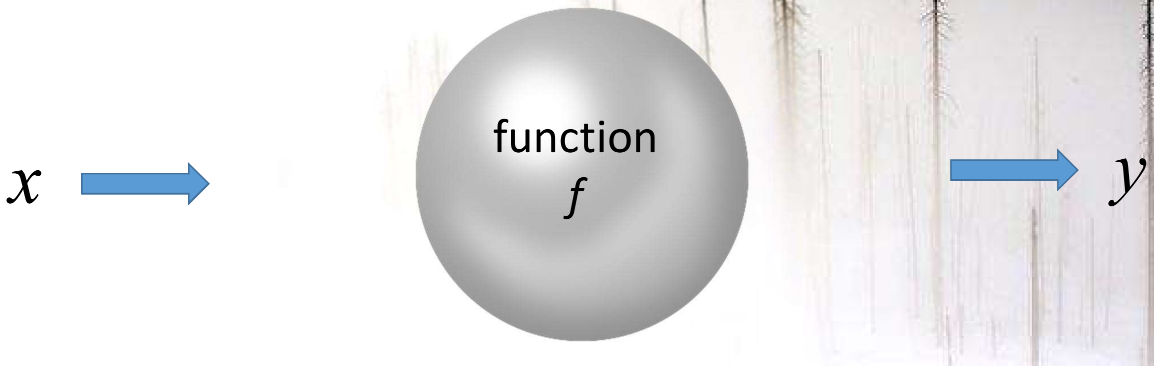
Lambda Calculus

Lambda Calculus

- Anonymous function calculus
- Functional model of computation
- **Alonzo Church** introduced **lambda calculus** in the 1930s.
  - Pure lambda calculus (‘30s)
  - Applied & Simply typed lambda calculus(‘40s)
  - Polymorphic lambda calculus (‘70s)

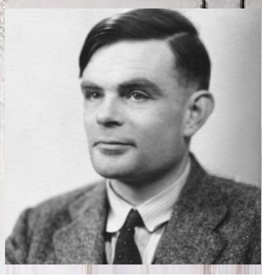
Functions

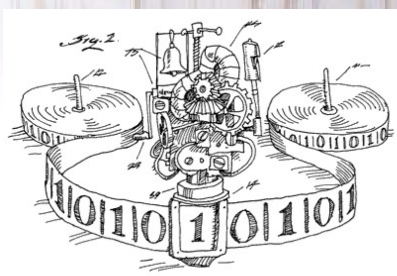


- Black Box
- Pure

Functional Model of Computation

- Express **computation** based on
  - (*anonymous*) function **abstraction** and
  - function **application** *via* **binding** and **substitution**.
- Equivalent to <u>*state-based*</u> **Turing Machine** by Alan Turing.
- Functional languages Lisp, ML, Haskell, F#, Clojure, Scala, etc. are derived from lambda calculus.





Syntax

Everything in lambda calculus is an expression (**E**).

Lambda expressions are composed of:

- variables ID such as x, y, z, ...
- the abstraction symbols lambda 'λ' and dot '.'
- parentheses ( )

if M, N ∈ E, then

rule 1: E ::= ID

rule 2: E ::= λ ID . M

rule 3: E ::= (M N)

rule 4: E ::= (E)

†Pure lambda calculus does not define constants, types, operators, etc.

Notation Simplification

- Outermost parentheses are dropped: **M N** instead of **(M N)**
- A sequence of abstractions is contracted: **λx.λy.λz.N** can be abbreviated as **λxyz.N**

Examples

if M, N ∈ E, then

rule 1: E ::= ID

rule 2: E ::= λ ID . M

rule 3: E ::= (M N)

rule 4: E ::= (E)

