# Sage Reference Manual: Category Framework

***Release 8.2***

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CHAPTER

ONE

# THE SAGE CATEGORY FRAMEWORK

## 1.1 Elements, parents, and categories in Sage: a (draft of) primer

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### 1.1.1 Abstract

The purpose of categories in Sage is to translate the mathematical concept of categories (category of groups, of vector spaces, . . . ) into a concrete software engineering design pattern for:

- organizing and promoting generic code
- fostering consistency across the Sage library (naming conventions, doc, tests)
- embedding more mathematical knowledge into the system

This design pattern is largely inspired from Axiom and its followers (Aldor, Fricas, MuPAD, . . . ). It differs from those by:

- blending in the Magma inspired concept of Parent/Element
- being built on top of (and not into) the standard Python object oriented and class hierarchy mechanism. This did not require changing the language, and could in principle be implemented in any language supporting the creation of new classes dynamically.

The general philosophy is that *Building mathematical information into the system yields more expressive, more conceptual and, at the end, easier to maintain and faster code* (within a programming realm; this would not necessarily apply to specialized libraries like gmp!).

#### One line pitch for mathematicians

Categories in Sage provide a library of interrelated bookshelves, with each bookshelf containing algorithms, tests, documentation, or some mathematical facts about the objects of a given category (e.g. groups).

#### One line pitch for programmers

Categories in Sage provide a large hierarchy of abstract classes for mathematical objects. To keep it maintainable, the inheritance information between the classes is not hardcoded but instead reconstructed dynamically from duplication free semantic information.

### 1.1.2 Introduction: Sage as a library of objects and algorithms

The Sage library, with more than one million lines of code, documentation, and tests, implements:

- Thousands of different kinds of objects (classes):

  Integers, polynomials, matrices, groups, number fields, elliptic curves, permutations, morphisms, languages, … and a few racoons …

- Tens of thousands methods and functions:

  Arithmetic, integer and polynomial factorization, pattern matching on words, …

#### Some challenges

- How to organize this library?

  One needs some bookshelves to group together related objects and algorithms.

- How to ensure consistency?

  Similar objects should behave similarly:

```
sage: Permutations(5).cardinality()
120

sage: GL(2,2).cardinality()
6

sage: A=random_matrix(ZZ,6,3,x=7)
sage: L=LatticePolytope(A.rows())
sage: L.npoints()                # oops!   # random
37
```

- How to ensure robustness?
- How to reduce duplication?

  Example: binary powering:

```
sage: m = 3
sage: m^8 == m*m*m*m*m*m*m*m == ((m^2)^2)^2
True
```

```
sage: m=random_matrix(QQ, 4, algorithm='echelonizable', rank=3, upper_bound=60)
sage: m^8 == m*m*m*m*m*m*m*m == ((m^2)^2)^2
True
```

We want to implement binary powering only once, as *generic* code that will apply in all cases.

### 1.1.3 A bit of help from abstract algebra

#### The hierarchy of categories

What makes binary powering work in the above examples? In both cases, we have *a set* endowed with a *multiplicative binary operation* which is *associative* and which has a unit element. Such a set is called a *monoid*, and binary powering (to a non-negative power) works generally for any monoid.

Sage knows about monoids:

```
sage: Monoids()
Category of monoids
```

and sure enough, binary powering is defined there:

```
sage: m._pow_int.__module__
'sage.categories.monoids'
```

That's our bookshelf! And it's used in many places:

```
sage: GL(2,ZZ) in Monoids()
True
sage: NN in Monoids()
True
```

For a less trivial bookshelf we can consider euclidean rings: once we know how to do euclidean division in some set $R$, we can compute gcd's in $R$ generically using the Euclidean algorithm.

We are in fact very lucky: abstract algebra provides us right away with a large and robust set of bookshelves which is the result of centuries of work of mathematicians to identify the important concepts. This includes for example:

```
sage: Sets()
Category of sets

sage: Groups()
Category of groups

sage: Rings()
Category of rings

sage: Fields()
Category of fields

sage: HopfAlgebras(QQ)
Category of hopf algebras over Rational Field
```

Each of the above is called a *category*. It typically specifies what are the operations on the elements, as well as the axioms satisfied by those operations. For example the category of groups specifies that a group is a set endowed with a binary operation (the multiplication) which is associative and admits a unit and inverses.

Each set in Sage knows which bookshelf of generic algorithms it can use, that is to which category it belongs:

```
sage: G = GL(2,ZZ)
sage: G.category()
Category of infinite groups
```

In fact a group is a semigroup, and Sage knows about this:

```
sage: Groups().is_subcategory(Semigroups())
True
sage: G in Semigroups()
True
```

Altogether, our group gets algorithms from a bunch of bookshelves:

```
sage: G.categories()
[Category of infinite groups, Category of groups, Category of monoids,
 ...,
 Category of magmas,
 Category of infinite sets, ...]
```

Those can be viewed graphically:

```
sage: g = Groups().category_graph()
sage: g.set_latex_options(format="dot2tex")
sage: view(g)                     # not tested
```

In case `dot2tex` is not available, you can use instead:

```
sage: g.show(vertex_shape=None, figsize=20)
```

Here is an overview of all categories in Sage:

```
sage: g = sage.categories.category.category_graph()
sage: g.set_latex_options(format="dot2tex")
sage: view(g)                     # not tested
```

Wrap-up: generic algorithms in Sage are organized in a hierarchy of bookshelves modelled upon the usual hierarchy of categories provided by abstract algebra.

### Elements, Parents, Categories

#### Parent

A *parent* is a Python instance modelling a set of mathematical elements together with its additional (algebraic) structure.

Examples include the ring of integers, the group $S_3$, the set of prime numbers, the set of linear maps between two given vector spaces, and a given finite semigroup.

These sets are often equipped with additional structure: the set of all integers forms a ring. The main way of encoding this information is specifying which categories a parent belongs to.

It is completely possible to have different Python instances modelling the same set of elements. For example, one might want to consider the ring of integers, or the poset of integers under their standard order, or the poset of integers under divisibility, or the semiring of integers under the operations of maximum and addition. Each of these would be a different instance, belonging to different categories.

For a given model, there should be a unique instance in Sage representing that parent:

```
sage: IntegerRing() is IntegerRing()
True
```

### Element

An *element* is a Python instance modelling a mathematical element of a set.

Examples of element include 5 in the integer ring, $x^3 - x$ in the polynomial ring in $x$ over the rationals, $4 + O(3^3)$ in the 3-adics, the transposition $(12)$ in $S_3$, and the identity morphism in the set of linear maps from $\mathbf{Q}^3$ to $\mathbf{Q}^3$.

Every element in Sage has a parent. The standard idiom in Sage for creating elements is to create their parent, and then provide enough data to define the element:

```
sage: R = PolynomialRing(ZZ, name='x')
sage: R([1,2,3])
3*x^2 + 2*x + 1
```

One can also create elements using various methods on the parent and arithmetic of elements:

```
sage: x = R.gen()
sage: 1 + 2*x + 3*x^2
3*x^2 + 2*x + 1
```

Unlike parents, elements in Sage are not necessarily unique:

```
sage: ZZ(5040) is ZZ(5040)
False
```

Many parents model algebraic structures, and their elements support arithmetic operations. One often further wants to do arithmetic by combining elements from different parents: adding together integers and rationals for example. Sage supports this feature using coercion (see `sage.structure.coerce` for more details).

It is possible for a parent to also have simultaneously the structure of an element. Consider for example the monoid of all finite groups, endowed with the Cartesian product operation. Then, every finite group (which is a parent) is also an element of this monoid. This is not yet implemented, and the design details are not yet fixed but experiments are underway in this direction.

**Todo:** Give a concrete example, typically using `ElementWrapper`.

### Category

A *category* is a Python instance modelling a mathematical category.

Examples of categories include the category of finite semigroups, the category of all (Python) objects, the category of $\mathbf{Z}$-algebras, and the category of Cartesian products of $\mathbf{Z}$-algebras:

```
sage: FiniteSemigroups()
Category of finite semigroups
sage: Objects()
Category of objects
sage: Algebras(ZZ)
Category of algebras over Integer Ring
sage: Algebras(ZZ).CartesianProducts()
Category of Cartesian products of algebras over Integer Ring
```

Mind the 's' in the names of the categories above; `GroupAlgebra` and `GroupAlgebras` are distinct things.

Every parent belongs to a collection of categories. Moreover, categories are interrelated by the *super categories* relation. For example, the category of rings is a super category of the category of fields, because every field is also a ring.

A category serves two roles:

- to provide a model for the mathematical concept of a category and the associated structures: homsets, morphisms, functorial constructions, axioms.
- to organize and promote generic code, naming conventions, documentation, and tests across similar mathematical structures.

### CategoryObject

Objects of a mathematical category are not necessarily parents. Parent has a superclass that provides a means of modeling such.

For example, the category of schemes does not have a faithful forgetful functor to the category of sets, so it does not make sense to talk about schemes as parents.

### Morphisms, Homsets

As category theorists will expect, *Morphisms* and *Homsets* will play an ever more important role, as support for them will improve.

---

Much of the mathematical information in Sage is encoded as relations between elements and their parents, parents and their categories, and categories and their super categories:

```
sage: 1.parent()
Integer Ring

sage: ZZ
Integer Ring

sage: ZZ.category()
Join of Category of euclidean domains
    and Category of infinite enumerated sets
    and Category of metric spaces

sage: ZZ.categories()
[Join of Category of euclidean domains
     and Category of infinite enumerated sets
     and Category of metric spaces,
 Category of euclidean domains, Category of principal ideal domains,
```

```
 Category of unique factorization domains, Category of gcd domains,
 Category of integral domains, Category of domains,
 Category of commutative rings, Category of rings, ...
 Category of magmas and additive magmas, ...
 Category of monoids, Category of semigroups,
 Category of commutative magmas, Category of unital magmas, Category of magmas,
 Category of commutative additive groups, ..., Category of additive magmas,
 Category of infinite enumerated sets, Category of enumerated sets,
 Category of infinite sets, Category of metric spaces,
 Category of topological spaces, Category of sets,
 Category of sets with partial maps,
 Category of objects]

sage: g = EuclideanDomains().category_graph()
sage: g.set_latex_options(format="dot2tex")
sage: view(g)                       # not tested
```

### 1.1.4 A bit of help from computer science

#### Hierarchy of classes

How are the bookshelves implemented in practice?

Sage uses the classical design paradigm of Object Oriented Programming (OOP). Its fundamental principle is that any object that a program is to manipulate should be modelled by an *instance* of a *class*. The class implements:

- a *data structure*: which describes how the object is stored,
- *methods*: which describe the operations on the object.

The instance itself contains the data for the given object, according to the specified data structure.

Hence, all the objects mentioned above should be instances of some classes. For example, an integer in Sage is an instance of the class `Integer` (and it knows about it!):

```
sage: i = 12
sage: type(i)
<type 'sage.rings.integer.Integer'>
```

Applying an operation is generally done by *calling a method*:

```
sage: i.factor()
2^2 * 3

sage: x = var('x')
sage: p = 6*x^2 + 12*x + 6
sage: type(p)
<type 'sage.symbolic.expression.Expression'>
sage: p.factor()
6*(x + 1)^2

sage: R.<x> = PolynomialRing(QQ, sparse=True)
sage: pQ = R ( p )
sage: type(pQ)
<class 'sage.rings.polynomial.polynomial_ring.PolynomialRing_field_with_category.
↪element_class'>
sage: pQ.factor()
```

```
(6) * (x + 1)^2

sage: pZ = ZZ['x'] ( p )
sage: type(pZ)
<type 'sage.rings.polynomial.polynomial_integer_dense_flint.Polynomial_integer_dense_
↪flint'>
sage: pZ.factor()
2 * 3 * (x + 1)^2
```

Factoring integers, expressions, or polynomials are distinct tasks, with completely different algorithms. Yet, from a user (or caller) point of view, all those objects can be manipulated alike. This illustrates the OOP concepts of *polymorphism*, *data abstraction*, and *encapsulation*.

Let us be curious, and see where some methods are defined. This can be done by introspection:

```
sage: i._mul_??                    # not tested
```

For plain Python methods, one can also just ask in which module they are implemented:

```
sage: i._pow_.__module__   # not tested (Trac #24275)
'sage.categories.semigroups'

sage: pQ._mul_.__module__
'sage.rings.polynomial.polynomial_element_generic'
sage: pQ._pow_.__module__  # not tested (Trac #24275)
'sage.categories.semigroups'
```

We see that integers and polynomials have each their own multiplication method: the multiplication algorithms are indeed unrelated and deeply tied to their respective datastructures. On the other hand, as we have seen above, they share the same powering method because the set $\mathbf{Z}$ of integers, and the set $\mathbf{Q}[x]$ of polynomials are both semigroups. Namely, the class for integers and the class for polynomials both derive from an *abstract class* for semigroup elements, which factors out the *generic* methods like `_pow_`. This illustrates the use of *hierarchy of classes* to share common code between classes having common behaviour.

OOP design is all about isolating the objects that one wants to model together with their operations, and designing an appropriate hierarchy of classes for organizing the code. As we have seen above, the design of the class hierarchy is easy since it can be modelled upon the hierarchy of categories (bookshelves). Here is for example a piece of the hierarchy of classes for an element of a group of permutations:

```
sage: P = Permutations(4)
sage: m = P.an_element()
sage: for cls in m.__class__.mro(): print(cls)
<class 'sage.combinat.permutation.StandardPermutations_n_with_category.element_class'>
<class 'sage.combinat.permutation.StandardPermutations_n.Element'>
<class 'sage.combinat.permutation.Permutation'>
...
<class 'sage.categories.groups.Groups.element_class'>
<class 'sage.categories.monoids.Monoids.element_class'>
...
<class 'sage.categories.semigroups.Semigroups.element_class'>
...
```

On the top, we see concrete classes that describe the data structure for matrices and provide the operations that are tied to this data structure. Then follow abstract classes that are attached to the hierarchy of categories and provide generic algorithms.

The full hierarchy is best viewed graphically:

```
sage: g = class_graph(m.__class__)
sage: g.set_latex_options(format="dot2tex")
sage: view(g)                    # not tested
```

### Parallel hierarchy of classes for parents

Let us recall that we do not just want to compute with elements of mathematical sets, but with the sets themselves:

```
sage: ZZ.one()
1

sage: R = QQ['x,y']
sage: R.krull_dimension()
2
sage: A = R.quotient( R.ideal(x^2 - 2) )
sage: A.krull_dimension() # todo: not implemented
```

Here are some typical operations that one may want to carry on various kinds of sets:

- The set of permutations of 5, the set of rational points of an elliptic curve: counting, listing, random generation
- A language (set of words): rationality testing, counting elements, generating series
- A finite semigroup: left/right ideals, center, representation theory
- A vector space, an algebra: Cartesian product, tensor product, quotient

Hence, following the OOP fundamental principle, parents should also be modelled by instances of some (hierarchy of) classes. For example, our group $G$ is an instance of the following class:

```
sage: G = GL(2,ZZ)
sage: type(G)
<class 'sage.groups.matrix_gps.linear.LinearMatrixGroup_gap_with_category'>
```

Here is a piece of the hierarchy of classes above it:

```
sage: for cls in G.__class__.mro(): print(cls)
<class 'sage.groups.matrix_gps.linear.LinearMatrixGroup_gap_with_category'>
...
<class 'sage.categories.groups.Groups.parent_class'>
<class 'sage.categories.monoids.Monoids.parent_class'>
<class 'sage.categories.semigroups.Semigroups.parent_class'>
...
```

Note that the hierarchy of abstract classes is again attached to categories and parallel to that we had seen for the elements. This is best viewed graphically:

```
sage: g = class_graph(m.__class__)
sage: g.relabel(lambda x: x.replace("_","\_"))
sage: g.set_latex_options(format="dot2tex")
sage: view(g)                    # not tested
```

**Note:** This is a progress upon systems like Axiom or MuPAD where a parent is modelled by the class of its elements; this oversimplification leads to confusion between methods on parents and elements, and makes parents special; in particular it prevents potentially interesting constructions like "groups of groups".

### 1.1.5 Sage categories

Why this business of categories? And to start with, why don't we just have a good old hierarchy of classes `Group`, `Semigroup`, `Magma`, `...` ?

#### Dynamic hierarchy of classes

As we have just seen, when we manipulate groups, we actually manipulate several kinds of objects:

- groups
- group elements
- morphisms between groups
- and even the category of groups itself!

Thus, on the group bookshelf, we want to put generic code for each of the above. We therefore need three, parallel hierarchies of abstract classes:

- Group, Monoid, Semigroup, Magma, ...
- GroupElement, MonoidElement, SemigroupElement, MagmaElement, ...
- GroupMorphism, SemigroupElement, SemigroupMorphism, MagmaMorphism, ...

(and in fact many more as we will see).

We could implement the above hierarchies as usual:

```
class Group(Monoid):
    # generic methods that apply to all groups

class GroupElement(MonoidElement):
    # generic methods that apply to all group elements

class GroupMorphism(MonoidMorphism):
    # generic methods that apply to all group morphisms
```

And indeed that's how it was done in Sage before 2009, and there are still many traces of this. The drawback of this approach is duplication: the fact that a group is a monoid is repeated three times above!

Instead, Sage now uses the following syntax, where the *Groups* bookshelf is structured into units with *nested classes*:

```
class Groups(Category):

    def super_categories(self):
        return [Monoids(), ...]

    class ParentMethods:
        # generic methods that apply to all groups

    class ElementMethods:
        # generic methods that apply to all group elements

    class MorphismMethods:
        # generic methods that apply to all group morphisms (not yet implemented)

    class SubcategoryMethods:
        # generic methods that apply to all subcategories of Groups()
```

With this syntax, the information that a group is a monoid is specified only once, in the *Category.super_categories()* method. And indeed, when the category of inverse unital magmas was introduced, there was a *single point of truth* to update in order to reflect the fact that a group is an inverse unital magma:

```
sage: Groups().super_categories()
[Category of monoids, Category of inverse unital magmas]
```

The price to pay (there is no free lunch) is that some magic is required to construct the actual hierarchy of classes for parents, elements, and morphisms. Namely, `Groups.ElementMethods` should be seen as just a bag of methods, and the actual class `Groups().element_class` is constructed from it by adding the appropriate super classes according to `Groups().super_categories()`:

```
sage: Groups().element_class
<class 'sage.categories.groups.Groups.element_class'>

sage: Groups().element_class.__bases__
(<class 'sage.categories.monoids.Monoids.element_class'>,
 <class 'sage.categories.magmas.Magmas.Unital.Inverse.element_class'>)
```

We now see that the hierarchy of classes for parents and elements is parallel to the hierarchy of categories:

```
sage: Groups().all_super_categories()
[Category of groups,
 Category of monoids,
 Category of semigroups,
 ...
 Category of magmas,
 Category of sets,
 ...]

sage: for cls in Groups().element_class.mro(): print(cls)
<class 'sage.categories.groups.Groups.element_class'>
<class 'sage.categories.monoids.Monoids.element_class'>
<class 'sage.categories.semigroups.Semigroups.element_class'>
...
<class 'sage.categories.magmas.Magmas.element_class'>
...
sage: for cls in Groups().parent_class.mro(): print(cls)
<class 'sage.categories.groups.Groups.parent_class'>
<class 'sage.categories.monoids.Monoids.parent_class'>
<class 'sage.categories.semigroups.Semigroups.parent_class'>
...
<class 'sage.categories.magmas.Magmas.parent_class'>
...
```

Another advantage of building the hierarchy of classes dynamically is that, for parametrized categories, the hierarchy may depend on the parameters. For example an algebra over $\mathbf{Q}$ is a $\mathbf{Q}$-vector space, but an algebra over $\mathbf{Z}$ is not (it is just a $\mathbf{Z}$-module)!

