2 f(x) \text{ on both sides} \\
\Rightarrow \frac{f(x + \alpha_1 d) - f(x)}{\alpha_1} \leqslant \frac{f(x + \alpha_2 d) - f(x)}{\alpha_2} & \quad \text{some algebra} \\
\Rightarrow g_d(\alpha_1) \leqslant g_d(\alpha_2). &
\end{aligned}$$

Therefore, $g_d(\alpha)$ is non-decreasing.

Q2: Non-convex Projections (similar to exercise 2.11 in the text). Grade:

Let $C \subset \mathbb{R}^n$ be a non-empty closed set (but possibly not convex), and consider any point $x \in \mathbb{R}^n$.

(a) Show that the function $g(w) \doteq \|w - x\|$ must have a nonempty, compact set of minima over C. Denote this set by $P_C(x)$.

(b) Show that $\operatorname{dist}_C(x) \doteq \inf_{w \in C} \|w - x\|$ is an everywhere finite-valued and continuous function of $x \in \mathbb{R}^n$. (If you like, you can show that it is Lipschitz continuous with modulus 1, which implies continuity.)

(c) Give an example showing that if C is not convex, dist_C need not be convex.

Solution

(a) *Proof.* To show that the function $g(w) = \|w - x\|$ must have a nonempty, compact set of minima over the closed set C, we can use the fact that C is nonempty and closed.

(i) If $x \in C$, then

$$\begin{aligned} \min_{w \in C} g(w) &= \|w - x\| \\ &= 0. \qquad \forall w = x \in C \end{aligned}$$

Therefore, $P_C(x) = \{ x \}$, which is nonempty and compact.

(ii) If $x \notin C$, then $g(w) = \|w - x\| > 0$ for all $w \in C$. We can then prove by contradiction. Assume that $g(w) = \|w - x\|$ does not have any minimum points within C. This means that for any point $w \in C$, there exists a sequence of points $\{ w_n \}$ such that

$$g(w_n) \leqslant g(w)$$

for all n (*i.e.* w_n gets arbitrary close to x). Since C is closed, the limit of this sequence, denoted as

$$w^* = \lim_{n \to \infty} w_n,$$

must also be in C because the limit of a sequence in a closed set belongs to that set. Moreover, since $g(w)$ is continuous, we have

$$\lim_{n \to \infty} g(w_n) = g(w^*).$$

But this would imply that $g(w^*) = 0$ (because $g(w_n)$ gets arbitrary close to 0), which means $w^* = x$. However, since $x \notin C$, we have a contradiction. Therefore, $g(w)$ must have a nonempty, compact set of minima over C (at least one minimum point).

□

(b) *Proof.* We show $\operatorname{dist}_C(x)$ is everywhere finiteness and continuous as follows:

(i) **Finiteness:** For any $x \in \mathbb{R}^n$, we have $\|w - x\| \geqslant 0$ for all $w \in C$ since the norm is always non-negative. Given that C is nonempty and closed, there exists some $w' \in C$, for any $x \in \mathbb{R}^n$, such that $\|w' - x\| \geqslant 0$. And because $g(w)$ is nonempty and compact, the $\|w' - x\|$ is finite. Since the infimum of a set of finite non-negative value is also finite non-negative, $\operatorname{dist}_C(x)$ must also be finite non-negative.

(ii) **Continuity:** Given two points $x, y \in \mathbb{R}^n$, let w^* be the point in C that achieves the infimum for x (*i.e.* $\|w^* - x\| = \operatorname{dist}_C(x)$). Then

$$\begin{aligned} \operatorname{dist}_C(y) &\leqslant \|w^* - y\| \\ &= \|(w^* - x) + (x - y)\| \\ &\leqslant \|w^* - x\| + \|x - y\| && \text{(by triangle inequality)} \\ &= \operatorname{dist}_C(x) + \|x - y\| \end{aligned}$$

By symmetry, we can also show that $\operatorname{dist}_C(x) \leqslant \operatorname{dist}_C(y) + \|x - y\|$. Therefore, we have $|\operatorname{dist}_C(x) - \operatorname{dist}_C(y)| \leqslant \|x - y\|$. This means that $\operatorname{dist}_C(x)$ is Lipschitz continuous with modulus 1, which implies continuity.

