HW 9 Solution

Group 1 **Grade:**

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability function of

- X_1, X_2.
- X_1, X_2, X_3.

Solution

$$P_{X_1X_2}(x_1x_2) = \begin{cases} \frac{5}{13}\frac{4}{12} & \text{if } x_1 = x_2 = 1 \\ \frac{5}{13}\frac{8}{12} & \text{if } x_1 = 1, x_2 = 0 \\ \frac{8}{13}\frac{5}{12} & \text{if } x_1 = 0, x_2 = 1 \\ \frac{8}{13}\frac{7}{12} & \text{if } x_1 = x_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{X_1X_2X_3}(x_1x_2x_3) = \begin{cases} \frac{5}{13}\frac{4}{12}\frac{3}{11} & \text{if } x_1 = x_2 = x_3 = 1 \\ \frac{5}{13}\frac{4}{12}\frac{8}{11} & \text{if } x_1 = x_2 = 1, x_3 = 0 \\ \frac{5}{13}\frac{8}{12}\frac{4}{11} & \text{if } x_1 = 1, x_2 = 0, x_3 = 1 \\ \frac{5}{13}\frac{8}{12}\frac{7}{11} & \text{if } x_1 = 1, x_2 = x_3 = 0 \\ \frac{8}{13}\frac{5}{12}\frac{4}{11} & \text{if } x_1 = 0, x_2 = x_3 = 1 \\ \frac{8}{13}\frac{5}{12}\frac{7}{11} & \text{if } x_1 = 0, x_2 = 1, x_3 = 0 \\ \frac{8}{13}\frac{7}{12}\frac{5}{11} & \text{if } x_1 = x_2 = 0, x_3 = 1 \\ \frac{8}{13}\frac{7}{12}\frac{6}{11} & \text{if } x_1 = x_2 = x_3 = 0 \\ 0 & \text{otherwise} \end{cases}$$

Group 2 **Grade:**

Consider a sequence of independent Bernoulli trails each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first and second successes. Find the joint probability mass function of X_1 and X_2.

Solution

$$P_{X_1X_2}(x_1x_2) = \begin{cases} p^2(1-p)^{x_1+x_2} & \text{if } x_1, x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Group 3 **Grade:**

The time that it takes to service a car is an exponential random variable with rate 1.

1. If A brings his car in at time 0 and B brings his car in at time t, what is the probability that B's car is ready before A's? Assume that the service times are independent, and the service begins upon arrival.
2. If both cars are brought in at time 0, with work starting on B's car only when A's car has been completely serviced, what is the probability that B's car is ready before 2?

Hint:

1. Find the Joint PDF of X_A and X_B.
2. $\mathbb{P}(X_A > X_B + t)$ can be written in a form of $\mathbb{P}(X_A \in (a, b), X_B \in (c, d))$.
3. Then $\mathbb{P}(X_A > X_B + t) = \int_a^b \int_c^d P_{X_A,Y_B}(x, y)\, \mathrm{d}y\, \mathrm{d}x$.
4. With the same logic, we can solve $\mathbb{P}(X_A + X_B < 2)$.

Solution

Let X_A be the service time of A's car and X_B be the service time of B's car. Due to independent, the joint PDF is

$$f_{X_A,X_B}(x_A, x_B) = \begin{cases} e^{-(x_A+x_B)} & x_A, x_B \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\mathbb{P}(X_A > X_B + t) &= \mathbb{P}(0 < X_B + t < X_A < \infty) \\
&= \int_0^\infty \int_{x_B+t}^\infty e^{-(x_A+x_B)}\, \mathrm{d}x_A\, \mathrm{d}x_B \\
&= \int_0^\infty e^{-x_B} \int_{x_B+t}^\infty e^{-x_A}\, \mathrm{d}x_A\, \mathrm{d}x_B \\
&= \int_0^\infty e^{-x_B} e^{-(x_B+t)}\, \mathrm{d}x_B \\
&= \int_0^\infty e^{-2x_B-t}\, \mathrm{d}x_B \\
&= \frac{1}{2}e^{-t}
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(X_A + X_B < 2) &= \mathbb{P}(0 < X_A < 2 - X_B < 2) \\
&= \int_0^2 \int_0^{2-x_B} e^{-(x_A+x_B)}\, \mathrm{d}x_A\, \mathrm{d}x_B \\
&= \int_0^2 e^{-x_B} \int_0^{2-x_B} e^{-x_A}\, \mathrm{d}x_A\, \mathrm{d}x_B \\
&= \int_0^2 e^{-x_B} - e^{-2}\, \mathrm{d}x_B \\
&= 1 - 3e^{-2}
\end{aligned}$$

Group 4 **Grade:**

Suppose X and Y are independent normal random variables with parameters (μ_X, σ_X) and (μ_Y, σ_Y), respectively. Find x such that $\mathbb{P}(X - Y > x) = \mathbb{P}(X + Y > a)$ for some constant a.

