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# **Sage 9.2 Reference Manual: Curves**

***Release 9.2***

**The Sage Development Team**

**Oct 25, 2020**



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## CURVE CONSTRUCTOR

Curves are constructed through the curve constructor, after an ambient space is defined either explicitly or implicitly.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: Curve([y - x^2], A)
Affine Plane Curve over Rational Field defined by -x^2 + y
```

```
sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: Curve(y^2*z^7 - x^9 - x*z^8)
Projective Plane Curve over Finite Field of size 5 defined by -x^9 + y^2*z^7 - x*z^8
```

AUTHORS:

- William Stein (2005-11-13)
- David Kohel (2006-01)
- Grayson Jorgenson (2016-06)

`sage.schemes.curves.constructor.Curve(F, A=None)`

Return the plane or space curve defined by  $F$ , where  $F$  can be either a multivariate polynomial, a list or tuple of polynomials, or an algebraic scheme.

If no ambient space is passed in for  $A$ , and if  $F$  is not an algebraic scheme, a new ambient space is constructed.

Also not specifying an ambient space will cause the curve to be defined in either affine or projective space based on properties of  $F$ . In particular, if  $F$  contains a nonhomogeneous polynomial, the curve is affine, and if  $F$  consists of homogeneous polynomials, then the curve is projective.

INPUT:

- $F$  – a multivariate polynomial, or a list or tuple of polynomials, or an algebraic scheme.
- $A$  – (default: None) an ambient space in which to create the curve.

EXAMPLES: A projective plane curve.

```
sage: x,y,z = QQ['x,y,z'].gens()
sage: C = Curve(x^3 + y^3 + z^3); C
Projective Plane Curve over Rational Field defined by x^3 + y^3 + z^3
sage: C.genus()
1
```

Affine plane curves.

```

sage: x,y = GF(7)['x,y'].gens()
sage: C = Curve(y^2 + x^3 + x^10); C
Affine Plane Curve over Finite Field of size 7 defined by x^10 + x^3 + y^2
sage: C.genus()
0
sage: x, y = QQ['x,y'].gens()
sage: Curve(x^3 + y^3 + 1)
Affine Plane Curve over Rational Field defined by x^3 + y^3 + 1

```

A projective space curve.

```

sage: x,y,z,w = QQ['x,y,z,w'].gens()
sage: C = Curve([x^3 + y^3 - z^3 - w^3, x^5 - y*z^4]); C
Projective Curve over Rational Field defined by x^3 + y^3 - z^3 - w^3, x^5 - y*z^4
sage: C.genus()
13

```

An affine space curve.

```

sage: x,y,z = QQ['x,y,z'].gens()
sage: C = Curve([y^2 + x^3 + x^10 + z^7, x^2 + y^2]); C
Affine Curve over Rational Field defined by x^10 + z^7 + x^3 + y^2, x^2 + y^2
sage: C.genus()
47

```

We can also make non-reduced non-irreducible curves.

```

sage: x,y,z = QQ['x,y,z'].gens()
sage: Curve((x-y)*(x+y))
Projective Conic Curve over Rational Field defined by x^2 - y^2
sage: Curve((x-y)^2*(x+y)^2)
Projective Plane Curve over Rational Field defined by x^4 - 2*x^2*y^2 + y^4

```

A union of curves is a curve.

```

sage: x,y,z = QQ['x,y,z'].gens()
sage: C = Curve(x^3 + y^3 + z^3)
sage: D = Curve(x^4 + y^4 + z^4)
sage: C.union(D)
Projective Plane Curve over Rational Field defined by
x^7 + x^4*y^3 + x^3*y^4 + y^7 + x^4*z^3 + y^4*z^3 + x^3*z^4 + y^3*z^4 + z^7

```

The intersection is not a curve, though it is a scheme.

```

sage: X = C.intersection(D); X
Closed subscheme of Projective Space of dimension 2 over Rational Field defined
by:
x^3 + y^3 + z^3,
x^4 + y^4 + z^4

```

Note that the intersection has dimension 0.

```

sage: X.dimension()
0
sage: I = X.defining_ideal(); I
Ideal (x^3 + y^3 + z^3, x^4 + y^4 + z^4) of Multivariate Polynomial Ring in x, y,
z over Rational Field

```

If only a polynomial in three variables is given, then it must be homogeneous such that a projective curve is constructed.

```
sage: x,y,z = QQ['x,y,z'].gens()
sage: Curve(x^2+y^2)
Projective Conic Curve over Rational Field defined by x^2 + y^2
sage: Curve(x^2+y^2+z)
Traceback (most recent call last):
...
TypeError: x^2 + y^2 + z is not a homogeneous polynomial
```

An ambient space can be specified to construct a space curve in an affine or a projective space.

```
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: C = Curve([y - x^2, z - x^3], A)
sage: C
Affine Curve over Rational Field defined by -x^2 + y, -x^3 + z
sage: A == C.ambient_space()
True
```

The defining polynomial must be nonzero unless the ambient space itself is of dimension 1.

```
sage: P1.<x,y> = ProjectiveSpace(1, GF(5))
sage: S = P1.coordinate_ring()
sage: Curve(S(0), P1)
Projective Line over Finite Field of size 5
sage: Curve(P1)
Projective Line over Finite Field of size 5
```

```
sage: A1.<x> = AffineSpace(1, QQ)
sage: R = A1.coordinate_ring()
sage: Curve(R(0), A1)
Affine Line over Rational Field
sage: Curve(A1)
Affine Line over Rational Field
```





## BASE CLASS OF CURVES

This module defines the base class of curves in Sage.

Curves in Sage are reduced subschemes of dimension 1 of an ambient space. The ambient space is either an affine space or a projective space.

EXAMPLES:

```
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: C = Curve([x - y, z - 2])
sage: C
Affine Curve over Rational Field defined by x - y, z - 2
sage: C.dimension()
1
```

AUTHORS:

- William Stein (2005)

**class** sage.schemes.curves.curve.**Curve\_generic**(*A, polynomials*)

Bases: sage.schemes.generic.algebraic\_scheme.AlgebraicScheme\_subscheme

Generic curve class.

EXAMPLES:

```
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: C = Curve([x-y, z-2])
sage: loads(C.dumps()) == C
True
```

**change\_ring**(*R*)

Return a new curve which is this curve coerced to R.

INPUT:

- R – ring or embedding

OUTPUT: a new curve which is this curve coerced to R

EXAMPLES:

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([x^2 - y^2, z*y - 4/5*w^2], P)
sage: C.change_ring(QuadraticField(-1))
Projective Curve over Number Field in a with defining polynomial x^2 + 1 with
↪ a = 1*I defined by x^2 - y^2, y*z - 4/5*w^2
```

```

sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^3 + a^2 - 1)
sage: A.<x,y> = AffineSpace(K, 2)
sage: C = Curve([K.0*x^2 - x + y^3 - 11], A)
sage: L = K.embeddings(QQbar)
sage: set_verbose(-1) # suppress warnings for slow computation
sage: C.change_ring(L[0])
Affine Plane Curve over Algebraic Field defined by y^3 +
(-0.8774388331233464? - 0.744861766619745?*I)*x^2 - x - 11

```

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = P.curve([y*x - 18*x^2 + 17*z^2])
sage: C.change_ring(GF(17))
Projective Plane Curve over Finite Field of size 17 defined by -x^2 + x*y

```

**defining\_polynomial()**

Return the defining polynomial of the curve.

EXAMPLES:

```

sage: x,y,z = PolynomialRing(QQ, 3, names='x,y,z').gens()
sage: C = Curve(y^2*z - x^3 - 17*x*z^2 + y*z^2)
sage: C.defining_polynomial()
-x^3 + y^2*z - 17*x*z^2 + y*z^2

```

**divisor(v, base\_ring=None, check=True, reduce=True)**

Return the divisor specified by  $v$ .

**Warning:** The coefficients of the divisor must be in the base ring and the terms must be reduced. If you set `check=False` and/or `reduce=False` it is your responsibility to pass a valid object  $v$ .

EXAMPLES:

```

sage: x,y,z = PolynomialRing(QQ, 3, names='x,y,z').gens()
sage: C = Curve(y^2*z - x^3 - 17*x*z^2 + y*z^2)

```

**divisor\_group(base\_ring=None)**

Return the divisor group of the curve.

INPUT:

- `base_ring` – the base ring of the divisor group. Usually, this is  $\mathbf{Z}$  (default) or  $\mathbf{Q}$ .

OUTPUT: the divisor group of the curve

EXAMPLES:

```

sage: x,y,z = PolynomialRing(QQ, 3, names='x,y,z').gens()
sage: C = Curve(y^2*z - x^3 - 17*x*z^2 + y*z^2)
sage: Cp = Curve(y^2*z - x^3 - 17*x*z^2 + y*z^2)
sage: C.divisor_group() is Cp.divisor_group()
True

```

**genus()**

Return the geometric genus of the curve.

EXAMPLES:

```

sage: x,y,z = PolynomialRing(QQ, 3, names='x,y,z').gens()
sage: C = Curve(y^2*z - x^3 - 17*x*z^2 + y*z^2)
sage: C.genus()
1

```

**geometric\_genus()**

Return the geometric genus of the curve.

This is by definition the genus of the normalization of the projective closure of the curve over the algebraic closure of the base field; the base field must be a prime field.

---

**Note:** This calls Singular's genus command.

---

**EXAMPLES:**

Examples of projective curves.

```

sage: P2 = ProjectiveSpace(2, GF(5), names=['x','y','z'])
sage: x, y, z = P2.coordinate_ring().gens()
sage: C = Curve(y^2*z - x^3 - 17*x*z^2 + y*z^2)
sage: C.geometric_genus()
1
sage: C = Curve(y^2*z - x^3)
sage: C.geometric_genus()
0
sage: C = Curve(x^10 + y^7*z^3 + z^10)
sage: C.geometric_genus()
3

```

Examples of affine curves.

```

sage: x, y = PolynomialRing(GF(5), 2, 'xy').gens()
sage: C = Curve(y^2 - x^3 - 17*x + y)
sage: C.geometric_genus()
1
sage: C = Curve(y^2 - x^3)
sage: C.geometric_genus()
0
sage: C = Curve(x^10 + y^7 + 1)
sage: C.geometric_genus()
3

```

**intersection\_points(C, F=None)**

Return the points in the intersection of this curve and the curve C.

If the intersection of these two curves has dimension greater than zero, and if the base ring of this curve is not a finite field, then an error is returned.

INPUT:

- C – a curve in the same ambient space as this curve
- F – (default: None); field over which to compute the intersection points; if not specified, the base ring of this curve is used

OUTPUT: a list of points in the ambient space of this curve

EXAMPLES:

```

sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^2 + a + 1)
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([y^2 - w*z, w^3 - y^3], P)
sage: D = Curve([x*y - w*z, z^3 - y^3], P)
sage: C.intersection_points(D, F=K)
[(-b - 1 : -b - 1 : b : 1), (b : b : -b - 1 : 1), (1 : 0 : 0 : 0),
(1 : 1 : 1 : 1)]

```

```

sage: A.<x,y> = AffineSpace(GF(7), 2)
sage: C = Curve([y^3 - x^3], A)
sage: D = Curve([-x*y^3 + y^4 - 2*x^3 + 2*x^2*y], A)
sage: C.intersection_points(D)
[(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 3), (5, 5), (5, 6),
(6, 6)]

```

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y^3 - x^3], A)
sage: D = Curve([-x*y^3 + y^4 - 2*x^3 + 2*x^2*y], A)
sage: C.intersection_points(D)
Traceback (most recent call last):
...
NotImplementedError: the intersection must have dimension zero or
(=Rational Field) must be a finite field

```

**intersects\_at(C, P)**

Return whether the point P is or is not in the intersection of this curve with the curve C.

INPUT:

- C – a curve in the same ambient space as this curve.
- P – a point in the ambient space of this curve.

EXAMPLES:

```

sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([x^2 - z^2, y^3 - w*x^2], P)
sage: D = Curve([w^2 - 2*x*y + z^2, y^2 - w^2], P)
sage: Q1 = P([1,1,-1,1])
sage: C.intersects_at(D, Q1)
True
sage: Q2 = P([0,0,1,-1])
sage: C.intersects_at(D, Q2)
False

```

```

sage: A.<x,y> = AffineSpace(GF(13), 2)
sage: C = Curve([y + 12*x^5 + 3*x^3 + 7], A)
sage: D = Curve([y^2 + 7*x^2 + 8], A)
sage: Q1 = A([9,6])
sage: C.intersects_at(D, Q1)
True
sage: Q2 = A([3,7])
sage: C.intersects_at(D, Q2)
False

```

**is\_singular(P=None)**

Return whether P is a singular point of this curve, or if no point is passed, whether this curve is singular

or not.

This just uses the `is_smooth` function for algebraic subschemes.

INPUT:

- `P` – (default: `None`) a point on this curve

OUTPUT:

A boolean. If a point `P` is provided, and if `P` lies on this curve, returns `True` if `P` is a singular point of this curve, and `False` otherwise. If no point is provided, returns `True` or `False` depending on whether this curve is or is not singular, respectively.

EXAMPLES:

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = P.curve([y^2 - x^2 - z^2, z - w])
sage: C.is_singular()
False
```

```
sage: A.<x,y,z> = AffineSpace(GF(11), 3)
sage: C = A.curve([y^3 - z^5, x^5 - y + 1])
sage: Q = A([7,0,0])
sage: C.is_singular(Q)
True
```

**`singular_points(F=None)`**

Return the set of singular points of this curve.

INPUT:

- `F` – (default: `None`) field over which to find the singular points; if not given, the base ring of this curve is used

OUTPUT: a list of points in the ambient space of this curve

EXAMPLES:

```
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: C = Curve([y^2 - x^5, x - z], A)
sage: C.singular_points()
[(0, 0, 0)]
```

```
sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^8 - a^4 + 1)
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([359/12*x*y^2*z^2 + 2*y*z^4 + 187/12*y^3*z^2 + x*z^4\
+ 67/3*x^2*y*z^2 + 117/4*y^5 + 9*x^5 + 6*x^3*z^2 + 393/4*x*y^4\
+ 145*x^2*y^3 + 115*x^3*y^2 + 49*x^4*y], P)
sage: sorted(C.singular_points(K), key=str)
[(-1/2*b^5 - 1/2*b^3 + 1/2*b - 1 : 1 : 0),
 (-2/3*b^4 + 1/3 : 0 : 1),
 (-b^6 : b^6 : 1),
 (1/2*b^5 + 1/2*b^3 - 1/2*b - 1 : 1 : 0),
 (2/3*b^4 - 1/3 : 0 : 1),
 (b^6 : -b^6 : 1)]
```

**`singular_subscheme()`**

Return the subscheme of singular points of this curve.

OUTPUT:

- a subscheme in the ambient space of this curve.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(CC, 2)
sage: C = Curve([y^4 - 2*x^5 - x^2*y], A)
sage: C.singular_subscheme()
Closed subscheme of Affine Space of dimension 2 over Complex Field
with 53 bits of precision defined by:
(-2.000000000000000)*x^5 + y^4 - x^2*y,
(-10.000000000000000)*x^4 + (-2.000000000000000)*x*y,
4.000000000000000*y^3 - x^2
```

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([y^8 - x^2*z*w^5, w^2 - 2*y^2 - x*z], P)
sage: C.singular_subscheme()
Closed subscheme of Projective Space of dimension 3 over Rational
Field defined by:
y^8 - x^2*z*w^5,
-2*y^2 - x*z + w^2,
-x^3*y*z^4 + 3*x^2*y*z^3*w^2 - 3*x*y*z^2*w^4 + 8*x*y*z*w^5 + y*z*w^6,
x^2*z*w^5,
-5*x^2*z^2*w^4 - 4*x*z*w^6,
x^4*y*z^3 - 3*x^3*y*z^2*w^2 + 3*x^2*y*z*w^4 - 4*x^2*y*w^5 - x*y*w^6,
-2*x^3*y*z^3*w + 6*x^2*y*z^2*w^3 - 20*x^2*y*z*w^4 - 6*x*y*z*w^5 +
2*y*w^7,
-5*x^3*z*w^4 - 2*x^2*w^6
```

**union** (*other*)

Return the union of self and other.

EXAMPLES:

```
sage: x,y,z = PolynomialRing(QQ, 3, names='x,y,z').gens()
sage: C1 = Curve(z - x)
sage: C2 = Curve(y - x)
sage: C1.union(C2).defining_polynomial()
x^2 - x*y - x*z + y*z
```

## AFFINE CURVES

Affine curves in Sage are curves in an affine space or an affine plane.

EXAMPLES:

We can construct curves in either an affine plane:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y - x^2], A); C
Affine Plane Curve over Rational Field defined by -x^2 + y
```

or in higher dimensional affine space:

```
sage: A.<x,y,z,w> = AffineSpace(QQ, 4)
sage: C = Curve([y - x^2, z - w^3, w - y^4], A); C
Affine Curve over Rational Field defined by -x^2 + y, -w^3 + z, -y^4 + w
```

### 3.1 Integral affine curves over finite fields

If the curve is defined over a finite field and integral, that is reduced and irreducible, its function field is tightly coupled with the curve so that advanced computations based on Sage's global function field machinery are available.

EXAMPLES:

```
sage: k.<a> = GF(2)
sage: A.<x,y,z> = AffineSpace(k, 3)
sage: C = Curve([x^2 + x - y^3, y^4 - y - z^3], A)
sage: C.genus()
10
sage: C.function_field()
Function field in z defined by z^9 + x^8 + x^6 + x^5 + x^4 + x^3 + x
```

Closed points of arbitrary degree can be computed:

```
sage: C.closed_points()
[Point (x, y, z), Point (x + 1, y, z)]
sage: C.closed_points(2)
[Point (x^2 + x + 1, y + 1, z),
 Point (y^2 + y + 1, x + y, z),
 Point (y^2 + y + 1, x + y + 1, z)]
sage: p = _[0]
sage: p.places()
[Place (x^2 + x + 1, (1/(x^4 + x^2 + 1))*z^7 + (1/(x^4 + x^2 + 1))*z^6 + 1)]
```

The places at infinity correspond to the extra closed points of the curve's projective closure:

```
sage: C.places_at_infinity()
[Place (1/x, 1/x*z)]
```

It is easy to transit to and from the function field of the curve:

```
sage: fx = C(x)
sage: fy = C(y)
sage: fx^2 + fx - fy^3
0
sage: fx.divisor()
-9*Place (1/x, 1/x*z)
+ 9*Place (x, z)
sage: p, = fx.zeros()
sage: C.place_to_closed_point(p)
Point (x, y, z)
sage: _.rational_point()
(0, 0, 0)
sage: _.closed_point()
Point (x, y, z)
sage: _.place()
Place (x, z)
```

## 3.2 Integral affine curves over $\mathbb{Q}$

An integral curve over  $\mathbb{Q}$  is equipped also with the function field. Unlike over finite fields, it is not possible to enumerate closed points.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve(x^2 + y^2 - 1)
sage: p = C(0,1)
sage: p
(0, 1)
sage: p.closed_point()
Point (x, y - 1)
sage: p1 = _.place()
sage: C.parametric_representation(p1)
(s + ..., 1 - 1/2*s^2 - 1/8*s^4 - 1/16*s^6 + ...)
sage: sx, sy = _
sage: sx = sx.polynomial(10); sx
s
sage: sy = sy.polynomial(10); sy
-7/256*s^10 - 5/128*s^8 - 1/16*s^6 - 1/8*s^4 - 1/2*s^2 + 1
sage: s = var('s')
sage: P1 = parametric_plot([sx, sy], (s, -1, 1), color='red')
sage: P2 = C.plot((x, -1, 1), (y, 0, 2)) # half circle
sage: P1 + P2
Graphics object consisting of 2 graphics primitives
```

AUTHORS:

- William Stein (2005-11-13)
- David Joyner (2005-11-13)



- David Kohel (2006-01)
- Grayson Jorgenson (2016-08)
- Kwankyu Lee (2019-05): added integral affine curves

**class** sage.schemes.curves.affine\_curve.**AffineCurve**(A, X)  
 Bases: `sage.schemes.curves.curve.Curve_generic`, `sage.schemes.affine.affine_subscheme.AlgebraicScheme_subscheme_affine`

Affine curves.

EXAMPLES:

```
sage: R.<v> = QQ[]
sage: K.<u> = NumberField(v^2 + 3)
sage: A.<x,y,z> = AffineSpace(K, 3)
sage: C = Curve([z - u*x^2, y^2], A); C
Affine Curve over Number Field in u with defining polynomial v^2 + 3
defined by (-u)*x^2 + z, y^2
```

```
sage: A.<x,y,z> = AffineSpace(GF(7), 3)
sage: C = Curve([x^2 - z, z - 8*x], A); C
Affine Curve over Finite Field of size 7 defined by x^2 - z, -x + z
```

**projective\_closure**(i=0, PP=None)

Return the projective closure of this affine curve.

INPUT:

- i – (default: 0) the index of the affine coordinate chart of the projective space that the affine ambient space of this curve embeds into.
- PP – (default: None) ambient projective space to compute the projective closure in. This is constructed if it is not given.

OUTPUT:

- a curve in projective space.

EXAMPLES:

```
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: C = Curve([y-x^2, z-x^3], A)
sage: C.projective_closure()
Projective Curve over Rational Field defined by x1^2 - x0*x2,
x1*x2 - x0*x3, x2^2 - x1*x3
```

```
sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: C = Curve([y - x^2, z - x^3], A)
sage: C.projective_closure()
Projective Curve over Rational Field defined by
x1^2 - x0*x2, x1*x2 - x0*x3, x2^2 - x1*x3
```

```
sage: A.<x,y> = AffineSpace(CC, 2)
sage: C = Curve(y - x^3 + x - 1, A)
sage: C.projective_closure(1)
Projective Plane Curve over Complex Field with 53 bits of precision defined by
x0^3 - x0*x1^2 + x1^3 - x1^2*x2
```

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: P.<u,v,w> = ProjectiveSpace(QQ, 2)
sage: C = Curve([y - x^2], A)
sage: C.projective_closure(1, P).ambient_space() == P
True

```

**class** sage.schemes.curves.affine\_curve.**AffineCurve\_field**(A, X)

Bases: `sage.schemes.curves.affine_curve.AffineCurve`, `sage.schemes.affine.affine_subscheme.AlgebraicScheme_subscheme_affine_field`

Affine curves over fields.

**blowup** (P=None)

Return the blow up of this affine curve at the point P.

The blow up is described by affine charts. This curve must be irreducible.

INPUT:

- P – (default: None) a point on this curve at which to blow up; if None, then P is taken to be the origin.

OUTPUT: a tuple of

- a tuple of curves in affine space of the same dimension as the ambient space of this curve, which define the blow up in each affine chart.
- a tuple of tuples such that the jth element of the ith tuple is the transition map from the ith affine patch to the jth affine patch.
- a tuple consisting of the restrictions of the projection map from the blow up back to the original curve, restricted to each affine patch. There the ith element will be the projection from the ith affine patch.

EXAMPLES:

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y^2 - x^3], A)
sage: C.blowup()
((Affine Plane Curve over Rational Field defined by s1^2 - x,
  Affine Plane Curve over Rational Field defined by y*s0^3 - 1),
 ([Scheme endomorphism of Affine Plane Curve over Rational Field defined by s1^
↪2 - x
  Defn: Defined on coordinates by sending (x, s1) to
      (x, s1), Scheme morphism:
  From: Affine Plane Curve over Rational Field defined by s1^2 - x
  To:   Affine Plane Curve over Rational Field defined by y*s0^3 - 1
  Defn: Defined on coordinates by sending (x, s1) to
      (x*s1, 1/s1)], [Scheme morphism:
  From: Affine Plane Curve over Rational Field defined by y*s0^3 - 1
  To:   Affine Plane Curve over Rational Field defined by s1^2 - x
  Defn: Defined on coordinates by sending (y, s0) to
      (y*s0, 1/s0),
  Scheme endomorphism of Affine Plane Curve over Rational Field defined by y
↪y*s0^3 - 1
  Defn: Defined on coordinates by sending (y, s0) to
      (y, s0)]),
 (Scheme morphism:
  From: Affine Plane Curve over Rational Field defined by s1^2 - x
  To:   Affine Plane Curve over Rational Field defined by -x^3 + y^2
  Defn: Defined on coordinates by sending (x, s1) to
      (x, x*s1), Scheme morphism:

```

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```

From: Affine Plane Curve over Rational Field defined by  $y*s0^3 - 1$ 
To:   Affine Plane Curve over Rational Field defined by  $-x^3 + y^2$ 
Defn: Defined on coordinates by sending  $(y, s0)$  to
       $(y*s0, y))$ 

```

```

sage: K.<a> = QuadraticField(2)
sage: A.<x,y,z> = AffineSpace(K, 3)
sage: C = Curve([y^2 - a*x^5, x - z], A)
sage: B = C.blowup()
sage: B[0]
(Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$  with a_
↪= 1.414213562373095? defined by  $s2 - 1, 2*x^3 + (-a)*s1^2$ ,
Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$  with a_
↪= 1.414213562373095? defined by  $s0 - s2, 2*y^3*s2^5 + (-a)$ ,
Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$  with a_
↪= 1.414213562373095? defined by  $s0 - 1, 2*z^3 + (-a)*s1^2$ )
sage: B[1][0][2]
Scheme morphism:
  From: Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $s2 - 1, 2*x^3 + (-a)*s1^2$ 
  To:   Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $s0 - 1, 2*z^3 + (-a)*s1^2$ 
  Defn: Defined on coordinates by sending  $(x, s1, s2)$  to
         $(x*s2, 1/s2, s1/s2)$ 
sage: B[1][2][0]
Scheme morphism:
  From: Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $s0 - 1, 2*z^3 + (-a)*s1^2$ 
  To:   Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $s2 - 1, 2*x^3 + (-a)*s1^2$ 
  Defn: Defined on coordinates by sending  $(z, s0, s1)$  to
         $(z*s0, s1/s0, 1/s0)$ 
sage: B[2]
(Scheme morphism:
  From: Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $s2 - 1, 2*x^3 + (-a)*s1^2$ 
  To:   Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $(-a)*x^5 + y^2, x - z$ 
  Defn: Defined on coordinates by sending  $(x, s1, s2)$  to
         $(x, x*s1, x*s2)$ , Scheme morphism:
  From: Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $s0 - s2, 2*y^3*s2^5 + (-a)$ 
  To:   Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $(-a)*x^5 + y^2, x - z$ 
  Defn: Defined on coordinates by sending  $(y, s0, s2)$  to
         $(y*s0, y, y*s2)$ , Scheme morphism:
  From: Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $s0 - 1, 2*z^3 + (-a)*s1^2$ 
  To:   Affine Curve over Number Field in a with defining polynomial  $x^2 - 2$ 
↪with a = 1.414213562373095? defined by  $(-a)*x^5 + y^2, x - z$ 
  Defn: Defined on coordinates by sending  $(z, s0, s1)$  to
         $(z*s0, z*s1, z)$ )

```

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve((y - 3/2)^3 - (x + 2)^5 - (x + 2)^6)
sage: Q = A([-2, 3/2])

```

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```

sage: C.blowup(Q)
((Affine Plane Curve over Rational Field defined by  $x^3 - s_1^3 + 7x^2 + 16x$ 
↪  $+ 12$ ,
  Affine Plane Curve over Rational Field defined by  $8y^3s_0^6 - 36y^2s_0^6$ 
↪  $+ 8y^2s_0^5 + 54ys_0^6 - 24ys_0^5 - 27s_0^6 + 18s_0^5 - 8$ ),
  ([Scheme endomorphism of Affine Plane Curve over Rational Field defined by  $x^3 - s_1^3 + 7x^2 + 16x + 12$ 
    Defn: Defined on coordinates by sending (x, s1) to
      (x, s1), Scheme morphism:
    From: Affine Plane Curve over Rational Field defined by  $x^3 - s_1^3 + 7x^2 + 16x + 12$ 
    To: Affine Plane Curve over Rational Field defined by  $8y^3s_0^6 - 36y^2s_0^6 + 8y^2s_0^5 + 54ys_0^6 - 24ys_0^5 - 27s_0^6 + 18s_0^5 - 8$ 
    Defn: Defined on coordinates by sending (x, s1) to
       $(xs_1 + 2s_1 + 3/2, 1/s_1)$ ], [Scheme morphism:
    From: Affine Plane Curve over Rational Field defined by  $8y^3s_0^6 - 36y^2s_0^6 + 8y^2s_0^5 + 54ys_0^6 - 24ys_0^5 - 27s_0^6 + 18s_0^5 - 8$ 
    To: Affine Plane Curve over Rational Field defined by  $x^3 - s_1^3 + 7x^2 + 16x + 12$ 
    Defn: Defined on coordinates by sending (y, s0) to
       $(ys_0 - 3/2s_0 - 2, 1/s_0)$ ,
    Scheme endomorphism of Affine Plane Curve over Rational Field defined by
↪  $8y^3s_0^6 - 36y^2s_0^6 + 8y^2s_0^5 + 54ys_0^6 - 24ys_0^5 - 27s_0^6 + 18s_0^5 - 8$ 
    Defn: Defined on coordinates by sending (y, s0) to
      (y, s0)]),
  (Scheme morphism:
    From: Affine Plane Curve over Rational Field defined by  $x^3 - s_1^3 + 7x^2 + 16x + 12$ 
    To: Affine Plane Curve over Rational Field defined by  $-x^6 - 13x^5 - 70x^4 - 200x^3 + y^3 - 320x^2 - 9/2y^2 - 272x + 27/4y - 795/8$ 
    Defn: Defined on coordinates by sending (x, s1) to
      (x,  $xs_1 + 2s_1 + 3/2$ ), Scheme morphism:
    From: Affine Plane Curve over Rational Field defined by  $8y^3s_0^6 - 36y^2s_0^6 + 8y^2s_0^5 + 54ys_0^6 - 24ys_0^5 - 27s_0^6 + 18s_0^5 - 8$ 
    To: Affine Plane Curve over Rational Field defined by  $-x^6 - 13x^5 - 70x^4 - 200x^3 + y^3 - 320x^2 - 9/2y^2 - 272x + 27/4y - 795/8$ 
    Defn: Defined on coordinates by sending (y, s0) to
       $(ys_0 - 3/2s_0 - 2, y)$ ))

```

```

sage: A.<x,y,z,w> = AffineSpace(QQ, 4)
sage: C = A.curve([(x + 1)^2 + y^2]^3 - 4*(x + 1)^2*y^2, y - z, w - 4])
sage: Q = C([-1,0,0,4])
sage: B = C.blowup(Q)
sage: B[0]
(Affine Curve over Rational Field defined by  $s_3, s_1 - s_2, x^2s_2^6 + 2xs_2^6 + 3x^2s_2^4 + s_2^6 + 6xs_2^4 + 3x^2s_2^2 + 3s_2^4 + 6xs_2^2 + x^2 - s_2^2 + 2x + 1$ ,
  Affine Curve over Rational Field defined by  $s_3, s_2 - 1, y^2s_0^6 +$ 

```

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```

3*y^2*s0^4 + 3*y^2*s0^2 + y^2 - 4*s0^2,
  Affine Curve over Rational Field defined by s3, s1 - 1, z^2*s0^6 +
3*z^2*s0^4 + 3*z^2*s0^2 + z^2 - 4*s0^2,
  Closed subscheme of Affine Space of dimension 4 over Rational Field
defined by:
  1)
sage: Q = A([6,2,3,1])
sage: B = C.blowup(Q)
Traceback (most recent call last):
...
TypeError: (=(6, 2, 3, 1)) must be a point on this curve

```

```

sage: A.<x,y> = AffineSpace(QuadraticField(-1), 2)
sage: C = A.curve([y^2 + x^2])
sage: C.blowup()
Traceback (most recent call last):
...
TypeError: this curve must be irreducible

```

**plane\_projection** (*AP=None*)

Return a projection of this curve into an affine plane so that the image of the projection is a plane curve.

INPUT:

- *AP* – (default: *None*) the affine plane to project this curve into. This space must be defined over the same base field as this curve, and must have dimension two. This space will be constructed if not specified.

OUTPUT: a tuple of

- a scheme morphism from this curve into an affine plane
- the plane curve that defines the image of that morphism

EXAMPLES:

```

sage: A.<x,y,z,w> = AffineSpace(QQ, 4)
sage: C = Curve([x^2 - y*z*w, z^3 - w, w + x*y - 3*z^3], A)
sage: C.plane_projection()
(Scheme morphism:
  From: Affine Curve over Rational Field defined by -y*z*w + x^2, z^3 -
w, -3*z^3 + x*y + w
  To:   Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y, z, w) to
        (x, y), Affine Plane Curve over Rational Field defined by
x0^2*x1^7 - 16*x0^4)

```

```

sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^2 + 2)
sage: A.<x,y,z> = AffineSpace(K, 3)
sage: C = A.curve([x - b, y - 2])
sage: B.<a,b> = AffineSpace(K, 2)
sage: proj1 = C.plane_projection(AP=B)
sage: proj1
(Scheme morphism:
  From: Affine Curve over Number Field in b with defining polynomial
a^2 + 2 defined by x + (-b), y - 2
  To:   Affine Space of dimension 2 over Number Field in b with

```

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```

defining polynomial a^2 + 2
  Defn: Defined on coordinates by sending (x, y, z) to
        (x, z),
  Affine Plane Curve over Number Field in b with defining polynomial a^2
+ 2 defined by a + (-b))
sage: proj1[1].ambient_space() is B
True
sage: proj2 = C.plane_projection()
sage: proj2[1].ambient_space() is B
False

```

**projection** (*indices*, *AS=None*)

Return the projection of this curve onto the coordinates specified by *indices*.

