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# **Sage 9.5 Reference Manual: Power Series Rings and Laurent Series Rings**

***Release 9.5***

**The Sage Development Team**

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## POWER SERIES RINGS

Power series rings are constructed in the standard Sage fashion. See also *Multivariate Power Series Rings*.

EXAMPLES:

Construct rings and elements:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: R.random_element(6) # random
-4 - 1/2*t^2 - 1/95*t^3 + 1/2*t^4 - 12*t^5 + 0(t^6)
```

```
sage: R.<t,u,v> = PowerSeriesRing(QQ); R
Multivariate Power Series Ring in t, u, v over Rational Field
sage: p = -t + 1/2*t^3*u - 1/4*t^4*u + 2/3*v^5 + R.0(6); p
-t + 1/2*t^3*u - 1/4*t^4*u + 2/3*v^5 + 0(t, u, v)^6
sage: p in R
True
```

The default precision is specified at construction, but does not bound the precision of created elements.

```
sage: R.<t> = PowerSeriesRing(QQ, default_prec=5)
sage: R.random_element(6) # random
1/2 - 1/4*t + 2/3*t^2 - 5/2*t^3 + 2/3*t^5 + 0(t^6)
```

Construct univariate power series from a list of coefficients:

```
sage: S = R([1, 3, 5, 7]); S
1 + 3*t + 5*t^2 + 7*t^3
```

The default precision of a power series ring stays fixed and cannot be changed. To work with different default precision, create a new power series ring:

```
sage: R.<x> = PowerSeriesRing(QQ, default_prec=10)
sage: sin(x)
x - 1/6*x^3 + 1/120*x^5 - 1/5040*x^7 + 1/362880*x^9 + 0(x^10)
sage: R.<x> = PowerSeriesRing(QQ, default_prec=15)
sage: sin(x)
x - 1/6*x^3 + 1/120*x^5 - 1/5040*x^7 + 1/362880*x^9 - 1/39916800*x^11 + 1/6227020800*x^13 + 0(x^15)
```

An iterated example:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: S.<t2> = PowerSeriesRing(R)
sage: S
Power Series Ring in t2 over Power Series Ring in t over Integer Ring
sage: S.base_ring()
Power Series Ring in t over Integer Ring
```

Sage can compute with power series over the symbolic ring.

```
sage: K.<t> = PowerSeriesRing(SR, default_prec=5)
sage: a, b, c = var('a,b,c')
sage: f = a + b*t + c*t^2 + O(t^3)
sage: f*f
a^2 + 2*a*b*t + (b^2 + 2*a*c)*t^2 + O(t^3)
sage: f = sqrt(2) + sqrt(3)*t + O(t^3)
sage: f^2
2 + 2*sqrt(3)*sqrt(2)*t + 3*t^2 + O(t^3)
```

Elements are first coerced to constants in `base_ring`, then coerced into the `PowerSeriesRing`:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: f = Mod(2, 3) * t; (f, f.parent())
(2*t, Power Series Ring in t over Ring of integers modulo 3)
```

We make a sparse power series.

```
sage: R.<x> = PowerSeriesRing(QQ, sparse=True); R
Sparse Power Series Ring in x over Rational Field
sage: f = 1 + x^1000000
sage: g = f*f
sage: g.degree()
2000000
```

We make a sparse Laurent series from a power series generator:

```
sage: R.<t> = PowerSeriesRing(QQ, sparse=True)
sage: latex(-2/3*(1/t^3) + 1/t + 3/5*t^2 + O(t^5))
\frac{-\frac{2}{3}}{t^3} + \frac{1}{t} + \frac{3}{5}t^2 + O(t^5)
sage: S = parent(1/t); S
Sparse Laurent Series Ring in t over Rational Field
```

Choose another implementation of the attached polynomial ring:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: type(t.polynomial())
<... 'sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_flint'>
sage: S.<s> = PowerSeriesRing(ZZ, implementation='NTL')
sage: type(s.polynomial())
<... 'sage.rings.polynomial.polynomial_integer_dense_ntl.Polynomial_integer_dense_ntl'>
```

AUTHORS:

- William Stein: the code
- Jeremy Cho (2006-05-17): some examples (above)

- Niles Johnson (2010-09): implement multivariate power series
- Simon King (2012-08): use category and coercion framework, [trac ticket #13412](#)

```
sage.rings.power_series_ring.PowerSeriesRing(base_ring, name=None, arg2=None, names=None,
                                              sparse=False, default_prec=None, order='negdeglex',
                                              num_gens=None, implementation=None)
```

Create a univariate or multivariate power series ring over a given (commutative) base ring.

INPUT:

- `base_ring` - a commutative ring
- `name, names` - name(s) of the indeterminate
- **`default_prec` - the default precision used if an exact object must** be changed to an approximate object in order to do an arithmetic operation. If left as `None`, it will be set to the global default (20) in the univariate case, and 12 in the multivariate case.
- `sparse` - (default: `False`) whether power series are represented as sparse objects.
- `order` - (default: `negdeglex`) term ordering, for multivariate case
- `num_gens` - number of generators, for multivariate case

There is a unique power series ring over each base ring with given variable name. Two power series over the same base ring with different variable names are not equal or isomorphic.

EXAMPLES (Univariate):

```
sage: R = PowerSeriesRing(QQ, 'x'); R
Power Series Ring in x over Rational Field
```

```
sage: S = PowerSeriesRing(QQ, 'y'); S
Power Series Ring in y over Rational Field
```

```
sage: R = PowerSeriesRing(QQ, 10)
Traceback (most recent call last):
...
ValueError: variable name '10' does not start with a letter
```

```
sage: S = PowerSeriesRing(QQ, 'x', default_prec = 15); S
Power Series Ring in x over Rational Field
sage: S.default_prec()
15
```

EXAMPLES (Multivariate) See also [Multivariate Power Series Rings](#):

```
sage: R = PowerSeriesRing(QQ, 't,u,v'); R
Multivariate Power Series Ring in t, u, v over Rational Field
```

```
sage: N = PowerSeriesRing(QQ, 'w', num_gens=5); N
Multivariate Power Series Ring in w0, w1, w2, w3, w4 over Rational Field
```

Number of generators can be specified before variable name without using keyword:

```
sage: M = PowerSeriesRing(QQ, 4, 'k'); M
Multivariate Power Series Ring in k0, k1, k2, k3 over Rational Field
```

Multivariate power series can be constructed using angle bracket or double square bracket notation:

```
sage: R.<t,u,v> = PowerSeriesRing(QQ, 't,u,v'); R
Multivariate Power Series Ring in t, u, v over Rational Field
```

```
sage: ZZ[['s,t,u']]
Multivariate Power Series Ring in s, t, u over Integer Ring
```

Sparse multivariate power series ring:

```
sage: M = PowerSeriesRing(QQ,4,'k',sparse=True); M
Sparse Multivariate Power Series Ring in k0, k1, k2, k3 over
Rational Field
```

Power series ring over polynomial ring:

```
sage: H = PowerSeriesRing(PolynomialRing(ZZ,3,'z'),4,'f'); H
Multivariate Power Series Ring in f0, f1, f2, f3 over Multivariate
Polynomial Ring in z0, z1, z2 over Integer Ring
```

Power series ring over finite field:

```
sage: S = PowerSeriesRing(GF(65537),'x,y'); S
Multivariate Power Series Ring in x, y over Finite Field of size
65537
```

Power series ring with many variables:

```
sage: R = PowerSeriesRing(ZZ, ['x%s'%p for p in primes(100)]); R
Multivariate Power Series Ring in x2, x3, x5, x7, x11, x13, x17, x19,
x23, x29, x31, x37, x41, x43, x47, x53, x59, x61, x67, x71, x73, x79,
x83, x89, x97 over Integer Ring
```

- Use `inject_variables()` to make the variables available for interactive use.

```
sage: R.inject_variables()
Defining x2, x3, x5, x7, x11, x13, x17, x19, x23, x29, x31, x37,
x41, x43, x47, x53, x59, x61, x67, x71, x73, x79, x83, x89, x97

sage: f = x47 + 3*x11*x29 - x19 + R.0(3)
sage: f in R
True
```

Variable ordering determines how series are displayed:

```
sage: T.<a,b> = PowerSeriesRing(ZZ,order='deglex'); T
Multivariate Power Series Ring in a, b over Integer Ring
sage: T.term_order()
Degree lexicographic term order
sage: p = - 2*b^6 + a^5*b^2 + a^7 - b^2 - a*b^3 + T.0(9); p
a^7 + a^5*b^2 - 2*b^6 - a*b^3 - b^2 + 0(a, b)^9

sage: U = PowerSeriesRing(ZZ, 'a,b',order='negdeglex'); U
Multivariate Power Series Ring in a, b over Integer Ring
```

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```
sage: U.term_order()
Negative degree lexicographic term order
sage: U(p)
-b^2 - a*b^3 - 2*b^6 + a^7 + a^5*b^2 + 0(a, b)^9
```

See also:

- `sage.misc.defaults.set_series_precision()`

```
class sage.rings.power_series_ring.PowerSeriesRing_domain(base_ring, name=None,
                                                           default_prec=None, sparse=False,
                                                           implementation=None, category=None)

Bases:      sage.rings.power_series_ring.PowerSeriesRing_generic,  sage.rings.ring.
IntegralDomain
```

**fraction\_field()**

Return the Laurent series ring over the fraction field of the base ring.

This is actually *not* the fraction field of this ring, but its completion with respect to the topology defined by the valuation. When we are working at finite precision, these two fields are indistinguishable; that is the reason why we allow ourselves to make this confusion here.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: R.fraction_field()
Laurent Series Ring in t over Rational Field
sage: Frac(R)
Laurent Series Ring in t over Rational Field
```

```
class sage.rings.power_series_ring.PowerSeriesRing_generic(base_ring, name=None,
                                                           default_prec=None, sparse=False,
                                                           implementation=None,
                                                           category=None)

Bases:      sage.structure.unique_representation.UniqueRepresentation,  sage.rings.ring.
CommutativeRing, sage.structure.nonexact.Nonexact
```

A power series ring.

**base\_extend(R)**

Return the power series ring over R in the same variable as self, assuming there is a canonical coerce map from the base ring of self to R.

EXAMPLES:

```
sage: R.<T> = GF(7)[[]]; R
Power Series Ring in T over Finite Field of size 7
sage: R.change_ring(ZZ)
Power Series Ring in T over Integer Ring
sage: R.base_extend(ZZ)
Traceback (most recent call last):
...
TypeError: no base extension defined
```

**change\_ring(R)**

Return the power series ring over R in the same variable as self.

EXAMPLES:

```
sage: R.<T> = QQ[[]]; R
Power Series Ring in T over Rational Field
sage: R.change_ring(GF(7))
Power Series Ring in T over Finite Field of size 7
sage: R.base_extend(GF(7))
Traceback (most recent call last):
...
TypeError: no base extension defined
sage: R.base_extend(QuadraticField(3, 'a'))
Power Series Ring in T over Number Field in a with defining polynomial x^2 - 3
↳with a = 1.732050807568878?
```

**change\_var**(var)

Return the power series ring in variable var over the same base ring.

EXAMPLES:

```
sage: R.<T> = QQ[[]]; R
Power Series Ring in T over Rational Field
sage: R.change_var('D')
Power Series Ring in D over Rational Field
```

**characteristic**()

Return the characteristic of this power series ring, which is the same as the characteristic of the base ring of the power series ring.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: R.characteristic()
0
sage: R.<w> = Integers(2^50)[[]]; R
Power Series Ring in w over Ring of integers modulo 1125899906842624
sage: R.characteristic()
1125899906842624
```

**construction**()

Return the functorial construction of self, namely, completion of the univariate polynomial ring with respect to the indeterminate (to a given precision).

EXAMPLES:

```
sage: R = PowerSeriesRing(ZZ, 'x')
sage: c, S = R.construction(); S
Univariate Polynomial Ring in x over Integer Ring
sage: R == c(S)
True
sage: R = PowerSeriesRing(ZZ, 'x', sparse=True)
sage: c, S = R.construction()
sage: R == c(S)
True
```

**gen**(n=0)

Return the generator of this power series ring.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: R.gen()
t
sage: R.gen(3)
Traceback (most recent call last):
...
IndexError: generator n>0 not defined
```

**is\_dense()**

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: t.is_dense()
True
sage: R.<t> = PowerSeriesRing(ZZ, sparse=True)
sage: t.is_dense()
False
```

**is\_exact()**

Return False since the ring of power series over any ring is not exact.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: R.is_exact()
False
```

**is\_field(*proof=True*)**

Return False since the ring of power series over any ring is never a field.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: R.is_field()
False
```

**is\_finite()**

Return False since the ring of power series over any ring is never finite.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: R.is_finite()
False
```

**is\_sparse()**

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: t.is_sparse()
False
sage: R.<t> = PowerSeriesRing(ZZ, sparse=True)
sage: t.is_sparse()
True
```

**laurent\_series\_ring()**

If this is the power series ring  $R[[t]]$ , return the Laurent series ring  $R((t))$ .

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ,default_prec=5)
sage: S = R.laurent_series_ring(); S
Laurent Series Ring in t over Integer Ring
sage: S.default_prec()
5
sage: f = 1+t; g=1/f; g
1 - t + t^2 - t^3 + t^4 + O(t^5)
```

**ngens()**

Return the number of generators of this power series ring.

This is always 1.

EXAMPLES:

```
sage: R.<t> = ZZ[[ ]]
sage: R.ngens()
1
```

**random\_element(*prec=None, \*args, \*\*kws*)**

Return a random power series.

INPUT:

- *prec* - Integer specifying precision of output (default: default precision of self)
- *\*args, \*\*kws* - Passed on to the `random_element` method for the base ring

OUTPUT:

- Power series with precision *prec* whose coefficients are random elements from the base ring, randomized subject to the arguments *\*args* and *\*\*kws*

ALGORITHM:

Call the `random_element` method on the underlying polynomial ring.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: R.random_element(5) # random
-4 - 1/2*t^2 - 1/95*t^3 + 1/2*t^4 + O(t^5)
sage: R.random_element(10) # random
-1/2 + 2*t - 2/7*t^2 - 25*t^3 - t^4 + 2*t^5 - 4*t^7 - 1/3*t^8 - t^9 + O(t^10)
```

If given no argument, `random_element` uses default precision of self:

```
sage: T = PowerSeriesRing(ZZ, 't')
sage: T.default_prec()
20
sage: T.random_element() # random
4 + 2*t - t^2 - t^3 + 2*t^4 + t^5 + t^6 - 2*t^7 - t^8 - t^9 + t^11 - 6*t^12 +
2*t^14 + 2*t^16 - t^17 - 3*t^18 + O(t^20)
sage: S = PowerSeriesRing(ZZ, 't', default_prec=4)
```

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```
sage: S.random_element() # random
2 - t - 5*t^2 + t^3 + O(t^4)
```

Further arguments are passed to the underlying base ring ([trac ticket #9481](#)):

```
sage: SZ = PowerSeriesRing(ZZ, 'v')
sage: SQ = PowerSeriesRing(QQ, 'v')
sage: SR = PowerSeriesRing(RR, 'v')

sage: SZ.random_element(x=4, y=6) # random
4 + 5*v + 5*v^2 + 5*v^3 + 4*v^4 + 5*v^5 + 5*v^6 + 5*v^7 + 4*v^8 + 5*v^9 + 4*v^
↪10 + 4*v^11 + 5*v^12 + 5*v^13 + 5*v^14 + 5*v^15 + 5*v^16 + 5*v^17 + 4*v^18 +
↪5*v^19 + O(v^20)
sage: SZ.random_element(3, x=4, y=6) # random
5 + 4*v + 5*v^2 + O(v^3)
sage: SQ.random_element(3, num_bound=3, den_bound=100) # random
1/87 - 3/70*v - 3/44*v^2 + O(v^3)
sage: SR.random_element(3, max=10, min=-10) # random
2.85948321262904 - 9.73071330911226*v - 6.60414378519265*v^2 + O(v^3)
```

### **residue\_field()**

Return the residue field of this power series ring.

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(GF(17))
sage: R.residue_field()
Finite Field of size 17
sage: R.<x> = PowerSeriesRing(Zp(5))
sage: R.residue_field()
Finite Field of size 5
```

### **uniformizer()**

Return a uniformizer of this power series ring if it is a discrete valuation ring (i.e., if the base ring is actually a field). Otherwise, an error is raised.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: R.uniformizer()
t

sage: R.<t> = PowerSeriesRing(ZZ)
sage: R.uniformizer()
Traceback (most recent call last):
...
TypeError: The base ring is not a field
```

### **variable\_names\_recursive(depth=None)**

Return the list of variable names of this and its base rings.

EXAMPLES:

```

sage: R = QQ[['x']][['y']][['z']]
sage: R.variable_names_recursive()
('x', 'y', 'z')
sage: R.variable_names_recursive(2)
('y', 'z')

```

```

class sage.rings.power_series_ring.PowerSeriesRing_over_field(base_ring, name=None,
                                                              default_prec=None, sparse=False,
                                                              implementation=None,
                                                              category=None)

```

Bases: `sage.rings.power_series_ring.PowerSeriesRing_domain`

**fraction\_field()**

Return the fraction field of this power series ring, which is defined since this is over a field.

This fraction field is just the Laurent series ring over the base field.

EXAMPLES:

```

sage: R.<t> = PowerSeriesRing(GF(7))
sage: R.fraction_field()
Laurent Series Ring in t over Finite Field of size 7
sage: Frac(R)
Laurent Series Ring in t over Finite Field of size 7

```

`sage.rings.power_series_ring.is_PowerSeriesRing(R)`

Return True if this is a *univariate* power series ring. This is in keeping with the behavior of `is_PolynomialRing` versus `is_MPolynomialRing`.

EXAMPLES:

```

sage: from sage.rings.power_series_ring import is_PowerSeriesRing
sage: is_PowerSeriesRing(10)
False
sage: is_PowerSeriesRing(QQ[['x']])
True

```

`sage.rings.power_series_ring.unpickle_power_series_ring_v0(base_ring, name, default_prec, sparse)`

Unpickle (deserialize) a univariate power series ring according to the given inputs.

EXAMPLES:

```

sage: P.<x> = PowerSeriesRing(QQ)
sage: loads(dumps(P)) == P # indirect doctest
True

```

## POWER SERIES

Sage provides an implementation of dense and sparse power series over any Sage base ring. This is the base class of the implementations of univariate and multivariate power series ring elements in Sage (see also *Power Series Methods*, *Multivariate Power Series*).

AUTHORS:

- William Stein
- David Harvey (2006-09-11): added `solve_linear_de()` method
- Robert Bradshaw (2007-04): `sqrt`, `rmul`, `lmul`, shifting
- Robert Bradshaw (2007-04): Cython version
- Simon King (2012-08): use category and coercion framework, [trac ticket #13412](#)

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(ZZ)
sage: TestSuite(R).run()
sage: R([1,2,3])
1 + 2*x + 3*x^2
sage: R([1,2,3], 10)
1 + 2*x + 3*x^2 + 0(x^10)
sage: f = 1 + 2*x - 3*x^3 + 0(x^4); f
1 + 2*x - 3*x^3 + 0(x^4)
sage: f^10
1 + 20*x + 180*x^2 + 930*x^3 + 0(x^4)
sage: g = 1/f; g
1 - 2*x + 4*x^2 - 5*x^3 + 0(x^4)
sage: g * f
1 + 0(x^4)
```

In Python (as opposed to Sage) create the power series ring and its generator as follows:

```
sage: R = PowerSeriesRing(ZZ, 'x')
sage: x = R.gen()
sage: parent(x)
Power Series Ring in x over Integer Ring
```

EXAMPLES:

This example illustrates that coercion for power series rings is consistent with coercion for polynomial rings.

```
sage: poly_ring1.<gen1> = PolynomialRing(QQ)
sage: poly_ring2.<gen2> = PolynomialRing(QQ)
sage: huge_ring.<x> = PolynomialRing(poly_ring1)
```

The generator of the first ring gets coerced in as itself, since it is the base ring.

```
sage: huge_ring(gen1)
gen1
```

The generator of the second ring gets mapped via the natural map sending one generator to the other.

```
sage: huge_ring(gen2)
x
```

With power series the behavior is the same.

```
sage: power_ring1.<gen1> = PowerSeriesRing(QQ)
sage: power_ring2.<gen2> = PowerSeriesRing(QQ)
sage: huge_power_ring.<x> = PowerSeriesRing(power_ring1)
sage: huge_power_ring(gen1)
gen1
sage: huge_power_ring(gen2)
x
```

**class** sage.rings.power\_series\_ring\_element.PowerSeries

Bases: sage.structure.element.AlgebraElement

A power series. Base class of univariate and multivariate power series. The following methods are available with both types of objects.

**O**(prec)

Return this series plus  $O(x^{\text{prec}})$ . Does not change self.

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(ZZ)
sage: p = 1 + x^2 + x^10; p
1 + x^2 + x^10
sage: p.O(15)
1 + x^2 + x^10 + O(x^15)
sage: p.O(5)
1 + x^2 + O(x^5)
sage: p.O(-5)
Traceback (most recent call last):
...
ValueError: prec (= -5) must be non-negative
```

**V**(n)

If  $f = \sum a_m x^m$ , then this function returns  $\sum a_m x^{nm}$ .

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(ZZ)
sage: p = 1 + x^2 + x^10; p
1 + x^2 + x^10
sage: p.V(3)
```

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```
1 + x^6 + x^30
sage: (p+O(x^20)).V(3)
1 + x^6 + x^30 + O(x^60)
```

### **add\_bigoh(*prec*)**

Return the power series of precision at most *prec* got by adding  $O(q^{\text{prec}})$  to *f*, where *q* is the variable.

EXAMPLES:

```
sage: R.<A> = RDF[[]]
sage: f = (1+A+O(A^5))^5; f
1.0 + 5.0*A + 10.0*A^2 + 10.0*A^3 + 5.0*A^4 + O(A^5)
sage: f.add_bigoh(3)
1.0 + 5.0*A + 10.0*A^2 + O(A^3)
sage: f.add_bigoh(5)
1.0 + 5.0*A + 10.0*A^2 + 10.0*A^3 + 5.0*A^4 + O(A^5)
```

### **base\_extend(*R*)**

Return a copy of this power series but with coefficients in *R*.

The following coercion uses `base_extend` implicitly:

```
sage: R.<t> = ZZ[['t']]
sage: (t - t^2) * Mod(1, 3)
t + 2*t^2
```

### **base\_ring()**

Return the base ring that this power series is defined over.

EXAMPLES:

```
sage: R.<t> = GF(49, 'alpha')[[]]
sage: (t^2 + O(t^3)).base_ring()
Finite Field in alpha of size 7^2
```

### **change\_ring(*R*)**

Change if possible the coefficients of self to lie in *R*.

EXAMPLES:

```
sage: R.<T> = QQ[[]]; R
Power Series Ring in T over Rational Field
sage: f = 1 - 1/2*T + 1/3*T^2 + O(T^3)
sage: f.base_extend(GF(5))
Traceback (most recent call last):
...
TypeError: no base extension defined
sage: f.change_ring(GF(5))
1 + 2*T + 2*T^2 + O(T^3)
sage: f.change_ring(GF(3))
Traceback (most recent call last):
...
ZeroDivisionError: inverse of Mod(0, 3) does not exist
```

We can only change the ring if there is a `__call__` coercion defined. The following succeeds because `ZZ(K(4))` is defined.

```
sage: K.<a> = NumberField(cyclotomic_polynomial(3), 'a')
sage: R.<t> = K[['t']]
sage: (4*t).change_ring(ZZ)
4*t
```

This does not succeed because  $\mathbb{Z}\mathbb{Z}(K(a+1))$  is not defined.

```
sage: K.<a> = NumberField(cyclotomic_polynomial(3), 'a')
sage: R.<t> = K[['t']]
sage: ((a+1)*t).change_ring(ZZ)
Traceback (most recent call last):
...
TypeError: Unable to coerce a + 1 to an integer
```

### **coefficients()**

Return the nonzero coefficients of self.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: f = t + t^2 - 10/3*t^3
sage: f.coefficients()
[1, 1, -10/3]
```

### **common\_prec(f)**

Return minimum precision of  $f$  and self.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ)
```

```
sage: f = t + t^2 + 0(t^3)
sage: g = t + t^3 + t^4 + 0(t^4)
sage: f.common_prec(g)
3
sage: g.common_prec(f)
3
```

```
sage: f = t + t^2 + 0(t^3)
sage: g = t^2
sage: f.common_prec(g)
3
sage: g.common_prec(f)
3
```

```
sage: f = t + t^2
sage: f = t^2
sage: f.common_prec(g)
+Infinity
```

### **cos(prec='infinity')**

Apply cos to the formal power series.

INPUT:

- `prec` – Integer or infinity. The degree to truncate the result to.

OUTPUT:

A new power series.

EXAMPLES:

For one variable:

```
sage: t = PowerSeriesRing(QQ, 't').gen()
sage: f = (t + t**2).O(4)
sage: cos(f)
1 - 1/2*t^2 - t^3 + O(t^4)
```

For several variables:

```
sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = a + b + a*b + T.O(3)
sage: cos(f)
1 - 1/2*a^2 - a*b - 1/2*b^2 + O(a, b)^3
sage: f.cos()
1 - 1/2*a^2 - a*b - 1/2*b^2 + O(a, b)^3
sage: f.cos(prec=2)
1 + O(a, b)^2
```

If the power series has a non-zero constant coefficient  $c$ , one raises an error:

```
sage: g = 2+f
sage: cos(g)
Traceback (most recent call last):
...
ValueError: can only apply cos to formal power series with zero constant term
```

If no precision is specified, the default precision is used:

```
sage: T.default_prec()
12
sage: cos(a)
1 - 1/2*a^2 + 1/24*a^4 - 1/720*a^6 + 1/40320*a^8 - 1/3628800*a^10 + O(a, b)^12
sage: a.cos(prec=5)
1 - 1/2*a^2 + 1/24*a^4 + O(a, b)^5
sage: cos(a + T.O(5))
1 - 1/2*a^2 + 1/24*a^4 + O(a, b)^5
```

**cosh**(*prec*='infinity')

Apply cosh to the formal power series.

INPUT:

- `prec` – Integer or infinity. The degree to truncate the result to.

OUTPUT:

A new power series.

EXAMPLES:

For one variable:

```
sage: t = PowerSeriesRing(QQ, 't').gen()
sage: f = (t + t**2).O(4)
sage: cosh(f)
1 + 1/2*t^2 + t^3 + O(t^4)
```

For several variables:

```
sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = a + b + a*b + T.O(3)
sage: cosh(f)
1 + 1/2*a^2 + a*b + 1/2*b^2 + O(a, b)^3
sage: f.cosh()
1 + 1/2*a^2 + a*b + 1/2*b^2 + O(a, b)^3
sage: f.cosh(prec=2)
1 + O(a, b)^2
```

If the power series has a non-zero constant coefficient  $c$ , one raises an error:

```
sage: g = 2+f
sage: cosh(g)
Traceback (most recent call last):
...
ValueError: can only apply cosh to formal power series with zero
constant term
```

If no precision is specified, the default precision is used:

```
sage: T.default_prec()
12
sage: cosh(a)
1 + 1/2*a^2 + 1/24*a^4 + 1/720*a^6 + 1/40320*a^8 + 1/3628800*a^10 +
O(a, b)^12
sage: a.cosh(prec=5)
1 + 1/2*a^2 + 1/24*a^4 + O(a, b)^5
sage: cosh(a + T.O(5))
1 + 1/2*a^2 + 1/24*a^4 + O(a, b)^5
```

### degree()

Return the degree of this power series, which is by definition the degree of the underlying polynomial.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ, sparse=True)
sage: f = t^1000000 + O(t^10000000)
sage: f.degree()
1000000
```

### derivative(\*args)

The formal derivative of this power series, with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied; see documentation for the global derivative() function for more details.

See also:

`_derivative()`

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(QQ)
sage: g = -x + x^2/2 - x^4 + O(x^6)
sage: g.derivative()
-1 + x - 4*x^3 + O(x^5)
sage: g.derivative(x)
-1 + x - 4*x^3 + O(x^5)
sage: g.derivative(x, x)
1 - 12*x^2 + O(x^4)
sage: g.derivative(x, 2)
1 - 12*x^2 + O(x^4)
```

#### **egf\_to\_ogf()**

Return the ordinary generating function power series, assuming self is an exponential generating function power series.

This function is known as `serlaplace` in PARI/GP.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: f = t + t^2/factorial(2) + 2*t^3/factorial(3)
sage: f.egf_to_ogf()
t + t^2 + 2*t^3
```

#### **exp(prec=None)**

Return exp of this power series to the indicated precision.

INPUT:

- `prec` - integer; default is `self.parent().default_prec`

ALGORITHM: See [solve\\_linear\\_de\(\)](#).

---

#### **Note:**

- Screwy things can happen if the coefficient ring is not a field of characteristic zero. See [solve\\_linear\\_de\(\)](#).
- 

AUTHORS:

- David Harvey (2006-09-08): rewrote to use simplest possible “lazy” algorithm.
- David Harvey (2006-09-10): rewrote to use divide-and-conquer strategy.
- David Harvey (2006-09-11): factored functionality out to `solve_linear_de()`.
- Sourav Sen Gupta, David Harvey (2008-11): handle constant term

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ, default_prec=10)
```

Check that  $\exp(t)$  is, well,  $\exp(t)$ :

```
sage: (t + O(t^10)).exp()
1 + t + 1/2*t^2 + 1/6*t^3 + 1/24*t^4 + 1/120*t^5 + 1/720*t^6 + 1/5040*t^7 + 1/
40320*t^8 + 1/362880*t^9 + O(t^10)
```

Check that  $\exp(\log(1+t))$  is  $1+t$ :

```
sage: (sum([(-t)^n/n for n in range(1, 10)]) + O(t^10)).exp()
1 + t + O(t^10)
```

Check that  $\exp(2t + t^2 - t^5)$  is whatever it is:

```
sage: (2*t + t^2 - t^5 + O(t^10)).exp()
1 + 2*t + 3*t^2 + 10/3*t^3 + 19/6*t^4 + 8/5*t^5 - 7/90*t^6 - 538/315*t^7 - 425/
↳ 168*t^8 - 30629/11340*t^9 + O(t^10)
```

Check requesting lower precision:

```
sage: (t + t^2 - t^5 + O(t^10)).exp(5)
1 + t + 3/2*t^2 + 7/6*t^3 + 25/24*t^4 + O(t^5)
```

Can't get more precision than the input:

```
sage: (t + t^2 + O(t^3)).exp(10)
1 + t + 3/2*t^2 + O(t^3)
```

Check some boundary cases:

```
sage: (t + O(t^2)).exp(1)
1 + O(t)
sage: (t + O(t^2)).exp(0)
O(t^0)
```

Handle nonzero constant term (fixes [trac ticket #4477](#)):

```
sage: R.<x> = PowerSeriesRing(RR)
sage: (1 + x + x^2 + O(x^3)).exp()
2.71828182845905 + 2.71828182845905*x + 4.07742274268857*x^2 + O(x^3)
```

```
sage: R.<x> = PowerSeriesRing(ZZ)
sage: (1 + x + O(x^2)).exp()
Traceback (most recent call last):
...
ArithmeticError: exponential of constant term does not belong to coefficient_
↳ ring (consider working in a larger ring)
```

```
sage: R.<x> = PowerSeriesRing(GF(5))
sage: (1 + x + O(x^2)).exp()
Traceback (most recent call last):
...
ArithmeticError: constant term of power series does not support exponentiation
```

### **exponents()**

Return the exponents appearing in self with nonzero coefficients.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: f = t + t^2 - 10/3*t^3
```

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```
sage: f.exponents()
[1, 2, 3]
```

**inverse()**

Return the inverse of self, i.e.,  $\text{self}^{-1}$ .

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ, sparse=True)
sage: t.inverse()
t^-1
sage: type(_)
<class 'sage.rings.laurent_series_ring_element.LaurentSeries'>
sage: (1-t).inverse()
1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 + ...
```

**is\_dense()**

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: t.is_dense()
True
sage: R.<t> = PowerSeriesRing(ZZ, sparse=True)
sage: t.is_dense()
False
```

**is\_gen()**

Return True if this is the generator (the variable) of the power series ring.

EXAMPLES:

```
sage: R.<t> = QQ[[t]]
sage: t.is_gen()
True
sage: (1 + 2*t).is_gen()
False
```

Note that this only returns True on the actual generator, not on something that happens to be equal to it.

```
sage: (1*t).is_gen()
False
sage: 1*t == t
True
```

**is\_monomial()**

Return True if this element is a monomial. That is, if self is  $x^n$  for some non-negative integer  $n$ .