**Note:** At this point this whole infrastructure may feel like overdesigning, right? We felt like this too! But we will see later that, once one gets used to it, this approach scales very naturally.

From a computer science point of view, this infrastructure implements, on top of standard multiple inheritance, a dynamic composition mechanism of mixin classes (Wikipedia article Mixin), governed by mathematical properties.

For implementation details on how the hierarchy of classes for parents and elements is constructed, see *Category*.

### On the category hierarchy: subcategories and super categories

We have seen above that, for example, the category of sets is a super category of the category of groups. This models the fact that a group can be unambiguously considered as a set by forgetting its group operation. In object-oriented parlance, we want the relation "a group *is a* set", so that groups can directly inherit code implemented on sets.

Formally, a category `Cs()` is a *super category* of a category `Ds()` if Sage considers any object of `Ds()` to be an object of `Cs()`, up to an implicit application of a canonical functor from `Ds()` to `Cs()`. This functor is normally an inclusion of categories or a forgetful functor. Reciprocally, `Ds()` is said to be a *subcategory* of `Cs()`.

> **Warning:** This terminology deviates from the usual mathematical definition of *subcategory* and is subject to change. Indeed, the forgetful functor from the category of groups to the category of sets is not an inclusion of categories, as it is not injective: a given set may admit more than one group structure. See trac ticket #16183 for more details. The name *supercategory* is also used with a different meaning in certain areas of mathematics.

### Categories are instances and have operations

Note that categories themselves are naturally modelled by instances because they can have operations of their own. An important one is:

```
sage: Groups().example()
General Linear Group of degree 4 over Rational Field
```

which gives an example of object of the category. Besides illustrating the category, the example provides a minimal template for implementing a new object in the category:

```
sage: S = Semigroups().example(); S
An example of a semigroup: the left zero semigroup
```

Its source code can be obtained by introspection:

```
sage: S??                                          # not tested
```

This example is also typically used for testing generic methods. See `Category.example()` for more.

Other operations on categories include querying the super categories or the axioms satisfied by the operations of a category:

```
sage: Groups().super_categories()
[Category of monoids, Category of inverse unital magmas]
sage: Groups().axioms()
frozenset({'Associative', 'Inverse', 'Unital'})
```

or constructing the intersection of two categories, or the smallest category containing them:

```
sage: Groups() & FiniteSets()
Category of finite groups
sage: Algebras(QQ) | Groups()
Category of monoids
```

### Specifications and generic documentation

Categories do not only contain code but also the specifications of the operations. In particular a list of mandatory and optional methods to be implemented can be found by introspection with:

```
sage: Groups().required_methods()
{'element': {'optional': ['_mul_'], 'required': []},
 'parent': {'optional': [], 'required': ['__contains__']}}
```

Documentation about those methods can be obtained with:

```
sage: G = Groups()
sage: G.element_class._mul_?         # not tested
sage: G.parent_class.one?            # not tested
```

See also the `abstract_method()` decorator.

> **Warning:** Well, more precisely, that's how things should be, but there is still some work to do in this direction. For example, the inverse operation is not specified above. Also, we are still missing a good programmatic syntax to specify the input and output types of the methods. Finally, in many cases the implementer must provide at least one of two methods, each having a default implementation using the other one (e.g. listing or iterating for a finite enumerated set); there is currently no good programmatic way to specify this.

### Generic tests

Another feature that parents and elements receive from categories is generic tests; their purpose is to check (at least to some extent) that the parent satisfies the required mathematical properties (is my semigroup indeed associative?) and is implemented according to the specifications (does the method `an_element` indeed return an element of the parent?):

```
sage: S = FiniteSemigroups().example(alphabet=('a', 'b'))
sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
    running ._test_elements_eq_reflexive() . . . pass
    running ._test_elements_eq_symmetric() . . . pass
    running ._test_elements_eq_transitive() . . . pass
    running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

Tests can be run individually:

```
sage: S._test_associativity()
```

Here is how to access the code of this test:

```
sage: S._test_associativity?? # not tested
```

Here is how to run the test on all elements:

```
sage: L = S.list()
sage: S._test_associativity(elements=L)
```

See TestSuite for more information.

Let us see what happens when a test fails. Here we redefine the product of $S$ to something definitely not associative:

```
sage: S.product = lambda x, y: S("("+x.value +y.value+")")
```

And rerun the test:

```
sage: S._test_associativity(elements=L)
Traceback (most recent call last):
...
  File ".../sage/categories/semigroups.py", line ..., in _test_associativity
    tester.assertTrue((x * y) * z == x * (y * z))
...
AssertionError: False is not true
```

We can recover instantly the actual values of x, y, z, that is, a counterexample to the associativity of our broken semigroup, using post mortem introspection with the Python debugger pdb (this does not work yet in the notebook):

```
sage: import pdb
sage: pdb.pm()                          # not tested
> /opt/sage-5.11.rc1/local/lib/python/unittest/case.py(424)assertTrue()
-> raise self.failureException(msg)
(Pdb) u
> /opt/sage-5.11.rc1/local/lib/python2.7/site-packages/sage/categories/semigroups.
↪py(145)_test_associativity()
-> tester.assertTrue((x * y) * z == x * (y * z))
(Pdb) p x, y, z
('a', 'a', 'a')
(Pdb) p (x * y) * z
'((aa)a)'
(Pdb) p x * (y * z)
'(a(aa))'
```

### Wrap-up

- Categories provide a natural hierarchy of bookshelves to organize not only code, but also specifications and testing tools.
- Everything about, say, algebras with a distinguished basis is gathered in *AlgebrasWithBasis* or its super categories. This includes properties and algorithms for elements, parents, morphisms, but also, as we will see, for constructions like Cartesian products or quotients.
- The mathematical relations between elements, parents, and categories translate dynamically into a traditional hierarchy of classes.

- This design enforces robustness and consistency, which is particularly welcome given that Python is an interpreted language without static type checking.

### 1.1.6 Case study

In this section, we study an existing parent in detail; a good followup is to go through the *sage.categories.tutorial* or the thematic tutorial on coercion and categories ("How to implement new algebraic structures in Sage") to learn how to implement a new one!

We consider the example of finite semigroup provided by the category:

```
sage: S = FiniteSemigroups().example(); S
An example of a finite semigroup: the left regular band generated by ('a', 'b', 'c',
↪'d')
sage: S?                        # not tested
```

Where do all the operations on `S` and its elements come from?

```
sage: x = S('a')
```

`_repr_` is a technical method which comes with the data structure (`ElementWrapper`); since it's implemented in Cython, we need to use Sage's introspection tools to recover where it's implemented:

```
sage: x._repr_.__module__
sage: sage.misc.sageinspect.sage_getfile(x._repr_)
'.../sage/structure/element_wrapper.pyx'
```

`_pow_int` is a generic method for all finite semigroups:

```
sage: x._pow_int.__module__
'sage.categories.semigroups'
```

`__mul__` is a generic method provided by the *Magmas* category (a *magma* is a set with an inner law $*$, not necessarily associative). If the two arguments are in the same parent, it will call the method `_mul_`, and otherwise let the `coercion model` try to discover how to do the multiplication:

```
sage: x.__mul__??                              # not tested
```

Since it is a speed critical method, it is implemented in Cython in a separate file:

```
sage: x._mul_.__module__
'sage.categories.coercion_methods'
```

`_mul_` is a default implementation, also provided by the *Magmas* category, that delegates the work to the method `product` of the parent (following the advice: if you do not know what to do, ask your parent); it's also a speed critical method:

```
sage: x._mul_??                                # not tested
sage: x._mul_.__module__
'sage.categories.coercion_methods'
sage: from six import get_method_function as gmf
sage: gmf(x._mul_) is gmf(Magmas.ElementMethods._mul_parent)
True
```

`product` is a mathematical method implemented by the parent:

```
sage: S.product.__module__
'sage.categories.examples.finite_semigroups'
```

`cayley_graph` is a generic method on the parent, provided by the *FiniteSemigroups* category:

```
sage: S.cayley_graph.__module__
'sage.categories.semigroups'
```

`multiplication_table` is a generic method on the parent, provided by the *Magmas* category (it does not require associativity):

```
sage: S.multiplication_table.__module__
'sage.categories.magmas'
```

Consider now the implementation of the semigroup:

```
sage: S??                                        # not tested
```

This implementation specifies a data structure for the parents and the elements, and makes a promise: the implemented parent is a finite semigroup. Then it fulfills the promise by implementing the basic operation `product`. It also implements the optional method `semigroup_generators`. In exchange, $S$ and its elements receive generic implementations of all the other operations. $S$ may override any of those by more efficient ones. It may typically implement the element method `is_idempotent` to always return `True`.

A (not yet complete) list of mandatory and optional methods to be implemented can be found by introspection with:

```
sage: FiniteSemigroups().required_methods()
{'element': {'optional': ['_mul_'], 'required': []},
 'parent': {'optional': ['semigroup_generators'],
  'required': ['__contains__']}}
```

`product` does not appear in the list because a default implementation is provided in term of the method `_mul_` on elements. Of course, at least one of them should be implemented. On the other hand, a default implementation for `__contains__` is provided by `Parent`.

Documentation about those methods can be obtained with:

```
sage: C = FiniteSemigroups().element_class
sage: C._mul_?                                   # not tested
```

See also the `abstract_method()` decorator.

Here is the code for the finite semigroups category:

```
sage: FiniteSemigroups??                         # not tested
```

### 1.1.7 Specifying the category of a parent

Some parent constructors (not enough!) allow to specify the desired category for the parent. This can typically be used to specify additional properties of the parent that we know to hold a priori. For example, permutation groups are by default in the category of finite permutation groups (no surprise):

```
sage: P = PermutationGroup([[(1,2,3)]]); P
Permutation Group with generators [(1,2,3)]
sage: P.category()
Category of finite enumerated permutation groups
```

In this case, the group is commutative, so we can specify this:

```
sage: P = PermutationGroup([[(1,2,3)]], category=PermutationGroups().Finite().
↪Commutative()); P
Permutation Group with generators [(1,2,3)]
sage: P.category()
Category of finite enumerated commutative permutation groups
```

This feature can even be used, typically in experimental code, to add more structure to existing parents, and in particular to add methods for the parents or the elements, without touching the code base:

```
sage: class Foos(Category):
....:     def super_categories(self):
....:          return [PermutationGroups().Finite().Commutative()]
....:     class ParentMethods:
....:         def foo(self): print("foo")
....:     class ElementMethods:
....:         def bar(self): print("bar")

sage: P = PermutationGroup([[(1,2,3)]], category=Foos())
sage: P.foo()
foo
sage: p = P.an_element()
sage: p.bar()
bar
```

In the long run, it would be thinkable to use this idiom to implement forgetful functors; for example the above group could be constructed as a plain set with:

```
sage: P = PermutationGroup([[(1,2,3)]], category=Sets()) # todo: not implemented
```

At this stage though, this is still to be explored for robustness and practicality. For now, most parents that accept a category argument only accept a subcategory of the default one.

### 1.1.8 Scaling further: functorial constructions, axioms, ...

In this section, we explore more advanced features of categories. Along the way, we illustrate that a large hierarchy of categories is desirable to model complicated mathematics, and that scaling to support such a large hierarchy is the driving motivation for the design of the category infrastructure.

#### Functorial constructions

Sage has support for a certain number of so-called *covariant functorial constructions* which can be used to construct new parents from existing ones while carrying over as much as possible of their algebraic structure. This includes:

- Cartesian products: See `cartesian_product`.
- Tensor products: See `tensor`.
- Subquotients / quotients / subobjects / isomorphic objects: See:
  - `Sets().Subquotients`,
  - `Sets().Quotients`,
  - `Sets().Subobjects`,
  - `Sets().IsomorphicObjects`

- Dual objects: See *Modules().DualObjects*.
- Algebras, as in group algebras, monoid algebras, ...: See: `Sets.ParentMethods.algebras()`.

Let for example $A$ and $B$ be two parents, and let us construct the Cartesian product $A \times B \times B$:

```
sage: A = AlgebrasWithBasis(QQ).example();     A.rename("A")
sage: B = HopfAlgebrasWithBasis(QQ).example(); B.rename("B")
sage: C = cartesian_product([A, B, B]); C
A (+) B (+) B
```

In which category should this new parent be? Since $A$ and $B$ are vector spaces, the result is, as a vector space, the direct sum $A \oplus B \oplus B$, hence the notation. Also, since both $A$ and $B$ are monoids, $A \times B \times B$ is naturally endowed with a monoid structure for pointwise multiplication:

```
sage: C in Monoids()
True
```

the unit being the Cartesian product of the units of the operands:

```
sage: C.one()
B[(0, word: )] + B[(1, ())] + B[(2, ())]
sage: cartesian_product([A.one(), B.one(), B.one()])
B[(0, word: )] + B[(1, ())] + B[(2, ())]
```

The pointwise product can be implemented generically for all magmas (i.e. sets endowed with a multiplicative operation) that are constructed as Cartesian products. It's thus implemented in the *Magmas* category:

```
sage: C.product.__module__
'sage.categories.magmas'
```

More specifically, keeping on using nested classes to structure the code, the product method is put in the nested class *Magmas.CartesianProducts.ParentMethods*:

```
class Magmas(Category):
    class ParentMethods:
        # methods for magmas
    class ElementMethods:
        # methods for elements of magmas
    class CartesianProduct(CartesianProductCategory):
        class ParentMethods:
            # methods for magmas that are constructed as Cartesian products
            def product(self, x, y):
                # ...
        class ElementMethods:
            # ...
```

**Note:** The support for nested classes in Python is relatively recent. Their intensive use for the category infrastructure did reveal some glitches in their implementation, in particular around class naming and introspection. Sage currently works around the more annoying ones but some remain visible. See e.g. `sage.misc.nested_class_test`.

Let us now look at the categories of `C`:

```
sage: C.categories()
[Category of finite dimensional Cartesian products of algebras with basis over␣
↪Rational Field, ...
 Category of Cartesian products of algebras over Rational Field, ...
```

```
Category of Cartesian products of semigroups, Category of semigroups, ...
Category of Cartesian products of magmas, ..., Category of magmas, ...
Category of Cartesian products of additive magmas, ..., Category of additive magmas,
Category of Cartesian products of sets, Category of sets, ...]
```

This reveals the parallel hierarchy of categories for Cartesian products of semigroups magmas, … We are thus glad that Sage uses its knowledge that a monoid is a semigroup to automatically deduce that a Cartesian product of monoids is a Cartesian product of semigroups, and build the hierarchy of classes for parents and elements accordingly.

In general, the Cartesian product of $A$ and $B$ can potentially be an algebra, a coalgebra, a differential module, and be finite dimensional, or graded, or …. This can only be decided at runtime, by introspection into the properties of $A$ and $B$; furthermore, the number of possible combinations (e.g. finite dimensional differential algebra) grows exponentially with the number of properties.

### Axioms

#### First examples

We have seen that Sage is aware of the axioms satisfied by, for example, groups:

```
sage: Groups().axioms()
frozenset({'Associative', 'Inverse', 'Unital'})
```

In fact, the category of groups can be *defined* by stating that a group is a magma, that is a set endowed with an internal binary multiplication, which satisfies the above axioms. Accordingly, we can construct the category of groups from the category of magmas:

```
sage: Magmas().Associative().Unital().Inverse()
Category of groups
```

In general, we can construct new categories in Sage by specifying the axioms that are satisfied by the operations of the super categories. For example, starting from the category of magmas, we can construct all the following categories just by specifying the axioms satisfied by the multiplication:

```
sage: Magmas()
Category of magmas
sage: Magmas().Unital()
Category of unital magmas
```

```
sage: Magmas().Commutative().Unital()
Category of commutative unital magmas
sage: Magmas().Unital().Commutative()
Category of commutative unital magmas
```

```
sage: Magmas().Associative()
Category of semigroups
```

```
sage: Magmas().Associative().Unital()
Category of monoids
```

```
sage: Magmas().Associative().Unital().Commutative()
Category of commutative monoids
```

```
sage: Magmas().Associative().Unital().Inverse()
Category of groups
```

### Axioms and categories with axioms

Here, `Associative`, `Unital`, `Commutative` are axioms. In general, any category `Cs` in Sage can declare a new axiom `A`. Then, the *category with axiom* `Cs.A()` models the subcategory of the objects of `Cs` satisfying the axiom `A`. Similarly, for any subcategory `Ds` of `Cs`, `Ds.A()` models the subcategory of the objects of `Ds` satisfying the axiom `A`. In most cases, it's a *full subcategory* (see Wikipedia article Subcategory).

For example, the category of sets defines the `Finite` axiom, and this axiom is available in the subcategory of groups:

```
sage: Sets().Finite()
Category of finite sets
sage: Groups().Finite()
Category of finite groups
```

The meaning of each axiom is described in the documentation of the corresponding method, which can be obtained as usual by instrospection:

```
sage: C = Groups()
sage: C.Finite?              # not tested
```

The purpose of categories with axioms is no different from other categories: to provide bookshelves of code, documentation, mathematical knowledge, tests, for their objects. The extra feature is that, when intersecting categories, axioms are automatically combined together:

```
sage: C = Magmas().Associative() & Magmas().Unital().Inverse() & Sets().Finite(); C
Category of finite groups
sage: sorted(C.axioms())
['Associative', 'Finite', 'Inverse', 'Unital']
```

For a more advanced example, Sage knows that a ring is a set $C$ endowed with a multiplication which distributes over addition, such that $(C, +)$ is a commutative additive group and $(C, *)$ is a monoid:

```
sage: C = (CommutativeAdditiveGroups() & Monoids()).Distributive(); C
Category of rings

sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive', 'Unital']
```

The infrastructure allows for specifying further deduction rules, in order to encode mathematical facts like Wedderburn's theorem:

```
sage: DivisionRings() & Sets().Finite()
Category of finite enumerated fields
```

**Note:** When an axiom specifies the properties of some operations in Sage, the notations for those operations are tied to this axiom. For example, as we have seen above, we need two distinct axioms for associativity: the axiom "AdditiveAssociative" is about the properties of the addition $+$, whereas the axiom "Associative" is about the properties of the multiplication $*$.

We are touching here an inherent limitation of the current infrastructure. There is indeed no support for providing generic code that is independent of the notations. In particular, the category hierarchy about additive structures (additive monoids, additive groups, . . . ) is completely duplicated by that for multiplicative structures (monoids, groups, . . . ).

As far as we know, none of the existing computer algebra systems has a good solution for this problem. The difficulty is that this is not only about a single notation but a bunch of operators and methods: `+, -, zero, summation, sum, ...` in one case, `*, /, one, product, prod, factor, ...` in the other. Sharing something between the two hierarchies of categories would only be useful if one could write generic code that applies in both cases; for that one needs to somehow automatically substitute the right operations in the right spots in the code. That's kind of what we are doing manually between e.g. `AdditiveMagmas.ParentMethods.addition_table()` and `Magmas.ParentMethods.multiplication_table()`, but doing this systematically is a different beast from what we have been doing so far with just usual inheritance.

### Single entry point and name space usage

A nice feature of the notation `Cs.A()` is that, from a single entry point (say the category `Magmas` as above), one can explore a whole range of related categories, typically with the help of introspection to discover which axioms are available, and without having to import new Python modules. This feature will be used in trac ticket #15741 to unclutter the global name space from, for example, the many variants of the category of algebras like:

```
sage: FiniteDimensionalAlgebrasWithBasis(QQ)
Category of finite dimensional algebras with basis over Rational Field
```

There will of course be a deprecation step, but it's recommended to prefer right away the more flexible notation:

```
sage: Algebras(QQ).WithBasis().FiniteDimensional()
Category of finite dimensional algebras with basis over Rational Field
```

**Design discussion**

How far should this be pushed? `Fields` should definitely stay, but should `FiniteGroups` or `DivisionRings` be removed from the global namespace? Do we want to further completely deprecate the notation `FiniteGroups()` in favor of ``Groups().Finite()`?

### On the potential combinatorial explosion of categories with axioms

Even for a very simple category like `Magmas`, there are about $2^5$ potential combinations of the axioms! Think about what this becomes for a category with two operations $+$ and $*$:

```
sage: C = (Magmas() & AdditiveMagmas()).Distributive(); C
Category of distributive magmas and additive magmas

sage: C.Associative().AdditiveAssociative().AdditiveCommutative().AdditiveUnital().
↪AdditiveInverse()
Category of rngs

sage: C.Associative().AdditiveAssociative().AdditiveCommutative().AdditiveUnital().
↪Unital()
Category of semirings

sage: C.Associative().AdditiveAssociative().AdditiveCommutative().AdditiveUnital().
↪AdditiveInverse().Unital()
```

```
Category of rings

sage: Rings().Division()
Category of division rings

sage: Rings().Division().Commutative()
Category of fields

sage: Rings().Division().Finite()
Category of finite enumerated fields
```

or for more advanced categories:

```
sage: g = HopfAlgebras(QQ).WithBasis().Graded().Connected().category_graph()
sage: g.set_latex_options(format="dot2tex")
sage: view(g)                     # not tested
```

### Difference between axioms and regressive covariant functorial constructions

Our running examples here will be the axiom `FiniteDimensional` and the regressive covariant functorial construction `Graded`. Let `Cs` be some subcategory of `Modules`, say the category of modules itself:

```
sage: Cs = Modules(QQ)
```

Then, `Cs.FiniteDimensional()` (respectively `Cs.Graded()`) is the subcategory of the objects `O` of `Cs` which are finite dimensional (respectively graded).

Let also `Ds` be a subcategory of `Cs`, say:

```
sage: Ds = Algebras(QQ)
```

A finite dimensional algebra is also a finite dimensional module:

```
sage: Algebras(QQ).FiniteDimensional().is_subcategory( Modules(QQ).
↪FiniteDimensional() )
True
```

Similarly a graded algebra is also a graded module:

```
sage: Algebras(QQ).Graded().is_subcategory( Modules(QQ).Graded() )
True
```

This is the *covariance* property: for `A` an axiom or a covariant functorial construction, if `Ds` is a subcategory of `Cs`, then `Ds.A()` is a subcategory of `Cs.A()`.

What happens if we consider reciprocally an object of `Cs.A()` which is also in `Ds`? A finite dimensional module which is also an algebra is a finite dimensional algebra:

```
sage: Modules(QQ).FiniteDimensional() & Algebras(QQ)
Category of finite dimensional algebras over Rational Field
```

On the other hand, a graded module $O$ which is also an algebra is not necessarily a graded algebra! Indeed, the grading on $O$ may not be compatible with the product on $O$:

```
sage: Modules(QQ).Graded() & Algebras(QQ)
Join of Category of algebras over Rational Field and Category of graded modules over␣
↪Rational Field
```

The relevant difference between FiniteDimensional and Graded is that FiniteDimensional is a statement about the properties of O seen as a module (and thus does not depend on the given category), whereas Graded is a statement about the properties of O and all its operations in the given category.

In general, if a category satisfies a given axiom, any subcategory also satisfies that axiom. Another formulation is that, for an axiom A defined in a super category Cs of Ds, Ds.A() is the intersection of the categories Ds and Cs.A():

```
sage: As = Algebras(QQ).FiniteDimensional(); As
Category of finite dimensional algebras over Rational Field
sage: Bs = Algebras(QQ) & Modules(QQ).FiniteDimensional(); As
Category of finite dimensional algebras over Rational Field
sage: As is Bs
True
```

An immediate consequence is that, as we have already noticed, axioms commute:

```
sage: As = Algebras(QQ).FiniteDimensional().WithBasis(); As
Category of finite dimensional algebras with basis over Rational Field
sage: Bs = Algebras(QQ).WithBasis().FiniteDimensional(); Bs
Category of finite dimensional algebras with basis over Rational Field
sage: As is Bs
True
```

On the other hand, axioms do not necessarily commute with functorial constructions, even if the current printout may missuggest so:

```
sage: As = Algebras(QQ).Graded().WithBasis(); As
Category of graded algebras with basis over Rational Field
sage: Bs = Algebras(QQ).WithBasis().Graded(); Bs
Category of graded algebras with basis over Rational Field
sage: As is Bs
False
```

This is because Bs is the category of algebras endowed with basis, which are further graded; in particular the basis must respect the grading (i.e. be made of homogeneous elements). On the other hand, As is the category of graded algebras, which are further endowed with some basis; that basis need not respect the grading. In fact As is really a join category:

```
sage: type(As)
<class 'sage.categories.category.JoinCategory_with_category'>
sage: As._repr_(as_join=True)
'Join of Category of algebras with basis over Rational Field and Category of graded␣
↪algebras over Rational Field'
```

**Todo:** Improve the printing of functorial constructions and joins to raise this potentially dangerous ambiguity.

### Further reading on axioms

We refer to *sage.categories.category_with_axiom* for how to implement axioms.

### Wrap-up

As we have seen, there is a combinatorial explosion of possible classes. Constructing by hand the full class hierarchy would not scale unless one would restrict to a very rigid subset. Even if it was possible to construct automatically the full hierarchy, this would not scale with respect to system resources.

When designing software systems with large hierarchies of abstract classes for business objects, the difficulty is usually to identify a proper set of key concepts. Here we are lucky, as the key concepts have been long identified and are relatively few:

- Operations ($+$, $*$, ...)
- Axioms on those operations (associativity, ...)
- Constructions (Cartesian products, ...)

Better, those concepts are sufficiently well known so that a user can reasonably be expected to be familiar with the concepts that are involved for his own needs.

Instead, the difficulty is concentrated in the huge number of possible combinations, an unpredictable large subset of which being potentially of interest; at the same time, only a small – but moving – subset has code naturally attached to it.

This has led to the current design, where one focuses on writing the relatively few classes for which there is actual code or mathematical information, and lets Sage *compose dynamically and lazily* those building blocks to construct the minimal hierarchy of classes needed for the computation at hand. This allows for the infrastructure to scale smoothly as bookshelves are added, extended, or reorganized.

## 1.1.9 Writing a new category

Each category $C$ **must** be provided with a method `C.super_categories()` and *can* be provided with a method `C._subcategory_hook_(D)`. Also, it may be needed to insert $C$ into the output of the `super_categories()` method of some other category. This determines the position of $C$ in the category graph.

A category *may* provide methods that can be used by all its objects, respectively by all elements of its objects.

Each category *should* come with a good example, in `sage.categories.examples`.

### Inserting the new category into the category graph

`C.super_categories()` *must* return a list of categories, namely the *immediate* super categories of $C$. Of course, if you know that your new category $C$ is an immediate super category of some existing category $D$, then you should also update the method `D.super_categories` to include $C$.

The immediate super categories of $C$ *should not* be *join categories*. Furthermore, one always should have:

```
Cs().is_subcategory( Category.join(Cs().super_categories()) )

Cs()._cmp_key  >  other._cmp_key  for other in Cs().super_categories()
```

This is checked by `_test_category()`.

In several cases, the category $C$ is directly provided with a generic implementation of `super_categories`; a typical example is when $C$ implements an axiom or a functorial construction; in such a case, $C$ may implement `C.extra_super_categories()` to complement the super categories discovered by the generic implementation. This method needs not return immediate super categories; instead it's usually best to specify the largest super category providing the desired mathematical information. For example, the category `Magmas.Commutative.Algebras` just states that the algebra of a commutative magma is a commutative magma. This is sufficient to let Sage deduce that it's in fact a commutative algebra.

### Methods for objects and elements

Different objects of the same category share some algebraic features, and very often these features can be encoded in a method, in a generic way. For example, for every commutative additive monoid, it makes sense to ask for the sum of a list of elements. Sage's category framework allows to provide a generic implementation for all objects of a category.

If you want to provide your new category with generic methods for objects (or elements of objects), then you simply add a nested class called `ParentMethods` (or `ElementMethods`). The methods of that class will automatically become methods of the objects (or the elements). For instance:

```
sage: P.<x,y> = ZZ[]
sage: P.prod([x,y,2])
2*x*y
sage: P.prod.__module__
'sage.categories.monoids'
sage: P.prod.__func__ is Monoids().ParentMethods.prod.__func__
True
```

We recommend to study the code of one example:

```
sage: C = CommutativeAdditiveMonoids()
sage: C??                               # not tested
```

### On the order of super categories

The generic method `C.all_super_categories()` determines recursively the list of *all* super categories of $C$.

The order of the categories in this list does influence the inheritance of methods for parents and elements. Namely, if $P$ is an object in the category $C$ and if $C_1$ and $C_2$ are both super categories of $C$ defining some method `foo` in `ParentMethods`, then $P$ will use $C_1$'s version of `foo` if and only if $C_1$ appears in `C.all_super_categories()` before $C_2$.

However this must be considered as an *implementation detail*: if $C_1$ and $C_2$ are incomparable categories, then the order in which they appear must be mathematically irrelevant: in particular, the methods `foo` in $C_1$ and $C_2$ must have the same semantic. Code should not rely on any specific order, as it is subject to later change. Whenever one of the implementations is preferred in some common subcategory of $C_1$ and $C_2$, for example for efficiency reasons, the ambiguity should be resolved explicitly by definining a method `foo` in this category. See the method `some_elements` in the code of the category *`FiniteCoxeterGroups`* for an example.

Since trac ticket #11943, `C.all_super_categories()` is computed by the so-called `C3` algorithm used by Python to compute Method Resolution Order of new-style classes. Thus the order in `C.all_super_categories()`, `C.parent_class.mro()` and `C.element_class.mro()` are guaranteed to be consistent.

Since trac ticket #13589, the `C3` algorithm is put under control of some total order on categories. This order is not necessarily meaningful, but it guarantees that `C3` always finds a consistent Method Resolution Order. For background, see `sage.misc.c3_controlled`. A visible effect is that the order in which categories are specified in `C.super_categories()`, or in a join category, no longer influences the result of `C.all_super_categories()`.

### Subcategory hook (advanced optimization feature)

The default implementation of the method `C.is_subcategory(D)` is to look up whether $D$ appears in `C.all_super_categories()`. However, building the list of all the super categories of $C$ is an expensive operation that is sometimes best avoided. For example, if both $C$ and $D$ are categories defined over a base, but the bases differ, then one knows right away that they can not be subcategories of each other.

When such a short-path is known, one can implement a method `_subcategory_hook_`. Then, `C.is_subcategory(D)` first calls `D._subcategory_hook_(C)`. If this returns `Unknown`, then `C.is_subcategory(D)` tries to find `D` in `C.all_super_categories()`. Otherwise, `C.is_subcategory(D)` returns the result of `D._subcategory_hook_(C)`.

By default, `D._subcategory_hook_(C)` tests whether `issubclass(C.parent_class,D.parent_class)`, which is very often giving the right answer:

```
sage: Rings()._subcategory_hook_(Algebras(QQ))
True
sage: HopfAlgebras(QQ)._subcategory_hook_(Algebras(QQ))
False
sage: Algebras(QQ)._subcategory_hook_(HopfAlgebras(QQ))
True
```

## 1.2 Categories

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Every Sage object lies in a category. Categories in Sage are modeled on the mathematical idea of category, and are distinct from Python classes, which are a programming construct.

In most cases, typing `x.category()` returns the category to which `x` belongs. If `C` is a category and `x` is any object, `C(x)` tries to make an object in `C` from `x`. Checking if `x` belongs to `C` is done as usually by `x in C`.

See *Category* and *sage.categories.primer* for more details.

EXAMPLES:

We create a couple of categories:

```
sage: Sets()
Category of sets
sage: GSets(AbelianGroup([2,4,9]))
Category of G-sets for Multiplicative Abelian group isomorphic to C2 x C4 x C9
sage: Semigroups()
Category of semigroups
sage: VectorSpaces(FiniteField(11))
Category of vector spaces over Finite Field of size 11
sage: Ideals(IntegerRing())
Category of ring ideals in Integer Ring
```

Let's request the category of some objects:

```
sage: V = VectorSpace(RationalField(), 3)
sage: V.category()
Category of finite dimensional vector spaces with basis
 over (number fields and quotient fields and metric spaces)
sage: G = SymmetricGroup(9)
```

```
sage: G.category()
Join of Category of finite enumerated permutation groups
 and Category of finite weyl groups
sage: P = PerfectMatchings(3)
sage: P.category()
Category of finite enumerated sets
```

Let's check some memberships:

```
sage: V in VectorSpaces(QQ)
True
sage: V in VectorSpaces(FiniteField(11))
False
sage: G in Monoids()
True
sage: P in Rings()
False
```

For parametrized categories one can use the following shorthand:

```
sage: V in VectorSpaces
True
sage: G in VectorSpaces
False
```

A parent `P` is in a category `C` if `P.category()` is a subcategory of `C`.

**Note:** Any object of a category should be an instance of `CategoryObject`.

For backward compatibility this is not yet enforced:

```
sage: class A:
....:   def category(self):
....:       return Fields()
sage: A() in Rings()
True
```

By default, the category of an element $x$ of a parent $P$ is the category of all objects of $P$ (this is dubious an may be deprecated):

```
sage: V = VectorSpace(RationalField(), 3)
sage: v = V.gen(1)
sage: v.category()
Category of elements of Vector space of dimension 3 over Rational Field
```

**class** sage.categories.category.**Category**(*s=None*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.sage_object.SageObject`

The base class for modeling mathematical categories, like for example:

- `Groups()`: the category of groups
- `EuclideanDomains()`: the category of euclidean rings
- `VectorSpaces(QQ)`: the category of vector spaces over the field of rationals

See *sage.categories.primer* for an introduction to categories in Sage, their relevance, purpose, and usage. The documentation below will focus on their implementation.

Technically, a category is an instance of the class *Category* or some of its subclasses. Some categories, like *VectorSpaces*, are parametrized: `VectorSpaces(QQ)` is one of many instances of the class *VectorSpaces*. On the other hand, `EuclideanDomains()` is the single instance of the class *EuclideanDomains*.

Recall that an algebraic structure (say, the ring $\mathbf{Q}[x]$) is modelled in Sage by an object which is called a parent. This object belongs to certain categories (here `EuclideanDomains()` and `Algebras()`). The elements of the ring are themselves objects.

The class of a category (say *EuclideanDomains*) can define simultaneously:

- Operations on the category itself (what is its super categories? its category of morphisms? its dual category?).
- Generic operations on parents in this category, like the ring $\mathbf{Q}[x]$.
- Generic operations on elements of such parents (e. g., the Euclidean algorithm for computing gcds).
- Generic operations on morphisms of this category.

This is achieved as follows:

```
sage: from sage.categories.all import Category
sage: class EuclideanDomains(Category):
....:     # operations on the category itself
....:     def super_categories(self):
....:         [Rings()]
....:
....:     def dummy(self): # TODO: find some good examples
....:          pass
....:
....:     class ParentMethods: # holds the generic operations on parents
....:          # TODO: find a good example of an operation
....:          pass
....:
....:     class ElementMethods:# holds the generic operations on elements
....:          def gcd(x,y):
....:              # Euclid algorithms
....:              pass
....:
....:     class MorphismMethods: # holds the generic operations on morphisms
....:          # TODO: find a good example of an operation
....:          pass
....:
```

Note that the nested class `ParentMethods` is merely a container of operations, and does not inherit from anything. Instead, the hierarchy relation is defined once at the level of the categories, and the actual hierarchy of classes is built in parallel from all the `ParentMethods` nested classes, and stored in the attributes `parent_class`. Then, a parent in a category `C` receives the appropriate operations from all the super categories by usual class inheritance from `C.parent_class`.

Similarly, two other hierarchies of classes, for elements and morphisms respectively, are built from all the `ElementMethods` and `MorphismMethods` nested classes.

EXAMPLES:

We define a hierarchy of four categories `As()`, `Bs()`, `Cs()`, `Ds()` with a diamond inheritance. Think for example:

- As(): the category of sets
- Bs(): the category of additive groups
- Cs(): the category of multiplicative monoids
- Ds(): the category of rings

```
sage: from sage.categories.all import Category
sage: from sage.misc.lazy_attribute import lazy_attribute
sage: class As (Category):
....:     def super_categories(self):
....:         return []
....:
....:     class ParentMethods:
....:         def fA(self):
....:             return "A"
....:         f = fA

sage: class Bs (Category):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class ParentMethods:
....:         def fB(self):
....:             return "B"

sage: class Cs (Category):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class ParentMethods:
....:         def fC(self):
....:             return "C"
....:         f = fC

sage: class Ds (Category):
....:     def super_categories(self):
....:         return [Bs(),Cs()]
....:
....:     class ParentMethods:
....:         def fD(self):
....:             return "D"
```

Categories should always have unique representation; by trac ticket trac ticket #12215, this means that it will be kept in cache, but only if there is still some strong reference to it.

We check this before proceeding:

```
sage: import gc
sage: idAs = id(As())
sage: _ = gc.collect()
sage: n == id(As())
False
sage: a = As()
sage: id(As()) == id(As())
True
sage: As().parent_class == As().parent_class
True
```

We construct a parent in the category `Ds()` (that, is an instance of `Ds().parent_class`), and check that it has access to all the methods provided by all the categories, with the appropriate inheritance order:

```
sage: D = Ds().parent_class()
sage: [ D.fA(), D.fB(), D.fC(), D.fD() ]
['A', 'B', 'C', 'D']
sage: D.f()
'C'
```

```
sage: C = Cs().parent_class()
sage: [ C.fA(), C.fC() ]
['A', 'C']
sage: C.f()
'C'
```

Here is the parallel hierarchy of classes which has been built automatically, together with the method resolution order (`.mro()`):

```
sage: As().parent_class
<class '__main__.As.parent_class'>
sage: As().parent_class.__bases__
(<... 'object'>,)
sage: As().parent_class.mro()
[<class '__main__.As.parent_class'>, <... 'object'>]
```

```
sage: Bs().parent_class
<class '__main__.Bs.parent_class'>
sage: Bs().parent_class.__bases__
(<class '__main__.As.parent_class'>,)
sage: Bs().parent_class.mro()
[<class '__main__.Bs.parent_class'>, <class '__main__.As.parent_class'>, <...
↪'object'>]
```

```
sage: Cs().parent_class
<class '__main__.Cs.parent_class'>
sage: Cs().parent_class.__bases__
(<class '__main__.As.parent_class'>,)
sage: Cs().parent_class.__mro__
(<class '__main__.Cs.parent_class'>, <class '__main__.As.parent_class'>, <...
↪'object'>)
```

```
sage: Ds().parent_class
<class '__main__.Ds.parent_class'>
sage: Ds().parent_class.__bases__
(<class '__main__.Cs.parent_class'>, <class '__main__.Bs.parent_class'>)
sage: Ds().parent_class.mro()
[<class '__main__.Ds.parent_class'>, <class '__main__.Cs.parent_class'>, <class '_
↪_main__.Bs.parent_class'>, <class '__main__.As.parent_class'>, <... 'object'>]
```

Note that that two categories in the same class need not have the same `super_categories`. For example, `Algebras(QQ)` has `VectorSpaces(QQ)` as super category, whereas `Algebras(ZZ)` only has `Modules(ZZ)` as super category. In particular, the constructed parent class and element class will differ (inheriting, or not, methods specific for vector spaces):

```
sage: Algebras(QQ).parent_class is Algebras(ZZ).parent_class
False
```

```
sage: issubclass(Algebras(QQ).parent_class, VectorSpaces(QQ).parent_class)
True
```

On the other hand, identical hierarchies of classes are, preferably, built only once (e.g. for categories over a base ring):

```
sage: Algebras(GF(5)).parent_class is Algebras(GF(7)).parent_class
True
sage: F = FractionField(ZZ['t'])
sage: Coalgebras(F).parent_class is Coalgebras(FractionField(F['x'])).parent_class
True
```

We now construct a parent in the usual way:

```
sage: class myparent(Parent):
....:     def __init__(self):
....:         Parent.__init__(self, category=Ds())
....:     def g(self):
....:         return "myparent"
....:     class Element:
....:         pass
sage: D = myparent()
sage: D.__class__
<class '__main__.myparent_with_category'>
sage: D.__class__.__bases__
(<class '__main__.myparent'>, <class '__main__.Ds.parent_class'>)
sage: D.__class__.mro()
[<class '__main__.myparent_with_category'>,
<class '__main__.myparent'>,
<type 'sage.structure.parent.Parent'>,
<type 'sage.structure.category_object.CategoryObject'>,
<type 'sage.structure.sage_object.SageObject'>,
<class '__main__.Ds.parent_class'>,
<class '__main__.Cs.parent_class'>,
<class '__main__.Bs.parent_class'>,
<class '__main__.As.parent_class'>,
<... 'object'>]
sage: D.fA()
'A'
sage: D.fB()
'B'
sage: D.fC()
'C'
sage: D.fD()
'D'
sage: D.f()
'C'
sage: D.g()
'myparent'
```

```
sage: D.element_class
<class '__main__.myparent_with_category.element_class'>
sage: D.element_class.mro()
[<class '__main__.myparent_with_category.element_class'>,
<class __main__.Element at ...>,
<class '__main__.Ds.element_class'>,
<class '__main__.Cs.element_class'>,
```

```
<class '__main__.Bs.element_class'>,
<class '__main__.As.element_class'>,
<... 'object'>]
```

**_super_categories**()

The immediate super categories of this category.

This lazy attribute caches the result of the mandatory method *super_categories()* for speed. It also does some mangling (flattening join categories, sorting, . . . ).

Whenever speed matters, developers are advised to use this lazy attribute rather than calling *super_categories()*.

**Note:** This attribute is likely to eventually become a tuple. When this happens, we might as well use *Category._sort()*, if not *Category._sort_uniq()*.

EXAMPLES:

```
sage: Rings()._super_categories
[Category of rngs, Category of semirings]
```

**_super_categories_for_classes**()

The super categories of this category used for building classes.

This is a close variant of *_super_categories()* used for constructing the list of the bases for *parent_class()*, *element_class()*, and friends. The purpose is ensure that Python will find a proper Method Resolution Order for those classes. For background, see sage.misc.c3_controlled.

**See also:**

_cmp_key().

**Note:** This attribute is calculated as a by-product of computing *_all_super_categories()*.

EXAMPLES:

```
sage: Rings()._super_categories_for_classes
[Category of rngs, Category of semirings]
```

**_all_super_categories**()

All the super categories of this category, including this category.

Since trac ticket #11943, the order of super categories is determined by Python's method resolution order C3 algorithm.

**See also:**

*all_super_categories()*

**Note:** this attribute is likely to eventually become a tuple.

**Note:** this sets *_super_categories_for_classes()* as a side effect

EXAMPLES:

```
sage: C = Rings(); C
Category of rings
sage: C._all_super_categories
[Category of rings, Category of rngs, Category of semirings, ...
 Category of monoids, ...
 Category of commutative additive groups, ...
 Category of sets, Category of sets with partial maps,
 Category of objects]
```

**_all_super_categories_proper**()

All the proper super categories of this category.

Since trac ticket #11943, the order of super categories is determined by Python's method resolution order C3 algorithm.

**See also:**

*all_super_categories()*

---

**Note:** this attribute is likely to eventually become a tuple.

---

EXAMPLES:

```
sage: C = Rings(); C
Category of rings
sage: C._all_super_categories_proper
[Category of rngs, Category of semirings, ...
 Category of monoids, ...
 Category of commutative additive groups, ...
 Category of sets, Category of sets with partial maps,
 Category of objects]
```

**_set_of_super_categories**()

The frozen set of all proper super categories of this category.

---

**Note:** this is used for speeding up category containment tests.

---

**See also:**

*all_super_categories()*

EXAMPLES:

```
sage: Groups()._set_of_super_categories
frozenset({Category of inverse unital magmas,
           Category of unital magmas,
           Category of magmas,
           Category of monoids,
           Category of objects,
           Category of semigroups,
           Category of sets with partial maps,
           Category of sets})
sage: sorted(Groups()._set_of_super_categories, key=str)
[Category of inverse unital magmas, Category of magmas, Category of monoids,
```

```
Category of objects, Category of semigroups, Category of sets,
Category of sets with partial maps, Category of unital magmas]
```

**_make_named_class** (*name*, *method_provider*, *cache=False*, *picklable=True*)

Construction of the parent/element/... class of `self`.

INPUT:

- `name` – a string; the name of the class as an attribute of `self`. E.g. "parent_class"
- `method_provider` – a string; the name of an attribute of `self` that provides methods for the new class (in addition to those coming from the super categories). E.g. "ParentMethods"
- `cache` – a boolean or `ignore_reduction` (default: `False`) (passed down to dynamic_class; for internal use only)
- `picklable` – a boolean (default: `True`)

ASSUMPTION:

It is assumed that this method is only called from a lazy attribute whose name coincides with the given `name`.

OUTPUT:

A dynamic class with bases given by the corresponding named classes of `self`'s super_categories, and methods taken from the class `getattr(self,method_provider)`.

**Note:**

- In this default implementation, the reduction data of the named class makes it depend on `self`. Since the result is going to be stored in a lazy attribute of `self` anyway, we may as well disable the caching in `dynamic_class` (hence the default value `cache=False`).
- *`CategoryWithParameters`* overrides this method so that the same parent/element/... classes can be shared between closely related categories.
- The bases of the named class may also contain the named classes of some indirect super categories, according to *`_super_categories_for_classes()`*. This is to guarantee that Python will build consistent method resolution orders. For background, see `sage.misc.c3_controlled`.

**See also:**

*`CategoryWithParameters._make_named_class()`*

EXAMPLES:

```
sage: PC = Rings()._make_named_class("parent_class", "ParentMethods"); PC
<class 'sage.categories.rings.Rings.parent_class'>
sage: type(PC)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
sage: PC.__bases__
(<class 'sage.categories.rngs.Rngs.parent_class'>,
 <class 'sage.categories.semirings.Semirings.parent_class'>)
```

Note that, by default, the result is not cached:

```
sage: PC is Rings()._make_named_class("parent_class", "ParentMethods")
False
```

Indeed this method is only meant to construct lazy attributes like `parent_class` which already handle this caching:

```
sage: Rings().parent_class
<class 'sage.categories.rings.Rings.parent_class'>
```

Reduction for pickling also assumes the existence of this lazy attribute:

```
sage: PC._reduction
(<built-in function getattr>, (Category of rings, 'parent_class'))
sage: loads(dumps(PC)) is Rings().parent_class
True
```

**_repr_**()

Return the print representation of this category.

EXAMPLES:

```
sage: Sets() # indirect doctest
Category of sets
```

**_repr_object_names**()

Return the name of the objects of this category.

EXAMPLES:

```
sage: FiniteGroups()._repr_object_names()
'finite groups'
sage: AlgebrasWithBasis(QQ)._repr_object_names()
'algebras with basis over Rational Field'
```

**_test_category**(***options*)

Run generic tests on this category

**See also:**

`TestSuite`.

EXAMPLES:

```
sage: Sets()._test_category()
```

Let us now write a couple broken categories:

```
sage: class MyObjects(Category):
....:     pass
sage: MyObjects()._test_category()
Traceback (most recent call last):
...
NotImplementedError: <abstract method super_categories at ...>

sage: class MyObjects(Category):
....:     def super_categories(self):
....:         return tuple()
sage: MyObjects()._test_category()
Traceback (most recent call last):
...
AssertionError: Category of my objects.super_categories() should return a list

sage: class MyObjects(Category):
```

```
....:       def super_categories(self):
....:           return []
sage: MyObjects()._test_category()
Traceback (most recent call last):
...
AssertionError: Category of my objects is not a subcategory of Objects()
```

**_with_axiom**(*axiom*)

Return the subcategory of the objects of `self` satisfying the given `axiom`.

INPUT:

- `axiom` – a string, the name of an axiom

EXAMPLES:

```
sage: Sets()._with_axiom("Finite")
Category of finite sets

sage: type(Magmas().Finite().Commutative())
<class 'sage.categories.category.JoinCategory_with_category'>
sage: Magmas().Finite().Commutative().super_categories()
[Category of commutative magmas, Category of finite sets]
sage: Algebras(QQ).WithBasis().Commutative() is Algebras(QQ).Commutative().
↪WithBasis()
True
```

When `axiom` is not defined for `self`, `self` is returned:

```
sage: Sets()._with_axiom("Associative")
Category of sets
```

**Warning:** This may be changed in the future to raising an error.

**_with_axiom_as_tuple**(*axiom*)

Return a tuple of categories whose join is `self._with_axiom()`.

INPUT:

- `axiom` – a string, the name of an axiom

This is a lazy version of *_with_axiom()* which is used to avoid recursion loops during join calculations.

**Note:** The order in the result is irrelevant.

EXAMPLES:

```
sage: Sets()._with_axiom_as_tuple('Finite')
(Category of finite sets,)
sage: Magmas()._with_axiom_as_tuple('Finite')
(Category of magmas, Category of finite sets)
sage: Rings().Division()._with_axiom_as_tuple('Finite')
(Category of division rings,
 Category of finite monoids,
 Category of commutative magmas,
 Category of finite additive groups)
```

```
sage: HopfAlgebras(QQ)._with_axiom_as_tuple('FiniteDimensional')
(Category of hopf algebras over Rational Field,
 Category of finite dimensional modules over Rational Field)
```

**_without_axioms** (*named=False*)

Return the category without the axioms that have been added to create it.

INPUT:

- `named` – a boolean (default: `False`)

---

**Todo:** Improve this explanation.

---

If `named` is `True`, then this stops at the first category that has an explicit name of its own. See *category_with_axiom.CategoryWithAxiom._without_axioms()*

EXAMPLES:

```
sage: Sets()._without_axioms()
Category of sets
sage: Semigroups()._without_axioms()
Category of magmas
sage: Algebras(QQ).Commutative().WithBasis()._without_axioms()
Category of magmatic algebras over Rational Field
sage: Algebras(QQ).Commutative().WithBasis()._without_axioms(named=True)
Category of algebras over Rational Field
```

**static _sort** (*categories*)

Return the categories after sorting them decreasingly according to their comparison key.

**See also:**

`_cmp_key()`

INPUT:

- `categories` – a list (or iterable) of non-join categories

OUTPUT:

A sorted tuple of categories, possibly with repeats.

---

**Note:** The auxiliary function $_flatten_categories$ used in the test below expects a second argument, which is a type such that instances of that type will be replaced by its super categories. Usually, this type is *JoinCategory*.

---

EXAMPLES:

```
sage: Category._sort([Sets(), Objects(), Coalgebras(QQ), Monoids(), Sets().
↪Finite()])
(Category of monoids,
 Category of coalgebras over Rational Field,
 Category of finite sets,
 Category of sets,
 Category of objects)
sage: Category._sort([Sets().Finite(), Semigroups().Finite(), Sets().Facade(),
↪Magmas().Commutative()])
(Category of finite semigroups,
```

```
 Category of commutative magmas,
 Category of finite sets,
 Category of facade sets)
sage: Category._sort(Category._flatten_categories([Sets().Finite(),␣
↪Algebras(QQ).WithBasis(), Semigroups().Finite(), Sets().Facade(),
↪Algebras(QQ).Commutative(), Algebras(QQ).Graded().WithBasis()], sage.
↪categories.category.JoinCategory))
(Category of algebras with basis over Rational Field,
 Category of algebras with basis over Rational Field,
 Category of graded algebras over Rational Field,
 Category of commutative algebras over Rational Field,
 Category of finite semigroups,
 Category of finite sets,
 Category of facade sets)
```

**_sort_uniq**(*categories*)

Return the categories after sorting them and removing redundant categories.

Redundant categories include duplicates and categories which are super categories of other categories in the input.

INPUT:

- `categories` – a list (or iterable) of categories

OUTPUT: a sorted tuple of mutually incomparable categories

EXAMPLES:

```
sage: Category._sort_uniq([Rings(), Monoids(), Coalgebras(QQ)])
(Category of rings, Category of coalgebras over Rational Field)
```

Note that, in the above example, `Monoids()` does not appear in the result because it is a super category of `Rings()`.

**static __classcall__**(**args*, ***options*)

Input mangling for unique representation.

Let `C = Cs(...)` be a category. Since trac ticket #12895, the class of `C` is a dynamic subclass `Cs_with_category` of `Cs` in order for `C` to inherit code from the `SubcategoryMethods` nested classes of its super categories.

The purpose of this `__classcall__` method is to ensure that reconstructing `C` from its class with `Cs_with_category(...)` actually calls properly `Cs(...)` and gives back `C`.

**See also:**

*subcategory_class()*

EXAMPLES:

```
sage: A = Algebras(QQ)
sage: A.__class__
<class 'sage.categories.algebras.Algebras_with_category'>
sage: A is Algebras(QQ)
True
sage: A is A.__class__(QQ)
True
```

**__init__**(*s=None*)

Initializes this category.

EXAMPLES:

```
sage: class SemiprimitiveRings(Category):
....:     def super_categories(self):
....:         return [Rings()]
....:
....:     class ParentMethods:
....:         def jacobson_radical(self):
....:             return self.ideal(0)
....:
sage: C = SemiprimitiveRings()
sage: C
Category of semiprimitive rings
sage: C.__class__
<class '__main__.SemiprimitiveRings_with_category'>
```

**Note:** Specifying the name of this category by passing a string is deprecated. If the default name (built from the name of the class) is not adequate, please use `_repr_object_names()` to customize it.

**Realizations**()

Return the category of realizations of the parent `self` or of objects of the category `self`

INPUT:

- `self` – a parent or a concrete category

**Note:** this *function* is actually inserted as a *method* in the class `Category` (see `Realizations()`). It is defined here for code locality reasons.

EXAMPLES:

The category of realizations of some algebra:

```
sage: Algebras(QQ).Realizations()
Join of Category of algebras over Rational Field and Category of realizations␣
↪of unital magmas
```

The category of realizations of a given algebra:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.Realizations()
Category of realizations of The subset algebra of {1, 2, 3} over Rational␣
↪Field

sage: C = GradedHopfAlgebrasWithBasis(QQ).Realizations(); C
Join of Category of graded hopf algebras with basis over Rational Field and␣
↪Category of realizations of hopf algebras over Rational Field
sage: C.super_categories()
[Category of graded hopf algebras with basis over Rational Field, Category of␣
↪realizations of hopf algebras over Rational Field]

sage: TestSuite(C).run()
```

**See also:**

- *Sets().WithRealizations*
- ClasscallMetaclass

**Todo:** Add an optional argument to allow for:

```
sage: Realizations(A, category = Blahs()) # todo: not implemented
```

**WithRealizations()**

Return the category of parents in `self` endowed with multiple realizations.

INPUT:

- `self` – a category

**See also:**

- The documentation and code (*sage.categories.examples.with_realizations*) of `Sets().WithRealizations().example()` for more on how to use and implement a parent with several realizations.
- Various use cases:
  - SymmetricFunctions
  - QuasiSymmetricFunctions
  - NonCommutativeSymmetricFunctions
  - SymmetricFunctionsNonCommutingVariables
  - DescentAlgebra
  - algebras.Moebius
  - IwahoriHeckeAlgebra
  - ExtendedAffineWeylGroup
- The Implementing Algebraic Structures thematic tutorial.
- *sage.categories.realizations*

**Note:** this *function* is actually inserted as a *method* in the class *Category* (see *WithRealizations()*). It is defined here for code locality reasons.