INPUT:

- *indices* – a list or tuple of distinct integers specifying the indices of the coordinates to use in the projection. Can also be a list or tuple consisting of variables of the coordinate ring of the ambient space of this curve. If integers are used to specify the coordinates, 0 denotes the first coordinate. The length of *indices* must be between two and one less than the dimension of the ambient space of this curve, inclusive.
- *AS* – (default: None) the affine space the projected curve will be defined in. This space must be defined over the same base field as this curve, and must have dimension equal to the length of *indices*. This space is constructed if not specified.

OUTPUT: a tuple of

- a scheme morphism from this curve to affine space of dimension equal to the number of coordinates specified in *indices*
- the affine subscheme that is the image of that morphism. If the image is a curve, the second element of the tuple will be a curve.

EXAMPLES:

```

sage: A.<x,y,z> = AffineSpace(QQ, 3)
sage: C = Curve([y^7 - x^2 + x^3 - 2*z, z^2 - x^7 - y^2], A)
sage: C.projection([0,1])
(Scheme morphism:
  From: Affine Curve over Rational Field defined by y^7 + x^3 - x^2 -
2*z, -x^7 - y^2 + z^2
  To:   Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x, y, z) to
        (x, y),
  Affine Plane Curve over Rational Field defined by x1^14 + 2*x0^3*x1^7 -
2*x0^2*x1^7 - 4*x0^7 + x0^6 - 2*x0^5 + x0^4 - 4*x1^2)
sage: C.projection([0,1,3,4])
Traceback (most recent call last):
...
ValueError: ([0, 1, 3, 4]) must be a list or tuple of length between 2
and (=2), inclusive

```

```

sage: A.<x,y,z,w> = AffineSpace(QQ, 4)
sage: C = Curve([x - 2, y - 3, z - 1], A)
sage: B.<a,b,c> = AffineSpace(QQ, 3)
sage: C.projection([0,1,2], AS=B)
(Scheme morphism:

```

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```

From: Affine Curve over Rational Field defined by  $x - 2, y - 3, z - 1$ 
To:   Affine Space of dimension 3 over Rational Field
Defn: Defined on coordinates by sending  $(x, y, z, w)$  to
       $(x, y, z),$ 
Closed subscheme of Affine Space of dimension 3 over Rational Field
defined by:
  c - 1,
  b - 3,
  a - 2)

```

```

sage: A.<x,y,z,w,u> = AffineSpace(GF(11), 5)
sage: C = Curve([x^3 - 5*y*z + u^2, x - y^2 + 3*z^2, w^2 + 2*u^3*y, y - u^2 +
↪z*x], A)
sage: B.<a,b,c> = AffineSpace(GF(11), 3)
sage: proj1 = C.projection([1,2,4], AS=B)
sage: proj1
(Scheme morphism:
  From: Affine Curve over Finite Field of size 11 defined by  $x^3 - 5*y*z + u^2, -y^2 + 3*z^2 + x, 2*y*u^3 + w^2, x*z - u^2 + y$ 
  To:   Affine Space of dimension 3 over Finite Field of size 11
  Defn: Defined on coordinates by sending  $(x, y, z, w, u)$  to
         $(y, z, u),$ 
  Affine Curve over Finite Field of size 11 defined by  $a^2*b - 3*b^3 - c^2 + a, c^6 - 5*a*b^4 + b^3*c^2 - 3*a*c^4 + 3*a^2*c^2 - a^3, a^2*c^4 - 3*b^2*c^4 - 2*a^3*c^2 - 5*a*b^2*c^2 + a^4 - 5*a*b^3 + 2*b^4 + b^2*c^2 - 3*b*c^2 + 3*a*b, a^4*c^2 + 2*b^4*c^2 - a^5 - 2*a*b^4 + 5*b*c^4 + a*b*c^2 - 5*a*b^2 + 4*b^3 + b*c^2 + 5*c^2 - 5*a, a^6 - 5*b^6 - 5*b^3*c^2 + 5*a*b^3 + 2*c^4 - 4*a*c^2 + 2*a^2 - 5*a*b + c^2)$ 
sage: proj1[1].ambient_space() is B
True
sage: proj2 = C.projection([1,2,4])
sage: proj2[1].ambient_space() is B
False
sage: C.projection([1,2,3,5], AS=B)
Traceback (most recent call last):
...
TypeError: (=Affine Space of dimension 3 over Finite Field of size 11)
must have dimension (=4)

```

```

sage: A.<x,y,z,w> = AffineSpace(QQ, 4)
sage: C = A.curve([x*y - z^3, x*z - w^3, w^2 - x^3])
sage: C.projection([y,z])
(Scheme morphism:
  From: Affine Curve over Rational Field defined by  $-z^3 + x*y, -w^3 + x*z, -x^3 + w^2$ 
  To:   Affine Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending  $(x, y, z, w)$  to
         $(y, z),$ 
  Affine Plane Curve over Rational Field defined by  $x^{123} - x^{0^7}*x^{1^4}$ )
sage: B.<x,y,z> = AffineSpace(QQ, 3)
sage: C.projection([x,y,z], AS=B)
(Scheme morphism:
  From: Affine Curve over Rational Field defined by  $-z^3 + x*y, -w^3 + x*z, -x^3 + w^2$ 
  To:   Affine Space of dimension 3 over Rational Field
  Defn: Defined on coordinates by sending  $(x, y, z, w)$  to

```

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```

(x, y, z),
Affine Curve over Rational Field defined by  $z^3 - x*y$ ,  $x^8 - x*z^2$ ,
 $x^7*z^2 - x*y*z$ )
sage: C.projection([y,z,z])
Traceback (most recent call last):
...
ValueError: ([y, z, z]) must be a list or tuple of distinct indices or
variables

```

**resolution\_of\_singularities** (*extend=False*)

Return a nonsingular model for this affine curve created by blowing up its singular points.

The nonsingular model is given as a collection of affine patches that cover it. If *extend* is *False* and if the base field is a number field, or if the base field is a finite field, the model returned may have singularities with coordinates not contained in the base field. An error is returned if this curve is already nonsingular, or if it has no singular points over its base field. This curve must be irreducible, and must be defined over a number field or finite field.

INPUT:

- *extend* – (default: *False*) specifies whether to extend the base field when necessary to find all singular points when this curve is defined over a number field. If *extend* is *False*, then only singularities with coordinates in the base field of this curve will be resolved. However, setting *extend* to *True* will slow down computations.

OUTPUT: a tuple of

- a tuple of curves in affine space of the same dimension as the ambient space of this curve, which represent affine patches of the resolution of singularities.
- a tuple of tuples such that the *j*th element of the *i*th tuple is the transition map from the *i*th patch to the *j*th patch.
- a tuple consisting of birational maps from the patches back to the original curve that were created by composing the projection maps generated from the blow up computations. There the *i*th element will be a map from the *i*th patch.

EXAMPLES:

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y^2 - x^3], A)
sage: C.resolution_of_singularities()
((Affine Plane Curve over Rational Field defined by  $s_1^2 - x$ ,
Affine Plane Curve over Rational Field defined by  $y*s_0^3 - 1$ ),
((Scheme endomorphism of Affine Plane Curve over Rational Field defined by  $s_1^2 - x$ 
↪  $s_1^2 - x$ 
Defn: Defined on coordinates by sending (x, s1) to
(x, s1), Scheme morphism:
From: Affine Plane Curve over Rational Field defined by  $s_1^2 - x$ 
To: Affine Plane Curve over Rational Field defined by  $y*s_0^3 - 1$ 
Defn: Defined on coordinates by sending (x, s1) to
(x*s1, 1/s1), (Scheme morphism:
From: Affine Plane Curve over Rational Field defined by  $y*s_0^3 - 1$ 
To: Affine Plane Curve over Rational Field defined by  $s_1^2 - x$ 
Defn: Defined on coordinates by sending (y, s0) to
(y*s0, 1/s0),
Scheme endomorphism of Affine Plane Curve over Rational Field defined by  $y*s_0^3 - 1$ 
↪  $y*s_0^3 - 1$ 

```

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```

Defn: Defined on coordinates by sending (y, s0) to
      (y, s0))),
(Scheme morphism:
  From: Affine Plane Curve over Rational Field defined by s1^2 - x
  To:   Affine Plane Curve over Rational Field defined by -x^3 + y^2
  Defn: Defined on coordinates by sending (x, s1) to
        (x, x*s1), Scheme morphism:
  From: Affine Plane Curve over Rational Field defined by y*s0^3 - 1
  To:   Affine Plane Curve over Rational Field defined by -x^3 + y^2
  Defn: Defined on coordinates by sending (y, s0) to
        (y*s0, y)))

```

```

sage: set_verbose(-1)
sage: K.<a> = QuadraticField(3)
sage: A.<x,y> = AffineSpace(K, 2)
sage: C = A.curve(x^4 + 2*x^2 + a*y^3 + 1)
sage: C.resolution_of_singularities(extend=True)[0] # long time (2 seconds)
(Affine Plane Curve over Number Field in a0 with defining polynomial y^4 -
  ↪ 4*y^2 + 16 defined by
  24*x^2*ss1^3 + 24*ss1^3 + (a0^3 - 8*a0),
  Affine Plane Curve over Number Field in a0 with defining polynomial y^4 -
  ↪ 4*y^2 + 16 defined by
  24*s1^2*ss0 + (a0^3 - 8*a0)*ss0^2 + (-6*a0^3)*s1,
  Affine Plane Curve over Number Field in a0 with defining polynomial y^4 -
  ↪ 4*y^2 + 16 defined by
  8*y^2*s0^4 + (4*a0^3)*y*s0^3 - 32*s0^2 + (a0^3 - 8*a0)*y)

```

```

sage: A.<x,y,z> = AffineSpace(GF(5), 3)
sage: C = Curve([y - x^3, (z - 2)^2 - y^3 - x^3], A)
sage: R = C.resolution_of_singularities()
sage: R[0]
(Affine Curve over Finite Field of size 5 defined by x^2 - s1, s1^4 - x*s2^2,
  ↪ + s1, x*s1^3 - s2^2 + x,
  Affine Curve over Finite Field of size 5 defined by y*s2^2 - y^2 - 1, s2^4 -
  ↪ s0^3 - y^2 - 2, y*s0^3
  - s2^2 + y, Affine Curve over Finite Field of size 5 defined by s0^3*s1 +
  ↪ z*s1^3 + s1^4 - 2*s1^3 - 1,
  z*s0^3 + z*s1^3 - 2*s0^3 - 2*s1^3 - 1, z^2*s1^3 + z*s1^3 - s1^3 - z + s1 + 2)

```

```

sage: A.<x,y,z,w> = AffineSpace(QQ, 4)
sage: C = A.curve([(x - 2)^2 + y^2)^2 - (x - 2)^2 - y^2 + (x - 2)^3, z - y -
  ↪ 7, w - 4])
sage: B = C.resolution_of_singularities()
sage: B[0]
(Affine Curve over Rational Field defined by s3, s1 - s2, x^2*s2^4 -
  4*x*s2^4 + 2*x^2*s2^2 + 4*s2^4 - 8*x*s2^2 + x^2 + 7*s2^2 - 3*x + 1,
  Affine Curve over Rational Field defined by s3, s2 - 1, y^2*s0^4 +
  2*y^2*s0^2 + y*s0^3 + y^2 - s0^2 - 1,
  Affine Curve over Rational Field defined by s3, s1 - 1, z^2*s0^4 -
  14*z*s0^4 + 2*z^2*s0^2 + z*s0^3 + 49*s0^4 - 28*z*s0^2 - 7*s0^3 + z^2 +
  97*s0^2 - 14*z + 48,
  Closed subscheme of Affine Space of dimension 4 over Rational Field
  defined by:
  1)

```

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y - x^2 + 1], A)
sage: C.resolution_of_singularities()
Traceback (most recent call last):
...
TypeError: this curve is already nonsingular

```

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve([(x^2 + y^2 - y - 2)*(y - x^2 + 2) + y^3])
sage: C.resolution_of_singularities()
Traceback (most recent call last):
...
TypeError: this curve has no singular points over its base field. If
working over a number field use extend=True

```

**tangent\_line(p)**

Return the tangent line at the point p.

INPUT:

- p – a rational point of the curve

EXAMPLES:

```

sage: A3.<x,y,z> = AffineSpace(3, QQ)
sage: C = Curve([x + y + z, x^2 - y^2*z^2 + z^3])
sage: p = C(0,0,0)
sage: C.tangent_line(p)
Traceback (most recent call last):
...
ValueError: the curve is not smooth at (0, 0, 0)
sage: p = C(1,0,-1)
sage: C.tangent_line(p)
Affine Curve over Rational Field defined by x + y + z, 2*x + 3*z + 1

```

We check that the tangent line at p is the tangent space at p, translated to p.

```

sage: Tp = C.tangent_space(p)
sage: Tp
Closed subscheme of Affine Space of dimension 3 over Rational Field defined
↳by:
  x + y + z,
  2*x + 3*z
sage: phi = A3.translation(A3.origin(), p)
sage: T = phi * Tp.embedding_morphism()
sage: T.image()
Closed subscheme of Affine Space of dimension 3 over Rational Field defined
↳by:
  -2*y + z + 1,
  x + y + z
sage: _ == C.tangent_line(p)
True

```

**class** sage.schemes.curves.affine\_curve.**AffinePlaneCurve**(A,f)

Bases: *sage.schemes.curves.affine\_curve.AffineCurve*

Affine plane curves.

**divisor\_of\_function**(*r*)

Return the divisor of a function on a curve.

INPUT: *r* is a rational function on *X*

OUTPUT:

- *list* - The divisor of *r* represented as a list of coefficients and points. (TODO: This will change to a more structural output in the future.)

EXAMPLES:

```
sage: F = GF(5)
sage: P2 = AffineSpace(2, F, names = 'xy')
sage: R = P2.coordinate_ring()
sage: x, y = R.gens()
sage: f = y^2 - x^9 - x
sage: C = Curve(f)
sage: K = FractionField(R)
sage: r = 1/x
sage: C.divisor_of_function(r)      # todo: not implemented (broken)
[[-1, (0, 0, 1)]]
sage: r = 1/x^3
sage: C.divisor_of_function(r)      # todo: not implemented (broken)
[[-3, (0, 0, 1)]]
```

**is\_ordinary\_singularity**(*P*)

Return whether the singular point *P* of this affine plane curve is an ordinary singularity.

The point *P* is an ordinary singularity of this curve if it is a singular point, and if the tangents of this curve at *P* are distinct.

INPUT:

- *P* – a point on this curve

OUTPUT:

True or False depending on whether *P* is or is not an ordinary singularity of this curve, respectively. An error is raised if *P* is not a singular point of this curve.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y^2 - x^3], A)
sage: Q = A([0,0])
sage: C.is_ordinary_singularity(Q)
False
```

```
sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^2 - 3)
sage: A.<x,y> = AffineSpace(K, 2)
sage: C = Curve([(x^2 + y^2 - 2*x)^2 - x^2 - y^2], A)
sage: Q = A([0,0])
sage: C.is_ordinary_singularity(Q)
True
```

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve([x^2*y - y^2*x + y^2 + x^3])
sage: Q = A([-1,-1])
```

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```
sage: C.is_ordinary_singularity(Q)
Traceback (most recent call last):
...
TypeError: ((-1, -1)) is not a singular point of (=Affine Plane Curve
over Rational Field defined by  $x^3 + x^2y - xy^2 + y^2$ )
```

**is\_transverse**(C, P)

Return whether the intersection of this curve with the curve C at the point P is transverse.

The intersection at P is transverse if P is a nonsingular point of both curves, and if the tangents of the curves at P are distinct.

INPUT:

- C – a curve in the ambient space of this curve.
- P – a point in the intersection of both curves.

OUTPUT: Boolean.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([x^2 + y^2 - 1], A)
sage: D = Curve([x - 1], A)
sage: Q = A([1,0])
sage: C.is_transverse(D, Q)
False
```

```
sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^3 + 2)
sage: A.<x,y> = AffineSpace(K, 2)
sage: C = A.curve([x*y])
sage: D = A.curve([y - b*x])
sage: Q = A([0,0])
sage: C.is_transverse(D, Q)
False
```

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y - x^3], A)
sage: D = Curve([y + x], A)
sage: Q = A([0,0])
sage: C.is_transverse(D, Q)
True
```

**local\_coordinates**(pt, n)

Return local coordinates to precision n at the given point.

Behaviour is flaky - some choices of n are worst than others.

INPUT:

- pt - an F-rational point on X which is not a point of ramification for the projection (x,y) - x.
- n - the number of terms desired

OUTPUT:  $x = x_0 + t$   $y = y_0 +$  power series in t

EXAMPLES:

```

sage: F = GF(5)
sage: pt = (2, 3)
sage: R = PolynomialRing(F, 2, names = ['x', 'y'])
sage: x, y = R.gens()
sage: f = y^2 - x^9 - x
sage: C = Curve(f)
sage: C.local_coordinates(pt, 9)
[t + 2, -2*t^12 - 2*t^11 + 2*t^9 + t^8 - 2*t^7 - 2*t^6 - 2*t^4 + t^3 - 2*t^2 -
↪ 2]

```

**multiplicity(P)**

Return the multiplicity of this affine plane curve at the point P.

In the special case of affine plane curves, the multiplicity of an affine plane curve at the point (0,0) can be computed as the minimum of the degrees of the homogeneous components of its defining polynomial. To compute the multiplicity of a different point, a linear change of coordinates is used.

This curve must be defined over a field. An error is raised if P is not a point on this curve.

INPUT:

- P – a point in the ambient space of this curve.

OUTPUT:

An integer.

EXAMPLES:

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y^2 - x^3], A)
sage: Q1 = A([1,1])
sage: C.multiplicity(Q1)
1
sage: Q2 = A([0,0])
sage: C.multiplicity(Q2)
2

```

```

sage: A.<x,y> = AffineSpace(QQbar, 2)
sage: C = Curve([-x^7 + (-7)*x^6 + y^6 + (-21)*x^5 + 12*y^5 + (-35)*x^4 +
↪ 60*y^4 + \
(-35)*x^3 + 160*y^3 + (-21)*x^2 + 240*y^2 + (-7)*x + 192*y + 63], A)
sage: Q = A([-1,-2])
sage: C.multiplicity(Q)
6

```

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve([y^3 - x^3 + x^6])
sage: Q = A([1,1])
sage: C.multiplicity(Q)
Traceback (most recent call last):
...
TypeError: (=(1, 1)) is not a point on (=Affine Plane Curve over
Rational Field defined by x^6 - x^3 + y^3)

```

**plot(\*args, \*\*kws)**

Plot the real points on this affine plane curve.

INPUT:

- `*args` - optional tuples (variable, minimum, maximum) for plotting dimensions
- `**kwargs` - optional keyword arguments passed on to `implicit_plot`

EXAMPLES:

A cuspidal curve:

```
sage: R.<x, y> = QQ[]
sage: C = Curve(x^3 - y^2)
sage: C.plot()
Graphics object consisting of 1 graphics primitive
```

A 5-nodal curve of degree 11. This example also illustrates some of the optional arguments:

```
sage: R.<x, y> = ZZ[]
sage: C = Curve(32*x^2 - 2097152*y^11 + 1441792*y^9 - 360448*y^7 + 39424*y^5 -
↪ 1760*y^3 + 22*y - 1)
sage: C.plot((x, -1, 1), (y, -1, 1), plot_points=400)
Graphics object consisting of 1 graphics primitive
```

A line over  $\mathbb{R}$ :

```
sage: R.<x, y> = RR[]
sage: C = Curve(R(y - sqrt(2)*x))
sage: C.plot()
Graphics object consisting of 1 graphics primitive
```

### `rational_parameterization()`

Return a rational parameterization of this curve.

This curve must have rational coefficients and be absolutely irreducible (i.e. irreducible over the algebraic closure of the rational field). The curve must also be rational (have geometric genus zero).

The rational parameterization may have coefficients in a quadratic extension of the rational field.

OUTPUT:

- a birational map between  $\mathbb{A}^1$  and this curve, given as a scheme morphism.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y^2 - x], A)
sage: C.rational_parameterization()
Scheme morphism:
  From: Affine Space of dimension 1 over Rational Field
  To:   Affine Plane Curve over Rational Field defined by y^2 - x
  Defn: Defined on coordinates by sending (t) to
        (t^2, t)
```

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([(x^2 + y^2 - 2*x)^2 - x^2 - y^2], A)
sage: C.rational_parameterization()
Scheme morphism:
  From: Affine Space of dimension 1 over Rational Field
  To:   Affine Plane Curve over Rational Field defined by x^4 +
        2*x^2*y^2 + y^4 - 4*x^3 - 4*x*y^2 + 3*x^2 - y^2
  Defn: Defined on coordinates by sending (t) to
        ((-12*t^4 + 6*t^3 + 4*t^2 - 2*t)/(-25*t^4 + 40*t^3 - 26*t^2 +
        8*t - 1), (-9*t^4 + 12*t^3 - 4*t + 1)/(-25*t^4 + 40*t^3 - 26*t^2 + 8*t - 1))
```

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([x^2 + y^2 + 7], A)
sage: C.rational_parameterization()
Scheme morphism:
  From: Affine Space of dimension 1 over Number Field in a with defining
  ↪ polynomial a^2 + 7
  To:   Affine Plane Curve over Number Field in a with defining
  polynomial a^2 + 7 defined by x^2 + y^2 + 7
  Defn: Defined on coordinates by sending (t) to
        ((-7*t^2 + 7)/((-a)*t^2 + (-a)), 14*t/((-a)*t^2 + (-a)))

```

**tangents** (*P*, *factor=True*)

Return the tangents of this affine plane curve at the point *P*.

The point *P* must be a point on this curve.

INPUT:

- *P* – a point on this curve
- *factor* – (default: *True*) whether to attempt computing the polynomials of the individual tangent lines over the base field of this curve, or to just return the polynomial corresponding to the union of the tangent lines (which requires fewer computations)

OUTPUT: a list of polynomials in the coordinate ring of the ambient space

EXAMPLES:

```

sage: set_verbose(-1)
sage: A.<x,y> = AffineSpace(QQbar, 2)
sage: C = Curve([x^5*y^3 + 2*x^4*y^4 + x^3*y^5 + 3*x^4*y^3 + 6*x^3*y^4 + 3*x^
  ↪ 2*y^5\
+ 3*x^3*y^3 + 6*x^2*y^4 + 3*x*y^5 + x^5 + 10*x^4*y + 40*x^3*y^2 + 81*x^2*y^3\
  ↪ + 82*x*y^4\
+ 33*y^5], A)
sage: Q = A([0,0])
sage: C.tangents(Q)
[x + 3.425299577684700?*y, x + (1.949159013086856? + 1.179307909383728?*I)*y,
x + (1.949159013086856? - 1.179307909383728?*I)*y, x + (1.338191198070795? +
  ↪ 0.2560234251008043?*I)*y,
x + (1.338191198070795? - 0.2560234251008043?*I)*y]
sage: C.tangents(Q, factor=False)
[120*x^5 + 1200*x^4*y + 4800*x^3*y^2 + 9720*x^2*y^3 + 9840*x*y^4 + 3960*y^5]

```

```

sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^2 - 3)
sage: A.<x,y> = AffineSpace(K, 2)
sage: C = Curve([(x^2 + y^2 - 2*x)^2 - x^2 - y^2], A)
sage: Q = A([0,0])
sage: C.tangents(Q)
[x + (-1/3*b)*y, x + (1/3*b)*y]

```

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve([y^2 - x^3 - x^2])
sage: Q = A([0,0])
sage: C.tangents(Q)
[x - y, x + y]

```

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve([y*x - x^4 + 2*x^2])
sage: Q = A([1,1])
sage: C.tangents(Q)
Traceback (most recent call last):
...
TypeError: (=(1, 1)) is not a point on (=Affine Plane Curve over
Rational Field defined by -x^4 + 2*x^2 + x*y)

```

**class** sage.schemes.curves.affine\_curve.**AffinePlaneCurve\_field**(A,f)  
 Bases: `sage.schemes.curves.affine_curve.AffinePlaneCurve`, `sage.schemes.curves.affine_curve.AffineCurve_field`

Affine plane curves over fields.

**fundamental\_group**()

Return a presentation of the fundamental group of the complement of self.

---

**Note:** The curve must be defined over the rationals or a number field with an embedding over  $\overline{\mathbb{Q}}$ .

---

EXAMPLES:

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve(y^2 - x^3 - x^2)
sage: C.fundamental_group() # optional - sirocco
Finitely presented group < x0 | >

```

In the case of number fields, they need to have an embedding to the algebraic field:

```

sage: a = QQ[x](x^2+5).roots(QQbar)[0][0]
sage: F = NumberField(a.minpoly(), 'a', embedding=a)
sage: F.inject_variables()
Defining a
sage: A.<x,y> = AffineSpace(F, 2)
sage: C = A.curve(y^2 - a*x^3 - x^2)
sage: C.fundamental_group() # optional - sirocco
Finitely presented group < x0 | >

```

**Warning:** This functionality requires the sirocco package to be installed.

**riemann\_surface**(\*\*kwargs)

Return the complex riemann surface determined by this curve

OUTPUT:

- RiemannSurface object

EXAMPLES:

```

sage: R.<x,y>=QQ[]
sage: C=Curve(x^3+3*y^3+5)
sage: C.riemann_surface()
Riemann surface defined by polynomial f = x^3 + 3*y^3 + 5 = 0, with 53 bits_
↳ of precision

```



**class** `sage.schemes.curves.affine_curve.AffinePlaneCurve_finite_field(A,f)`  
 Bases: `sage.schemes.curves.affine_curve.AffinePlaneCurve_field`

Affine plane curves over finite fields.

**rational\_points** (*algorithm='enum'*)

Return sorted list of all rational points on this curve.

INPUT:

- *algorithm* – possible choices:
  - 'enum' – use *very* naive point enumeration to find all rational points on this curve over a finite field.
  - 'bn' – via Singular's Brill-Noether package.
  - 'all' – use all implemented algorithms and verify that they give the same answer, then return it

---

**Note:** The Brill-Noether package does not always work. When it fails, a `RuntimeError` exception is raised.

---

EXAMPLES:

```
sage: x, y = (GF(5)['x,y']).gens()
sage: f = y^2 - x^9 - x
sage: C = Curve(f); C
Affine Plane Curve over Finite Field of size 5 defined by -x^9 + y^2 - x
sage: C.rational_points(algorithm='bn')
[(0, 0), (2, 2), (2, 3), (3, 1), (3, 4)]
sage: C = Curve(x - y + 1)
sage: C.rational_points()
[(0, 1), (1, 2), (2, 3), (3, 4), (4, 0)]
```

We compare Brill-Noether and enumeration:

```
sage: x, y = (GF(17)['x,y']).gens()
sage: C = Curve(x^2 + y^5 + x*y - 19)
sage: v = C.rational_points(algorithm='bn')
sage: w = C.rational_points(algorithm='enum')
sage: len(v)
20
sage: v == w
True

sage: A.<x,y> = AffineSpace(2,GF(9,'a'))
sage: C = Curve(x^2 + y^2 - 1)
sage: C
Affine Plane Curve over Finite Field in a of size 3^2 defined by x^2 + y^2 - 1
sage: C.rational_points()
[(0, 1), (0, 2), (1, 0), (2, 0), (a + 1, a + 1), (a + 1, 2*a + 2), (2*a + 2, a + 1), (2*a + 2, 2*a + 2)]
```

**riemann\_roch\_basis** (*D*)

Return a basis of the Riemann-Roch space of the divisor *D*.

This interfaces with Singular's Brill-Noether command.

This curve is assumed to be a plane curve defined by a polynomial equation  $f(x, y) = 0$  over a prime finite field  $F = GF(p)$  in 2 variables  $x, y$  representing a curve  $X : f(x, y) = 0$  having  $n$   $F$ -rational points (see

the Sage function `places_on_curve`)

INPUT:

- $D$  – an  $n$ -tuple of integers  $(d_1, \dots, d_n)$  representing the divisor  $Div = d_1P_1 + \dots + d_nP_n$ , where  $X(F) = \{P_1, \dots, P_n\}$ . The ordering is that dictated by `places_on_curve`.

OUTPUT: a basis of  $L(Div)$

EXAMPLES:

```
sage: R = PolynomialRing(GF(5), 2, names = ["x", "y"])
sage: x, y = R.gens()
sage: f = y^2 - x^9 - x
sage: C = Curve(f)
sage: D = [6, 0, 0, 0, 0, 0]
sage: C.riemann_roch_basis(D)
[1, (-x*z^5 + y^2*z^4)/x^6, (-x*z^6 + y^2*z^5)/x^7, (-x*z^7 + y^2*z^6)/x^8]
```

**class** `sage.schemes.curves.affine_curve.IntegralAffineCurve` ( $A, X$ )

Bases: `sage.schemes.curves.affine_curve.AffineCurve_field`

Base class for integral affine curves.

**coordinate\_functions**()

Return the coordinate functions.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(8), 2)
sage: C = Curve(x^5 + y^5 + x*y + 1)
sage: x, y = C.coordinate_functions()
sage: x^5 + y^5 + x*y + 1
0
```

**function** ( $f$ )

Return the function field element coerced from  $f$ .