EXAMPLES:

```
sage: k.<z> = PowerSeriesRing(QQ, 'z')
sage: z.is_monomial()
True
sage: k(1).is_monomial()
True
```

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```

sage: (z+1).is_monomial()
False
sage: (z^2909).is_monomial()
True
sage: (3*z^2909).is_monomial()
False

```

**is\_sparse()**

EXAMPLES:

```

sage: R.<t> = PowerSeriesRing(ZZ)
sage: t.is_sparse()
False
sage: R.<t> = PowerSeriesRing(ZZ, sparse=True)
sage: t.is_sparse()
True

```

**is\_square()**

Return True if this function has a square root in this ring, e.g., there is an element  $y$  in `self.parent()` such that  $y^2$  equals `self`.

ALGORITHM: If the base ring is a field, this is true whenever the power series has even valuation and the leading coefficient is a perfect square.

For an integral domain, it attempts the square root in the fraction field and tests whether or not the result lies in the original ring.

EXAMPLES:

```

sage: K.<t> = PowerSeriesRing(QQ, 't', 5)
sage: (1+t).is_square()
True
sage: (2+t).is_square()
False
sage: (2+t.change_ring(RR)).is_square()
True
sage: t.is_square()
False
sage: K.<t> = PowerSeriesRing(ZZ, 't', 5)
sage: (1+t).is_square()
False
sage: f = (1+t)^100
sage: f.is_square()
True

```

**is\_unit()**

Return True if this power series is invertible.

A power series is invertible precisely when the constant term is invertible.

EXAMPLES:

```

sage: R.<t> = PowerSeriesRing(ZZ)
sage: (-1 + t - t^5).is_unit()
True

```

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```
sage: (3 + t - t^5).is_unit()
False
sage: 0(t^0).is_unit()
False
```

AUTHORS:

- David Harvey (2006-09-03)

**jacobi\_continued\_fraction()**Return the Jacobi continued fraction of `self`.

The J-fraction or Jacobi continued fraction of a power series is a continued fraction expansion with steps of size two. We use the following convention

$$1/(1 + A_0t + B_0t^2/(1 + A_1t + B_1t^2/(1 + \cdots)))$$

OUTPUT:

tuple of pairs  $(A_n, B_n)$  for  $n \geq 0$ 

The expansion is done as long as possible given the precision. Whenever the expansion is not well-defined, because it would require to divide by zero, an exception is raised.

See section 2.7 of [Kra1999det] for the close relationship of this kind of expansion with Hankel determinants and orthogonal polynomials.

EXAMPLES:

```
sage: t = PowerSeriesRing(QQ, 't').gen()
sage: s = sum(factorial(k) * t**k for k in range(12)).0(12)
sage: s.jacobi_continued_fraction()
((-1, -1), (-3, -4), (-5, -9), (-7, -16), (-9, -25))
```

Another example:

```
sage: (log(1+t)/t).jacobi_continued_fraction()
((1/2, -1/12),
 (1/2, -1/15),
 (1/2, -9/140),
 (1/2, -4/63),
 (1/2, -25/396),
 (1/2, -9/143),
 (1/2, -49/780),
 (1/2, -16/255),
 (1/2, -81/1292))
```

**laurent\_series()**

Return the Laurent series associated to this power series, i.e., this series considered as a Laurent series.

EXAMPLES:

```
sage: k.<w> = QQ[[]]
sage: f = 1+17*w+15*w^3+0(w^5)
sage: parent(f)
Power Series Ring in w over Rational Field
sage: g = f.laurent_series(); g
1 + 17*w + 15*w^3 + 0(w^5)
```

**lift\_to\_precision**(*absprec=None*)Return a congruent power series with absolute precision at least *absprec*.

INPUT:

- *absprec* – an integer or None (default: None), the absolute precision of the result. If None, lifts to an exact element.

EXAMPLES:

```
sage: A.<t> = PowerSeriesRing(GF(5))
sage: x = t + t^2 + O(t^5)
sage: x.lift_to_precision(10)
t + t^2 + O(t^10)
sage: x.lift_to_precision()
t + t^2
```

**list**()

See this method in derived classes:

- `sage.rings.power_series_poly.PowerSeries_poly.list()`,
- `sage.rings.multi_power_series_ring_element.MPowerSeries.list()`

Implementations *MUST* override this in the derived class.

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(ZZ)
sage: PowerSeries.list(1+x^2)
Traceback (most recent call last):
...
NotImplementedError
```

**log**(*prec=None*)

Return log of this power series to the indicated precision.

This works only if the constant term of the power series is 1 or the base ring can take the logarithm of the constant coefficient.

INPUT:

- *prec* – integer; default is `self.parent().default_prec()`

ALGORITHM: See `solve_linear_de()`.

**Warning:** Screwy things can happen if the coefficient ring is not a field of characteristic zero. See `solve_linear_de()`.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ, default_prec=10)
sage: (1 + t + O(t^10)).log()
t - 1/2*t^2 + 1/3*t^3 - 1/4*t^4 + 1/5*t^5 - 1/6*t^6 + 1/7*t^7 - 1/8*t^8 + 1/9*t^
↪ 9 + O(t^10)

sage: t.exp().log()
t + O(t^10)
```

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```

sage: (1+t).log().exp()
1 + t + O(t^10)

sage: (-1 + t + O(t^10)).log()
Traceback (most recent call last):
...
ArithmeticError: constant term of power series is not 1

sage: R.<t> = PowerSeriesRing(RR)
sage: (2+t).log().exp()
2.000000000000000 + 1.000000000000000*t + O(t^20)

```

**map\_coefficients**(*f*, *new\_base\_ring*=None)

Return the series obtained by applying *f* to the non-zero coefficients of *self*.

If *f* is a `sage.categories.map.Map`, then the resulting series will be defined over the codomain of *f*. Otherwise, the resulting polynomial will be over the same ring as *self*. Set *new\_base\_ring* to override this behaviour.

INPUT:

- *f* – a callable that will be applied to the coefficients of *self*.
- *new\_base\_ring* (optional) – if given, the resulting polynomial will be defined over this ring.

EXAMPLES:

```

sage: R.<x> = SR[[]]
sage: f = (1+I)*x^2 + 3*x - I
sage: f.map_coefficients(lambda z: z.conjugate())
I + 3*x + (-I + 1)*x^2
sage: R.<x> = ZZ[[]]
sage: f = x^2 + 2
sage: f.map_coefficients(lambda a: a + 42)
44 + 43*x^2

```

Examples with different base ring:

```

sage: R.<x> = ZZ[[]]
sage: k = GF(2)
sage: residue = lambda x: k(x)
sage: f = 4*x^2+x+3
sage: g = f.map_coefficients(residue); g
1 + x
sage: g.parent()
Power Series Ring in x over Integer Ring
sage: g = f.map_coefficients(residue, new_base_ring = k); g
1 + x
sage: g.parent()
Power Series Ring in x over Finite Field of size 2
sage: residue = k.coerce_map_from(ZZ)
sage: g = f.map_coefficients(residue); g
1 + x

```

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```
sage: g.parent()
Power Series Ring in x over Finite Field of size 2
```

Tests other implementations:

```
sage: R.<q> = PowerSeriesRing(GF(11), implementation='pari')
sage: f = q - q^3 + O(q^10)
sage: f.map_coefficients(lambda c: c - 2)
10*q + 8*q^3 + O(q^10)
```

**nth\_root**(*n*, *prec*=None)

Return the *n*-th root of this power series.

INPUT:

- *n* – integer
- *prec* – integer (optional) - precision of the result. Though, if this series has finite precision, then the result cannot have larger precision.

EXAMPLES:

```
sage: R.<x> = QQ[[[]]]
sage: (1+x).nth_root(5)
1 + 1/5*x - 2/25*x^2 + ... + 12039376311816/2384185791015625*x^19 + O(x^20)

sage: (1 + x + O(x^5)).nth_root(5)
1 + 1/5*x - 2/25*x^2 + 6/125*x^3 - 21/625*x^4 + O(x^5)
```

Check that the results are consistent with taking log and exponential:

```
sage: R.<x> = PowerSeriesRing(QQ, default_prec=100)
sage: p = (1 + 2*x - x^4)**200
sage: p1 = p.nth_root(1000, prec=100)
sage: p2 = (p.log()/1000).exp()
sage: p1.prec() == p2.prec() == 100
True
sage: p1.polynomial() == p2.polynomial()
True
```

Positive characteristic:

```
sage: R.<u> = GF(3)[[]]
sage: p = 1 + 2 * u^2
sage: p.nth_root(4)
1 + 2*u^2 + u^6 + 2*u^8 + u^12 + 2*u^14 + O(u^20)
sage: p.nth_root(4)**4
1 + 2*u^2 + O(u^20)
```

**ogf\_to\_egf**()

Return the exponential generating function power series, assuming self is an ordinary generating function power series.

This can also be computed as `serconvol(f, exp(t))` in PARI/GP.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: f = t + t^2 + 2*t^3
sage: f.ogf_to_egf()
t + 1/2*t^2 + 1/3*t^3
```

**padded\_list**(*n=None*)

Return a list of coefficients of self up to (but not including)  $q^n$ .

Includes 0's in the list on the right so that the list has length  $n$ .

INPUT:

- $n$  - (optional) an integer that is at least 0. If  $n$  is not given, it will be taken to be the precision of self, unless this is  $+\infty$ , in which case we just return `self.list()`.

EXAMPLES:

```
sage: R.<q> = PowerSeriesRing(QQ)
sage: f = 1 - 17*q + 13*q^2 + 10*q^4 + O(q^7)
sage: f.list()
[1, -17, 13, 0, 10]
sage: f.padded_list(7)
[1, -17, 13, 0, 10, 0, 0]
sage: f.padded_list(10)
[1, -17, 13, 0, 10, 0, 0, 0, 0, 0]
sage: f.padded_list(3)
[1, -17, 13]
sage: f.padded_list()
[1, -17, 13, 0, 10, 0, 0]
sage: g = 1 - 17*q + 13*q^2 + 10*q^4
sage: g.list()
[1, -17, 13, 0, 10]
sage: g.padded_list()
[1, -17, 13, 0, 10]
sage: g.padded_list(10)
[1, -17, 13, 0, 10, 0, 0, 0, 0, 0]
```

**polynomial()**

See this method in derived classes:

- `sage.rings.power_series_poly.PowerSeries_poly.polynomial()`,
- `sage.rings.multi_power_series_ring_element.MPowerSeries.polynomial()`

Implementations *MUST* override this in the derived class.

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(ZZ)
sage: PowerSeries.polynomial(1+x^2)
Traceback (most recent call last):
...
NotImplementedError
```

**prec()**

The precision of  $\dots + O(x^r)$  is by definition  $r$ .

EXAMPLES:

```
sage: R.<t> = ZZ[[t]]
sage: (t^2 + O(t^3)).prec()
3
sage: (1 - t^2 + O(t^100)).prec()
100
```

**precision\_absolute()**

Return the absolute precision of this series.

By definition, the absolute precision of  $\dots + O(x^r)$  is  $r$ .

EXAMPLES:

```
sage: R.<t> = ZZ[[t]]
sage: (t^2 + O(t^3)).precision_absolute()
3
sage: (1 - t^2 + O(t^100)).precision_absolute()
100
```

**precision\_relative()**

Return the relative precision of this series, that is the difference between its absolute precision and its valuation.

By convention, the relative precision of 0 (or  $O(x^r)$  for any  $r$ ) is 0.

EXAMPLES:

```
sage: R.<t> = ZZ[[t]]
sage: (t^2 + O(t^3)).precision_relative()
1
sage: (1 - t^2 + O(t^100)).precision_relative()
100
sage: O(t^4).precision_relative()
0
```

**shift( $n$ )**

Return this power series multiplied by the power  $t^n$ .

If  $n$  is negative, terms below  $t^{-n}$  are discarded.

This power series is left unchanged.

---

**Note:** Despite the fact that higher order terms are printed to the right in a power series, right shifting decreases the powers of  $t$ , while left shifting increases them. This is to be consistent with polynomials, integers, etc.

---

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ['y'], 't', 5)
sage: f = ~(1+t); f
1 - t + t^2 - t^3 + t^4 + O(t^5)
sage: f.shift(3)
t^3 - t^4 + t^5 - t^6 + t^7 + O(t^8)
sage: f >> 2
1 - t + t^2 + O(t^3)
```

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```
sage: f << 10
t^10 - t^11 + t^12 - t^13 + t^14 + O(t^15)
sage: t << 29
t^30
```

AUTHORS:

- Robert Bradshaw (2007-04-18)

**sin**(prec='infinity')

Apply sin to the formal power series.

INPUT:

- prec – Integer or infinity. The degree to truncate the result to.

OUTPUT:

A new power series.

EXAMPLES:

For one variable:

```
sage: t = PowerSeriesRing(QQ, 't').gen()
sage: f = (t + t**2).O(4)
sage: sin(f)
t + t^2 - 1/6*t^3 + O(t^4)
```

For several variables:

```
sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = a + b + a*b + T.O(3)
sage: sin(f)
a + b + a*b + O(a, b)^3
sage: f.sin()
a + b + a*b + O(a, b)^3
sage: f.sin(prec=2)
a + b + O(a, b)^2
```

If the power series has a non-zero constant coefficient  $c$ , one raises an error:

```
sage: g = 2+f
sage: sin(g)
Traceback (most recent call last):
...
ValueError: can only apply sin to formal power series with zero constant term
```

If no precision is specified, the default precision is used:

```
sage: T.default_prec()
12
sage: sin(a)
a - 1/6*a^3 + 1/120*a^5 - 1/5040*a^7 + 1/362880*a^9 - 1/39916800*a^11 + O(a, b)^
↪12
sage: a.sin(prec=5)
a - 1/6*a^3 + O(a, b)^5
```

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```
sage: sin(a + T.O(5))
a - 1/6*a^3 + O(a, b)^5
```

**sinh**(prec='infinity')

Apply sinh to the formal power series.

INPUT:

- prec – Integer or infinity. The degree to truncate the result to.

OUTPUT:

A new power series.

EXAMPLES:

For one variable:

```
sage: t = PowerSeriesRing(QQ, 't').gen()
sage: f = (t + t**2).O(4)
sage: sinh(f)
t + t^2 + 1/6*t^3 + O(t^4)
```

For several variables:

```
sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = a + b + a*b + T.O(3)
sage: sin(f)
a + b + a*b + O(a, b)^3
sage: f.sinh()
a + b + a*b + O(a, b)^3
sage: f.sinh(prec=2)
a + b + O(a, b)^2
```

If the power series has a non-zero constant coefficient  $c$ , one raises an error:

```
sage: g = 2+f
sage: sinh(g)
Traceback (most recent call last):
...
ValueError: can only apply sinh to formal power series with zero
constant term
```

If no precision is specified, the default precision is used:

```
sage: T.default_t_prec()
12
sage: sinh(a)
a + 1/6*a^3 + 1/120*a^5 + 1/5040*a^7 + 1/362880*a^9 +
1/39916800*a^11 + O(a, b)^12
sage: a.sinh(prec=5)
a + 1/6*a^3 + O(a, b)^5
sage: sinh(a + T.O(5))
a + 1/6*a^3 + O(a, b)^5
```



**solve\_linear\_de**(*prec*='infinity', *b*=None, *f0*=None)

Obtain a power series solution to an inhomogeneous linear differential equation of the form:

$$f'(t) = a(t)f(t) + b(t).$$

INPUT:

- **self** - the power series  $a(t)$
- **b** - the power series  $b(t)$  (default is zero)
- **f0** - the constant term of  $f$  ("initial condition") (default is 1)
- **prec** - desired precision of result (this will be reduced if either  $a$  or  $b$  have less precision available)

OUTPUT: the power series  $f$ , to indicated precision

ALGORITHM: A divide-and-conquer strategy; see the source code. Running time is approximately  $M(n) \log n$ , where  $M(n)$  is the time required for a polynomial multiplication of length  $n$  over the coefficient ring. (If you're working over something like  $\mathbf{Q}$ , running time analysis can be a little complicated because the coefficients tend to explode.)

**Note:**

- If the coefficient ring is a field of characteristic zero, then the solution will exist and is unique.
- For other coefficient rings, things are more complicated. A solution may not exist, and if it does it may not be unique. Generally, by the time the  $n$ th term has been computed, the algorithm will have attempted divisions by  $n!$  in the coefficient ring. So if your coefficient ring has enough 'precision', and if your coefficient ring can perform divisions even when the answer is not unique, and if you know in advance that a solution exists, then this function will find a solution (otherwise it will probably crash).

AUTHORS:

- David Harvey (2006-09-11): factored functionality out from `exp()` function, cleaned up precision tests a bit

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ, default_prec=10)
```

```
sage: a = 2 - 3*t + 4*t^2 + O(t^10)
sage: b = 3 - 4*t^2 + O(t^7)
sage: f = a.solve_linear_de(prec=5, b=b, f0=3/5)
sage: f
3/5 + 21/5*t + 33/10*t^2 - 38/15*t^3 + 11/24*t^4 + O(t^5)
sage: f.derivative() - a*f - b
O(t^4)
```

```
sage: a = 2 - 3*t + 4*t^2
sage: b = b = 3 - 4*t^2
sage: f = a.solve_linear_de(b=b, f0=3/5)
Traceback (most recent call last):
...
ValueError: cannot solve differential equation to infinite precision
```

```
sage: a.solve_linear_de(prec=5, b=b, f0=3/5)
3/5 + 21/5*t + 33/10*t^2 - 38/15*t^3 + 11/24*t^4 + O(t^5)
```

**sqrt**(prec=None, extend=False, all=False, name=None)

Return a square root of self.

INPUT:

- **prec** - integer (default: None): if not None and the series has infinite precision, truncates series at precision **prec**.
- **extend** - bool (default: False); if True, return a square root in an extension ring, if necessary. Otherwise, raise a ValueError if the square root is not in the base power series ring. For example, if **extend** is True the square root of a power series with odd degree leading coefficient is defined as an element of a formal extension ring.
- **name** - string; if **extend** is True, you must also specify the print name of the formal square root.
- **all** - bool (default: False); if True, return all square roots of self, instead of just one.

ALGORITHM: Newton's method

$$x_{i+1} = \frac{1}{2}(x_i + \text{self}/x_i)$$

EXAMPLES:

```
sage: K.<t> = PowerSeriesRing(QQ, 't', 5)
sage: sqrt(t^2)
t
sage: sqrt(1+t)
1 + 1/2*t - 1/8*t^2 + 1/16*t^3 - 5/128*t^4 + O(t^5)
sage: sqrt(4+t)
2 + 1/4*t - 1/64*t^2 + 1/512*t^3 - 5/16384*t^4 + O(t^5)
sage: u = sqrt(2+t, prec=2, extend=True, name = 'alpha'); u
alpha
sage: u^2
2 + t
sage: u.parent()
Univariate Quotient Polynomial Ring in alpha over Power Series Ring in t over
Rational Field with modulus x^2 - 2 - t
sage: K.<t> = PowerSeriesRing(QQ, 't', 50)
sage: sqrt(1+2*t+t^2)
1 + t
sage: sqrt(t^2 + 2*t^4 + t^6)
t + t^3
sage: sqrt(1 + t + t^2 + 7*t^3)^2
1 + t + t^2 + 7*t^3 + O(t^50)
sage: sqrt(K(0))
0
sage: sqrt(t^2)
t
```

```
sage: K.<t> = PowerSeriesRing(CDF, 5)
sage: v = sqrt(-1 + t + t^3, all=True); v
[1.0*I - 0.5*I*t - 0.125*I*t^2 - 0.5625*I*t^3 - 0.2890625*I*t^4 + O(t^5),
 -1.0*I + 0.5*I*t + 0.125*I*t^2 + 0.5625*I*t^3 + 0.2890625*I*t^4 + O(t^5)]
```

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```
sage: [a^2 for a in v]
[-1.0 + 1.0*t + 0.0*t^2 + 1.0*t^3 + 0(t^5), -1.0 + 1.0*t + 0.0*t^2 + 1.0*t^3 +
  ↪ 0(t^5)]
```

A formal square root:

```
sage: K.<t> = PowerSeriesRing(QQ, 5)
sage: f = 2*t + t^3 + 0(t^4)
sage: s = f.sqrt(extend=True, name='sqrtf'); s
sqrtf
sage: s^2
2*t + t^3 + 0(t^4)
sage: parent(s)
Univariate Quotient Polynomial Ring in sqrtf over Power Series Ring in t over
  ↪ Rational Field with modulus x^2 - 2*t - t^3 + 0(t^4)
```

AUTHORS:

- Robert Bradshaw
- William Stein

### **square\_root()**

Return the square root of self in this ring. If this cannot be done then an error will be raised.

This function succeeds if and only if `self.is_square()`

EXAMPLES:

```
sage: K.<t> = PowerSeriesRing(QQ, 't', 5)
sage: (1+t).square_root()
1 + 1/2*t - 1/8*t^2 + 1/16*t^3 - 5/128*t^4 + 0(t^5)
sage: (2+t).square_root()
Traceback (most recent call last):
...
ValueError: Square root does not live in this ring.
sage: (2+t.change_ring(RR)).square_root()
1.41421356237309 + 0.353553390593274*t - 0.0441941738241592*t^2 + 0.
  ↪ 0110485434560398*t^3 - 0.00345266983001244*t^4 + 0(t^5)
sage: t.square_root()
Traceback (most recent call last):
...
ValueError: Square root not defined for power series of odd valuation.
sage: K.<t> = PowerSeriesRing(ZZ, 't', 5)
sage: f = (1+t)^20
sage: f.square_root()
1 + 10*t + 45*t^2 + 120*t^3 + 210*t^4 + 0(t^5)
sage: f = 1+t
sage: f.square_root()
Traceback (most recent call last):
...
ValueError: Square root does not live in this ring.
```

AUTHORS:

- Robert Bradshaw

**stieltjes\_continued\_fraction()**

Return the Stieltjes continued fraction of `self`.

The S-fraction or Stieltjes continued fraction of a power series is a continued fraction expansion with steps of size one. We use the following convention

$$1/(1 - A_1 t / (1 - A_2 t / (1 - A_3 t / (1 - \dots))))$$

OUTPUT:

$A_n$  for  $n \geq 1$

The expansion is done as long as possible given the precision. Whenever the expansion is not well-defined, because it would require to divide by zero, an exception is raised.

EXAMPLES:

```
sage: t = PowerSeriesRing(QQ, 't').gen()
sage: s = sum(catalan_number(k) * t**k for k in range(12)).0(12)
sage: s.stieltjes_continued_fraction()
(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)
```

Another example:

```
sage: (exp(t)).stieltjes_continued_fraction()
(1,
 -1/2,
 1/6,
 -1/6,
 1/10,
 -1/10,
 1/14,
 -1/14,
 1/18,
 -1/18,
 1/22,
 -1/22,
 1/26,
 -1/26,
 1/30,
 -1/30,
 1/34,
 -1/34,
 1/38)
```

**tan(*prec*='infinity')**

Apply tan to the formal power series.

INPUT:

- *prec* – Integer or infinity. The degree to truncate the result to.

OUTPUT:

A new power series.

EXAMPLES:

For one variable:

```
sage: t = PowerSeriesRing(QQ, 't').gen()
sage: f = (t + t**2).O(4)
sage: tan(f)
t + t^2 + 1/3*t^3 + O(t^4)
```

For several variables:

```
sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = a + b + a*b + T.O(3)
sage: tan(f)
a + b + a*b + O(a, b)^3
sage: f.tan()
a + b + a*b + O(a, b)^3
sage: f.tan(prec=2)
a + b + O(a, b)^2
```

If the power series has a non-zero constant coefficient  $c$ , one raises an error:

```
sage: g = 2+f
sage: tan(g)
Traceback (most recent call last):
...
ValueError: can only apply tan to formal power series with zero constant term
```

If no precision is specified, the default precision is used:

```
sage: T.default_prec()
12
sage: tan(a)
a + 1/3*a^3 + 2/15*a^5 + 17/315*a^7 + 62/2835*a^9 + 1382/155925*a^11 + O(a, b)^
↪12
sage: a.tan(prec=5)
a + 1/3*a^3 + O(a, b)^5
sage: tan(a + T.O(5))
a + 1/3*a^3 + O(a, b)^5
```

**tanh**(*prec*='infinity')

Apply tanh to the formal power series.

INPUT:

- *prec* – Integer or infinity. The degree to truncate the result to.

OUTPUT:

A new power series.

EXAMPLES:

For one variable:

```
sage: t = PowerSeriesRing(QQ, 't').gen()
sage: f = (t + t**2).O(4)
sage: tanh(f)
t + t^2 - 1/3*t^3 + O(t^4)
```

For several variables:

```

sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = a + b + a*b + T.O(3)
sage: tanh(f)
a + b + a*b + O(a, b)^3
sage: f.tanh()
a + b + a*b + O(a, b)^3
sage: f.tanh(prec=2)
a + b + O(a, b)^2

```

If the power series has a non-zero constant coefficient  $c$ , one raises an error:

```

sage: g = 2+f
sage: tanh(g)
Traceback (most recent call last):
...
ValueError: can only apply tanh to formal power series with zero
constant term

```

If no precision is specified, the default precision is used:

```

sage: T.default_prec()
12
sage: tanh(a)
a - 1/3*a^3 + 2/15*a^5 - 17/315*a^7 + 62/2835*a^9 -
1382/155925*a^11 + O(a, b)^12
sage: a.tanh(prec=5)
a - 1/3*a^3 + O(a, b)^5
sage: tanh(a + T.O(5))
a - 1/3*a^3 + O(a, b)^5

```

**truncate**(*prec*='infinity')

The polynomial obtained from power series by truncation.

EXAMPLES:

```

sage: R.<I> = GF(2)[[]]
sage: f = 1/(1+I+O(I^8)); f
1 + I + I^2 + I^3 + I^4 + I^5 + I^6 + I^7 + O(I^8)
sage: f.truncate(5)
I^4 + I^3 + I^2 + I + 1

```

**valuation**()

Return the valuation of this power series.

This is equal to the valuation of the underlying polynomial.

EXAMPLES:

Sparse examples:

```

sage: R.<t> = PowerSeriesRing(QQ, sparse=True)
sage: f = t^1000000 + O(t^10000000)
sage: f.valuation()
1000000
sage: R(0).valuation()
+Infinity

```

Dense examples:

```
sage: R.<t> = PowerSeriesRing(ZZ)
sage: f = 17*t^100 + 0(t^110)
sage: f.valuation()
100
sage: t.valuation()
1
```

#### **valuation\_zero\_part()**

Factor self as  $q^n \cdot (a_0 + a_1q + \cdots)$  with  $a_0$  nonzero. Then this function returns  $a_0 + a_1q + \cdots$ .

---

**Note:** This valuation zero part need not be a unit if, e.g.,  $a_0$  is not invertible in the base ring.

---

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ)
sage: ((1/3)*t^5*(17-2/3*t^3)).valuation_zero_part()
17/3 - 2/9*t^3
```

In this example the valuation 0 part is not a unit:

```
sage: R.<t> = PowerSeriesRing(ZZ, sparse=True)
sage: u = (-2*t^5*(17-t^3)).valuation_zero_part(); u
-34 + 2*t^3
sage: u.is_unit()
False
sage: u.valuation()
0
```

#### **variable()**

Return a string with the name of the variable of this power series.

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(Rationals())
sage: f = x^2 + 3*x^4 + 0(x^7)
sage: f.variable()
'x'
```

AUTHORS:

- David Harvey (2006-08-08)

#### **sage.rings.power\_series\_ring\_element.is\_PowerSeries(x)**

Return True if x is an instance of a univariate or multivariate power series.

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(ZZ)
sage: from sage.rings.power_series_ring_element import is_PowerSeries
sage: is_PowerSeries(1+x^2)
True
sage: is_PowerSeries(x-x)
True
```

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```
sage: is_PowerSeries(0)
False
sage: var('x')
x
sage: is_PowerSeries(1+x^2)
False
```

`sage.rings.power_series_ring_element.make_element_from_parent_v0(parent, *args)`

`sage.rings.power_series_ring_element.make_powerseries_poly_v0(parent, f, prec, is_gen)`



## POWER SERIES METHODS

The class `PowerSeries_poly` provides additional methods for univariate power series.

```
class sage.rings.power_series_poly.PowerSeries_poly
    Bases: sage.rings.power\_series\_ring\_element.PowerSeries
```

EXAMPLES:

```
sage: R.<q> = PowerSeriesRing(CC)
sage: R
Power Series Ring in q over Complex Field with 53 bits of precision
sage: loads(q.dumps()) == q
True

sage: R.<t> = QQ[[[]]]
sage: f = 3 - t^3 + 0(t^5)
sage: a = f^3; a
27 - 27*t^3 + 0(t^5)
sage: b = f^-3; b
1/27 + 1/27*t^3 + 0(t^5)
sage: a*b
1 + 0(t^5)
```

Check that [trac ticket #22216](#) is fixed:

```
sage: R.<T> = PowerSeriesRing(QQ)
sage: R(pari('1 + 0(T)'))
1 + 0(T)
sage: R(pari('1/T + 0(T)'))
Traceback (most recent call last):
...
ValueError: series has negative valuation
```

### `degree()`

Return the degree of the underlying polynomial of `self`.

That is, if `self` is of the form  $f(x) + O(x^n)$ , we return the degree of  $f(x)$ . Note that if  $f(x)$  is 0, we return  $-1$ , just as with polynomials.

EXAMPLES:

```
sage: R.<t> = ZZ[[[]]]
sage: (5 + t^3 + 0(t^4)).degree()
3
```

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```
sage: (5 + 0(t^4)).degree()
0
sage: 0(t^4).degree()
-1
```

**dict()**

Return a dictionary of coefficients for `self`.

This is simply a dict for the underlying polynomial, so need not have keys corresponding to every number smaller than `self.prec()`.

EXAMPLES:

```
sage: R.<t> = ZZ[[t]]
sage: f = 1 + t^10 + 0(t^12)
sage: f.dict()
{0: 1, 10: 1}
```

**integral** (*var=None*)

Return the integral of this power series.

By default, the integration variable is the variable of the power series.

Otherwise, the integration variable is the optional parameter `var`

---

**Note:** The integral is always chosen so the constant term is 0.

---

EXAMPLES:

```
sage: k.<w> = QQ[[w]]
sage: (1+17*w+15*w^3+0(w^5)).integral()
w + 17/2*w^2 + 15/4*w^4 + 0(w^6)
sage: (w^3 + 4*w^4 + 0(w^7)).integral()
1/4*w^4 + 4/5*w^5 + 0(w^8)
sage: (3*w^2).integral()
w^3
```

**list()**

Return the list of known coefficients for `self`.

This is just the list of coefficients of the underlying polynomial, so in particular, need not have length equal to `self.prec()`.

EXAMPLES:

```
sage: R.<t> = ZZ[[t]]
sage: f = 1 - 5*t^3 + t^5 + 0(t^7)
sage: f.list()
[1, 0, 0, -5, 0, 1]
```

**pade** (*m, n*)

Return the Padé approximant of `self` of index  $(m, n)$ .

The Padé approximant of index  $(m, n)$  of a formal power series  $f$  is the quotient  $Q/P$  of two polynomials

$Q$  and  $P$  such that  $\deg(Q) \leq m$ ,  $\deg(P) \leq n$  and

$$f(z) - Q(z)/P(z) = O(z^{m+n+1}).$$

The formal power series  $f$  must be known up to order  $n + m$ .

See [Wikipedia article Padé approximant](#)

INPUT:

- $m, n$  – integers, describing the degrees of the polynomials

OUTPUT:

a ratio of two polynomials

**Warning:** The current implementation uses a very slow algorithm and is not suitable for high orders.

ALGORITHM:

This method uses the formula as a quotient of two determinants.

See also:

- `sage.matrix.berlekamp_massey`,
- `sage.rings.polynomial.polynomial_zmod_flint.Polynomial_zmod_flint.rational_reconstruct()`

EXAMPLES:

```
sage: z = PowerSeriesRing(QQ, 'z').gen()
sage: exp(z).pade(4, 0)
1/24*z^4 + 1/6*z^3 + 1/2*z^2 + z + 1
sage: exp(z).pade(1, 1)
(-z - 2)/(z - 2)
sage: exp(z).pade(3, 3)
(-z^3 - 12*z^2 - 60*z - 120)/(z^3 - 12*z^2 + 60*z - 120)
sage: log(1-z).pade(4, 4)
(25/6*z^4 - 130/3*z^3 + 105*z^2 - 70*z)/(z^4 - 20*z^3 + 90*z^2 - 140*z + 70)
sage: sqrt(1+z).pade(3, 2)
(1/6*z^3 + 3*z^2 + 8*z + 16/3)/(z^2 + 16/3*z + 16/3)
sage: exp(2*z).pade(3, 3)
(-z^3 - 6*z^2 - 15*z - 15)/(z^3 - 6*z^2 + 15*z - 15)
```

**polynomial()**

Return the underlying polynomial of `self`.

EXAMPLES:

```
sage: R.<t> = GF(7)[[]]
sage: f = 3 - t^3 + 0(t^5)
sage: f.polynomial()
6*t^3 + 3
```

**reverse**(*precision=None*)

Return the reverse of  $f$ , i.e., the series  $g$  such that  $g(f(x)) = x$ .

Given an optional argument *precision*, return the reverse with given precision (note that the reverse can have precision at most  $f.\text{prec}()$ ). If  $f$  has infinite precision, and the argument *precision* is not given, then the precision of the reverse defaults to the default precision of  $f.\text{parent}()$ .

Note that this is only possible if the valuation of self is exactly 1.