x

λx . x

x y

λ λx . y

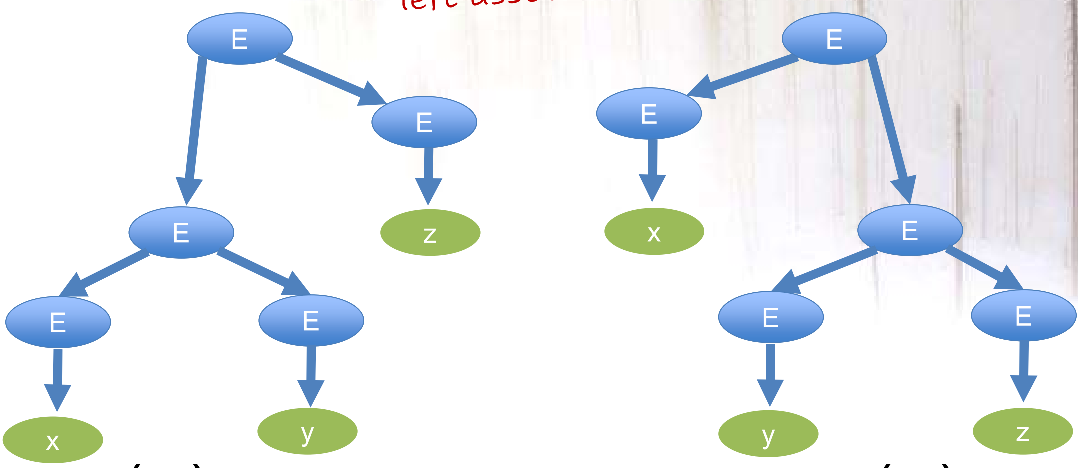
λx . y z

foo λ bar . (foo (bar baz))

Ambiguous Syntax (1)

How to parse **x y z** ?

Application is
left associative

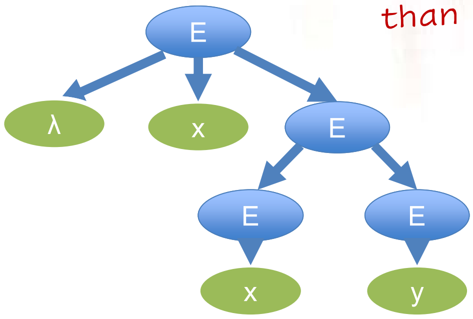


(x y) z **x (y z)**

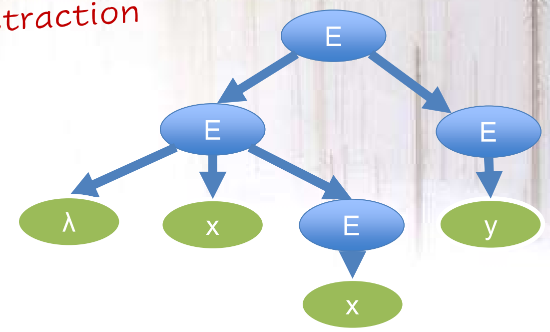
Ambiguous Syntax (2)

How to parse **λx . x y** ?

application has higher precedence than abstraction



λx . (x y)



(λx . x) y

Disambiguation Rules

Applications are assumed to be **left associative**:

M N P may be written instead of **((M N) P)**

The body of an abstraction **extends as far right** as possible:

λx . M N means …

λx . (M N) and not **(λx . M) N**

λx . λx . x is …

λx. (λx . (x))

Examples

(λx . y) x is the same as **λx . y x** ?

- No!
- **λx . y x = λx . (y x)**

λx . (x) y is the same as **λx . ((x) y)** ?

- Yes!

λa . λb . λc . a b c is the same as …

- **λa . (λb . (λc . ((a b) c)))**

Semantics

Every **ID** that we see in lambda calculus is called a **variable**.

E → λ ID . M is called an **abstraction**

The **ID** is the **variable** of the abstraction (also **bind variable**)

M is called the **body** of the abstraction

E → M N

This is called an **application**

Semantics (Cont’d)

λ ID . M defines a new **anonymous function**

- This is the reason why called "***Lambda Expressions***" in Java 8 etc.
- **ID** is the **formal parameter** of the function
- **M** is the **body** of the function

E → M N, function **application**, is similar to calling function **M** and setting its formal parameter to be **N**

Application has higher precedence than abstraction:

- **λx . A B** means **λx . (A B)**, not **(λx . A) B**

Examples

Assume that we have the function + and the constant 1.

λx . + x 1

- represents a function that adds one to its argument

Computation by Rewriting

For now, think of rewriting as *replacing all occurrences of the formal parameter x in the function with the argument*:

(λx . + x 1) 2

=> (+ 2 1)

=> 3

This process is called **β-reduction**

How can + function be defined if abstractions only accept 1 parameter?