EXAMPLES:

```
sage: Sets().WithRealizations()
Category of sets with realizations
```

## Parent with realizations

Let us now explain the concept of realizations. A *parent with realizations* is a facade parent (see *Sets.Facade*) admitting multiple concrete realizations where its elements are represented. Consider for example an algebra $A$ which admits several natural bases:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
```

For each such basis $B$ one implements a parent $P_B$ which realizes $A$ with its elements represented by expanding them on the basis $B$:

```
sage: A.F()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
sage: A.Out()
The subset algebra of {1, 2, 3} over Rational Field in the Out basis
sage: A.In()
The subset algebra of {1, 2, 3} over Rational Field in the In basis

sage: A.an_element()
F[{}] + 2*F[{1}] + 3*F[{2}] + F[{1, 2}]
```

If $B$ and $B'$ are two bases, then the change of basis from $B$ to $B'$ is implemented by a canonical coercion between $P_B$ and $P_{B'}$:

```
sage: F = A.F(); In = A.In(); Out = A.Out()
sage: i = In.an_element(); i
In[{}] + 2*In[{1}] + 3*In[{2}] + In[{1, 2}]
sage: F(i)
7*F[{}] + 3*F[{1}] + 4*F[{2}] + F[{1, 2}]
sage: F.coerce_map_from(Out)
Generic morphism:
  From: The subset algebra of {1, 2, 3} over Rational Field in the Out basis
  To:   The subset algebra of {1, 2, 3} over Rational Field in the␣
↪Fundamental basis
```

allowing for mixed arithmetic:

```
sage: (1 + Out.from_set(1)) * In.from_set(2,3)
Out[{}] + 2*Out[{1}] + 2*Out[{2}] + 2*Out[{3}] + 2*Out[{1, 2}] + 2*Out[{1, 3}
↪] + 4*Out[{2, 3}] + 4*Out[{1, 2, 3}]
```

In our example, there are three realizations:

```
sage: A.realizations()
[The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis,
 The subset algebra of {1, 2, 3} over Rational Field in the In basis,
 The subset algebra of {1, 2, 3} over Rational Field in the Out basis]
```

Instead of manually defining the shorthands `F`, `In`, and `Out`, as above one can just do:

```
sage: A.inject_shorthands()
Defining F as shorthand for The subset algebra of {1, 2, 3} over Rational␣
↪Field in the Fundamental basis
Defining In as shorthand for The subset algebra of {1, 2, 3} over Rational␣
↪Field in the In basis
Defining Out as shorthand for The subset algebra of {1, 2, 3} over Rational␣
↪Field in the Out basis
```

### Rationale

Besides some goodies described below, the role of $A$ is threefold:

- To provide, as illustrated above, a single entry point for the algebra as a whole: documentation, access to its properties and different realizations, etc.
- To provide a natural location for the initialization of the bases and the coercions between, and other methods that are common to all bases.
- To let other objects refer to $A$ while allowing elements to be represented in any of the realizations.

We now illustrate this second point by defining the polynomial ring with coefficients in $A$:

```
sage: P = A['x']; P
Univariate Polynomial Ring in x over The subset algebra of {1, 2, 3} over␣
↪Rational Field
sage: x = P.gen()
```

In the following examples, the coefficients turn out to be all represented in the $F$ basis:

```
sage: P.one()
F[{}]
sage: (P.an_element() + 1)^2
F[{}]*x^2 + 2*F[{}]*x + F[{}]
```

However we can create a polynomial with mixed coefficients, and compute with it:

```
sage: p = P([1, In[{1}], Out[{2}] ]); p
Out[{2}]*x^2 + In[{1}]*x + F[{}]
sage: p^2
Out[{2}]*x^4
+ (-8*In[{}] + 4*In[{1}] + 8*In[{2}] + 4*In[{3}] - 4*In[{1, 2}] - 2*In[{1, 3}
↪] - 4*In[{2, 3}] + 2*In[{1, 2, 3}])*x^3
+ (F[{}] + 3*F[{1}] + 2*F[{2}] - 2*F[{1, 2}] - 2*F[{2, 3}] + 2*F[{1, 2, 3}
↪])*x^2
+ (2*F[{}] + 2*F[{1}])*x
+ F[{}]
```

Note how each coefficient involves a single basis which need not be that of the other coefficients. Which basis is used depends on how coercion happened during mixed arithmetic and needs not be deterministic.

One can easily coerce all coefficient to a given basis with:

```
sage: p.map_coefficients(In)
(-4*In[{}] + 2*In[{1}] + 4*In[{2}] + 2*In[{3}] - 2*In[{1, 2}] - In[{1, 3}] -␣
↪2*In[{2, 3}] + In[{1, 2, 3}])*x^2 + In[{1}]*x + In[{}]
```

Alas, the natural notation for constructing such polynomials does not yet work:

```
sage: In[{1}] * x
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *: 'The subset algebra of {1, 2,␣
↪3} over Rational Field in the In basis' and 'Univariate Polynomial Ring in␣
↪x over The subset algebra of {1, 2, 3} over Rational Field'
```

### The category of realizations of $A$

The set of all realizations of $A$, together with the coercion morphisms is a category (whose class inherits from *Category_realization_of_parent*):

```
sage: A.Realizations()
Category of realizations of The subset algebra of {1, 2, 3} over Rational␣
↪Field
```

The various parent realizing $A$ belong to this category:

```
sage: A.F() in A.Realizations()
True
```

$A$ itself is in the category of algebras with realizations:

```
sage: A in Algebras(QQ).WithRealizations()
True
```

The (mostly technical) `WithRealizations` categories are the analogs of the `*WithSeveralBases` categories in MuPAD-Combinat. They provide support tools for handling the different realizations and the morphisms between them.

Typically, `VectorSpaces(QQ).FiniteDimensional().WithRealizations()` will eventually be in charge, whenever a coercion $\phi : A \mapsto B$ is registered, to register $\phi^{-1}$ as coercion $B \mapsto A$ if there is none defined yet. To achieve this, `FiniteDimensionalVectorSpaces` would provide a nested class `WithRealizations` implementing the appropriate logic.

`WithRealizations` is a *`regressive covariant functorial construction`*. On our example, this simply means that $A$ is automatically in the category of rings with realizations (covariance):

```
sage: A in Rings().WithRealizations()
True
```

and in the category of algebras (regressiveness):

```
sage: A in Algebras(QQ)
True
```

---

**Note:** For `C` a category, `C.WithRealizations()` in fact calls `sage.categories.with_realizations.WithRealizations(C)`. The later is responsible for building the hierarchy of the categories with realizations in parallel to that of their base categories, optimizing away those categories that do not provide a `WithRealizations` nested class. See *`sage.categories.covariant_functorial_construction`* for the technical details.

---

**Note:** Design question: currently `WithRealizations` is a regressive construction. That is `self.WithRealizations()` is a subcategory of `self` by default:

```
sage: Algebras(QQ).WithRealizations().super_categories()
[Category of algebras over Rational Field,
 Category of monoids with realizations,
 Category of additive unital additive magmas with realizations]
```

Is this always desirable? For example, `AlgebrasWithBasis(QQ).WithRealizations()` should certainly be a subcategory of `Algebras(QQ)`, but not of `AlgebrasWithBasis(QQ)`. This is because `AlgebrasWithBasis(QQ)` is specifying something about the concrete realization.

---

**additional_structure**()

Return whether `self` defines additional structure.

OUTPUT:

- `self` if `self` defines additional structure and `None` otherwise. This default implementation returns `self`.

A category $C$ *defines additional structure* if $C$-morphisms shall preserve more structure (e.g. operations) than that specified by the super categories of $C$. For example, the category of magmas defines additional structure, namely the operation $*$ that shall be preserved by magma morphisms. On the other hand the category of rings does not define additional structure: a function between two rings that is both a unital magma morphism and a unital additive magma morphism is automatically a ring morphism.

Formally speaking $C$ *defines additional structure*, if $C$ is *not* a full subcategory of the join of its super categories: the morphisms need to preserve more structure, and thus the homsets are smaller.

By default, a category is considered as defining additional structure, unless it is a *category with axiom*.

EXAMPLES:

Here are some typical structure categories, with the additional structure they define:

```
sage: Sets().additional_structure()
Category of sets
sage: Magmas().additional_structure()          # `*`
Category of magmas
sage: AdditiveMagmas().additional_structure() # `+`
Category of additive magmas
sage: LeftModules(ZZ).additional_structure()  # left multiplication by scalar
Category of left modules over Integer Ring
sage: Coalgebras(QQ).additional_structure()   # coproduct
Category of coalgebras over Rational Field
sage: Crystals().additional_structure()       # crystal operators
Category of crystals
```

On the other hand, the category of semigroups is not a structure category, since its operation $+$ is already defined by the category of magmas:

```
sage: Semigroups().additional_structure()
```

Most *categories with axiom* don't define additional structure:

```
sage: Sets().Finite().additional_structure()
sage: Rings().Commutative().additional_structure()
sage: Modules(QQ).FiniteDimensional().additional_structure()
sage: from sage.categories.magmatic_algebras import MagmaticAlgebras
sage: MagmaticAlgebras(QQ).Unital().additional_structure()
```

As of Sage 6.4, the only exceptions are the category of unital magmas or the category of unital additive magmas (both define a unit which shall be preserved by morphisms):

```
sage: Magmas().Unital().additional_structure()
Category of unital magmas
sage: AdditiveMagmas().AdditiveUnital().additional_structure()
Category of additive unital additive magmas
```

Similarly, *functorial construction categories* don't define additional structure, unless the construction is actually defined by their base category. For example, the category of graded modules defines a grading which shall be preserved by morphisms:

```
sage: Modules(ZZ).Graded().additional_structure()
Category of graded modules over Integer Ring
```

On the other hand, the category of graded algebras does not define additional structure; indeed an algebra morphism which is also a module morphism is a graded algebra morphism:

```
sage: Algebras(ZZ).Graded().additional_structure()
```

Similarly, morphisms are requested to preserve the structure given by the following constructions:

```
sage: Sets().Quotients().additional_structure()
Category of quotients of sets
sage: Sets().CartesianProducts().additional_structure()
Category of Cartesian products of sets
sage: Modules(QQ).TensorProducts().additional_structure()
```

This might change, as we are lacking enough data points to guarantee that this was the correct design decision.

---

**Note:** In some cases a category defines additional structure, where the structure can be useful to manipulate morphisms but where, in most use cases, we don't want the morphisms to necessarily preserve it. For example, in the context of finite dimensional vector spaces, having a distinguished basis allows for representing morphisms by matrices; yet considering only morphisms that preserve that distinguished basis would be boring.

In such cases, we might want to eventually have two categories, one where the additional structure is preserved, and one where it's not necessarily preserved (we would need to find an idiom for this).

At this point, a choice is to be made each time, according to the main use cases. Some of those choices are yet to be settled. For example, should by default:

- an euclidean domain morphism preserve euclidean division?

  ```
  sage: EuclideanDomains().additional_structure()
  Category of euclidean domains
  ```

- an enumerated set morphism preserve the distinguished enumeration?

  ```
  sage: EnumeratedSets().additional_structure()
  ```

- a module with basis morphism preserve the distinguished basis?

  ```
  sage: Modules(QQ).WithBasis().additional_structure()
  ```

---

**See also:**

This method together with the methods overloading it provide the basic data to determine, for a given category, the super categories that define some structure (see `structure()`), and to test whether a category is a full subcategory of some other category (see `is_full_subcategory()`). For example, the category of Coxeter groups is not full subcategory of the category of groups since morphisms need to preserve the distinguished generators:

```
sage: CoxeterGroups().is_full_subcategory(Groups())
False
```

The support for modeling full subcategories has been introduced in trac ticket #16340.

**all_super_categories**(*proper=False*)

Returns the list of all super categories of this category.

INPUT:

- proper – a boolean (default: False); whether to exclude this category.

Since trac ticket #11943, the order of super categories is determined by Python's method resolution order C3 algorithm.

**Note:** Whenever speed matters, the developers are advised to use instead the lazy attributes *_all_super_categories()*, *_all_super_categories_proper()*, or *_set_of_super_categories()*, as appropriate. Simply because lazy attributes are much faster than any method.

EXAMPLES:

```
sage: C = Rings(); C
Category of rings
sage: C.all_super_categories()
[Category of rings, Category of rngs, Category of semirings, ...
 Category of monoids, ...
 Category of commutative additive groups, ...
 Category of sets, Category of sets with partial maps,
 Category of objects]

sage: C.all_super_categories(proper = True)
[Category of rngs, Category of semirings, ...
 Category of monoids, ...
 Category of commutative additive groups, ...
 Category of sets, Category of sets with partial maps,
 Category of objects]

sage: Sets().all_super_categories()
[Category of sets, Category of sets with partial maps, Category of objects]
sage: Sets().all_super_categories(proper=True)
[Category of sets with partial maps, Category of objects]
sage: Sets().all_super_categories() is Sets()._all_super_categories
True
sage: Sets().all_super_categories(proper=True) is Sets()._all_super_
↪categories_proper
True
```

**classmethod an_instance()**

Return an instance of this class.

EXAMPLES:

```
sage: Rings.an_instance()
Category of rings
```

Parametrized categories should overload this default implementation to provide appropriate arguments:

```
sage: Algebras.an_instance()
Category of algebras over Rational Field
sage: Bimodules.an_instance()
Category of bimodules over Rational Field on the left and Real Field with 53␣
↪bits of precision on the right
sage: AlgebraIdeals.an_instance()
Category of algebra ideals in Univariate Polynomial Ring in x over Rational␣
↪Field
```

**axioms**()

Return the axioms known to be satisfied by all the objects of `self`.

Technically, this is the set of all the axioms `A` such that, if `Cs` is the category defining `A`, then `self` is a subcategory of `Cs().A()`. Any additional axiom `A` would yield a strict subcategory of `self`, at the very least `self & Cs().A()` where `Cs` is the category defining `A`.

EXAMPLES:

```
sage: Monoids().axioms()
frozenset({'Associative', 'Unital'})
sage: (EnumeratedSets().Infinite() & Sets().Facade()).axioms()
frozenset({'Enumerated', 'Facade', 'Infinite'})
```

**category**()

Return the category of this category. So far, all categories are in the category of objects.

EXAMPLES:

```
sage: Sets().category()
Category of objects
sage: VectorSpaces(QQ).category()
Category of objects
```

**category_graph**()

Returns the graph of all super categories of this category

EXAMPLES:

```
sage: C = Algebras(QQ)
sage: G = C.category_graph()
sage: G.is_directed_acyclic()
True
sage: G.girth()
4
```

**element_class**()

A common super class for all elements of parents in this category (and its subcategories).

This class contains the methods defined in the nested class `self.ElementMethods` (if it exists), and has as bases the element classes of the super categories of `self`.

**See also:**

- *parent_class()*, *morphism_class()*
- *Category* for details

EXAMPLES:

```
sage: C = Algebras(QQ).element_class; C
<class 'sage.categories.algebras.Algebras.element_class'>
sage: type(C)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
```

By trac ticket #11935, some categories share their element classes. For example, the element class of an algebra only depends on the category of the base. A typical example is the category of algebras over a field versus algebras over a non-field:

```
sage: Algebras(GF(5)).element_class is Algebras(GF(3)).element_class
True
sage: Algebras(QQ).element_class is Algebras(ZZ).element_class
False
sage: Algebras(ZZ['t']).element_class is Algebras(ZZ['t','x']).element_class
True
```

These classes are constructed with `__slots__ = []`, so they behave like extension types:

```
sage: E = FiniteEnumeratedSets().element_class
sage: from sage.structure.misc import is_extension_type
sage: is_extension_type(E)
True
```

**See also:**

*`parent_class()`*

**`example`**(**args*, ***keywords*)

Returns an object in this category. Most of the time, this is a parent.

This serves three purposes:

- Give a typical example to better explain what the category is all about. (and by the way prove that the category is non empty :-) )
- Provide a minimal template for implementing other objects in this category
- Provide an object on which to test generic code implemented by the category

For all those applications, the implementation of the object shall be kept to a strict minimum. The object is therefore not meant to be used for other applications; most of the time a full featured version is available elsewhere in Sage, and should be used insted.

Technical note: by default `FooBar(...).example()` is constructed by looking up `sage.categories.examples.foo_bar.Example` and calling it as `Example()`. Extra positional or named parameters are also passed down. For a category over base ring, the base ring is further passed down as an optional argument.

Categories are welcome to override this default implementation.

EXAMPLES:

```
sage: Semigroups().example()
An example of a semigroup: the left zero semigroup

sage: Monoids().Subquotients().example()
NotImplemented
```

**`full_super_categories`**()

Return the *immediate* full super categories of `self`.

**See also:**

- *`super_categories()`*
- *`is_full_subcategory()`*

**Warning:** The current implementation selects the full subcategories among the immediate super categories of `self`. This assumes that, if $C \subset B \subset A$ is a chain of categories and $C$ is a full subcategory of $A$, then $C$ is a full subcategory of $B$ and $B$ is a full subcategory of $A$.

This assumption is guaranteed to hold with the current model and implementation of full subcategories in Sage. However, mathematically speaking, this is too restrictive. This indeed prevents the complete modelling of situations where any $A$ morphism between elements of $C$ automatically preserves the $B$ structure. See below for an example.

EXAMPLES:

A semigroup morphism between two finite semigroups is a finite semigroup morphism:

```
sage: Semigroups().Finite().full_super_categories()
[Category of semigroups]
```

On the other hand, a semigroup morphism between two monoids is not necessarily a monoid morphism (which must map the unit to the unit):

```
sage: Monoids().super_categories()
[Category of semigroups, Category of unital magmas]
sage: Monoids().full_super_categories()
[Category of unital magmas]
```

Any semigroup morphism between two groups is automatically a monoid morphism (in a group the unit is the unique idempotent, so it has to be mapped to the unit). Yet, due to the limitation of the model advertised above, Sage currently can't be taught that the category of groups is a full subcategory of the category of semigroups:

```
sage: Groups().full_super_categories()     # todo: not implemented
[Category of monoids, Category of semigroups, Category of inverse unital␣
↪magmas]
sage: Groups().full_super_categories()
[Category of monoids, Category of inverse unital magmas]
```

**is_abelian**()

Returns whether this category is abelian.

An abelian category is a category satisfying:

- It has a zero object;
- It has all pullbacks and pushouts;
- All monomorphisms and epimorphisms are normal.

Equivalently, one can define an increasing sequence of conditions:

- A category is pre-additive if it is enriched over abelian groups (all homsets are abelian groups and composition is bilinear);
- A pre-additive category is additive if every finite set of objects has a biproduct (we can form direct sums and direct products);
- An additive category is pre-abelian if every morphism has both a kernel and a cokernel;
- A pre-abelian category is abelian if every monomorphism is the kernel of some morphism and every epimorphism is the cokernel of some morphism.

EXAMPLES:

```
sage: Modules(ZZ).is_abelian()
True
sage: FreeModules(ZZ).is_abelian()
False
sage: FreeModules(QQ).is_abelian()
True
sage: CommutativeAdditiveGroups().is_abelian()
True
sage: Semigroups().is_abelian()
Traceback (most recent call last):
NotImplementedError: is_abelian
```

**is_full_subcategory** (*other*)

Return whether `self` is a full subcategory of `other`.

A subcategory $B$ of a category $A$ is a *full subcategory* if any $A$-morphism between two objects of $B$ is also a $B$-morphism (the reciprocal always holds: any $B$-morphism between two objects of $B$ is an $A$-morphism).

This is computed by testing whether `self` is a subcategory of `other` and whether they have the same structure, as determined by *structure()* from the result of *additional_structure()* on the super categories.

> **Warning:** A positive answer is guaranteed to be mathematically correct. A negative answer may mean that Sage has not been taught enough information (or can not yet within the current model) to derive this information. See *full_super_categories()* for a discussion.

**See also:**

- *is_subcategory()*
- *full_super_categories()*

EXAMPLES:

```
sage: Magmas().Associative().is_full_subcategory(Magmas())
True
sage: Magmas().Unital().is_full_subcategory(Magmas())
False
sage: Rings().is_full_subcategory(Magmas().Unital() & AdditiveMagmas().
↪AdditiveUnital())
True
```

Here are two typical examples of false negatives:

```
sage: Groups().is_full_subcategory(Semigroups())
False
sage: Groups().is_full_subcategory(Semigroups()) # todo: not implemented
True
sage: Fields().is_full_subcategory(Rings())
False
sage: Fields().is_full_subcategory(Rings())      # todo: not implemented
True
```

**Todo:** The latter is a consequence of *EuclideanDomains* currently being a structure category. Is this what we want?