ALGORITHM:

We first attempt to pass the computation to pari; if this fails, we use Lagrange inversion. Using `sage: set_verbose(1)` will print a message if passing to pari fails.

If the base ring has positive characteristic, then we attempt to lift to a characteristic zero ring and perform the reverse there. If this fails, an error is raised.

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(QQ)
sage: f = 2*x + 3*x^2 - x^4 + 0(x^5)
sage: g = f.reverse()
sage: g
1/2*x - 3/8*x^2 + 9/16*x^3 - 131/128*x^4 + 0(x^5)
sage: f(g)
x + 0(x^5)
sage: g(f)
x + 0(x^5)

sage: A.<t> = PowerSeriesRing(ZZ)
sage: a = t - t^2 - 2*t^4 + t^5 + 0(t^6)
sage: b = a.reverse(); b
t + t^2 + 2*t^3 + 7*t^4 + 25*t^5 + 0(t^6)
sage: a(b)
t + 0(t^6)
sage: b(a)
t + 0(t^6)

sage: B.<b,c> = PolynomialRing(ZZ)
sage: A.<t> = PowerSeriesRing(B)
sage: f = t + b*t^2 + c*t^3 + 0(t^4)
sage: g = f.reverse(); g
t - b*t^2 + (2*b^2 - c)*t^3 + 0(t^4)
sage: f(g)
t + 0(t^4)
sage: g(f)
t + 0(t^4)

sage: A.<t> = PowerSeriesRing(ZZ)
sage: B.<s> = A[[]]
sage: f = (1 - 3*t + 4*t^3 + 0(t^4))*s + (2 + t + t^2 + 0(t^3))*s^2 + 0(s^3)
sage: from sage.misc.verbose import set_verbose
sage: set_verbose(1)
sage: g = f.reverse(); g
verbose 1 (<module>) passing to pari failed; trying Lagrange inversion
(1 + 3*t + 9*t^2 + 23*t^3 + 0(t^4))*s + (-2 - 19*t - 118*t^2 + 0(t^3))*s^2 +
0(s^3)
```

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```
sage: set_verbose(0)
sage: f(g) == g(f) == s
True
```

If the leading coefficient is not a unit, we pass to its fraction field if possible:

```
sage: A.<t> = PowerSeriesRing(ZZ)
sage: a = 2*t - 4*t^2 + t^4 - t^5 + 0(t^6)
sage: a.reverse()
1/2*t + 1/2*t^2 + t^3 + 79/32*t^4 + 437/64*t^5 + 0(t^6)

sage: B.<b> = PolynomialRing(ZZ)
sage: A.<t> = PowerSeriesRing(B)
sage: f = 2*b*t + b*t^2 + 3*b^2*t^3 + 0(t^4)
sage: g = f.reverse(); g
1/(2*b)*t - 1/(8*b^2)*t^2 + ((-3*b + 1)/(16*b^3))*t^3 + 0(t^4)
sage: f(g)
t + 0(t^4)
sage: g(f)
t + 0(t^4)
```

We can handle some base rings of positive characteristic:

```
sage: A8.<t> = PowerSeriesRing(Zmod(8))
sage: a = t - 15*t^2 - 2*t^4 + t^5 + 0(t^6)
sage: b = a.reverse(); b
t + 7*t^2 + 2*t^3 + 5*t^4 + t^5 + 0(t^6)
sage: a(b)
t + 0(t^6)
sage: b(a)
t + 0(t^6)
```

The optional argument `precision` sets the precision of the output:

```
sage: R.<x> = PowerSeriesRing(QQ)
sage: f = 2*x + 3*x^2 - 7*x^3 + x^4 + 0(x^5)
sage: g = f.reverse(precision=3); g
1/2*x - 3/8*x^2 + 0(x^3)
sage: f(g)
x + 0(x^3)
sage: g(f)
x + 0(x^3)
```

If the input series has infinite precision, the precision of the output is automatically set to the default precision of the parent ring:

```
sage: R.<x> = PowerSeriesRing(QQ, default_prec=20)
sage: (x - x^2).reverse() # get some Catalan numbers
x + x^2 + 2*x^3 + 5*x^4 + 14*x^5 + 42*x^6 + 132*x^7 + 429*x^8 + 1430*x^9 +
4862*x^10 + 16796*x^11 + 58786*x^12 + 208012*x^13 + 742900*x^14 + 2674440*x^
15 + 9694845*x^16 + 35357670*x^17 + 129644790*x^18 + 477638700*x^19 + 0(x^20)
sage: (x - x^2).reverse(precision=3)
x + x^2 + 0(x^3)
```

**truncate**(*prec*='infinity')The polynomial obtained from power series by truncation at precision *prec*.

EXAMPLES:

```

sage: R.<I> = GF(2)[[]]
sage: f = 1/(1+I+O(I^8)); f
1 + I + I^2 + I^3 + I^4 + I^5 + I^6 + I^7 + O(I^8)
sage: f.truncate(5)
I^4 + I^3 + I^2 + I + 1

```

**truncate\_powerseries**(*prec*)Given input *prec* = *n*, returns the power series of degree < *n* which is equivalent to self modulo  $x^n$ .

EXAMPLES:

```

sage: R.<I> = GF(2)[[]]
sage: f = 1/(1+I+O(I^8)); f
1 + I + I^2 + I^3 + I^4 + I^5 + I^6 + I^7 + O(I^8)
sage: f.truncate_powerseries(5)
1 + I + I^2 + I^3 + I^4 + O(I^5)

```

**valuation**()Return the valuation of *self*.

EXAMPLES:

```

sage: R.<t> = QQ[[]]
sage: (5 - t^8 + O(t^11)).valuation()
0
sage: (-t^8 + O(t^11)).valuation()
8
sage: O(t^7).valuation()
7
sage: R(0).valuation()
+Infinity

```

**sage.rings.power\_series\_poly.make\_powerseries\_poly\_v0**(*parent*, *f*, *prec*, *is\_gen*)Return the power series specified by *f*, *prec*, and *is\_gen*.

This function exists for the purposes of pickling. Do not delete this function – if you change the internal representation, instead make a new function and make sure that both kinds of objects correctly unpickle as the new type.

EXAMPLES:

```

sage: R.<t> = QQ[[]]
sage: sage.rings.power_series_poly.make_powerseries_poly_v0(R, t, infinity, True)
t

```

## POWER SERIES IMPLEMENTED USING PARI

### EXAMPLES:

This implementation can be selected for any base ring supported by PARI by passing the keyword `implementation='pari'` to the *PowerSeriesRing()* constructor:

```
sage: R.<q> = PowerSeriesRing(ZZ, implementation='pari'); R
Power Series Ring in q over Integer Ring
sage: S.<t> = PowerSeriesRing(CC, implementation='pari'); S
Power Series Ring in t over Complex Field with 53 bits of precision
```

Note that only the type of the elements depends on the implementation, not the type of the parents:

```
sage: type(R)
<class 'sage.rings.power_series_ring.PowerSeriesRing_domain_with_category'>
sage: type(q)
<class 'sage.rings.power_series_pari.PowerSeries_pari'>
sage: type(S)
<class 'sage.rings.power_series_ring.PowerSeriesRing_over_field_with_category'>
sage: type(t)
<class 'sage.rings.power_series_pari.PowerSeries_pari'>
```

If  $k$  is a finite field implemented using PARI, this is the default implementation for power series over  $k$ :

```
sage: k.<c> = GF(5^12)
sage: type(c)
<class 'sage.rings.finite_rings.element_pari_ffelt.FiniteFieldElement_pari_ffelt'>
sage: A.<x> = k[][]
sage: type(x)
<class 'sage.rings.power_series_pari.PowerSeries_pari'>
```

**Warning:** Because this implementation uses the PARI interface, the PARI variable ordering must be respected in the sense that the variable name of the power series ring must have higher priority than any variable names occurring in the base ring:

```
sage: R.<y> = QQ[]
sage: S.<x> = PowerSeriesRing(R, implementation='pari'); S
Power Series Ring in x over Univariate Polynomial Ring in y over Rational Field
```

Reversing the variable ordering leads to errors:

```

sage: R.<x> = QQ[]
sage: S.<y> = PowerSeriesRing(R, implementation='pari')
Traceback (most recent call last):
...
PariError: incorrect priority in gtopoly: variable x <= y

```

AUTHORS:

- Peter Bruin (December 2013): initial version

**class** sage.rings.power\_series\_pari.PowerSeries\_pari

Bases: *sage.rings.power\_series\_ring\_element.PowerSeries*

A power series implemented using PARI.

INPUT:

- **parent** – the power series ring to use as the parent
- **f** – object from which to construct a power series
- **prec** – (default: infinity) precision of the element to be constructed
- **check** – ignored, but accepted for compatibility with *PowerSeries\_poly*

**dict()**

Return a dictionary of coefficients for **self**.

This is simply a dict for the underlying polynomial; it need not have keys corresponding to every number smaller than **self.prec()**.

EXAMPLES:

```

sage: R.<t> = PowerSeriesRing(ZZ, implementation='pari')
sage: f = 1 + t^10 + O(t^12)
sage: f.dict()
{0: 1, 10: 1}

```

**integral**(*var=None*)

Return the formal integral of **self**.

By default, the integration variable is the variable of the power series. Otherwise, the integration variable is the optional parameter **var**.

---

**Note:** The integral is always chosen so the constant term is 0.

---

EXAMPLES:

```

sage: k.<w> = PowerSeriesRing(QQ, implementation='pari')
sage: (1+17*w+15*w^3+O(w^5)).integral()
w + 17/2*w^2 + 15/4*w^4 + O(w^6)
sage: (w^3 + 4*w^4 + O(w^7)).integral()
1/4*w^4 + 4/5*w^5 + O(w^8)
sage: (3*w^2).integral()
w^3

```



**list()**

Return the list of known coefficients for `self`.

This is just the list of coefficients of the underlying polynomial; it need not have length equal to `self.prec()`.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(ZZ, implementation='pari')
sage: f = 1 - 5*t^3 + t^5 + O(t^7)
sage: f.list()
[1, 0, 0, -5, 0, 1]

sage: S.<u> = PowerSeriesRing(pAdicRing(5), implementation='pari')
sage: (2 + u).list()
[2 + O(5^20), 1 + O(5^20)]
```

**padded\_list(*n=None*)**

Return a list of coefficients of `self` up to (but not including)  $q^n$ .

The list is padded with zeroes on the right so that it has length  $n$ .

INPUT:

- **$n$  – a non-negative integer (optional); if  $n$  is not given, it will be taken to be the precision of `self`, unless this is `Infinity`, in which case we just return `self.list()`**

EXAMPLES:

```
sage: R.<q> = PowerSeriesRing(QQ, implementation='pari')
sage: f = 1 - 17*q + 13*q^2 + 10*q^4 + O(q^7)
sage: f.list()
[1, -17, 13, 0, 10]
sage: f.padded_list(7)
[1, -17, 13, 0, 10, 0, 0]
sage: f.padded_list(10)
[1, -17, 13, 0, 10, 0, 0, 0, 0, 0]
sage: f.padded_list(3)
[1, -17, 13]
sage: f.padded_list()
[1, -17, 13, 0, 10, 0, 0]
sage: g = 1 - 17*q + 13*q^2 + 10*q^4
sage: g.list()
[1, -17, 13, 0, 10]
sage: g.padded_list()
[1, -17, 13, 0, 10]
sage: g.padded_list(10)
[1, -17, 13, 0, 10, 0, 0, 0, 0, 0]
```

**polynomial()**

Convert `self` to a polynomial.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(GF(7), implementation='pari')
sage: f = 3 - t^3 + O(t^5)
sage: f.polynomial()
6*t^3 + 3
```

**reverse**(*precision=None*)

Return the reverse of `self`.

The reverse of a power series  $f$  is the power series  $g$  such that  $g(f(x)) = x$ . This exists if and only if the valuation of `self` is exactly 1 and the coefficient of  $x$  is a unit.

If the optional argument `precision` is given, the reverse is returned with this precision. If `f` has infinite precision and the argument `precision` is not given, then the reverse is returned with the default precision of `f.parent()`.

EXAMPLES:

```
sage: R.<x> = PowerSeriesRing(QQ, implementation='pari')
sage: f = 2*x + 3*x^2 - x^4 + O(x^5)
sage: g = f.reverse()
sage: g
1/2*x - 3/8*x^2 + 9/16*x^3 - 131/128*x^4 + O(x^5)
sage: f(g)
x + O(x^5)
sage: g(f)
x + O(x^5)

sage: A.<t> = PowerSeriesRing(ZZ, implementation='pari')
sage: a = t - t^2 - 2*t^4 + t^5 + O(t^6)
sage: b = a.reverse(); b
t + t^2 + 2*t^3 + 7*t^4 + 25*t^5 + O(t^6)
sage: a(b)
t + O(t^6)
sage: b(a)
t + O(t^6)

sage: B.<b,c> = PolynomialRing(ZZ)
sage: A.<t> = PowerSeriesRing(B, implementation='pari')
sage: f = t + b*t^2 + c*t^3 + O(t^4)
sage: g = f.reverse(); g
t - b*t^2 + (2*b^2 - c)*t^3 + O(t^4)
sage: f(g)
t + O(t^4)
sage: g(f)
t + O(t^4)

sage: A.<t> = PowerSeriesRing(ZZ, implementation='pari')
sage: B.<x> = PowerSeriesRing(A, implementation='pari')
sage: f = (1 - 3*t + 4*t^3 + O(t^4))*x + (2 + t + t^2 + O(t^3))*x^2 + O(x^3)
sage: g = f.reverse(); g
(1 + 3*t + 9*t^2 + 23*t^3 + O(t^4))*x + (-2 - 19*t - 118*t^2 + O(t^3))*x^2 +
O(x^3)
```

The optional argument `precision` sets the precision of the output:

```
sage: R.<x> = PowerSeriesRing(QQ, implementation='pari')
sage: f = 2*x + 3*x^2 - 7*x^3 + x^4 + O(x^5)
sage: g = f.reverse(precision=3); g
1/2*x - 3/8*x^2 + O(x^3)
sage: f(g)
```

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```
x + 0(x^3)
sage: g(f)
x + 0(x^3)
```

If the input series has infinite precision, the precision of the output is automatically set to the default precision of the parent ring:

```
sage: R.<x> = PowerSeriesRing(QQ, default_prec=20, implementation='pari')
sage: (x - x^2).reverse() # get some Catalan numbers
x + x^2 + 2*x^3 + 5*x^4 + 14*x^5 + 42*x^6 + 132*x^7 + 429*x^8
+ 1430*x^9 + 4862*x^10 + 16796*x^11 + 58786*x^12 + 208012*x^13
+ 742900*x^14 + 2674440*x^15 + 9694845*x^16 + 35357670*x^17
+ 129644790*x^18 + 477638700*x^19 + 0(x^20)
sage: (x - x^2).reverse(precision=3)
x + x^2 + 0(x^3)
```

**valuation()**

Return the valuation of self.

EXAMPLES:

```
sage: R.<t> = PowerSeriesRing(QQ, implementation='pari')
sage: (5 - t^8 + 0(t^11)).valuation()
0
sage: (-t^8 + 0(t^11)).valuation()
8
sage: 0(t^7).valuation()
7
sage: R(0).valuation()
+Infinity
```



## MULTIVARIATE POWER SERIES RINGS

Construct a multivariate power series ring (in finitely many variables) over a given (commutative) base ring.

EXAMPLES:

Construct rings and elements:

```
sage: R.<t,u,v> = PowerSeriesRing(QQ); R
Multivariate Power Series Ring in t, u, v over Rational Field
sage: TestSuite(R).run()
sage: p = -t + 1/2*t^3*u - 1/4*t^4*u + 2/3*v^5 + R.0(6); p
-t + 1/2*t^3*u - 1/4*t^4*u + 2/3*v^5 + O(t, u, v)^6
sage: p in R
True

sage: g = 1 + v + 3*u*t^2 - 2*v^2*t^2; g
1 + v + 3*t^2*u - 2*t^2*v^2
sage: g in R
True
```

Add big O as with single variable power series:

```
sage: g.add_bigoh(3)
1 + v + O(t, u, v)^3
sage: g = g.O(5); g
1 + v + 3*t^2*u - 2*t^2*v^2 + O(t, u, v)^5
```

Sage keeps track of total-degree precision:

```
sage: f = (g-1)^2 - g + 1; f
-v + v^2 - 3*t^2*u + 6*t^2*u*v + 2*t^2*v^2 + O(t, u, v)^5
sage: f in R
True
sage: f.prec()
5
sage: ((g-1-v)^2).prec()
8
```

Construct multivariate power series rings over various base rings.

```
sage: M = PowerSeriesRing(QQ, 4, 'k'); M
Multivariate Power Series Ring in k0, k1, k2, k3 over Rational Field
sage: loads(dumps(M)) is M
```

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```

True
sage: TestSuite(M).run()

sage: H = PowerSeriesRing(PolynomialRing(ZZ,3,'z'),4,'f'); H
Multivariate Power Series Ring in f0, f1, f2, f3 over Multivariate
Polynomial Ring in z0, z1, z2 over Integer Ring
sage: TestSuite(H).run()
sage: loads(dumps(H)) is H
True

sage: z = H.base_ring().gens()
sage: f = H.gens()
sage: h = 4*z[1]^2 + 2*z[0]*z[2] + z[1]*z[2] + z[2]^2 \
+ (-z[2]^2 - 2*z[0] + z[2])*f[0]*f[2] \
+ (-22*z[0]^2 + 2*z[1]^2 - z[0]*z[2] + z[2]^2 - 1955*z[2])*f[1]*f[2] \
+ (-z[0]*z[1] - 2*z[1]^2)*f[2]*f[3] \
+ (2*z[0]*z[1] + z[1]*z[2] - z[2]^2 - z[1] + 3*z[2])*f[3]^2 \
+ H.O(3)
sage: h in H
True
sage: h
4*z1^2 + 2*z0*z2 + z1*z2 + z2^2 + (-z2^2 - 2*z0 + z2)*f0*f2
+ (-22*z0^2 + 2*z1^2 - z0*z2 + z2^2 - 1955*z2)*f1*f2
+ (-z0*z1 - 2*z1^2)*f2*f3 + (2*z0*z1 + z1*z2 - z2^2 - z1 + 3*z2)*f3^2
+ O(f0, f1, f2, f3)^3

```

- Use angle-bracket notation:

```

sage: S.<x,y> = PowerSeriesRing(GF(65537)); S
Multivariate Power Series Ring in x, y over Finite Field of size 65537
sage: s = -30077*x + 9485*x*y - 6260*y^3 + 12870*x^2*y^2 - 20289*y^4 + S.O(5); s
-30077*x + 9485*x*y - 6260*y^3 + 12870*x^2*y^2 - 20289*y^4 + O(x, y)^5
sage: s in S
True
sage: TestSuite(S).run()
sage: loads(dumps(S)) is S
True

```

- Use double square bracket notation:

```

sage: ZZ[['s,t,u']]
Multivariate Power Series Ring in s, t, u over Integer Ring
sage: GF(127931)[['x,y']]
Multivariate Power Series Ring in x, y over Finite Field of size 127931

```

Variable ordering determines how series are displayed.

```

sage: T.<a,b> = PowerSeriesRing(ZZ,order='deglex'); T
Multivariate Power Series Ring in a, b over Integer Ring
sage: TestSuite(T).run()
sage: loads(dumps(T)) is T
True
sage: T.term_order()

```

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Degree lexicographic term order

```
sage: p = - 2*b^6 + a^5*b^2 + a^7 - b^2 - a*b^3 + T.O(9); p
a^7 + a^5*b^2 - 2*b^6 - a*b^3 - b^2 + O(a, b)^9
```

```
sage: U = PowerSeriesRing(ZZ, 'a,b', order='negdeglex'); U
```

Multivariate Power Series Ring in a, b over Integer Ring

```
sage: U.term_order()
```

Negative degree lexicographic term order

```
sage: U(p)
-b^2 - a*b^3 - 2*b^6 + a^7 + a^5*b^2 + O(a, b)^9
```

Change from one base ring to another:

```
sage: R.<t,u,v> = PowerSeriesRing(QQ); R
```

Multivariate Power Series Ring in t, u, v over Rational Field

```
sage: R.base_extend(RR)
```

Multivariate Power Series Ring in t, u, v over Real Field with 53 bits of precision

```
sage: R.change_ring(IntegerModRing(10))
```

Multivariate Power Series Ring in t, u, v over Ring of integers modulo 10

```
sage: S = PowerSeriesRing(GF(65537), 2, 'x,y'); S
```

Multivariate Power Series Ring in x, y over Finite Field of size 65537

```
sage: S.change_ring(GF(5))
```

Multivariate Power Series Ring in x, y over Finite Field of size 5

Coercion from polynomial ring:

```
sage: R.<t,u,v> = PowerSeriesRing(QQ); R
```

Multivariate Power Series Ring in t, u, v over Rational Field

```
sage: A = PolynomialRing(ZZ, 3, 't,u,v')
```

```
sage: g = A.gens()
```

```
sage: a = 2*g[0]*g[2] - 2*g[0] - 2; a
```

$2*t*v - 2*t - 2$

```
sage: R(a)
```

$-2 - 2*t + 2*t*v$

```
sage: R(a).O(4)
```

$-2 - 2*t + 2*t*v + O(t, u, v)^4$

```
sage: a.parent()
```

Multivariate Polynomial Ring in t, u, v over Integer Ring

```
sage: a in R
```

True

Coercion from polynomial ring in subset of variables:

```
sage: R.<t,u,v> = PowerSeriesRing(QQ); R
```

Multivariate Power Series Ring in t, u, v over Rational Field

```
sage: A = PolynomialRing(QQ, 2, 't,v')
```

```
sage: g = A.gens()
```

```
sage: a = -2*g[0]*g[1] - 1/27*g[1]^2 + g[0] - 1/2*g[1]; a
```

$-2*t*v - 1/27*v^2 + t - 1/2*v$

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```
sage: a in R
True
```

Coercion from symbolic ring:

```
sage: x,y = var('x,y')
sage: S = PowerSeriesRing(GF(11),2,'x,y'); S
Multivariate Power Series Ring in x, y over Finite Field of size 11
sage: type(x)
<class 'sage.symbolic.expression.Expression'>
sage: type(S(x))
<class 'sage.rings.multi_power_series_ring.MPowerSeriesRing_generic_with_category.
  ↳element_class'>

sage: f = S(2/7 - 100*x^2 + 1/3*x*y + y^2).O(3); f
5 - x^2 + 4*x*y + y^2 + O(x, y)^3
sage: f.parent()
Multivariate Power Series Ring in x, y over Finite Field of size 11
sage: f.parent() == S
True
```

The implementation of the multivariate power series ring uses a combination of multivariate polynomials and univariate power series. Namely, in order to construct the multivariate power series ring  $R[[x_1, x_2, \dots, x_n]]$ , we consider the univariate power series ring  $S[[T]]$  over the multivariate polynomial ring  $S := R[x_1, x_2, \dots, x_n]$ , and in it we take the subring formed by all power series whose  $i$ -th coefficient has degree  $i$  for all  $i \geq 0$ . This subring is isomorphic to  $R[[x_1, x_2, \dots, x_n]]$ . This is how  $R[[x_1, x_2, \dots, x_n]]$  is implemented in this class. The ring  $S$  is called the foreground polynomial ring, and the ring  $S[[T]]$  is called the background univariate power series ring.

AUTHORS:

- Niles Johnson (2010-07): initial code
- Simon King (2012-08, 2013-02): Use category and coercion framework, [trac ticket #13412](#) and [trac ticket #14084](#)

```
class sage.rings.multi_power_series_ring.MPowerSeriesRing_generic(base_ring, num_gens,
                                                                    name_list, order='negdeglex',
                                                                    default_prec=10,
                                                                    sparse=False)
```

Bases: `sage.rings.power_series_ring.PowerSeriesRing_generic`, `sage.structure.nonexact.Nonexact`

A multivariate power series ring. This class is implemented as a single variable power series ring in the variable  $T$  over a multivariable polynomial ring in the specified generators. Each generator  $g$  of the multivariable polynomial ring (called the “foreground ring”) is mapped to  $g*T$  in the single variable power series ring (called the “background ring”). The background power series ring is used to do arithmetic and track total-degree precision. The foreground polynomial ring is used to display elements.

For usage and examples, see above, and [PowerSeriesRing\(\)](#).

#### Element

alias of `sage.rings.multi_power_series_ring_element.MPowerSeries`

#### $O(\text{prec})$

Return big oh with precision `prec`. This function is an alias for `bigoh`.

EXAMPLES:



```

sage: T.<a,b> = PowerSeriesRing(ZZ,2); T
Multivariate Power Series Ring in a, b over Integer Ring
sage: T.O(10)
0 + O(a, b)^10
sage: T.bigoh(10)
0 + O(a, b)^10

```

**bigoh(*prec*)**

Return big oh with precision *prec*. The function `O` does the same thing.

EXAMPLES:

```

sage: T.<a,b> = PowerSeriesRing(ZZ,2); T
Multivariate Power Series Ring in a, b over Integer Ring
sage: T.bigoh(10)
0 + O(a, b)^10
sage: T.O(10)
0 + O(a, b)^10

```

**change\_ring(*R*)**

Returns the power series ring over *R* in the same variable as self. This function ignores the question of whether the base ring of self is or can extend to the base ring of *R*; for the latter, use `base_extend`.

EXAMPLES:

```

sage: R.<t,u,v> = PowerSeriesRing(QQ); R
Multivariate Power Series Ring in t, u, v over Rational Field
sage: R.base_extend(RR)
Multivariate Power Series Ring in t, u, v over Real Field with
53 bits of precision
sage: R.change_ring(IntegerModRing(10))
Multivariate Power Series Ring in t, u, v over Ring of integers
modulo 10
sage: R.base_extend(IntegerModRing(10))
Traceback (most recent call last):
...
TypeError: no base extension defined

sage: S = PowerSeriesRing(GF(65537),2,'x,y'); S
Multivariate Power Series Ring in x, y over Finite Field of size
65537
sage: S.change_ring(GF(5))
Multivariate Power Series Ring in x, y over Finite Field of size 5

```

**characteristic()**

Return characteristic of base ring, which is characteristic of self.

EXAMPLES:

```

sage: H = PowerSeriesRing(GF(65537),4,'f'); H
Multivariate Power Series Ring in f0, f1, f2, f3 over
Finite Field of size 65537
sage: H.characteristic()
65537

```

**construction()**

Returns a functor  $F$  and base ring  $R$  such that  $F(R) == \text{self}$ .

EXAMPLES:

```
sage: M = PowerSeriesRing(QQ,4,'f'); M
Multivariate Power Series Ring in f0, f1, f2, f3 over Rational Field

sage: (c,R) = M.construction(); (c,R)
(Completion[('f0', 'f1', 'f2', 'f3'), prec=12],
Multivariate Polynomial Ring in f0, f1, f2, f3 over Rational Field)
sage: c
Completion[('f0', 'f1', 'f2', 'f3'), prec=12]
sage: c(R)
Multivariate Power Series Ring in f0, f1, f2, f3 over Rational Field
sage: c(R) == M
True
```

**gen( $n=0$ )**

Return the  $n$ th generator of self.

EXAMPLES:

```
sage: M = PowerSeriesRing(ZZ,10,'v')
sage: M.gen(6)
v6
```

**is\_dense()**

Is self dense? (opposite of sparse)

EXAMPLES:

```
sage: M = PowerSeriesRing(ZZ,3,'s,t,u'); M
Multivariate Power Series Ring in s, t, u over Integer Ring
sage: M.is_dense()
True
sage: N = PowerSeriesRing(ZZ,3,'s,t,u',sparse=True); N
Sparse Multivariate Power Series Ring in s, t, u over Integer Ring
sage: N.is_dense()
False
```

**is\_integral\_domain( $proof=False$ )**

Return True if the base ring is an integral domain; otherwise return False.

EXAMPLES:

```
sage: M = PowerSeriesRing(QQ,4,'v'); M
Multivariate Power Series Ring in v0, v1, v2, v3 over Rational Field
sage: M.is_integral_domain()
True
```

**is\_noetherian( $proof=False$ )**

Power series over a Noetherian ring are Noetherian.

EXAMPLES:

```

sage: M = PowerSeriesRing(QQ,4,'v'); M
Multivariate Power Series Ring in v0, v1, v2, v3 over Rational Field
sage: M.is_noetherian()
True

sage: W = PowerSeriesRing(InfinitePolynomialRing(ZZ,'a'),2,'x,y')
sage: W.is_noetherian()
False

```

**is\_sparse()**

Is self sparse?

EXAMPLES:

```

sage: M = PowerSeriesRing(ZZ,3,'s,t,u'); M
Multivariate Power Series Ring in s, t, u over Integer Ring
sage: M.is_sparse()
False

sage: N = PowerSeriesRing(ZZ,3,'s,t,u',sparse=True); N
Sparse Multivariate Power Series Ring in s, t, u over Integer Ring
sage: N.is_sparse()
True

```

**laurent\_series\_ring()**

Laurent series not yet implemented for multivariate power series rings

**ngens()**

Return number of generators of self.

EXAMPLES:

```

sage: M = PowerSeriesRing(ZZ,10,'v')
sage: M.ngens()
10

```

**prec\_ideal()**

Return the ideal which determines precision; this is the ideal generated by all of the generators of our background polynomial ring.

EXAMPLES:

```

sage: A.<s,t,u> = PowerSeriesRing(ZZ)
sage: A.prec_ideal()
Ideal (s, t, u) of Multivariate Polynomial Ring in s, t, u over
Integer Ring

```

**remove\_var(\*var)**

Remove given variable or sequence of variables from self.

EXAMPLES:

```

sage: A.<s,t,u> = PowerSeriesRing(ZZ)
sage: A.remove_var(t)
Multivariate Power Series Ring in s, u over Integer Ring
sage: A.remove_var(s,t)
Power Series Ring in u over Integer Ring

```

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```
sage: M = PowerSeriesRing(GF(5), 5, 't'); M
Multivariate Power Series Ring in t0, t1, t2, t3, t4 over
Finite Field of size 5
sage: M.remove_var(M.gens()[3])
Multivariate Power Series Ring in t0, t1, t2, t4 over Finite
Field of size 5
```

Removing all variables results in the base ring:

```
sage: M.remove_var(*M.gens())
Finite Field of size 5
```

**term\_order()**

Print term ordering of self. Term orderings are implemented by the TermOrder class.

EXAMPLES:

```
sage: M.<x,y,z> = PowerSeriesRing(ZZ, 3)
sage: M.term_order()
Negative degree lexicographic term order
sage: m = y*z^12 - y^6*z^8 - x^7*y^5*z^2 + x*y^2*z + M.O(15); m
x*y^2*z + y*z^12 - x^7*y^5*z^2 - y^6*z^8 + O(x, y, z)^15

sage: N = PowerSeriesRing(ZZ, 3, 'x,y,z', order="deglex")
sage: N.term_order()
Degree lexicographic term order
sage: N(m)
-x^7*y^5*z^2 - y^6*z^8 + y*z^12 + x*y^2*z + O(x, y, z)^15
```

`sage.rings.multi_power_series_ring.is_MPowerSeriesRing(x)`

Return true if input is a multivariate power series ring.

`sage.rings.multi_power_series_ring.unpickle_multi_power_series_ring_v0(base_ring, num_gens, names, order, default_prec, sparse)`

Unpickle (deserialize) a multivariate power series ring according to the given inputs.

EXAMPLES:

```
sage: P.<x,y> = PowerSeriesRing(QQ)
sage: loads(dumps(P)) == P # indirect doctest
True
```

## MULTIVARIATE POWER SERIES

Construct and manipulate multivariate power series (in finitely many variables) over a given commutative ring. Multivariate power series are implemented with total-degree precision.