Currying

Translate a function that takes *multiple* arguments ***into a sequence of functions*** that each take a **single** argument.

λ (x, y) . (+ x y) *// invalid λ-expression*

➔ **λx . λy . ((+ x) y)**

(λx . λy . ((+ x) y)) 1

=> λy . ((+ 1) y)

(λx . λy . ((+ x) y)) 10 20

=> (λy . ((+ 10) y)) 20

=> ((+ 10) 20) = 30

Abstract Syntax Tree

Example AST:

λ : *abstraction operator*
apply : *application operator*

(λx . x+1)3

```
        apply
        /   \
       λ     3
      / \
     x   +
        / \
       x   1
```

- The right subtree of the apply node is the **actual** argument.
- The left subtree of the apply node (with a lambda at its root) is the function.
- The left child of the lambda is the **formal** parameter.
- The right child of the lambda is the function **body**.

AST Rewriting Examples

Rewrite the AST by finding applications of functions to arguments, and for each, replacing the formal parameter with the argument in the function body.

```
      apply        =>        +
      /   \                 / \
     λ     3               3   1
    / \
   x   +
      / \
     x   1
```

AST Rewriting Examples

(λx . x + 1)((λy . y + 2) 3)

```
     apply          =>   apply     =>  apply    =>  +  =>  6
    /     \             /   \          /   \        / \
   λ       apply       λ     +        λ     5      5   1
  / \       / \       / \   / \      / \
 x   +     λ   3     x   + 3   2    x   +
    / \   / \           / \            / \
   x   1 y   +         x   1          x   1
            / \
           y   2
```

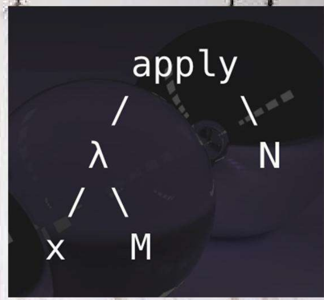
Note that this is not the only possible sequence.
More on this later!

AST Rewriting Examples

(λf . λx .f x) λy . y + 1

```
      apply       =>   λ         =>   λ       λx.x+1
     /     \          / \            / \
    λ       λ        x  apply       x   +
   / \     / \          /   \          / \
  f   λ   y   +        λ     x        x   1
     / \     / \      / \
    x apply y   1    y   +
      /  \              / \
     f    x            y   1
```

Problems with the naive rewriting rule



Simple rule for rewriting **(λx . M)N** was

*"Replace all occurrences of **x** in **M** with **N**".*

However, there are two problems with this rule.

Problem #1:

"*scope escape*" problem

=> don't want to replace all occurrences of x

Problem #2:

"*capture*" or "*name clash*" problem

Problem #1 (Scope Escape)

Consider the following:

(λx . (x + ((λx . x + 1) 3))) 2

Let's rewrite the ***inner*** expression first.

(λx . (x + ((λx . x + 1) 3))) 2

=> (λx . (x + (3 + 1))) 2

=> (λx . (x + 4)) 2

=> (2 + 4)

=> 6

Problem # 1 (Scope Escape) (Cont'd)

(λx . (x + ((λx . x + 1) 3))) 2

This time, let's rewrite the ***outer*** expression first.

(λx . (x + ((λx . x + 1) 3))) 2

=> (2 + ((λx . 2 + 1) 3))

=> (2 + (2 + 1))

=> (2 + 3)

=> 5

Problem #2

What should be the correct answer?

Consider the following:

((λx . λy . x) y) z

((λx . λy . x) y) z

=> (λy . y) z

=> z

```
          apply       ==>      apply       ==>   z
          /   \                /   \
       apply   z              λ     z
       /   \                 / \
      λ     y               y   y
     / \
    x   λ
       / \
      y   x
```

The Issue behind Problem #1 is Scoping

To understand how to fix the first problem, we need to understand **scoping**, which involves the following:

- **Bound Variable**: a variable that is associated with some lambda.
- **Free Variable**: a variable that is *not* associated with any lambda.
- Intuitively, in lambda-expression **M**, variable **x** is **bound** if, in the AST, **x** is in the subtree of a lambda with left child **x**:

```
      λ
     / \
    x  /\
      /  \
     /    \
    /..x...\
       |
     this x is bound
```


Free & Bound Variables

1. **FV(x) = { x }**

   In the expression **x**, variable **x** is free.