```
sage: EuclideanDomains().is_full_subcategory(Rings())
False
```

**is_subcategory** (*c*)

Returns True if self is naturally embedded as a subcategory of c.

EXAMPLES:

```
sage: AbGrps = CommutativeAdditiveGroups()
sage: Rings().is_subcategory(AbGrps)
True
sage: AbGrps.is_subcategory(Rings())
False
```

The `is_subcategory` function takes into account the base.

```
sage: M3 = VectorSpaces(FiniteField(3))
sage: M9 = VectorSpaces(FiniteField(9, 'a'))
sage: M3.is_subcategory(M9)
False
```

Join categories are properly handled:

```
sage: CatJ = Category.join((CommutativeAdditiveGroups(), Semigroups()))
sage: Rings().is_subcategory(CatJ)
True
```

```
sage: V3 = VectorSpaces(FiniteField(3))
sage: POSet = PartiallyOrderedSets()
sage: PoV3 = Category.join((V3, POSet))
sage: A3 = AlgebrasWithBasis(FiniteField(3))
sage: PoA3 = Category.join((A3, POSet))
sage: PoA3.is_subcategory(PoV3)
True
sage: PoV3.is_subcategory(PoV3)
True
sage: PoV3.is_subcategory(PoA3)
False
```

**static join** (*categories*, *as_list=False*, *ignore_axioms=()*, *axioms=()*)

Return the join of the input categories in the lattice of categories.

At the level of objects and morphisms, this operation corresponds to intersection: the objects and morphisms of a join category are those that belong to all its super categories.

INPUT:

- `categories` – a list (or iterable) of categories
- `as_list` – a boolean (default: `False`); whether the result should be returned as a list
- `axioms` – a tuple of strings; the names of some supplementary axioms

**See also:**

`__and__()` for a shortcut

EXAMPLES:

```
sage: J = Category.join((Groups(), CommutativeAdditiveMonoids())); J
Join of Category of groups and Category of commutative additive monoids
sage: J.super_categories()
[Category of groups, Category of commutative additive monoids]
sage: J.all_super_categories(proper=True)
[Category of groups, ..., Category of magmas,
 Category of commutative additive monoids, ..., Category of additive magmas,
 Category of sets, ...]
```

As a short hand, one can use:

```
sage: Groups() & CommutativeAdditiveMonoids()
Join of Category of groups and Category of commutative additive monoids
```

This is a commutative and associative operation:

```
sage: Groups() & Posets()
Join of Category of groups and Category of posets
sage: Posets() & Groups()
Join of Category of groups and Category of posets

sage: Groups() & (CommutativeAdditiveMonoids() & Posets())
Join of Category of groups
    and Category of commutative additive monoids
    and Category of posets
sage: (Groups() & CommutativeAdditiveMonoids()) & Posets()
Join of Category of groups
    and Category of commutative additive monoids
    and Category of posets
```

The join of a single category is the category itself:

```
sage: Category.join([Monoids()])
Category of monoids
```

Similarly, the join of several mutually comparable categories is the smallest one:

```
sage: Category.join((Sets(), Rings(), Monoids()))
Category of rings
```

In particular, the unit is the top category `Objects`:

```
sage: Groups() & Objects()
Category of groups
```

If the optional parameter `as_list` is `True`, this returns the super categories of the join as a list, without constructing the join category itself:

```
sage: Category.join((Groups(), CommutativeAdditiveMonoids()), as_list=True)
[Category of groups, Category of commutative additive monoids]
sage: Category.join((Sets(), Rings(), Monoids()), as_list=True)
[Category of rings]
sage: Category.join((Modules(ZZ), FiniteFields()), as_list=True)
[Category of finite enumerated fields, Category of modules over Integer Ring]
sage: Category.join([], as_list=True)
[]
```

```
sage: Category.join([Groups()], as_list=True)
[Category of groups]
sage: Category.join([Groups() & Posets()], as_list=True)
[Category of groups, Category of posets]
```

Support for axiom categories (TODO: put here meaningfull examples):

```
sage: Sets().Facade() & Sets().Infinite()
Category of facade infinite sets
sage: Magmas().Infinite() & Sets().Facade()
Category of facade infinite magmas

sage: FiniteSets() & Monoids()
Category of finite monoids
sage: Rings().Commutative() & Sets().Finite()
Category of finite commutative rings
```

Note that several of the above examples are actually join categories; they are just nicely displayed:

```
sage: AlgebrasWithBasis(QQ) & FiniteSets().Algebras(QQ)
Join of Category of finite dimensional algebras with basis over Rational Field
    and Category of finite set algebras over Rational Field

sage: UniqueFactorizationDomains() & Algebras(QQ)
Join of Category of unique factorization domains
    and Category of commutative algebras over Rational Field
```

**static meet** (*categories*)

Returns the meet of a list of categories

INPUT:

- `categories` - a non empty list (or iterable) of categories

**See also:**

`__or__()` for a shortcut

EXAMPLES:

```
sage: Category.meet([Algebras(ZZ), Algebras(QQ), Groups()])
Category of monoids
```

That meet of an empty list should be a category which is a subcategory of all categories, which does not make practical sense:

```
sage: Category.meet([])
Traceback (most recent call last):
...
ValueError: The meet of an empty list of categories is not implemented
```

**morphism_class** ()

A common super class for all morphisms between parents in this category (and its subcategories).

This class contains the methods defined in the nested class `self.MorphismMethods` (if it exists), and has as bases the morphism classes of the super categories of `self`.

**See also:**

- *parent_class()*, *element_class()*

- *Category* for details

EXAMPLES:

```
sage: C = Algebras(QQ).morphism_class; C
<class 'sage.categories.algebras.Algebras.morphism_class'>
sage: type(C)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
```

**or_subcategory**(*category=None*, *join=False*)

Return `category` or `self` if `category` is `None`.

INPUT:

- `category` – a sub category of `self`, tuple/list thereof, or `None`
- `join` – a boolean (default: `False`)

OUTPUT:

- a category

EXAMPLES:

```
sage: Monoids().or_subcategory(Groups())
Category of groups
sage: Monoids().or_subcategory(None)
Category of monoids
```

If category is a list/tuple, then a join category is returned:

```
sage: Monoids().or_subcategory((CommutativeAdditiveMonoids(), Groups()))
Join of Category of groups and Category of commutative additive monoids
```

If `join` is `False`, an error if raised if category is not a subcategory of `self`:

```
sage: Monoids().or_subcategory(EnumeratedSets())
Traceback (most recent call last):
...
ValueError: Subcategory of `Category of monoids` required; got `Category of␣
↪enumerated sets`
```

Otherwise, the two categories are joined together:

```
sage: Monoids().or_subcategory(EnumeratedSets(), join=True)
Category of enumerated monoids
```

**parent_class**()

A common super class for all parents in this category (and its subcategories).

This class contains the methods defined in the nested class `self.ParentMethods` (if it exists), and has as bases the parent classes of the super categories of `self`.

**See also:**

- *element_class()*, *morphism_class()*
- *Category* for details

EXAMPLES:

```
sage: C = Algebras(QQ).parent_class; C
<class 'sage.categories.algebras.Algebras.parent_class'>
sage: type(C)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
```

By trac ticket #11935, some categories share their parent classes. For example, the parent class of an algebra only depends on the category of the base ring. A typical example is the category of algebras over a finite field versus algebras over a non-field:

```
sage: Algebras(GF(7)).parent_class is Algebras(GF(5)).parent_class
True
sage: Algebras(QQ).parent_class is Algebras(ZZ).parent_class
False
sage: Algebras(ZZ['t']).parent_class is Algebras(ZZ['t','x']).parent_class
True
```

See `CategoryWithParameters` for an abstract base class for categories that depend on parameters, even though the parent and element classes only depend on the parent or element classes of its super categories. It is used in `Bimodules`, `Category_over_base` and `sage.categories.category.JoinCategory`.

**required_methods**()

Returns the methods that are required and optional for parents in this category and their elements.

EXAMPLES:

```
sage: Algebras(QQ).required_methods()
{'element': {'optional': ['_add_', '_mul_'], 'required': ['__nonzero__']},
 'parent': {'optional': ['algebra_generators'], 'required': ['__contains__']}}
```

**structure**()

Return the structure `self` is endowed with.

This method returns the structure that morphisms in this category shall be preserving. For example, it tells that a ring is a set endowed with a structure of both a unital magma and an additive unital magma which satisfies some further axioms. In other words, a ring morphism is a function that preserves the unital magma and additive unital magma structure.

In practice, this returns the collection of all the super categories of `self` that define some additional structure, as a frozen set.

EXAMPLES:

```
sage: Objects().structure()
frozenset()

sage: def structure(C):
....:     return Category._sort(C.structure())

sage: structure(Sets())
(Category of sets, Category of sets with partial maps)
sage: structure(Magmas())
(Category of magmas, Category of sets, Category of sets with partial maps)
```

In the following example, we only list the smallest structure categories to get a more readable output:

```
sage: def structure(C):
....:     return Category._sort_uniq(C.structure())
```

```
sage: structure(Magmas())
(Category of magmas,)
sage: structure(Rings())
(Category of unital magmas, Category of additive unital additive magmas)
sage: structure(Fields())
(Category of euclidean domains,)
sage: structure(Algebras(QQ))
(Category of unital magmas,
 Category of right modules over Rational Field,
 Category of left modules over Rational Field)
sage: structure(HopfAlgebras(QQ).Graded().WithBasis().Connected())
(Category of hopf algebras over Rational Field,
 Category of graded modules over Rational Field)
```

This method is used in *is_full_subcategory()* for deciding whether a category is a full subcategory of some other category, and for documentation purposes. It is computed recursively from the result of *additional_structure()* on the super categories of `self`.

**subcategory_class**()

A common superclass for all subcategories of this category (including this one).

This class derives from `D.subcategory_class` for each super category $D$ of `self`, and includes all the methods from the nested class `self.SubcategoryMethods`, if it exists.

**See also:**

- trac ticket #12895
- *parent_class()*
- *element_class()*
- *_make_named_class()*

EXAMPLES:

```
sage: cls = Rings().subcategory_class; cls
<class 'sage.categories.rings.Rings.subcategory_class'>
sage: type(cls)
<class 'sage.structure.dynamic_class.DynamicMetaclass'>
```

`Rings()` is an instance of this class, as well as all its subcategories:

```
sage: isinstance(Rings(), cls)
True
sage: isinstance(AlgebrasWithBasis(QQ), cls)
True
```

**super_categories**()

Return the *immediate* super categories of `self`.

OUTPUT:

- a duplicate-free list of categories.

Every category should implement this method.

EXAMPLES:

```
sage: Groups().super_categories()
[Category of monoids, Category of inverse unital magmas]
sage: Objects().super_categories()
[]
```

**Note:** Since trac ticket #10963, the order of the categories in the result is irrelevant. For details, see *On the order of super categories*.

**Note:** Whenever speed matters, developers are advised to use the lazy attribute `_super_categories()` instead of calling this method.

**class** sage.categories.category.**CategoryWithParameters**(*s=None*)
Bases: `sage.categories.category.Category`

A parametrized category whose parent/element classes depend only on its super categories.

Many categories in Sage are parametrized, like `C = Algebras(K)` which takes a base ring as parameter. In many cases, however, the operations provided by `C` in the parent class and element class depend only on the super categories of `C`. For example, the vector space operations are provided if and only if `K` is a field, since `VectorSpaces(K)` is a super category of `C` only in that case. In such cases, and as an optimization (see trac ticket #11935), we want to use the same parent and element class for all fields. This is the purpose of this abstract class.

Currently, `JoinCategory`, `Category_over_base` and `Bimodules` inherit from this class.

EXAMPLES:

```
sage: C1 = Algebras(GF(5))
sage: C2 = Algebras(GF(3))
sage: C3 = Algebras(ZZ)
sage: from sage.categories.category import CategoryWithParameters
sage: isinstance(C1, CategoryWithParameters)
True
sage: C1.parent_class is C2.parent_class
True
sage: C1.parent_class is C3.parent_class
False
```

**_make_named_class**(*name*, *method_provider*, *cache=False*, ***options*)
Return the parent/element/… class of `self`.

INPUT:

- `name` – a string; the name of the class as an attribute of `self`
- `method_provider` – a string; the name of an attribute of `self` that provides methods for the new class (in addition to what comes from the super categories)
- `**options` – other named options to pass down to `Category._make_named_class()`.

ASSUMPTION:

It is assumed that this method is only called from a lazy attribute whose name coincides with the given `name`.

OUTPUT:

A dynamic class that has the corresponding named classes of the super categories of `self` as bases and contains the methods provided by `getattr(self, method_provider)`.

---

**Note:** This method overrides *Category._make_named_class()* so that the returned class *only* depends on the corresponding named classes of the super categories and on the provided methods. This allows for sharing the named classes across closely related categories providing the same code to their parents, elements and so on.

---

EXAMPLES:

The categories of bimodules over the fields `CC` or `RR` provide the same methods to their parents and elements:

```
sage: Bimodules(ZZ,RR).parent_class is Bimodules(ZZ,RDF).parent_class
↪#indirect doctest
True
sage: Bimodules(CC,ZZ).element_class is Bimodules(RR,ZZ).element_class
True
```

On the other hand, modules over a field have more methods than modules over a ring:

```
sage: Modules(GF(3)).parent_class is Modules(ZZ).parent_class
False
sage: Modules(GF(3)).element_class is Modules(ZZ).element_class
False
```

For a more subtle example, one could possibly share the classes for `GF(3)` and `GF(2^3, 'x')`, but this is not currently the case:

```
sage: Modules(GF(3)).parent_class is Modules(GF(2^3,'x')).parent_class
False
```

This is because those two fields do not have the exact same category:

```
sage: GF(3).category()
Join of Category of finite enumerated fields
 and Category of subquotients of monoids
 and Category of quotients of semigroups
sage: GF(2^3,'x').category()
Category of finite enumerated fields
```

Similarly for `QQ` and `RR`:

```
sage: QQ.category()
Join of Category of number fields
 and Category of quotient fields
 and Category of metric spaces
sage: RR.category()
Join of Category of fields and Category of complete metric spaces
sage: Modules(QQ).parent_class is Modules(RR).parent_class
False
```

Some other cases where one could potentially share those classes:

```
sage: Modules(GF(3),dispatch=False).parent_class  is Modules(ZZ).parent_class
False
```

```
sage: Modules(GF(3),dispatch=False).element_class is Modules(ZZ).element_class
False
```

**class** sage.categories.category.**JoinCategory**(*super_categories*, ***kwds*)
Bases: *sage.categories.category.CategoryWithParameters*

A class for joins of several categories. Do not use directly; see Category.join instead.

EXAMPLES:

```
sage: from sage.categories.category import JoinCategory
sage: J = JoinCategory((Groups(), CommutativeAdditiveMonoids())); J
Join of Category of groups and Category of commutative additive monoids
sage: J.super_categories()
[Category of groups, Category of commutative additive monoids]
sage: J.all_super_categories(proper=True)
[Category of groups, ..., Category of magmas,
 Category of commutative additive monoids, ..., Category of additive magmas,
 Category of sets, Category of sets with partial maps, Category of objects]
```

By trac ticket #11935, join categories and categories over base rings inherit from *CategoryWithParameters*. This allows for sharing parent and element classes between similar categories. For example, since group algebras belong to a join category and since the underlying implementation is the same for all finite fields, we have:

```
sage: G = SymmetricGroup(10)
sage: A3 = G.algebra(GF(3))
sage: A5 = G.algebra(GF(5))
sage: type(A3.category())
<class 'sage.categories.category.JoinCategory_with_category'>
sage: type(A3) is type(A5)
True
```

**_repr_object_names**()
Return the name of the objects of this category.

**See also:**

*Category._repr_object_names()*, *_repr_()*, *_without_axioms()*

EXAMPLES:

```
sage: Groups().Finite().Commutative()._repr_(as_join=True)
'Join of Category of finite groups and Category of commutative groups'
sage: Groups().Finite().Commutative()._repr_object_names()
'finite commutative groups'
```

This uses *_without_axioms()* which may fail if this category is not obtained by adjoining axioms to some super categories:

```
sage: Category.join((Groups(), CommutativeAdditiveMonoids()))._repr_object_
↪names()
Traceback (most recent call last):
...
ValueError: This join category isn't built by adding axioms to a single␣
↪category
```

**_repr_**(*as_join=False*)
Print representation.

INPUT:

- as_join – a boolean (default: False)

EXAMPLES:

```
sage: Category.join((Groups(), CommutativeAdditiveMonoids())) #indirect␣
↪doctest
Join of Category of groups and Category of commutative additive monoids
```

By default, when a join category is built from category by adjoining axioms, a nice name is printed out:

```
sage: Groups().Facade().Finite()
Category of facade finite groups
```

But this is in fact really a join category:

```
sage: Groups().Facade().Finite()._repr_(as_join = True)
'Join of Category of finite groups and Category of facade sets'
```

The rationale is to make it more readable, and hide the technical details of how this category is constructed internally, especially since this construction is likely to change over time when new axiom categories are implemented.

This join category may possibly be obtained by adding axioms to different categories; so the result is not guaranteed to be unique; when this is not the case the first found is used.

**See also:**

*Category._repr_()*, *_repr_object_names()*

**_without_axioms**(*named=False*)

When adjoining axioms to a category, one often gets a join category; this method tries to recover the original category from this join category.

INPUT:

- named – a boolean (default: False)

See *Category._without_axioms()* for the description of the named parameter.

EXAMPLES:

```
sage: C = Category.join([Monoids(), Posets()]).Finite()
sage: C._repr_(as_join=True)
'Join of Category of finite monoids and Category of finite posets'
sage: C._without_axioms()
Traceback (most recent call last):
...
ValueError: This join category isn't built by adding axioms to a single␣
↪category
sage: C = Monoids().Infinite()
sage: C._repr_(as_join=True)
'Join of Category of monoids and Category of infinite sets'
sage: C._without_axioms()
Category of magmas
sage: C._without_axioms(named=True)
Category of monoids
```

**additional_structure**()

Return None.

Indeed, a join category defines no additional structure.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: Modules(ZZ).additional_structure()
```

**is_subcategory**(*C*)

Check whether this join category is subcategory of another category C.