EXAMPLES:

Power series arithmetic, tracking precision:

```
sage: R.<s,t> = PowerSeriesRing(ZZ); R
Multivariate Power Series Ring in s, t over Integer Ring

sage: f = 1 + s + 3*s^2; f
1 + s + 3*s^2
sage: g = t^2*s + 3*t^2*s^2 + R.O(5); g
s*t^2 + 3*s^2*t^2 + 0(s, t)^5
sage: g = t^2*s + 3*t^2*s^2 + 0(s, t)^5; g
s*t^2 + 3*s^2*t^2 + 0(s, t)^5
sage: f = f.O(7); f
1 + s + 3*s^2 + 0(s, t)^7
sage: f += s; f
1 + 2*s + 3*s^2 + 0(s, t)^7
sage: f*g
s*t^2 + 5*s^2*t^2 + 0(s, t)^5
sage: (f-1)*g
2*s^2*t^2 + 9*s^3*t^2 + 0(s, t)^6
sage: f*g - g
2*s^2*t^2 + 0(s, t)^5

sage: f*=s; f
s + 2*s^2 + 3*s^3 + 0(s, t)^8
sage: f%2
s + s^3 + 0(s, t)^8
sage: (f%2).parent()
Multivariate Power Series Ring in s, t over Ring of integers modulo 2
```

As with univariate power series, comparison of  $f$  and  $g$  is done up to the minimum precision of  $f$  and  $g$ :

```
sage: f = 1 + t + s + s*t + R.O(3); f
1 + s + t + s*t + 0(s, t)^3
sage: g = s^2 + 2*s^4 - s^5 + s^2*t^3 + R.O(6); g
s^2 + 2*s^4 - s^5 + s^2*t^3 + 0(s, t)^6
sage: f == g
False
```

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```
sage: g == g.add_bigoh(3)
True
sage: f < g
False
sage: f > g
True
```

Calling:

```
sage: f = s^2 + s*t + s^3 + s^2*t + 3*s^4 + 3*s^3*t + R.O(5); f
s^2 + s*t + s^3 + s^2*t + 3*s^4 + 3*s^3*t + 0(s, t)^5
sage: f(t,s)
s*t + t^2 + s*t^2 + t^3 + 3*s*t^3 + 3*t^4 + 0(s, t)^5
sage: f(t^2,s^2)
s^2*t^2 + t^4 + s^2*t^4 + t^6 + 3*s^2*t^6 + 3*t^8 + 0(s, t)^10
```

Substitution is defined only for elements of positive valuation, unless  $f$  has infinite precision:

```
sage: f(t^2,s^2+1)
Traceback (most recent call last):
...
TypeError: Substitution defined only for elements of positive valuation,
unless self has infinite precision.

sage: g = f.truncate()
sage: g(t^2,s^2+1)
t^2 + s^2*t^2 + 2*t^4 + s^2*t^4 + 4*t^6 + 3*s^2*t^6 + 3*t^8
sage: g(t^2,(s^2+1).O(3))
t^2 + s^2*t^2 + 2*t^4 + 0(s, t)^5
```

0 has valuation +Infinity:

```
sage: f(t^2,0)
t^4 + t^6 + 3*t^8 + 0(s, t)^10
sage: f(t^2,s^2+s)
s*t^2 + s^2*t^2 + t^4 + 0(s, t)^5
```

Substitution of power series with finite precision works too:

```
sage: f(s.O(2),t)
s^2 + s*t + 0(s, t)^3
sage: f(f,f)
2*s^4 + 4*s^3*t + 2*s^2*t^2 + 4*s^5 + 8*s^4*t + 4*s^3*t^2 + 16*s^6 +
34*s^5*t + 20*s^4*t^2 + 2*s^3*t^3 + 0(s, t)^7
sage: t(f,f)
s^2 + s*t + s^3 + s^2*t + 3*s^4 + 3*s^3*t + 0(s, t)^5
sage: t(0,f) == s(f,0)
True
```

The subs syntax works as expected:

```
sage: r0 = -t^2 - s*t^3 - 2*t^6 + s^7 + s^5*t^2 + R.O(10)
sage: r1 = s^4 - s*t^4 + s^6*t - 4*s^2*t^5 - 6*s^3*t^5 + R.O(10)
```

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```

sage: r2 = 2*s^3*t^2 - 2*s*t^4 - 2*s^3*t^4 + s*t^7 + R.O(10)
sage: r0.subs({t:r2,s:r1})
-4*s^6*t^4 + 8*s^4*t^6 - 4*s^2*t^8 + 8*s^6*t^6 - 8*s^4*t^8 - 4*s^4*t^9
+ 4*s^2*t^11 - 4*s^6*t^8 + O(s, t)^15
sage: r0.subs({t:r2,s:r1}) == r0(r1,r2)
True

```

Construct ring homomorphisms from one power series ring to another:

```

sage: A.<a,b> = PowerSeriesRing(QQ)
sage: X.<x,y> = PowerSeriesRing(QQ)

sage: phi = Hom(A,X)([x,2*y]); phi
Ring morphism:
  From: Multivariate Power Series Ring in a, b over Rational Field
  To:   Multivariate Power Series Ring in x, y over Rational Field
  Defn: a |--> x
        b |--> 2*y

sage: phi(a+b+3*a*b^2 + A.O(5))
x + 2*y + 12*x*y^2 + O(x, y)^5

```

Multiplicative inversion of power series:

```

sage: h = 1 + s + t + s*t + s^2*t^2 + 3*s^4 + 3*s^3*t + R.O(5)
sage: k = h^-1; k
1 - s - t + s^2 + s*t + t^2 - s^3 - s^2*t - s*t^2 - t^3 - 2*s^4 -
2*s^3*t + s*t^3 + t^4 + O(s, t)^5
sage: h*k
1 + O(s, t)^5

sage: f = 1 - 5*s^29 - 5*s^28*t + 4*s^18*t^35 + \
.....: 4*s^17*t^36 - s^45*t^25 - s^44*t^26 + s^7*t^83 + \
.....: s^6*t^84 + R.O(101)
sage: h = ~f; h
1 + 5*s^29 + 5*s^28*t - 4*s^18*t^35 - 4*s^17*t^36 + 25*s^58 + 50*s^57*t
+ 25*s^56*t^2 + s^45*t^25 + s^44*t^26 - 40*s^47*t^35 - 80*s^46*t^36
- 40*s^45*t^37 + 125*s^87 + 375*s^86*t + 375*s^85*t^2 + 125*s^84*t^3
- s^7*t^83 - s^6*t^84 + 10*s^74*t^25 + 20*s^73*t^26 + 10*s^72*t^27
+ O(s, t)^101
sage: h*f
1 + O(s, t)^101

```

AUTHORS:

- Niles Johnson (07/2010): initial code
- Simon King (08/2012): Use category and coercion framework, [trac ticket #13412](#)

```

class sage.rings.multi_power_series_ring_element.MO(x)
    Bases: object

```

Object representing a zero element with given precision.

EXAMPLES:

```

sage: R.<u,v> = QQ[[]]
sage: m = 0(u, v)
sage: m^4
0 + 0(u, v)^4
sage: m^1
0 + 0(u, v)^1

sage: T.<a,b,c> = PowerSeriesRing(ZZ,3)
sage: z = 0(a, b, c)
sage: z^1
0 + 0(a, b, c)^1
sage: 1 + a + z^1
1 + 0(a, b, c)^1

sage: w = 1 + a + 0(a, b, c)^2; w
1 + a + 0(a, b, c)^2
sage: w^2
1 + 2*a + 0(a, b, c)^2

```

**class** sage.rings.multi\_power\_series\_ring\_element.**MPowerSeries**(parent, x=0, prec=+ Infinity, is\_gen=False, check=False)

Bases: [sage.rings.power\\_series\\_ring\\_element.PowerSeries](#)

Multivariate power series; these are the elements of Multivariate Power Series Rings.

INPUT:

- parent – A multivariate power series.
- x – The element (default: 0). This can be another [MPowerSeries](#) object, or an element of one of the following:
  - the background univariate power series ring
  - the foreground polynomial ring
  - a ring that coerces to one of the above two
- prec – (default: infinity) The precision
- is\_gen – (default: False) Is this element one of the generators?
- check – (default: False) Needed by univariate power series class

EXAMPLES:

Construct multivariate power series from generators:

```

sage: S.<s,t> = PowerSeriesRing(ZZ)
sage: f = s + 4*t + 3*s*t
sage: f in S
True
sage: f = f.add_bigoh(4); f
s + 4*t + 3*s*t + 0(s, t)^4
sage: g = 1 + s + t - s*t + S.0(5); g
1 + s + t - s*t + 0(s, t)^5

sage: T = PowerSeriesRing(GF(3),5,'t'); T

```

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Multivariate Power Series Ring in  $t_0, t_1, t_2, t_3, t_4$  over Finite  
Field of size 3

```
sage: t = T.gens()
sage: w = t[0] - 2*t[1]*t[3] + 5*t[4]^3 - t[0]^3*t[2]^2; w
t0 + t1*t3 - t4^3 - t0^3*t2^2
sage: w = w.add_bigoh(5); w
t0 + t1*t3 - t4^3 + O(t0, t1, t2, t3, t4)^5
sage: w in T
True

sage: w = t[0] - 2*t[0]*t[2] + 5*t[4]^3 - t[0]^3*t[2]^2 + T.O(6)
sage: w
t0 + t0*t2 - t4^3 - t0^3*t2^2 + O(t0, t1, t2, t3, t4)^6
```

Get random elements:

```
sage: S.random_element(4) # random
-2*t + t^2 - 12*s^3 + O(s, t)^4

sage: T.random_element(10) # random
-t1^2*t3^2*t4^2 + t1^5*t3^3*t4 + O(t0, t1, t2, t3, t4)^10
```

Convert elements from polynomial rings:

```
sage: R = PolynomialRing(ZZ, 5, T.variable_names())
sage: t = R.gens()
sage: r = -t[2]*t[3] + t[3]^2 + t[4]^2
sage: T(r)
-t2*t3 + t3^2 + t4^2
sage: r.parent()
Multivariate Polynomial Ring in t0, t1, t2, t3, t4 over Integer Ring
sage: r in T
True
```

$O(\text{prec})$

Return a multivariate power series of precision `prec` obtained by truncating `self` at precision `prec`.

This is the same as `add_bigoh()`.

EXAMPLES:

```
sage: B.<x,y> = PowerSeriesRing(QQ); B
Multivariate Power Series Ring in x, y over Rational Field
sage: r = 1 - x*y + x^2
sage: r.O(4)
1 + x^2 - x*y + O(x, y)^4
sage: r.O(2)
1 + O(x, y)^2
```

Note that this does not change `self`:

```
sage: r
1 + x^2 - x*y
```

$V(n)$ 

If

$$f = \sum a_{m_0, \dots, m_k} x_0^{m_0} \cdots x_k^{m_k},$$

then this function returns

$$\sum a_{m_0, \dots, m_k} x_0^{nm_0} \cdots x_k^{nm_k}.$$

The total-degree precision of the output is  $n$  times the precision of `self`.

EXAMPLES:

```
sage: H = QQ[['x,y,z']]
sage: (x,y,z) = H.gens()
sage: h = -x*y^4*z^7 - 1/4*y*z^12 + 1/2*x^7*y^5*z^2 \
+ 2/3*y^6*z^8 + H.0(15)
sage: h.V(3)
-x^3*y^12*z^21 - 1/4*y^3*z^36 + 1/2*x^21*y^15*z^6 + 2/3*y^18*z^24 + 0(x, y, z)^45
```

**add\_bigoh(*prec*)**Return a multivariate power series of precision *prec* obtained by truncating `self` at precision *prec*.This is the same as `O()`.

EXAMPLES:

```
sage: B.<x,y> = PowerSeriesRing(QQ); B
Multivariate Power Series Ring in x, y over Rational Field
sage: r = 1 - x*y + x^2
sage: r.add_bigoh(4)
1 + x^2 - x*y + 0(x, y)^4
sage: r.add_bigoh(2)
1 + 0(x, y)^2
```

Note that this does not change `self`:

```
sage: r
1 + x^2 - x*y
```

**coefficients()**

Return a dict of monomials and coefficients.

EXAMPLES:

```
sage: R.<s,t> = PowerSeriesRing(ZZ); R
Multivariate Power Series Ring in s, t over Integer Ring
sage: f = 1 + t + s + s*t + R.0(3)
sage: f.coefficients()
{s*t: 1, t: 1, s: 1, 1: 1}
sage: (f^2).coefficients()
{t^2: 1, s*t: 4, s^2: 1, t: 2, s: 2, 1: 1}

sage: g = f^2 + f - 2; g
3*s + 3*t + s^2 + 5*s*t + t^2 + 0(s, t)^3
```

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```

sage: cd = g.coefficients()
sage: g2 = sum(k*v for (k,v) in cd.items()); g2
3*s + 3*t + s^2 + 5*s*t + t^2
sage: g2 == g.truncate()
True

```

**constant\_coefficient()**

Return constant coefficient of self.

EXAMPLES:

```

sage: R.<a,b,c> = PowerSeriesRing(ZZ); R
Multivariate Power Series Ring in a, b, c over Integer Ring
sage: f = 3 + a + b - a*b - b*c - a*c + R.0(4)
sage: f.constant_coefficient()
3
sage: f.constant_coefficient().parent()
Integer Ring

```

**degree()**

Return degree of underlying polynomial of self.

EXAMPLES:

```

sage: B.<x,y> = PowerSeriesRing(QQ)
sage: B
Multivariate Power Series Ring in x, y over Rational Field
sage: r = 1 - x*y + x^2
sage: r = r.add_bigoh(4); r
1 + x^2 - x*y + O(x, y)^4
sage: r.degree()
2

```

**derivative(\*args)**

The formal derivative of this power series, with respect to variables supplied in args.

EXAMPLES:

```

sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = a + b + a^2*b + T.0(5)
sage: f.derivative(a)
1 + 2*a*b + O(a, b)^4
sage: f.derivative(a,2)
2*b + O(a, b)^3
sage: f.derivative(a,a)
2*b + O(a, b)^3
sage: f.derivative([a,a])
2*b + O(a, b)^3
sage: f.derivative(a,5)
0 + O(a, b)^0
sage: f.derivative(a,6)
0 + O(a, b)^0

```

**dict()**

Return underlying dictionary with keys the exponents and values the coefficients of this power series.

EXAMPLES:

```
sage: M = PowerSeriesRing(QQ,4,'t',sparse=True); M
Sparse Multivariate Power Series Ring in t0, t1, t2, t3 over
Rational Field

sage: M.inject_variables()
Defining t0, t1, t2, t3

sage: m = 2/3*t0*t1^15*t3^48 - t0^15*t1^21*t2^28*t3^5
sage: m2 = 1/2*t0^12*t1^29*t2^46*t3^6 - 1/4*t0^39*t1^5*t2^23*t3^30 + M.O(100)
sage: s = m + m2
sage: s.dict()
{(1, 15, 0, 48): 2/3,
 (12, 29, 46, 6): 1/2,
 (15, 21, 28, 5): -1,
 (39, 5, 23, 30): -1/4}
```

**egf()**

Method from univariate power series not yet implemented

**exp**(*prec*=+Infinity)

Exponentiate the formal power series.

INPUT:

- *prec* – Integer or infinity. The degree to truncate the result to.

OUTPUT:

The exponentiated multivariate power series as a new multivariate power series.

EXAMPLES:

```
sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = a + b + a*b + T.O(3)
sage: exp(f)
1 + a + b + 1/2*a^2 + 2*a*b + 1/2*b^2 + O(a, b)^3
sage: f.exp()
1 + a + b + 1/2*a^2 + 2*a*b + 1/2*b^2 + O(a, b)^3
sage: f.exp(prec=2)
1 + a + b + O(a, b)^2
sage: log(exp(f)) - f
0 + O(a, b)^3
```

If the power series has a constant coefficient  $c$  and  $\exp(c)$  is transcendental, then  $\exp(f)$  would have to be a power series over the [SymbolicRing](#). These are not yet implemented and therefore such cases raise an error:

```
sage: g = 2+f
sage: exp(g)
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *: 'Symbolic Ring' and
'Power Series Ring in Tbg over Multivariate Polynomial Ring in a, b
over Rational Field'
```

Another workaround for this limitation is to change base ring to one which is closed under exponentiation, such as  $\mathbf{R}$  or  $\mathbf{C}$ :

```
sage: exp(g.change_ring(RDF))
7.38905609... + 7.38905609...*a + 7.38905609...*b + 3.69452804...*a^2 +
14.7781121...*a*b + 3.69452804...*b^2 + O(a, b)^3
```

If no precision is specified, the default precision is used:

```
sage: T.default_prec()
12
sage: exp(a)
1 + a + 1/2*a^2 + 1/6*a^3 + 1/24*a^4 + 1/120*a^5 + 1/720*a^6 + 1/5040*a^7 +
1/40320*a^8 + 1/362880*a^9 + 1/3628800*a^10 + 1/39916800*a^11 + O(a, b)^12
sage: a.exp(prec=5)
1 + a + 1/2*a^2 + 1/6*a^3 + 1/24*a^4 + O(a, b)^5
sage: exp(a + T.O(5))
1 + a + 1/2*a^2 + 1/6*a^3 + 1/24*a^4 + O(a, b)^5
```

### exponents()

Return a list of tuples which hold the exponents of each monomial of `self`.

EXAMPLES:

```
sage: H = QQ[['x,y']]
sage: (x,y) = H.gens()
sage: h = -y^2 - x*y^3 - 6/5*y^6 - x^7 + 2*x^5*y^2 + H.O(10)
sage: h
-y^2 - x*y^3 - 6/5*y^6 - x^7 + 2*x^5*y^2 + O(x, y)^10
sage: h.exponents()
[(0, 2), (1, 3), (0, 6), (7, 0), (5, 2)]
```

### integral(\*args)

The formal integral of this multivariate power series, with respect to variables supplied in `args`.

The variable sequence `args` can contain both variables and counts; for the syntax, see `derivative_parse()`.

EXAMPLES:

```
sage: T.<a,b> = PowerSeriesRing(QQ,2)
sage: f = a + b + a^2*b + T.O(5)
sage: f.integral(a, 2)
1/6*a^3 + 1/2*a^2*b + 1/12*a^4*b + O(a, b)^7
sage: f.integral(a, b)
1/2*a^2*b + 1/2*a*b^2 + 1/6*a^3*b^2 + O(a, b)^7
sage: f.integral(a, 5)
1/720*a^6 + 1/120*a^5*b + 1/2520*a^7*b + O(a, b)^10
```

Only integration with respect to variables works:

```
sage: f.integral(a+b)
Traceback (most recent call last):
...
ValueError: a + b is not a variable
```

**Warning:** Coefficient division.

If the base ring is not a field (e.g.  $\mathbb{ZZ}$ ), or if it has a non-zero characteristic, (e.g.  $\mathbb{ZZ}/3\mathbb{ZZ}$ ), integration is not always possible while staying with the same base ring. In the first case, Sage will report that it has not been able to coerce some coefficient to the base ring:

```
sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = a + T.O(5)
sage: f.integral(a)
Traceback (most recent call last):
...
TypeError: no conversion of this rational to integer
```

One can get the correct result by changing the base ring first:

```
sage: f.change_ring(QQ).integral(a)
1/2*a^2 + O(a, b)^6
```

However, a correct result is returned even without base change if the denominator cancels:

```
sage: f = 2*b + T.O(5)
sage: f.integral(b)
b^2 + O(a, b)^6
```

In non-zero characteristic, Sage will report that a zero division occurred

```
sage: T.<a,b> = PowerSeriesRing(Zmod(3),2)
sage: (a^3).integral(a)
a^4
sage: (a^2).integral(a)
Traceback (most recent call last):
...
ZeroDivisionError: inverse of Mod(0, 3) does not exist
```

### **is\_nilpotent()**

Return True if `self` is nilpotent. This occurs if

- `self` has finite precision and positive valuation, or
- `self` is constant and nilpotent in base ring.

Otherwise, return False.

**Warning:** This is so far just a sufficient condition, so don't trust a False output to be legit!

**Todo:** What should we do about this method? Is nilpotency of a power series even decidable (assuming a nilpotency oracle in the base ring)? And I am not sure that returning True just because the series has finite precision and zero constant term is a good idea.

### EXAMPLES:

```
sage: R.<a,b,c> = PowerSeriesRing(Zmod(8)); R
Multivariate Power Series Ring in a, b, c over Ring of integers
modulo 8
```

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```

sage: f = a + b + c + a^2*c
sage: f.is_nilpotent()
False
sage: f = f.0(4); f
a + b + c + a^2*c + 0(a, b, c)^4
sage: f.is_nilpotent()
True

sage: g = R(2)
sage: g.is_nilpotent()
True
sage: (g.0(4)).is_nilpotent()
True

sage: S = R.change_ring(QQ)
sage: S(g).is_nilpotent()
False
sage: S(g.0(4)).is_nilpotent()
False

```

**is\_square()**

Method from univariate power series not yet implemented.

**is\_unit()**

A multivariate power series is a unit if and only if its constant coefficient is a unit.

EXAMPLES:

```

sage: R.<a,b> = PowerSeriesRing(ZZ); R
Multivariate Power Series Ring in a, b over Integer Ring
sage: f = 2 + a^2 + a*b + a^3 + R.0(9)
sage: f.is_unit()
False
sage: f.base_extend(QQ).is_unit()
True
sage: (0(a,b)^0).is_unit()
False

```

**laurent\_series()**

Not implemented for multivariate power series.

**list()**

Doesn't make sense for multivariate power series. Multivariate polynomials don't have list of coefficients either.

**log(*prec*=+ Infinity)**

Return the logarithm of the formal power series.

INPUT:

- *prec* – Integer or infinity. The degree to truncate the result to.

OUTPUT:

The logarithm of the multivariate power series as a new multivariate power series.

EXAMPLES:

```

sage: T.<a,b> = PowerSeriesRing(ZZ,2)
sage: f = 1 + a + b + a*b + T.O(5)
sage: f.log()
a + b - 1/2*a^2 - 1/2*b^2 + 1/3*a^3 + 1/3*b^3 - 1/4*a^4 - 1/4*b^4 + O(a, b)^5
sage: log(f)
a + b - 1/2*a^2 - 1/2*b^2 + 1/3*a^3 + 1/3*b^3 - 1/4*a^4 - 1/4*b^4 + O(a, b)^5
sage: exp(log(f)) - f
0 + O(a, b)^5

```

If the power series has a constant coefficient  $c$  and  $\exp(c)$  is transcendental, then  $\exp(f)$  would have to be a power series over the [SymbolicRing](#). These are not yet implemented and therefore such cases raise an error:

```

sage: g = 2+f
sage: log(g)
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for -: 'Symbolic Ring' and 'Power
Series Ring in Tbg over Multivariate Polynomial Ring in a, b over Rational Field
↪ '

```

Another workaround for this limitation is to change base ring to one which is closed under exponentiation, such as  $\mathbf{R}$  or  $\mathbf{C}$ :

```

sage: log(g.change_ring(RDF))
1.09861228... + 0.333333333...*a + 0.333333333...*b - 0.055555555...*a^2
+ 0.222222222...*a*b - 0.055555555...*b^2 + 0.0123456790...*a^3
- 0.0740740740...*a^2*b - 0.0740740740...*a*b^2 + 0.0123456790...*b^3
- 0.00308641975...*a^4 + 0.0246913580...*a^3*b + 0.0246913580...*a*b^3
- 0.00308641975...*b^4 + O(a, b)^5

```

### **monomials()**

Return a list of monomials of `self`.

These are the keys of the dict returned by [coefficients\(\)](#).

EXAMPLES:

```

sage: R.<a,b,c> = PowerSeriesRing(ZZ); R
Multivariate Power Series Ring in a, b, c over Integer Ring
sage: f = 1 + a + b - a*b - b*c - a*c + R.O(4)
sage: sorted(f.monomials())
[b*c, a*c, a*b, b, a, 1]
sage: f = 1 + 2*a + 7*b - 2*a*b - 4*b*c - 13*a*c + R.O(4)
sage: sorted(f.monomials())
[b*c, a*c, a*b, b, a, 1]
sage: f = R.zero()
sage: f.monomials()
[]

```

### **ogf()**

Method from univariate power series not yet implemented

### **padded\_list()**

Method from univariate power series not yet implemented.



**polynomial()**

Return the underlying polynomial of `self` as an element of the underlying multivariate polynomial ring (the “foreground polynomial ring”).

EXAMPLES:

```
sage: M = PowerSeriesRing(QQ,4,'t'); M
Multivariate Power Series Ring in t0, t1, t2, t3 over Rational
Field
sage: t = M.gens()
sage: f = 1/2*t[0]^3*t[1]^3*t[2]^2 + 2/3*t[0]*t[2]^6*t[3] - t[0]^
3*t[1]^3*t[3]^3 - 1/4*t[0]*t[1]*t[2]^7 + M.O(10)
sage: f
1/2*t0^3*t1^3*t2^2 + 2/3*t0*t2^6*t3 - t0^3*t1^3*t3^3
- 1/4*t0*t1*t2^7 + O(t0, t1, t2, t3)^10
sage: f.polynomial()
1/2*t0^3*t1^3*t2^2 + 2/3*t0*t2^6*t3 - t0^3*t1^3*t3^3
- 1/4*t0*t1*t2^7
sage: f.polynomial().parent()
Multivariate Polynomial Ring in t0, t1, t2, t3 over Rational Field
```

Contrast with `truncate()`:

```
sage: f.truncate()
1/2*t0^3*t1^3*t2^2 + 2/3*t0*t2^6*t3 - t0^3*t1^3*t3^3 - 1/4*t0*t1*t2^7
sage: f.truncate().parent()
Multivariate Power Series Ring in t0, t1, t2, t3 over Rational Field
```

**prec()**

Return precision of `self`.

EXAMPLES:

```
sage: R.<a,b,c> = PowerSeriesRing(ZZ); R
Multivariate Power Series Ring in a, b, c over Integer Ring
sage: f = 3 + a + b - a*b - b*c - a*c + R.O(4)
sage: f.prec()
4
sage: f.truncate().prec()
+Infinity
```

**quo\_rem(*other*, *precision=None*)**

Return the pair of quotient and remainder for the increasing power division of `self` by `other`.

If  $a$  and  $b$  are two elements of a power series ring  $R[[x_1, x_2, \dots, x_n]]$  such that the trailing term of  $b$  is invertible in  $R$ , then the pair of quotient and remainder for the increasing power division of  $a$  by  $b$  is the unique pair  $(u, v) \in R[[x_1, x_2, \dots, x_n]] \times R[[x_1, x_2, \dots, x_n]]$  such that  $a = bu + v$  and such that no monomial appearing in  $v$  divides the trailing monomial (`trailing_monomial()`) of  $b$ . Note that this depends on the order of the variables.

This method returns both quotient and remainder as power series, even though in mathematics, the remainder for the increasing power division of two power series is a polynomial. This is because Sage’s power series come with a precision, and that precision is not always sufficient to determine the remainder completely. Disregarding this issue, the `polynomial()` method can be used to recast the remainder as an actual polynomial.

INPUT:

- **other** – an element of the same power series ring as **self** such that the trailing term of **other** is invertible in **self** (this is automatically satisfied if the base ring is a field, unless **other** is zero)
- **precision** – (default: the default precision of the parent of **self**) nonnegative integer, determining the precision to be cast on the resulting quotient and remainder if both **self** and **other** have infinite precision (ignored otherwise); note that the resulting precision might be lower than this integer

EXAMPLES:

```
sage: R.<a,b,c> = PowerSeriesRing(ZZ)
sage: f = 1 + a + b - a*b + R.O(3)
sage: g = 1 + 2*a - 3*a*b + R.O(3)
sage: q, r = f.quo_rem(g); q, r
(1 - a + b + 2*a^2 + 0(a, b, c)^3, 0 + 0(a, b, c)^3)
sage: f == q*g+r
True

sage: q, r = (a*f).quo_rem(g); q, r
(a - a^2 + a*b + 2*a^3 + 0(a, b, c)^4, 0 + 0(a, b, c)^4)
sage: a*f == q*g+r
True

sage: q, r = (a*f).quo_rem(a*g); q, r
(1 - a + b + 2*a^2 + 0(a, b, c)^3, 0 + 0(a, b, c)^4)
sage: a*f == q*(a*g)+r
True

sage: q, r = (a*f).quo_rem(b*g); q, r
(a - 3*a^2 + 0(a, b, c)^3, a + a^2 + 0(a, b, c)^4)
sage: a*f == q*(b*g)+r
True
```

Trying to divide two polynomials, we run into the issue that there is no natural setting for the precision of the quotient and remainder (and if we wouldn't set a precision, the algorithm would never terminate). Here, default precision comes to our help:

```
sage: (1+a^3).quo_rem(a+a^2)
(a^2 - a^3 + a^4 - a^5 + a^6 - a^7 + a^8 - a^9 + a^10 + 0(a, b, c)^11, 1 + 0(a, b, c)^12)

sage: (1+a^3+a*b).quo_rem(b+c)
(a + 0(a, b, c)^11, 1 - a*c + a^3 + 0(a, b, c)^12)
sage: (1+a^3+a*b).quo_rem(b+c, precision=17)
(a + 0(a, b, c)^16, 1 - a*c + a^3 + 0(a, b, c)^17)

sage: (a^2+b^2+c^2).quo_rem(a+b+c)
(a - b - c + 0(a, b, c)^11, 2*b^2 + 2*b*c + 2*c^2 + 0(a, b, c)^12)

sage: (a^2+b^2+c^2).quo_rem(1/(1+a+b+c))
(a^2 + b^2 + c^2 + a^3 + a^2*b + a^2*c + a*b^2 + a*c^2 + b^3 + b^2*c + b*c^2 + a^4 + a^3*b + a^3*c + a^2*b^2 + a^2*b*c + a^2*c^2 + a*b^3 + a*b^2*c + a*b*c^2 + b^4 + b^3*c + b^2*c^2 + b*c^3 + 0(a, b, c)^14, 0)
```

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```
sage: (a^2+b^2+c^2).quo_rem(a/(1+a+b+c))
(a + a^2 + a*b + a*c + 0(a, b, c)^13, b^2 + c^2)

sage: (1+a+a^15).quo_rem(a^2)
(0 + 0(a, b, c)^10, 1 + a + 0(a, b, c)^12)

sage: (1+a+a^15).quo_rem(a^2, precision=15)
(0 + 0(a, b, c)^13, 1 + a + 0(a, b, c)^15)

sage: (1+a+a^15).quo_rem(a^2, precision=16)
(a^13 + 0(a, b, c)^14, 1 + a + 0(a, b, c)^16)
```

Illustrating the dependency on the ordering of variables:

```
sage: (1+a+b).quo_rem(b+c)
(1 + 0(a, b, c)^11, 1 + a - c + 0(a, b, c)^12)

sage: (1+b+c).quo_rem(c+a)
(0 + 0(a, b, c)^11, 1 + b + c + 0(a, b, c)^12)

sage: (1+c+a).quo_rem(a+b)
(1 + 0(a, b, c)^11, 1 - b + c + 0(a, b, c)^12)
```

#### **shift(*n*)**

Doesn't make sense for multivariate power series.

#### **solve\_linear\_de(*prec*=+Infinity, *b*=None, *f0*=None)**

Not implemented for multivariate power series.

#### **sqrt()**

Method from univariate power series not yet implemented. Depends on square root method for multivariate polynomials.

#### **square\_root()**

Method from univariate power series not yet implemented. Depends on square root method for multivariate polynomials.

#### **trailing\_monomial()**

Return the trailing monomial of *self*.

This is defined here as the lowest term of the underlying polynomial.

EXAMPLES:

```
sage: R.<a,b,c> = PowerSeriesRing(ZZ)
sage: f = 1 + a + b - a*b + R.0(3)
sage: f.trailing_monomial()
1
sage: f = a^2*b^3*f; f
a^2*b^3 + a^3*b^3 + a^2*b^4 - a^3*b^4 + 0(a, b, c)^8
sage: f.trailing_monomial()
a^2*b^3
```

#### **truncate(*prec*=+Infinity)**

Return infinite precision multivariate power series formed by truncating *self* at precision *prec*.

EXAMPLES:

```
sage: M = PowerSeriesRing(QQ,4,'t'); M
Multivariate Power Series Ring in t0, t1, t2, t3 over Rational Field
```

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```

sage: t = M.gens()
sage: f = 1/2*t[0]^3*t[1]^3*t[2]^2 + 2/3*t[0]*t[2]^6*t[3] - t[0]^
3*t[1]^3*t[3]^3 - 1/4*t[0]*t[1]*t[2]^7 + M.O(10)
sage: f
1/2*t0^3*t1^3*t2^2 + 2/3*t0*t2^6*t3 - t0^3*t1^3*t3^3
- 1/4*t0*t1*t2^7 + O(t0, t1, t2, t3)^10

sage: f.truncate()
1/2*t0^3*t1^3*t2^2 + 2/3*t0*t2^6*t3 - t0^3*t1^3*t3^3
- 1/4*t0*t1*t2^7
sage: f.truncate().parent()
Multivariate Power Series Ring in t0, t1, t2, t3 over Rational Field

```

Contrast with polynomial:

```

sage: f.polynomial()
1/2*t0^3*t1^3*t2^2 + 2/3*t0*t2^6*t3 - t0^3*t1^3*t3^3 - 1/4*t0*t1*t2^7
sage: f.polynomial().parent()
Multivariate Polynomial Ring in t0, t1, t2, t3 over Rational Field

```

### valuation()

Return the valuation of `self`.

The valuation of a power series  $f$  is the highest nonnegative integer  $k$  less or equal to the precision of  $f$  and such that the coefficient of  $f$  before each term of degree  $< k$  is zero. (If such an integer does not exist, then the valuation is the precision of  $f$  itself.)

EXAMPLES:

```

sage: R.<a,b> = PowerSeriesRing(GF(4949717)); R
Multivariate Power Series Ring in a, b over Finite Field of
size 4949717
sage: f = a^2 + a*b + a^3 + R.O(9)
sage: f.valuation()
2
sage: g = 1 + a + a^3
sage: g.valuation()
0
sage: R.zero().valuation()
+Infinity

```

### valuation\_zero\_part()

Doesn't make sense for multivariate power series; valuation zero with respect to which variable?

### variable()

Doesn't make sense for multivariate power series.

### variables()

Return tuple of variables occurring in `self`.