2. **FV(M N) = FV(M) ∪ FV(N)**

   In the expression **M N**:

   a. The free variables of **M N** are the **union** of two sets

   b. The bound variables of **M N** are also the **union** of two sets

3. **FV(λx . M) = FV(M) - x**

   In the expression **λx . M**, every **x** in **M** is bound; Every variable **y** ≠ **x** that is free in **M** is free in **λx . M**; Every variable that is bound in **M** is bound in **λx . M**.

```
      λ
     / \
    x  /\
      /  \
     /    \
    /..x...\
       |
     this x is bound
```

Free & Bound Variables (Cont'd)

The same variable can appear many times in different contexts. Some instances may be bound, others free.

```
(λx.y)(λy.yx)
     |      ||
     |      |free
   free     |
          bound
```


Free & Bound Variables (Cont'd)

λ x. ((λ y. ((λ x. * x y) 2)) x) y

free

Free Variables?

x free in λ x . x y z?

y free in λ x . x y z?

x free in (λ x . (+ x 1)) x?

z free in λ x . λ y . λ z . z y x?

x free in (λ x . z foo) (λ y . y x)?

x free in x λ x . x ?

x free in (λ x . x y) x ?

x free in λ x . y x ?

Consideration for Bound Variables

If a variable is not free, it is bound.

Bound by what abstraction?

What is the scope of a bind variable?

More on Bound Variable Rules

If **x** is free in **M**, then it is bound by **λ x .** in **λ x . M**

If **x** is bound by a particular **λ x** . in **M**, then **x** is bound by the same **λ x .** in **λ z . M** **=>** **λ z . (… (λ x . x)…)**

- Even if **z == x**
- **λ x . λ x . x**
  - Which lambda expression binds **x**?

Examples

(λ x . x (λ y . x y z y) x) x y

=> (λ **x** . **x** (λ ***y*** . **x** ***y*** <u>**z**</u> ***y***) **x**) <u>**x**</u> <u>**y**</u>

(λ x . λ y . x y) (λ z . x z)

=> (λ **x** . λ ***y*** . **x** ***y***) (λ **z** . <u>**x**</u> **z**)

(λ x . x λ x . z x)

=> (λ **x** . **x** λ **x** . <u>**z**</u> **x**)

Combinators

An expression is a **combinator** if it does not have any free variables. The expression is also said to be **closed**.

λx . λy . x y x combinator?

λx . x combinator?

λz . λx . x y z combinator?

Well-Known Combinators

I = $\lambda x.x$

K = $\lambda x.\lambda y.x$

B = $\lambda f.\lambda g.\lambda x.f\ (g\ x)$

W = $\lambda f.\lambda x.f\ x\ x$

S = $\lambda x.\lambda y.\lambda z.x\ z\ (y\ z)$

C = $\lambda f.\lambda x.\lambda y.f\ y\ x$

Y = $\lambda f.(\lambda x.f\ (x\ x))\ (\lambda x.f\ (x\ x))$

Revised Rewriting Rule

To solve problem #1 above, given lambda expression **(λx . M) N**

"**Replace all occurrences of x in M with N**."

Revised Rewriting Rule

To solve problem #1 above, given lambda expression **(λx . M) N**

"**Replace all occurrences of x <u>that are *free*</u> in M with N**."

For example:

```
      +----- M ---------+
      |                 |
(λx.  x + ((λx.x + 1)3)) 2
       |        |
       |        |
      free     bound
      in M     in M

  => 2 + ((λx.x + 1)3)
```

The Issue behind Problem #2 is Name Clash

The variable **y** that is ***free* in the argument** to a λ-expression ***becomes bound*** after rewriting, because it is put into the scope of a λ-expression with a formal parameter named **y**:

```
((λx.λy.x)y)z
           |
           |
          free, but gets bound after application
```

***free* argument becomes bound after application!**

Equivalence

What does it mean for two functions to be equivalent?

- λ y . y = λ x . x ?
- λ x . x y = λ y . y x ?
- λ x . x = λ x . x ?

Two expressions that are α-equivalent must have the same set of free variables.