□

(c) Consider two disjoint closed balls in $\mathbb{R}^2$,

$$B_1 = \{ w \mid \|w - (0,0)\| \leqslant 1 \}$$

$$B_2 = \{ w \mid \|w - (4,0)\| \leqslant 1 \}.$$

Let $C = B_1 \cup B_2$. C is not convex since the line segment between any point in B_1 and any point in B_2 is not entirely contained in C. Consider three points: $x_1 = (0,0)$, $x_2 = (4,0)$, and $x_{mid} = (2,0)$. Clearly, $\text{dist}_C(x_1) = \text{dist}_C(x_2) = 0$, and $\text{dist}_C(x_{mid}) = 1$. Since

$$\text{dist}_C(x_{mid}) = \text{dist}_C(\frac{1}{2}x_1 + \frac{1}{2}x_2) = 1 \geqslant 0 = \frac{1}{2}\text{dist}_C(x_1) + \frac{1}{2}\text{dist}_C(x_2),$$

$\text{dist}_C(x)$ is not convex.

Q3 Grade:

Given a set $X \subseteq \mathbb{R}^n$, its *indicator function* is the function $\delta_X : \mathbb{R}^n \to (-\infty, +\infty]$ given by

$$\delta_X(x) = \begin{cases} 0 & \text{if } x \in X \\ +\infty & \text{if } x \notin X \end{cases}$$

(a) Show that if X is a closed set, δ_X is a closed function.

(b) Show that if X is a convex set, δ_X is a convex function.

Solution

(a) *Proof.* To show that δ_X is a closed function, we need to show that the epigraph of δ_X is a closed set. The epigraph of δ_X is defined as

$$\text{epi}(\delta_X) = \{ (x, \alpha) \mid \alpha \geqslant \delta_X(x) \}.$$

Since X is a closed set, we have $\delta_X(x) = 0$ for all $x \in X$ and $\delta_X(x) = +\infty$ for all $x \notin X$. Therefore, the epigraph of δ_X can be written as

$$\text{epi}(\delta_X) = \{ (x, \alpha) \mid \alpha \geqslant 0, x \in X \} \cup \{ (x, \alpha) \mid \alpha \geqslant +\infty, x \notin X \}.$$

The first set is the product of a closed set and a closed interval, which is closed. The second set is an empty set $\emptyset$. Therefore, $\text{epi}(\delta_X)$ is a union of a closed set and an empty set, which is closed. This means that δ_X is a closed function. □

(b) *Proof.* To show that δ_X is a convex function, we need to show that

$$\delta_X(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda\delta_X(x_1) + (1-\lambda)\delta_X(x_2) \quad \forall x_1, x_2 \in \mathbb{R}^n, \lambda \in [0,1]$$

(i) If $x_1, x_2 \in X$, then $\delta_X(x_1) = \delta_X(x_2) = 0$. Therefore, with X is a convex set, we have

$$\delta_X(\lambda x_1 + (1-\lambda)x_2) = 0 \leqslant 0 = \lambda\delta_X(x_1) + (1-\lambda)\delta_X(x_2).$$

(ii) If either x_1 or x_2 (or both) is not in X, then the right side of the inequality becomes infinite. Therefore, the inequality holds trivially.

This concludes that δ_X is a convex function. □

Q4 **Grade:**

Suppose $K \subset \mathbb{R}^n$ is a nonempty closed convex cone and $y \notin K$. Using the separating hyperplane theorem, show that there exists a vector $a \in \mathbb{R}^n$ such that $\langle a, x \rangle \leqslant 0$ for all $x \in K$ and $\langle a, y \rangle > 0$ (this is equivalent to showing that there is a hyperplane separating y from K that passes through the origin).

Solution

Since K is a nonempty closed convex cone and $y \notin K$, we have $K \cap \{ y \} = \emptyset$. Therefore, by the separating hyperplane theorem, there exists a vector $a \in \mathbb{R}^n$ such that $\langle a, x \rangle \leqslant 0$ for all $x \in K$ and $\langle a, y \rangle > 0$. This is equivalent to showing that there is a hyperplane separating y from K that passes through the origin. This concludes the proof.