EXAMPLES:

```

sage: T = PowerSeriesRing(GF(3),5,'t'); T
Multivariate Power Series Ring in t0, t1, t2, t3, t4 over
Finite Field of size 3

```

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```
sage: t = T.gens()
sage: w = t[0] - 2*t[0]*t[2] + 5*t[4]^3 - t[0]^3*t[2]^2 + T.O(6)
sage: w
t0 + t0*t2 - t4^3 - t0^3*t2^2 + O(t0, t1, t2, t3, t4)^6
sage: w.variables()
(t0, t2, t4)
```

`sage.rings.multi_power_series_ring_element.is_MPowerSeries(f)`

Return True if *f* is a multivariate power series.



## LAURENT SERIES RINGS

EXAMPLES:

```
sage: R = LaurentSeriesRing(QQ, "x")
sage: R.base_ring()
Rational Field
sage: S = LaurentSeriesRing(GF(17)['x'], 'y')
sage: S
Laurent Series Ring in y over Univariate Polynomial Ring in x over
Finite Field of size 17
sage: S.base_ring()
Univariate Polynomial Ring in x over Finite Field of size 17
```

See also:

- `sage.misc.defaults.set_series_precision()`

```
class sage.rings.laurent_series_ring.LaurentSeriesRing(power_series)
    Bases: sage.structure.unique_representation.UniqueRepresentation, sage.rings.ring.
            CommutativeRing
    Univariate Laurent Series Ring.
```

EXAMPLES:

```
sage: R = LaurentSeriesRing(QQ, 'x'); R
Laurent Series Ring in x over Rational Field
sage: x = R.0
sage: g = 1 - x + x^2 - x^4 + 0(x^8); g
1 - x + x^2 - x^4 + 0(x^8)
sage: g = 10*x^(-3) + 2006 - 19*x + x^2 - x^4 + 0(x^8); g
10*x^-3 + 2006 - 19*x + x^2 - x^4 + 0(x^8)
```

You can also use more mathematical notation when the base is a field:

```
sage: Frac(QQ[['x']])
Laurent Series Ring in x over Rational Field
sage: Frac(GF(5)['y'])
Fraction Field of Univariate Polynomial Ring in y over Finite Field of size 5
```

When the base ring is a domain, the fraction field is the Laurent series ring over the fraction field of the base ring:

```
sage: Frac(ZZ[['t']])
Laurent Series Ring in t over Rational Field
```

Laurent series rings are determined by their variable and the base ring, and are globally unique:

```
sage: K = Qp(5, prec = 5)
sage: L = Qp(5, prec = 200)
sage: R.<x> = LaurentSeriesRing(K)
sage: S.<y> = LaurentSeriesRing(L)
sage: R is S
False
sage: T.<y> = LaurentSeriesRing(Qp(5,prec=200))
sage: S is T
True
sage: W.<y> = LaurentSeriesRing(Qp(5,prec=199))
sage: W is T
False

sage: K = LaurentSeriesRing(CC, 'q')
sage: K
Laurent Series Ring in q over Complex Field with 53 bits of precision
sage: loads(K.dumps()) == K
True
sage: P = QQ[['x']]
sage: F = Frac(P)
sage: TestSuite(F).run()
```

When the base ring  $k$  is a field, the ring  $k((x))$  is a CDVF, that is a field equipped with a discrete valuation for which it is complete. The appropriate (sub)category is automatically set in this case:

```
sage: k = GF(11)
sage: R.<x> = k[[[]]]
sage: F = Frac(R)
sage: F.category()
Join of Category of complete discrete valuation fields and Category of commutative_
↪algebras over (finite enumerated fields and subquotients of monoids and quotients_
↪of semigroups) and Category of infinite sets
sage: TestSuite(F).run()
```

### Element

alias of `sage.rings.laurent_series_ring_element.LaurentSeries`

### base\_extend(R)

Return the Laurent series ring over  $R$  in the same variable as self, assuming there is a canonical coerce map from the base ring of self to  $R$ .

EXAMPLES:

```
sage: K.<x> = LaurentSeriesRing(QQ, default_prec=4)
sage: K.base_extend(QQ[['t']])
Laurent Series Ring in x over Univariate Polynomial Ring in t over Rational_
↪Field
```

### change\_ring(R)

EXAMPLES:



```

sage: K.<x> = LaurentSeriesRing(QQ, default_prec=4)
sage: R = K.change_ring(ZZ); R
Laurent Series Ring in x over Integer Ring
sage: R.default_prec()
4

```

**characteristic()**

EXAMPLES:

```

sage: R.<x> = LaurentSeriesRing(GF(17))
sage: R.characteristic()
17

```

**construction()**

Return the functorial construction of this Laurent power series ring.

The construction is given as the completion of the Laurent polynomials.

EXAMPLES:

```

sage: L.<t> = LaurentSeriesRing(ZZ, default_prec=42)
sage: phi, arg = L.construction()
sage: phi
Completion[t, prec=42]
sage: arg
Univariate Laurent Polynomial Ring in t over Integer Ring
sage: phi(arg) is L
True

```

Because of this construction, pushout is automatically available:

```

sage: 1/2 * t
1/2*t
sage: parent(1/2 * t)
Laurent Series Ring in t over Rational Field

sage: QQbar.gen() * t
I*t
sage: parent(QQbar.gen() * t)
Laurent Series Ring in t over Algebraic Field

```

**default\_prec()**

Get the precision to which exact elements are truncated when necessary (most frequently when inverting).

EXAMPLES:

```

sage: R.<x> = LaurentSeriesRing(QQ, default_prec=5)
sage: R.default_prec()
5

```

**fraction\_field()**

Return the fraction field of this ring of Laurent series.

If the base ring is a field, then Laurent series are already a field. If the base ring is a domain, then the Laurent series over its fraction field is returned. Otherwise, raise a `ValueError`.

EXAMPLES:

```

sage: R = LaurentSeriesRing(ZZ, 't', 30).fraction_field()
sage: R
Laurent Series Ring in t over Rational Field
sage: R.default_prec()
30

sage: LaurentSeriesRing(Zmod(4), 't').fraction_field()
Traceback (most recent call last):
...
ValueError: must be an integral domain

```

**gen**( $n=0$ )

EXAMPLES:

```

sage: R = LaurentSeriesRing(QQ, "x")
sage: R.gen()
x

```

**is\_dense**()

EXAMPLES:

```

sage: K.<x> = LaurentSeriesRing(QQ, sparse=True)
sage: K.is_dense()
False

```

**is\_exact**()

Laurent series rings are inexact.

EXAMPLES:

```

sage: R = LaurentSeriesRing(QQ, "x")
sage: R.is_exact()
False

```

**is\_field**(*proof=True*)

A Laurent series ring is a field if and only if the base ring is a field.

**is\_sparse**()

Return if self is a sparse implementation.

EXAMPLES:

```

sage: K.<x> = LaurentSeriesRing(QQ, sparse=True)
sage: K.is_sparse()
True

```

**laurent\_polynomial\_ring**()

If this is the Laurent series ring  $R((t))$ , return the Laurent polynomial ring  $R[t, 1/t]$ .

EXAMPLES:

```

sage: R = LaurentSeriesRing(QQ, "x")
sage: R.laurent_polynomial_ring()
Univariate Laurent Polynomial Ring in x over Rational Field

```

**ngens**()

Laurent series rings are univariate.

EXAMPLES:

```
sage: R = LaurentSeriesRing(QQ, "x")
sage: R.ngens()
1
```

### **polynomial\_ring()**

If this is the Laurent series ring  $R((t))$ , return the polynomial ring  $R[t]$ .

EXAMPLES:

```
sage: R = LaurentSeriesRing(QQ, "x")
sage: R.polynomial_ring()
Univariate Polynomial Ring in x over Rational Field
```

### **power\_series\_ring()**

If this is the Laurent series ring  $R((t))$ , return the power series ring  $R[[t]]$ .

EXAMPLES:

```
sage: R = LaurentSeriesRing(QQ, "x")
sage: R.power_series_ring()
Power Series Ring in x over Rational Field
```

### **random\_element(*algorithm='default'*)**

Return a random element of this Laurent series ring.

The optional *algorithm* parameter decides how elements are generated. Algorithms currently implemented:

- 'default': Choose an integer *shift* using the standard distribution on the integers. Then choose a list of coefficients using the *random\_element* function of the base ring, and construct a new element based on those coefficients, so that the *i*-th coefficient corresponds to the (*i*+*shift*)-th power of the uniformizer. The amount of coefficients is determined by the *default\_prec* of the ring. Note that this method only creates non-exact elements.

EXAMPLES:

```
sage: S.<s> = LaurentSeriesRing(GF(3))
sage: S.random_element() # random
s^-8 + s^-7 + s^-6 + s^-5 + s^-1 + s + s^3 + s^4
+ s^5 + 2*s^6 + s^7 + s^11 + 0(s^12)
```

### **residue\_field()**

Return the residue field of this Laurent series field if it is a complete discrete valuation field (i.e. if the base ring is a field, in which case it is also the residue field).

EXAMPLES:

```
sage: R.<x> = LaurentSeriesRing(GF(17))
sage: R.residue_field()
Finite Field of size 17

sage: R.<x> = LaurentSeriesRing(ZZ)
sage: R.residue_field()
Traceback (most recent call last):
...
TypeError: the base ring is not a field
```

**uniformizer()**

Return a uniformizer of this Laurent series field if it is a discrete valuation field (i.e. if the base ring is actually a field). Otherwise, an error is raised.

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ)
sage: R.uniformizer()
t

sage: R.<t> = LaurentSeriesRing(ZZ)
sage: R.uniformizer()
Traceback (most recent call last):
...
TypeError: the base ring is not a field
```

**sage.rings.laurent\_series\_ring.is\_LaurentSeriesRing(*x*)**

Return True if this is a *univariate* Laurent series ring.

This is in keeping with the behavior of `is_PolynomialRing` versus `is_MPolynomialRing`.

## LAURENT SERIES

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(GF(7), 't'); R
Laurent Series Ring in t over Finite Field of size 7
sage: f = 1/(1-t+O(t^10)); f
1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + O(t^10)
```

Laurent series are immutable:

```
sage: f[2]
1
sage: f[2] = 5
Traceback (most recent call last):
...
IndexError: Laurent series are immutable
```

We compute with a Laurent series over the complex mpfr numbers.

```
sage: K.<q> = Frac(CC[['q']])
sage: K
Laurent Series Ring in q over Complex Field with 53 bits of precision
sage: q
1.0000000000000000*q
```

Saving and loading.

```
sage: loads(q.dumps()) == q
True
sage: loads(K.dumps()) == K
True
```

IMPLEMENTATION: Laurent series in Sage are represented internally as a power of the variable times the unit part (which need not be a unit - it's a polynomial with nonzero constant term). The zero Laurent series has unit part 0.

AUTHORS:

- William Stein: original version
- David Joyner (2006-01-22): added examples
- Robert Bradshaw (2007-04): optimizations, shifting
- Robert Bradshaw: Cython version

**class** sage.rings.laurent\_series\_ring\_element.LaurentSeries

Bases: [sage.structure.element.AlgebraElement](#)

A Laurent Series.

We consider a Laurent series of the form  $t^n \cdot f$  where  $f$  is a power series.

INPUT:

- **parent** – a Laurent series ring
- **f** – a power series (or something can be coerced to one); note that **f** does *not* have to be a unit
- **n** – (default: 0) integer

**O(prec)**

Return the Laurent series of precision at most **prec** obtained by adding  $O(q^{\text{prec}})$ , where  $q$  is the variable.

The precision of **self** and the integer **prec** can be arbitrary. The resulting Laurent series will have precision equal to the minimum of the precision of **self** and **prec**. The term  $O(q^{\text{prec}})$  is the zero series with precision **prec**.

See also [add\\_bigoh\(\)](#).

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ)
sage: f = t^-5 + t^-4 + t^3 + O(t^10); f
t^-5 + t^-4 + t^3 + O(t^10)
sage: f.O(-4)
t^-5 + O(t^-4)
sage: f.O(15)
t^-5 + t^-4 + t^3 + O(t^10)
```

**V(n)**

Return the  $n$ -th Verschiebung of **self**.

If  $f = \sum a_m x^m$  then this function returns  $\sum a_m x^{mn}$ .

EXAMPLES:

```
sage: R.<x> = LaurentSeriesRing(QQ)
sage: f = -1/x + 1 + 2*x^2 + 5*x^5
sage: f.V(2)
-x^-2 + 1 + 2*x^4 + 5*x^10
sage: f.V(-1)
5*x^-5 + 2*x^-2 + 1 - x
sage: h = f.add_bigoh(7)
sage: h.V(2)
-x^-2 + 1 + 2*x^4 + 5*x^10 + O(x^14)
sage: h.V(-2)
Traceback (most recent call last):
...
ValueError: For finite precision only positive arguments allowed
```

**add\_bigoh(prec)**

Return the truncated series at chosen precision **prec**.

See also [O\(\)](#).

INPUT:

- `prec` – the precision of the series as an integer

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ)
sage: f = t^2 + t^3 + O(t^10); f
t^2 + t^3 + O(t^10)
sage: f.add_bigoh(5)
t^2 + t^3 + O(t^5)
```

### `change_ring(R)`

Change the base ring of `self`.

EXAMPLES:

```
sage: R.<q> = LaurentSeriesRing(ZZ)
sage: p = R([1,2,3]); p
1 + 2*q + 3*q^2
sage: p.change_ring(GF(2))
1 + q^2
```

### `coefficients()`

Return the nonzero coefficients of `self`.

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ)
sage: f = -5/t^(2) + t + t^2 - 10/3*t^3
sage: f.coefficients()
[-5, 1, 1, -10/3]
```

### `common_prec(other)`

Return the minimum precision of `self` and `other`.

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ)
```

```
sage: f = t^(-1) + t + t^2 + O(t^3)
sage: g = t + t^3 + t^4 + O(t^4)
sage: f.common_prec(g)
3
sage: g.common_prec(f)
3
```

```
sage: f = t + t^2 + O(t^3)
sage: g = t^(-3) + t^2
sage: f.common_prec(g)
3
sage: g.common_prec(f)
3
```

```
sage: f = t + t^2
sage: g = t^2
sage: f.common_prec(g)
+Infinity
```

```
sage: f = t^(-3) + O(t^(-2))
sage: g = t^(-5) + O(t^(-1))
sage: f.common_prec(g)
-2
```

```
sage: f = O(t^2)
sage: g = O(t^5)
sage: f.common_prec(g)
2
```

**common\_valuation(*other*)**

Return the minimum valuation of *self* and *other*.

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ)
```

```
sage: f = t^(-1) + t + t^2 + O(t^3)
sage: g = t + t^3 + t^4 + O(t^4)
sage: f.common_valuation(g)
-1
sage: g.common_valuation(f)
-1
```

```
sage: f = t + t^2 + O(t^3)
sage: g = t^(-3) + t^2
sage: f.common_valuation(g)
-3
sage: g.common_valuation(f)
-3
```

```
sage: f = t + t^2
sage: g = t^2
sage: f.common_valuation(g)
1
```

```
sage: f = t^(-3) + O(t^(-2))
sage: g = t^(-5) + O(t^(-1))
sage: f.common_valuation(g)
-5
```

```
sage: f = O(t^2)
sage: g = O(t^5)
sage: f.common_valuation(g)
+Infinity
```

**degree()**

Return the degree of a polynomial equivalent to this power series modulo big oh of the precision.

EXAMPLES:



```

sage: x = Frac(QQ[['x']]).0
sage: g = x^2 - x^4 + 0(x^8)
sage: g.degree()
4
sage: g = -10/x^5 + x^2 - x^4 + 0(x^8)
sage: g.degree()
4
sage: (x^-2 + 0(x^0)).degree()
-2

```

**derivative(\*args)**

The formal derivative of this Laurent series, with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied; see documentation for the global derivative() function for more details.

See also:

`_derivative()`

EXAMPLES:

```

sage: R.<x> = LaurentSeriesRing(QQ)
sage: g = 1/x^10 - x + x^2 - x^4 + 0(x^8)
sage: g.derivative()
-10*x^-11 - 1 + 2*x - 4*x^3 + 0(x^7)
sage: g.derivative(x)
-10*x^-11 - 1 + 2*x - 4*x^3 + 0(x^7)

```

```

sage: R.<t> = PolynomialRing(ZZ)
sage: S.<x> = LaurentSeriesRing(R)
sage: f = 2*t/x + (3*t^2 + 6*t)*x + 0(x^2)
sage: f.derivative()
-2*t*x^-2 + (3*t^2 + 6*t) + 0(x)
sage: f.derivative(x)
-2*t*x^-2 + (3*t^2 + 6*t) + 0(x)
sage: f.derivative(t)
2*x^-1 + (6*t + 6)*x + 0(x^2)

```

**exponents()**

Return the exponents appearing in self with nonzero coefficients.

EXAMPLES:

```

sage: R.<t> = LaurentSeriesRing(QQ)
sage: f = -5/t^2 + t + t^2 - 10/3*t^3
sage: f.exponents()
[-2, 1, 2, 3]

```

**integral()**

The formal integral of this Laurent series with 0 constant term.

EXAMPLES: The integral may or may not be defined if the base ring is not a field.

```

sage: t = LaurentSeriesRing(ZZ, 't').0
sage: f = 2*t^-3 + 3*t^2 + 0(t^4)

```

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```
sage: f.integral()
-t^-2 + t^3 + O(t^5)
```

```
sage: f = t^3
sage: f.integral()
Traceback (most recent call last):
...
ArithmeticError: Coefficients of integral cannot be coerced into the base ring
```

The integral of  $1/t$  is  $\log(t)$ , which is not given by a Laurent series:

```
sage: t = Frac(QQ[['t']]).0
sage: f = -1/t^3 - 31/t + O(t^3)
sage: f.integral()
Traceback (most recent call last):
...
ArithmeticError: The integral of is not a Laurent series, since t^-1 has
↪ nonzero coefficient.
```

Another example with just one negative coefficient:

```
sage: A.<t> = QQ[[]]
sage: f = -2*t^(-4) + O(t^8)
sage: f.integral()
2/3*t^-3 + O(t^9)
sage: f.integral().derivative() == f
True
```

### **inverse()**

Return the inverse of self, i.e.,  $\text{self}^{-1}$ .

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(ZZ)
sage: t.inverse()
t^-1
sage: (1-t).inverse()
1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 + ...
```

### **is\_monomial()**

Return True if this element is a monomial. That is, if self is  $x^n$  for some integer  $n$ .

EXAMPLES:

```
sage: k.<z> = LaurentSeriesRing(QQ, 'z')
sage: (30*z).is_monomial()
False
sage: k(1).is_monomial()
True
sage: (z+1).is_monomial()
False
sage: (z^-2909).is_monomial()
True
```

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```
sage: (3*z^-2909).is_monomial()
False
```

**is\_unit()**

Return True if this is Laurent series is a unit in this ring.

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ)
sage: (2+t).is_unit()
True
sage: f = 2+t^2+O(t^10); f.is_unit()
True
sage: 1/f
1/2 - 1/4*t^2 + 1/8*t^4 - 1/16*t^6 + 1/32*t^8 + O(t^10)
sage: R(0).is_unit()
False
sage: R.<s> = LaurentSeriesRing(ZZ)
sage: f = 2 + s^2 + O(s^10)
sage: f.is_unit()
False
sage: 1/f
Traceback (most recent call last):
...
ValueError: constant term 2 is not a unit
```

ALGORITHM: A Laurent series is a unit if and only if its “unit part” is a unit.

**is\_zero()**

EXAMPLES:

```
sage: x = Frac(QQ[['x']]).0
sage: f = 1/x + x + x^2 + 3*x^4 + O(x^7)
sage: f.is_zero()
0
sage: z = 0*f
sage: z.is_zero()
1
```

**laurent\_polynomial()**

Return the corresponding Laurent polynomial.

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ)
sage: f = t^-3 + t + 7*t^2 + O(t^5)
sage: g = f.laurent_polynomial(); g
t^-3 + t + 7*t^2
sage: g.parent()
Univariate Laurent Polynomial Ring in t over Rational Field
```

**lift\_to\_precision(absprec=None)**

Return a congruent Laurent series with absolute precision at least absprec.

INPUT:

- `absprec` – an integer or `None` (default: `None`), the absolute precision of the result. If `None`, lifts to an exact element.

EXAMPLES:

```
sage: A.<t> = LaurentSeriesRing(GF(5))
sage: x = t^(-1) + t^2 + O(t^5)
sage: x.lift_to_precision(10)
t^-1 + t^2 + O(t^10)
sage: x.lift_to_precision()
t^-1 + t^2
```

**list()**

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ)
sage: f = -5/t^2 + t + t^2 - 10/3*t^3
sage: f.list()
[-5, 0, 0, 1, 1, -10/3]
```

**nth\_root(*n*, *prec*=None)**

Return the *n*-th root of this Laurent power series.

INPUT:

- *n* – integer
- *prec* – integer (optional) - precision of the result. Though, if this series has finite precision, then the result cannot have larger precision.

EXAMPLES:

```
sage: R.<x> = LaurentSeriesRing(QQ)
sage: (x^-2 + 1 + x).nth_root(2)
x^-1 + 1/2*x + 1/2*x^2 - ... - 19437/65536*x^18 + O(x^19)
sage: (x^-2 + 1 + x).nth_root(2)**2
x^-2 + 1 + x + O(x^18)

sage: j = j_invariant_qexp()
sage: q = j.parent().gen()
sage: j(q^3).nth_root(3)
q^-1 + 248*q^2 + 4124*q^5 + ... + O(q^29)
sage: (j(q^2) - 1728).nth_root(2)
q^-1 - 492*q - 22590*q^3 - ... + O(q^19)
```

**power\_series()**

Convert this Laurent series to a power series.

An error is raised if the Laurent series has a term (or an error term  $O(x^k)$ ) whose exponent is negative.

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(ZZ)
sage: f = 1/(1-t+O(t^10)); f.parent()
Laurent Series Ring in t over Integer Ring
sage: g = f.power_series(); g
1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + O(t^10)
```

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```

sage: parent(g)
Power Series Ring in t over Integer Ring
sage: f = 3/t^2 + t^2 + t^3 + O(t^10)
sage: f.power_series()
Traceback (most recent call last):
...
TypeError: self is not a power series

```

**prec()**

This function returns the  $n$  so that the Laurent series is of the form (stuff) +  $O(t^n)$ . It doesn't matter how many negative powers appear in the expansion. In particular, `prec` could be negative.

EXAMPLES:

```

sage: x = Frac(QQ[['x']]).0
sage: f = x^2 + 3*x^4 + O(x^7)
sage: f.prec()
7
sage: g = 1/x^10 - x + x^2 - x^4 + O(x^8)
sage: g.prec()
8

```

**precision\_absolute()**

Return the absolute precision of this series.

By definition, the absolute precision of  $\dots + O(x^r)$  is  $r$ .

EXAMPLES:

```

sage: R.<t> = ZZ[[t]]
sage: (t^2 + O(t^3)).precision_absolute()
3
sage: (1 - t^2 + O(t^100)).precision_absolute()
100

```

**precision\_relative()**

Return the relative precision of this series, that is the difference between its absolute precision and its valuation.

By convention, the relative precision of 0 (or  $O(x^r)$  for any  $r$ ) is 0.

EXAMPLES:

```

sage: R.<t> = ZZ[[t]]
sage: (t^2 + O(t^3)).precision_relative()
1
sage: (1 - t^2 + O(t^100)).precision_relative()
100
sage: O(t^4).precision_relative()
0

```

**residue()**

Return the residue of `self`.

Consider the Laurent series

$$f = \sum_{n \in \mathbb{Z}} a_n t^n = \cdots + \frac{a_{-2}}{t^2} + \frac{a_{-1}}{t} + a_0 + a_1 t + a_2 t^2 + \cdots,$$

then the residue of  $f$  is  $a_{-1}$ . Alternatively this is the coefficient of  $1/t$ .

EXAMPLES:

```
sage: t = LaurentSeriesRing(ZZ, 't').gen()
sage: f = 1/t**2 + 2/t + 3 + 4*t
sage: f.residue()
2
sage: f = t + t**2
sage: f.residue()
0
sage: f.residue().parent()
Integer Ring
```

**reverse**(*precision=None*)

Return the reverse of  $f$ , i.e., the series  $g$  such that  $g(f(x)) = x$ . Given an optional argument `precision`, return the reverse with given precision (note that the reverse can have precision at most `f.prec()`). If  $f$  has infinite precision, and the argument `precision` is not given, then the precision of the reverse defaults to the default precision of `f.parent()`.

Note that this is only possible if the valuation of self is exactly 1.

The implementation depends on the underlying power series element implementing a reverse method.

EXAMPLES:

```
sage: R.<x> = Frac(QQ[['x']])
sage: f = 2*x + 3*x^2 - x^4 + 0(x^5)
sage: g = f.reverse()
sage: g
1/2*x - 3/8*x^2 + 9/16*x^3 - 131/128*x^4 + 0(x^5)
sage: f(g)
x + 0(x^5)
sage: g(f)
x + 0(x^5)

sage: A.<t> = LaurentSeriesRing(ZZ)
sage: a = t - t^2 - 2*t^4 + t^5 + 0(t^6)
sage: b = a.reverse(); b
t + t^2 + 2*t^3 + 7*t^4 + 25*t^5 + 0(t^6)
sage: a(b)
t + 0(t^6)
sage: b(a)
t + 0(t^6)

sage: B.<b,c> = ZZ[ ]
sage: A.<t> = LaurentSeriesRing(B)
sage: f = t + b*t^2 + c*t^3 + 0(t^4)
sage: g = f.reverse(); g
t - b*t^2 + (2*b^2 - c)*t^3 + 0(t^4)
sage: f(g)
```

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```
t + 0(t^4)
sage: g(f)
t + 0(t^4)

sage: A.<t> = PowerSeriesRing(ZZ)
sage: B.<s> = LaurentSeriesRing(A)
sage: f = (1 - 3*t + 4*t^3 + 0(t^4))*s + (2 + t + t^2 + 0(t^3))*s^2 + 0(s^3)
sage: set_verbose(1)
sage: g = f.reverse(); g
verbose 1 (<module>) passing to pari failed; trying Lagrange inversion
(1 + 3*t + 9*t^2 + 23*t^3 + 0(t^4))*s + (-2 - 19*t - 118*t^2 + 0(t^3))*s^2 +
↳ 0(s^3)
sage: set_verbose(0)
sage: f(g) == g(f) == s
True
```

If the leading coefficient is not a unit, we pass to its fraction field if possible:

```
sage: A.<t> = LaurentSeriesRing(ZZ)
sage: a = 2*t - 4*t^2 + t^4 - t^5 + 0(t^6)
sage: a.reverse()
1/2*t + 1/2*t^2 + t^3 + 79/32*t^4 + 437/64*t^5 + 0(t^6)

sage: B.<b> = PolynomialRing(ZZ)
sage: A.<t> = LaurentSeriesRing(B)
sage: f = 2*b*t + b*t^2 + 3*b^2*t^3 + 0(t^4)
sage: g = f.reverse(); g
1/(2*b)*t - 1/(8*b^2)*t^2 + ((-3*b + 1)/(16*b^3))*t^3 + 0(t^4)
sage: f(g)
t + 0(t^4)
sage: g(f)
t + 0(t^4)
```

We can handle some base rings of positive characteristic:

```
sage: A8.<t> = LaurentSeriesRing(Zmod(8))
sage: a = t - 15*t^2 - 2*t^4 + t^5 + 0(t^6)
sage: b = a.reverse(); b
t + 7*t^2 + 2*t^3 + 5*t^4 + t^5 + 0(t^6)
sage: a(b)
t + 0(t^6)
sage: b(a)
t + 0(t^6)
```

The optional argument `precision` sets the precision of the output:

```
sage: R.<x> = LaurentSeriesRing(QQ)
sage: f = 2*x + 3*x^2 - 7*x^3 + x^4 + 0(x^5)
sage: g = f.reverse(precision=3); g
1/2*x - 3/8*x^2 + 0(x^3)
sage: f(g)
x + 0(x^3)
```

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```
sage: g(f)
x + 0(x^3)
```

If the input series has infinite precision, the precision of the output is automatically set to the default precision of the parent ring:

```
sage: R.<x> = LaurentSeriesRing(QQ, default_prec=20)
sage: (x - x^2).reverse() # get some Catalan numbers
x + x^2 + 2*x^3 + 5*x^4 + 14*x^5 + 42*x^6 + 132*x^7 + 429*x^8 + 1430*x^9 +
↪ 4862*x^10 + 16796*x^11 + 58786*x^12 + 208012*x^13 + 742900*x^14 + 2674440*x^
↪ 15 + 9694845*x^16 + 35357670*x^17 + 129644790*x^18 + 477638700*x^19 + 0(x^20)
sage: (x - x^2).reverse(precision=3)
x + x^2 + 0(x^3)
```

**shift(*k*)**

Returns this Laurent series multiplied by the power  $t^n$ . Does not change this series.

**Note:** Despite the fact that higher order terms are printed to the right in a power series, right shifting decreases the powers of  $t$ , while left shifting increases them. This is to be consistent with polynomials, integers, etc.

EXAMPLES:

```
sage: R.<t> = LaurentSeriesRing(QQ['y'])
sage: f = (t+t^(-1))^4; f
t^-4 + 4*t^-2 + 6 + 4*t^2 + t^4
sage: f.shift(10)
t^6 + 4*t^8 + 6*t^10 + 4*t^12 + t^14
sage: f >> 10
t^-14 + 4*t^-12 + 6*t^-10 + 4*t^-8 + t^-6
sage: t << 4
t^5
sage: t + 0(t^3) >> 4
t^-3 + 0(t^-1)
```

AUTHORS:

- Robert Bradshaw (2007-04-18)

**truncate(*n*)**

Return the Laurent series of degree  $< n$  which is equivalent to self modulo  $x^n$ .

EXAMPLES:

```
sage: A.<x> = LaurentSeriesRing(ZZ)
sage: f = 1/(1-x)
sage: f
1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^10 + x^11 + x^12 + x^
↪ 13 + x^14 + x^15 + x^16 + x^17 + x^18 + x^19 + 0(x^20)
sage: f.truncate(10)
1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9
```

**truncate\_laurentseries(*n*)**

Replace any terms of degree  $\geq n$  by big oh.



EXAMPLES:

```
sage: A.<x> = LaurentSeriesRing(ZZ)
sage: f = 1/(1-x)
sage: f
1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^10 + x^11 + x^12 + x^
↪13 + x^14 + x^15 + x^16 + x^17 + x^18 + x^19 + 0(x^20)
sage: f.truncate_laurentseries(10)
1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + 0(x^10)
```

**truncate\_neg(*n*)**

Return the Laurent series equivalent to `self` except without any degree *n* terms.

This is equivalent to:

```
self - self.truncate(n)
```

EXAMPLES:

```
sage: A.<t> = LaurentSeriesRing(ZZ)
sage: f = 1/(1-t)
sage: f.truncate_neg(15)
t^15 + t^16 + t^17 + t^18 + t^19 + 0(t^20)
```

**valuation()**

EXAMPLES:

```
sage: R.<x> = LaurentSeriesRing(QQ)
sage: f = 1/x + x^2 + 3*x^4 + 0(x^7)
sage: g = 1 - x + x^2 - x^4 + 0(x^8)
sage: f.valuation()
-1
sage: g.valuation()
0
```

Note that the valuation of an element undistinguishable from zero is infinite:

```
sage: h = f - f; h
0(x^7)
sage: h.valuation()
+Infinity
```

**valuation\_zero\_part()**

EXAMPLES:

```
sage: x = Frac(QQ[['x']]).0
sage: f = x + x^2 + 3*x^4 + 0(x^7)
sage: f/x
1 + x + 3*x^3 + 0(x^6)
sage: f.valuation_zero_part()
1 + x + 3*x^3 + 0(x^6)
sage: g = 1/x^7 - x + x^2 - x^4 + 0(x^8)
sage: g.valuation_zero_part()
1 - x^8 + x^9 - x^11 + 0(x^15)
```

**variable()**

EXAMPLES:

```

sage: x = Frac(QQ[['x']]).0
sage: f = 1/x + x^2 + 3*x^4 + O(x^7)
sage: f.variable()
'x'

```

**verschiebung(*n*)**Return the *n*-th Verschiebung of *self*.If  $f = \sum a_m x^m$  then this function returns  $\sum a_m x^{mn}$ .

EXAMPLES:

```

sage: R.<x> = LaurentSeriesRing(QQ)
sage: f = -1/x + 1 + 2*x^2 + 5*x^5
sage: f.V(2)
-x^-2 + 1 + 2*x^4 + 5*x^10
sage: f.V(-1)
5*x^-5 + 2*x^-2 + 1 - x
sage: h = f.add_bigoh(7)
sage: h.V(2)
-x^-2 + 1 + 2*x^4 + 5*x^10 + O(x^14)
sage: h.V(-2)
Traceback (most recent call last):
...
ValueError: For finite precision only positive arguments allowed

```

`sage.rings.laurent_series_ring_element.is_LaurentSeries(x)`

## LAZY SERIES

A lazy series is a series whose coefficients are computed on demand. Therefore, unlike the usual Laurent/power/etc. series in Sage, lazy series have infinite precision.