INPUT:

- $f$  – an element of the coordinate ring of either the curve or its ambient space.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(8), 2)
sage: C = Curve(x^5 + y^5 + x*y + 1)
sage: f = C.function(x/y)
sage: f
(x/(x^5 + 1))*y^4 + x^2/(x^5 + 1)
sage: df = f.differential(); df
((1/(x^10 + 1))*y^4 + x^6/(x^10 + 1)) d(x)
sage: df.divisor()
2*Place (1/x, 1/x^4*y^4 + 1/x^3*y^3 + 1/x^2*y^2 + 1/x*y + 1)
+ 2*Place (1/x, 1/x*y + 1)
- 2*Place (x + 1, y)
- 2*Place (x^4 + x^3 + x^2 + x + 1, y)
```

**function\_field**()

Return the function field of the curve.

EXAMPLES:

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve(x^3 - y^2 - x^4 - y^4)
sage: C.function_field()
Function field in y defined by y^4 + y^2 + x^4 - x^3

```

```

sage: A.<x,y> = AffineSpace(GF(8), 2)
sage: C = Curve(x^5 + y^5 + x*y + 1)
sage: C.function_field()
Function field in y defined by y^5 + x*y + x^5 + 1

```

### **parametric\_representation** (*place, name=None*)

Return a power series representation of the branch of the curve given by *place*.

INPUT:

- *place* – a place on the curve

EXAMPLES:

```

sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve(x^2 + y^2 - 1)
sage: p = C(0,1)
sage: p.closed_point()
Point (x, y - 1)
sage: pl = _.place()
sage: C.parametric_representation(pl)
(s + ..., 1 - 1/2*s^2 - 1/8*s^4 - 1/16*s^6 + ...)

```

```

sage: A.<x,y> = AffineSpace(GF(7^2), 2)
sage: C = Curve(x^2 - x^4 - y^4)
sage: p, = C.singular_closed_points()
sage: b1, b2 = p.places()
sage: xs, ys = C.parametric_representation(b1)
sage: f = xs^2 - xs^4 - ys^4
sage: [f.coefficient(i) for i in range(5)]
[0, 0, 0, 0, 0]
sage: xs, ys = C.parametric_representation(b2)
sage: f = xs^2 - xs^4 - ys^4
sage: [f.coefficient(i) for i in range(5)]
[0, 0, 0, 0, 0]

```

### **place\_to\_closed\_point** (*place*)

Return the closed point on the place.

INPUT:

- *place* – a place of the function field of the curve

EXAMPLES:

```

sage: A.<x,y> = AffineSpace(GF(4), 2)
sage: C = Curve(x^5 + y^5 + x*y + 1)
sage: F = C.function_field()
sage: pls = F.places(1)
sage: C.place_to_closed_point(pls[-1])
Point (x + 1, y + 1)
sage: C.place_to_closed_point(pls[-2])
Point (x + 1, y + 1)

```

**places\_at\_infinity()**

Return the places of the curve at infinity.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve(x^3 - y^2 - x^4 - y^4)
sage: C.places_at_infinity()
[Place (1/x, 1/x^2*y, 1/x^3*y^2, 1/x^4*y^3)]
```

```
sage: F = GF(9)
sage: A2.<x,y> = AffineSpace(F, 2)
sage: C = A2.curve(y^3 + y - x^4)
sage: C.places_at_infinity()
[Place (1/x, 1/x^3*y^2)]
```

```
sage: A.<x,y,z> = AffineSpace(GF(11), 3)
sage: C = Curve([x*z-y^2,y-z^2,x-y*z], A)
sage: C.places_at_infinity()
[Place (1/x, 1/x*z^2)]
```

**places\_on(point)**

Return the places on the closed point.

INPUT:

- *point* – a closed point of the curve

OUTPUT: a list of the places of the function field of the curve

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve(x^3 - y^2 - x^4 - y^4)
sage: C.singular_closed_points()
[Point (x, y)]
sage: p, = _
sage: C.places_on(p)
[Place (x, y, y^2, 1/x*y^3 + 1/x*y)]
```

```
sage: k.<a> = GF(9)
sage: A.<x,y> = AffineSpace(k, 2)
sage: C = Curve(y^2 - x^5 - x^4 - 2*x^3 - 2*x - 2)
sage: pts = C.closed_points()
sage: pts
[Point (x, y + (a + 1)),
 Point (x, y + (-a - 1)),
 Point (x + (a + 1), y + (a - 1)),
 Point (x + (a + 1), y + (-a + 1)),
 Point (x - 1, y + (a + 1)),
 Point (x - 1, y + (-a - 1)),
 Point (x + (-a - 1), y + (a)),
 Point (x + (-a - 1), y + (-a)),
 Point (x + 1, y + 1),
 Point (x + 1, y - 1)]
sage: p1, p2, p3 = pts[:3]
sage: C.places_on(p1)
[Place (x, y + a + 1)]
```

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```
sage: C.places_on(p2)
[Place (x, y + 2*a + 2)]
sage: C.places_on(p3)
[Place (x + a + 1, y + a + 2)]
```

```
sage: F.<a> = GF(8)
sage: P.<x,y,z> = ProjectiveSpace(F, 2)
sage: Cp = Curve(x^3*y + y^3*z + x*z^3)
sage: C = Cp.affine_patch(0)
```

**singular\_closed\_points()**

Return the singular closed points of the curve.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(7^2), 2)
sage: C = Curve(x^2 - x^4 - y^4)
sage: C.singular_closed_points()
[Point (x, y)]
```

```
sage: A.<x,y,z> = AffineSpace(GF(11), 3)
sage: C = Curve([x*z - y^2, y - z^2, x - y*z], A)
sage: C.singular_closed_points()
[]
```

**class** sage.schemes.curves.affine\_curve.**IntegralAffineCurve\_finite\_field**(A, X)

Bases: *sage.schemes.curves.affine\_curve.IntegralAffineCurve*

Integral affine curves.

INPUT:

- A – an ambient space in which the curve lives
- X – list of polynomials that define the curve

EXAMPLES:

```
sage: A.<x,y,z> = AffineSpace(GF(11), 3)
sage: C = Curve([x*z - y^2, y - z^2, x - y*z], A); C
Affine Curve over Finite Field of size 11 defined by -y^2 + x*z, -z^2 + y, -y*z +
↪ x
sage: C.function_field()
Function field in z defined by z^3 + 10*x
```

**closed\_points(degree=1)**

Return a list of the closed points of degree of the curve.

INPUT:

- degree – a positive integer

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(7), 2)
sage: C = Curve(x^2 - x^4 - y^4)
sage: C.closed_points()
[Point (x, y),
```

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```

Point (x + 1, y),
Point (x + 2, y + 2),
Point (x + 2, y - 2),
Point (x - 2, y + 2),
Point (x - 2, y - 2),
Point (x - 1, y)]

```

**places** (*degree=1*)

Return all places on the curve of the degree.

INPUT:

- degree – positive integer

EXAMPLES:

```

sage: F = GF(9)
sage: A2.<x,y> = AffineSpace(F, 2)
sage: C = A2.curve(y^3 + y - x^4)
sage: C.places()
[Place (1/x, 1/x^3*y^2),
 Place (x, y),
 Place (x, y + z2 + 1),
 Place (x, y + 2*z2 + 2),
 Place (x + z2, y + 2),
 Place (x + z2, y + z2),
 Place (x + z2, y + 2*z2 + 1),
 Place (x + z2 + 1, y + 1),
 Place (x + z2 + 1, y + z2 + 2),
 Place (x + z2 + 1, y + 2*z2),
 Place (x + 2*z2 + 1, y + 2),
 Place (x + 2*z2 + 1, y + z2),
 Place (x + 2*z2 + 1, y + 2*z2 + 1),
 Place (x + 2, y + 1),
 Place (x + 2, y + z2 + 2),
 Place (x + 2, y + 2*z2),
 Place (x + 2*z2, y + 2),
 Place (x + 2*z2, y + z2),
 Place (x + 2*z2, y + 2*z2 + 1),
 Place (x + 2*z2 + 2, y + 1),
 Place (x + 2*z2 + 2, y + z2 + 2),
 Place (x + 2*z2 + 2, y + 2*z2),
 Place (x + z2 + 2, y + 2),
 Place (x + z2 + 2, y + z2),
 Place (x + z2 + 2, y + 2*z2 + 1),
 Place (x + 1, y + 1),
 Place (x + 1, y + z2 + 2),
 Place (x + 1, y + 2*z2)]

```

**class** sage.schemes.curves.affine\_curve.**IntegralAffinePlaneCurve** (*A, f*)

Bases: *sage.schemes.curves.affine\_curve.IntegralAffineCurve*, *sage.schemes.curves.affine\_curve.AffinePlaneCurve\_field*

**class** sage.schemes.curves.affine\_curve.**IntegralAffinePlaneCurve\_finite\_field** (*A, f*)

Bases: *sage.schemes.curves.affine\_curve.AffinePlaneCurve\_finite\_field*, *sage.schemes.curves.affine\_curve.IntegralAffineCurve\_finite\_field*

Integral affine plane curve over a finite field.

## EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(8), 2)
sage: C = Curve(x^5 + y^5 + x*y + 1); C
Affine Plane Curve over Finite Field in z3 of size 2^3 defined by x^5 + y^5 + x*y
↪ + 1
sage: C.function_field()
Function field in y defined by y^5 + x*y + x^5 + 1
```





## PROJECTIVE CURVES

Projective curves in Sage are curves in a projective space or a projective plane.

EXAMPLES:

We can construct curves in either a projective plane:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([y*z^2 - x^3], P); C
Projective Plane Curve over Rational Field defined by -x^3 + y*z^2
```

or in higher dimensional projective spaces:

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([y*w^3 - x^4, z*w^3 - x^4], P); C
Projective Curve over Rational Field defined by -x^4 + y*w^3, -x^4 + z*w^3
```

### 4.1 Integral projective curves over finite fields

If the curve is defined over a finite field and integral, that is reduced and irreducible, its function field is tightly coupled with the curve so that advanced computations based on Sage's global function field machinery are available.

EXAMPLES:

```
sage: k = GF(2)
sage: P.<x,y,z> = ProjectiveSpace(k, 2)
sage: C = Curve(x^2*z - y^3, P)
sage: C.genus()
0
sage: C.function_field()
Function field in z defined by z + y^3
```

Closed points of arbitrary degree can be computed:

```
sage: C.closed_points()
[Point (x, y), Point (y, z), Point (x + z, y + z)]
sage: C.closed_points(2)
[Point (y^2 + y*z + z^2, x + z)]
sage: C.closed_points(3)
[Point (y^3 + y^2*z + z^3, x + y + z),
 Point (x^2 + y*z + z^2, x*y + x*z + y*z, y^2 + x*z + y*z + z^2)]
```

All singular closed points can be found:

```

sage: C.singular_closed_points()
[Point (x, y)]
sage: p = _[0]
sage: p.places() # a unibranch singularity, that is, a cusp
[Place (1/y)]
sage: pls = _[0]
sage: C.place_to_closed_point(pls)
Point (x, y)

```

It is easy to transit to and from the function field of the curve:

```

sage: fx = C(x/z)
sage: fy = C(y/z)
sage: fx^2 - fy^3
0
sage: fx.divisor()
3*Place (1/y)
- 3*Place (y)
sage: p, = fx.poles()
sage: p
Place (y)
sage: C.place_to_closed_point(p)
Point (y, z)
sage: _.rational_point()
(1 : 0 : 0)
sage: _.closed_point()
Point (y, z)
sage: _.place()
Place (y)

```

## 4.2 Integral projective curves over $\mathbb{Q}$

An integral curve over  $\mathbb{Q}$  is also equipped with the function field. Unlike over finite fields, it is not possible to enumerate closed points.

EXAMPLES:

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve(x^2*z^2 - x^4 - y^4, P)
sage: C.singular_closed_points()
[Point (x, y)]
sage: p, = _
sage: p.places()
[Place (1/y, 1/y^2*z - 1), Place (1/y, 1/y^2*z + 1)]
sage: fy = C.function(y/z)
sage: fy.divisor()
Place (1/y, 1/y^2*z - 1)
+ Place (1/y, 1/y^2*z + 1)
+ Place (y, z - 1)
+ Place (y, z + 1)
- Place (y^4 + 1, z)
sage: supp = _.support()
sage: pl = supp[0]
sage: C.place_to_closed_point(pl)
Point (x, y)

```

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```
sage: p1 = supp[1]
sage: C.place_to_closed_point(p1)
Point (x, y)
sage: _.rational_point()
(0 : 0 : 1)
sage: _ in C
True
```

AUTHORS:

- William Stein (2005-11-13)
- David Joyner (2005-11-13)
- David Kohel (2006-01)
- Moritz Minzloff (2010-11)
- Grayson Jorgenson (2016-08)
- Kwankyu Lee (2019-05): added integral projective curves

```
sage.schemes.curves.projective_curve.Hasse_bounds (q, genus=1)
```

Return the Hasse-Weil bounds for the cardinality of a nonsingular curve defined over  $\mathbf{F}_q$  of given genus.

INPUT:

- `q (int)` – a prime power
- `genus (int, default 1)` – a non-negative integer,

OUTPUT:

(tuple) The Hasse bounds (lb,ub) for the cardinality of a curve of genus `genus` defined over  $\mathbf{F}_q$ .

EXAMPLES:

```
sage: Hasse_bounds(2)
(1, 5)
sage: Hasse_bounds(next_prime(10^30))
(999999999999980000000000000058, 1000000000000002000000000000058)
```

```
class sage.schemes.curves.projective_curve.IntegralProjectiveCurve ( $A, f$ )
```

Bases: `sage.schemes.curves.projective_curve.ProjectiveCurve_field`

Integral projective curve.

**coordinate\_functions** (*i=None*)

Return the coordinate functions for the `i`-th affine patch.

If `i` is `None`, return the homogeneous coordinate functions.

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(GF(4), 2)
sage: C = Curve(x^5 + y^5 + x*y*z^3 + z^5)
sage: C.coordinate_functions(0)
(y, z)
sage: C.coordinate_functions(1)
(1/y, 1/y*z)
```

**function** ( $f$ )

Return the function field element coerced from  $x$ .

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(GF(4), 2)
sage: C = Curve(x^5 + y^5 + x*y*z^3 + z^5)
sage: f = C.function(x/y); f
1/y
sage: f.divisor()
Place (1/y, 1/y^2*z^2 + z^2/y*z + 1)
+ Place (1/y, 1/y^2*z^2 + ((z^2 + 1)/y)*z + 1)
+ Place (1/y, 1/y*z + 1)
- Place (y, z^2 + z^2*z + 1)
- Place (y, z^2 + (z^2 + 1)*z + 1)
- Place (y, z + 1)
```

**function\_field()**

Return the function field of this curve.

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve(x^2 + y^2 + z^2, P)
sage: C.function_field()
Function field in z defined by z^2 + y^2 + 1
```

```
sage: P.<x,y,z> = ProjectiveSpace(GF(4), 2)
sage: C = Curve(x^5 + y^5 + x*y*z^3 + z^5)
sage: C.function_field()
Function field in z defined by z^5 + y*z^3 + y^5 + 1
```

**place\_to\_closed\_point(place)**

Return the closed point at the place.

INPUT:

- place – a place of the function field of the curve

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: C = Curve(y^2*z^7 - x^9 - x*z^8)
sage: pls = C.places()
sage: C.place_to_closed_point(pls[-1])
Point (x - 2*z, y - 2*z)
sage: pls2 = C.places(2)
sage: C.place_to_closed_point(pls2[0])
Point (y^2 + y*z + z^2, x + y)
```

**places\_on(point)**

Return the places on the closed point.

INPUT:

- point – a closed point of the curve

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve(x*y*z^4 - x^6 - y^6)
sage: C.singular_closed_points()
[Point (x, y)]
```

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```

sage: p, = _
sage: C.places_on(p)
[Place (1/y, 1/y^2*z, 1/y^3*z^2, 1/y^4*z^3), Place (y, y*z, y*z^2, y*z^3)]
sage: pl1, pl2 = _
sage: C.place_to_closed_point(pl1)
Point (x, y)
sage: C.place_to_closed_point(pl2)
Point (x, y)

```

```

sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: C = Curve(x^2*z - y^3)
sage: [C.places_on(p) for p in C.closed_points()]
[[Place (1/y)],
 [Place (y)],
 [Place (y + 1)],
 [Place (y + 2)],
 [Place (y + 3)],
 [Place (y + 4)]]

```

**singular\_closed\_points()**

Return the singular closed points of the curve.

EXAMPLES:

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve(y^2*z - x^3, P)
sage: C.singular_closed_points()
[Point (x, y)]

```

```

sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: C = Curve(y^2*z^7 - x^9 - x*z^8)
sage: C.singular_closed_points()
[Point (x, z)]

```

**class** sage.schemes.curves.projective\_curve.**IntegralProjectiveCurve\_finite\_field**(*A*, *f*)

Bases: *sage.schemes.curves.projective\_curve.IntegralProjectiveCurve*

Integral projective curve over a finite field.

INPUT:

- *A* – an ambient projective space
- *f* – homogeneous polynomials defining the curve

EXAMPLES:

```

sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: C = Curve(y^2*z^7 - x^9 - x*z^8)
sage: C.function_field()
Function field in z defined by z^8 + 4*y^2*z^7 + 1
sage: C.closed_points()
[Point (x, z),
 Point (x, y),
 Point (x - 2*z, y + 2*z),
 Point (x + 2*z, y + z),

```

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```
Point (x + 2*z, y - z),
Point (x - 2*z, y - 2*z)]
```

**L\_polynomial** (*name='t'*)

Return the L-polynomial of this possibly singular curve.

INPUT:

- *name* – (default: *t*) name of the variable of the polynomial

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(3), 2)
sage: C = Curve(y^2 - x^5 - x^4 - 2*x^3 - 2*x - 2)
sage: Cbar = C.projective_closure()
sage: Cbar.L_polynomial()
9*t^4 - 3*t^3 + t^2 - t + 1
```

**closed\_points** (*degree=1*)

Return a list of closed points of degree of the curve.

INPUT:

- *degree* – a positive integer

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(9), 2)
sage: C = Curve(y^2 - x^5 - x^4 - 2*x^3 - 2*x-2)
sage: Cp = C.projective_closure()
sage: Cp.closed_points()
[Point (x0, x1),
 Point (x0 + (-z2 - 1)*x2, x1),
 Point (x0 + (z2 + 1)*x2, x1),
 Point (x0 + (z2)*x2, x1 + (z2 - 1)*x2),
 Point (x0 + (-z2)*x2, x1 + (-z2 + 1)*x2),
 Point (x0 + (-z2 - 1)*x2, x1 + (-z2 - 1)*x2),
 Point (x0 + (z2 + 1)*x2, x1 + (z2 + 1)*x2),
 Point (x0 + (z2 - 1)*x2, x1 + (z2)*x2),
 Point (x0 + (-z2 + 1)*x2, x1 + (-z2)*x2),
 Point (x0 + x2, x1 - x2),
 Point (x0 - x2, x1 + x2)]
```

**number\_of\_rational\_points** (*r=1*)Return the number of rational points of the curve with constant field extended by degree *r*.

INPUT:

- *r* – positive integer (default: 1)

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(3), 2)
sage: C = Curve(y^2 - x^5 - x^4 - 2*x^3 - 2*x - 2)
sage: Cbar = C.projective_closure()
sage: Cbar.number_of_rational_points(3)
21
sage: D = Cbar.change_ring(Cbar.base_ring().extension(3))
sage: D.base_ring()
```

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```
Finite Field in z3 of size 3^3
sage: len(D.closed_points())
21
```

**places** (*degree=1*)

Return all places on the curve of the degree.

INPUT:

- degree – positive integer

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(GF(5), 2)
sage: C = Curve(x^2*z - y^3)
sage: C.places()
[Place (1/y),
 Place (y),
 Place (y + 1),
 Place (y + 2),
 Place (y + 3),
 Place (y + 4)]
sage: C.places(2)
[Place (y^2 + 2),
 Place (y^2 + 3),
 Place (y^2 + y + 1),
 Place (y^2 + y + 2),
 Place (y^2 + 2*y + 3),
 Place (y^2 + 2*y + 4),
 Place (y^2 + 3*y + 3),
 Place (y^2 + 3*y + 4),
 Place (y^2 + 4*y + 1),
 Place (y^2 + 4*y + 2)]
```

**class** sage.schemes.curves.projective\_curve.**IntegralProjectivePlaneCurve** (*A*, *f*)

Bases: *sage.schemes.curves.projective\_curve.IntegralProjectiveCurve*, *sage.schemes.curves.projective\_curve.ProjectivePlaneCurve\_field*

**class** sage.schemes.curves.projective\_curve.**IntegralProjectivePlaneCurve\_finite\_field** (*A*, *f*)

Bases: *sage.schemes.curves.projective\_curve.IntegralProjectiveCurve\_finite\_field*, *sage.schemes.curves.projective\_curve.ProjectivePlaneCurve\_finite\_field*

Integral projective plane curve over a finite field.

INPUT:

- A – ambient projective plane
- f – a homogeneous equation that defines the curve

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(9), 2)
sage: C = Curve(y^2-x^5-x^4-2*x^3-2*x-2)
sage: Cb = C.projective_closure()
sage: Cb.singular_closed_points()
[Point (x0, x1)]
```

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```
sage: Cb.function_field()
Function field in y defined by  $y^2 + 2x^5 + 2x^4 + x^3 + x + 1$ 
```

**class** sage.schemes.curves.projective\_curve.**ProjectiveCurve**(A,X)

Bases: `sage.schemes.curves.curve.Curve_generic`, `sage.schemes.projective.projective_subscheme.AlgebraicScheme_subscheme_projective`

Curves in projective spaces.

INPUT:

- A – ambient projective space
- X – list of multivariate polynomials; defining equations of the curve

EXAMPLES:

```
sage: P.<x,y,z,w,u> = ProjectiveSpace(GF(7), 4)
sage: C = Curve([y*u^2 - x^3, z*u^2 - x^3, w*u^2 - x^3, y^3 - x^3], P); C
Projective Curve over Finite Field of size 7 defined by  $-x^3 + yu^2$ ,
 $-x^3 + zu^2$ ,  $-x^3 + wu^2$ ,  $-x^3 + y^3$ 
```

```
sage: K.<u> = CyclotomicField(11)
sage: P.<x,y,z,w> = ProjectiveSpace(K, 3)
sage: C = Curve([y*w - u*z^2 - x^2, x*w - 3*u^2*z*w], P); C
Projective Curve over Cyclotomic Field of order 11 and degree 10 defined
by  $-x^2 + (-u)*z^2 + y*w$ ,  $x*w + (-3*u^2)*z*w$ 
```

**affine\_patch**(i, AA=None)

Return the  $i$ -th affine patch of this projective curve.

INPUT:

- i – affine coordinate chart of the projective ambient space of this curve to compute affine patch with respect to
- AA – (default: None) ambient affine space, this is constructed if it is not given

OUTPUT: a curve in affine space

EXAMPLES:

```
sage: P.<x,y,z,w> = ProjectiveSpace(CC, 3)
sage: C = Curve([y*z - x^2, w^2 - x*y], P)
sage: C.affine_patch(0)
Affine Curve over Complex Field with 53 bits of precision defined by
 $y*z - 1.000000000000000$ ,  $w^2 - y$ 
```

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve(x^3 - x^2*y + y^3 - x^2*z, P)
sage: C.affine_patch(1)
Affine Plane Curve over Rational Field defined by  $x^3 - x^2*z - x^2 + 1$ 
```

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: P.<u,v,w> = ProjectiveSpace(QQ, 2)
sage: C = Curve([u^2 - v^2], P)
sage: C.affine_patch(1, A).ambient_space() == A
True
```



**plane\_projection** (*PP=None*)

Return a projection of this curve into a projective plane.

INPUT:

- *PP* – (default: *None*) the projective plane the projected curve will be defined in. This space must be defined over the same base field as this curve, and must have dimension two. This space is constructed if not specified.

OUTPUT: a tuple of

- a scheme morphism from this curve into a projective plane
- the projective curve that is the image of that morphism

EXAMPLES:

```
sage: P.<x,y,z,w,u,v> = ProjectiveSpace(QQ, 5)
sage: C = P.curve([x*u - z*v, w - y, w*y - x^2, y^3*u^2*z - w^4*w])
sage: L.<a,b,c> = ProjectiveSpace(QQ, 2)
sage: proj1 = C.plane_projection(PP=L)
sage: proj1
(Scheme morphism:
  From: Projective Curve over Rational Field defined by x*u - z*v, -y +
w, -x^2 + y*w, -w^5 + 2*y^3*z*u
  To:   Projective Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z : w : u : v) to
      (x : -z + u : -z + v),
  Projective Plane Curve over Rational Field defined by a^8 + 6*a^7*b +
4*a^5*b^3 - 4*a^7*c - 2*a^6*b*c - 4*a^5*b^2*c + 2*a^6*c^2)
sage: proj1[1].ambient_space() is L
True
sage: proj2 = C.projection()
sage: proj2[1].ambient_space() is L
False
```

```
sage: P.<x,y,z,w,u> = ProjectiveSpace(GF(7), 4)
sage: C = P.curve([x^2 - 6*y^2, w*z*u - y^3 + 4*y^2*z, u^2 - x^2])
sage: C.plane_projection()
(Scheme morphism:
  From: Projective Curve over Finite Field of size 7 defined by x^2 +
y^2, -y^3 - 3*y^2*z + z*w*u, -x^2 + u^2
  To:   Projective Space of dimension 2 over Finite Field of size 7
  Defn: Defined on coordinates by sending (x : y : z : w : u) to
      (y : z : -x + w),
  Projective Plane Curve over Finite Field of size 7 defined by x0^10 -
2*x0^9*x1 + 3*x0^8*x1^2 - 2*x0^7*x1^3 + x0^6*x1^4 + 2*x0^6*x1^2*x2^2 -
2*x0^5*x1^3*x2^2 - x0^4*x1^4*x2^2 + x0^2*x1^4*x2^4)
```

```
sage: P.<x,y,z> = ProjectiveSpace(GF(17), 2)
sage: C = P.curve(x^2 - y*z - z^2)
sage: C.plane_projection()
Traceback (most recent call last):
...
TypeError: this curve is already a plane curve
```

**projection** (*P=None, PS=None*)

Return a projection of this curve into projective space of dimension one less than the dimension of the ambient space of this curve.

This curve must not already be a plane curve. Over finite fields, if this curve contains all points in its ambient space, then an error will be returned.

INPUT:

- $P$  – (default: None) a point not on this curve that will be used to define the projection map; this is constructed if not specified.
- $PS$  – (default: None) the projective space the projected curve will be defined in. This space must be defined over the same base ring as this curve, and must have dimension one less than that of the ambient space of this curve. This space will be constructed if not specified.

OUTPUT: a tuple of

- a scheme morphism from this curve into a projective space of dimension one less than that of the ambient space of this curve
- the projective curve that is the image of that morphism

EXAMPLES:

```
sage: K.<a> = CyclotomicField(3)
sage: P.<x,y,z,w> = ProjectiveSpace(K, 3)
sage: C = Curve([y*w - x^2, z*w^2 - a*x^3], P)
sage: L.<a,b,c> = ProjectiveSpace(K, 2)
sage: proj1 = C.projection(PS=L)
sage: proj1
(Scheme morphism:
  From: Projective Curve over Cyclotomic Field of order 3 and degree 2
  defined by -x^2 + y*w, (-a)*x^3 + z*w^2
  To: Projective Space of dimension 2 over Cyclotomic Field of order
  3 and degree 2
  Defn: Defined on coordinates by sending (x : y : z : w) to
      (x : y : -z + w),
  Projective Plane Curve over Cyclotomic Field of order 3 and degree 2
  defined by a^6 + (-a)*a^3*b^3 - a^4*b*c)
sage: proj1[1].ambient_space() is L
True
sage: proj2 = C.projection()
sage: proj2[1].ambient_space() is L
False
```

```
sage: P.<x,y,z,w,a,b,c> = ProjectiveSpace(QQ, 6)
sage: C = Curve([y - x, z - a - b, w^2 - c^2, z - x - a, x^2 - w*z], P)
sage: C.projection()
(Scheme morphism:
  From: Projective Curve over Rational Field defined by -x + y, z - a -
  b, w^2 - c^2, -x + z - a, x^2 - z*w
  To: Projective Space of dimension 5 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z : w : a : b : c)
  to
      (x : y : -z + w : a : b : c),
  Projective Curve over Rational Field defined by x1 - x4, x0 - x4, x2*x3
  + x3^2 + x2*x4 + 2*x3*x4, x2^2 - x3^2 - 2*x3*x4 + x4^2 - x5^2, x2*x4^2 +
  x3*x4^2 + x4^3 - x3*x5^2 - x4*x5^2, x4^4 - x3^2*x5^2 - 2*x3*x4*x5^2 -
  x4^2*x5^2)
```

```
sage: P.<x,y,z,w> = ProjectiveSpace(GF(2), 3)
sage: C = P.curve([(x - y)*(x - z)*(x - w)*(y - z)*(y - w),
↪ x*y*z*w*(x+y+z+w)])
```

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```

sage: C.projection()
Traceback (most recent call last):
...
NotImplementedError: this curve contains all points of its ambient space

```

```

sage: P.<x,y,z,w,u> = ProjectiveSpace(GF(7), 4)
sage: C = P.curve([x^3 - y*z*u, w^2 - u^2 + 2*x*z, 3*x*w - y^2])
sage: L.<a,b,c,d> = ProjectiveSpace(GF(7), 3)
sage: C.projection(PS=L)
(Scheme morphism:
  From: Projective Curve over Finite Field of size 7 defined by x^3 -
y*z*u, 2*x*z + w^2 - u^2, -y^2 + 3*x*w
  To:   Projective Space of dimension 3 over Finite Field of size 7
  Defn: Defined on coordinates by sending (x : y : z : w : u) to
        (x : y : z : w),
  Projective Curve over Finite Field of size 7 defined by b^2 - 3*a*d,
a^5*b + a*b*c^3*d - 3*b*c^2*d^3, a^6 + a^2*c^3*d - 3*a*c^2*d^3)
sage: Q.<a,b,c> = ProjectiveSpace(GF(7), 2)
sage: C.projection(PS=Q)
Traceback (most recent call last):
...
TypeError: (=Projective Space of dimension 2 over Finite Field of size
7) must have dimension (=3)

```

```

sage: PP.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = PP.curve([x^3 - z^2*y, w^2 - z*x])
sage: Q = PP([1,0,1,1])
sage: C.projection(P=Q)
(Scheme morphism:
  From: Projective Curve over Rational Field defined by x^3 - y*z^2, -x*z +
↪w^2
  To:   Projective Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z : w) to
        (y : -x + z : -x + w),
  Projective Plane Curve over Rational Field defined by x0*x1^5 -
6*x0*x1^4*x2 + 14*x0*x1^3*x2^2 - 16*x0*x1^2*x2^3 + 9*x0*x1*x2^4 -
2*x0*x2^5 - x2^6)
sage: LL.<a,b,c> = ProjectiveSpace(QQ, 2)
sage: Q = PP([0,0,0,1])
sage: C.projection(PS=LL, P=Q)
(Scheme morphism:
  From: Projective Curve over Rational Field defined by x^3 - y*z^2,
-x*z + w^2
  To:   Projective Space of dimension 2 over Rational Field
  Defn: Defined on coordinates by sending (x : y : z : w) to
        (x : y : z),
  Projective Plane Curve over Rational Field defined by a^3 - b*c^2)
sage: Q = PP([0,0,1,0])
sage: C.projection(P=Q)
Traceback (most recent call last):
...
TypeError: (=(0 : 0 : 1 : 0)) must be a point not on this curve

```

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = P.curve(y^2 - x^2 + z^2)

```

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```
sage: C.projection()
Traceback (most recent call last):
...
TypeError: this curve is already a plane curve
```

**class** sage.schemes.curves.projective\_curve.**ProjectiveCurve\_field**( $A, X$ )

Bases: `sage.schemes.curves.projective_curve.ProjectiveCurve`, `sage.schemes.projective.projective_subscheme.AlgebraicScheme_subscheme_projective_field`

Projective curves over fields.

**arithmetic\_genus**()

Return the arithmetic genus of this projective curve.

This is the arithmetic genus  $g_a(C)$  as defined in [Har1977]. If  $P$  is the Hilbert polynomial of the defining ideal of this curve, then the arithmetic genus of this curve is  $1 - P(0)$ . This curve must be irreducible.

EXAMPLES:

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = P.curve([w*z - x^2, w^2 + y^2 + z^2])
sage: C.arithmetic_genus()
1
```

```
sage: P.<x,y,z,w,t> = ProjectiveSpace(GF(7), 4)
sage: C = P.curve([t^3 - x*y*w, x^3 + y^3 + z^3, z - w])
sage: C.arithmetic_genus()
10
```

**is\_complete\_intersection**()

Return whether this projective curve is a complete intersection.