α-equivalence

α-equivalence is when two functions vary only by the names of the bound variables: $M =_\alpha N$

To solve Problem #2, we need a way to rename variables in an expression:

– Simple find and replace?

– λx. x λy. x y z

- Can we rename x to foo?
- Can we rename y to bar?
- Can we rename y to x?
- Can we rename x to z?

α-conversion

as long as there is no name conflict with other variables

The basic idea is that formal parameter names are unimportant; so rename them as needed to avoid capture.

α-conversion modifies expressions of the form **λx . M** to **λz . M′ .**

Renames all the occurrences of **x** that are ***free*** in **M** to some other variable **z** that does not occur in **M** (and then **λx** is changed to **λz**).

For example,

λx . λy . x + y *alpha-reduces* to **λz . λy . z + y**

Renaming Operation (α-conversion)

E {y/x}

1. x {y/x} = y
2. z {y/x} = z, if z ≠ x
3. (**M N**) {y/x} = (**M** {y/x}) (**N** {y/x})
4. (λ x . **E**) {y/x} = (λ y . **E** {y/x}) if no y in **E**
5. (λ z . **E**) {y/x} = (λ z . **E** {y/x}), if z ≠ x

Examples

(λ x . x) {foo/x}
=> (λ foo . (x) {foo/x})
=> (λ foo . foo)

1. x {y/x} = y
2. z {y/x} = z, if z ≠ x
3. (**M N**) {y/x} = (**M** {y/x}) (**N** {y/x})
4. (λ x . **E**) {y/x} = (λ y . **E** {y/x}) if no y in **E**
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Examples

1. x {y/x} = y
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3. (**M N**) {y/x} = (**M** {y/x}) (**N** {y/x})
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5. (λ z . **E**) {y/x} = (λ z . **E** {y/x}), if z ≠ x

((λ x . x (λ y . x y z y) x) x y) {bar/x}

=> (λ x . x (λ y . x y z y) x) {bar/x} (x) {bar/x} (y) {bar/x}

=> (λ x . x (λ y . x y z y) x) {bar/x} (x) {bar/x} y

=> (λ x . x (λ y . x y z y) x) {bar/x} bar y

=> (λ bar . (x (λ y . x y z y) x) {bar/x}) bar y

=> (λ bar . (bar (λ y . x y z y) {bar/x} bar)) bar y

=> (λ bar . (bar (λ y . (x y z y) {bar/x}) bar)) bar y

=> (λ bar . (bar (λ y . (bar y z y)) bar)) bar y

α-equivalence

For all expressions **E** and all variables **y** that do not occur in **E**

$$\lambda\, x\, .\, E \;=_{\alpha}\; \lambda\, y\, .\, (E\ \{y/x\})$$

λ y . y = λ x . x ?

((λ x . x (λ y . x y z y) x) x y) = ((λ y . y (λ z . y z w z) y) y x) ?

((λ x . x (λ y . x y w y) x) x y)

((λ x. x (λ z . x z w z) x) x y)

((λ y. y (λ z . y z w z) y) x y)

Substitution

Renaming allows us to replace one variable name with another.

However, our goal is to reduce **(λ x . + x 1) 2** to **(+ 2 1)**, which replaces **x** with the expression **2**.

We need another operator, called **substitution**, to replace a variable by a lambda expression.

- **E[x→N]**, where **E** and **N** are lambda expressions and **x** is a name

Substitution

(λx . + x 1) 2

=> (+ x 1) [x→2]

=> (+ 2 1)

(λx . (λx . + x 1)) 2

=> (λx . + x 1) [x→2]

=> (λx . + x 1)

(λy . λx . y x) (λz . x z)

=> (λx . y x) [y→ λ z . x z]

=> (λx . (λz . x z) x) => trouble!

=> (λw . (λz . x z) w)

// substitution after alpha-reduction!