EXAMPLES:

Laurent series over the integer ring are particularly useful as generating functions for sequences arising in combinatorics.

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ)
```

The generating function of the Fibonacci sequence is:

```
sage: f = 1 / (1 - z - z^2)
sage: f
1 + z + 2*z^2 + 3*z^3 + 5*z^4 + 8*z^5 + 13*z^6 + O(z^7)
```

In principle, we can now compute any coefficient of  $f$ :

```
sage: f.coefficient(100)
573147844013817084101
```

Which coefficients are actually computed depends on the type of implementation. For the sparse implementation, only the coefficients which are needed are computed.

```
sage: s = L(lambda n: n, valuation=0); s
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + O(z^7)
sage: s.coefficient(10)
10
sage: s._coeff_stream._cache
{0: 0, 1: 1, 2: 2, 3: 3, 4: 4, 5: 5, 6: 6, 10: 10}
```

Using the dense implementation, all coefficients up to the required coefficient are computed.

```
sage: L.<x> = LazyLaurentSeriesRing(ZZ, sparse=False)
sage: s = L(lambda n: n, valuation=0); s
x + 2*x^2 + 3*x^3 + 4*x^4 + 5*x^5 + 6*x^6 + O(x^7)
sage: s.coefficient(10)
10
sage: s._coeff_stream._cache
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

We can do arithmetic with lazy power series:

```

sage: f
1 + z + 2*z^2 + 3*z^3 + 5*z^4 + 8*z^5 + 13*z^6 + 0(z^7)
sage: f^-1
1 - z - z^2 + 0(z^7)
sage: f + f^-1
2 + z^2 + 3*z^3 + 5*z^4 + 8*z^5 + 13*z^6 + 0(z^7)
sage: g = (f + f^-1)*(f - f^-1); g
4*z + 6*z^2 + 8*z^3 + 19*z^4 + 38*z^5 + 71*z^6 + 0(z^7)

```

We can change the base ring:

```

sage: h = g.change_ring(QQ)
sage: h.parent()
Lazy Laurent Series Ring in z over Rational Field
sage: h
4*z + 6*z^2 + 8*z^3 + 19*z^4 + 38*z^5 + 71*z^6 + 0(z^7)
sage: hinv = h^-1; hinv
1/4*z^-1 - 3/8 + 1/16*z - 17/32*z^2 + 5/64*z^3 - 29/128*z^4 + 165/256*z^5 + 0(z^6)
sage: hinv.valuation()
-1

```

AUTHORS:

- Kwankyu Lee (2019-02-24): initial version
- Tejasvi Chebrolu, Martin Rubey, Travis Scrimshaw (2021-08): refactored and expanded functionality

**class** `sage.rings.lazy_series.LazyCauchyProductSeries`(*parent, coeff\_stream*)  
 Bases: `sage.rings.lazy_series.LazyModuleElement`

A class for series where multiplication is the Cauchy product.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: f = 1 / (1 - z)
sage: f
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + 0(z^7)
sage: f * (1 - z)
1 + 0(z^7)

sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=True)
sage: f = 1 / (1 - z)
sage: f
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + 0(z^7)

```

**valuation()**

Return the valuation of `self`.

This method determines the valuation of the series by looking for a nonzero coefficient. Hence if the series happens to be zero, then it may run forever.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: s = 1/(1 - z) - 1/(1 - 2*z)
sage: s.valuation()

```

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```

1
sage: t = z - z
sage: t.valuation()
+Infinity
sage: M = L(lambda n: n^2, 0)
sage: M.valuation()
1
sage: (M - M).valuation()
+Infinity

```

**class** sage.rings.lazy\_series.LazyDirichletSeries(*parent*, *coeff\_stream*)

Bases: [sage.rings.lazy\\_series.LazyModuleElement](#)

A Dirichlet series where the coefficients are computed lazily.

EXAMPLES:

```

sage: L = LazyDirichletSeriesRing(ZZ, "z")
sage: f = L(constant=1)^2; f
1 + 2/2^z + 2/3^z + 3/4^z + 2/5^z + 4/6^z + 2/7^z + 0(1/(8^z))
sage: f.coefficient(100) == number_of_divisors(100)
True

```

Lazy Dirichlet series is picklable:

```

sage: g = loads(dumps(f))
sage: g
1 + 2/2^z + 2/3^z + 3/4^z + 2/5^z + 4/6^z + 2/7^z + 0(1/(8^z))
sage: g == f
True

```

**valuation()**

Return the valuation of *self*.

This method determines the valuation of the series by looking for a nonzero coefficient. Hence if the series happens to be zero, then it may run forever.

EXAMPLES:

```

sage: L = LazyDirichletSeriesRing(ZZ, "z")
sage: mu = L(moebius); mu.valuation()
0
sage: (mu - mu).valuation()
+Infinity
sage: g = L(constant=1, valuation=2)
sage: g.valuation()
log(2)
sage: (g*g).valuation()
2*log(2)

```

**class** sage.rings.lazy\_series.LazyLaurentSeries(*parent*, *coeff\_stream*)

Bases: [sage.rings.lazy\\_series.LazyCauchyProductSeries](#)

A Laurent series where the coefficients are computed lazily.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ)
```

We can build a series from a function and specify if the series eventually takes a constant value:

```
sage: f = L(lambda i: i, valuation=-3, constant=-1, degree=3)
sage: f
-3*z^-3 - 2*z^-2 - z^-1 + z + 2*z^2 - z^3 - z^4 - z^5 + 0(z^6)
sage: f[-2]
-2
sage: f[10]
-1
sage: f[-5]
0

sage: f = L(lambda i: i, valuation=-3)
sage: f
-3*z^-3 - 2*z^-2 - z^-1 + z + 2*z^2 + 3*z^3 + 0(z^4)
sage: f[20]
20
```

Anything that converts into a polynomial can be input, where we can also specify the valuation or if the series eventually takes a constant value:

```
sage: L([-5,2,0,5])
-5 + 2*z + 5*z^3
sage: L([-5,2,0,5], constant=6)
-5 + 2*z + 5*z^3 + 6*z^4 + 6*z^5 + 6*z^6 + 0(z^7)
sage: L([-5,2,0,5], degree=6, constant=6)
-5 + 2*z + 5*z^3 + 6*z^6 + 6*z^7 + 6*z^8 + 0(z^9)
sage: L([-5,2,0,5], valuation=-2, degree=3, constant=6)
-5*z^-2 + 2*z^-1 + 5*z + 6*z^3 + 6*z^4 + 6*z^5 + 0(z^6)
sage: L([-5,2,0,5], valuation=5)
-5*z^5 + 2*z^6 + 5*z^8
sage: L({-2:9, 3:4}, constant=2, degree=5)
9*z^-2 + 4*z^3 + 2*z^5 + 2*z^6 + 2*z^7 + 0(z^8)
```

We can also perform arithmetic:

```
sage: f = 1 / (1 - z - z^2)
sage: f
1 + z + 2*z^2 + 3*z^3 + 5*z^4 + 8*z^5 + 13*z^6 + 0(z^7)
sage: f.coefficient(100)
573147844013817084101
sage: f = (z^-2 - 1 + 2*z) / (z^-1 - z + 3*z^2)
sage: f
z^-1 - z^2 - z^4 + 3*z^5 + 0(z^6)
```

However, we may not always be able to know when a result is exactly a polynomial:

```
sage: f * (z^-1 - z + 3*z^2)
z^-2 - 1 + 2*z + 0(z^5)
```

**approximate\_series(*prec*, *name=None*)**

Return the Laurent series with absolute precision *prec* approximated from this series.

INPUT:

- `prec` – an integer
- `name` – name of the variable; if it is `None`, the name of the variable of the series is used

OUTPUT: a Laurent series with absolute precision `prec`

EXAMPLES:

```
sage: L = LazyLaurentSeriesRing(ZZ, 'z')
sage: z = L.gen()
sage: f = (z - 2*z^3)^5/(1 - 2*z)
sage: f
z^5 + 2*z^6 - 6*z^7 - 12*z^8 + 16*z^9 + 32*z^10 - 16*z^11 + 0(z^12)
sage: g = f.approximate_series(10)
sage: g
z^5 + 2*z^6 - 6*z^7 - 12*z^8 + 16*z^9 + 0(z^10)
sage: g.parent()
Power Series Ring in z over Integer Ring
sage: h = (f^-1).approximate_series(3)
sage: h
z^-5 - 2*z^-4 + 10*z^-3 - 20*z^-2 + 60*z^-1 - 120 + 280*z - 560*z^2 + 0(z^3)
sage: h.parent()
Laurent Series Ring in z over Integer Ring
```

**compose(g)**

Return the composition of `self` with `g`.

Given two Laurent Series  $f$  and  $g$  over the same base ring, the composition  $(f \circ g)(z) = f(g(z))$  is defined if and only if:

- $g = 0$  and  $\text{val}(f) \geq 0$ ,
- $g$  is non-zero and  $f$  has only finitely many non-zero coefficients,
- $g$  is non-zero and  $\text{val}(g) > 0$ .

INPUT:

- `g` – other series

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: f = z^2 + 1 + z
sage: f(0)
1
sage: f(L(0))
1
sage: f(f)
3 + 3*z + 4*z^2 + 2*z^3 + z^4
sage: g = z^-3/(1-2*z); g
z^-3 + 2*z^-2 + 4*z^-1 + 8 + 16*z + 32*z^2 + 64*z^3 + 0(z^4)
sage: f(g)
z^-6 + 4*z^-5 + 12*z^-4 + 33*z^-3 + 82*z^-2 + 196*z^-1 + 457 + 0(z)
sage: g^2 + 1 + g
z^-6 + 4*z^-5 + 12*z^-4 + 33*z^-3 + 82*z^-2 + 196*z^-1 + 457 + 0(z)
sage: f(int(2))
```

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```

sage: f = z^-2 + z + 4*z^3
sage: f(f)
4*z^-6 + 12*z^-3 + z^-2 + 48*z^-1 + 12 + 0(z)
sage: f^-2 + f + 4*f^3
4*z^-6 + 12*z^-3 + z^-2 + 48*z^-1 + 12 + 0(z)
sage: f(g)
4*z^-9 + 24*z^-8 + 96*z^-7 + 320*z^-6 + 960*z^-5 + 2688*z^-4 + 7169*z^-3 + 0(z^-
↪2)
sage: g^-2 + g + 4*g^3
4*z^-9 + 24*z^-8 + 96*z^-7 + 320*z^-6 + 960*z^-5 + 2688*z^-4 + 7169*z^-3 + 0(z^-
↪2)

sage: f = z^-3 + z^-2 + 1 / (1 + z^2); f
z^-3 + z^-2 + 1 - z^2 + 0(z^4)
sage: g = z^3 / (1 + z - z^3); g
z^3 - z^4 + z^5 - z^7 + 2*z^8 - 2*z^9 + 0(z^10)
sage: f(g)
z^-9 + 3*z^-8 + 3*z^-7 - z^-6 - 4*z^-5 - 2*z^-4 + z^-3 + 0(z^-2)
sage: g^-3 + g^-2 + 1 / (1 + g^2)
z^-9 + 3*z^-8 + 3*z^-7 - z^-6 - 4*z^-5 - 2*z^-4 + z^-3 + 0(z^-2)

sage: f = z^-3
sage: g = z^-2 + z^-1
sage: g^(-3)
z^6 - 3*z^7 + 6*z^8 - 10*z^9 + 15*z^10 - 21*z^11 + 28*z^12 + 0(z^13)
sage: f(g)
z^6 - 3*z^7 + 6*z^8 - 10*z^9 + 15*z^10 - 21*z^11 + 28*z^12 + 0(z^13)

sage: f = z^2 + z^3
sage: g = z^-3 + z^-2
sage: f^-3 + f^-2
z^-6 - 3*z^-5 + 7*z^-4 - 12*z^-3 + 18*z^-2 - 25*z^-1 + 33 + 0(z)
sage: g(f)
z^-6 - 3*z^-5 + 7*z^-4 - 12*z^-3 + 18*z^-2 - 25*z^-1 + 33 + 0(z)
sage: g^2 + g^3
z^-9 + 3*z^-8 + 3*z^-7 + 2*z^-6 + 2*z^-5 + z^-4
sage: f(g)
z^-9 + 3*z^-8 + 3*z^-7 + 2*z^-6 + 2*z^-5 + z^-4

sage: f = L(lambda n: n, valuation=0); f
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + 0(z^7)
sage: f(z^2)
z^2 + 2*z^4 + 3*z^6 + 0(z^7)

sage: f = L(lambda n: n, valuation=-2); f
-2*z^-2 - z^-1 + z + 2*z^2 + 3*z^3 + 4*z^4 + 0(z^5)
sage: f3 = f(z^3); f3
-2*z^-6 - z^-3 + 0(z)
sage: [f3[i] for i in range(-6,13)]
[-2, 0, 0, -1, 0, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 3, 0, 0, 4]

```



We compose a Laurent polynomial with a generic element:

```
sage: R.<x> = QQ[]
sage: f = z^2 + 1 + z^-1
sage: g = x^2 + x + 3
sage: f(g)
(x^6 + 3*x^5 + 12*x^4 + 19*x^3 + 37*x^2 + 28*x + 31)/(x^2 + x + 3)
sage: f(g) == g^2 + 1 + g^-1
True
```

We compose with another lazy Laurent series:

```
sage: LS.<y> = LazyLaurentSeriesRing(QQ)
sage: f = z^2 + 1 + z^-1
sage: fy = f(y); fy
y^-1 + 1 + y^2
sage: fy.parent() is LS
True
sage: g = y - y
sage: f(g)
Traceback (most recent call last):
...
ZeroDivisionError: the valuation of the series must be nonnegative

sage: g = 1 - y
sage: f(g)
3 - y + 2*y^2 + y^3 + y^4 + y^5 + 0(y^6)
sage: g^2 + 1 + g^-1
3 - y + 2*y^2 + y^3 + y^4 + y^5 + 0(y^6)

sage: f = L(lambda n: n, valuation=0); f
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + 0(z^7)
sage: f(0)
0
sage: f(y)
y + 2*y^2 + 3*y^3 + 4*y^4 + 5*y^5 + 6*y^6 + 0(y^7)
sage: fp = f(y - y)
sage: fp == 0
True
sage: fp.parent() is LS
True

sage: f = z^2 + 3 + z
sage: f(y - y)
3
```

With both of them sparse:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ, sparse=True)
sage: LS.<y> = LazyLaurentSeriesRing(QQ, sparse=True)
sage: f = L(lambda n: 1, valuation=0); f
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + 0(z^7)
sage: f(y^2)
1 + y^2 + y^4 + y^6 + 0(y^7)
```

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```

sage: fp = f - 1 + z^-2; fp
z^-2 + z + z^2 + z^3 + z^4 + 0(z^5)
sage: fpy = fp(y^2); fpy
y^-4 + y^2 + 0(y^3)
sage: fpy.parent() is LS
True
sage: [fpy[i] for i in range(-4,11)]
[1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1]

sage: g = LS(valuation=2, constant=1); g
y^2 + y^3 + y^4 + 0(y^5)
sage: fg = f(g); fg
1 + y^2 + y^3 + 2*y^4 + 3*y^5 + 5*y^6 + 0(y^7)
sage: 1 + g + g^2 + g^3 + g^4 + g^5 + g^6
1 + y^2 + y^3 + 2*y^4 + 3*y^5 + 5*y^6 + 0(y^7)

sage: h = LS(lambda n: 1 if n % 2 else 0, valuation=2); h
y^3 + y^5 + y^7 + 0(y^9)
sage: fgh = fg(h); fgh
1 + y^6 + 0(y^7)
sage: [fgh[i] for i in range(0, 15)]
[1, 0, 0, 0, 0, 0, 1, 0, 2, 1, 3, 3, 6, 6, 13]
sage: t = 1 + h^2 + h^3 + 2*h^4 + 3*h^5 + 5*h^6
sage: [t[i] for i in range(0, 15)]
[1, 0, 0, 0, 0, 0, 1, 0, 2, 1, 3, 3, 6, 6, 13]

```

We look at mixing the sparse and the dense:

```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: f = L(lambda n: 1, valuation=0); f
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + 0(z^7)
sage: g = LS(lambda n: 1, valuation=1); g
y + y^2 + y^3 + y^4 + y^5 + y^6 + y^7 + 0(y^8)
sage: f(g)
1 + y + 2*y^2 + 4*y^3 + 8*y^4 + 16*y^5 + 32*y^6 + 0(y^7)

sage: f = z^-2 + 1 + z
sage: g = 1/(y*(1-y)); g
y^-1 + 1 + y + y^2 + y^3 + y^4 + y^5 + 0(y^6)
sage: f(g)
y^-1 + 2 + y + 2*y^2 - y^3 + 2*y^4 + y^5 + 0(y^6)
sage: g^-2 + 1 + g
y^-1 + 2 + y + 2*y^2 - y^3 + 2*y^4 + y^5 + 0(y^6)

sage: f = z^-3 + z^-2 + 1
sage: g = 1/(y^2*(1-y)); g
y^-2 + y^-1 + 1 + y + y^2 + y^3 + y^4 + 0(y^5)
sage: f(g)
1 + y^4 - 2*y^5 + 2*y^6 + 0(y^7)
sage: g^-3 + g^-2 + 1
1 + y^4 - 2*y^5 + 2*y^6 + 0(y^7)

```

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```
sage: z(y)
y
```

We look at cases where the composition does not exist.  $g = 0$  and  $\text{val}(f) < 0$ :

```
sage: g = L(0)
sage: f = z^-1 + z^-2
sage: f.valuation() < 0
True
sage: f(g)
Traceback (most recent call last):
...
ZeroDivisionError: the valuation of the series must be nonnegative
```

$g \neq 0$  and  $\text{val}(g) \leq 0$  and  $f$  has infinitely many non-zero coefficients`:

```
sage: g = z^-1 + z^-2
sage: g.valuation() <= 0
True
sage: f = L(lambda n: n, valuation=0)
sage: f(g)
Traceback (most recent call last):
...
ValueError: can only compose with a positive valuation series

sage: f = L(lambda n: n, valuation=1)
sage: f(1 + z)
Traceback (most recent call last):
...
ValueError: can only compose with a positive valuation series
```

We compose the exponential with a Dirichlet series:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: e = L(lambda n: 1/factorial(n), 0)
sage: D = LazyDirichletSeriesRing(QQ, "s")
sage: g = D(constant=1)-1; g
1/(2^s) + 1/(3^s) + 1/(4^s) + 0(1/(5^s))

sage: e(g)[0:10]
[0, 1, 1, 1, 3/2, 1, 2, 1, 13/6, 3/2]

sage: sum(g^k/factorial(k) for k in range(10))[0:10]
[0, 1, 1, 1, 3/2, 1, 2, 1, 13/6, 3/2]

sage: g = D([0,1,0,1,1,2]); g
1/(2^s) + 1/(4^s) + 1/(5^s) + 2/6^s
sage: e(g)[0:10]
[0, 1, 1, 0, 3/2, 1, 2, 0, 7/6, 0]
sage: sum(g^k/factorial(k) for k in range(10))[0:10]
[0, 1, 1, 0, 3/2, 1, 2, 0, 7/6, 0]

sage: e(D([1,0,1]))
```

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```

Traceback (most recent call last):
...
ValueError: can only compose with a positive valuation series

sage: e5 = L(e, degree=5); e5
1 + z + 1/2*z^2 + 1/6*z^3 + 1/24*z^4
sage: e5(g)
1 + 1/(2^s) + 3/2/4^s + 1/(5^s) + 2/6^s + 0(1/(8^s))
sage: sum(e5[k] * g^k for k in range(5))
1 + 1/(2^s) + 3/2/4^s + 1/(5^s) + 2/6^s + 0(1/(8^s))

```

The output parent is always the common parent between the base ring of  $f$  and the parent of  $g$  or extended to the corresponding lazy series:

```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: R.<x> = ZZ[]
sage: parent(z(x))
Univariate Polynomial Ring in x over Rational Field
sage: parent(z(R.zero()))
Univariate Polynomial Ring in x over Rational Field
sage: parent(z(0))
Rational Field
sage: f = 1 / (1 - z)
sage: f(x)
1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + 0(x^7)
sage: three = L(3)(x^2); three
3
sage: parent(three)
Univariate Polynomial Ring in x over Rational Field

```

**polynomial**(*degree=None, name=None*)

Return self as a Laurent polynomial if self is actually so.

INPUT:

- *degree* – None or an integer
- *name* – name of the variable; if it is None, the name of the variable of the series is used

OUTPUT:

A Laurent polynomial if the valuation of the series is negative or a polynomial otherwise.

If *degree* is not None, the terms of the series of degree greater than *degree* are truncated first. If *degree* is None and the series is not a polynomial or a Laurent polynomial, a `ValueError` is raised.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: f = L([1,0,0,2,0,0,3], valuation=-5); f
z^5 + 2*z^8 + 3*z^12
sage: f.polynomial()
3*z^12 + 2*z^8 + z^5

```

**revert**()

Return the compositional inverse of self.

Given a Laurent Series  $f$ , the compositional inverse is a Laurent Series  $g$  over the same base ring, such that  $(f \circ g)(z) = f(g(z)) = z$ .

The compositional inverse exists if and only if:

- $\text{val}(f) = 1$ , or
- $f = a + bz$  with  $ab \neq 0$ , or
- $f = a/z$  with  $a \neq 0$

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: z.revert()
z + 0(z^8)
sage: (1/z).revert()
z^-1
sage: (z-z^2).revert()
z + z^2 + 2*z^3 + 5*z^4 + 14*z^5 + 42*z^6 + 132*z^7 + 0(z^8)
```

**class** sage.rings.lazy\_series.LazyModuleElement(parent, coeff\_stream)

Bases: sage.structure.element.Element

A lazy sequence with a module structure given by term-wise addition and scalar multiplication.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: M = L(lambda n: n, valuation=0)
sage: N = L(lambda n: 1, valuation=0)
sage: M[:10]
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
sage: N[:10]
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

Two sequences can be added:

```
sage: O = M + N
sage: O[:10]
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

Two sequences can be subtracted:

```
sage: P = M - N
sage: P[:10]
[-1, 0, 1, 2, 3, 4, 5, 6, 7, 8]
```

A sequence can be multiplied by a scalar:

```
sage: Q = 2 * M
sage: Q[:10]
[0, 2, 4, 6, 8, 10, 12, 14, 16, 18]
```

The negation of a sequence can also be found:

```
sage: R = -M
sage: R[:10]
[0, -1, -2, -3, -4, -5, -6, -7, -8, -9]
```

**arccos()**

Return the arccos of self.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(RR)
sage: arccos(z)
1.57079632679490 - 1.000000000000000*z + 0.000000000000000*z^2
- 0.166666666666667*z^3 + 0.000000000000000*z^4
- 0.075000000000000*z^5 + 0(1.000000000000000*z^7)

sage: L.<z> = LazyLaurentSeriesRing(SR)
sage: arccos(z/(1-z))
1/2*pi - z - z^2 - 7/6*z^3 - 3/2*z^4 - 83/40*z^5 - 73/24*z^6 + 0(z^7)

sage: L.<x, y> = LazyTaylorSeriesRing(SR) # not tested
sage: arccos(x/(1-y)) # not tested
1/2*pi + (-x) + (-x*y) + ((-1/6)*x^3-x*y^2) + ((-1/2)*x^3*y-x*y^3)
+ ((-3/40)*x^5-x^3*y^2-x*y^4) + ((-3/8)*x^5*y+(-5/3)*x^3*y^3-x*y^5) + 0(x,y)^7
```

**arccot()**

Return the arctangent of self.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(RR)
sage: arccot(z)
1.57079632679490 - 1.000000000000000*z + 0.000000000000000*z^2
+ 0.333333333333333*z^3 + 0.000000000000000*z^4
- 0.200000000000000*z^5 + 0(1.000000000000000*z^7)

sage: L.<z> = LazyLaurentSeriesRing(SR)
sage: arccot(z/(1-z))
1/2*pi - z - z^2 - 2/3*z^3 + 4/5*z^5 + 4/3*z^6 + 0(z^7)

sage: L.<x, y> = LazyTaylorSeriesRing(SR) # not tested
sage: acot(x/(1-y)) # not tested
1/2*pi + (-x) + (-x*y) + (1/3*x^3-x*y^2) + (x^3*y-x*y^3)
+ ((-1/5)*x^5+2*x^3*y^2-x*y^4) + (-x^5*y+10/3*x^3*y^3-x*y^5) + 0(x,y)^7
```

**arcsin()**

Return the arcsin of self.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: arcsin(z)
z + 1/6*z^3 + 3/40*z^5 + 5/112*z^7 + 0(z^8)

sage: L.<x, y> = LazyTaylorSeriesRing(QQ) # not tested
sage: asin(x/(1-y)) # not tested
```

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```
x + x*y + (1/6*x^3+x*y^2) + (1/2*x^3*y+x*y^3)
+ (3/40*x^5+x^3*y^2+x*y^4) + (3/8*x^5*y+5/3*x^3*y^3+x*y^5)
+ (5/112*x^7+9/8*x^5*y^2+5/2*x^3*y^4+x*y^6) + 0(x,y)^8
```

### arcsinh()

Return the inverse of the hyperbolic sine of `self`.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: asinh(z)
z - 1/6*z^3 + 3/40*z^5 - 5/112*z^7 + 0(z^8)

sage: L.<x, y> = LazyTaylorSeriesRing(QQ) # not tested
sage: asinh(x/(1-y)) # not tested
x + x*y + ((-1/6)*x^3+x*y^2) + ((-1/2)*x^3*y+x*y^3)
+ (3/40*x^5-x^3*y^2+x*y^4) + (3/8*x^5*y+(-5/3)*x^3*y^3+x*y^5)
+ ((-5/112)*x^7+9/8*x^5*y^2+(-5/2)*x^3*y^4+x*y^6) + 0(x,y)^8
```

### arctan()

Return the arctangent of `self`.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: arctan(z)
z - 1/3*z^3 + 1/5*z^5 - 1/7*z^7 + 0(z^8)

sage: L.<x, y> = LazyTaylorSeriesRing(QQ) # not tested
sage: atan(x/(1-y)) # not tested
x + x*y + ((-1/3)*x^3+x*y^2) + (-x^3*y+x*y^3)
+ (1/5*x^5+(-2)*x^3*y^2+x*y^4) + (x^5*y+(-10/3)*x^3*y^3+x*y^5)
+ ((-1/7)*x^7+3*x^5*y^2+(-5)*x^3*y^4+x*y^6) + 0(x,y)^8
```

### arctanh()

Return the inverse of the hyperbolic tangent of `self`.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: atanh(z)
z + 1/3*z^3 + 1/5*z^5 + 1/7*z^7 + 0(z^8)

sage: L.<x, y> = LazyTaylorSeriesRing(QQ) # not tested
sage: atanh(x/(1-y)) # not tested
x + x*y + (1/3*x^3+x*y^2) + (x^3*y+x*y^3) + (1/5*x^5+2*x^3*y^2+x*y^4)
+ (x^5*y+10/3*x^3*y^3+x*y^5) + (1/7*x^7+3*x^5*y^2+5*x^3*y^4+x*y^6) + 0(x,y)^8
```

### change\_ring(ring)

Return `self` with coefficients converted to elements of `ring`.

INPUT:

- `ring` – a ring

EXAMPLES:

Dense Implementation:

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=False)
sage: s = 2 + z
sage: t = s.change_ring(QQ)
sage: t^-1
1/2 - 1/4*z + 1/8*z^2 - 1/16*z^3 + 1/32*z^4 - 1/64*z^5 + 1/128*z^6 + O(z^7)
sage: M = L(lambda n: n, valuation=0); M
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + O(z^7)
sage: N = M.change_ring(QQ)
sage: N.parent()
Lazy Laurent Series Ring in z over Rational Field
sage: M.parent()
Lazy Laurent Series Ring in z over Integer Ring
```

Sparse Implementation:

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=True)
sage: M = L(lambda n: n, valuation=0); M
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + O(z^7)
sage: M.parent()
Lazy Laurent Series Ring in z over Integer Ring
sage: N = M.change_ring(QQ)
sage: N.parent()
Lazy Laurent Series Ring in z over Rational Field
sage: M^-1
z^-1 - 2 + z + O(z^6)
```

A Dirichlet series example:

```
sage: L = LazyDirichletSeriesRing(ZZ, 'z')
sage: s = L(constant=2)
sage: t = s.change_ring(QQ)
sage: t.parent()
Lazy Dirichlet Series Ring in z over Rational Field
sage: t^-1
1/2 - 1/2/2^z - 1/2/3^z - 1/2/5^z + 1/2/6^z - 1/2/7^z + O(1/(8^z))
```

### **coefficient**(*n*)

Return the coefficient of the term with exponent *n* of the series.

INPUT:

- *n* – integer; the exponent

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=False)
sage: f = z / (1 - 2*z^3)
sage: [f[n] for n in range(20)]
[0, 1, 0, 0, 2, 0, 0, 4, 0, 0, 8, 0, 0, 16, 0, 0, 32, 0, 0, 64]
sage: f[0:20]
[0, 1, 0, 0, 2, 0, 0, 4, 0, 0, 8, 0, 0, 16, 0, 0, 32, 0, 0, 64]

sage: M = L(lambda n: n, valuation=0)
```

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```

sage: [M[n] for n in range(20)]
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]

sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=True)
sage: M = L(lambda n: n, valuation=0)
sage: [M[n] for n in range(20)]
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]

```

Similarly for Dirichlet series:

```

sage: L = LazyDirichletSeriesRing(ZZ, "z")
sage: f = L(lambda n: n)
sage: [f[n] for n in range(1, 11)]
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
sage: f[1:11]
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

sage: M = L(lambda n: n)
sage: [M[n] for n in range(1, 11)]
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
sage: L = LazyDirichletSeriesRing(ZZ, "z", sparse=True)
sage: M = L(lambda n: n)
sage: [M[n] for n in range(1, 11)]
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

```

### cos()

Return the cosine of self.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: cos(z)
1 - 1/2*z^2 + 1/24*z^4 - 1/720*z^6 + O(z^7)

sage: L.<x,y> = LazyTaylorSeriesRing(QQ) # not tested
sage: cos(x/(1-y)).finite_part(4) # not tested
1/24*x^4 + (-3/2)*x^2*y^2 - x^2*y + (-1/2)*x^2 + 1

```

### cosh()

Return the cosh of self.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: cosh(z)
1 + 1/2*z^2 + 1/24*z^4 + 1/720*z^6 + O(z^7)

sage: L.<x,y> = LazyTaylorSeriesRing(QQ) # not tested
sage: cosh(x/(1-y)) # not tested
1 + 1/2*x^2 + x^2*y + (1/24*x^4+3/2*x^2*y^2) + (1/6*x^4*y+2*x^2*y^3)
+ (1/720*x^6+5/12*x^4*y^2+5/2*x^2*y^4) + O(x,y)^7

```

### cot()

Return the cotangent of self.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: cot(z)
z^-1 - 1/3*z - 1/45*z^3 - 2/945*z^5 + 0(z^6)

sage: L.<x> = LazyLaurentSeriesRing(QQ)
sage: cot(x/(1-x)).polynomial(4)
x^-1 - 1 - 1/3*x - 1/3*x^2 - 16/45*x^3 - 2/5*x^4
```

### coth()

Return the hyperbolic cotangent of `self`.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: coth(z)
z^-1 + 1/3*z - 1/45*z^3 + 2/945*z^5 + 0(z^6)

sage: coth(z + z^2)
z^-1 - 1 + 4/3*z - 2/3*z^2 + 44/45*z^3 - 16/15*z^4 + 884/945*z^5 + 0(z^6)
```

### csc()

Return the cosecant of `self`.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: csc(z)
z^-1 + 1/6*z + 7/360*z^3 + 31/15120*z^5 + 0(z^6)

sage: L.<x> = LazyLaurentSeriesRing(QQ)
sage: csc(x/(1-x)).polynomial(4)
x^-1 - 1 + 1/6*x + 1/6*x^2 + 67/360*x^3 + 9/40*x^4
```

### csch()

Return the hyperbolic cosecant of `self`.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: csch(z)
z^-1 - 1/6*z + 7/360*z^3 - 31/15120*z^5 + 0(z^6)

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: csch(z/(1-z))
z^-1 - 1 - 1/6*z - 1/6*z^2 - 53/360*z^3 - 13/120*z^4 - 787/15120*z^5 + 0(z^6)
```

### define(s)

Define an equation by `self = s`.

INPUT:

- `s` – a Laurent polynomial

EXAMPLES:

We begin by constructing the Catalan numbers:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: C = L(None, valuation=0)
sage: C.define(1 + z*C^2)
sage: C
1 + z + 2*z^2 + 5*z^3 + 14*z^4 + 42*z^5 + 132*z^6 + 0(z^7)

```

The Catalan numbers but with a valuation 1:

```

sage: B = L(None, valuation=1)
sage: B.define(z + B^2)
sage: B
z + z^2 + 2*z^3 + 5*z^4 + 14*z^5 + 42*z^6 + 132*z^7 + 0(z^8)

```

We can define multiple series that are linked:

```

sage: s = L(None, valuation=0)
sage: t = L(None, valuation=0)
sage: s.define(1 + z*t^3)
sage: t.define(1 + z*s^2)
sage: s[:9]
[1, 1, 3, 9, 34, 132, 546, 2327, 10191]
sage: t[:9]
[1, 1, 2, 7, 24, 95, 386, 1641, 7150]

```

A bigger example:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: A = L(None, valuation=5)
sage: B = L(None, valuation=0)
sage: C = L(None, valuation=2)
sage: A.define(z^5 + B^2)
sage: B.define(z^5 + C^2)
sage: C.define(z^2 + C^2 + A^2)
sage: A[0:15]
[0, 0, 0, 0, 0, 1, 0, 0, 1, 2, 5, 4, 14, 10, 48]
sage: B[0:15]
[0, 0, 0, 0, 1, 1, 2, 0, 5, 0, 14, 0, 44, 0, 138]
sage: C[0:15]
[0, 0, 1, 0, 1, 0, 2, 0, 5, 0, 15, 0, 44, 2, 142]

```

Counting binary trees:

```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: s = L(None, valuation=1)
sage: s.define(z + (s^2+s(z^2))/2)
sage: [s[i] for i in range(9)]
[0, 1, 1, 1, 2, 3, 6, 11, 23]

```

The  $q$ -Catalan numbers:

```

sage: R.<q> = ZZ[]
sage: L.<z> = LazyLaurentSeriesRing(R)
sage: s = L(None, valuation=0)
sage: s.define(1+z*s*s(q*z))

```

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```

sage: s
1 + z + (q + 1)*z^2 + (q^3 + q^2 + 2*q + 1)*z^3
+ (q^6 + q^5 + 2*q^4 + 3*q^3 + 3*q^2 + 3*q + 1)*z^4
+ (q^10 + q^9 + 2*q^8 + 3*q^7 + 5*q^6 + 5*q^5 + 7*q^4 + 7*q^3 + 6*q^2 + 4*q +
↪1)*z^5
+ (q^15 + q^14 + 2*q^13 + 3*q^12 + 5*q^11 + 7*q^10 + 9*q^9 + 11*q^8
+ 14*q^7 + 16*q^6 + 16*q^5 + 17*q^4 + 14*q^3 + 10*q^2 + 5*q + 1)*z^6 + 0(z^
↪7)

```

We count unlabeled ordered trees by total number of nodes and number of internal nodes:

```

sage: R.<q> = QQ[]
sage: Q.<z> = LazyLaurentSeriesRing(R)
sage: leaf = z
sage: internal_node = q * z
sage: L = Q(constant=1, degree=1)
sage: T = Q(None, valuation=1)
sage: T.define(leaf + internal_node * L(T))
sage: [T[i] for i in range(6)]
[0, 1, q, q^2 + q, q^3 + 3*q^2 + q, q^4 + 6*q^3 + 6*q^2 + q]

```

Similarly for Dirichlet series:

```

sage: L = LazyDirichletSeriesRing(ZZ, "z")
sage: g = L(constant=1, valuation=2)
sage: F = L(None); F.define(1 + g*F)
sage: [F[i] for i in range(1, 16)]
[1, 1, 1, 2, 1, 3, 1, 4, 2, 3, 1, 8, 1, 3, 3]
sage: oeis(_) # optional, ↪
↪internet
0: A002033: Number of perfect partitions of n.
1: A074206: Kalmár's [Kalmar's] problem: number of ordered factorizations of n.
...

sage: F = L(None); F.define(1 + g*F*F)
sage: [F[i] for i in range(1, 16)]
[1, 1, 1, 3, 1, 5, 1, 10, 3, 5, 1, 24, 1, 5, 5]

```

### exp()

Return the exponential series of self.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: exp(z)
1 + z + 1/2*z^2 + 1/6*z^3 + 1/24*z^4 + 1/120*z^5 + 1/720*z^6 + 0(z^7)
sage: exp(z + z^2)
1 + z + 3/2*z^2 + 7/6*z^3 + 25/24*z^4 + 27/40*z^5 + 331/720*z^6 + 0(z^7)
sage: exp(0)
1
sage: exp(1 + z)
Traceback (most recent call last):
...

```

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```
ValueError: can only compose with a positive valuation series
```

```
sage: L.<x,y> = LazyTaylorSeriesRing(QQ) # not tested
sage: exp(x+y)[4].factor() # not tested
(1/24) * (x + y)^4
sage: exp(x/(1-y)).finite_part(3) # not tested
1/6*x^3 + x^2*y + x*y^2 + 1/2*x^2 + x*y + x + 1
```

**hypergeometric(a, b)**

Return the  ${}_pF_q$ -hypergeometric function  ${}_pF_q$  where  $(p, q)$  is the parameterization of `self`.