EXAMPLES:

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([x*y - z*w, x^2 - y*w, y^2*w - x*z*w], P)
sage: C.is_complete_intersection()
False
```

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([y*w - x^2, z*w^2 - x^3], P)
sage: C.is_complete_intersection()
True

sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([z^2 - y*w, y*z - x*w, y^2 - x*z], P)
sage: C.is_complete_intersection()
False
```

**tangent\_line**( $p$ )

Return the tangent line at the point  $p$ .

INPUT:

- $p$  – a rational point of the curve

EXAMPLES:

```

sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([x*y - z*w, x^2 - y*w, y^2*w - x*z*w], P)
sage: p = C(1,1,1,1)
sage: C.tangent_line(p)
Projective Curve over Rational Field defined by -2*x + y + w, -3*x + z + 2*w

```

**class** sage.schemes.curves.projective\_curve.**ProjectivePlaneCurve**(A,f)

Bases: *sage.schemes.curves.projective\_curve.ProjectiveCurve*

Curves in projective planes.

INPUT:

- A – projective plane
- f – homogeneous polynomial in the homogeneous coordinate ring of the plane

EXAMPLES:

A projective plane curve defined over an algebraic closure of  $\mathbb{Q}$ :

```

sage: P.<x,y,z> = ProjectiveSpace(QQbar, 2)
sage: set_verbosity(-1) # suppress warnings for slow computation
sage: C = Curve([y*z - x^2 - QQbar.gen()*z^2], P); C
Projective Plane Curve over Algebraic Field defined by
-x^2 + y*z + (-1)*z^2

```

A projective plane curve defined over a finite field:

```

sage: P.<x,y,z> = ProjectiveSpace(GF(5^2, 'v'), 2)
sage: C = Curve([y^2*z - x*z^2 - z^3], P); C
Projective Plane Curve over Finite Field in v of size 5^2 defined by y^2*z - x*z^
↪ 2 - z^3

```

**degree** ()

Return the degree of this projective curve.

For a plane curve, this is just the degree of its defining polynomial.

OUTPUT: integer.

EXAMPLES:

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = P.curve([y^7 - x^2*z^5 + 7*z^7])
sage: C.degree()
7

```

**divisor\_of\_function** (r)

Return the divisor of a function on a curve.

INPUT: r is a rational function on X

OUTPUT:

- list – The divisor of r represented as a list of coefficients and points. (TODO: This will change to a more structural output in the future.)

EXAMPLES:

```

sage: FF = FiniteField(5)
sage: P2 = ProjectiveSpace(2, FF, names = ['x', 'y', 'z'])
sage: R = P2.coordinate_ring()
sage: x, y, z = R.gens()
sage: f = y^2*z^7 - x^9 - x*z^8
sage: C = Curve(f)
sage: K = FractionField(R)
sage: r = 1/x
sage: C.divisor_of_function(r)      # todo: not implemented !!!!
[[-1, (0, 0, 1)]]
sage: r = 1/x^3
sage: C.divisor_of_function(r)      # todo: not implemented !!!!
[[-3, (0, 0, 1)]]

```

**excellent\_position( $Q$ )**

Return a transformation of this curve into one in excellent position with respect to the point  $Q$ .

Here excellent position is defined as in [Ful1989]. A curve  $C$  of degree  $d$  containing the point  $(0 : 0 : 1)$  with multiplicity  $r$  is said to be in excellent position if none of the coordinate lines are tangent to  $C$  at any of the fundamental points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ , and  $(0 : 0 : 1)$ , and if the two coordinate lines containing  $(0 : 0 : 1)$  intersect  $C$  transversally in  $d - r$  distinct non-fundamental points, and if the other coordinate line intersects  $C$  transversally at  $d$  distinct, non-fundamental points.

INPUT:

- $Q$  – a point on this curve.

OUTPUT:

- a scheme morphism from this curve to a curve in excellent position that is a restriction of a change of coordinates map of the projective plane.

EXAMPLES:

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([x*y - z^2], P)
sage: Q = P([1,1,1])
sage: C.excellent_position(Q)
Scheme morphism:
  From: Projective Plane Curve over Rational Field defined by x*y - z^2
  To:   Projective Plane Curve over Rational Field defined by -x^2 -
3*x*y - 4*y^2 - x*z - 3*y*z
  Defn: Defined on coordinates by sending (x : y : z) to
        (-x + 1/2*y + 1/2*z : -1/2*y + 1/2*z : x + 1/2*y - 1/2*z)

```

```

sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^2 - 3)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: C = P.curve([z^2*y^3*x^4 - y^6*x^3 - 4*z^2*y^4*x^3 - 4*z^4*y^2*x^3 +
↪ 3*y^7*x^2 + 10*z^2*y^5*x^2\
+ 9*z^4*y^3*x^2 + 5*z^6*y*x^2 - 3*y^8*x - 9*z^2*y^6*x - 11*z^4*y^4*x - 7*z^
↪ 6*y^2*x - 2*z^8*x + y^9 +\
2*z^2*y^7 + 3*z^4*y^5 + 4*z^6*y^3 + 2*z^8*y])
sage: Q = P([1,0,0])
sage: C.excellent_position(Q)
Scheme morphism:
  From: Projective Plane Curve over Number Field in b with defining
polynomial a^2 - 3 defined by -x^3*y^6 + 3*x^2*y^7 - 3*x*y^8 + y^9 +
x^4*y^3*z^2 - 4*x^3*y^4*z^2 + 10*x^2*y^5*z^2 - 9*x*y^6*z^2 + 2*y^7*z^2 -

```

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```

4*x^3*y^2*z^4 + 9*x^2*y^3*z^4 - 11*x*y^4*z^4 + 3*y^5*z^4 + 5*x^2*y*z^6 -
7*x*y^2*z^6 + 4*y^3*z^6 - 2*x*z^8 + 2*y*z^8

```

```

To: Projective Plane Curve over Number Field in b with defining
polynomial a^2 - 3 defined by 900*x^9 - 7410*x^8*y + 29282*x^7*y^2 -
69710*x^6*y^3 + 110818*x^5*y^4 - 123178*x^4*y^5 + 96550*x^3*y^6 -
52570*x^2*y^7 + 18194*x*y^8 - 3388*y^9 - 1550*x^8*z + 9892*x^7*y*z -
30756*x^6*y^2*z + 58692*x^5*y^3*z - 75600*x^4*y^4*z + 67916*x^3*y^5*z -
42364*x^2*y^6*z + 16844*x*y^7*z - 3586*y^8*z + 786*x^7*z^2 -
3958*x^6*y*z^2 + 9746*x^5*y^2*z^2 - 14694*x^4*y^3*z^2 +
15174*x^3*y^4*z^2 - 10802*x^2*y^5*z^2 + 5014*x*y^6*z^2 - 1266*y^7*z^2 -
144*x^6*z^3 + 512*x^5*y*z^3 - 912*x^4*y^2*z^3 + 1024*x^3*y^3*z^3 -
816*x^2*y^4*z^3 + 512*x*y^5*z^3 - 176*y^6*z^3 + 8*x^5*z^4 - 8*x^4*y*z^4
- 16*x^3*y^2*z^4 + 16*x^2*y^3*z^4 + 8*x*y^4*z^4 - 8*y^5*z^4

```

```

Defn: Defined on coordinates by sending (x : y : z) to
      (1/4*y + 1/2*z : -1/4*y + 1/2*z : x + 1/4*y - 1/2*z)

```

```

sage: set_verbose(-1)
sage: a = QQbar(sqrt(2))
sage: P.<x,y,z> = ProjectiveSpace(QQbar, 2)
sage: C = Curve([(1/4*a)*x^3 + (-3/4*a)*x^2*y + (-3/4*a)*x*y^2 + (-1/4*a)*y^
↪ 3 - 2*x*y*z], P)
sage: Q = P([0,0,1])
sage: C.excellent_position(Q)

```

Scheme morphism:

```

From: Projective Plane Curve over Algebraic Field defined by
(-0.3535533905932738?)*x^3 + (-1.060660171779822?)*x^2*y +
(-1.060660171779822?)*x*y^2 + (-0.3535533905932738?)*y^3 + (-2)*x*y*z
To: Projective Plane Curve over Algebraic Field defined by
(-2.828427124746190?)*x^3 + (-2)*x^2*y + 2*y^3 + (-2)*x^2*z + 2*y^2*z
Defn: Defined on coordinates by sending (x : y : z) to
      (1/2*x + 1/2*y : (-1/2)*x + 1/2*y : 1/2*x + (-1/2)*y + z)

```

### **is\_ordinary\_singularity(P)**

Return whether the singular point  $P$  of this projective plane curve is an ordinary singularity.

The point  $P$  is an ordinary singularity of this curve if it is a singular point, and if the tangents of this curve at  $P$  are distinct.

INPUT:

- $P$  – a point on this curve.

OUTPUT:

- Boolean. True or False depending on whether  $P$  is or is not an ordinary singularity of this curve, respectively. An error is raised if  $P$  is not a singular point of this curve.

EXAMPLES:

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([y^2*z^3 - x^5], P)
sage: Q = P([0,0,1])
sage: C.is_ordinary_singularity(Q)
False

```

```

sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^2 - 3)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)

```

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```

sage: C = P.curve([x^2*y^3*z^4 - y^6*z^3 - 4*x^2*y^4*z^3 - 4*x^4*y^2*z^3 +
↪ 3*y^7*z^2 + 10*x^2*y^5*z^2\
+ 9*x^4*y^3*z^2 + 5*x^6*y*z^2 - 3*y^8*z - 9*x^2*y^6*z - 11*x^4*y^4*z - 7*x^
↪ 6*y^2*z - 2*x^8*z + y^9 +\
2*x^2*y^7 + 3*x^4*y^5 + 4*x^6*y^3 + 2*x^8*y])
sage: Q = P([0,1,1])
sage: C.is_ordinary_singularity(Q)
True

```

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = P.curve([z^5 - y^5 + x^5 + x*y^2*z^2])
sage: Q = P([0,1,1])
sage: C.is_ordinary_singularity(Q)
Traceback (most recent call last):
...
TypeError: (=(0 : 1 : 1)) is not a singular point of (=(Projective Plane
Curve over Rational Field defined by x^5 - y^5 + x*y^2*z^2 + z^5)

```

**is\_singular** (*P=None*)

Return whether this curve is singular or not, or if a point *P* is provided, whether *P* is a singular point of this curve.

INPUT:

- *P* – (default: None) a point on this curve

OUTPUT:

If no point *P* is provided, return True or False depending on whether this curve is singular or not. If a point *P* is provided, return True or False depending on whether *P* is or is not a singular point of this curve.

EXAMPLES:

Over  $\mathbb{Q}$ :

```

sage: F = QQ
sage: P2.<X,Y,Z> = ProjectiveSpace(F, 2)
sage: C = Curve(X^3-Y^2*Z)
sage: C.is_singular()
True

```

Over a finite field:

```

sage: F = GF(19)
sage: P2.<X,Y,Z> = ProjectiveSpace(F, 2)
sage: C = Curve(X^3+Y^3+Z^3)
sage: C.is_singular()
False
sage: D = Curve(X^4-X*Z^3)
sage: D.is_singular()
True
sage: E = Curve(X^5+19*Y^5+Z^5)
sage: E.is_singular()
True
sage: E = Curve(X^5+9*Y^5+Z^5)
sage: E.is_singular()
False

```



Over  $\mathbb{C}$ :

```
sage: F = CC
sage: P2.<X,Y,Z> = ProjectiveSpace(F, 2)
sage: C = Curve(X)
sage: C.is_singular()
False
sage: D = Curve(Y^2*Z-X^3)
sage: D.is_singular()
True
sage: E = Curve(Y^2*Z-X^3+Z^3)
sage: E.is_singular()
False
```

Showing that [trac ticket #12187](#) is fixed:

```
sage: F.<X,Y,Z> = GF(2)[]
sage: G = Curve(X^2+Y*Z)
sage: G.is_singular()
False
```

```
sage: P.<x,y,z> = ProjectiveSpace(CC, 2)
sage: C = Curve([y^4 - x^3*z], P)
sage: Q = P([0,0,1])
sage: C.is_singular()
True
```

### **is\_transverse**( $C, P$ )

Return whether the intersection of this curve with the curve  $C$  at the point  $P$  is transverse.

The intersection at  $P$  is transverse if  $P$  is a nonsingular point of both curves, and if the tangents of the curves at  $P$  are distinct.

INPUT:

- $C$  – a curve in the ambient space of this curve.
- $P$  – a point in the intersection of both curves.

OUTPUT: Boolean.

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([x^2 - y^2], P)
sage: D = Curve([x - y], P)
sage: Q = P([1,1,0])
sage: C.is_transverse(D, Q)
False
```

```
sage: K = QuadraticField(-1)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: C = Curve([y^2*z - K.0*x^3], P)
sage: D = Curve([z*x + y^2], P)
sage: Q = P([0,0,1])
sage: C.is_transverse(D, Q)
False
```

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([x^2 - 2*y^2 - 2*z^2], P)
sage: D = Curve([y - z], P)
sage: Q = P([2,1,1])
sage: C.is_transverse(D, Q)
True

```

**local\_coordinates** (*pt*, *n*)

Return local coordinates to precision *n* at the given point.

Behaviour is flaky - some choices of *n* are worse than others.

INPUT:

- **pt** – a rational point on **X** which is not a point of ramification for the projection  $(x, y) \rightarrow x$ .
- *n* – the number of terms desired

OUTPUT:  $x = x_0 + t, y = y_0 +$  power series in  $t$

EXAMPLES:

```

sage: FF = FiniteField(5)
sage: P2 = ProjectiveSpace(2, FF, names = ['x', 'y', 'z'])
sage: x, y, z = P2.coordinate_ring().gens()
sage: C = Curve(y^2*z^7 - x^9 - x*z^8)
sage: pt = C([2,3,1])
sage: C.local_coordinates(pt, 9)      # todo: not implemented !!!!
      [2 + t, 3 + 3*t^2 + t^3 + 3*t^4 + 3*t^6 + 3*t^7 + t^8 + 2*t^9 + 3*t^11,
      ↪ + 3*t^12]

```

**ordinary\_model** ()

Return a birational map from this curve to a plane curve with only ordinary singularities.

Currently only implemented over number fields. If not all of the coordinates of the non-ordinary singularities of this curve are contained in its base field, then the domain and codomain of the map returned will be defined over an extension. This curve must be irreducible.

OUTPUT:

- a scheme morphism from this curve to a curve with only ordinary singularities that defines a birational map between the two curves.

EXAMPLES:

```

sage: set_verbose(-1)
sage: K = QuadraticField(3)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: C = Curve([x^5 - K.0*y*z^4], P)
sage: C.ordinary_model()
Scheme morphism:
  From: Projective Plane Curve over Number Field in a with defining
  ↪ polynomial x^2 - 3 with a = 1.732050807568878? defined by x^5 + (-a)*y*z^4
  To:   Projective Plane Curve over Number Field in a with defining
  ↪ polynomial x^2 - 3 with a = 1.732050807568878? defined by (-a)*x^5*y + (-
  ↪ 4*a)*x^4*y^2 + (-6*a)*x^3*y^3 + (-4*a)*x^2*y^4 + (-a)*x*y^5 + (-a - 1)*x^
  ↪ 5*z + (-4*a + 5)*x^4*y*z + (-6*a - 10)*x^3*y^2*z + (-4*a + 10)*x^2*y^3*z +
  ↪ (-a - 5)*x*y^4*z + y^5*z
  Defn: Defined on coordinates by sending (x : y : z) to
      (-1/4*x^2 - 1/2*x*y + 1/2*x*z + 1/2*y*z - 1/4*z^2 : 1/4*x^2 + 1/2*x*y
  ↪ + 1/2*y*z - 1/4*z^2 : -1/4*x^2 + 1/4*z^2)

```

```

sage: set_verbose(-1)
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([y^2*z^2 - x^4 - x^3*z], P)
sage: D = C.ordinary_model(); D # long time (2 seconds)
Scheme morphism:
  From: Projective Plane Curve over Rational Field defined by -x^4 -
x^3*z + y^2*z^2
  To:   Projective Plane Curve over Rational Field defined by 4*x^6*y^3
- 24*x^5*y^4 + 36*x^4*y^5 + 8*x^6*y^2*z - 40*x^5*y^3*z + 24*x^4*y^4*z +
72*x^3*y^5*z - 4*x^6*y*z^2 + 8*x^5*y^2*z^2 - 56*x^4*y^3*z^2 +
104*x^3*y^4*z^2 + 44*x^2*y^5*z^2 + 8*x^6*z^3 - 16*x^5*y*z^3 -
24*x^4*y^2*z^3 + 40*x^3*y^3*z^3 + 48*x^2*y^4*z^3 + 8*x*y^5*z^3 -
8*x^5*z^4 + 36*x^4*y*z^4 - 56*x^3*y^2*z^4 + 20*x^2*y^3*z^4 +
40*x*y^4*z^4 - 16*y^5*z^4
  Defn: Defined on coordinates by sending (x : y : z) to
      (-3/64*x^4 + 9/64*x^2*y^2 - 3/32*x*y^3 - 1/16*x^3*z -
1/8*x^2*y*z + 1/4*x*y^2*z - 1/16*y^3*z - 1/8*x*y*z^2 + 1/16*y^2*z^2 :
-1/64*x^4 + 3/64*x^2*y^2 - 1/32*x*y^3 + 1/16*x*y^2*z - 1/16*y^3*z +
1/16*y^2*z^2 : 3/64*x^4 - 3/32*x^3*y + 3/64*x^2*y^2 + 1/16*x^3*z -
3/16*x^2*y*z + 1/8*x*y^2*z - 1/8*x*y*z^2 + 1/16*y^2*z^2)
sage: all(D.codomain().is_ordinary_singularity(Q) for Q in D.codomain()).
↪singular_points() # long time
True

```

```

sage: set_verbose(-1)
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([(x^2 + y^2 - y*z - 2*z^2)*(y*z - x^2 + 2*z^2)*z + y^5], P)
sage: C.ordinary_model() # long time (5 seconds)
Scheme morphism:
  From: Projective Plane Curve over Number Field in a with defining
polynomial y^2 - 2 defined by y^5 - x^4*z - x^2*y^2*z + 2*x^2*y*z^2 +
y^3*z^2 + 4*x^2*z^3 + y^2*z^3 - 4*y*z^4 - 4*z^5
  To:   Projective Plane Curve over Number Field in a with defining
polynomial y^2 - 2 defined by (-29*a + 1)*x^8*y^6 + (10*a + 158)*x^7*y^7
+ (-109*a - 31)*x^6*y^8 + (-80*a - 198)*x^8*y^5*z + (531*a +
272)*x^7*y^6*z + (170*a - 718)*x^6*y^7*z + (19*a - 636)*x^5*y^8*z +
(-200*a - 628)*x^8*y^4*z^2 + (1557*a - 114)*x^7*y^5*z^2 + (2197*a -
2449)*x^6*y^6*z^2 + (1223*a - 3800)*x^5*y^7*z^2 + (343*a -
1329)*x^4*y^8*z^2 + (-323*a - 809)*x^8*y^3*z^3 + (1630*a -
631)*x^7*y^4*z^3 + (4190*a - 3126)*x^6*y^5*z^3 + (3904*a -
7110)*x^5*y^6*z^3 + (1789*a - 5161)*x^4*y^7*z^3 + (330*a -
1083)*x^3*y^8*z^3 + (-259*a - 524)*x^8*y^2*z^4 + (720*a -
605)*x^7*y^3*z^4 + (3082*a - 2011)*x^6*y^4*z^4 + (4548*a -
5462)*x^5*y^5*z^4 + (2958*a - 6611)*x^4*y^6*z^4 + (994*a -
2931)*x^3*y^7*z^4 + (117*a - 416)*x^2*y^8*z^4 + (-108*a - 184)*x^8*y*z^5
+ (169*a - 168)*x^7*y^2*z^5 + (831*a - 835)*x^6*y^3*z^5 + (2225*a -
1725)*x^5*y^4*z^5 + (1970*a - 3316)*x^4*y^5*z^5 + (952*a -
2442)*x^3*y^6*z^5 + (217*a - 725)*x^2*y^7*z^5 + (16*a - 77)*x*y^8*z^5 +
(-23*a - 35)*x^8*z^6 + (43*a + 24)*x^7*y*z^6 + (21*a - 198)*x^6*y^2*z^6
+ (377*a - 179)*x^5*y^3*z^6 + (458*a - 537)*x^4*y^4*z^6 + (288*a -
624)*x^3*y^5*z^6 + (100*a - 299)*x^2*y^6*z^6 + (16*a - 67)*x*y^7*z^6 -
5*y^8*z^6
  Defn: Defined on coordinates by sending (x : y : z) to
      ((-5/128*a - 5/128)*x^4 + (-5/32*a + 5/32)*x^3*y + (-1/16*a +
3/32)*x^2*y^2 + (1/16*a - 1/16)*x*y^3 + (1/32*a - 1/32)*y^4 - 1/32*x^3*z
+ (3/16*a - 5/8)*x^2*y*z + (1/8*a - 5/16)*x*y^2*z + (1/8*a +
5/32)*x^2*z^2 + (-3/16*a + 5/16)*x*y*z^2 + (-3/16*a - 1/16)*y^2*z^2 +

```

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$$1/16*x*z^3 + (1/4*a + 1/4)*y*z^3 + (-3/32*a - 5/32)*z^4 : (-5/128*a - 5/128)*x^4 + (5/32*a)*x^3*y + (3/32*a + 3/32)*x^2*y^2 + (-1/16*a)*x*y^3 + (-1/32*a - 1/32)*y^4 - 1/32*x^3*z + (-11/32*a)*x^2*y*z + (1/8*a + 5/16)*x*y^2*z + (3/16*a + 1/4)*y^3*z + (1/8*a + 5/32)*x^2*z^2 + (-1/16*a - 3/8)*x*y*z^2 + (-3/8*a - 9/16)*y^2*z^2 + 1/16*x*z^3 + (5/16*a + 1/2)*y*z^3 + (-3/32*a - 5/32)*z^4 : (1/64*a + 3/128)*x^4 + (-1/32*a - 1/32)*x^3*y + (3/32*a - 9/32)*x^2*y^2 + (1/16*a - 3/16)*x*y^3 - 1/32*y^4 + (3/32*a + 1/8)*x^2*y*z + (-1/8*a + 1/8)*x*y^2*z + (-1/16*a)*y^3*z + (-1/16*a - 3/32)*x^2*z^2 + (1/16*a + 1/16)*x*y*z^2 + (3/16*a + 3/16)*y^2*z^2 + (-3/16*a - 1/4)*y*z^3 + (1/16*a + 3/32)*z^4$$
**plot** (\*args, \*\*kws)

Plot the real points of an affine patch of this projective plane curve.

INPUT:

- self - an affine plane curve
- patch - (optional) the affine patch to be plotted; if not specified, the patch corresponding to the last projective coordinate being nonzero
- \*args - optional tuples (variable, minimum, maximum) for plotting dimensions
- \*\*kws - optional keyword arguments passed on to `implicit_plot`

EXAMPLES:

A cuspidal curve:

```
sage: R.<x, y, z> = QQ[]
sage: C = Curve(x^3 - y^2*z)
sage: C.plot()
Graphics object consisting of 1 graphics primitive
```

The other affine patches of the same curve:

```
sage: C.plot(patch=0)
Graphics object consisting of 1 graphics primitive
sage: C.plot(patch=1)
Graphics object consisting of 1 graphics primitive
```

An elliptic curve:

```
sage: E = EllipticCurve('101a')
sage: C = Curve(E)
sage: C.plot()
Graphics object consisting of 1 graphics primitive
sage: C.plot(patch=0)
Graphics object consisting of 1 graphics primitive
sage: C.plot(patch=1)
Graphics object consisting of 1 graphics primitive
```

A hyperelliptic curve:

```
sage: P.<x> = QQ[]
sage: f = 4*x^5 - 30*x^3 + 45*x - 22
sage: C = HyperellipticCurve(f)
sage: C.plot()
Graphics object consisting of 1 graphics primitive
```

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```

sage: C.plot(patch=0)
Graphics object consisting of 1 graphics primitive
sage: C.plot(patch=1)
Graphics object consisting of 1 graphics primitive

```

**quadratic\_transform()**

Return a birational map from this curve to the proper transform of this curve with respect to the standard Cremona transformation.

The standard Cremona transformation is the birational automorphism of  $\mathbb{P}^2$  defined  $(x : y : z) \mapsto (yz : xz : xy)$ .

OUTPUT:

- a scheme morphism representing the restriction of the standard Cremona transformation from this curve to the proper transform.

EXAMPLES:

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve(x^3*y - z^4 - z^2*x^2, P)
sage: C.quadratic_transform()
Scheme morphism:
  From: Projective Plane Curve over Rational Field defined by x^3*y -
x^2*z^2 - z^4
  To:   Projective Plane Curve over Rational Field defined by -x^3*y -
x*y*z^2 + z^4
  Defn: Defined on coordinates by sending (x : y : z) to
      (y*z : x*z : x*y)

```

```

sage: P.<x,y,z> = ProjectiveSpace(GF(17), 2)
sage: C = P.curve([y^7*z^2 - 16*x^9 + x*y*z^7 + 2*z^9])
sage: C.quadratic_transform()
Scheme morphism:
  From: Projective Plane Curve over Finite Field of size 17 defined by
x^9 + y^7*z^2 + x*y*z^7 + 2*z^9
  To:   Projective Plane Curve over Finite Field of size 17 defined by
2*x^9*y^7 + x^8*y^6*z^2 + x^9*z^7 + y^7*z^9
  Defn: Defined on coordinates by sending (x : y : z) to
      (y*z : x*z : x*y)

```

**tangents(P, factor=True)**

Return the tangents of this projective plane curve at the point P.

These are found by homogenizing the tangents of an affine patch of this curve containing P. The point P must be a point on this curve.

INPUT:

- P – a point on this curve.
- factor – (default: True) whether to attempt computing the polynomials of the individual tangent lines over the base field of this curve, or to just return the polynomial corresponding to the union of the tangent lines (which requires fewer computations).

OUTPUT:

a list of polynomials in the coordinate ring of the ambient space of this curve.

EXAMPLES:

```

sage: set_verbose(-1)
sage: P.<x,y,z> = ProjectiveSpace(QQbar, 2)
sage: C = Curve([x^3*y + 2*x^2*y^2 + x*y^3 + x^3*z + 7*x^2*y*z + 14*x*y^2*z +
↪ 9*y^3*z], P)
sage: Q = P([0,0,1])
sage: C.tangents(Q)
[x + 4.147899035704788?*y, x + (1.426050482147607? + 0.3689894074818041?*I)*y,
x + (1.426050482147607? - 0.3689894074818041?*I)*y]
sage: C.tangents(Q, factor=False)
[6*x^3 + 42*x^2*y + 84*x*y^2 + 54*y^3]

```

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = P.curve([x^2*y^3*z^4 - y^6*z^3 - 4*x^2*y^4*z^3 - 4*x^4*y^2*z^3 +
↪ 3*y^7*z^2 + \
10*x^2*y^5*z^2 + 9*x^4*y^3*z^2 + 5*x^6*y*z^2 - 3*y^8*z - 9*x^2*y^6*z - 11*x^
↪ 4*y^4*z - \
7*x^6*y^2*z - 2*x^8*z + y^9 + 2*x^2*y^7 + 3*x^4*y^5 + 4*x^6*y^3 + 2*x^8*y])
sage: Q = P([0,1,1])
sage: C.tangents(Q)
[-y + z, 3*x^2 - y^2 + 2*y*z - z^2]

```

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = P.curve([z^3*x + y^4 - x^2*z^2])
sage: Q = P([1,1,1])
sage: C.tangents(Q)
Traceback (most recent call last):
...
TypeError: (=(1 : 1 : 1)) is not a point on (=Projective Plane Curve
over Rational Field defined by y^4 - x^2*z^2 + x*z^3)

```

**class** sage.schemes.curves.projective\_curve.**ProjectivePlaneCurve\_field**(A,f)  
 Bases: *sage.schemes.curves.projective\_curve.ProjectivePlaneCurve*, *sage.schemes.curves.projective\_curve.ProjectiveCurve\_field*

Projective plane curves over fields.

**arithmetic\_genus**()

Return the arithmetic genus of this projective curve.

This is the arithmetic genus  $g_a(C)$  as defined in [Har1977]. For a projective plane curve of degree  $d$ , this is simply  $(d-1)(d-2)/2$ . It need *not* equal the geometric genus (the genus of the normalization of the curve). This curve must be irreducible.

EXAMPLES:

```

sage: x,y,z = PolynomialRing(GF(5), 3, 'xyz').gens()
sage: C = Curve(y^2*z^7 - x^9 - x*z^8); C
Projective Plane Curve over Finite Field of size 5 defined by -x^9 + y^2*z^7 -
↪ x*z^8
sage: C.arithmetic_genus()
28
sage: C.genus()
4

```

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([y^3*x - x^2*y*z - 7*z^4])
sage: C.arithmetic_genus()
3

```

**fundamental\_group()**

Return a presentation of the fundamental group of the complement of `self`.

**Note:** The curve must be defined over the rationals or a number field with an embedding over  $\overline{\mathbb{Q}}$ .

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = P.curve(x^2*z-y^3)
sage: C.fundamental_group() # optional - sirocco
Finitely presented group < x0 | x0^3 >
```

In the case of number fields, they need to have an embedding into the algebraic field:

```
sage: a = QQ[x](x^2+5).roots(QQbar)[0][0]
sage: a
-2.236067977499790? * I
sage: F = NumberField(a.minpoly(), 'a', embedding=a)
sage: P.<x,y,z> = ProjectiveSpace(F, 2)
sage: F.inject_variables()
Defining a
sage: C = P.curve(x^2 + a * y^2)
sage: C.fundamental_group() # optional - sirocco
Finitely presented group < x0 | >
```

**Warning:** This functionality requires the `sirocco` package to be installed.

**rational\_parameterization()**

Return a rational parameterization of this curve.