Substitution Rule

E [x→N]

1. x [x→N] = N
2. y [x→N] = y, if x ≠ y
3. (E_1 E_2) [x→N] = (E_1 [x→N]) (E_2 [x→N])
4. (λ x . E) [x→N] = (λ x . E)
5. (λ y . E) [x→N] = (λ y . E [x→N]) when y ≠x and y $\notin FV(N)$
6. (λ y . E) [x→N] = (λ z . E {z/y} [x→N]) when y ≠x and y $\in FV(N)$, and
   z ≠x and z $\notin FV(E, N)$

Substitution Example

1. x [x→N] = N
2. y [x→N] = y, if x ≠ y
3. (E_1 E_2) [x→N] = (E_1 [x→N]) (E_2 [x→N])
4. (λ x . E) [x→N] = (λ x . E)
5. (λ y . E) [x→N] = (λ y . E [x→N]) when y ≠x and y $\notin FV(N)$
6. (λ y . E) [x→N] = (λ z . E {z/y} [x→N]) when y ≠x and y $\in FV(N)$, and
   z ≠x and z $\notin FV(E, N)$

(λ x . x) [x→foo]
=> (λ x . x) by (4)

(+ 1 x) [x→2]
=> (+ [x→2] 1 [x→2] x [x→2]) by (3)
=> (+ 1 2) by (2) and (1)

Substitution Example

1. x [x→N] = N
2. y [x→N] = y, if x ≠ y
3. (E_1 E_2) [x→N] = (E_1 [x→N]) (E_2 [x→N])
4. (λ x . E) [x→N] = (λ x . E)
5. (λ y . E) [x→N] = (λ y . E [x→N]) when y ≠x and y ∉ $FV(N)$
6. (λ y . E) [x→N] = (λ z . E {z/y} [x→N]) when y ≠x and y ∈ $FV(N)$, and z ≠x and z ∉ $FV(E,N)$

(λ x . y x) [y→λ z . x z]

=> (λ w . (y w)) [y→λ z . x z] by (6) since x ∈ FV(λ z . x z)

=> (λ w . (y w) [y→λ z . x z]) by (5)

=> (λ w . (y [y→λ z . x z] w [y→λ z . x z]) by (3)

=> (λ w . (λ z . x z) w [y→λ z . x z]) by (1)

=> (λ w . (λ z . x z) w) by (2)

Substitution Examples

1. x [x→N] = N
2. y [x→N] = y, if x ≠ y
3. (E_1 E_2) [x→N] = (E_1 [x→N]) (E_2 [x→N])
4. (λ x . E) [x→N] = (λ x . E)
5. (λ y . E) [x→N] = (λ y . E [x→N]) when y ≠x and y ∉ $FV(N)$
6. (λ y . E) [x→N] = (λ z . E {z/y} [x→N]) when y ≠x and y ∈ $FV(N)$, and z ≠x and z ∉ $FV(E,N)$

(λy . (λf . f x) y) [x→f y]

=> λw . ((λf . f x) w) [x→f y] by (6) since y ∈ $FV(f\ y)$

=> λw . ((λf . f x) [x→f y] w [x→f y]) by (3)

=> λw . ((λf . f x) [x→f y] w) by (2)

=> λw . (λz . (z x) [x→f y]) w by (6) since f ∈ $FV(f\ y)$

=> λw . (λz . z (f y)) w by (3), (2) and (1)

Precise Meaning of Rewriting: β-reduction

Defined using **substitution** (which in turn uses **α-reduction**).

Denoted by **(λx . M) N $\rightarrow_\beta$ M [x -> N]**

The left-hand side **(λx . M) N** is called the **redex**.

The right-hand side **M[x -> N]** is called the **contractum**

The notation means **M** with all ***free*** occurrences of **x** replaced with **N** in a way that avoids capture.

We say that **(λx . M) N** beta-reduces to **M** with **N** substituted for **x**.

Normal Form

Computing with λ-expressions involves rewriting them using β-reduction.

A computation is finished when there are **no more redexes**.

A λ-expression without redexes is in normal form,

A λ-expression has a normal form *iff* there is some sequence of β-reduction (and/or expansions) that that leads to a normal form.

$$E_1 =>^* E_2$$

Examples

$(\lambda x . x)\ y$

$\quad \Rightarrow x\ [x \rightarrow y]$

$\quad \Rightarrow y$

$(\lambda x . x\ (\lambda x . x))\ (u\ r)$

$\quad \Rightarrow (x\ (\lambda x . x))\ [x \rightarrow (u\ r)]$

$\quad \Rightarrow (u\ r)\ (\lambda x . x)$

Examples

Let's try inner redex first!