Hint:

1. $X - Y$ and $X + Y$ are normal random variables as well. Find their parameters.
2. Convert $X - Y$ and $X + Y$ to standard normal random variables.
3. Solve the problem with Φ function, the CDF of standard normal random variable.

Solution

$X - Y \sim \mathsf{N}(\mu_X - \mu_Y, \sqrt{\sigma_X^2 + \sigma_Y^2})$, $X + Y \sim \mathsf{N}(\mu_X + \mu_Y, \sqrt{\sigma_X^2 + \sigma_Y^2})$.

$$
\begin{aligned}
&\mathbb{P}(X - Y > x) = \mathbb{P}(X + Y > a)\\
\Rightarrow&\mathbb{P}(X - Y \leq x) = \mathbb{P}(X + Y \leq a)\\
\Rightarrow&\mathbb{P}\left(\frac{X - Y - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}} \leq \frac{x - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) = \mathbb{P}\left(\frac{X + Y - (\mu_X + \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}} \leq \frac{a - (\mu_X + \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)\\
\Rightarrow&\Phi\left(\frac{x - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) = \Phi\left(\frac{a - (\mu_X + \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)\\
\Rightarrow&\frac{x - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}} = \frac{a - (\mu_X + \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\\
\Rightarrow&x = a - 2\mu_Y
\end{aligned}
$$

Group 5 **Grade:**

Let Y_1 and Y_2 be uncorrelated random variables and consider $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$.

1. Find the $\mathrm{Cov}(U_1, U_2)$ in terms of the variances of Y_1 and Y_2.
2. Find an expression for the coefficient of correlation between U_1 and U_2.
3. Is it possible that $\mathrm{Cov}(U_1, U_2) = 0$? When does this occur?

Hint:

1. $\mathrm{Cov}(U_1, U_2) = \mathbb{E}[U_1 U_2] - \mathbb{E}[U_1]\mathbb{E}[U_2] = \ldots$.
2. $\rho = \frac{\sigma_{U_1 U_2}}{\sigma_{U_1}\sigma_{U_2}}$

Solution

$$
\begin{aligned}
\mathbb{E}[U_1] &= \mathbb{E}[Y_1 + Y_2] = \mathbb{E}[Y_1] + \mathbb{E}[Y_2]\\
\mathbb{E}[U_2] &= \mathbb{E}[Y_1 - Y_2] = \mathbb{E}[Y_1] - \mathbb{E}[Y_2]\\
\mathbb{E}[U_1 U_2] &= \mathbb{E}[(Y_1 + Y_2)(Y_1 - Y_2)] = \mathbb{E}\left[Y_1^2\right] - \mathbb{E}\left[Y_2^2\right]
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Cov}(U_1 U_2) &= \mathbb{E}[U_1 U_2] - \mathbb{E}[U_1]\mathbb{E}[U_2] \\
&= \mathbb{E}\left[Y_1^2\right] - \mathbb{E}\left[Y_2^2\right] - (\mathbb{E}[Y_1] + \mathbb{E}[Y_2])(\mathbb{E}[Y_1] - \mathbb{E}[Y_2]) \\
&= \mathbb{E}\left[Y_1^2\right] - \mathbb{E}\left[Y_2^2\right] - (\mathbb{E}[Y_1]^2 - \mathbb{E}[Y_2]^2) \\
&= \operatorname{Var}(Y_1) - \operatorname{Var}(Y_2) \\
\rho &= \frac{\operatorname{Cov}(U_1, U_2)}{\sqrt{\operatorname{Var}(U_1)\operatorname{Var}(U_2)}} \\
&= \frac{\operatorname{Var}(Y_1) - \operatorname{Var}(Y_2)}{\sqrt{\operatorname{Var}(Y_1)\operatorname{Var}(Y_2)}}
\end{aligned}
$$

When $\operatorname{Var}(Y_1) = \operatorname{Var}(Y_2)$, $\rho = 0$.