This curve must have rational coefficients and be absolutely irreducible (i.e. irreducible over the algebraic closure of the rational field). The curve must also be rational (have geometric genus zero).

The rational parameterization may have coefficients in a quadratic extension of the rational field.

OUTPUT:

- a birational map between  $\mathbb{P}^1$  and this curve, given as a scheme morphism.

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([y^2*z - x^3], P)
sage: C.rational_parameterization()
Scheme morphism:
  From: Projective Space of dimension 1 over Rational Field
  To:   Projective Plane Curve over Rational Field defined by -x^3 + y^2*z
  Defn: Defined on coordinates by sending (s : t) to
        (s^2*t : s^3 : t^3)
```

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([x^3 - 4*y*z^2 + x*z^2 - x*y*z], P)
sage: C.rational_parameterization()
Scheme morphism:
  From: Projective Space of dimension 1 over Rational Field
```

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```

To: Projective Plane Curve over Rational Field defined by  $x^3 - x*y*z + x*z^2 - 4*y*z^2$ 
Defn: Defined on coordinates by sending  $(s : t)$  to
       $(4*s^2*t + s*t^2 : s^2*t + t^3 : 4*s^3 + s^2*t)$ 

```

```

sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([x^2 + y^2 + z^2], P)
sage: C.rational_parameterization()
Scheme morphism:
  From: Projective Space of dimension 1 over Number Field in a with defining
  ↪ polynomial  $a^2 + 1$ 
  To: Projective Plane Curve over Number Field in a with defining
  ↪ polynomial  $a^2 + 1$  defined by  $x^2 + y^2 + z^2$ 
  Defn: Defined on coordinates by sending  $(s : t)$  to
         $((-a)*s^2 + (-a)*t^2 : s^2 - t^2 : 2*s*t)$ 

```

**riemann\_surface** (\*\*kwargs)

Return the complex Riemann surface determined by this curve

OUTPUT:

- RiemannSurface object

EXAMPLES:

```

sage: R.<x,y,z>=QQ[]
sage: C=Curve(x^3+3*y^3+5*z^3)
sage: C.riemann_surface()
Riemann surface defined by polynomial  $f = x^3 + 3*y^3 + 5 = 0$ , with 53 bits
  ↪ of precision

```

**class** sage.schemes.curves.projective\_curve.**ProjectivePlaneCurve\_finite\_field**(A, f)

Bases: *sage.schemes.curves.projective\_curve.ProjectivePlaneCurve\_field*

Projective plane curves over finite fields

**rational\_points** (algorithm='enum', sort=True)

Return the rational points on this curve.

INPUT:

- algorithm – one of
  - 'enum' – straightforward enumeration
  - 'bn' – via Singular's brnoeth package.
- sort – boolean (default: True); whether the output points should be sorted. If False, the order of the output is non-deterministic.

OUTPUT: a list of all the rational points on the curve, possibly sorted.

---

**Note:** The Brill-Noether package does not always work (i.e., the 'bn' algorithm. When it fails a RuntimeError exception is raised.

---

EXAMPLES:



```

sage: x, y, z = PolynomialRing(GF(5), 3, 'xyz').gens()
sage: f = y^2*z^7 - x^9 - x*z^8
sage: C = Curve(f); C
Projective Plane Curve over Finite Field of size 5 defined by
-x^9 + y^2*z^7 - x*z^8
sage: C.rational_points()
[(0 : 0 : 1), (0 : 1 : 0), (2 : 2 : 1), (2 : 3 : 1),
 (3 : 1 : 1), (3 : 4 : 1)]
sage: C = Curve(x - y + z)
sage: C.rational_points()
[(0 : 1 : 1), (1 : 1 : 0), (1 : 2 : 1), (2 : 3 : 1),
 (3 : 4 : 1), (4 : 0 : 1)]
sage: C = Curve(x*z+z^2)
sage: C.rational_points('all')
[(0 : 1 : 0), (1 : 0 : 0), (1 : 1 : 0), (2 : 1 : 0),
 (3 : 1 : 0), (4 : 0 : 1), (4 : 1 : 0), (4 : 1 : 1),
 (4 : 2 : 1), (4 : 3 : 1), (4 : 4 : 1)]

```

```

sage: F = GF(7)
sage: P2.<X,Y,Z> = ProjectiveSpace(F,2)
sage: C = Curve(X^3+Y^3-Z^3)
sage: C.rational_points()
[(0 : 1 : 1), (0 : 2 : 1), (0 : 4 : 1), (1 : 0 : 1), (2 : 0 : 1),
 (3 : 1 : 0), (4 : 0 : 1), (5 : 1 : 0), (6 : 1 : 0)]

```

```

sage: F = GF(1237)
sage: P2.<X,Y,Z> = ProjectiveSpace(F,2)
sage: C = Curve(X^7+7*Y^6*Z+Z^4*X^2*Y*89)
sage: len(C.rational_points())
1237

```

```

sage: F = GF(2^6, 'a')
sage: P2.<X,Y,Z> = ProjectiveSpace(F,2)
sage: C = Curve(X^5+11*X*Y*Z^3 + X^2*Y^3 - 13*Y^2*Z^3)
sage: len(C.rational_points())
104

```

```

sage: R.<x,y,z> = GF(2)[]
sage: f = x^3*y + y^3*z + x*z^3
sage: C = Curve(f); pts = C.rational_points()
sage: pts
[(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)]

```

#### **rational\_points\_iterator()**

Return a generator object for the rational points on this curve.

INPUT:

- `self` – a projective curve

OUTPUT:

A generator of all the rational points on the curve defined over its base field.

EXAMPLES:

```

sage: F = GF(37)
sage: P2.<X,Y,Z> = ProjectiveSpace(F,2)
sage: C = Curve(X^7+Y*X*Z^5+Y^7*12)
sage: len(list(C.rational_points_iterator()))
37

```

```

sage: F = GF(2)
sage: P2.<X,Y,Z> = ProjectiveSpace(F,2)
sage: C = Curve(X*Y*Z)
sage: a = C.rational_points_iterator()
sage: next(a)
(1 : 0 : 0)
sage: next(a)
(0 : 1 : 0)
sage: next(a)
(1 : 1 : 0)
sage: next(a)
(0 : 0 : 1)
sage: next(a)
(1 : 0 : 1)
sage: next(a)
(0 : 1 : 1)
sage: next(a)
Traceback (most recent call last):
...
StopIteration

```

```

sage: F = GF(3^2, 'a')
sage: P2.<X,Y,Z> = ProjectiveSpace(F,2)
sage: C = Curve(X^3+5*Y^2*Z-33*X*Y*X)
sage: b = C.rational_points_iterator()
sage: next(b)
(0 : 1 : 0)
sage: next(b)
(0 : 0 : 1)
sage: next(b)
(2*a + 2 : a : 1)
sage: next(b)
(2 : a + 1 : 1)
sage: next(b)
(a + 1 : 2*a + 1 : 1)
sage: next(b)
(1 : 2 : 1)
sage: next(b)
(2*a + 2 : 2*a : 1)
sage: next(b)
(2 : 2*a + 2 : 1)
sage: next(b)
(a + 1 : a + 2 : 1)
sage: next(b)
(1 : 1 : 1)
sage: next(b)
Traceback (most recent call last):
...
StopIteration

```

**riemann\_roch\_basis(D)**

Return a basis for the Riemann-Roch space corresponding to  $D$ .

This uses Singular's Brill-Noether implementation.

INPUT:

- $D$  - a divisor

OUTPUT: a list of function field elements that form a basis of the Riemann-Roch space

EXAMPLES:

```
sage: R.<x,y,z> = GF(2)[]
sage: f = x^3*y + y^3*z + x*z^3
sage: C = Curve(f); pts = C.rational_points()
sage: D = C.divisor([ (4, pts[0]), (4, pts[2]) ])
sage: C.riemann_roch_basis(D)
[x/y, 1, z/y, z^2/y^2, z/x, z^2/(x*y)]
```

```
sage: R.<x,y,z> = GF(5)[]
sage: f = x^7 + y^7 + z^7
sage: C = Curve(f); pts = C.rational_points()
sage: D = C.divisor([ (3, pts[0]), (-1, pts[1]), (10, pts[5]) ])
sage: C.riemann_roch_basis(D)
[(-x - 2*y)/(-2*x - 2*y), (-x + z)/(x + y)]
```

---

**Note:** Currently this only works over prime field and divisors supported on rational points.

---



## RATIONAL POINTS OF CURVES

We can create points on projective curves:

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([x^3 - 2*x*z^2 - y^3, z^3 - w^3 - x*y*z], P)
sage: Q = C([1,1,0,0])
sage: Q.parent()
Set of rational points of Projective Curve over Rational Field defined
by x^3 - y^3 - 2*x*z^2, -x*y*z + z^3 - w^3
```

or on affine curves:

```
sage: A.<x,y> = AffineSpace(GF(23), 2)
sage: C = Curve([y - y^4 + 17*x^2 - 2*x + 22], A)
sage: Q = C([22,21])
sage: Q.parent()
Set of rational points of Affine Plane Curve over Finite Field of size
23 defined by -y^4 - 6*x^2 - 2*x + y - 1
```

AUTHORS:

- Grayson Jorgenson (2016-6): initial version

```
class sage.schemes.curves.point.AffineCurvePoint_field(X, v, check=True)
    Bases: sage.schemes.affine.affine_point.SchemeMorphism_point_affine_field
    is_singular()
        Return whether this point is a singular point of the affine curve it is on.
```

EXAMPLES:

```
sage: K = QuadraticField(-1)
sage: A.<x,y,z> = AffineSpace(K, 3)
sage: C = Curve([(x^4 + 2*z + 2)*y, z - y + 1])
sage: Q1 = C([0,0,-1])
sage: Q1.is_singular()
True
sage: Q2 = C([-K.gen(),0,-1])
sage: Q2.is_singular()
False
```

```
class sage.schemes.curves.point.AffinePlaneCurvePoint_field(X, v, check=True)
    Bases: sage.schemes.curves.point.AffineCurvePoint_field
    Point of an affine plane curve over a field.
```

**is\_ordinary\_singularity()**

Return whether this point is an ordinary singularity of the affine plane curve it is on.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve([x^5 - x^3*y^2 + 5*x^4 - x^3*y - 3*x^2*y^2 +
....: x*y^3 + 10*x^3 - 3*x^2*y - 3*x*y^2 + y^3 + 10*x^2 - 3*x*y - y^2 +
....: 5*x - y + 1])
sage: Q = C([-1,0])
sage: Q.is_ordinary_singularity()
True
```

```
sage: A.<x,y> = AffineSpace(GF(7), 2)
sage: C = A.curve([y^2 - x^7 - 6*x^3])
sage: Q = C([0,0])
sage: Q.is_ordinary_singularity()
False
```

**is\_transverse(D)**

Return whether the intersection of the curve D at this point with the curve this point is on is transverse or not.

INPUT:

- D – a curve in the same ambient space as the curve this point is on.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = Curve([y - x^2], A)
sage: D = Curve([y], A)
sage: Q = C([0,0])
sage: Q.is_transverse(D)
False
```

```
sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^2 - 2)
sage: A.<x,y> = AffineSpace(K, 2)
sage: C = Curve([y^2 + x^2 - 1], A)
sage: D = Curve([y - x], A)
sage: Q = C([-1/2*b, -1/2*b])
sage: Q.is_transverse(D)
True
```

**multiplicity()**

Return the multiplicity of this point with respect to the affine curve it is on.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve([2*x^7 - 3*x^6*y + x^5*y^2 + 31*x^6 - 40*x^5*y +
....: 13*x^4*y^2 - x^3*y^3 + 207*x^5 - 228*x^4*y + 70*x^3*y^2 - 7*x^2*y^3
....: + 775*x^4 - 713*x^3*y + 193*x^2*y^2 - 19*x*y^3 + y^4 + 1764*x^3 -
....: 1293*x^2*y + 277*x*y^2 - 22*y^3 + 2451*x^2 - 1297*x*y + 172*y^2 +
....: 1935*x - 570*y + 675])
sage: Q = C([-2,1])
sage: Q.multiplicity()
4
```

**tangents()**

Return the tangents at this point of the affine plane curve this point is on.

OUTPUT: a list of polynomials in the coordinate ring of the ambient space of the curve this point is on.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(QQ, 2)
sage: C = A.curve([x^5 - x^3*y^2 + 5*x^4 - x^3*y - 3*x^2*y^2 +
....: x*y^3 + 10*x^3 - 3*x^2*y - 3*x*y^2 + y^3 + 10*x^2 - 3*x*y - y^2 +
....: 5*x - y + 1])
sage: Q = C([-1,0])
sage: Q.tangents()
[y, x + 1, x - y + 1, x + y + 1]
```

```
class sage.schemes.curves.point.AffinePlaneCurvePoint_finite_field(X, v,
                                                                    check=True)
Bases: sage.schemes.curves.point.AffinePlaneCurvePoint_field, sage.schemes.
affine.affine_point.SchemeMorphism_point_affine_finite_field
```

Point of an affine plane curve over a finite field.

```
class sage.schemes.curves.point.IntegralAffineCurvePoint(X, v, check=True)
```

```
Bases: sage.schemes.curves.point.AffineCurvePoint_field
```

Point of an integral affine curve.

**closed\_point()**

Return the closed point that corresponds to this rational point.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(8), 2)
sage: C = Curve(x^5 + y^5 + x*y + 1)
sage: p = C([1,1])
sage: p.closed_point()
Point (x + 1, y + 1)
```

**place()**

Return a place on this point.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(2), 2)
sage: C = Curve(x^5 + y^5 + x*y + 1)
sage: p = C(-1,-1)
sage: p
(1, 1)
sage: p.closed_point()
Point (x + 1, y + 1)
sage: _.place()
Place (x + 1, (1/(x^5 + 1))*y^4 + ((x^5 + x^4 + 1)/(x^5 + 1))*y^3 +
((x^5 + x^3 + 1)/(x^5 + 1))*y^2 + (x^2/(x^5 + 1))*y)
```

**places()**

Return all places on this point.

EXAMPLES:

```
sage: A.<x,y> = AffineSpace(GF(2), 2)
sage: C = Curve(x^5 + y^5 + x*y + 1)
```

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```

sage: p = C(-1,-1)
sage: p
(1, 1)
sage: p.closed_point()
Point (x + 1, y + 1)
sage: _.places()
[Place (x + 1, (1/(x^5 + 1))*y^4 + ((x^5 + x^4 + 1)/(x^5 + 1))*y^3
+ ((x^5 + x^3 + 1)/(x^5 + 1))*y^2 + (x^2/(x^5 + 1))*y), Place (x +
1, (1/(x^5 + 1))*y^4 + ((x^5 + x^4 + 1)/(x^5 + 1))*y^3 + (x^3/(x^5
+ 1))*y^2 + (x^2/(x^5 + 1))*y + x + 1)]

```

```

class sage.schemes.curves.point.IntegralAffineCurvePoint_finite_field(X, v,
                                                                    check=True)

```

Bases: `sage.schemes.curves.point.IntegralAffineCurvePoint`

Point of an integral affine curve over a finite field.

```

class sage.schemes.curves.point.IntegralAffinePlaneCurvePoint (X, v,
                                                                check=True)
Bases: sage.schemes.curves.point.IntegralAffineCurvePoint, sage.schemes.
curves.point.AffinePlaneCurvePoint_field

```

Point of an integral affine plane curve over a finite field.

```

class sage.schemes.curves.point.IntegralAffinePlaneCurvePoint_finite_field(X,
                                                                              v,
                                                                              check=True)
Bases: sage.schemes.curves.point.AffinePlaneCurvePoint_finite_field, sage.
schemes.curves.point.IntegralAffineCurvePoint_finite_field

```

Point of an integral affine plane curve over a finite field.

```

class sage.schemes.curves.point.IntegralProjectiveCurvePoint (X, v, check=True)
Bases: sage.schemes.curves.point.ProjectiveCurvePoint_field

```

```

closed_point ()
Return the closed point corresponding to this rational point.

```

EXAMPLES:

```

sage: P.<x,y,z> = ProjectiveSpace(GF(17), 2)
sage: C = Curve([x^4 - 16*y^3*z], P)
sage: C.singular_points()
[(0 : 0 : 1)]
sage: p = _[0]
sage: p.closed_point()
Point (x, y)

```

```

place ()
Return a place on this point.

```

EXAMPLES:

```

sage: P.<x,y,z> = ProjectiveSpace(GF(17), 2)
sage: C = Curve([x^4 - 16*y^3*z], P)
sage: C.singular_points()
[(0 : 0 : 1)]
sage: p = _[0]
sage: p.place()
Place (y)

```



**places()**

Return all places on this point.

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(GF(17), 2)
sage: C = Curve([x^4 - 16*y^3*z], P)
sage: C.singular_points()
[(0 : 0 : 1)]
sage: p = _[0]
sage: p.places()
[Place (y)]
```

```
class sage.schemes.curves.point.IntegralProjectiveCurvePoint_finite_field(X,
                                                                           v,
                                                                           check=True)
```

Bases: *sage.schemes.curves.point.IntegralProjectiveCurvePoint*

Point of an integral projective curve over a finite field.

```
class sage.schemes.curves.point.IntegralProjectivePlaneCurvePoint(X, v,
                                                                    check=True)
Bases: sage.schemes.curves.point.IntegralProjectiveCurvePoint, sage.schemes.
curves.point.ProjectivePlaneCurvePoint_field
```

Point of an integral projective plane curve over a field.

```
class sage.schemes.curves.point.IntegralProjectivePlaneCurvePoint_finite_field(X,
                                                                                v,
                                                                                check=True)
Bases: sage.schemes.curves.point.ProjectivePlaneCurvePoint_finite_field,
sage.schemes.curves.point.IntegralProjectiveCurvePoint_finite_field
```

Point of an integral projective plane curve over a finite field.

```
class sage.schemes.curves.point.ProjectiveCurvePoint_field(X, v, check=True)
Bases: sage.schemes.projective.projective_point.SchemeMorphism_point_projective_field
```

Point of a projective curve over a field.

**is\_singular()**

Return whether this point is a singular point of the projective curve it is on.

EXAMPLES:

```
sage: P.<x,y,z,w> = ProjectiveSpace(QQ, 3)
sage: C = Curve([x^2 - y^2, z - w], P)
sage: Q1 = C([0,0,1,1])
sage: Q1.is_singular()
True
sage: Q2 = C([1,1,1,1])
sage: Q2.is_singular()
False
```

```
class sage.schemes.curves.point.ProjectivePlaneCurvePoint_field(X, v,
                                                                check=True)
```

Bases: *sage.schemes.curves.point.ProjectiveCurvePoint\_field*

Point of a projective plane curve over a field.

**is\_ordinary\_singularity()**

Return whether this point is an ordinary singularity of the projective plane curve it is on.

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([z^6 - x^6 - x^3*z^3 - x^3*y^3])
sage: Q = C([0,1,0])
sage: Q.is_ordinary_singularity()
False
```

```
sage: R.<a> = QQ[]
sage: K.<b> = NumberField(a^2 - 3)
sage: P.<x,y,z> = ProjectiveSpace(K, 2)
sage: C = P.curve([x^2*y^3*z^4 - y^6*z^3 - 4*x^2*y^4*z^3 -
....: 4*x^4*y^2*z^3 + 3*y^7*z^2 + 10*x^2*y^5*z^2 + 9*x^4*y^3*z^2 +
....: 5*x^6*y*z^2 - 3*y^8*z - 9*x^2*y^6*z - 11*x^4*y^4*z - 7*x^6*y^2*z -
....: 2*x^8*z + y^9 + 2*x^2*y^7 + 3*x^4*y^5 + 4*x^6*y^3 + 2*x^8*y])
sage: Q = C([-1/2, 1/2, 1])
sage: Q.is_ordinary_singularity()
True
```

### **is\_transverse(D)**

Return whether the intersection of the curve D at this point with the curve this point is on is transverse or not.

INPUT:

- D – a curve in the same ambient space as the curve this point is on

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([x^2 - 2*y^2 - 2*z^2], P)
sage: D = Curve([y - z], P)
sage: Q = C([2,1,1])
sage: Q.is_transverse(D)
True
```

```
sage: P.<x,y,z> = ProjectiveSpace(GF(17), 2)
sage: C = Curve([x^4 - 16*y^3*z], P)
sage: D = Curve([y^2 - z*x], P)
sage: Q = C([0,0,1])
sage: Q.is_transverse(D)
False
```

### **multiplicity()**

Return the multiplicity of this point with respect to the projective curve it is on.

EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(GF(17), 2)
sage: C = Curve([y^3*z - 16*x^4], P)
sage: Q = C([0,0,1])
sage: Q.multiplicity()
3
```

### **tangents()**

Return the tangents at this point of the projective plane curve this point is on.

OUTPUT:

A list of polynomials in the coordinate ring of the ambient space of the curve this point is on.

## EXAMPLES:

```
sage: P.<x,y,z> = ProjectiveSpace(QQ, 2)
sage: C = Curve([y^2*z^3 - x^5 + 18*y*x*z^3])
sage: Q = C([0,0,1])
sage: Q.tangents()
[y, 18*x + y]
```

```
class sage.schemes.curves.point.ProjectivePlaneCurvePoint_finite_field(X, v,
                                                                    check=True)
Bases:      sage.schemes.curves.point.ProjectivePlaneCurvePoint_field, sage.
schemes.projective.projective_point.SchemeMorphism_point_projective_finite_field
```

Point of a projective plane curve over a finite field.



## CLOSED POINTS OF INTEGRAL CURVES

A rational point of a curve in Sage is represented by its coordinates. If the curve is defined over finite field and integral, that is reduced and irreducible, then it is empowered by the global function field machinery of Sage. Thus closed points of the curve are computable, as represented by maximal ideals of the coordinate ring of the ambient space.

EXAMPLES:

```
sage: F.<a> = GF(2)
sage: P.<x,y> = AffineSpace(F, 2);
sage: C = Curve(y^2 + y - x^3)
sage: C.closed_points()
[Point (x, y), Point (x, y + 1)]
sage: C.closed_points(2)
[Point (y^2 + y + 1, x + 1),
 Point (y^2 + y + 1, x + y),
 Point (y^2 + y + 1, x + y + 1)]
sage: C.closed_points(3)
[Point (x^2 + x + y, x*y + 1, y^2 + x + 1),
 Point (x^2 + x + y + 1, x*y + x + 1, y^2 + x)]
```

Closed points of projective curves are represented by homogeneous maximal ideals:

```
sage: F.<a> = GF(2)
sage: P.<x,y,z> = ProjectiveSpace(F, 2)
sage: C = Curve(x^3*y + y^3*z + x*z^3)
sage: C.closed_points()
[Point (x, z), Point (x, y), Point (y, z)]
sage: C.closed_points(2)
[Point (y^2 + y*z + z^2, x + y + z)]
sage: C.closed_points(3)
[Point (y^3 + y^2*z + z^3, x + y),
 Point (y^3 + y*z^2 + z^3, x + z),
 Point (x^2 + x*z + y*z + z^2, x*y + x*z + z^2, y^2 + x*z),
 Point (x^2 + y*z, x*y + x*z + z^2, y^2 + x*z + y*z),
 Point (x^3 + x*z^2 + z^3, y + z),
 Point (x^2 + y*z + z^2, x*y + x*z + y*z, y^2 + x*z + y*z + z^2),
 Point (x^2 + y*z + z^2, x*y + z^2, y^2 + x*z + y*z)]
```

Rational points are easily converted to closed points and vice versa if the closed point is of degree one:

```
sage: F.<a> = GF(2)
sage: P.<x,y,z> = ProjectiveSpace(F, 2)
sage: C = Curve(x^3*y + y^3*z + x*z^3)
sage: p1, p2, p3 = C.closed_points()
sage: p1.rational_point()
```

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```

(0 : 1 : 0)
sage: p2.rational_point()
(0 : 0 : 1)
sage: p3.rational_point()
(1 : 0 : 0)
sage: _.closed_point()
Point (y, z)
sage: _ == p3
True

```

## AUTHORS:

- Kwankyu Lee (2019-03): initial version

**class** `sage.schemes.curves.closed_point.CurveClosedPoint` (*S, P, check=False*)  
 Bases: `sage.schemes.generic.point.SchemeTopologicalPoint_prime_ideal`

Base class of closed points of curves.

**class** `sage.schemes.curves.closed_point.IntegralAffineCurveClosedPoint` (*curve, prime\_ideal, degree*)  
 Bases: `sage.schemes.curves.closed_point.IntegralCurveClosedPoint`

Closed points of affine curves.

**projective** (*i=0*)

Return the point in the projective closure of the curve, of which this curve is the *i*-th affine patch.

INPUT:

- *i* – an integer

EXAMPLES:

```

sage: F.<a> = GF(2)
sage: A.<x,y> = AffineSpace(F, 2)
sage: C = Curve(y^2 + y - x^3, A)
sage: p1, p2 = C.closed_points()
sage: p1
Point (x, y)
sage: p2
Point (x, y + 1)
sage: p1.projective()
Point (x1, x2)
sage: p2.projective(0)
Point (x1, x0 + x2)
sage: p2.projective(1)
Point (x0, x1 + x2)
sage: p2.projective(2)
Point (x0, x1 + x2)

```

**rational\_point** ()

Return the rational point if this closed point is of degree 1.

EXAMPLES:

```

sage: A.<x,y> = AffineSpace(GF(3^2), 2)
sage: C = Curve(y^2 - x^5 - x^4 - 2*x^3 - 2*x-2)

```

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```

sage: C.closed_points()
[Point (x, y + (z2 + 1)),
 Point (x, y + (-z2 - 1)),
 Point (x + (z2 + 1), y + (z2 - 1)),
 Point (x + (z2 + 1), y + (-z2 + 1)),
 Point (x - 1, y + (z2 + 1)),
 Point (x - 1, y + (-z2 - 1)),
 Point (x + (-z2 - 1), y + (z2)),
 Point (x + (-z2 - 1), y + (-z2)),
 Point (x + 1, y + 1),
 Point (x + 1, y - 1)]
sage: [p.rational_point() for p in _]
[(0, 2*z2 + 2),
 (0, z2 + 1),
 (2*z2 + 2, 2*z2 + 1),
 (2*z2 + 2, z2 + 2),
 (1, 2*z2 + 2),
 (1, z2 + 1),
 (z2 + 1, 2*z2),
 (z2 + 1, z2),
 (2, 2),
 (2, 1)]
sage: set(_) == set(C.rational_points())
True

```

**class** sage.schemes.curves.closed\_point.**IntegralCurveClosedPoint** (*curve*,  
*prime\_ideal*,  
*degree*)

Bases: *sage.schemes.curves.closed\_point.CurveClosedPoint*

Closed points of integral curves.

INPUT:

- *curve* – the curve to which the closed point belongs
- *prime\_ideal* – a prime ideal
- *degree* – degree of the closed point

EXAMPLES:

```

sage: F.<a> = GF(4)
sage: P.<x,y> = AffineSpace(F, 2);
sage: C = Curve(y^2 + y - x^3)
sage: C.closed_points()
[Point (x, y),
 Point (x, y + 1),
 Point (x + (a), y + (a)),
 Point (x + (a), y + (a + 1)),
 Point (x + (a + 1), y + (a)),
 Point (x + (a + 1), y + (a + 1)),
 Point (x + 1, y + (a)),
 Point (x + 1, y + (a + 1))]

```

**curve** ()

Return the curve to which this point belongs.

EXAMPLES:

```

sage: F.<a> = GF(4)
sage: P.<x,y> = AffineSpace(F, 2);
sage: C = Curve(y^2 + y - x^3)
sage: pts = C.closed_points()
sage: p = pts[0]
sage: p.curve()
Affine Plane Curve over Finite Field in a of size 2^2 defined by x^3 + y^2 + y

```

**degree()**

Return the degree of the point.

EXAMPLES:

```

sage: F.<a> = GF(4)
sage: P.<x,y> = AffineSpace(F, 2);
sage: C = Curve(y^2 + y - x^3)
sage: pts = C.closed_points()
sage: p = pts[0]
sage: p.degree()
1

```

**place()**

Return a place on this closed point.

If there are more than one, arbitrary one is chosen.

EXAMPLES:

```

sage: F.<a> = GF(4)
sage: P.<x,y> = AffineSpace(F, 2);
sage: C = Curve(y^2 + y - x^3)
sage: pts = C.closed_points()
sage: p = pts[0]
sage: p.place()
Place (x, y)

```

**places()**

Return all places on this closed point.

EXAMPLES:

```

sage: F.<a> = GF(4)
sage: P.<x,y> = AffineSpace(F, 2);
sage: C = Curve(y^2 + y - x^3)
sage: pts = C.closed_points()
sage: p = pts[0]
sage: p.places()
[Place (x, y)]

```

**class** sage.schemes.curves.closed\_point.**IntegralProjectiveCurveClosedPoint** (*curve*,  
*prime\_ideal*,  
*de-  
gree*

Bases: *sage.schemes.curves.closed\_point.IntegralCurveClosedPoint*

Closed points of projective plane curves.

**affine** (*i=None*)

Return the point in the *i*-th affine patch of the curve.



INPUT:

- $i$  – an integer; if not specified, it is chosen automatically.

EXAMPLES:

```
sage: F.<a> = GF(2)
sage: P.<x,y,z> = ProjectiveSpace(F, 2)
sage: C = Curve(x^3*y + y^3*z + x*z^3)
sage: p1, p2, p3 = C.closed_points()
sage: p1.affine()
Point (x, z)
sage: p2.affine()
Point (x, y)
sage: p3.affine()
Point (y, z)
sage: p3.affine(0)
Point (y, z)
sage: p3.affine(1)
Traceback (most recent call last):
...
ValueError: not in the affine patch
```

**rational\_point()**

Return the rational point if this closed point is of degree 1.

EXAMPLES:

```
sage: F.<a> = GF(4)
sage: P.<x,y,z> = ProjectiveSpace(F, 2)
sage: C = Curve(x^3*y + y^3*z + x*z^3)
sage: C.closed_points()
[Point (x, z),
 Point (x, y),
 Point (y, z),
 Point (x + (a)*z, y + (a + 1)*z),
 Point (x + (a + 1)*z, y + (a)*z)]
sage: [p.rational_point() for p in _]
[(0 : 1 : 0), (0 : 0 : 1), (1 : 0 : 0), (a : a + 1 : 1), (a + 1 : a : 1)]
sage: set(_) == set(C.rational_points())
True
```



## JACOBIANS OF CURVES

This module defines the base class of Jacobians as an abstract scheme.

AUTHORS:

- William Stein (2005)

```
sage.schemes.jacobians.abstract_jacobian.Jacobian(C)
```

EXAMPLES:

```
sage: from sage.schemes.jacobians.abstract_jacobian import Jacobian
sage: P2.<x, y, z> = ProjectiveSpace(QQ, 2)
sage: C = Curve(x^3 + y^3 + z^3)
sage: J = Jacobian(C)
Jacobian of Projective Plane Curve over Rational Field defined by x^3 + y^3 + z^3
```

```
class sage.schemes.jacobians.abstract_jacobian.Jacobian_generic(C)
```

Bases: `sage.schemes.generic.scheme.Scheme`

Base class for Jacobians of projective curves.

The input must be a projective curve over a field.

EXAMPLES:

```
sage: from sage.schemes.jacobians.abstract_jacobian import Jacobian
sage: P2.<x, y, z> = ProjectiveSpace(QQ, 2)
sage: C = Curve(x^3 + y^3 + z^3)
sage: J = Jacobian(C); J
Jacobian of Projective Plane Curve over Rational Field defined by x^3 + y^3 + z^3
```

```
base_extend(R)
```

Return the natural extension of self over  $R$

INPUT:

- $R$  – a field. The new base field.

OUTPUT:

The Jacobian over the ring  $R$ .

EXAMPLES:

```
sage: R.<x> = QQ['x']
sage: H = HyperellipticCurve(x^3-10*x+9)
sage: Jac = H.jacobian(); Jac
Jacobian of Hyperelliptic Curve over Rational Field defined by y^2 = x^3 -
↪ 10*x + 9
```

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```

sage: F.<a> = QQ.extension(x^2+1)
sage: Jac.base_extend(F)
Jacobian of Hyperelliptic Curve over Number Field in a with defining
polynomial x^2 + 1 defined by y^2 = x^3 - 10*x + 9

```

**change\_ring(*R*)**Return the Jacobian over the ring *R*.

INPUT:

- *R* – a field. The new base ring.