EXAMPLES:

```
sage: Category.join([Rings(),Modules(QQ)]).is_subcategory(Category.
↪join([Rngs(),Bimodules(QQ,QQ)]))
True
```

**super_categories**()

Returns the immediate super categories, as per *Category.super_categories()*.

EXAMPLES:

```
sage: from sage.categories.category import JoinCategory
sage: JoinCategory((Semigroups(), FiniteEnumeratedSets())).super_categories()
[Category of semigroups, Category of finite enumerated sets]
```

sage.categories.category.**category_graph**(*categories=None*)

Return the graph of the categories in Sage.

INPUT:

- categories – a list (or iterable) of categories

If categories is specified, then the graph contains the mentioned categories together with all their super categories. Otherwise the graph contains (an instance of) each category in sage.categories.all (e.g. Algebras(QQ) for algebras).

For readability, the names of the category are shortened.

**Todo:** Further remove the base ring (see also trac ticket #15801).

EXAMPLES:

```
sage: G = sage.categories.category.category_graph(categories = [Groups()])
sage: G.vertices()
['groups', 'inverse unital magmas', 'magmas', 'monoids', 'objects',
 'semigroups', 'sets', 'sets with partial maps', 'unital magmas']
sage: G.plot()
Graphics object consisting of 20 graphics primitives

sage: sage.categories.category.category_graph().plot()
Graphics object consisting of ... graphics primitives
```

sage.categories.category.**category_sample**()

Return a sample of categories.

It is constructed by looking for all concrete category classes declared in sage.categories.all, calling *Category.an_instance()* on those and taking all their super categories.

EXAMPLES:

```
sage: from sage.categories.category import category_sample
sage: sorted(category_sample(), key=str)
[Category of G-sets for Symmetric group of order 8! as a permutation group,
 Category of Hecke modules over Rational Field,
 Category of Lie algebras over Rational Field,
 Category of additive magmas, ...,
 Category of fields, ...,
 Category of graded hopf algebras with basis over Rational Field, ...,
 Category of modular abelian varieties over Rational Field, ...,
 Category of simplicial complexes, ...,
 Category of vector spaces over Rational Field, ...,
 Category of weyl groups, ...
```

sage.categories.category.**is_Category**(*x*)

Returns True if x is a category.

EXAMPLES:

```
sage: sage.categories.category.is_Category(CommutativeAdditiveSemigroups())
True
sage: sage.categories.category.is_Category(ZZ)
False
```

## 1.3 Axioms

This documentation covers how to implement axioms and proceeds with an overview of the implementation of the axiom infrastructure. It assumes that the reader is familiar with the *category primer*, and in particular its *section about axioms*.

### 1.3.1 Implementing axioms

#### Simple case involving a single predefined axiom

Suppose that one wants to provide code (and documentation, tests, . . . ) for the objects of some existing category `Cs()` that satisfy some predefined axiom `A`.

The first step is to open the hood and check whether there already exists a class implementing the category `Cs().A()`. For example, taking `Cs=Semigroups` and the `Finite` axiom, there already exists a class for the category of finite semigroups:

```
sage: Semigroups().Finite()
Category of finite semigroups
sage: type(Semigroups().Finite())
<class 'sage.categories.finite_semigroups.FiniteSemigroups_with_category'>
```

In this case, we say that the category of semigroups *implements* the axiom `Finite`, and code about finite semigroups should go in the class *`FiniteSemigroups`* (or, as usual, in its nested classes `ParentMethods`, `ElementMethods`, and so on).

On the other hand, there is no class for the category of infinite semigroups:

```
sage: Semigroups().Infinite()
Category of infinite semigroups
sage: type(Semigroups().Infinite())
<class 'sage.categories.category.JoinCategory_with_category'>
```

This category is indeed just constructed as the intersection of the categories of semigroups and of infinite sets respectively:

```
sage: Semigroups().Infinite().super_categories()
[Category of semigroups, Category of infinite sets]
```

In this case, one needs to create a new class to implement the axiom `Infinite` for this category. This boils down to adding a nested class `Semigroups.Infinite` inheriting from *`CategoryWithAxiom`*.

In the following example, we implement a category `Cs`, with a subcategory for the objects satisfying the `Finite` axiom defined in the super category `Sets` (we will see later on how to *define* new axioms):

```
sage: from sage.categories.category_with_axiom import CategoryWithAxiom
sage: class Cs(Category):
....:     def super_categories(self):
....:         return [Sets()]
....:     class Finite(CategoryWithAxiom):
....:         class ParentMethods:
....:             def foo(self):
....:                 print("I am a method on finite C's")
```

```
sage: Cs().Finite()
Category of finite cs
sage: Cs().Finite().super_categories()
[Category of finite sets, Category of cs]
sage: Cs().Finite().all_super_categories()
[Category of finite cs, Category of finite sets,
 Category of cs, Category of sets, ...]
sage: Cs().Finite().axioms()
frozenset({'Finite'})
```

Now a parent declared in the category `Cs().Finite()` inherits from all the methods of finite sets and of finite $C$'s, as desired:

```
sage: P = Parent(category=Cs().Finite())
sage: P.is_finite()            # Provided by Sets.Finite.ParentMethods
True
sage: P.foo()                  # Provided by Cs.Finite.ParentMethods
I am a method on finite C's
```

**Note:**

- This follows the same idiom as for *Covariant Functorial Constructions*.
- From an object oriented point of view, any subcategory `Cs()` of *`Sets`* inherits a `Finite` method. Usually `Cs` could complement this method by overriding it with a method `Cs.Finite` which would make a super call to `Sets.Finite` and then do extra stuff.

  In the above example, `Cs` also wants to complement `Sets.Finite`, though not by doing more stuff, but by providing it with an additional mixin class containing the code for finite `Cs`. To keep the analogy, this mixin class is to be put in `Cs.Finite`.

- By defining the axiom `Finite`, *`Sets`* fixes the semantic of `Cs.Finite()` for all its subcategories `Cs`: namely "the category of `Cs` which are finite as sets". Hence, for example, `Modules.Free.Finite` cannot be used to model the category of free modules of finite rank, even though their traditional name "finite free modules" might suggest it.
- It may come as a surprise that we can actually use the same name `Finite` for the mixin class and for the method defining the axiom; indeed, by default a class does not have a binding behavior and would completely override the method. See the section *Defining a new axiom* for details and the rationale behind it.

  An alternative would have been to give another name to the mixin class, like `FiniteCategory`. However this would have resulted in more namespace pollution, whereas using `Finite` is already clear, explicit, and easier to remember.
- Under the hood, the category `Cs().Finite()` is aware that it has been constructed from the category `Cs()` by adding the axiom `Finite`:

```
sage: Cs().Finite().base_category()
Category of cs
sage: Cs().Finite()._axiom
'Finite'
```

---

Over time, the nested class `Cs.Finite` may become large and too cumbersome to keep as a nested subclass of `Cs`. Or the category with axiom may have a name of its own in the literature, like *semigroups* rather than *associative magmas*, or *fields* rather than *commutative division rings*. In this case, the category with axiom can be put elsewhere, typically in a separate file, with just a link from `Cs`:

```
sage: class Cs(Category):
....:     def super_categories(self):
....:         return [Sets()]
sage: class FiniteCs(CategoryWithAxiom):
....:     class ParentMethods:
....:         def foo(self):
....:             print("I am a method on finite C's")
sage: Cs.Finite = FiniteCs
sage: Cs().Finite()
Category of finite cs
```

For a real example, see the code of the class *`FiniteGroups`* and the link to it in *`Groups`*. Note that the link is implemented using `LazyImport`; this is highly recommended: it makes sure that *`FiniteGroups`* is imported after *`Groups`* it depends upon, and makes it explicit that the class *`Groups`* can be imported and is fully functional without importing *`FiniteGroups`*.

---

**Note:** Some categories with axioms are created upon Sage's startup. In such a case, one needs to pass the `at_startup=True` option to `LazyImport`, in order to quiet the warning about that lazy import being resolved upon startup. See for example `Sets.Finite`.

This is undoubtedly a code smell. Nevertheless, it is preferable to stick to lazy imports, first to resolve the import order properly, and more importantly as a reminder that the category would be best not constructed upon Sage's startup. This is to spur developers to reduce the number of parents (and therefore categories) that are constructed upon startup. Each `at_startup=True` that will be removed will be a measure of progress in this direction.

---

**Note:** In principle, due to a limitation of `LazyImport` with nested classes (see trac ticket #15648), one should pass the option `as_name` to `LazyImport`:

```
Finite = LazyImport('sage.categories.finite_groups', 'FiniteGroups', as_name='Finite')
```

in order to prevent `Groups.Finite` to keep on reimporting `FiniteGroups`.

Given that passing this option introduces some redundancy and is error prone, the axiom infrastructure includes a little workaround which makes the `as_name` unnecessary in this case.

### Making the category with axiom directly callable

If desired, a category with axiom can be constructed directly through its class rather than through its base category:

```
sage: Semigroups()
Category of semigroups
sage: Semigroups() is Magmas().Associative()
True

sage: FiniteGroups()
Category of finite groups
sage: FiniteGroups() is Groups().Finite()
True
```

For this notation to work, the class *Semigroups* needs to be aware of the base category class (here, *Magmas*) and of the axiom (here, `Associative`):

```
sage: Semigroups._base_category_class_and_axiom
(<class 'sage.categories.magmas.Magmas'>, 'Associative')
sage: Fields._base_category_class_and_axiom
(<class 'sage.categories.division_rings.DivisionRings'>, 'Commutative')
sage: FiniteGroups._base_category_class_and_axiom
(<class 'sage.categories.groups.Groups'>, 'Finite')
sage: FiniteDimensionalAlgebrasWithBasis._base_category_class_and_axiom
(<class 'sage.categories.algebras_with_basis.AlgebrasWithBasis'>, 'FiniteDimensional')
```

In our example, the attribute `_base_category_class_and_axiom` was set upon calling `Cs().Finite()`, which makes the notation seemingly work:

```
sage: FiniteCs()
Category of finite cs
sage: FiniteCs._base_category_class_and_axiom
(<class '__main__.Cs'>, 'Finite')
sage: FiniteCs._base_category_class_and_axiom_origin
'set by __classget__'
```

But calling `FiniteCs()` right after defining the class would have failed (try it!). In general, one needs to set the attribute explicitly:

```
sage: class FiniteCs(CategoryWithAxiom):
....:     _base_category_class_and_axiom = (Cs, 'Finite')
....:     class ParentMethods:
....:         def foo(self):
....:             print("I am a method on finite C's")
```

Having to set explicitly this link back from `FiniteCs` to `Cs` introduces redundancy in the code. It would therefore be desirable to have the infrastructure set the link automatically instead (a difficulty is to achieve this while supporting lazy imported categories with axiom).

As a first step, the link is set automatically upon accessing the class from the base category class:

```
sage: Algebras.WithBasis._base_category_class_and_axiom
(<class 'sage.categories.algebras.Algebras'>, 'WithBasis')
sage: Algebras.WithBasis._base_category_class_and_axiom_origin
'set by __classget__'
```

Hence, for whatever this notation is worth, one can currently do:

```
sage: Algebras.WithBasis(QQ)
Category of algebras with basis over Rational Field
```

We don't recommend using syntax like `Algebras.WithBasis(QQ)`, as it may eventually be deprecated.

As a second step, Sage tries some obvious heuristics to deduce the link from the name of the category with axiom (see *base_category_class_and_axiom()* for the details). This typically covers the following examples:

```
sage: FiniteCoxeterGroups()
Category of finite coxeter groups
sage: FiniteCoxeterGroups() is CoxeterGroups().Finite()
True
sage: FiniteCoxeterGroups._base_category_class_and_axiom_origin
'deduced by base_category_class_and_axiom'

sage: FiniteDimensionalAlgebrasWithBasis(QQ)
Category of finite dimensional algebras with basis over Rational Field
sage: FiniteDimensionalAlgebrasWithBasis(QQ) is Algebras(QQ).FiniteDimensional().
↪WithBasis()
True
```

If the heuristic succeeds, the result is guaranteed to be correct. If it fails, typically because the category has a name of its own like *Fields*, the attribute `_base_category_class_and_axiom` should be set explicitly. For more examples, see the code of the classes *Semigroups* or *Fields*.

**Note:** When printing out a category with axiom, the heuristic determines whether a category has a name of its own by checking out how `_base_category_class_and_axiom` was set:

```
sage: Fields._base_category_class_and_axiom_origin
'hardcoded'
```

See *CategoryWithAxiom._without_axioms()*, *CategoryWithAxiom._repr_object_names_static()*.

In our running example `FiniteCs`, Sage failed to deduce automatically the base category class and axiom because the class `Cs` is not in the standard location `sage.categories.cs`.

**Design discussion**

The above deduction, based on names, is undoubtedly inelegant. But it's safe (either the result is guaranteed to be correct, or an error is raised), it saves on some redundant information, and it is only used for the simple shorthands like `FiniteGroups()` for `Groups().Finite()`. Finally, most if not all of these shorthands are likely to eventually disappear (see trac ticket #15741 and the *related discussion in the primer*).

### Defining a new axiom

We describe now how to define a new axiom. The first step is to figure out the largest category where the axiom makes sense. For example `Sets` for `Finite`, `Magmas` for `Associative`, or `Modules` for `FiniteDimensional`. Here we define the axiom `Green` for the category `Cs` and its subcategories:

```
sage: from sage.categories.category_with_axiom import CategoryWithAxiom
sage: class Cs(Category):
....:     def super_categories(self):
....:         return [Sets()]
....:     class SubcategoryMethods:
....:         def Green(self):
....:             '<documentation of the axiom Green>'
....:             return self._with_axiom("Green")
....:     class Green(CategoryWithAxiom):
....:         class ParentMethods:
....:             def foo(self):
....:                 print("I am a method on green C's")
```

With the current implementation, the name of the axiom must also be added to a global container:

```
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: all_axioms += ("Green",)
```

We can now use the axiom as usual:

```
sage: Cs().Green()
Category of green cs

sage: P = Parent(category=Cs().Green())
sage: P.foo()
I am a method on green C's
```

Compared with our first example, the only newcomer is the method `.Green()` that can be used by any subcategory `Ds()` of `Cs()` to add the axiom `Green`. Note that the expression `Ds().Green` always evaluates to this method, regardless of whether `Ds` has a nested class `Ds.Green` or not (an implementation detail):

```
sage: Cs().Green
<bound method Cs_with_category.Green of Category of cs>
```

Thanks to this feature (implemented in *`CategoryWithAxiom.__classget__()`*), the user is systematically referred to the documentation of this method when doing introspection on `Ds().Green`:

```
sage: C = Cs()
sage: C.Green?              # not tested
sage: Cs().Green.__doc__
'<documentation of the axiom Green>'
```

It is therefore the natural spot for the documentation of the axiom.

**Note:** The presence of the nested class `Green` in `Cs` is currently mandatory even if it is empty.

**Todo:** Specify whether or not one should systematically use @cached_method in the definition of the axiom. And make sure all the definition of axioms in Sage are consistent in this respect!

**Todo:** We could possibly define an @axiom decorator? This could hide two little implementation details: whether or not to make the method a cached method, and the call to _with_axiom(…) under the hood. It could do possibly do some more magic. The gain is not obvious though.

**Note:** `all_axioms` is only used marginally, for sanity checks and when trying to derive automatically the base category class. The order of the axioms in this tuple also controls the order in which they appear when printing out categories with axioms (see `CategoryWithAxiom._repr_object_names_static()`).

During a Sage session, new axioms should only be added at the *end* of `all_axioms`, as above, so as to not break the cache of `axioms_rank()`. Otherwise, they can be inserted statically anywhere in the tuple. For axioms defined within the Sage library, the name is best inserted by editing directly the definition of `all_axioms` in `sage.categories.category_with_axiom`.

**Design note**

Let us state again that, unlike what the existence of `all_axioms` might suggest, the definition of an axiom is local to a category and its subcategories. In particular, two independent categories `Cs()` and `Ds()` can very well define axioms with the same name and different semantics. As long as the two hierarchies of subcategories don't intersect, this is not a problem. And if they do intersect naturally (that is if one is likely to create a parent belonging to both categories), this probably means that the categories `Cs` and `Ds` are about related enough areas of mathematics that one should clear the ambiguity by having either the same semantic or different names.

This caveat is no different from that of name clashes in hierarchy of classes involving multiple inheritance.

**Todo:** Explore ways to get rid of this global `all_axioms` tuple, and/or have automatic registration there, and/or having a register_axiom(…) method.

### Special case: defining an axiom depending on several categories

In some cases, the largest category where the axiom makes sense is the intersection of two categories. This is typically the case for axioms specifying compatibility conditions between two otherwise unrelated operations, like `Distributive` which specifies a compatibility between $*$ and $+$. Ideally, we would want the `Distributive` axiom to be defined by:

```
sage: Magmas() & AdditiveMagmas()
Join of Category of magmas and Category of additive magmas
```

The current infrastructure does not support this perfectly: indeed, defining an axiom for a category $C$ requires $C$ to have a class of its own; hence a `JoinCategory` as above won't do; we need to implement a new class like `MagmasAndAdditiveMagmas`; furthermore, we cannot yet model the fact that `MagmasAndAdditiveMagmas()` *is* the intersection of `Magmas()` and `AdditiveMagmas()` rather than a mere subcategory:

```
sage: from sage.categories.magmas_and_additive_magmas import MagmasAndAdditiveMagmas
sage: Magmas() & AdditiveMagmas() is MagmasAndAdditiveMagmas()
False
sage: Magmas() & AdditiveMagmas()             # todo: not implemented
Category of magmas and additive magmas
```

Still, there is a workaround to get the natural notations:

```
sage: (Magmas() & AdditiveMagmas()).Distributive()
Category of distributive magmas and additive magmas
sage: (Monoids() & CommutativeAdditiveGroups()).Distributive()
Category of rings
```

The trick is to define `Distributive` as usual in *MagmasAndAdditiveMagmas*, and to add a method *Magmas.SubcategoryMethods.Distributive()* which checks that `self` is a subcategory of both `Magmas()` and `AdditiveMagmas()`, complains if not, and otherwise takes the intersection of `self` with `MagmasAndAdditiveMagmas()` before calling `Distributive`.

The downsides of this workaround are:

- Creation of an otherwise empty class *MagmasAndAdditiveMagmas*.
- Pollution of the namespace of `Magmas()` (and subcategories like `Groups()`) with a method that is irrelevant (but safely complains if called).
- `C._with_axiom('Distributive')` is not strictly equivalent to `C.Distributive()`, which can be unpleasantly surprising:

```
sage: (Monoids() & CommutativeAdditiveGroups()).Distributive()
Category of rings

sage: (Monoids() & CommutativeAdditiveGroups())._with_axiom('Distributive')
Join of Category of monoids and Category of commutative additive groups
```

---

**Todo:** Other categories that would be better implemented via an axiom depending on a join category include:

- *Algebras*: defining an associative unital algebra as a ring and a module satisfying the suitable compatibility axiom between inner multiplication and multiplication by scalars (bilinearity). Of course this should be implemented at the level of *MagmaticAlgebras*, if not higher.
- *Bialgebras*: defining an bialgebra as an algebra and coalgebra where the coproduct is a morphism for the product.
- *Bimodules*: defining a bimodule as a left and right module where the two actions commute.