INPUT:

- `a` – the first parameter of the hypergeometric function
- `b` – the second parameter of the hypergeometric function

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: z.hypergeometric([1, 1], [1])
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + 0(z^7)
sage: z.hypergeometric([], []) - exp(z)
0(z^7)
```

**log()**

Return the series for the natural logarithm of `self`.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: log(1/(1-z))
z + 1/2*z^2 + 1/3*z^3 + 1/4*z^4 + 1/5*z^5 + 1/6*z^6 + 1/7*z^7 + 0(z^8)

sage: L.<x, y> = LazyTaylorSeriesRing(QQ) # not tested
sage: log((1 + x/(1-y))).finite_part(3) # not tested
1/3*x^3 - x^2*y + x*y^2 + (-1/2)*x^2 + x*y + x
```

**map\_coefficients(func, ring=None)**

Return the series with `func` applied to each nonzero coefficient of `self`.

INPUT:

- `func` – function that takes in a coefficient and returns a new coefficient

EXAMPLES:

Dense Implementation:

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=False)
sage: s = z/(1 - 2*z^2)
sage: t = s.map_coefficients(lambda c: c + 1)
sage: s
z + 2*z^3 + 4*z^5 + 8*z^7 + 0(z^8)
sage: t
2*z + 3*z^3 + 5*z^5 + 9*z^7 + 0(z^8)
sage: m = L(lambda n: n, valuation=0); m
```

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```

z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + 0(z^7)
sage: m.map_coefficients(lambda c: c + 1)
2*z + 3*z^2 + 4*z^3 + 5*z^4 + 6*z^5 + 7*z^6 + 0(z^7)

```

Sparse Implementation:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=True)
sage: m = L(lambda n: n, valuation=0); m
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + 0(z^7)
sage: m.map_coefficients(lambda c: c + 1)
2*z + 3*z^2 + 4*z^3 + 5*z^4 + 6*z^5 + 7*z^6 + 0(z^7)

```

An example where the series is known to be exact:

```

sage: f = z + z^2 + z^3
sage: f.map_coefficients(lambda c: c + 1)
2*z + 2*z^2 + 2*z^3

```

Similarly for Dirichlet series:

```

sage: L = LazyDirichletSeriesRing(ZZ, "z")
sage: s = L(lambda n: n-1); s
1/(2^z) + 2/3^z + 3/4^z + 4/5^z + 5/6^z + 6/7^z + 0(1/(8^z))
sage: s.map_coefficients(lambda c: c + 1)
2/2^z + 3/3^z + 4/4^z + 5/5^z + 6/6^z + 7/7^z + 0(1/(8^z))

```

### prec()

Return the precision of the series, which is infinity.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: f = 1/(1 - z)
sage: f.prec()
+Infinity

```

### sec()

Return the secant of self.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: sec(z)
1 + 1/2*z^2 + 5/24*z^4 + 61/720*z^6 + 0(z^7)

sage: L.<x, y> = LazyTaylorSeriesRing(QQ) # not tested
sage: sec(x/(1-y)).finite_part(4) # not tested
5/24*x^4 + 3/2*x^2*y^2 + x^2*y + 1/2*x^2 + 1

```

### sech()

Return the hyperbolic secant of self.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: sech(z)
1 - 1/2*z^2 + 5/24*z^4 - 61/720*z^6 + O(z^7)

sage: L.<x, y> = LazyTaylorSeriesRing(QQ) # not tested
sage: sech(x/(1-y)) # not tested
1 + ((-1/2)*x^2) + (-x^2*y) + (5/24*x^4+(-3/2)*x^2*y^2)
+ (5/6*x^4*y+(-2)*x^2*y^3) + ((-61/720)*x^6+25/12*x^4*y^2+(-5/2)*x^2*y^4)
+ O(x,y)^7

```

### shift(n)

Return self with the indices shifted by n.

For example, a Laurent series is multiplied by the power  $z^n$ , where  $z$  is the variable of self.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: f = 1 / (1 + 2*z)
sage: f
1 - 2*z + 4*z^2 - 8*z^3 + 16*z^4 - 32*z^5 + 64*z^6 + O(z^7)
sage: f.shift(3)
z^3 - 2*z^4 + 4*z^5 - 8*z^6 + 16*z^7 - 32*z^8 + 64*z^9 + O(z^10)
sage: f << -3 # shorthand
z^-3 - 2*z^-2 + 4*z^-1 - 8 + 16*z - 32*z^2 + 64*z^3 + O(z^4)
sage: g = z^-3 + 3 + z^2
sage: g.shift(5)
z^2 + 3*z^5 + z^7
sage: L([2,0,3], valuation=2, degree=7, constant=1) << -2
2 + 3*z^2 + z^5 + z^6 + z^7 + O(z^8)

sage: D = LazyDirichletSeriesRing(QQ, 't')
sage: f = D([0,1,2]); f
1/(2^t) + 2/3^t
sage: f.shift(3)
1/(5^t) + 2/6^t

```

### sin()

Return the sine of self.

EXAMPLES:

```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: sin(z)
z - 1/6*z^3 + 1/120*z^5 - 1/5040*z^7 + O(z^8)

sage: sin(1 + z)
Traceback (most recent call last):
...
ValueError: can only compose with a positive valuation series

sage: L.<x,y> = LazyTaylorSeriesRing(QQ) # not tested
sage: sin(x/(1-y)).finite_part(3) # not tested
(-1/6)*x^3 + x*y^2 + x*y + x

```

**sinh()**

Return the sinh of self.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: sinh(z)
z + 1/6*z^3 + 1/120*z^5 + 1/5040*z^7 + O(z^8)

sage: L.<x, y> = LazyTaylorSeriesRing(QQ) # not tested
sage: sinh(x/(1-y)) # not tested
x + x*y + (1/6*x^3+x*y^2) + (1/2*x^3*y+x*y^3)
+ (1/120*x^5+x^3*y^2+x*y^4) + (1/24*x^5*y+5/3*x^3*y^3+x*y^5)
+ (1/5040*x^7+1/8*x^5*y^2+5/2*x^3*y^4+x*y^6) + O(x,y)^8
```

**sqrt()**

Return self^(1/2).

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: sqrt(1+z)
1 + 1/2*z - 1/8*z^2 + 1/16*z^3 - 5/128*z^4 + 7/256*z^5 - 21/1024*z^6 + O(z^7)

sage: L.<x,y> = LazyTaylorSeriesRing(QQ) # not tested
sage: sqrt(1+x/(1-y)) # not tested
1 + 1/2*x + ((-1/8)*x^2+1/2*x*y) + (1/16*x^3+(-1/4)*x^2*y+1/2*x*y^2)
+ ((-5/128)*x^4+3/16*x^3*y+(-3/8)*x^2*y^2+1/2*x*y^3)
+ (7/256*x^5+(-5/32)*x^4*y+3/8*x^3*y^2+(-1/2)*x^2*y^3+1/2*x*y^4)
+ ((-21/1024)*x^6+35/256*x^5*y+(-25/64)*x^4*y^2+5/8*x^3*y^3+(-5/8)*x^2*y^4+1/
↪ 2*x*y^5)
+ O(x,y)^7
```

This also works for Dirichlet series:

```
sage: D = LazyDirichletSeriesRing(SR, "s")
sage: Z = D(constant=1)
sage: f = sqrt(Z)
sage: f
1 + 1/2/2^s + 1/2/3^s + 3/8/4^s + 1/2/5^s + 1/4/6^s + 1/2/7^s + O(1/(8^s))
sage: f*f - Z
O(1/(8^s))
```

**tan()**

Return the tangent of self.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ) sage: tan(z) z + 1/3*z^3 + 2/15*z^5 + 17/315*z^7 +
O(z^8)

sage: L.<x,y> = LazyTaylorSeriesRing(QQ) # not tested sage: tan(x/(1-y)).finite_part(5) # not
tested 2/15*x^5 + 2*x^3*y^2 + x*y^4 + x^3*y + x*y^3 + 1/3*x^3 + x*y^2 + x*y + x
```

**tanh()**

Return the tanh of self.

EXAMPLES:



```

sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: tanh(z)
z - 1/3*z^3 + 2/15*z^5 - 17/315*z^7 + 0(z^8)

sage: L.<x, y> = LazyTaylorSeriesRing(QQ) # not tested
sage: tanh(x/(1-y)) # not tested
x + x*y + ((-1/3)*x^3+x*y^2) + (-x^3*y+x*y^3)
+ (2/15*x^5+(-2)*x^3*y^2+x*y^4) + (2/3*x^5*y+(-10/3)*x^3*y^3+x*y^5)
+ ((-17/315)*x^7+2*x^5*y^2+(-5)*x^3*y^4+x*y^6) + 0(x,y)^8

```

**truncate(d)**

Return this series with its terms of degree  $\geq d$  truncated.

INPUT:

- $d$  – integer; the degree from which the series is truncated

EXAMPLES:

Dense Implementation:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=False)
sage: alpha = 1/(1-z)
sage: alpha
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + 0(z^7)
sage: beta = alpha.truncate(5)
sage: beta
1 + z + z^2 + z^3 + z^4
sage: alpha - beta
z^5 + z^6 + 0(z^7)
sage: M = L(lambda n: n, valuation=0); M
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + 0(z^7)
sage: M.truncate(4)
z + 2*z^2 + 3*z^3

```

Sparse Implementation:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=True)
sage: M = L(lambda n: n, valuation=0); M
z + 2*z^2 + 3*z^3 + 4*z^4 + 5*z^5 + 6*z^6 + 0(z^7)
sage: M.truncate(4)
z + 2*z^2 + 3*z^3

```

Series which are known to be exact can also be truncated:

```

sage: M = z + z^2 + z^3 + z^4
sage: M.truncate(4)
z + z^2 + z^3

```



## LAZY SERIES RINGS

AUTHORS:

- Kwankyu Lee (2019-02-24): initial version
- Tejasvi Chebrolu, Martin Rubey, Travis Scrimshaw (2021-08): refactored and expanded functionality

**class** `sage.rings.lazy_series_ring.LazyDirichletSeriesRing`(*base\_ring*, *names*, *sparse=True*,  
*category=None*)

Bases: `sage.rings.lazy_series_ring.LazySeriesRing`

Lazy Dirichlet series ring.

INPUT:

- *base\_ring* – base ring of this Dirichlet series ring
- *names* – name of the generator of this Dirichlet series ring
- *sparse* – (default: True) whether this series is sparse or not

EXAMPLES:

```
sage: LazyDirichletSeriesRing(ZZ, 't')
Lazy Dirichlet Series Ring in t over Integer Ring
```

**Element**

alias of `sage.rings.lazy_series.LazyDirichletSeries`

**characteristic()**

Return the characteristic of this lazy power series ring, which is the same as the characteristic of its base ring.

EXAMPLES:

```
sage: L = LazyDirichletSeriesRing(ZZ, "s")
sage: L.characteristic()
0
```

**one()**

Return the constant series 1.

EXAMPLES:

```
sage: L = LazyDirichletSeriesRing(ZZ, 'z')
sage: L.one()
1
```

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```
sage: ~L.one()
1 + 0(1/(8^z))
```

**options**(\*get\_value, \*\*set\_value)

Set and display the options for Lazy Laurent series.

If no parameters are set, then the function returns a copy of the options dictionary.

The options to Lazy Laurent series can be accessed as using [LazyLaurentSeriesRing.options](#) of [LazyLaurentSeriesRing](#).

OPTIONS:

- `constant_length` – (default: 3) the number of coefficients to display for nonzero constant series
- `display_length` – (default: 7) the number of coefficients to display from the valuation

EXAMPLES:

```
sage: LLS.<z> = LazyLaurentSeriesRing(QQ)
sage: LLS.options.display_length
7
sage: f = 1/(1-z)
sage: f
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + 0(z^7)
sage: LLS.options.display_length = 10
sage: f
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9 + 0(z^10)
sage: g = LLS(lambda n: n^2, valuation=-2, degree=5, constant=42)
sage: g
4*z^-2 + z^-1 + z + 4*z^2 + 9*z^3 + 16*z^4 + 42*z^5 + 42*z^6 + 42*z^7 + 0(z^8)
sage: LLS.options.constant_length = 1
sage: g
4*z^-2 + z^-1 + z + 4*z^2 + 9*z^3 + 16*z^4 + 42*z^5 + 0(z^6)
sage: LazyLaurentSeriesRing.options._reset()
sage: LazyLaurentSeriesRing.options.display_length
7
```

See [GlobalOptions](#) for more features of these options.

**zero()**

Return the zero series.

EXAMPLES:

```
sage: L = LazyDirichletSeriesRing(ZZ, 'z')
sage: L.zero()
0
```

```
class sage.rings.lazy_series_ring.LazyLaurentSeriesRing(base_ring, names, sparse=True,
                                                         category=None)
```

Bases: [sage.rings.lazy\\_series\\_ring.LazySeriesRing](#)

The ring of lazy Laurent series.

The ring of Laurent series over a ring with the usual arithmetic where the coefficients are computed lazily.

INPUT:

- `base_ring` – base ring
- `names` – name of the generator
- `sparse` – (default: `True`) whether the implementation of the series is sparse or not

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(QQ)
sage: 1 / (1 - z)
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + O(z^7)
sage: 1 / (1 - z) == 1 / (1 - z)
True
sage: L in Fields
True
```

Lazy Laurent series ring over a finite field:

```
sage: L.<z> = LazyLaurentSeriesRing(GF(3)); L
Lazy Laurent Series Ring in z over Finite Field of size 3
sage: e = 1 / (1 + z)
sage: e.coefficient(100)
1
sage: e.coefficient(100).parent()
Finite Field of size 3
```

Series can be defined by specifying a coefficient function along with a valuation or a degree where after the series is eventually constant:

```
sage: R.<x,y> = QQ[]
sage: L.<z> = LazyLaurentSeriesRing(R)
sage: def coeff(n):
.....:     if n < 0:
.....:         return -2 + n
.....:     if n == 0:
.....:         return 6
.....:     return x + y^n
sage: f = L(coeff, valuation=-5)
sage: f
-7*z^-5 - 6*z^-4 - 5*z^-3 - 4*z^-2 - 3*z^-1 + 6 + (x + y)*z + O(z^2)
sage: 1 / (1 - f)
1/7*z^5 - 6/49*z^6 + 1/343*z^7 + 8/2401*z^8 + 64/16807*z^9
+ 17319/117649*z^10 + (1/49*x + 1/49*y - 180781/823543)*z^11 + O(z^12)
sage: L(coeff, valuation=-3, degree=3, constant=x)
-5*z^-3 - 4*z^-2 - 3*z^-1 + 6 + (x + y)*z + (y^2 + x)*z^2
+ x*z^3 + x*z^4 + x*z^5 + O(z^6)
```

Similarly, we can specify a polynomial or the initial coefficients with anything that converts into the corresponding Laurent polynomial ring:

```
sage: L([1, x, y, 0, x+y])
1 + x*z + y*z^2 + (x + y)*z^4
sage: L([1, x, y, 0, x+y], constant=2)
1 + x*z + y*z^2 + (x + y)*z^4 + 2*z^5 + 2*z^6 + 2*z^7 + O(z^8)
sage: L([1, x, y, 0, x+y], degree=7, constant=2)
```

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```

1 + x*z + y*z^2 + (x + y)*z^4 + 2*z^7 + 2*z^8 + 2*z^9 + 0(z^10)
sage: L([1, x, y, 0, x+y], valuation=-2)
z^-2 + x*z^-1 + y + (x + y)*z^2
sage: L([1, x, y, 0, x+y], valuation=-2, constant=3)
z^-2 + x*z^-1 + y + (x + y)*z^2 + 3*z^3 + 3*z^4 + 3*z^5 + 0(z^6)
sage: L([1, x, y, 0, x+y], valuation=-2, degree=4, constant=3)
z^-2 + x*z^-1 + y + (x + y)*z^2 + 3*z^4 + 3*z^5 + 3*z^6 + 0(z^7)

```

Some additional examples over the integer ring:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: L in Fields
False
sage: 1 / (1 - 2*z)^3
1 + 6*z + 24*z^2 + 80*z^3 + 240*z^4 + 672*z^5 + 1792*z^6 + 0(z^7)

sage: R.<x> = LaurentPolynomialRing(ZZ)
sage: L(x^-2 + 3 + x)
z^-2 + 3 + z
sage: L(x^-2 + 3 + x, valuation=-5, constant=2)
z^-5 + 3*z^-3 + z^-2 + 2*z^-1 + 2 + 2*z + 0(z^2)
sage: L(x^-2 + 3 + x, valuation=-5, degree=0, constant=2)
z^-5 + 3*z^-3 + z^-2 + 2 + 2*z + 2*z^2 + 0(z^3)

```

We can also truncate, shift, and make eventually constant any Laurent series:

```

sage: f = 1 / (z + z^2)
sage: f
z^-1 - 1 + z - z^2 + z^3 - z^4 + z^5 + 0(z^6)
sage: L(f, valuation=2)
z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + z^8 + 0(z^9)
sage: L(f, degree=3)
z^-1 - 1 + z - z^2
sage: L(f, degree=3, constant=2)
z^-1 - 1 + z - z^2 + 2*z^3 + 2*z^4 + 2*z^5 + 0(z^6)
sage: L(f, valuation=1, degree=4)
z - z^2 + z^3
sage: L(f, valuation=1, degree=4, constant=5)
z - z^2 + z^3 + 5*z^4 + 5*z^5 + 5*z^6 + 0(z^7)

```

Power series can be defined recursively (see `define()` for more examples):

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: s = L(None, valuation=0)
sage: s.define(1 + z*s^2)
sage: s
1 + z + 2*z^2 + 5*z^3 + 14*z^4 + 42*z^5 + 132*z^6 + 0(z^7)

```

If we do not explicitly know the exact value of every coefficient, then equality checking will depend on the computed coefficients. If at a certain point we cannot prove two series are different (which involves the coefficients we have computed), then we will raise an error:

```

sage: f = 1 / (z + z^2); f
z^-1 - 1 + z - z^2 + z^3 - z^4 + z^5 + 0(z^6)
sage: f2 = f * 2 # currently no coefficients computed
sage: f3 = f * 3 # currently no coefficients computed
sage: f2 == f3
Traceback (most recent call last):
...
ValueError: undecidable
sage: f2 # computes some of the coefficients of f2
2*z^-1 - 2 + 2*z - 2*z^2 + 2*z^3 - 2*z^4 + 2*z^5 + 0(z^6)
sage: f3 # computes some of the coefficients of f3
3*z^-1 - 3 + 3*z - 3*z^2 + 3*z^3 - 3*z^4 + 3*z^5 + 0(z^6)
sage: f2 == f3
False

```

The implementation of the ring can be either be a sparse or a dense one. The default is a sparse implementation:

```

sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: L.is_sparse()
True
sage: L.<z> = LazyLaurentSeriesRing(ZZ, sparse=False)
sage: L.is_sparse()
False

```

### Element

alias of `sage.rings.lazy_series.LazyLaurentSeries`

### characteristic()

Return the characteristic of this lazy power series ring, which is the same as the characteristic of its base ring.

EXAMPLES:

```

sage: L.<t> = LazyLaurentSeriesRing(ZZ)
sage: L.characteristic()
0
sage: R.<w> = LazyLaurentSeriesRing(GF(11)); R
Lazy Laurent Series Ring in w over Finite Field of size 11
sage: R.characteristic()
11

```

### gen( $n=0$ )

Return the  $n$ -th generator of self.

EXAMPLES:

```

sage: L = LazyLaurentSeriesRing(ZZ, 'z')
sage: L.gen()
z
sage: L.gen(3)
Traceback (most recent call last):
...
IndexError: there is only one generator

```

### gens()

Return the generators of self.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: L.gens()
(z,)
sage: 1/(1 - z)
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + 0(z^7)
```

**is\_sparse()**

Return whether `self` is sparse or not.

EXAMPLES:

```
sage: L = LazyLaurentSeriesRing(ZZ, 'z', sparse=False)
sage: L.is_sparse()
False

sage: L = LazyLaurentSeriesRing(ZZ, 'z', sparse=True)
sage: L.is_sparse()
True
```

**ngens()**

Return the number of generators of `self`.

This is always 1.

EXAMPLES:

```
sage: L.<z> = LazyLaurentSeriesRing(ZZ)
sage: L.ngens()
1
```

**one()**

Return the constant series 1.

EXAMPLES:

```
sage: L = LazyLaurentSeriesRing(ZZ, 'z')
sage: L.one()
1
```

**options(\*get\_value, \*\*set\_value)**

Set and display the options for Lazy Laurent series.

If no parameters are set, then the function returns a copy of the options dictionary.

The options to Lazy Laurent series can be accessed as using [\*LazyLaurentSeriesRing.options\*](#) of [\*LazyLaurentSeriesRing\*](#).

OPTIONS:

- `constant_length` – (default: 3) the number of coefficients to display for nonzero constant series
- `display_length` – (default: 7) the number of coefficients to display from the valuation

EXAMPLES:

```
sage: LLS.<z> = LazyLaurentSeriesRing(QQ)
sage: LLS.options.display_length
```

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```

7
sage: f = 1/(1-z)
sage: f
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + 0(z^7)
sage: LLS.options.display_length = 10
sage: f
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9 + 0(z^10)
sage: g = LLS(lambda n: n^2, valuation=-2, degree=5, constant=42)
sage: g
4*z^-2 + z^-1 + z + 4*z^2 + 9*z^3 + 16*z^4 + 42*z^5 + 42*z^6 + 42*z^7 + 0(z^8)
sage: LLS.options.constant_length = 1
sage: g
4*z^-2 + z^-1 + z + 4*z^2 + 9*z^3 + 16*z^4 + 42*z^5 + 0(z^6)
sage: LazyLaurentSeriesRing.options._reset()
sage: LazyLaurentSeriesRing.options.display_length
7

```

See [GlobalOptions](#) for more features of these options.

**series**(*coefficient*, *valuation*, *degree=None*, *constant=None*)

Return a lazy Laurent series.

INPUT:

- **coefficient** – Python function that computes coefficients or a list
- **valuation** – integer; approximate valuation of the series
- **degree** – (optional) integer
- **constant** – (optional) an element of the base ring

Let the coefficient of index  $i$  mean the coefficient of the term of the series with exponent  $i$ .

Python function **coefficient** returns the value of the coefficient of index  $i$  from input  $s$  and  $i$  where  $s$  is the series itself.

Let valuation be  $n$ . All coefficients of index below  $n$  are zero. If **constant** is not specified, then the **coefficient** function is responsible to compute the values of all coefficients of index  $\geq n$ . If **degree** or **constant** is a pair  $(c, m)$ , then the **coefficient** function is responsible to compute the values of all coefficients of index  $\geq n$  and  $< m$  and all the coefficients of index  $\geq m$  is the constant  $c$ .

EXAMPLES:

```

sage: L = LazyLaurentSeriesRing(ZZ, 'z')
sage: L.series(lambda s, i: i, 5, (1,10))
5*z^5 + 6*z^6 + 7*z^7 + 8*z^8 + 9*z^9 + z^10 + z^11 + z^12 + 0(z^13)

sage: def g(s, i):
....:     if i < 0:
....:         return 1
....:     else:
....:         return s.coefficient(i - 1) + i
sage: e = L.series(g, -5); e
z^-5 + z^-4 + z^-3 + z^-2 + z^-1 + 1 + 2*z + 0(z^2)
sage: f = e^-1; f
z^5 - z^6 - z^11 + 0(z^12)

```

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```
sage: f.coefficient(10)
0
sage: f.coefficient(20)
9
sage: f.coefficient(30)
-219
```

Alternatively, the `coefficient` can be a list of elements of the base ring. Then these elements are read as coefficients of the terms of degrees starting from the valuation. In this case, `constant` may be just an element of the base ring instead of a tuple or can be simply omitted if it is zero.

```
sage: L = LazyLaurentSeriesRing(ZZ, 'z')
sage: f = L.series([1,2,3,4], -5); f
z^-5 + 2*z^-4 + 3*z^-3 + 4*z^-2
sage: g = L.series([1,3,5,7,9], 5, constant=-1); g
z^5 + 3*z^6 + 5*z^7 + 7*z^8 + 9*z^9 - z^10 - z^11 - z^12 + 0(z^13)
```

**some\_elements()**

Return a list of elements of `self`.

EXAMPLES:

```
sage: L = LazyLaurentSeriesRing(ZZ, 'z')
sage: L.some_elements()
[0, 1, z,
 -3*z^-4 + z^-3 - 12*z^-2 - 2*z^-1 - 10 - 8*z + z^2 + z^3,
 z^-2 + 3*z^-1 + 2*z + z^2 + z^3 + z^4 + z^5 + 0(z^6),
 -2*z^-3 - 2*z^-2 + 4*z^-1 + 11 - z - 34*z^2 - 31*z^3 + 0(z^4),
 4*z^-2 + z^-1 + z + 4*z^2 + 9*z^3 + 16*z^4 + 0(z^5)]

sage: L = LazyLaurentSeriesRing(GF(2), 'z')
sage: L.some_elements()
[0, 1, z,
 z^-4 + z^-3 + z^2 + z^3,
 z^-1 + z^2 + z^3 + z^4 + z^5 + 0(z^6),
 1 + z + z^3 + z^4 + z^6 + 0(z^7),
 z^-1 + z + z^3 + 0(z^5)]

sage: L = LazyLaurentSeriesRing(GF(3), 'z')
sage: L.some_elements()
[0, 1, z,
 z^-3 + z^-1 + 2 + z + z^2 + z^3,
 z^2 + z^3 + z^4 + z^5 + 0(z^6),
 z^-3 + z^-2 + z^-1 + 2 + 2*z + 2*z^2 + 2*z^3 + 0(z^4),
 z^-2 + z^-1 + z + z^2 + z^4 + 0(z^5)]
```

**zero()**

Return the zero series.

EXAMPLES:

```
sage: L = LazyLaurentSeriesRing(ZZ, 'z')
sage: L.zero()
0
```

**class** sage.rings.lazy\_series\_ring.LazySeriesRing

Bases: sage.structure.unique\_representation.UniqueRepresentation, sage.structure.parent.Parent

Abstract base class for lazy series.



## PUISEUX SERIES RING

The ring of Puiseux series.

AUTHORS:

- Chris Swierczewski 2016: initial version on <https://github.com/abelfunctions/abelfunctions/tree/master/abelfunctions>
- Frédéric Chapoton 2016: integration of code
- Travis Scrimshaw, Sebastian Oehms 2019-2020: basic improvements and completions

REFERENCES:

- [Wikipedia article Puiseux\\_series](#)

```
class sage.rings.puiseux_series_ring.PuiseuxSeriesRing(laurent_series)
    Bases:      sage.structure.unique_representation.UniqueRepresentation,  sage.rings.ring.
                CommutativeRing
```

Rings of Puiseux series.

EXAMPLES:

```
sage: P = PuiseuxSeriesRing(QQ, 'y')
sage: y = P.gen()
sage: f = y**(4/3) + y**(-5/6); f
y^(-5/6) + y^(4/3)
sage: f.add_bigoh(2)
y^(-5/6) + y^(4/3) + O(y^2)
sage: f.add_bigoh(1)
y^(-5/6) + O(y)
```

**Element**

alias of `sage.rings.puiseux_series_ring_element.PuiseuxSeries`

**base\_extend(R)**

Extend the coefficients.

INPUT:

- R – a ring

EXAMPLES:

```
sage: A = PuiseuxSeriesRing(ZZ, 'y')
sage: A.base_extend(QQ)
Puiseux Series Ring in y over Rational Field
```

**change\_ring(*R*)**

Return a Puiseux series ring over another ring.

INPUT:

- *R* – a ring

EXAMPLES:

```
sage: A = PuiseuxSeriesRing(ZZ, 'y')
sage: A.change_ring(QQ)
Puisseux Series Ring in y over Rational Field
```

**default\_prec()**

Return the default precision of *self*.

EXAMPLES:

```
sage: A = PuiseuxSeriesRing(AA, 'z')
sage: A.default_prec()
20
```

**fraction\_field()**

Return the fraction field of this ring of Laurent series.

If the base ring is a field, then Puiseux series are already a field. If the base ring is a domain, then the Puiseux series over its fraction field is returned. Otherwise, raise a `ValueError`.

