2. Mathematical Preliminaries (Types and Programming Languages)

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Week 2

2.1 Sets, Relations, and Functions: Sets

Standard notation for sets

- by enumeration, e.g., as $\{ 0, 2, 4, 6, \cdots \}$, or
- by *comprehension*, e.g., as $\{x \in \mathbb{N} \mid x \textit{ is even}\}$ where $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, 3, \cdots\}$.

$\emptyset$ for the empty set

$S \backslash T$ for the set difference of S and T

$|S|$ for the size of a set S

$\mathcal{P}(S)$ for the powerset of S, i.e., the set of all the subsets of S.

- Q. $\mathcal{P}(\{1, 2, 3\}) =$

2.1 Sets, Relations, and Functions: Relations

An *n*-place *relation* over sets S_1, S_2, ... , S_n is a set $R \subseteq S_1 \times S_2 \times \cdots \times S_n$ of tuples of elements from S_1 through S_n.

- We say that the elements $s_1 \in S_1$ through $s_n \in S_n$ are *related by R* if the tuple $(s_1, \cdots, s_n) \in R$.

Q. By $MyRelation = \{(1,2),(2,4),(3,6),(4,8),(5,10),\cdots\}$, how the elements are related?

Q. Define your relation and explain how elements are related by your relation?

2.1 Sets, Relations, and Functions: Relations

A 1-place relation on a set S is called *a predicate on S*.

- $Even = \{(0), (2), (4), \cdots\} \simeq \{0, 2, 4, \cdots\} \subseteq \mathbb{N}$ is a relation.
- $Even$ is regarded as a predicate on $\mathbb{N}$ as $Even(elem)$ is true iff $elem \in Even$. E.g., $Even(2)$ is true, $Even(3)$ is false, and so on.

A 2-place relation on S and T is called *a binary relation*. We often write $s\ R\ t$ instead of $(s, t) \in R$.

A 2-place relation on the same sets S and S is simply called *a binary relation on S*.

For readability, three- or more place relations are often written using a "mixfix" concrete syntax. For example, for the typing relation for the simply typed lambda calculus in Ch.9, we write $\Gamma \vdash s : T$ to mean "the triple (Γ, s, T) is in the typing relation."
E.g., $(x : int, x + 1, int) \in TypeSystem$ but
$(x : bool, x + 1, ?) \notin TypeSystem$

2.1 Sets, Relations, and Functions: Functions

Suppose a relation R on sets S and T. The *domain* of R, written $dom(R)$, is $\{s \mid (s,t) \in R\}$, and the *range* of R, written $range(R)$, is $\{t \mid (s,t) \in R\}$. E.g., $dom(MyRelation)$ and $range(MyRelation)$.

Such a relation R is called a *(partial) function* from S to T if $(s,t_1) \in R$ and $(s,t_2) \in R$, we have $t_1 = t_2$.

Such a relation R is called a *total function* if R is a partial function and $dom(R) = S$.

A partial function R is is *defined* on s if $s \in dom(R)$, which is denoted by $R(x)\downarrow$. Otherewise it is undefined, denoted by $R(x)\uparrow$.

Q. Show an example of a partial function that is not a total one.

2.2 Ordered Sets

Suppose a binary relation R on a set S.

Q. Find out three definitions that R is *reflexive*, R is *symmetric*, and R is *transitive*, respectively.

R is an *equivalence* on S if it is reflexive, symmetric, and transitive.

The *reflexive closure* of R is $R \cup \{(s, s) \mid s \in S\}$.

The *transitive closure* of R is $R^+ = \bigcup_i R_i$ where

$$\begin{array}{rcl} R_0 & = & R \\ R_{i+1} & = & R_i \cup \{(s, u) \mid (s, t) \in R_i and (t, u) \in R_i \ for\ some\ t\ \} \end{array}$$

The *reflexive and transitive closure* of R is R^*.

Q. $MyRelation^*$

2.2 Ordered Sets

A predicate P on a set S is *preserved by* a binary relation R on S

$$\begin{array}{rcl} P & : & S \\ R & : & S \times S \end{array}$$

for all $s_1 \ R \ s_2$, if $\mathcal{P}(s_1)$ then $\mathcal{P}(s_2)$.