OUTPUT:

The Jacobian over the ring *R*.

EXAMPLES:

```

sage: R.<x> = QQ['x']
sage: H = HyperellipticCurve(x^3-10*x+9)
sage: Jac = H.jacobian(); Jac
Jacobian of Hyperelliptic Curve over Rational
Field defined by y^2 = x^3 - 10*x + 9
sage: Jac.change_ring(RDF)
Jacobian of Hyperelliptic Curve over Real Double
Field defined by y^2 = x^3 - 10.0*x + 9.0

```

**curve()**

Return the curve of which self is the Jacobian.

EXAMPLES:

```

sage: from sage.schemes.jacobians.abstract_jacobian import Jacobian
sage: P2.<x, y, z> = ProjectiveSpace(QQ, 2)
sage: J = Jacobian(Curve(x^3 + y^3 + z^3))
sage: J.curve()
Projective Plane Curve over Rational Field defined by x^3 + y^3 + z^3

```

sage.schemes.jacobians.abstract\_jacobian.**is\_Jacobian(*J*)**Return True if *J* is of type Jacobian\_generic.

EXAMPLES:

```

sage: from sage.schemes.jacobians.abstract_jacobian import Jacobian, is_Jacobian
sage: P2.<x, y, z> = ProjectiveSpace(QQ, 2)
sage: C = Curve(x^3 + y^3 + z^3)
sage: J = Jacobian(C)
sage: is_Jacobian(J)
True

```

```

sage: E = EllipticCurve('37a1')
sage: is_Jacobian(E)
False

```

## PLANE CONICS

### 8.1 Plane conic constructor

AUTHORS:

- Marco Streng (2010-07-20)
- Nick Alexander (2008-01-08)

`sage.schemes.plane_conics.constructor.Conic` (*base\_field*, *F=None*, *names=None*,  
*unique=True*)

Return the plane projective conic curve defined by  $F$  over  $\text{base\_field}$ .

The input form `Conic(F, names=None)` is also accepted, in which case the fraction field of the base ring of  $F$  is used as base field.

INPUT:

- `base_field` – The base field of the conic.
- `names` – a list, tuple, or comma separated string of three variable names specifying the names of the coordinate functions of the ambient space  $\mathbf{P}^3$ . If not specified or read off from  $F$ , then this defaults to `'x, y, z'`.
- $F$  – a polynomial, list, matrix, ternary quadratic form, or list or tuple of 5 points in the plane.
  - If  $F$  is a polynomial or quadratic form, then the output is the curve in the projective plane defined by  $F = 0$ .
  - If  $F$  is a polynomial, then it must be a polynomial of degree at most 2 in 2 variables, or a homogeneous polynomial in of degree 2 in 3 variables.
  - If  $F$  is a matrix, then the output is the zero locus of  $(x, y, z)F(x, y, z)^t$ .
  - If  $F$  is a list of coefficients, then it has length 3 or 6 and gives the coefficients of the monomials  $x^2, y^2, z^2$  or all 6 monomials  $x^2, xy, xz, y^2, yz, z^2$  in lexicographic order.
  - If  $F$  is a list of 5 points in the plane, then the output is a conic through those points.
- `unique` – Used only if  $F$  is a list of points in the plane. If the conic through the points is not unique, then raise `ValueError` if and only if `unique` is `True`

OUTPUT:

A plane projective conic curve defined by  $F$  over a field.

EXAMPLES:

Conic curves given by polynomials

```

sage: X,Y,Z = QQ['X,Y,Z'].gens()
sage: Conic(X^2 - X*Y + Y^2 - Z^2)
Projective Conic Curve over Rational Field defined by X^2 - X*Y + Y^2 - Z^2
sage: x,y = GF(7)['x,y'].gens()
sage: Conic(x^2 - x + 2*y^2 - 3, 'U,V,W')
Projective Conic Curve over Finite Field of size 7 defined by U^2 + 2*V^2 - U*W -
↪ 3*W^2

```

### Conic curves given by matrices

```

sage: Conic(matrix(QQ, [[1, 2, 0], [4, 0, 0], [7, 0, 9]]), 'x,y,z')
Projective Conic Curve over Rational Field defined by x^2 + 6*x*y + 7*x*z + 9*z^2

sage: x,y,z = GF(11)['x,y,z'].gens()
sage: C = Conic(x^2+y^2-2*z^2); C
Projective Conic Curve over Finite Field of size 11 defined by x^2 + y^2 - 2*z^2
sage: Conic(C.symmetric_matrix(), 'x,y,z')
Projective Conic Curve over Finite Field of size 11 defined by x^2 + y^2 - 2*z^2

```

### Conics given by coefficients

```

sage: Conic(QQ, [1,2,3])
Projective Conic Curve over Rational Field defined by x^2 + 2*y^2 + 3*z^2
sage: Conic(GF(7), [1,2,3,4,5,6], 'X')
Projective Conic Curve over Finite Field of size 7 defined by X0^2 + 2*X0*X1 -
↪ 3*X1^2 + 3*X0*X2 - 2*X1*X2 - X2^2

```

### The conic through a set of points

```

sage: C = Conic(QQ, [[10,2],[3,4],[-7,6],[7,8],[9,10]]); C
Projective Conic Curve over Rational Field defined by x^2 + 13/4*x*y - 17/4*y^2 -
↪ 35/2*x*z + 91/4*y*z - 37/2*z^2
sage: C.rational_point()
(10 : 2 : 1)
sage: C.point([3,4])
(3 : 4 : 1)

sage: a=AffineSpace(GF(13),2)
sage: Conic([a([x,x^2]) for x in range(5)])
Projective Conic Curve over Finite Field of size 13 defined by x^2 - y*z

```

## 8.2 Projective plane conics over a field

### AUTHORS:

- Marco Streng (2010-07-20)
- Nick Alexander (2008-01-08)

**class** sage.schemes.plane\_conics.con\_field.**ProjectiveConic\_field**(A,f)

Bases: *sage.schemes.curves.projective\_curve.ProjectivePlaneCurve*

Create a projective plane conic curve over a field. See `Conic` for full documentation.

### EXAMPLES:

```

sage: K = FractionField(PolynomialRing(QQ, 't'))
sage: P.<X, Y, Z> = K[]
sage: Conic(X^2 + Y^2 - Z^2)
Projective Conic Curve over Fraction Field of Univariate Polynomial Ring in t
↳ over Rational Field defined by X^2 + Y^2 - Z^2

```

**base\_extend(S)**

Returns the conic over  $S$  given by the same equation as `self`.

EXAMPLES:

```

sage: c = Conic([1, 1, 1]); c
Projective Conic Curve over Rational Field defined by x^2 + y^2 + z^2
sage: c.has_rational_point()
False
sage: d = c.base_extend(QuadraticField(-1, 'i')); d
Projective Conic Curve over Number Field in i with defining polynomial x^2 +
↳ 1 with i = 1*I defined by x^2 + y^2 + z^2
sage: d.rational_point(algorithm = 'rnfisnorm')
(i : 1 : 0)

```

**cache\_point(p)**

Replace the point in the cache of `self` by  $p$  for use by `self.rational_point()` and `self.parametrization()`.

EXAMPLES:

```

sage: c = Conic([1, -1, 1])
sage: c.point([15, 17, 8])
(15/8 : 17/8 : 1)
sage: c.rational_point()
(15/8 : 17/8 : 1)
sage: c.cache_point(c.rational_point(read_cache = False))
sage: c.rational_point()
(-1 : 1 : 0)

```

**coefficients()**

Gives a the 6 coefficients of the conic `self` in lexicographic order.

EXAMPLES:

```

sage: Conic(QQ, [1, 2, 3, 4, 5, 6]).coefficients()
[1, 2, 3, 4, 5, 6]

sage: P.<x,y,z> = GF(13)[]
sage: a = Conic(x^2+5*x*y+y^2+z^2).coefficients(); a
[1, 5, 0, 1, 0, 1]
sage: Conic(a)
Projective Conic Curve over Finite Field of size 13 defined by x^2 + 5*x*y +
↳ y^2 + z^2

```

**derivative\_matrix()**

Gives the derivative of the defining polynomial of the conic `self`, which is a linear map, as a  $3 \times 3$  matrix.

EXAMPLES:

In characteristic different from 2, the derivative matrix is twice the symmetric matrix:

```

sage: c = Conic(QQ, [1, 1, 1, 1, 1, 0])
sage: c.symmetric_matrix()
[ 1 1/2 1/2]
[1/2  1 1/2]
[1/2 1/2  0]
sage: c.derivative_matrix()
[2 1 1]
[1 2 1]
[1 1 0]

```

An example in characteristic 2:

```

sage: P.<t> = GF(2)[]
sage: c = Conic([t, 1, t^2, 1, 1, 0]); c
Projective Conic Curve over Fraction Field of Univariate Polynomial Ring in t
↳over Finite Field of size 2 (using GF2X) defined by t*x^2 + x*y + y^2 + t^
↳2*x*z + y*z
sage: c.is_smooth()
True
sage: c.derivative_matrix()
[ 0  1 t^2]
[ 1  0  1]
[t^2  1  0]

```

### **determinant()**

Returns the determinant of the symmetric matrix that defines the conic `self`.

This is defined only if the base field has characteristic different from 2.

EXAMPLES:

```

sage: C = Conic([1, 2, 3, 4, 5, 6])
sage: C.determinant()
41/4
sage: C.symmetric_matrix().determinant()
41/4

```

Determinants are only defined in characteristic different from 2:

```

sage: C = Conic(GF(2), [1, 1, 1, 1, 1, 0])
sage: C.is_smooth()
True
sage: C.determinant()
Traceback (most recent call last):
...
ValueError: The conic self (= Projective Conic Curve over Finite Field of
↳size 2 defined by x^2 + x*y + y^2 + x*z + y*z) has no symmetric matrix
↳because the base field has characteristic 2

```

### **diagonal\_matrix()**

Returns a diagonal matrix  $D$  and a matrix  $T$  such that  $T^t A T = D$  holds, where  $(x, y, z)A(x, y, z)^t$  is the defining polynomial of the conic `self`.

EXAMPLES:

```

sage: c = Conic(QQ, [1, 2, 3, 4, 5, 6])
sage: d, t = c.diagonal_matrix(); d, t
(

```

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```

[ 1 0 0] [ 1 -1 -7/6]
[ 0 3 0] [ 0 1 -1/3]
[ 0 0 41/12], [ 0 0 1]
)
sage: t.transpose()*c.symmetric_matrix()*t
[ 1 0 0]
[ 0 3 0]
[ 0 0 41/12]

```

Diagonal matrices are only defined in characteristic different from 2:

```

sage: c = Conic(GF(4, 'a'), [0, 1, 1, 1, 1, 1])
sage: c.is_smooth()
True
sage: c.diagonal_matrix()
Traceback (most recent call last):
...
ValueError: The conic self (= Projective Conic Curve over Finite Field in a_
↳ of size 2^2 defined by x*y + y^2 + x*z + y*z + z^2) has no symmetric matrix_
↳ because the base field has characteristic 2

```

**diagonalization** (*names=None*)

Returns a diagonal conic  $C$ , an isomorphism of schemes  $M : C \rightarrow \text{self}$  and the inverse  $N$  of  $M$ .

EXAMPLES:

```

sage: Conic(GF(5), [1,0,1,1,0,1]).diagonalization()
(Projective Conic Curve over Finite Field of size 5 defined by x^2 + y^2 +
↳ 2*z^2,
Scheme morphism:
  From: Projective Conic Curve over Finite Field of size 5 defined by x^2 + y^
↳ 2 + 2*z^2
  To:   Projective Conic Curve over Finite Field of size 5 defined by x^2 + y^
↳ 2 + x*z + z^2
  Defn: Defined on coordinates by sending (x : y : z) to
        (x + 2*z : y : z),
Scheme morphism:
  From: Projective Conic Curve over Finite Field of size 5 defined by x^2 + y^
↳ 2 + x*z + z^2
  To:   Projective Conic Curve over Finite Field of size 5 defined by x^2 + y^
↳ 2 + 2*z^2
  Defn: Defined on coordinates by sending (x : y : z) to
        (x - 2*z : y : z))

```

The diagonalization is only defined in characteristic different from 2:

```

sage: Conic(GF(2), [1,1,1,1,1,0]).diagonalization()
Traceback (most recent call last):
...
ValueError: The conic self (= Projective Conic Curve over Finite Field of
↳ size 2 defined by x^2 + x*y + y^2 + x*z + y*z) has no symmetric matrix_
↳ because the base field has characteristic 2

```

An example over a global function field:

```

sage: K = FractionField(PolynomialRing(GF(7), 't'))
sage: (t,) = K.gens()

```

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```

sage: C = Conic(K, [t/2, 0, 1, 2, 0, 3])
sage: C.diagonalization()
(Projective Conic Curve over Fraction Field of Univariate Polynomial
↪Ring in t over Finite Field of size 7 defined by 4*t*x^2 + 2*y^2 + ((3*t +
↪3)/t)*z^
↪2,
Scheme morphism:
  From: Projective Conic Curve over Fraction Field of Univariate Polynomial
↪Ring in t over Finite Field of size 7 defined by 4*t*x^2 + 2*y^2 + ((3*t +
↪3)/t)*z^2
  To: Projective Conic Curve over Fraction Field of Univariate Polynomial
↪Ring in t over Finite Field of size 7 defined by 4*t*x^2 + 2*y^2 + x*z +
↪3*z^2
  Defn: Defined on coordinates by sending (x : y : z) to
      (x + 6/t*z : y : z),
Scheme morphism:
  From: Projective Conic Curve over Fraction Field of Univariate Polynomial
↪Ring in t over Finite Field of size 7 defined by 4*t*x^2 + 2*y^2 + x*z +
↪3*z^2
  To: Projective Conic Curve over Fraction Field of Univariate Polynomial
↪Ring in t over Finite Field of size 7 defined by 4*t*x^2 + 2*y^2 + ((3*t +
↪3)/t)*z^2
  Defn: Defined on coordinates by sending (x : y : z) to
      (x + 1/t*z : y : z))

```

**gens()**

Returns the generators of the coordinate ring of self.

**EXAMPLES:**

```

sage: P.<x,y,z> = QQ[]
sage: c = Conic(x^2+y^2+z^2)
sage: c.gens()
(xbar, ybar, zbar)
sage: c.defined_polynomial()(c.gens())
0

```

The function `gens()` is required for the following construction:

```

sage: C.<a,b,c> = Conic(GF(3), [1, 1, 1])
sage: C
Projective Conic Curve over Finite Field of size 3 defined by a^2 + b^2 + c^2

```

**has\_rational\_point** (*point=False, algorithm='default', read\_cache=True*)Returns True if and only if the conic `self` has a point over its base field `B`.If `point` is True, then returns a second output, which is a rational point if one exists.Points are cached whenever they are found. Cached information is used if and only if `read_cache` is True.**ALGORITHM:**The parameter `algorithm` specifies the algorithm to be used:

- 'default' – If the base field is real or complex, use an elementary native Sage implementation.
- 'magma' (requires Magma to be installed) – delegates the task to the Magma computer algebra system.

**EXAMPLES:**

```

sage: Conic(RR, [1, 1, 1]).has_rational_point()
False
sage: Conic(CC, [1, 1, 1]).has_rational_point()
True

sage: Conic(RR, [1, 2, -3]).has_rational_point(point = True)
(True, (1.73205080756888 : 0.0000000000000000 : 1.0000000000000000))

```

Conics over polynomial rings can be solved internally:

```

sage: R.<t> = QQ[]
sage: C = Conic([-2, t^2+1, t^2-1])
sage: C.has_rational_point()
True

```

And they can also be solved with Magma:

```

sage: C.has_rational_point(algorithm='magma') # optional - magma
True
sage: C.has_rational_point(algorithm='magma', point=True) # optional - magma
(True, (-t : 1 : 1))

sage: D = Conic([t, 1, t^2])
sage: D.has_rational_point(algorithm='magma') # optional - magma
False

```

**has\_singular\_point** (*point=False*)

Return True if and only if the conic self has a rational singular point.

If point is True, then also return a rational singular point (or None if no such point exists).

EXAMPLES:

```

sage: c = Conic(QQ, [1, 0, 1]); c
Projective Conic Curve over Rational Field defined by x^2 + z^2
sage: c.has_singular_point(point = True)
(True, (0 : 1 : 0))

sage: P.<x,y,z> = GF(7)[]
sage: e = Conic((x+y+z)*(x-y+2*z)); e
Projective Conic Curve over Finite Field of size 7 defined by x^2 - y^2 +
↪ 3*x*z + y*z + 2*z^2
sage: e.has_singular_point(point = True)
(True, (2 : 4 : 1))

sage: Conic([1, 1, -1]).has_singular_point()
False
sage: Conic([1, 1, -1]).has_singular_point(point = True)
(False, None)

```

has\_singular\_point is not implemented over all fields of characteristic 2. It is implemented over finite fields.

```

sage: F.<a> = FiniteField(8)
sage: Conic([a, a+1, 1]).has_singular_point(point = True)
(True, (a + 1 : 0 : 1))

sage: P.<t> = GF(2)[]

```

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```

sage: C = Conic(P, [t,t,1]); C
Projective Conic Curve over Fraction Field of Univariate Polynomial Ring in t
↳over Finite Field of size 2 (using GF2X) defined by t*x^2 + t*y^2 + z^2
sage: C.has_singular_point(point = False)
Traceback (most recent call last):
...
NotImplementedError: Sorry, find singular point on conics not implemented
↳over all fields of characteristic 2.

```

**hom** (*x*, *Y=None*)

Return the scheme morphism from *self* to *Y* defined by *x*. Here *x* can be a matrix or a sequence of polynomials. If *Y* is omitted, then a natural image is found if possible.

EXAMPLES:

Here are a few Morphisms given by matrices. In the first example, *Y* is omitted, in the second example, *Y* is specified.

```

sage: c = Conic([-1, 1, 1])
sage: h = c.hom(Matrix([[1,1,0],[0,1,0],[0,0,1]])); h
Scheme morphism:
  From: Projective Conic Curve over Rational Field defined by -x^2 + y^2 + z^2
  To:   Projective Conic Curve over Rational Field defined by -x^2 + 2*x*y +
↳z^2
  Defn: Defined on coordinates by sending (x : y : z) to
        (x + y : y : z)
sage: h([-1, 1, 0])
(0 : 1 : 0)

sage: c = Conic([-1, 1, 1])
sage: d = Conic([4, 1, -1])
sage: c.hom(Matrix([[0, 0, 1/2], [0, 1, 0], [1, 0, 0]]), d)
Scheme morphism:
  From: Projective Conic Curve over Rational Field defined by -x^2 + y^2 + z^2
  To:   Projective Conic Curve over Rational Field defined by 4*x^2 + y^2 - z^
↳2
  Defn: Defined on coordinates by sending (x : y : z) to
        (1/2*z : y : x)

```

ValueError is raised if the wrong codomain *Y* is specified:

```

sage: c = Conic([-1, 1, 1])
sage: c.hom(Matrix([[0, 0, 1/2], [0, 1, 0], [1, 0, 0]]), c)
Traceback (most recent call last):
...
ValueError: The matrix x (= [ 0  0 1/2]
[ 0  1  0]
[ 1  0  0]) does not define a map from self (= Projective Conic Curve over
↳Rational Field defined by -x^2 + y^2 + z^2) to Y (= Projective Conic Curve
↳over Rational Field defined by -x^2 + y^2 + z^2)

```

The identity map between two representations of the same conic:

```

sage: C = Conic([1,2,3,4,5,6])
sage: D = Conic([2,4,6,8,10,12])
sage: C.hom(identity_matrix(3), D)
Scheme morphism:

```

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```

From: Projective Conic Curve over Rational Field defined by  $x^2 + 2xy + 4y^2 + 3xz + 5yz + 6z^2$ 
To: Projective Conic Curve over Rational Field defined by  $2x^2 + 4xy + 8y^2 + 6xz + 10yz + 12z^2$ 
Defn: Defined on coordinates by sending  $(x : y : z)$  to
       $(x : y : z)$ 

```

An example not over the rational numbers:

```

sage: P.<t> = QQ[]
sage: C = Conic([1,0,0,t,0,1/t])
sage: D = Conic([1/t^2, 0, -2/t^2, t, 0, (t+1)/t^2])
sage: T = Matrix([[t,0,1],[0,1,0],[0,0,1]])
sage: C.hom(T, D)
Scheme morphism:
  From: Projective Conic Curve over Fraction Field of Univariate Polynomial
  Ring in t over Rational Field defined by  $x^2 + ty^2 + 1/tz^2$ 
  To: Projective Conic Curve over Fraction Field of Univariate Polynomial
  Ring in t over Rational Field defined by  $1/t^2x^2 + ty^2 + (-2/t^2)xz + (t+1)/t^2z^2$ 
  Defn: Defined on coordinates by sending  $(x : y : z)$  to
         $(tx + z : y : z)$ 

```

### **is\_diagonal()**

Return True if and only if the conic has the form  $a * x^2 + b * y^2 + c * z^2$ .

EXAMPLES:

```

sage: c=Conic([1,1,0,1,0,1]); c
Projective Conic Curve over Rational Field defined by  $x^2 + xy + y^2 + z^2$ 
sage: d,t = c.diagonal_matrix()
sage: c.is_diagonal()
False
sage: c.diagonalization()[0].is_diagonal()
True

```

### **is\_smooth()**

Returns True if and only if self is smooth.

EXAMPLES:

```

sage: Conic([1,-1,0]).is_smooth()
False
sage: Conic(GF(2), [1,1,1,1,1,0]).is_smooth()
True

```

### **matrix()**

Returns a matrix  $M$  such that  $(x, y, z)M(x, y, z)^t$  is the defining equation of self.

The matrix  $M$  is upper triangular if the base field has characteristic 2 and symmetric otherwise.

EXAMPLES:

```

sage: R.<x, y, z> = QQ[]
sage: C = Conic(x^2 + x*y + y^2 + z^2)
sage: C.matrix()
[ 1 1/2  0]
[1/2  1  0]

```

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```
[ 0 0 1]

sage: R.<x, y, z> = GF(2)[]
sage: C = Conic(x^2 + x*y + y^2 + x*z + z^2)
sage: C.matrix()
[1 1 1]
[0 1 0]
[0 0 1]
```

**parametrization** (*point=None, morphism=True*)Return a parametrization  $f$  of `self` together with the inverse of  $f$ .If `point` is specified, then that point is used for the parametrization. Otherwise, use `self.rational_point()` to find a point.If `morphism` is `True`, then  $f$  is returned in the form of a Scheme morphism. Otherwise, it is a tuple of polynomials that gives the parametrization.

EXAMPLES:

An example over a finite field

```
sage: c = Conic(GF(2), [1,1,1,1,1,0])
sage: c.parametrization()
(Scheme morphism:
  From: Projective Space of dimension 1 over Finite Field of size 2
  To:   Projective Conic Curve over Finite Field of size 2 defined by x^2 +
↪x*y
+ y^2 + x*z + y*z
  Defn: Defined on coordinates by sending (x : y) to
        (x*y + y^2 : x^2 + x*y : x^2 + x*y + y^2),
Scheme morphism:
  From: Projective Conic Curve over Finite Field of size 2 defined by x^2 +
↪x*y
+ y^2 + x*z + y*z
  To:   Projective Space of dimension 1 over Finite Field of size 2
  Defn: Defined on coordinates by sending (x : y : z) to
        (y : x))
```

An example with `morphism = False`

```
sage: R.<x,y,z> = QQ[]
sage: C = Curve(7*x^2 + 2*y*z + z^2)
sage: (p, i) = C.parametrization(morphism = False); (p, i)
([-2*x*y, x^2 + 7*y^2, -2*x^2], [-1/2*x, 1/7*y + 1/14*z])
sage: C.defined_polynomial()(p)
0
sage: i[0](p) / i[1](p)
x/y
```

A `ValueError` is raised if `self` has no rational point

```
sage: C = Conic(x^2 + y^2 + 7*z^2)
sage: C.parametrization()
Traceback (most recent call last):
...
ValueError: Conic Projective Conic Curve over Rational Field defined by x^2 +
↪y^2 + 7*z^2 has no rational points over Rational Field!
```

A `ValueError` is raised if `self` is not smooth

```
sage: C = Conic(x^2 + y^2)
sage: C.parametrization()
Traceback (most recent call last):
...
ValueError: The conic self (=Projective Conic Curve over Rational Field_
↪ defined by x^2 + y^2) is not smooth, hence does not have a parametrization.
```

**point** (*v*, *check=True*)

Constructs a point on `self` corresponding to the input `v`.

If `check` is `True`, then checks if `v` defines a valid point on `self`.

If no rational point on `self` is known yet, then also caches the point for use by `self.rational_point()` and `self.parametrization()`.

EXAMPLES

```
sage: c = Conic([1, -1, 1])
sage: c.point([15, 17, 8])
(15/8 : 17/8 : 1)
sage: c.rational_point()
(15/8 : 17/8 : 1)
sage: d = Conic([1, -1, 1])
sage: d.rational_point()
(-1 : 1 : 0)
```

**random\_rational\_point** (*\*args1*, *\*\*args2*)

Return a random rational point of the conic `self`.

ALGORITHM:

1. Compute a parametrization  $f$  of `self` using `self.parametrization()`.
2. Computes a random point  $(x : y)$  on the projective line.
3. Output  $f(x : y)$ .

The coordinates  $x$  and  $y$  are computed using `B.random_element`, where  $B$  is the base field of `self` and additional arguments to `random_rational_point` are passed to `random_element`.

If the base field is a finite field, then the output is uniformly distributed over the points of `self`.

EXAMPLES

```
sage: c = Conic(GF(2), [1, 1, 1, 1, 1, 0])
sage: [c.random_rational_point() for i in range(10)] # output is random
[(1 : 0 : 1), (1 : 0 : 1), (1 : 0 : 1), (0 : 1 : 1), (1 : 0 : 1), (0 : 0 : 1),
↪ (1 : 0 : 1), (1 : 0 : 1), (0 : 0 : 1), (1 : 0 : 1)]

sage: d = Conic(QQ, [1, 1, -1])
sage: d.random_rational_point(den_bound = 1, num_bound = 5) # output is random
(-24/25 : 7/25 : 1)

sage: Conic(QQ, [1, 1, 1]).random_rational_point()
Traceback (most recent call last):
...
ValueError: Conic Projective Conic Curve over Rational Field defined by x^2 +
↪ y^2 + z^2 has no rational points over Rational Field!
```

**rational\_point** (*algorithm='default', read\_cache=True*)

Return a point on self defined over the base field.

Raises ValueError if no rational point exists.

See self.has\_rational\_point for the algorithm used and for the use of the parameters algorithm and read\_cache.

EXAMPLES:

Examples over  $\mathbb{Q}$

```
sage: R.<x,y,z> = QQ[]
sage: C = Conic(7*x^2 + 2*y*z + z^2)
sage: C.rational_point()
(0 : 1 : 0)

sage: C = Conic(x^2 + 2*y^2 + z^2)
sage: C.rational_point()
Traceback (most recent call last):
...
ValueError: Conic Projective Conic Curve over Rational Field defined by x^2 +
↪ 2*y^2 + z^2 has no rational points over Rational Field!

sage: C = Conic(x^2 + y^2 + 7*z^2)
sage: C.rational_point(algorithm = 'rnfnorm')
Traceback (most recent call last):
...
ValueError: Conic Projective Conic Curve over Rational Field defined by x^2 +
↪ y^2 + 7*z^2 has no rational points over Rational Field!
```

Examples over number fields

```
sage: P.<x> = QQ[]
sage: L.<b> = NumberField(x^3-5)
sage: C = Conic(L, [3, 2, -b])
sage: p = C.rational_point(algorithm = 'rnfnorm')
sage: p                                     # output is random
(1/3*b^2 - 4/3*b + 4/3 : b^2 - 2 : 1)
sage: C.defining_polynomial() (list(p))
0

sage: K.<i> = QuadraticField(-1)
sage: D = Conic(K, [3, 2, 5])
sage: D.rational_point(algorithm = 'rnfnorm') # output is random
(-3 : 4*i : 1)

sage: L.<s> = QuadraticField(2)
sage: Conic(QQ, [1, 1, -3]).has_rational_point()
False
sage: E = Conic(L, [1, 1, -3])
sage: E.rational_point()                   # output is random
(-1 : -s : 1)
```

Currently Magma is better at solving conics over number fields than Sage, so it helps to use the algorithm 'magma' if Magma is installed:

```
sage: q = C.rational_point(algorithm = 'magma', read_cache=False) # optional -
↪ magma
```

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```

sage: q                                     # output is random, optional - magma
(1/5*b^2 : 1/5*b^2 : 1)
sage: C.defined_polynomial()(list(p))      # optional - magma
0
sage: len(str(p)) > 1.5*len(str(q))        # optional - magma
True

sage: D.rational_point(algorithm='magma', read_cache=False) # random,
↳optional - magma
(1 : 2*i : 1)

sage: E.rational_point(algorithm='magma', read_cache=False) # random,
↳optional - magma
(-s : 1 : 1)

sage: F = Conic([L.gen(), 30, -20])
sage: q = F.rational_point(algorithm='magma') # optional - magma
sage: q                                     # output is random, optional - magma
(-10/7*s + 40/7 : 5/7*s - 6/7 : 1)
sage: p = F.rational_point(read_cache=False)
sage: p                                     # output is random
(788210*s - 1114700 : -171135*s + 242022 : 1)
sage: len(str(p)) > len(str(q))            # optional - magma
True

sage: Conic([L.gen(), 30, -21]).has_rational_point(algorithm='magma') #
↳optional - magma
False

```

### Examples over finite fields

```

sage: F.<a> = FiniteField(7^20)
sage: C = Conic([1, a, -5]); C
Projective Conic Curve over Finite Field in a of size 7^20 defined by x^2 +
↳(a)*y^2 + 2*z^2
sage: C.rational_point() # output is random
(4*a^19 + 5*a^18 + 4*a^17 + a^16 + 6*a^15 + 3*a^13 + 6*a^11 + a^9 + 3*a^8 +
↳2*a^7 + 4*a^6 + 3*a^5 + 3*a^4 + a^3 + a + 6 : 5*a^18 + a^17 + a^16 + 6*a^15
↳+ 4*a^14 + a^13 + 5*a^12 + 5*a^10 + 2*a^9 + 6*a^8 + 6*a^7 + 6*a^6 + 2*a^4 +
↳3 : 1)

```

### Examples over $\mathbf{R}$ and $\mathbf{C}$

```

sage: Conic(CC, [1, 2, 3]).rational_point()
(0 : 1.22474487139159*I : 1)

sage: Conic(RR, [1, 1, 1]).rational_point()
Traceback (most recent call last):
...
ValueError: Conic Projective Conic Curve over Real Field with 53 bits of
↳precision defined by x^2 + y^2 + z^2 has no rational points over Real Field
↳with 53 bits of precision!