EXAMPLES:

```
sage: R = PuiseuxSeriesRing(ZZ, 't', 30).fraction_field()
sage: R
Puisseux Series Ring in t over Rational Field
sage: R.default_prec()
30

sage: PuiseuxSeriesRing(Zmod(4), 't').fraction_field()
Traceback (most recent call last):
...
ValueError: must be an integral domain
```

**gen(*n=0*)**

Return the generator of *self*.

EXAMPLES:

```
sage: A = PuiseuxSeriesRing(AA, 'z')
sage: A.gen()
z
```

**is\_dense()**

Return whether *self* is dense.

EXAMPLES:

```
sage: A = PuiseuxSeriesRing(ZZ, 'y')
sage: A.is_dense()
True
```

**is\_field**(*proof=True*)Return whether `self` is a field.

A Puiseux series ring is a field if and only its base ring is a field.

EXAMPLES:

```
sage: A = PuiseuxSeriesRing(ZZ, 'y')
sage: A.is_field()
False
sage: A.change_ring(QQ).is_field()
True
```

**is\_sparse**()Return whether `self` is sparse.

EXAMPLES:

```
sage: A = PuiseuxSeriesRing(ZZ, 'y')
sage: A.is_sparse()
False
```

**laurent\_series\_ring**()

Return the underlying Laurent series ring.

EXAMPLES:

```
sage: A = PuiseuxSeriesRing(AA, 'z')
sage: A.laurent_series_ring()
Laurent Series Ring in z over Algebraic Real Field
```

**ngens**()Return the number of generators of `self`, namely 1.

EXAMPLES:

```
sage: A = PuiseuxSeriesRing(AA, 'z')
sage: A.ngens()
1
```

**residue\_field**()

Return the residue field of this Puiseux series field if it is a complete discrete valuation field (i.e. if the base ring is a field, in which case it is also the residue field).

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(GF(17))
sage: R.residue_field()
Finite Field of size 17

sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: R.residue_field()
Traceback (most recent call last):
...
TypeError: the base ring is not a field
```

**uniformizer()**

Return a uniformizer of this Puiseux series field if it is a discrete valuation field (i.e. if the base ring is actually a field). Otherwise, an error is raised.

EXAMPLES:

```
sage: R.<t> = PuiseuxSeriesRing(QQ)
sage: R.uniformizer()
t

sage: R.<t> = PuiseuxSeriesRing(ZZ)
sage: R.uniformizer()
Traceback (most recent call last):
...
TypeError: the base ring is not a field
```



## PUISEUX SERIES RING ELEMENT

A Puiseux series is a series of the form

$$p(x) = \sum_{n=N}^{\infty} a_n (x - a)^{n/e},$$

where the integer  $e$  is called the *ramification index* of the series and the number  $a$  is the *center*. A Puiseux series is essentially a Laurent series but with fractional exponents.

EXAMPLES:

We begin by constructing the ring of Puiseux series in  $x$  with coefficients in the rationals:

```
sage: R.<x> = PuiseuxSeriesRing(QQ)
```

This command also defines  $x$  as the generator of this ring.

When constructing a Puiseux series, the ramification index is automatically determined from the greatest common divisor of the exponents:

```
sage: p = x^(1/2); p
x^(1/2)
sage: p.ramification_index()
2
sage: q = x^(1/2) + x**(1/3); q
x^(1/3) + x^(1/2)
sage: q.ramification_index()
6
```

Other arithmetic can be performed with Puiseux Series:

```
sage: p + q
x^(1/3) + 2*x^(1/2)
sage: p - q
-x^(1/3)
sage: p * q
x^(5/6) + x
sage: (p / q).add_bigoh(4/3)
x^(1/6) - x^(1/3) + x^(1/2) - x^(2/3) + x^(5/6) - x + x^(7/6) + O(x^(4/3))
```

Mind the base ring. However, the base ring can be changed:

```
sage: I*q
Traceback (most recent call last):
```

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```
...
TypeError: unsupported operand parent(s) for *: 'Number Field in I with defining_
↳ polynomial x^2 + 1 with I = 1*I' and 'Puisseux Series Ring in x over Rational Field'
sage: qz = q.change_ring(ZZ); qz
x^(1/3) + x^(1/2)
sage: qz.parent()
Puisseux Series Ring in x over Integer Ring
```

Other properties of the Puiseux series can be easily obtained:

```
sage: r = (3*x^(-1/5) + 7*x^(2/5) + (1/2)*x).add_bigoh(6/5); r
3*x^(-1/5) + 7*x^(2/5) + 1/2*x + O(x^(6/5))
sage: r.valuation()
-1/5
sage: r.prec()
6/5
sage: r.precision_absolute()
6/5
sage: r.precision_relative()
7/5
sage: r.exponents()
[-1/5, 2/5, 1]
sage: r.coefficients()
[3, 7, 1/2]
```

Finally, Puiseux series are compatible with other objects in Sage. For example, you can perform arithmetic with Laurent series:

```
sage: L.<x> = LaurentSeriesRing(ZZ)
sage: l = 3*x^(-2) + x^(-1) + 2 + x**3
sage: r + l
3*x^-2 + x^-1 + 3*x^(-1/5) + 2 + 7*x^(2/5) + 1/2*x + O(x^(6/5))
```

#### AUTHORS:

- Chris Swierczewski 2016: initial version on <https://github.com/abelfunctions/abelfunctions/tree/master/abelfunctions>
- Frédéric Chapoton 2016: integration of code
- Travis Scrimshaw, Sebastian Oehms 2019-2020: basic improvements and completions

#### REFERENCES:

- [Wikipedia article Puiseux\\_series](#)

**class** sage.rings.puisseux\_series\_ring\_element.**PuisseuxSeries**

Bases: sage.structure.element.AlgebraElement

A Puiseux series.

$$\sum_{n=-N}^{\infty} a_n x^{n/e}$$

It is stored as a Laurent series:

$$\sum_{n=-N}^{\infty} a_n t^n$$

where  $t = x^{1/e}$ .

INPUT:

- `parent` – the parent ring
- `f` – one of the following types of inputs:
  - instance of *PuiseuxSeries*
  - instance that can be coerced into the Laurent series ring of the parent
- `e` – integer (default: 1) the ramification index

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(QQ)
sage: p = x^(1/2) + x**3; p
x^(1/2) + x^3
sage: q = x**(1/2) - x**(-1/2)
sage: r = q.add_bigoh(7/2); r
-x^(-1/2) + x^(1/2) + O(x^(7/2))
sage: r**2
x^-1 - 2 + x + O(x^3)
```

**add\_bigoh(*prec*)**

Return the truncated series at chosen precision *prec*.

INPUT:

- `prec` – the precision of the series as a rational number

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(QQ)
sage: p = x^(-7/2) + 3 + 5*x^(1/2) - 7*x**3
sage: p.add_bigoh(2)
x^(-7/2) + 3 + 5*x^(1/2) + O(x^2)
sage: p.add_bigoh(0)
x^(-7/2) + O(1)
sage: p.add_bigoh(-1)
x^(-7/2) + O(x^-1)
```

---

**Note:** The precision passed to the method is adapted to the common ramification index:

```
sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = x**(-1/3) + 2*x**(1/5)
sage: p.add_bigoh(1/2)
x^(-1/3) + 2*x^(1/5) + O(x^(7/15))
```

---

**change\_ring(*R*)**

Return self over a the new ring *R*.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = x^(-7/2) + 3 + 5*x^(1/2) - 7*x**3
```

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```
sage: q = p.change_ring(QQ); q
x^(-7/2) + 3 + 5*x^(1/2) - 7*x^3
sage: q.parent()
Puiseux Series Ring in x over Rational Field
```

**coefficients()**

Return the list of coefficients.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = x^(3/4) + 2*x^(4/5) + 3*x^(5/6)
sage: p.coefficients()
[1, 2, 3]
```

**common\_prec(p)**Return the minimum precision of  $p$  and self.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = (x**(-1/3) + 2*x**3)**2
sage: q5 = p.add_bigoh(5); q5
x^(-2/3) + 4*x^(8/3) + 0(x^5)
sage: q7 = p.add_bigoh(7); q7
x^(-2/3) + 4*x^(8/3) + 4*x^6 + 0(x^7)
sage: q5.common_prec(q7)
5
sage: q7.common_prec(q5)
5
```

**degree()**

Return the degree of self.

EXAMPLES:

```
sage: P.<y> = PolynomialRing(GF(5))
sage: R.<x> = PuiseuxSeriesRing(P)
sage: p = 3*y*x**(-2/3) + 2*y**2*x**(1/5); p
3*y*x^(-2/3) + 2*y^2*x^(1/5)
sage: p.degree()
1/5
```

**exponents()**

Return the list of exponents.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = x^(3/4) + 2*x^(4/5) + 3*x^(5/6)
sage: p.exponents()
[3/4, 4/5, 5/6]
```

**inverse()**

Return the inverse of self.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(QQ)
sage: p = x^(-7/2) + 3 + 5*x^(1/2) - 7*x**3
sage: 1/p
x^(7/2) - 3*x^7 - 5*x^(15/2) + 7*x^10 + 9*x^(21/2) + 30*x^11 +
25*x^(23/2) + O(x^(27/2))
```

**is\_monomial()**

Return whether `self` is a monomial.

This is True if and only if `self` is  $x^p$  for some rational  $p$ .

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(QQ)
sage: p = x^(1/2) + 3/4 * x^(2/3)
sage: p.is_monomial()
False
sage: q = x**(11/13)
sage: q.is_monomial()
True
sage: q = 4*x**(11/13)
sage: q.is_monomial()
False
```

**is\_unit()**

Return whether `self` is a unit.

A Puiseux series is a unit if and only if its leading coefficient is.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = x^(-7/2) + 3 + 5*x^(1/2) - 7*x**3
sage: p.is_unit()
True
sage: q = 4 * x^(-7/2) + 3 * x**4
sage: q.is_unit()
False
```

**is\_zero()**

Return whether `self` is zero.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(QQ)
sage: p = x^(1/2) + 3/4 * x^(2/3)
sage: p.is_zero()
False
sage: R.zero().is_zero()
True
```

**laurent\_part()**

Return the underlying Laurent series.

EXAMPLES:

```

sage: R.<x> = PuiseuxSeriesRing(QQ)
sage: p = x^(1/2) + 3/4 * x^(2/3)
sage: p.laurent_part()
x^3 + 3/4*x^4

```

**laurent\_series()**

If self is a Laurent series, return it as a Laurent series.

EXAMPLES:

```

sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = x**(1/2) - x**(-1/2)
sage: p.laurent_series()
Traceback (most recent call last):
...
ArithmeticError: self is not a Laurent series
sage: q = p**2
sage: q.laurent_series()
x^-1 - 2 + x

```

**list()**

Return the list of coefficients indexed by the exponents of the the corresponding Laurent series.

EXAMPLES:

```

sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = x^(3/4) + 2*x^(4/5) + 3*x^(5/6)
sage: p.list()
[1, 0, 0, 2, 0, 3]

```

**power\_series()**

If self is a power series, return it as a power series.

EXAMPLES:

```

sage: R.<x> = PuiseuxSeriesRing(QQbar)
sage: p = x**(3/2) - QQbar(I)*x**(1/2)
sage: p.power_series()
Traceback (most recent call last):
...
ArithmeticError: self is not a power series
sage: q = p**2
sage: q.power_series()
-x - 2*I*x^2 + x^3

```

**prec()**

Return the precision of self.

EXAMPLES:

```

sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = (x**(-1/3) + 2*x**3)**2; p
x^(-2/3) + 4*x^(8/3) + 4*x^6
sage: q = p.add_bigoh(5); q
x^(-2/3) + 4*x^(8/3) + 0(x^5)

```

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```
sage: q.prec()
5
```

**precision\_absolute()**

Return the precision of self.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = (x**(-1/3) + 2*x**3)**2; p
x^(-2/3) + 4*x^(8/3) + 4*x^6
sage: q = p.add_bigoh(5); q
x^(-2/3) + 4*x^(8/3) + 0(x^5)
sage: q.prec()
5
```

**precision\_relative()**

Return the relative precision of the series.

The relative precision of the Puiseux series is the difference between its absolute precision and its valuation.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(GF(3))
sage: p = (x**(-1/3) + 2*x**3)**2; p
x^(-2/3) + x^(8/3) + x^6
sage: q = p.add_bigoh(7); q
x^(-2/3) + x^(8/3) + x^6 + 0(x^7)
sage: q.precision_relative()
23/3
```

**ramification\_index()**

Return the ramification index.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(QQ)
sage: p = x^(1/2) + 3/4 * x^(2/3)
sage: p.ramification_index()
6
```

**shift(r)**Return this Puiseux series multiplied by  $x^r$ .

EXAMPLES:

```
sage: P.<y> = LaurentPolynomialRing(ZZ)
sage: R.<x> = PuiseuxSeriesRing(P)
sage: p = y*x**(-1/3) + 2*y^(-2)*x**(1/2); p
y*x^(-1/3) + (2*y^(-2))*x^(1/2)
sage: p.shift(3)
y*x^(8/3) + (2*y^(-2))*x^(7/2)
```

**truncate(r)**Return the Puiseux series of degree  $< r$ .This is equivalent to self modulo  $x^r$ .

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(ZZ)
sage: p = (x**(-1/3) + 2*x**3)**2; p
x^(-2/3) + 4*x^(8/3) + 4*x^6
sage: q = p.truncate(5); q
x^(-2/3) + 4*x^(8/3)
sage: q == p.add_bigoh(5)
True
```

**valuation()**

Return the valuation of `self`.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(QQ)
sage: p = x^(-7/2) + 3 + 5*x^(1/2) - 7*x**3
sage: p.valuation()
-7/2
```

**variable()**

Return the variable of `self`.

EXAMPLES:

```
sage: R.<x> = PuiseuxSeriesRing(QQ)
sage: p = x^(-7/2) + 3 + 5*x^(1/2) - 7*x**3
sage: p.variable()
'x'
```



## TATE ALGEBRAS

Let  $K$  be a finite extension of  $\mathbf{Q}_p$  for some prime number  $p$  and let  $(v_1, \dots, v_n)$  be a tuple of real numbers.

The associated Tate algebra consists of series of the form

$$\sum_{i_1, \dots, i_n \in \mathbf{N}} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

for which the quantity

$$\text{val}(a_{i_1, \dots, i_n}) - (v_1 i_1 + \cdots + v_n i_n)$$

goes to infinity when the multi-index  $(i_1, \dots, i_n)$  goes to infinity.

These series converge on the closed disc defined by the inequalities  $\text{val}(x_i) \geq -v_i$  for all  $i \in \{1, \dots, n\}$ . The  $v_i$ 's are then the logarithms of the radii of convergence of the series in the above Tate algebra; they will be called the log radii of convergence.

We can create Tate algebras using the constructor `sage.rings.tate_algebra.TateAlgebra()`:

```
sage: K = Qp(2, 5, print_mode='digits')
sage: A.<x,y> = TateAlgebra(K)
sage: A
Tate Algebra in x (val >= 0), y (val >= 0) over 2-adic Field with capped relative
↳precision 5
```

As we observe, the default value for the log radii of convergence is 0 (the series then converge on the closed unit disc).

We can specify different log radii using the following syntax:

```
sage: B.<u,v> = TateAlgebra(K, log_radii=[1,2]); B
Tate Algebra in u (val >= -1), v (val >= -2) over 2-adic Field with capped relative
↳precision 5
```

Note that if we pass in the ring of integers of  $p$ -adic field, the same Tate algebra is returned:

```
sage: A1.<x,y> = TateAlgebra(K.integer_ring()); A1
Tate Algebra in x (val >= 0), y (val >= 0) over 2-adic Field with capped relative
↳precision 5
sage: A is A1
True
```

However the method `integer_ring()` constructs the integer ring of a Tate algebra, that is the subring consisting of series bounded by 1 on the domain of convergence:

```
sage: Ao = A.integer_ring()
sage: Ao
Integer ring of the Tate Algebra in x (val >= 0), y (val >= 0) over 2-adic Field with
↳ capped relative precision 5
```

Now we can build elements:

```
sage: f = 5 + 2*x*y^3 + 4*x^2*y^2; f
...00101 + ...000010*x*y^3 + ...0000100*x^2*y^2
sage: g = x^3*y + 2*x*y; g
...00001*x^3*y + ...000010*x*y
```

and perform all usual arithmetic operations on them:

```
sage: f + g
...00001*x^3*y + ...00101 + ...000010*x*y^3 + ...000010*x*y + ...0000100*x^2*y^2
sage: f * g
...00101*x^3*y + ...000010*x^4*y^4 + ...001010*x*y + ...0000100*x^5*y^3 + ...0000100*x^
↳ 2*y^4 + ...00001000*x^3*y^3
```

An element in the integer ring is invertible if and only if its reduction modulo  $p$  is a nonzero constant. In our example,  $f$  is invertible (its reduction modulo 2 is 1) but  $g$  is not:

```
sage: f.inverse_of_unit()
...01101 + ...01110*x*y^3 + ...10100*x^2*y^6 + ... + 0(2^5 * <x, y>)
sage: g.inverse_of_unit()
Traceback (most recent call last):
...
ValueError: this series is not invertible
```

The notation  $O(2^5)$  in the result above hides a series which lies in  $2^5$  times the integer ring of  $A$ , that is a series which is bounded by  $|2^5|$  (2-adic norm) on the domain of convergence.

We can also evaluate series in a point of the domain of convergence (in the base field or in an extension):

```
sage: L.<a> = Qq(2^3, 5)
sage: f(a^2, 2*a)
1 + 2^2 + a*2^4 + 0(2^5)

sage: var('u')
u
sage: L.<pi> = K.change(print_mode="series").extension(u^3 - 2)
sage: g(pi, 2*pi)
pi^7 + pi^8 + pi^19 + pi^20 + 0(pi^21)
```

Computations with ideals in Tate algebras are also supported:

```
sage: f = 7*x^3*y + 2*x*y - x*y^2 - 6*y^5
sage: g = x*y^4 + 8*x^3 - 3*y^3 + 1
sage: I = A.ideal([f, g])
sage: I.groebner_basis()
[...00001*x^2*y^3 + ...00001*y^4 + ...10001*x^2 + ... + 0(2^5 * <x, y>),
 ...00001*x*y^4 + ...11101*y^3 + ...00001 + ... + 0(2^5 * <x, y>),
 ...00001*y^5 + ...11111*x*y^3 + ...01001*x^2*y + ... + 0(2^5 * <x, y>),
```

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```
...00001*x^3 + ...01001*x*y + ...10110*y^4 + ...01110*x + O(2^5 * <x, y>)]
sage: (x^2 + 3*y)*f + 1/2*(x^3*y + x*y)*g in I
True
```

AUTHORS:

- Xavier Caruso, Thibaut Verron (2018-09)

**class** sage.rings.tate\_algebra.TateAlgebraFactory

Bases: sage.structure.factory.UniqueFactory

Construct a Tate algebra over a  $p$ -adic field.

Given a  $p$ -adic field  $K$ , variables  $X_1, \dots, X_k$  and convergence log radii  $v_1, \dots, v_n$  in  $\mathbf{R}$ , the corresponding Tate algebra  $KX_1, \dots, X_k$  consists of power series with coefficients  $a_{i_1, \dots, i_n}$  in  $K$  such that

$$\text{val}(a_{i_1, \dots, i_n}) - (i_1 v_1 + \dots + i_n v_n)$$

tends to infinity as  $i_1, \dots, i_n$  go towards infinity.

INPUT:

- **base** – a  $p$ -adic ring or field; if a ring is given, the Tate algebra over its fraction field will be constructed
- **prec** – an integer or None (default: None), the precision cap; it is used if an exact object must be truncated in order to do an arithmetic operation. If left as None, it will be set to the precision cap of the base field.
- **log\_radII** – an integer or a list or a tuple of integers (default: 0), the value(s)  $v_i$ . If an integer is given, this will be the common value for all  $v_i$ .
- **names** – names of the indeterminates
- **order** – the monomial ordering (default: degrevlex) used to break ties when comparing terms with the same coefficient valuation

EXAMPLES:

```
sage: R = Zp(2, 10, print_mode='digits'); R
2-adic Ring with capped relative precision 10
sage: A.<x,y> = TateAlgebra(R, order='lex'); A
Tate Algebra in x (val >= 0), y (val >= 0) over 2-adic Field with capped relative_
->precision 10
```

We observe that the result is the Tate algebra over the fraction field of  $R$  and not  $R$  itself:

```
sage: A.base_ring()
2-adic Field with capped relative precision 10
sage: A.base_ring() is R.fraction_field()
True
```

If we want to construct the ring of integers of the Tate algebra, we must use the method `integer_ring()`:

```
sage: Ao = A.integer_ring(); Ao
Integer ring of the Tate Algebra in x (val >= 0), y (val >= 0) over 2-adic Field_
->with capped relative precision 10
sage: Ao.base_ring()
2-adic Ring with capped relative precision 10
```

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```
sage: Ao.base_ring() is R
True
```

The term ordering is used (in particular) to determine how series are displayed. Terms are compared first according to the valuation of their coefficient, and ties are broken using the monomial ordering:

```
sage: A.term_order()
Lexicographic term order
sage: f = 2 + y^5 + x^2; f
...0000000001*x^2 + ...0000000001*y^5 + ...00000000010
sage: B.<x,y> = TateAlgebra(R); B
Tate Algebra in x (val >= 0), y (val >= 0) over 2-adic Field with capped relative_
↳precision 10
sage: B.term_order()
Degree reverse lexicographic term order
sage: B(f)
...0000000001*y^5 + ...0000000001*x^2 + ...00000000010
```

Here are examples of Tate algebra with smaller radii of convergence:

```
sage: B.<x,y> = TateAlgebra(R, log_radii=-1); B
Tate Algebra in x (val >= 1), y (val >= 1) over 2-adic Field with capped relative_
↳precision 10
sage: C.<x,y> = TateAlgebra(R, log_radii=[-1,-2]); C
Tate Algebra in x (val >= 1), y (val >= 2) over 2-adic Field with capped relative_
↳precision 10
```

#### AUTHORS:

- Xavier Caruso, Thibaut Verron (2018-09)

**create\_key**(*base*, *prec*=None, *log\_radii*=0, *names*=None, *order*='degrevlex')

Create a key from the input parameters.

#### INPUT:

- *base* – a  $p$ -adic ring or field
- *prec* – an integer or None (default: None)
- *log\_radii* – an integer or a list or a tuple of integers (default: 0)
- *names* – names of the indeterminates
- *order* – a monomial ordering (default: degrevlex)

#### EXAMPLES:

```
sage: TateAlgebra.create_key(Zp(2), names=['x', 'y'])
(2-adic Field with capped relative precision 20,
 20,
 (0, 0),
 ('x', 'y'),
 Degree reverse lexicographic term order)
```

**create\_object**(*version*, *key*)

Create an object using the given key.

**class** sage.rings.tate\_algebra.TateAlgebra\_generic(*field, prec, log\_radII, names, order,*  
*integral=False*)

Bases: sage.rings.ring.CommutativeAlgebra

Initialize the Tate algebra

**absolute\_e()**

Return the absolute index of ramification of this Tate algebra.

It is equal to the absolute index of ramification of the field of coefficients.

EXAMPLES:

```
sage: R = Zp(2)
sage: A.<u,v> = TateAlgebra(R)
sage: A.absolute_e()
1

sage: R.<a> = Zq(2^3)
sage: A.<u,v> = TateAlgebra(R)
sage: A.absolute_e()
1

sage: S.<a> = R.extension(x^2 - 2)
sage: A.<u,v> = TateAlgebra(S)
sage: A.absolute_e()
2
```

**characteristic()**

Return the characteristic of this algebra.

EXAMPLES:

```
sage: R = Zp(2, 10, print_mode='digits')
sage: A.<x,y> = TateAlgebra(R)
sage: A.characteristic()
0
```

**gen(*n=0*)**

Return the *n*-th generator of this Tate algebra.

INPUT:

- *n* - an integer (default: 0), the index of the requested generator

EXAMPLES:

```
sage: R = Zp(2, 10, print_mode='digits')
sage: A.<x,y> = TateAlgebra(R)
sage: A.gen()
...0000000001*x
sage: A.gen(0)
...0000000001*x
sage: A.gen(1)
...0000000001*y
sage: A.gen(2)
Traceback (most recent call last):
```

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```
...
ValueError: generator not defined
```

**gens()**

Return the list of generators of this Tate algebra.

EXAMPLES:

```
sage: R = Zp(2, 10, print_mode='digits')
sage: A.<x,y> = TateAlgebra(R)
sage: A.gens()
(...0000000001*x, ...0000000001*y)
```

**integer\_ring()**

Return the ring of integers (consisting of series bounded by 1 in the domain of convergence) of this Tate algebra.

EXAMPLES:

```
sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: Ao = A.integer_ring()
sage: Ao
Integer ring of the Tate Algebra in x (val >= 0), y (val >= 0) over 2-adic
Field with capped relative precision 10

sage: x in Ao
True
sage: x/2 in Ao
False
```

**is\_integral\_domain()**

Return True since any Tate algebra is an integral domain.

EXAMPLES:

```
sage: A.<x,y> = TateAlgebra(Zp(3))
sage: A.is_integral_domain()
True
```

**log\_radII()**

Return the list of the log-radii of convergence radii defining this Tate algebra.

EXAMPLES:

```
sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: A.log_radII()
(0, 0)

sage: B.<x,y> = TateAlgebra(R, log_radII=1)
sage: B.log_radII()
(1, 1)

sage: C.<x,y> = TateAlgebra(R, log_radII=(1,-1))
```

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```
sage: C.log_radII()
(1, -1)
```

**monoid\_of\_terms()**

Return the monoid of terms of this Tate algebra.

EXAMPLES:

```
sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: A.monoid_of_terms()
Monoid of terms in x (val >= 0), y (val >= 0) over 2-adic Field with capped_
relative precision 10
```

**ngens()**

Return the number of generators of this algebra.

EXAMPLES:

```
sage: R = Zp(2, 10, print_mode='digits')
sage: A.<x,y> = TateAlgebra(R)
sage: A.ngens()
2
```

**precision\_cap()**

Return the precision cap of this Tate algebra.

NOTE:

The precision cap is the truncation precision used for arithmetic operations computed by successive approximations (as inversion).

EXAMPLES:

By default the precision cap is the precision cap of the field of coefficients:

```
sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: A.precision_cap()
10
```

But it could be different (either smaller or larger) if we ask to:

```
sage: A.<x,y> = TateAlgebra(R, prec=5)
sage: A.precision_cap()
5

sage: A.<x,y> = TateAlgebra(R, prec=20)
sage: A.precision_cap()
20
```

**prime()**

Return the prime, that is the characteristic of the residue field.

EXAMPLES:

```

sage: R = Zp(3)
sage: A.<x,y> = TateAlgebra(R)
sage: A.prime()
3

```

**random\_element**(*degree=2, terms=5, integral=False, prec=None*)

Return a random element of this Tate algebra.

INPUT:

- *degree* – an integer (default: 2), an upper bound on the total degree of the result
- *terms* – an integer (default: 5), the maximal number of terms of the result
- *integral* – a boolean (default: False); if True the result will be in the ring of integers
- *prec* – (optional) an integer, the precision of the result

EXAMPLES:

```

sage: R = Zp(2, prec=10, print_mode="digits")
sage: A.<x,y> = TateAlgebra(R)
sage: A.random_element() # random
(...00101000.01)*y + ...1111011111*x^2 + ...0010010001*x*y + ...110000011 + ...
↪010100100*y^2

sage: A.random_element(degree=5, terms=3) # random
(...0101100.01)*x^2*y + (...01000011.11)*y^2 + ...00111011*x*y

sage: A.random_element(integral=True) # random
...000111101*x + ...1101110101 + ...00010010110*y + ...101110001100*x*y + ...
↪000001100100*y^2

```

Note that if we are already working on the ring of integers, specifying *integral=False* has no effect:

```

sage: Ao = A.integer_ring()
sage: f = Ao.random_element(integral=False); f # random
...1100111011*x^2 + ...1110100101*x + ...1100001101*y + ...1110110001 + ...
↪01011010110*y^2
sage: f in Ao
True

```

When the log radii are negative, integral series may have non integral coefficients:

```

sage: B.<x,y> = TateAlgebra(R, log_radii=[-1,-2])
sage: B.random_element(integral=True) # random
(...1111111.001)*x*y + (...111000101.1)*x + (...11010111.01)*y^2 + ...
↪0010011011*y + ...0010100011000

```

**some\_elements**()

Return a list of elements in this Tate algebra.

EXAMPLES:

```

sage: R = Zp(2, 10, print_mode='digits')
sage: A.<x,y> = TateAlgebra(R)
sage: A.some_elements()

```

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```
[0,
 ...000000000010,
 ...00000000001*x,
 ...00000000001*y,
 ...000000000010*x*y,
 ...000000000100,
 ...00000000001*x + ...000000000010,
 ...00000000001*y + ...000000000010,
 ...000000000010*x*y + ...000000000010,
 ...00000000010*x,
 ...00000000001*x + ...00000000001*y,
 ...00000000001*x + ...000000000010*x*y,
 ...00000000010*y,
 ...00000000001*y + ...000000000010*x*y,
 ...000000000100*x*y]
```

**term\_order()**

Return the monomial order used in this algebra.

EXAMPLES:

```
sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: A.term_order()
Degree reverse lexicographic term order

sage: A.<x,y> = TateAlgebra(R, order='lex')
sage: A.term_order()
Lexicographic term order
```

**variable\_names()**

Return the names of the variables of this algebra.

EXAMPLES:

```
sage: R = Zp(2, 10, print_mode='digits')
sage: A.<x,y> = TateAlgebra(R)
sage: A.variable_names()
('x', 'y')
```

**class sage.rings.tate\_algebra.TateTermMonoid(A)**

Bases: `sage.monoids.monoid.Monoid_class`, `sage.structure.unique_representation.UniqueRepresentation`

A base class for Tate algebra terms

A term in a Tate algebra  $K\{X_1, \dots, X_n\}$  (resp. in its ring of integers) is a monomial in this ring.

Those terms form a pre-ordered monoid, with term multiplication and the term order of the parent Tate algebra.

**Element**

alias of `sage.rings.tate_algebra_element.TateAlgebraTerm`

**algebra\_of\_series()**

Return the Tate algebra corresponding to this Tate term monoid.

EXAMPLES:

```

sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.algebra_of_series()
Tate Algebra in x (val >= 0), y (val >= 0) over 2-adic Field with capped_
relative precision 10
sage: T.algebra_of_series() is A
True

```

**base\_ring()**

Return the base ring of this Tate term monoid.

EXAMPLES:

```

sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.base_ring()
2-adic Field with capped relative precision 10

```

We observe that the base field is not R but its fraction field:

```

sage: T.base_ring() is R
False
sage: T.base_ring() is R.fraction_field()
True

```

If we really want to create an integral Tate algebra, we have to invoke the method `integer_ring()`:

```

sage: Ao = A.integer_ring(); Ao
Integer ring of the Tate Algebra in x (val >= 0), y (val >= 0) over 2-adic_
Field with capped relative precision 10
sage: Ao.base_ring()
2-adic Ring with capped relative precision 10
sage: Ao.base_ring() is R
True

```

**gen( $n=0$ )**

Return the  $n$ -th generator of this monoid of terms.

INPUT:

- $n$  - an integer (default: 0), the index of the requested generator

EXAMPLES:

```

sage: R = Zp(2, 10, print_mode='digits')
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.gen()
...0000000001*x
sage: T.gen(0)
...0000000001*x
sage: T.gen(1)
...0000000001*y

```

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```
sage: T.gen(2)
Traceback (most recent call last):
...
ValueError: generator not defined
```

**gens()**

Return the list of generators of this monoid of terms.

EXAMPLES:

```
sage: R = Zp(2, 10, print_mode='digits')
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.gens()
(...0000000001*x, ...0000000001*y)
```

**log\_radII()**

Return the log radii of convergence of this Tate term monoid.

EXAMPLES:

```
sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.log_radII()
(0, 0)

sage: B.<x,y> = TateAlgebra(R, log_radII=[1,2])
sage: B.monoid_of_terms().log_radII()
(1, 2)
```

**ngens()**

Return the number of variables in the Tate term monoid

EXAMPLES:

```
sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.ngens()
2
```

**prime()**

Return the prime, that is the characteristic of the residue field.

EXAMPLES:

```
sage: R = Zp(3)
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.prime()
3
```

**some\_elements()**

Return a list of elements in this monoid of terms.

EXAMPLES:

```
sage: R = Zp(2, 10, print_mode='digits')
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.some_elements()
[...000000000010, ...00000000001*x, ...00000000001*y, ...000000000010*x*y]
```

**term\_order()**

Return the term order on this Tate term monoid.

EXAMPLES:

```
sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.term_order() # default term order is grevlex
Degree reverse lexicographic term order

sage: A.<x,y> = TateAlgebra(R, order='lex')
sage: T = A.monoid_of_terms()
sage: T.term_order()
Lexicographic term order
```

**variable\_names()**

Return the names of the variables of this Tate term monoid.

EXAMPLES:

```
sage: R = Zp(2, 10)
sage: A.<x,y> = TateAlgebra(R)
sage: T = A.monoid_of_terms()
sage: T.variable_names()
('x', 'y')
```

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