$(\lambda x . y)\ ((\lambda z . z\ z)\ (\lambda w . w))$

$\quad \Rightarrow (\lambda x . y)\ ((z\ z)\ [z \rightarrow (\lambda w . w)])$

$\quad \Rightarrow (\lambda x . y)\ ((\lambda w . w)\ (\lambda w . w))$

$\quad \Rightarrow (\lambda x . y)\ ((w)\ [w \rightarrow (\lambda w . w)])$

$\quad \Rightarrow (\lambda x . y)\ (\lambda w . w)$

$\quad \Rightarrow y\ [x \rightarrow (\lambda w . w)]$

$\quad \Rightarrow y$

$(\lambda x . y)\ ((\lambda z . z\ z)\ (\lambda w . w))$

$\quad \Rightarrow y\ [x \rightarrow ((\lambda z . z\ z)\ (\lambda w . w))]$

$\quad \Rightarrow y$

η-Reduction

If **v** is a variable, **E** is a lambda expression (denoting a function), and **v** has no free occurrence in **E**,

$$\lambda v \,.\, (E\ v) \Rightarrow_{\eta} E$$

λx . (sqr x) $\Rightarrow_{\eta}$ sqr

λx . (add 5 x) $\Rightarrow_{\eta}$ (add 5).

δ-Reduction

If the lambda calculus has predefined constants (that is, if it is not pure), **rules associated with those predefined values and functions** are called δ rules.

For axample,

(add 3 5) $\Rightarrow_{\delta}$ 8 *and* (not true) $\Rightarrow_{\delta}$ false

Interesting Questions

Q1: Does every λ-expression have a normal form?

Q2: If a λ-expression does have a normal form, can we get there using only β-reductions?

Q3: If a λ-expression does have a normal form, do all choices of reduction sequences get there?

Q4: Is there a strategy for choosing β-reductions that is guaranteed to result in a normal form if one exists?

Q1: Does every λ-expression have a normal form?

- No!

(λ x . x x) (λ x . x x)

=> (x x) [x→(λ x . x x)]

=> (λ x . x x) (λ x . x x)

=> (x x) [x→(λ x . x x)]

=> (λ x . x x) (λ x . x x)

…

Q2: If a λ-expression does have a normal form, can we get there using only beta-reductions?

- Yes!
- See the "**Church-Rosser Theorem**".

Q3: If a λ-expression does have a normal form, do all choices of reduction sequences get there?

Consider the following lambda expression:

(λx.λy.y)((λz.zz)(λz.zz))

The sequence of choices that we make **can** determine whether or not we get to a normal form.

Q4: Is there a strategy for choosing β-reductions that is guaranteed to result in a normal form if one exists?

- Yes!
- **leftmost-outermost** *aka* **normal-order-reduction** **(NOR)**

Outermost and Innermost Redexes

Definition: An ***outermost redex*** is a redex that is not contained inside another one. (Similarly, an ***innermost* redex** is one that has no redexes inside it.)

In terms of the AST, an "**apply**" node represents an outermost redex *iff*

1. it represents a redex (its left child is a lambda), and
2. it has no ancestor "**apply**" node in the tree that also represents a redex.

Normal Order Reduction (NOR)

```
                            apply  <-- not a redex
                           /     \
an outermost redex --> apply      apply  <-- another outermost redex
                       /    \     /    \
                      λ     ...  λ      apply  <-- redex, but not outermost
                     / \        / \     /   \
                   ... ...    ... ... λ    ...
```

To do a normal-order reduction, always choose the **leftmost** of the **outermost** redexes.

NOR is like **call-by-name** parameter passing, where you evaluate an actual parameter only when the corresponding formal is used.

Applicative-Order Reduction (AOR)

Choose the **leftmost** of the **innermost** redexes.

AOR corresponds to **call-by-value** parameter passing: the arguments are reduced before applying the function.

The advantage of AOR is **efficiency**.

The disadvantage is that AOR **may fail to terminate** on a lambda expression that has a normal form.

Call-by-Need: Best of Both World

- **Call-by-need** is like call-by-name in that **an actual parameter is only evaluated when the corresponding formal is used** …
- However, the difference is that when using call-by-need, **the result of the evaluation is saved and is then reused** for each subsequent use of the formal.