```

#### **singular\_point()**

Returns a singular rational point of self

EXAMPLES:

```
sage: Conic(GF(2), [1,1,1,1,1,1]).singular_point()
(1 : 1 : 1)
```

ValueError is raised if the conic has no rational singular point

```
sage: Conic(QQ, [1,1,1,1,1,1]).singular_point()
Traceback (most recent call last):
...
ValueError: The conic self (= Projective Conic Curve over Rational Field_
↪defined by x^2 + x*y + y^2 + x*z + y*z + z^2) has no rational singular point
```

### **symmetric\_matrix()**

The symmetric matrix  $M$  such that  $(xyz)M(xyz)^t$  is the defining equation of self.

#### EXAMPLES

```
sage: R.<x, y, z> = QQ[]
sage: C = Conic(x^2 + x*y/2 + y^2 + z^2)
sage: C.symmetric_matrix()
[ 1 1/4  0]
[1/4  1  0]
[ 0  0  1]

sage: C = Conic(x^2 + 2*x*y + y^2 + 3*x*z + z^2)
sage: v = vector([x, y, z])
sage: v * C.symmetric_matrix() * v
x^2 + 2*x*y + y^2 + 3*x*z + z^2
```

### **upper\_triangular\_matrix()**

The upper-triangular matrix  $M$  such that  $(xyz)M(xyz)^t$  is the defining equation of self.

#### EXAMPLES:

```
sage: R.<x, y, z> = QQ[]
sage: C = Conic(x^2 + x*y + y^2 + z^2)
sage: C.upper_triangular_matrix()
[1 1 0]
[0 1 0]
[0 0 1]

sage: C = Conic(x^2 + 2*x*y + y^2 + 3*x*z + z^2)
sage: v = vector([x, y, z])
sage: v * C.upper_triangular_matrix() * v
x^2 + 2*x*y + y^2 + 3*x*z + z^2
```

### **variable\_names()**

Returns the variable names of the defining polynomial of self.

#### EXAMPLES:

```
sage: c=Conic([1,1,0,1,0,1], 'x,y,z')
sage: c.variable_names()
('x', 'y', 'z')
sage: c.variable_name()
'x'
```

The function `variable_names()` is required for the following construction:

```
sage: C.<p,q,r> = Conic(QQ, [1, 1, 1])
sage: C
Projective Conic Curve over Rational Field defined by p^2 + q^2 + r^2
```

### 8.3 Projective plane conics over a number field

AUTHORS:

- Marco Streng (2010-07-20)

**class** `sage.schemes.plane_conics.con_number_field.ProjectiveConic_number_field`(*A*, *f*)

Bases: `sage.schemes.plane_conics.con_field.ProjectiveConic_field`

Create a projective plane conic curve over a number field. See `Conic` for full documentation.

EXAMPLES:

```
sage: K.<a> = NumberField(x^3 - 2, 'a')
sage: P.<X, Y, Z> = K[]
sage: C = Conic(X^2 + Y^2 - a*Z^2)
Projective Conic Curve over Number Field in a with defining polynomial x^3 - 2
↳ defined by X^2 + Y^2 + (-a)*Z^2
```

**has\_rational\_point** (*point=False*, *obstruction=False*, *algorithm='default'*, *read\_cache=True*)

Returns True if and only if `self` has a point defined over its base field  $B$ .

If `point` and `obstruction` are both False (default), then the output is a boolean out saying whether `self` has a rational point.

If `point` or `obstruction` is True, then the output is a pair (`out`,  $S$ ), where `out` is as above and:

- if `point` is True and `self` has a rational point, then  $S$  is a rational point,
- if `obstruction` is True, `self` has no rational point, then  $S$  is a prime or infinite place of  $B$  such that no rational point exists over the completion at  $S$ .

Points and obstructions are cached whenever they are found. Cached information is used for the output if available, but only if `read_cache` is True.

ALGORITHM:

The parameter `algorithm` specifies the algorithm to be used:

- `'rnfisnorm'` – Use PARI's `rnfisnorm` (cannot be combined with `obstruction = True`)
- `'local'` – Check if a local solution exists for all primes and infinite places of  $B$  and apply the Hasse principle. (Cannot be combined with `point = True`.)
- `'default'` – Use algorithm `'rnfisnorm'` first. Then, if no point exists and obstructions are requested, use algorithm `'local'` to find an obstruction.
- `'magma'` (requires Magma to be installed) – delegates the task to the Magma computer algebra system.

EXAMPLES:

An example over  $\mathbb{Q}$

```

sage: C = Conic(QQ, [1, 113922743, -310146482690273725409])
sage: C.has_rational_point(point = True)
(True, (-76842858034579/5424 : -5316144401/5424 : 1))
sage: C.has_rational_point(algorithm = 'local', read_cache = False)
True

```

Examples over number fields:

```

sage: K.<i> = QuadraticField(-1)
sage: C = Conic(K, [1, 3, -5])
sage: C.has_rational_point(point = True, obstruction = True)
(False, Fractional ideal (-i - 2))
sage: C.has_rational_point(algorithm = "rnfisnorm")
False
sage: C.has_rational_point(algorithm = "rnfisnorm", obstruction = True, read_
↪cache=False)
Traceback (most recent call last):
...
ValueError: Algorithm rnfisnorm cannot be combined with obstruction = True in_
↪has_rational_point

sage: P.<x> = QQ[]
sage: L.<b> = NumberField(x^3-5)
sage: C = Conic(L, [1, 2, -3])
sage: C.has_rational_point(point = True, algorithm = 'rnfisnorm')
(True, (5/3 : -1/3 : 1))

sage: K.<a> = NumberField(x^4+2)
sage: Conic(QQ, [4,5,6]).has_rational_point()
False
sage: Conic(K, [4,5,6]).has_rational_point()
True
sage: Conic(K, [4,5,6]).has_rational_point(algorithm='magma', read_
↪cache=False) # optional - magma
True

```

### `is_locally_solvable(p)`

Returns True if and only if `self` has a solution over the completion of the base field  $B$  of `self` at  $p$ . Here  $p$  is a finite prime or infinite place of  $B$ .

EXAMPLES:

```

sage: P.<x> = QQ[]
sage: K.<a> = NumberField(x^3 + 5)
sage: C = Conic(K, [1, 2, 3 - a])
sage: [p1, p2] = K.places()
sage: C.is_locally_solvable(p1)
False

sage: C.is_locally_solvable(p2)
True

sage: O = K.maximal_order()
sage: f = (2*O).factor()
sage: C.is_locally_solvable(f[0][0])
True

```

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```
sage: C.is_locally_solvable(f[1][0])
False
```

**local\_obstructions** (*finite=True, infinite=True, read\_cache=True*)

Returns the sequence of finite primes and/or infinite places such that `self` is locally solvable at those primes and places.

If the base field is  $\mathbb{Q}$ , then the infinite place is denoted  $-1$ .

The parameters `finite` and `infinite` (both `True` by default) are used to specify whether to look at finite and/or infinite places. Note that `finite = True` involves factorization of the determinant of `self`, hence may be slow.

Local obstructions are cached. The parameter `read_cache` specifies whether to look at the cache before computing anything.

#### EXAMPLES

```
sage: K.<i> = QuadraticField(-1)
sage: Conic(K, [1, 2, 3]).local_obstructions()
[]

sage: L.<a> = QuadraticField(5)
sage: Conic(L, [1, 2, 3]).local_obstructions()
[Ring morphism:
  From: Number Field in a with defining polynomial x^2 - 5 with a = 2.
  ↪236067977499790?
  To:   Algebraic Real Field
  Defn: a |--> -2.236067977499790?, Ring morphism:
  From: Number Field in a with defining polynomial x^2 - 5 with a = 2.
  ↪236067977499790?
  To:   Algebraic Real Field
  Defn: a |--> 2.236067977499790?]
```

## 8.4 Projective plane conics over $\mathbb{Q}$

AUTHORS:

- Marco Streng (2010-07-20)
- Nick Alexander (2008-01-08)

**class** `sage.schemes.plane_conics.con_rational_field.ProjectiveConic_rational_field` (*A, f*)

Bases: `sage.schemes.plane_conics.con_number_field.ProjectiveConic_number_field`

Create a projective plane conic curve over  $\mathbb{Q}$ . See `Conic` for full documentation.

EXAMPLES:

```
sage: P.<X, Y, Z> = QQ[]
sage: Conic(X^2 + Y^2 - 3*Z^2)
Projective Conic Curve over Rational Field defined by X^2 + Y^2 - 3*Z^2
```

**has\_rational\_point** (*point=False, obstruction=False, algorithm='default', read\_cache=True*)

Returns `True` if and only if `self` has a point defined over  $\mathbb{Q}$ .

If `point` and `obstruction` are both `False` (default), then the output is a boolean out saying whether `self` has a rational point.

If `point` or `obstruction` is `True`, then the output is a pair `(out, S)`, where `out` is as above and the following holds:

- if `point` is `True` and `self` has a rational point, then `S` is a rational point,
- if `obstruction` is `True` and `self` has no rational point, then `S` is a prime such that no rational point exists over the completion at `S` or `-1` if no point exists over  $\mathbf{R}$ .

Points and obstructions are cached, whenever they are found. Cached information is used if and only if `read_cache` is `True`.

#### ALGORITHM:

The parameter `algorithm` specifies the algorithm to be used:

- `'qfsolve'` – Use PARI/GP function `qfsolve`
- `'rnfnorm'` – Use PARI's function `rnfnorm` (cannot be combined with `obstruction = True`)
- `'local'` – Check if a local solution exists for all primes and infinite places of  $\mathbf{Q}$  and apply the Hasse principle (cannot be combined with `point = True`)
- `'default'` – Use `'qfsolve'`
- `'magma'` (requires Magma to be installed) – delegates the task to the Magma computer algebra system.

#### EXAMPLES:

```
sage: C = Conic(QQ, [1, 2, -3])
sage: C.has_rational_point(point = True)
(True, (1 : 1 : 1))
sage: D = Conic(QQ, [1, 3, -5])
sage: D.has_rational_point(point = True)
(False, 3)
sage: P.<X,Y,Z> = QQ[]
sage: E = Curve(X^2 + Y^2 + Z^2); E
Projective Conic Curve over Rational Field defined by X^2 + Y^2 + Z^2
sage: E.has_rational_point(obstruction = True)
(False, -1)
```

The following would not terminate quickly with `algorithm = 'rnfnorm'`

```
sage: C = Conic(QQ, [1, 113922743, -310146482690273725409])
sage: C.has_rational_point(point = True)
(True, (-76842858034579/5424 : -5316144401/5424 : 1))
sage: C.has_rational_point(algorithm = 'local', read_cache = False)
True
sage: C.has_rational_point(point=True, algorithm='magma', read_cache=False) #_
↪ optional - magma
(True, (30106379962113/7913 : 12747947692/7913 : 1))
```

#### `is_locally_solvable(p)`

Returns `True` if and only if `self` has a solution over the  $p$ -adic numbers. Here  $p$  is a prime number or equals `-1`, infinity, or  $\mathbf{R}$  to denote the infinite place.

#### EXAMPLES:

```

sage: C = Conic(QQ, [1, 2, 3])
sage: C.is_locally_solvable(-1)
False
sage: C.is_locally_solvable(2)
False
sage: C.is_locally_solvable(3)
True
sage: C.is_locally_solvable(QQ.hom(RR))
False
sage: D = Conic(QQ, [1, 2, -3])
sage: D.is_locally_solvable(infinity)
True
sage: D.is_locally_solvable(RR)
True

```

**local\_obstructions** (*finite=True, infinite=True, read\_cache=True*)

Returns the sequence of finite primes and/or infinite places such that self is locally solvable at those primes and places.

The infinite place is denoted  $-1$ .

The parameters *finite* and *infinite* (both *True* by default) are used to specify whether to look at finite and/or infinite places. Note that *finite* = *True* involves factorization of the determinant of self, hence may be slow.

Local obstructions are cached. The parameter *read\_cache* specifies whether to look at the cache before computing anything.

#### EXAMPLES

```

sage: Conic(QQ, [1, 1, 1]).local_obstructions()
[2, -1]
sage: Conic(QQ, [1, 2, -3]).local_obstructions()
[]
sage: Conic(QQ, [1, 2, 3, 4, 5, 6]).local_obstructions()
[41, -1]

```

**parametrization** (*point=None, morphism=True*)

Return a parametrization  $f$  of self together with the inverse of  $f$ .

If *point* is specified, then that point is used for the parametrization. Otherwise, use *self.rational\_point()* to find a point.

If *morphism* is *True*, then  $f$  is returned in the form of a Scheme morphism. Otherwise, it is a tuple of polynomials that gives the parametrization.

#### ALGORITHM:

Uses the PARI/GP function *qfparam*.

#### EXAMPLES

```

sage: c = Conic([1, 1, -1])
sage: c.parametrization()
(Scheme morphism:
  From: Projective Space of dimension 1 over Rational Field
  To:   Projective Conic Curve over Rational Field defined by x^2 + y^2 - z^2
  Defn: Defined on coordinates by sending (x : y) to
        (2*x*y : x^2 - y^2 : x^2 + y^2),
  Scheme morphism:

```

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```

From: Projective Conic Curve over Rational Field defined by  $x^2 + y^2 - z^2$ 
To:   Projective Space of dimension 1 over Rational Field
Defn: Defined on coordinates by sending  $(x : y : z)$  to
       $(1/2*x : -1/2*y + 1/2*z)$ 

```

An example with `morphism = False`

```

sage: R.<x,y,z> = QQ[]
sage: C = Curve(7*x^2 + 2*y*z + z^2)
sage: (p, i) = C.parametrization(morphism = False); (p, i)
([-2*x*y, x^2 + 7*y^2, -2*x^2], [-1/2*x, 1/7*y + 1/14*z])
sage: C.defining_polynomial()(p)
0
sage: i[0](p) / i[1](p)
x/y

```

A `ValueError` is raised if `self` has no rational point

```

sage: C = Conic(x^2 + 2*y^2 + z^2)
sage: C.parametrization()
Traceback (most recent call last):
...
ValueError: Conic Projective Conic Curve over Rational Field defined by  $x^2 + 2y^2 + z^2$  has no rational points over Rational Field!

```

A `ValueError` is raised if `self` is not smooth

```

sage: C = Conic(x^2 + y^2)
sage: C.parametrization()
Traceback (most recent call last):
...
ValueError: The conic self (=Projective Conic Curve over Rational Field defined by  $x^2 + y^2$ ) is not smooth, hence does not have a parametrization.

```

## 8.5 Projective plane conics over finite fields

AUTHORS:

- Marco Streng (2010-07-20)

```

class sage.schemes.plane_conics.con_finite_field.ProjectiveConic_finite_field(A, f)
Bases: sage.schemes.plane_conics.con_field.ProjectiveConic_field, sage.schemes.curves.projective_curve.ProjectivePlaneCurve_finite_field

```

Create a projective plane conic curve over a finite field.

See `Conic` for full documentation.

EXAMPLES:

```

sage: K.<a> = FiniteField(9, 'a')
sage: P.<X, Y, Z> = K[]
sage: C = Conic(X^2 + Y^2 - a*Z^2)
Projective Conic Curve over Finite Field in a of size 3^2 defined by  $X^2 + Y^2 + (-a)*Z^2$ 

```



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[illegible]

## 8.6 Projective plane conics over a rational function field

The class `ProjectiveConic_rational_function_field` represents a projective plane conic over a rational function field  $F(t)$ , where  $F$  is any field. Instances can be created using `Conic()`.

AUTHORS:

- Lennart Ackermans (2016-02-07): initial version

EXAMPLES:

Create a conic:

```
sage: K = FractionField(PolynomialRing(QQ, 't'))
sage: P.<X, Y, Z> = K[]
sage: Conic(X^2 + Y^2 - Z^2)
Projective Conic Curve over Fraction Field of Univariate
Polynomial Ring in t over Rational Field defined by
X^2 + Y^2 - Z^2
```

Points can be found using `has_rational_point()`:

```
sage: K.<t> = FractionField(QQ['t'])
sage: C = Conic([1,-t,t])
sage: C.has_rational_point(point = True)
(True, (0 : 1 : 1))
```

**class** `sage.schemes.plane_conics.con_rational_function_field.ProjectiveConic_rational_funct.`

Bases: `sage.schemes.plane_conics.con_field.ProjectiveConic_field`

Create a projective plane conic curve over a rational function field  $F(t)$ , where  $F$  is any field.

The algorithms used in this class come mostly from [HC2006].

EXAMPLES:

```
sage: K = FractionField(PolynomialRing(QQ, 't'))
sage: P.<X, Y, Z> = K[]
sage: Conic(X^2 + Y^2 - Z^2)
Projective Conic Curve over Fraction Field of Univariate
Polynomial Ring in t over Rational Field defined by
X^2 + Y^2 - Z^2
```

REFERENCES:

- [HC2006]
- [Ack2016]

**find\_point** (*supports, roots, case, solution=0*)

Given a solubility certificate like in [HC2006], find a point on `self`. Assumes `self` is in reduced form (see [HC2006] for a definition).

If you don't have a solubility certificate and just want to find a point, use the function `has_rational_point()` instead.

INPUT:

- `self` – conic in reduced form.
- `supports` – 3-tuple where `supports[i]` is a list of all monic irreducible  $p \in F[t]$  that divide the  $i$ 'th of the 3 coefficients.

- `roots` – 3-tuple containing lists of roots of all elements of `supports[i]`, in the same order.
- `case` – 1 or 0, as in [HC2006].
- `solution` – (default: 0) a solution of (5) in [HC2006], if `case = 0`, 0 otherwise.

OUTPUT:

A point  $(x, y, z) \in F(t)$  of `self`. Output is undefined when the input solubility certificate is incorrect.

ALGORITHM:

The algorithm used is the algorithm `FindPoint` in [HC2006], with a simplification from [Ack2016].

EXAMPLES:

```
sage: K.<t> = FractionField(QQ['t'])
sage: C = Conic(K, [t^2-2, 2*t^3, -2*t^3-13*t^2-2*t+18])
sage: C.has_rational_point(point=True) # indirect test
(True, (-3 : (t + 1)/t : 1))
```

Different solubility certificates give different points:

```
sage: K.<t> = PolynomialRing(QQ, 't')
sage: C = Conic(K, [t^2-2, 2*t, -2*t^3-13*t^2-2*t+18])
sage: supp = [[t^2 - 2], [t], [t^3 + 13/2*t^2 + t - 9]]
sage: tbar1 = QQ.extension(supp[0][0], 'tbar').gens()[0]
sage: tbar2 = QQ.extension(supp[1][0], 'tbar').gens()[0]
sage: tbar3 = QQ.extension(supp[2][0], 'tbar').gens()[0]
sage: roots = [[tbar1 + 1], [1/3*tbar2^0], [2/3*tbar3^2 + 11/3*tbar3 - 3]]
sage: C.find_point(supp, roots, 1)
(3 : t + 1 : 1)
sage: roots = [[-tbar1 - 1], [-1/3*tbar2^0], [-2/3*tbar3^2 - 11/3*tbar3 + 3]]
sage: C.find_point(supp, roots, 1)
(3 : -t - 1 : 1)
```

**has\_rational\_point** (*point=False, algorithm='default', read\_cache=True*)

Returns True if and only if the conic `self` has a point over its base field  $F(t)$ , which is a field of rational functions.

If `point` is True, then returns a second output, which is a rational point if one exists.

Points are cached whenever they are found. Cached information is used if and only if `read_cache` is True.

The default algorithm does not (yet) work for all base fields  $F$ . In particular, sage is required to have:

- an algorithm for finding the square root of elements in finite extensions of  $F$ ;
- a factorization and gcd algorithm for  $F[t]$ ;
- an algorithm for solving conics over  $F$ .

ALGORITHM:

The parameter `algorithm` specifies the algorithm to be used:

- 'default' – use a native Sage implementation, based on the algorithm `Conic` in [HC2006].
- 'magma' (requires Magma to be installed) – delegates the task to the Magma computer algebra system.

EXAMPLES:

We can find points for function fields over (extensions of)  $\mathbb{Q}$  and finite fields:

```

sage: K.<t> = FractionField(PolynomialRing(QQ, 't'))
sage: C = Conic(K, [t^2-2, 2*t^3, -2*t^3-13*t^2-2*t+18])
sage: C.has_rational_point(point=True)
(True, (-3 : (t + 1)/t : 1))
sage: R.<t> = FiniteField(23)[]
sage: C = Conic([2, t^2+1, t^2+5])
sage: C.has_rational_point()
True
sage: C.has_rational_point(point=True)
(True, (5*t : 8 : 1))
sage: F.<i> = QuadraticField(-1)
sage: R.<t> = F[]
sage: C = Conic([1, i*t, -t^2+4])
sage: C.has_rational_point(point = True)
(True, (-t - 2*i : -2*i : 1))

```

It works on non-diagonal conics as well:

```

sage: K.<t> = QQ[]
sage: C = Conic([4, -4, 8, 1, -4, t + 4])
sage: C.has_rational_point(point=True)
(True, (1/2 : 1 : 0))

```

If no point exists output still depends on the argument point:

```

sage: K.<t> = QQ[]
sage: C = Conic(K, [t^2, (t-1), -2*(t-1)])
sage: C.has_rational_point()
False
sage: C.has_rational_point(point=True)
(False, None)

```

Due to limitations in Sage of algorithms we depend on, it is not yet possible to find points on conics over multivariate function fields (see the requirements above):

```

sage: F.<t1> = FractionField(QQ['t1'])
sage: K.<t2> = FractionField(F['t2'])
sage: a = K(1)
sage: b = 2*t2^2+2*t1*t2-t1^2
sage: c = -3*t2^4-4*t1*t2^3+8*t1^2*t2^2+16*t1^3-t2-48*t1^4
sage: C = Conic([a,b,c])
sage: C.has_rational_point()
Traceback (most recent call last):
...
NotImplementedError: is_square() not implemented for elements of
Univariate Quotient Polynomial Ring in tbar over Fraction Field
of Univariate Polynomial Ring in t1 over Rational Field with
modulus tbar^2 + t1*tbar - 1/2*t1^2

```

In some cases, the algorithm requires us to be able to solve conics over  $F$ . In particular, the following does not work:

```

sage: P.<u> = QQ[]
sage: E = P.fraction_field()
sage: Q.<Y> = E[]
sage: F.<v> = E.extension(Y^2 - u^3 - 1)
sage: R.<t> = F[]

```

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```

sage: K = R.fraction_field()
sage: C = Conic(K, [u, v, 1])
sage: C.has_rational_point()
Traceback (most recent call last):
...
NotImplementedError: has_rational_point not implemented for conics
over base field Univariate Quotient Polynomial Ring in v over
Fraction Field of Univariate Polynomial Ring in u over Rational
Field with modulus v^2 - u^3 - 1

```

`has_rational_point` fails for some conics over function fields over finite fields, due to [trac ticket #20003](#):

```

sage: K.<t> = PolynomialRing(GF(7))
sage: C = Conic([5*t^2+4, t^2+3*t+3, 6*t^2+3*t+2, 5*t^2+5, 4*t+3, 4*t^2+t+5])
sage: C.has_rational_point()
Traceback (most recent call last):
...
TypeError: self (=Scheme morphism:
  From: Projective Conic Curve over Fraction Field of Univariate Polynomial
  ↪ Ring in t over Finite Field of size 7 defined by (5*t^2 + 4)*x^2 + ((6*t^3
  ↪ + 3*t^2 + 5*t + 5)/(t + 3))*y^2 + ((6*t^6 + 3*t^5 + t^3 + 6*t^2 + 6*t + 2)/
  ↪ (t^4 + t^3 + 4*t^2 + 3*t + 1))*z^2
  To: Projective Conic Curve over Fraction Field of Univariate Polynomial
  ↪ Ring in t over Finite Field of size 7 defined by (5*t^2 + 4)*x^2 + (t^2 +
  ↪ 3*t + 3)*x*y + (5*t^2 + 5)*y^2 + (6*t^2 + 3*t + 2)*x*z + (4*t + 3)*y*z +
  ↪ (4*t^2 + t + 5)*z^2
  Defn: Defined on coordinates by sending (x : y : z) to
      (x + ((2*t + 5)/(t + 3))*y + ((3*t^4 + 2*t^3 + 5*t^2 + 5*t + 3)/(t^4
  ↪ + t^3 + 4*t^2 + 3*t + 1))*z : y + ((6*t^3 + 6*t^2 + 3*t + 6)/(t^3 + 4*t^2 +
  ↪ 2*t + 2))*z : z)) domain must equal right (=Scheme morphism:
  From: Projective Conic Curve over Fraction Field of Univariate Polynomial
  ↪ Ring in t over Finite Field of size 7 defined by (5*t^3 + 6*t^2 + 3*t +
  ↪ 3)*x^2 + (t + 4)*y^2 + (6*t^7 + 2*t^5 + t^4 + 2*t^3 + 3*t^2 + 6*t + 6)*z^2
  To: Projective Conic Curve over Fraction Field of Univariate Polynomial
  ↪ Ring in t over Finite Field of size 7 defined by (5/(t^3 + 4*t^2 + 2*t +
  ↪ 2))*x^2 + (1/(t^3 + 3*t^2 + 5*t + 1))*y^2 + ((6*t^6 + 3*t^5 + t^3 + 6*t^2 +
  ↪ 6*t + 2)/(t^9 + 5*t^8 + t^7 + 6*t^6 + 3*t^5 + 4*t^3 + t^2 + 5*t + 3))*z^2
  Defn: Defined on coordinates by sending (x : y : z) to
      ((t^3 + 4*t^2 + 2*t + 2)*x : (t^2 + 5)*y : (t^5 + 4*t^4 + t^2 + 3*t +
  ↪ 3)*z)) codomain

```



## PLANE QUARTICS

## 9.1 Quartic curve constructor

`sage.schemes.plane_quartics.quartic_constructor.QuarticCurve` ( $F$ ,  $PP=None$ ,  $check=False$ )

Returns the quartic curve defined by the polynomial  $F$ .

INPUT:

- $F$  – a polynomial in three variables, homogeneous of degree 4
- $PP$  – a projective plane (default:None)
- $check$  – whether to check for smoothness or not (default:False)

EXAMPLES:

```
sage: x,y,z=PolynomialRing(QQ,['x','y','z']).gens()
sage: QuarticCurve(x**4+y**4+z**4)
Quartic Curve over Rational Field defined by  $x^4 + y^4 + z^4$ 
```

## 9.2 Plane quartic curves over a general ring

These are generic genus 3 curves, as distinct from hyperelliptic curves of genus 3.

EXAMPLES:

```
sage: PP.<X,Y,Z> = ProjectiveSpace(2, QQ)
sage: f = X^4 + Y^4 + Z^4 - 3*X*Y*Z*(X+Y+Z)
sage: C = QuarticCurve(f); C
Quartic Curve over Rational Field defined by  $X^4 + Y^4 - 3X^2YZ - 3XY^2Z - 3XYZ^2 + Z^4$ 
```

**class** `sage.schemes.plane_quartics.quartic_generic.QuarticCurve_generic` ( $A$ ,  $f$ )

Bases: `sage.schemes.curves.projective_curve.ProjectivePlaneCurve`

**genus** ()

Returns the genus of self

EXAMPLES:

```
sage: x,y,z=PolynomialRing(QQ,['x','y','z']).gens()
sage: Q = QuarticCurve(x**4+y**4+z**4)
sage: Q.genus()
3
```

`sage.schemes.plane_quartics.quartic_generic.is_QuarticCurve(C)`

Checks whether C is a Quartic Curve

EXAMPLES:

```
sage: from sage.schemes.plane_quartics.quartic_generic import is_QuarticCurve
sage: x,y,z=PolynomialRing(QQ,['x','y','z']).gens()
sage: Q = QuarticCurve(x**4+y**4+z**4)
sage: is_QuarticCurve(Q)
True
```



## RIEMANN SURFACES

### 10.1 Riemann matrices and endomorphism rings of algebraic Riemann surfaces

This module provides a class, `RiemannSurface`, to model the Riemann surface determined by a plane algebraic curve over a subfield of the complex numbers.

A homology basis is derived from the edges of a Voronoi cell decomposition based on the branch locus. The pull-back of these edges to the Riemann surface provides a graph on it that contains a homology basis.

The class provides methods for computing the Riemann period matrix of the surface numerically, using a certified homotopy continuation method due to [Kr2016].

The class also provides facilities for computing the endomorphism ring of the period lattice numerically, by determining integer (near) solutions to the relevant approximate linear equations.

AUTHORS:

- Alexandre Zotine, Nils Bruin (2017-06-10): initial version
- Nils Bruin, Jeroen Sijsling (2018-01-05): algebraization, isomorphisms

EXAMPLES:

We compute the Riemann matrix of a genus 3 curve:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<x,y>=QQ[]
sage: f=x^4-x^3*y+2*x^3+2*x^2*y+2*x^2-2*x*y^2+4*x*y-y^3+3*y^2+2*y+1
sage: S=RiemannSurface(f,prec=100)
sage: M=S.riemann_matrix()
```

We test the usual properties, i.e., that the period matrix is symmetric and that the imaginary part is positive definite:

```
sage: all(abs(a) < 1e-20 for a in (M-M.T).list())
True
sage: iM = Matrix(RDF,3,3,[a.imag_part() for a in M.list()])
sage: iM.is_positive_definite()
True
```

We compute the endomorphism ring and check it has  $\mathbb{Z}$ -rank 6:

```
sage: A=S.endomorphism_basis(80,8)
sage: len(A) == 6
True
```

In fact it is an order in a number field:

```
sage: T.<t>=QQ[]
sage: K.<a>=NumberField(t^6 - t^5 + 2*t^4 + 8*t^3 - t^2 - 5*t + 7)
sage: all(len(a.minpoly().roots(K)) == a.minpoly().degree() for a in A)
True
```

**exception** `sage.schemes.riemann_surfaces.riemann_surface.ConvergenceError`  
 Bases: `ValueError`

Error object suitable for raising and catching when Newton iteration fails.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import ConvergenceError
sage: raise ConvergenceError("test")
Traceback (most recent call last):
...
ConvergenceError: test
sage: isinstance(ConvergenceError(), ValueError)
True
```

**class** `sage.schemes.riemann_surfaces.riemann_surface.RiemannSurface` (*f*,  
`prec=53`,  
`certification=True`,  
`differentials=None`)

Bases: `object`

Construct a Riemann Surface. This is specified by the zeroes of a bivariate polynomial with rational coefficients  $f(z, w) = 0$ .

INPUT:

- *f* – a bivariate polynomial with rational coefficients. The surface is interpreted as the covering space of the coordinate plane in the first variable.
- *prec* – the desired precision of computations on the surface in bits (default: 53)
- *certification* – a boolean (default: True) value indicating whether homotopy continuation is certified or not. Uncertified homotopy continuation can be faster.
- *differentials* – (default: None). If specified, provides a list of polynomials *h* such that  $h/(df/dw)dz$  is a regular differential on the Riemann surface. This is taken as a basis of the regular differentials, so the genus is assumed to be equal to the length of this list. The results from the homology basis computation are checked against this value. Providing this parameter makes the computation independent from Singular. For a nonsingular plane curve of degree *d*, an appropriate set is given by the monomials of degree up to  $d - 3$ .

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = w^2 - z^3 + 1
sage: RiemannSurface(f)
Riemann surface defined by polynomial f = -z^3 + w^2 + 1 = 0, with 53 bits of
↳ precision
```

Another Riemann surface with 100 bits of precision:

We can also work with Riemann surfaces that are defined over fields with a complex embedding, but since the current interface for computing genus and regular differentials in Singular presently does not support extensions of `QQ`, we need to specify a description of the differentials ourselves. We give an example of a CM elliptic curve:

**cohomology\_basis** (*option=1*)

INPUT:

- OUTPUT:

EXAMPLES:

`downstairs_edges()`

OUTPUT:

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = w^2 + z^3 - z^2
sage: S = RiemannSurface(f)
```

Compute the edges:

```
sage: S.downstairs_edges()
[(0, 1), (0, 5), (1, 4), (2, 3), (2, 4), (3, 5), (4, 5)]
```

This now gives an edgeset which one could use to form a graph.

---

**Note:** The numbering of the vertices is given by the Voronoi package.

---

### **downstairs\_graph()**

Return the Voronoi decomposition as a planar graph.

The result of this routine can be useful to interpret the labelling of the vertices.

OUTPUT:

The Voronoi decomposition as a graph, with appropriate planar embedding.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = w^2 - z^4 + 1
sage: S = RiemannSurface(f)
sage: S.downstairs_graph()
Graph on 11 vertices
```

Similarly one can form the graph of the upstairs edges, which is visually rather less attractive but can be instructive to verify that a homology basis is likely correctly computed.:

```
sage: G=Graph(S.upstairs_edges()); G
Graph on 22 vertices
sage: G.is_planar()
False
sage: G.genus()
1
sage: G.is_connected()
True
```

### **edge\_permutations()**

Compute the permutations of branches associated to each edge.