Q. Show a concrete example of P and S.

Q. Show that if P is preserved by R, then P is also preserved by R^*.

2.3 Sequences

A *sequence* is written by listing its elements, separated by commas, such as 1,2,3.

Suppose a is 3,2,1 and b is 5,6.

- 0,a denotes 0,3,2,1
- a,0 denotes 3,2,1,0
- b,a denotes 5,6,3,2,1

$1..n$ is the sequence of numbers from 1 to n

$|a|$ is the length of the sequence a.

The empty sequence is written either as • or as a blank.

Q. Write down all *permutation*s of the sequence 1,2,3.

2.4 Induction

To prove the infinite cases of a predicate, principles of induction may be useful.

[*Principle of Induction on Natural Numbers*] Suppose a predicate $\mathcal{P}$ on the natural numbers $\mathbb{N}$.

If $\mathcal{P}(0)$

and, for all $i \in \mathbb{N}$, $\mathcal{P}(i)$ implies $\mathcal{P}(i+1)$,

then $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

2.4 Induction

To prove "infinite cases", *principles of induction* may be useful.

Q. Let $\mathcal{P}_j \triangleq \sum_{i=0}^{j} i = \frac{j(j+1)}{2}$. We want to show $\mathcal{P}_j$ is true for all $j \in \mathbb{N}$.

1 (Base case) Prove $\mathcal{P}_0$ is true.

2-1 Assume $\mathcal{P}_4$ is true. Then show $\mathcal{P}_5$ is provable using the assumption.

2-2 (Inductive case) Generalize (2-1). Assume $\mathcal{P}_k$ is true for an arbitrary natural number k. Then show $\mathcal{P}_{k+1}$ is true using the assumption.

3 Verify you have proved $\mathcal{P}_j$ for all (infinite cases) $j \in \mathbb{N}$ by (1) and (2-2).

It is enough to prove the base and the inductive cases rather than to prove the infinite ones!

2.4 Induction

[*Principle of Induction on Natural Numbers*] Suppose a predicate $\mathcal{P}$ on the natural numbers $\mathbb{N}$.

If $\mathcal{P}(0)$

and, for all $i \in \mathbb{N}$, $\mathcal{P}(i)$ implies $\mathcal{P}(i+1)$,

then $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

Q. Explain why a proof in the previous exercise can be viewed as the principle of induction.

2.4 Induction

Sometimes we need another principle stronger than the "plain" principle of induction.

Q. We want to show that any positive number integer n greater than or equal to 2 is either a prime or a product of primes.

- Let $\mathcal{P}_j$ be either j is a prime or j is a product of primes.

1 (Base case) Show $\mathcal{P}_2$ is true.

2-1 Assume $\mathcal{P}_{16}$ and $\mathcal{P}_{64}$ are true. Then show $\mathcal{P}_{1024}$ is provable using the assumptions.

2-2 (Inductive case) Generalize (2-1). Assume $\mathcal{P}_i$ is true for an arbitrary natural number $i \leq k$. Then show $\mathcal{P}_{k+1}$ is true using the assumptions.

3 Verify you have proved $\mathcal{P}_j$ for all (infinite cases) $2 \leq j$ by (1) and (2-2).

To prove $\mathcal{P}_{k+1}$, not only we may need $\mathcal{P}_k$ but we may also need more as: $\mathcal{P}_k$, $\mathcal{P}_{k-1}$, $\mathcal{P}_{k-2}$, $\mathcal{P}_{k-3}$, ... , $\mathcal{P}_2$.

2.4 Induction

[*Principle of the complete induction on Natural Numbers*] Suppose a predicate $\mathcal{P}$ on the natural numbers $\mathbb{N}$.

If $\mathcal{P}(0)$

and, for all $i \in \mathbb{N}$, $\mathcal{P}(j)$ for all $j \leq i$ implies $\mathcal{P}(i+1)$,

then $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

More induction principles exist, for example, as

- the principle of lexicographic induction
- the principle of structural induction