Homework 1

Kailong Wang

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Q1: Affine images and preimages of convex sets. Grade:

Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, C \subset \mathbb{R}^n, D \subset \mathbb{R}^m$ be convex sets. Show that following sets are convex.

(a) The image of C under the affine map $x \mapsto Ax + b$. That is

$$\{Ax + b \mid x \in C\} \subset \mathbb{R}^m .$$

(b) The preimage of D under the affine map $x \mapsto Ax + b$. That is

$$\{x \mid Ax + b \in D\} \subset \mathbb{R}^n .$$

Solution

(a) *Proof.* Let $x_1, x_2 \in C$ and $\lambda \in [0, 1]$, then we have

$$\begin{aligned}\lambda(Ax_1 + b) + (1-\lambda)(Ax_2 + b) &= A(\lambda x_1 + (1-\lambda)x_2) + b \\ &\in A(C) + b.\end{aligned}$$

Thus, the image of C, $A(C) + b$ is convex. □

(b) *Proof.* Let $y_1, y_2 \in A^{-1}(D - b)$ so that $Ay_1 + b \in D, Ay_2 + b \in D$ and $\lambda \in [0, 1]$, then we have

$$\begin{aligned}A(\lambda y_1 + (1-\lambda)y_2) + b &= \lambda(Ay_1 + b) + (1-\lambda)(Ay_2 + b) \\ &\in \lambda D + (1-\lambda)D \\ &= D.\end{aligned}$$

Thus, The preimages of D, $A^{-1}(D - b)$ is convex. □

Q2: Affine functions. Grade:

Suppose that $f : \mathbb{R}^n \to \mathbb{R} \setminus \{-\infty, \infty\}$ always obey the convex function relation at equality, that is,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall x, y \in \mathbb{R}^n, \lambda \in [0, 1]. \tag{1}$$

Show that

(a) If eq. (1) holds as stated for all $\lambda \in [0, 1]$, it in fact holds for all $\lambda \in \mathbb{R}$.

(b) Any f for which eq. (1) holds must be of the form $f(x) = \langle a, x\rangle + b$ for $\lambda \in \mathbb{R}^n, b \in \mathbb{R}$ (that is, f is an inner product with some fixed vector plus a constant).

(c) Any function of this form has the property eq. (1).

Hint: given f satisfying the condition above, show that $g : x \mapsto f(x) \to f(0)$ is linear. You may then use (without proof, although the proof is very easy) that a linear function $g : \mathbb{R}^n \to \mathbb{R}$ must be of the form $x \mapsto \langle a, x\rangle$ for some $a \in \mathbb{R}^n$.

Solution

Proof. (a) Let $\lambda \in \mathbb{R}$, then we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= f(\lambda x + (1-\lambda)y) \\ &\leq \lambda f(x) + (1-\lambda)f(y) \\ &= \lambda f(x) + (1-\lambda)f(y) \\ &\leq \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

Thus, eq. (1) holds for all $\lambda \in \mathbb{R}$.

(b) Let $x = 0$, then we have $f(0) \leq \lambda f(x) + (1-\lambda)f(y)$, which implies $f(0) \leq (1-\lambda)f(y)$. Let $\lambda = 0$, then we have $f(0) \leq f(y)$. Thus, $f(0) \leq f(y)$ for all $y \in \mathbb{R}^n$. Let $y = 0$, then we have $f(x) \leq \lambda f(x) + (1-\lambda)f(0)$, which implies $(1-\lambda)f(0) \geq f(x)$. Let $\lambda = 0$, then we have $f(0) \geq f(x)$. Thus, $f(0) \geq f(x)$ for all $x \in \mathbb{R}^n$. Therefore, $f(0) = f(x)$ for all $x \in \mathbb{R}^n$. Let $y = 0$, then we have $f(\lambda x) \leq \lambda f(x) + (1-\lambda)f(0) = \lambda f(x) + (1-\lambda)f(x) = f(x)$. Thus, $f(\lambda x) \leq f(x)$ for all $x \in \mathbb{R}^n$. Let $x = 0$, then we have $f(\lambda x) \leq f(0) = f(x)$. Thus, $f(\lambda x) \leq f(x)$ for all $x \in \mathbb{R}^n$. Therefore, $f(\lambda x) = f(x)$ for all $x \in \mathbb{R}^n$. Thus, f is a constant function. Let $f(x) = b$, then we have $f(x) = \langle 0, x\rangle + b$. Thus, $f(x) = \langle a, x\rangle + b$ for $\lambda \in \mathbb{R}^n, b \in \mathbb{R}$.

(c) Let $f(x) = \langle a, x\rangle + b$ for $\lambda \in \mathbb{R}^n, b \in \mathbb{R}$, then we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \langle a, \lambda x + (1-\lambda)y\rangle + b \\ &= \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle + b \\ &= \lambda(\langle a, x\rangle + b) + (1-\lambda)(\langle a, y\rangle + b) \\ &= \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

Thus, f has the property eq. (1).

□