Over the vertices of the Voronoi decomposition around the branch locus, we label the fibres. By following along an edge, the lifts of the edge induce a permutation of that labelling.

OUTPUT:

A dictionary with as keys the edges of the Voronoi decomposition and as values the corresponding permutations.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = w^2 + z^2+1
sage: S = RiemannSurface(f)
sage: S.edge_permutations()
{(0, 2): (),
 (0, 4): (),
```

(continues on next page)

(continued from previous page)

```

(1, 2): (),
(1, 3): (0, 1),
(1, 6): (),
(2, 0): (),
(2, 1): (),
(2, 5): (0, 1),
(3, 1): (0, 1),
(3, 4): (),
(4, 0): (),
(4, 3): (),
(5, 2): (0, 1),
(5, 7): (),
(6, 1): (),
(6, 7): (),
(7, 5): (),
(7, 6): () }

```

**endomorphism\_basis** (*b=None, r=None*)

Numerically compute a  $\mathbf{Z}$ -basis for the endomorphism ring.

Let  $(I|M)$  be the normalized period matrix ( $M$  is the  $g \times g$  `riemann_matrix()`). We consider the system of matrix equations  $MA + C = (MB + D)M$  where  $A, B, C, D$  are  $g \times g$  integer matrices. We determine small integer (near) solutions using LLL reductions. These solutions are returned as  $2g \times 2g$  integer matrices obtained by stacking  $(D|B)$  on top of  $(C|A)$ .

INPUT:

- *b* – integer (default provided). The equation coefficients are scaled by  $2^b$  before rounding to integers.
- *r* – integer (default:  $b/4$ ). Solutions that have all coefficients smaller than  $2^r$  in absolute value are reported as actual solutions.

OUTPUT:

A list of  $2g \times 2g$  integer matrices that, for large enough *r* and  $b-r$ , generate the endomorphism ring.

EXAMPLES:

```

sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<x,y> = QQ[]
sage: S = RiemannSurface(x^3 + y^3 + 1)
sage: B = S.endomorphism_basis(); B #random
[
[1 0] [0 -1]
[0 1], [1 1]
]
sage: sorted([b.minpoly().disc() for b in B])
[-3, 1]

```

**homology\_basis** ()

Compute the homology basis of the Riemann surface.

OUTPUT:

A list of paths  $L = [P_1, \dots, P_n]$ . Each path  $P_i$  is of the form  $(k, [p_1 \dots p_m, p_1])$ , where  $k$  is the number of times to traverse the path (if negative, to traverse it backwards), and the  $p_i$  are vertices of the upstairs graph.

EXAMPLES:

In this example, there are two paths that form the homology basis:

```

sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: g = w^2 - z^4 + 1
sage: S = RiemannSurface(g)
sage: S.homology_basis() #random
[[ (1, [(3, 1), (5, 0), (9, 0), (10, 0), (2, 0), (4, 0),
      (7, 1), (10, 1), (3, 1)]),
  (1, [(8, 0), (6, 0), (7, 0), (10, 0), (2, 0), (4, 0),
      (7, 1), (10, 1), (9, 1), (8, 0)])]]

```

In order to check that the answer returned above is reasonable, we test some basic properties. We express the faces of the downstairs graph as  $\mathbb{Z}\mathbb{Z}$ -linear combinations of the edges and check that the projection of the homology basis upstairs projects down to independent linear combinations of an even number of faces:

```

sage: dg = S.downstairs_graph()
sage: edges = dg.edges()
sage: E = ZZ^len(edges)
sage: edge_to_E = { e[:2]: E.gen(i) for i,e in enumerate(edges) }
sage: edge_to_E.update({ (e[1],e[0]): -E.gen(i) for i,e in enumerate(edges) })
sage: face_span = E.submodule([sum(edge_to_E[e] for e in f) for f in dg.
    ↪ faces()])
sage: def path_to_E(path):
....:     k,P = path
....:     return k*sum(edge_to_E[(P[i][0],P[i+1][0])] for i in range(len(P)-
    ↪ 1))
sage: hom_basis = [sum(path_to_E(p) for p in loop) for loop in S.homology_
    ↪ basis()]
sage: face_span.submodule(hom_basis).rank()
2
sage: [sum(face_span.coordinate_vector(b))%2 for b in hom_basis]
[0, 0]

```

### homomorphism\_basis (other, b=None, r=None)

Numerically compute a  $\mathbb{Z}$ -basis for module of homomorphisms to a given complex torus.

Given another complex torus (given as the analytic Jacobian of a Riemann surface), numerically compute a basis for the homomorphism module. The answer is returned as a list of  $2g \times 2g$  integer matrices  $T=(D, B; C, A)$  such that if the columns of (IIM1) generate the lattice defining the Jacobian of the Riemann surface and the columns of (IIM2) do this for the codomain, then approximately we have  $(\text{IIM2})T=(D+M2C)(\text{IIM1})$ , i.e., up to a choice of basis for  $\mathbb{C}^g$  as a complex vector space, we realize (IIM1) as a sublattice of (IIM2).

INPUT:

- $b$  – integer (default provided). The equation coefficients are scaled by  $2^b$  before rounding to integers.
- $r$  – integer (default:  $b/4$ ). Solutions that have all coefficients smaller than  $2^r$  in absolute value are reported as actual solutions.

OUTPUT:

A list of  $2g \times 2g$  integer matrices that, for large enough  $r$  and  $b-r$ , generate the homomorphism module.

EXAMPLES:

```

sage: S1 = EllipticCurve("11a1").riemann_surface()
sage: S2 = EllipticCurve("11a3").riemann_surface()
sage: [m.det() for m in S1.homomorphism_basis(S2)]
[5]

```

**homotopy\_continuation** (*edge*)

Perform homotopy continuation along an edge of the Voronoi diagram using Newton iteration.

INPUT:

- *edge* – a tuple of integers indicating an edge of the Voronoi diagram

OUTPUT:

A list of complex numbers corresponding to the points which are reached when traversing along the direction of the edge. The ordering of these points indicates how they have been permuted due to the weaving of the curve.

EXAMPLES:

We check that continued values along an edge correspond (up to the appropriate permutation) to what is stored. Note that the permutation was originally computed from this data:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = z^3*w + w^3 + z
sage: S = RiemannSurface(f)
sage: edge1 = sorted(S.edge_permutations())[0]
sage: sigma = S.edge_permutations()[edge1]
sage: continued_values = S.homotopy_continuation(edge1)
sage: stored_values = S.w_values(S._vertices[edge1[1]])
sage: all(abs(continued_values[i]-stored_values[sigma(i)]) < 1e-8 for i in
↪ range(3))
True
```

**make\_zw\_interpolator** (*upstairs\_edge*)

Given an upstairs edge for which continuation data has been stored, return a function that computes  $z(t), w(t)$ , where  $t$  in  $[0, 1]$  is a parametrization of the edge.

INPUT:

- *upstairs\_edge* – a pair of integer tuples indicating an edge on the upstairs graph of the surface

OUTPUT:

A tuple (g, d), where g is the function that computes the interpolation along the edge and d is the difference of the z-values of the end and start point.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = w^2 - z^4 + 1
sage: S = RiemannSurface(f)
sage: _ = S.homology_basis()
sage: g,d = S.make_zw_interpolator([(0,0),(1,0)]);
sage: all(f(*g(i*0.1)).abs() < 1e-13 for i in range(10))
True
sage: abs((g(1)[0]-g(0)[0]) - d) < 1e-13
True
```

**matrix\_of\_integral\_values** (*differentials*)

Compute the path integrals of the given differentials along the homology basis.

The returned answer has a row for each differential. If the Riemann surface is given by the equation  $f(z,w) = 0$ , then the differentials are encoded by polynomials g, signifying the differential  $g(z,w)/(df/dw)dz$ .

INPUT:

- `differentials` – a list of polynomials.

OUTPUT:

A matrix, one row per differential, containing the values of the path integrals along the homology basis of the Riemann surface.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<x,y> = QQ[]
sage: S = RiemannSurface(x^3 + y^3 + 1)
sage: B = S.cohomology_basis()
sage: m = S.matrix_of_integral_values(B)
sage: parent(m)
Full MatrixSpace of 1 by 2 dense matrices over Complex Field with 53 bits of
precision
sage: (m[0,0]/m[0,1]).algdep(3).degree() #curve is CM, so the period is
quadratic
2
```

**`monodromy_group()`**

Compute local monodromy generators of the Riemann surface.

For each branch point, the local monodromy is encoded by a permutation. The permutations returned correspond to positively oriented loops around each branch point, with a fixed base point. This means the generators are properly conjugated to ensure that together they generate the global monodromy. The list has an entry for every finite point stored in `self.branch_locus`, plus an entry for the ramification above infinity.

OUTPUT:

A list of permutations, encoding the local monodromy at each branch point.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z, w> = QQ[]
sage: f = z^3*w + w^3 + z
sage: S = RiemannSurface(f)
sage: G = S.monodromy_group(); G
[(0,1,2), (0,1), (0,2), (1,2), (1,2), (1,2), (0,1), (0,2), (0,2)]
```

The permutations give the local monodromy generators for the branch points:

```
sage: list(zip(S.branch_locus + [unsigned_infinity], G)) #abs tol 0.0000001
[(0.000000000000000, (0,1,2)),
 (-1.31362670141929, (0,1)),
 (-0.819032851784253 - 1.02703471138023*I, (0,2)),
 (-0.819032851784253 + 1.02703471138023*I, (1,2)),
 (0.292309440469772 - 1.28069133740100*I, (1,2)),
 (0.292309440469772 + 1.28069133740100*I, (1,2)),
 (1.18353676202412 - 0.569961265016465*I, (0,1)),
 (1.18353676202412 + 0.569961265016465*I, (0,2)),
 (Infinity, (0,2))]
```

We can check the ramification by looking at the cycle lengths and verify it agrees with the Riemann-Hurwitz formula:



```
sage: 2*S.genus-2 == -2*S.degree + sum(e-1 for g in G for e in g.cycle_type())
True
```

**period\_matrix()**

Compute the period matrix of the surface.

OUTPUT:

A matrix of complex values.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = z^3*w + w^3 + z
sage: S = RiemannSurface(f, prec=30)
sage: M = S.period_matrix()
```

The results are highly arbitrary, so it is hard to check if the result produced is correct. The closely related `riemann_matrix` is somewhat easier to test.:

```
sage: parent(M)
Full MatrixSpace of 3 by 6 dense matrices over Complex Field with 30 bits of_
↪precision
sage: M.rank()
3
```

**plot\_paths()**

Make a graphical representation of the integration paths.

Returns a two dimensional plot containing the branch points (in red) and the integration paths (obtained from the Voronoi cells of the branch points). The integration paths are plotted by plotting the points that have been computed for homotopy continuation, so the density gives an indication of where numerically sensitive features occur.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<x,y> = QQ[]
sage: S = RiemannSurface(y^2 - x^3 - x)
sage: S.plot_paths()
Graphics object consisting of 2 graphics primitives
```

**plot\_paths3d(thickness=0.01)**

Return the homology basis as a graph in 3-space.

The homology basis of the surface is constructed by taking the Voronoi cells around the branch points and taking the inverse image of the edges on the Riemann surface. If the surface is given by the equation  $f(z, w)$ , the returned object gives the image of this graph in 3-space with coordinates  $(\operatorname{Re}(z), \operatorname{Im}(z), \operatorname{Im}(w))$ .

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<x,y> = QQ[]
sage: S = RiemannSurface(y^2-x^3-x)
sage: S.plot_paths3d()
Graphics3d Object
```

**riemann\_matrix()**

Compute the Riemann matrix.

OUTPUT:

A matrix of complex values.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = z^3*w + w^3 + z
sage: S = RiemannSurface(f, prec=60)
sage: M = S.riemann_matrix()
```

The Klein quartic has a Riemann matrix with values in a quadratic field:

```
sage: x = polygen(QQ)
sage: K.<a> = NumberField(x^2-x+2)
sage: all(len(m.algdep(6).roots(K)) > 0 for m in M.list())
True
```

**rosati\_involution(R)**

Computes the Rosati involution of an endomorphism.

The endomorphism in question should be given by its homology representation with respect to the symplectic basis of the Jacobian.

INPUT:

- $R$  – integral matrix.

OUTPUT:

The result of applying the Rosati involution to  $R$ .

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: A.<x,y> = QQ[]
sage: S = RiemannSurface(y^2 - (x^6 + 2*x^4 + 4*x^2 + 8), prec = 100)
sage: Rs = S.endomorphism_basis()
sage: S.rosati_involution(S.rosati_involution(Rs[1])) == Rs[1]
True
```

**simple\_vector\_line\_integral(upstairs\_edge, differentials)**

Perform vectorized integration along a straight path.

INPUT:

- *upstairs\_edge* – a pair of integer tuples corresponding to an edge of the upstairs graph.
- *differentials* – a list of polynomials; a polynomial  $g$  represents the differential  $g(z, w)/(df/dw)dz$  where  $f(z, w) = 0$  is the equation defining the Riemann surface.

OUTPUT:

A complex number, the value of the line integral.

EXAMPLES:

```

sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = w^2 - z^4 + 1
sage: S = RiemannSurface(f); S
Riemann surface defined by polynomial f = -z^4 + w^2 + 1 = 0, with 53 bits of
↪precision

```

Since we make use of data from homotopy continuation, we need to compute the necessary data:

```

sage: M = S.riemann_matrix()
sage: differentials = S.cohomology_basis()
sage: S.simple_vector_line_integral([(0,0), (1,0)], differentials) #abs tol 0.
↪00000001
(1.14590610929717e-16 - 0.352971844594760*I)

```

**Note:** Uses data that “homology\_basis” initializes.

**symplectic\_automorphism\_group** (*endo\_basis=None, b=None, r=None*)

Numerically compute the symplectic automorphism group as a permutation group.

INPUT:

- *endo\_basis* (default: None) – a **Z**-basis of the endomorphisms of *self*, as obtained from *endomorphism\_basis()*. If you have already calculated this basis, it saves time to pass it via this keyword argument. Otherwise the method will calculate it.
- *b* – integer (default provided): as for *homomorphism\_basis()*, and used in its invocation if (re)calculating said basis.
- *r* – integer (default:  $b/4$ ). as for *homomorphism\_basis()*, and used in its invocation if (re)calculating said basis.

OUTPUT:

The symplectic automorphism group of the Jacobian of the Riemann surface. The automorphism group of the Riemann surface itself can be recovered from this; if the curve is hyperelliptic, then it is identical, and if not, then one divides out by the central element corresponding to multiplication by -1.

EXAMPLES:

```

sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: A.<x,y> = QQ[]
sage: S = RiemannSurface(y^2 - (x^6 + 2*x^4 + 4*x^2 + 8), prec = 100)
sage: G = S.symplectic_automorphism_group()
sage: G.as_permutation_group().is_isomorphic(DihedralGroup(4))
True

```

**symplectic\_isomorphisms** (*other=None, hom\_basis=None, b=None, r=None*)

Numerically compute symplectic isomorphisms.

INPUT:

- *other* (default: *self*) – the codomain, another Riemann surface.
- *hom\_basis* (default: None) – a **Z**-basis of the homomorphisms from *self* to *other*, as obtained from *homomorphism\_basis()*. If you have already calculated this basis, it saves time to pass it via this keyword argument. Otherwise the method will calculate it.

- `b` – integer (default provided): as for `homomorphism_basis()`, and used in its invocation if (re)calculating said basis.
- `r` – integer (default: `b/4`). as for `homomorphism_basis()`, and used in its invocation if (re)calculating said basis.

OUTPUT:

Returns the combinations of the elements of `homomorphism_basis()` that correspond to symplectic isomorphisms between the Jacobians of `self` and `other`.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<x,y> = QQ[]
sage: f = y^2 - (x^6 + 2*x^4 + 4*x^2 + 8)
sage: X = RiemannSurface(f, prec = 100)
sage: P = X.period_matrix()
sage: g = y^2 - (x^6 + x^4 + x^2 + 1)
sage: Y = RiemannSurface(g, prec = 100)
sage: Q = Y.period_matrix()
sage: Rs = X.symplectic_isomorphisms(Y)
sage: Ts = X.tangent_representation_numerical(Rs, other = Y)
sage: test1 = all(((T*P - Q*R).norm() < 2^(-80)) for [T, R] in zip(Ts, Rs))
sage: test2 = all(det(R) == 1 for R in Rs)
sage: test1 and test2
True
```

**tangent\_representation\_algebraic** (*Rs*, *other=None*, *epscomp=None*)

Compute the algebraic tangent representations corresponding to the homology representations in *Rs*.

The representations on homology *Rs* have to be given with respect to the symplectic homology basis of the Jacobian of `self` and `other`. Such matrices can for example be obtained via `endomorphism_basis()`.

Let *P* and *Q* be the period matrices of `self` and `other`. Then for a homology representation *R*, the corresponding tangential representation *T* satisfies  $TP = QR$ .

INPUT:

- *Rs* – a set of matrices on homology to be converted to their tangent representations.
- *other* (default: `self`) – the codomain, another Riemann surface.
- *epscomp* – real number (default:  $2^{(-\text{prec} + 30)}$ ). Used to determine whether a complex number is close enough to a root of a polynomial.

OUTPUT:

The algebraic tangent representations of the matrices in *Rs*.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: A.<x,y> = QQ[]
sage: S = RiemannSurface(y^2 - (x^6 + 2*x^4 + 4*x^2 + 8), prec = 100)
sage: Rs = S.endomorphism_basis()
sage: Ts = S.tangent_representation_algebraic(Rs)
sage: Ts[0].base_ring().maximal_order().discriminant() == 8
True
```

**tangent\_representation\_numerical** (*Rs*, *other=None*)

Compute the numerical tangent representations corresponding to the homology representations in *Rs*.

The representations on homology  $R_s$  have to be given with respect to the symplectic homology basis of the Jacobian of `self` and `other`. Such matrices can for example be obtained via `endomorphism_basis()`.

Let  $P$  and  $Q$  be the period matrices of `self` and `other`. Then for a homology representation  $R$ , the corresponding tangential representation  $T$  satisfies  $TP = QR$ .

INPUT:

- $R_s$  – a set of matrices on homology to be converted to their tangent representations.
- `other` (default: `self`) – the codomain, another Riemann surface.

OUTPUT:

The numerical tangent representations of the matrices in  $R_s$ .

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: A.<x,y> = QQ[]
sage: S = RiemannSurface(y^2 - (x^6 + 2*x^4 + 4*x^2 + 8), prec = 100)
sage: P = S.period_matrix()
sage: Rs = S.endomorphism_basis()
sage: Ts = S.tangent_representation_numerical(Rs)
sage: all((T*P - P*R).norm() < 2^(-80)) for [T, R] in zip(Ts, Rs)
True
```

**upstairs\_edges()**

Compute the edgeset of the lift of the downstairs graph onto the Riemann surface.

OUTPUT:

An edgeset between vertices  $(i, j)$ , where  $i$  corresponds to the  $i$ -th point in the Voronoi diagram vertices, and  $j$  is the  $j$ -th  $w$ -value associated with that point.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = w^2 + z^3 - z^2
sage: S = RiemannSurface(f)
sage: edgeset = S.upstairs_edges()
sage: len(edgeset) == S.degree*len(S.downstairs_edges())
True
sage: {(v[0],w[0]) for v,w in edgeset} == set(S.downstairs_edges())
True
```

**w\_values(z0)**

Returns the points lying on the surface above  $z_0$ .

INPUT:

- $z_0$  – (complex) a point in the complex  $z$ -plane.

OUTPUT:

A set of complex numbers corresponding to solutions of  $f(z_0, w) = 0$ .

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import RiemannSurface
sage: R.<z,w> = QQ[]
sage: f = w^2 - z^4 + 1
sage: S = RiemannSurface(f)
```

Find the  $w$ -values above the origin, i.e. the solutions of  $w^2 + 1 = 0$ :

```
sage: S.w_values(0) # abs tol 1e-14
[-1.000000000000000*I, 1.000000000000000*I]
```

**class** `sage.schemes.riemann_surfaces.riemann_surface.RiemannSurfaceSum(L)`

Bases: `sage.schemes.riemann_surfaces.riemann_surface.RiemannSurface`

Represent the disjoint union of finitely many Riemann surfaces.

Rudimentary class to represent disjoint unions of Riemann surfaces. Exists mainly (and this is the only functionality actually implemented) to represent direct products of the complex tori that arise as analytic Jacobians of Riemann surfaces.

INPUT:

- $L$  – list of `RiemannSurface` objects

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: SC = HyperellipticCurve(x^6-2*x^4+3*x^2-7).riemann_surface(prec=60)
sage: S1 = HyperellipticCurve(x^3-2*x^2+3*x-7).riemann_surface(prec=60)
sage: S2 = HyperellipticCurve(1-2*x+3*x^2-7*x^3).riemann_surface(prec=60)
sage: len(SC.homomorphism_basis(S1+S2))
2
```

**period\_matrix()**

Return the period matrix of the surface.

This is just the diagonal block matrix constructed from the period matrices of the constituents.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import _
      ↪ RiemannSurface, RiemannSurfaceSum
sage: R.<x,y>=QQ[]
sage: S1 = RiemannSurface(y^2-x^3-x-1)
sage: S2 = RiemannSurface(y^2-x^3-x-5)
sage: S = RiemannSurfaceSum([S1,S2])
sage: S1S2 = S1.period_matrix().block_sum(S2.period_matrix())
sage: S.period_matrix() == S1S2[[0,1],[0,2,1,3]]
True
```

**riemann\_matrix()**

Return the normalized period matrix of the surface.

This is just the diagonal block matrix constructed from the Riemann matrices of the constituents.

EXAMPLES:

```
sage: from sage.schemes.riemann_surfaces.riemann_surface import _
      ↪ RiemannSurface, RiemannSurfaceSum
sage: R.<x,y>=QQ[]
sage: S1 = RiemannSurface(y^2-x^3-x-1)
```

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```

sage: S2 = RiemannSurface(y^2-x^3-x-5)
sage: S = RiemannSurfaceSum([S1,S2])
sage: S.riemann_matrix() == S1.riemann_matrix().block_sum(S2.riemann_matrix())
True

```

`sage.schemes.riemann_surfaces.riemann_surface.bisect(L, t)`

Find position in a sorted list using bisection.

Given a list  $L = [(t_0, \dots), (t_1, \dots), \dots, (t_n, \dots)]$  with increasing  $t_i$ , find the index  $i$  such that  $t_i \leq t < t_{i+1}$  using bisection. The rest of the tuple is available for whatever use required.

INPUT:

- $L$  – A list of tuples such that the first term of each tuple is a real number between 0 and 1. These real numbers must be increasing.
- $t$  – A real number between  $t_0$  and  $t_n$ .

OUTPUT:

An integer  $i$ , giving the position in  $L$  where  $t$  would be in

EXAMPLES:

Form a list of the desired form, and pick a real number between 0 and 1:

```

sage: from sage.schemes.riemann_surfaces.riemann_surface import bisect
sage: L = [(0.0, 'a'), (0.3, 'b'), (0.7, 'c'), (0.8, 'd'), (0.9, 'e'), (1.0, 'f')]
sage: t = 0.5
sage: bisect(L,t)
1

```

Another example which demonstrates that if  $t$  is equal to one of the  $t_i$ , it returns that index:

```

sage: L = [(0.0, 'a'), (0.1, 'b'), (0.45, 'c'), (0.5, 'd'), (0.65, 'e'), (1.0, 'f
↪')]
sage: t = 0.5
sage: bisect(L,t)
3

```

`sage.schemes.riemann_surfaces.riemann_surface.differential_basis_baker(f)`

Compute a differential bases for a curve that is nonsingular outside  $(1:0:0), (0:1:0), (0:0:1)$

Baker's theorem tells us that if a curve has its singularities at the coordinate vertices and meets some further easily tested genericity criteria, then we can read off a basis for the regular differentials from the interior of the Newton polygon spanned by the monomials. While this theorem only applies to special plane curves it is worth implementing because the analysis is relatively cheap and it applies to a lot of commonly encountered curves (e.g., curves given by a hyperelliptic model). Other advantages include that we can do the computation over any exact base ring (the alternative Singular based method for computing the adjoint ideal requires the rationals), and that we can avoid being affected by subtle bugs in the Singular code.

`None` is returned when  $f$  does not describe a curve of the relevant type. If  $f$  is of the relevant type, but is of genus 0 then `[]` is returned (which are both `False` values, but they are not equal).

INPUT:

- $f$  – a bivariate polynomial

EXAMPLES:

```

sage: from sage.schemes.riemann_surfaces.riemann_surface import differential_
      ↪basis_baker
sage: R.<x,y>=QQ[]
sage: f=x^3+y^3+x^5*y^5
sage: differential_basis_baker(f)
[y^2, x*y, x*y^2, x^2, x^2*y, x^2*y^2, x^2*y^3, x^3*y^2, x^3*y^3]
sage: f=y^2-(x-3)^2*x
sage: differential_basis_baker(f) is None
True
sage: differential_basis_baker(x^2+y^2-1)
[]

```

`sage.schemes.riemann_surfaces.riemann_surface.integer_matrix_relations` (*M1*,  
*M2*,  
*b=None*,  
*r=None*)

Determine integer relations between complex matrices.

Given two square matrices with complex entries of size  $g$ ,  $h$  respectively, numerically determine an (approximate) ZZ-basis for the  $2g \times 2h$  matrices with integer entries of the shape  $(D, B; C, A)$  such that  $B+M1*A=(D+M1*C)*M2$ . By considering real and imaginary parts separately we obtain  $2gh$  equations with real coefficients in  $4gh$  variables. We scale the coefficients by a constant  $2^b$  and round them to integers, in order to obtain an integer system of equations. Standard application of LLL allows us to determine near solutions.

The user can specify the parameter  $b$ , but by default the system will choose a  $b$  based on the size of the coefficients and the precision with which they are given.

INPUT:

- $M1$  – square complex valued matrix
- $M2$  – square complex valued matrix of same size as  $M1$
- $b$  – integer (default provided). The equation coefficients are scaled by  $2^b$  before rounding to integers.
- $r$  – integer (default:  $b/4$ ). The vectors found by LLL that satisfy the scaled equations to within  $2^r$  are reported as solutions.

OUTPUT:

A list of  $2g \times 2h$  integer matrices that, for large enough  $r$ ,  $b - r$ , generate the ZZ-module of relevant transformations.

EXAMPLES:

```

sage: from sage.schemes.riemann_surfaces.riemann_surface import integer_matrix_
      ↪relations
sage: M1=M2=matrix(CC,2,2,[sqrt(d) for d in [2,-3,-3,-6]])
sage: T=integer_matrix_relations(M1,M2)
sage: id=parent(M1)(1)
sage: Mlt=[id.augment(M1) * t for t in T]
sage: [(m[:, :2]^(-1)*m)[:, 2:]-M2).norm() < 1e-13 for m in Mlt]
[True, True]

```

`sage.schemes.riemann_surfaces.riemann_surface.numerical_inverse` (*C*)

Compute numerical inverse of a matrix via LU decomposition

INPUT:

- $C$  – A real or complex invertible square matrix

EXAMPLES:



```

sage: C=matrix(CC,3,3,[-4.5606e-31 + 1.2326e-31*I,
.....: -0.21313 + 0.24166*I,
.....: -3.4513e-31 + 0.16111*I,
.....: -1.0175 + 9.8608e-32*I,
.....: 0.30912 + 0.19962*I,
.....: -4.9304e-32 + 0.39923*I,
.....: 0.96793 - 3.4513e-31*I,
.....: -0.091587 + 0.19276*I,
.....: 3.9443e-31 + 0.38552*I])
sage: from sage.schemes.riemann_surfaces.riemann_surface import numerical_inverse
sage: max(abs(c) for c in (C^(-1)*C-C^0).list()) < 1e-10
False
sage: max(abs(c) for c in (numerical_inverse(C)*C-C^0).list()) < 1e-10
True

```

```

sage.schemes.riemann_surfaces.riemann_surface.voronoi_ghost (cpoints,          n=6,
                                                                CC=Complex Double Field)

```

Convert a set of complex points to a list of real tuples  $(x, y)$ , and appends  $n$  points in a big circle around them.

The effect is that, with  $n \geq 3$ , a Voronoi decomposition will have only finite cells around the original points. Furthermore, because the extra points are placed on a circle centered on the average of the given points, with a radius  $3/2$  times the largest distance between the center and the given points, these finite cells form a simply connected region.

INPUT:

- `cpoints` – a list of complex numbers

OUTPUT:

A list of real tuples  $(x, y)$  consisting of the original points and a set of points which surround them.

EXAMPLES:

```

sage: from sage.schemes.riemann_surfaces.riemann_surface import voronoi_ghost
sage: L = [1 + 1*I, 1 - 1*I, -1 + 1*I, -1 - 1*I]
sage: voronoi_ghost(L) # abs tol 1e-6
[(1.0, 1.0),
 (1.0, -1.0),
 (-1.0, 1.0),
 (-1.0, -1.0),
 (2.121320343559643, 0.0),
 (1.0606601717798216, 1.8371173070873836),
 (-1.060660171779821, 1.8371173070873839),
 (-2.121320343559643, 2.59786816870648e-16),
 (-1.0606601717798223, -1.8371173070873832),
 (1.06066017177982, -1.8371173070873845)]

```



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