---

**Todo:**

- Design and implement an idiom for the definition of an axiom by a join category.
- Or support more advanced joins, through some hook or registration process to specify that a given category *is* the intersection of two (or more) categories.
- Or at least improve the above workaround to avoid the last issue; this possibly could be achieved using a class `Magmas.Distributive` with a bit of `__classcall__` magic.

---

### Handling multiple axioms, arborescence structure of the code

#### Prelude

Let us consider the category of magmas, together with two of its axioms, namely `Associative` and `Unital`. An associative magma is a *semigroup* and a unital semigroup is a *monoid*. We have also seen that axioms commute:

```
sage: Magmas().Unital()
Category of unital magmas
sage: Magmas().Associative()
Category of semigroups
sage: Magmas().Associative().Unital()
Category of monoids
sage: Magmas().Unital().Associative()
Category of monoids
```

At the level of the classes implementing these categories, the following comes as a general naturalization of the previous section:

```
sage: Magmas.Unital
<class 'sage.categories.magmas.Magmas.Unital'>
sage: Magmas.Associative
<class 'sage.categories.semigroups.Semigroups'>
sage: Magmas.Associative.Unital
<class 'sage.categories.monoids.Monoids'>
```

However, the following may look suspicious at first:

```
sage: Magmas.Unital.Associative
Traceback (most recent call last):
...
AttributeError: type object 'Magmas.Unital' has no attribute 'Associative'
```

The purpose of this section is to explain the design of the code layout and the rationale for this mismatch.

### Abstract model

As we have seen in the *Primer*, the objects of a category `Cs()` can usually satisfy, or not, many different axioms. Out of all combinations of axioms, only a small number are relevant in practice, in the sense that we actually want to provide features for the objects satisfying these axioms.

Therefore, in the context of the category class `Cs`, we want to provide the system with a collection $(D_S)_{S\in\mathcal{S}}$ where each $S$ is a subset of the axioms and the corresponding $D_S$ is a class for the subcategory of the objects of `Cs()` satisfying the axioms in $S$. For example, if `Cs()` is the category of magmas, the pairs $(S, D_S)$ would include:

```
{Associative}                : Semigroups
{Associative, Unital}        : Monoids
{Associative, Unital, Inverse}: Groups
{Associative, Commutative}    : Commutative Semigroups
{Unital,      Inverse}        : Loops
```

Then, given a subset $T$ of axioms, we want the system to be able to select automatically the relevant classes $(D_S)_{S\in\mathcal{S},S\subset T}$, and build from them a category for the objects of `Cs` satisfying the axioms in $T$, together with its hierarchy of super categories. If $T$ is in the indexing set $\mathcal{S}$, then the class of the resulting category is directly $D_T$:

```
sage: C = Magmas().Unital().Inverse().Associative(); C
Category of groups
sage: type(C)
<class 'sage.categories.groups.Groups_with_category'>
```

Otherwise, we get a join category:

```
sage: C = Magmas().Infinite().Unital().Associative(); C
Category of infinite monoids
sage: type(C)
<class 'sage.categories.category.JoinCategory_with_category'>
sage: C.super_categories()
[Category of monoids, Category of infinite sets]
```

#### Concrete model as an arborescence of nested classes

We further want the construction to be efficient and amenable to laziness. This led us to the following design decision: the collection $(D_S)_{S\in\mathcal{S}}$ of classes should be structured as an arborescence (or equivalently a *rooted forest*). The root is `Cs`, corresponding to $S = \emptyset$. Any other class $D_S$ should be the child of a single class $D_{S'}$ where $S'$ is obtained from $S$ by removing a single axiom $A$. Of course, $D_{S'}$ and $A$ are respectively the base category class and axiom of the category with axiom $D_S$ that we have met in the first section.

At this point, we urge the reader to explore the code of *`Magmas`* and *`DistributiveMagmasAndAdditiveMagmas`* and see how the arborescence structure on the categories with axioms is reflected by the nesting of category classes.

#### Discussion of the design

##### Performance

Thanks to the arborescence structure on subsets of axioms, constructing the hierarchy of categories and computing intersections can be made efficient with, roughly speaking, a linear/quadratic complexity in the size of the involved category hierarchy multiplied by the number of axioms (see Section *Algorithms*). This is to be put in perspective with the manipulation of arbitrary collections of subsets (aka boolean functions) which can easily raise NP-hard problems.

Furthermore, thanks to its locality, the algorithms can be made suitably lazy: in particular, only the involved category classes need to be imported.

##### Flexibility

This design also brings in quite some flexibility, with the possibility to support features such as defining new axioms depending on other axioms and deduction rules. See below.

##### Asymmetry

As we have seen at the beginning of this section, this design introduces an asymmetry. It's not so bad in practice, since in most practical cases, we want to work incrementally. It's for example more natural to describe *`FiniteFields`* as *`Fields`* with the axiom `Finite` rather than *`Magmas`* and *`AdditiveMagmas`* with all (or at least sufficiently many) of the following axioms:

```
sage: sorted(Fields().axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Commutative', 'Distributive',
 'Division', 'NoZeroDivisors', 'Unital']
```

The main limitation is that the infrastructure currently imposes to be incremental by steps of a single axiom.

In practice, among the roughly 60 categories with axioms that are currently implemented in Sage, most admitted a (rather) natural choice of a base category and single axiom to add. For example, one usually thinks more naturally of a monoid as a semigroup which is unital rather than as a unital magma which is associative. Modeling this asymmetry in the code actually brings a bonus: it is used for printing out categories in a (heuristically) mathematician-friendly way:

```
sage: Magmas().Commutative().Associative()
Category of commutative semigroups
```

Only in a few cases is a choice made that feels mathematically arbitrary. This is essentially in the chain of nested classes *distributive_magmas_and_additive_magmas.DistributiveMagmasAndAdditiveMagmas.AdditiveAssociative.AdditiveCommutative.AdditiveUnital.Associative*.

### Placeholder classes

Given that we can only add a single axiom at a time when implementing a *CategoryWithAxiom*, we need to create a few category classes that are just placeholders. For the worst example, see the chain of nested classes *distributive_magmas_and_additive_magmas.DistributiveMagmasAndAdditiveMagmas.AdditiveAssociative.AdditiveCommutative.AdditiveUnital.Associative*.

This is suboptimal, but fits within the scope of the axiom infrastructure which is to reduce a potentially exponential number of placeholder category classes to just a couple.

Note also that, in the above example, it's likely that some of the intermediate classes will grow to non placeholder ones, as people will explore more weaker variants of rings.

### Mismatch between the arborescence of nested classes and the hierarchy of categories

The fact that the hierarchy relation between categories is not reflected directly as a relation between the classes may sound suspicious at first! However, as mentioned in the primer, this is actually a big selling point of the axioms infrastructure: by calculating automatically the hierarchy relation between categories with axioms one avoids the nightmare of maintaining it by hand. Instead, only a rather minimal number of links needs to be maintainted in the code (one per category with axiom).

Besides, with the flexibility introduced by runtime deduction rules (see below), the hierarchy of categories may depend on the parameters of the categories and not just their class. So it's fine to make it clear from the onset that the two relations do not match.

### Evolutivity

At this point, the arborescence structure has to be hardcoded by hand with the annoyances we have seen. This does not preclude, in a future iteration, to design and implement some idiom for categories with axioms that adds several axioms at once to a base category; maybe some variation around:

```
class DistributiveMagmasAndAdditiveMagmas:
    ...

    @category_with_axiom(
        AdditiveAssociative,
        AdditiveCommutative,
        AdditiveUnital,
        AdditiveInverse,
        Associative)
    def _(): return LazyImport('sage.categories.rngs', 'Rngs', at_startup=True)
```

or:

```
register_axiom_category(DistributiveMagmasAndAdditiveMagmas,
                        {AdditiveAssociative,
                         AdditiveCommutative,
                         AdditiveUnital,
                         AdditiveInverse,
                         Associative},
                        'sage.categories.rngs', 'Rngs', at_startup=True)
```

The infrastructure would then be in charge of building the appropriate arborescence under the hood. Or rely on some database (see discussion on trac ticket #10963, in particular at the end of comment 332).

### Axioms defined upon other axioms

Sometimes an axiom can only be defined when some other axiom holds. For example, the axiom `NoZeroDivisors` only makes sense if there is a zero, that is if the axiom `AdditiveUnital` holds. Hence, for the category *MagmasAndAdditiveMagmas*, we consider in the abstract model only those subsets of axioms where the presence of `NoZeroDivisors` implies that of `AdditiveUnital`. We also want the axiom to be only available if meaningful:

```
sage: Rings().NoZeroDivisors()
Category of domains
sage: Rings().Commutative().NoZeroDivisors()
Category of integral domains
sage: Semirings().NoZeroDivisors()
Traceback (most recent call last):
...
AttributeError: 'Semirings_with_category' object has no attribute 'NoZeroDivisors'
```

Concretely, this is to be implemented by defining the new axiom in the (`SubcategoryMethods` nested class of the) appropriate category with axiom. For example the axiom `NoZeroDivisors` would be naturally defined in `magmas_and_additive_magmas.MagmasAndAdditiveMagmas.Distributive.AdditiveUnital`.

**Note:** The axiom `NoZeroDivisors` is currently defined in *Rings*, by simple lack of need for the feature; it should be lifted up as soon as relevant, that is when some code will be available for parents with no zero divisors that are not necessarily rings.

### Deduction rules

A similar situation is when an axiom `A` of a category `Cs` implies some other axiom `B`, with the same consequence as above on the subsets of axioms appearing in the abstract model. For example, a division ring necessarily has no zero divisors:

```
sage: 'NoZeroDivisors' in Rings().Division().axioms()
True
sage: 'NoZeroDivisors' in Rings().axioms()
False
```

This deduction rule is implemented by the method `Rings.Division.extra_super_categories()`:

```
sage: Rings().Division().extra_super_categories()
(Category of domains,)
```

In general, this is to be implemented by a method `Cs.A.extra_super_categories` returning a tuple `(Cs().B(),)`, or preferably `(Ds().B(),)` where `Ds` is the category defining the axiom `B`.

This follows the same idiom as for deduction rules about functorial constructions (see `covariant_functorial_construction.CovariantConstructionCategory.extra_super_categories()`). For example, the fact that a Cartesian product of associative magmas (i.e. of semigroups) is an associative magma is implemented in *`Semigroups.CartesianProducts.extra_super_categories()`*:

```
sage: Magmas().Associative()
Category of semigroups
sage: Magmas().Associative().CartesianProducts().extra_super_categories()
[Category of semigroups]
```

Similarly, the fact that the algebra of a commutative magma is commutative is implemented in *`Magmas.Commutative.Algebras.extra_super_categories()`*:

```
sage: Magmas().Commutative().Algebras(QQ).extra_super_categories()
[Category of commutative magmas]
```

**Warning:** In some situations this idiom is inapplicable as it would require to implement two classes for the same category. This is the purpose of the next section.

### Special case

In the previous examples, the deduction rule only had an influence on the super categories of the category with axiom being constructed. For example, when constructing `Rings().Division()`, the rule `Rings.Division.extra_super_categories()` simply adds `Rings().NoZeroDivisors()` as a super category thereof.

In some situations this idiom is inapplicable because a class for the category with axiom under construction already exists elsewhere. Take for example Wedderburn's theorem: any finite division ring is commutative, i.e. is a finite field. In other words, `DivisionRings().Finite()` *coincides* with `Fields().Finite()`:

```
sage: DivisionRings().Finite()
Category of finite enumerated fields
sage: DivisionRings().Finite() is Fields().Finite()
True
```

Therefore we cannot create a class `DivisionRings.Finite` to hold the desired `extra_super_categories` method, because there is already a class for this category with axiom, namely `Fields.Finite`.

A natural idiom would be to have `DivisionRings.Finite` be a link to `Fields.Finite` (locally introducing an undirected cycle in the arborescence of nested classes). It would be a bit tricky to implement though, since one would need to detect, upon constructing `DivisionRings().Finite()`, that `DivisionRings.Finite` is actually `Fields.Finite`, in order to construct appropriately `Fields().Finite()`; and reciprocally, upon computing the super categories of `Fields().Finite()`, to not try to add `DivisionRings().Finite()` as a super category.

Instead the current idiom is to have a method `DivisionRings.Finite_extra_super_categories` which mimicks the behavior of the would-be `DivisionRings.Finite.extra_super_categories`:

```
sage: DivisionRings().Finite_extra_super_categories()
(Category of commutative magmas,)
```

This idiom is admittedly rudimentary, but consistent with how mathematical facts specifying non trivial inclusion relations between categories are implemented elsewhere in the various `extra_super_categories` methods of axiom categories and covariant functorial constructions. Besides, it gives a natural spot (the docstring of the method) to document and test the modeling of the mathematical fact. Finally, Wedderburn's theorem is arguably a theorem about division rings (in the context of division rings, finiteness implies commutativity) and therefore lives naturally in *DivisionRings*.

An alternative would be to implement the category of finite division rings (i.e. finite fields) in a class `DivisionRings.Finite` rather than `Fields.Finite`:

```
sage: from sage.categories.category_with_axiom import CategoryWithAxiom

sage: class MyDivisionRings(Category):
....:     def super_categories(self):
....:         return [Rings()]

sage: class MyFields(Category):
....:     def super_categories(self):
....:         return [MyDivisionRings()]

sage: class MyFiniteFields(CategoryWithAxiom):
....:     _base_category_class_and_axiom = (MyDivisionRings, "Finite")
....:     def extra_super_categories(self): # Wedderburn's theorem
....:         return [MyFields()]

sage: MyDivisionRings.Finite = MyFiniteFields

sage: MyDivisionRings().Finite()
Category of my finite fields
sage: MyFields().Finite() is MyDivisionRings().Finite()
True
```

In general, if several categories `C1s()`, `C2s()`, `...` are mapped to the same category when applying some axiom `A` (that is `C1s().A() == C2s().A() == ...`), then one should be careful to implement this category in a single class `Cs.A`, and set up methods `extra_super_categories` or `A_extra_super_categories` methods as appropriate. Each such method should return something like `[C2s()]` and not `[C2s().A()]` for the latter would likely lead to an infinite recursion.

**Design discussion**

Supporting similar deduction rules will be an important feature in the future, with quite a few occurrences already implemented in upcoming tickets. For the time being though there is a single occurrence of this idiom outside of the tests. So this would be an easy thing to refactor after trac ticket #10963 if a better idiom is found.

### Larger synthetic examples

We now consider some larger synthetic examples to check that the machinery works as expected. Let us start with a category defining a bunch of axioms, using *axiom()* for conciseness (don't do it for real axioms; they deserve a full documentation!):

```
sage: from sage.categories.category_singleton import Category_singleton
sage: from sage.categories.category_with_axiom import axiom
sage: import sage.categories.category_with_axiom
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: all_axioms += ("B","C","D","E","F")

sage: class As(Category_singleton):
....:     def super_categories(self):
....:         return [Objects()]
....:
....:     class SubcategoryMethods:
....:         B = axiom("B")
....:         C = axiom("C")
....:         D = axiom("D")
....:         E = axiom("E")
....:         F = axiom("F")
....:
....:     class B(CategoryWithAxiom):
....:         pass
....:     class C(CategoryWithAxiom):
....:         pass
....:     class D(CategoryWithAxiom):
....:         pass
....:     class E(CategoryWithAxiom):
....:         pass
....:     class F(CategoryWithAxiom):
....:         pass
```

Now we construct a subcategory where, by some theorem of William, axioms B and C together are equivalent to E and F together:

```
sage: class A1s(Category_singleton):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class B(CategoryWithAxiom):
....:         def C_extra_super_categories(self):
....:             return [As().E(), As().F()]
....:
....:     class E(CategoryWithAxiom):
....:         def F_extra_super_categories(self):
....:             return [As().B(), As().C()]

sage: A1s().B().C()
Category of e f a1s
```

The axioms B and C do not show up in the name of the obtained category because, for concision, the printing uses some heuristics to not show axioms that are implied by others. But they are satisfied:

```
sage: sorted(A1s().B().C().axioms())
['B', 'C', 'E', 'F']
```

Note also that this is a join category:

```
sage: type(A1s().B().C())
<class 'sage.categories.category.JoinCategory_with_category'>
sage: A1s().B().C().super_categories()
[Category of e a1s,
```

```
Category of f as,
 Category of b a1s,
 Category of c as]
```

As desired, William's theorem holds:

```
sage: A1s().B().C() is A1s().E().F()
True
```

and propagates appropriately to subcategories:

```
sage: C =  A1s().E().F().D().B().C()
sage: C is A1s().B().C().E().F().D()   # commutativity
True
sage: C is A1s().E().F().E().F().D()   # William's theorem
True
sage: C is A1s().E().E().F().F().D()   # commutativity
True
sage: C is A1s().E().F().D()           # idempotency
True
sage: C is A1s().D().E().F()
True
```

In this quick variant, we actually implement the category of `b c a2s`, and choose to do so in `A2s.B.C`:

```
sage: class A2s(Category_singleton):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class B(CategoryWithAxiom):
....:         class C(CategoryWithAxiom):
....:             def extra_super_categories(self):
....:                 return [As().E(), As().F()]
....:
....:     class E(CategoryWithAxiom):
....:         def F_extra_super_categories(self):
....:             return [As().B(), As().C()]


sage: A2s().B().C()
Category of e f a2s
sage: sorted(A2s().B().C().axioms())
['B', 'C', 'E', 'F']
sage: type(A2s().B().C())
<class '__main__.A2s.B.C_with_category'>
```

As desired, William's theorem and its consequences hold:

```
sage: A2s().B().C() is A2s().E().F()
True
sage: C =  A2s().E().F().D().B().C()
sage: C is A2s().B().C().E().F().D()   # commutativity
True
sage: C is A2s().E().F().E().F().D()   # William's theorem
True
sage: C is A2s().E().E().F().F().D()   # commutativity
True
sage: C is A2s().E().F().D()           # idempotency
```

```
True
sage: C is A2s().D().E().F()
True
```

Finally, we "accidentally" implement the category of `b c a1s`, both in `A3s.B.C` and `A3s.E.F`:

```
sage: class A3s(Category_singleton):
....:     def super_categories(self):
....:         return [As()]
....:
....:     class B(CategoryWithAxiom):
....:         class C(CategoryWithAxiom):
....:             def extra_super_categories(self):
....:                 return [As().E(), As().F()]
....:
....:     class E(CategoryWithAxiom):
....:         class F(CategoryWithAxiom):
....:             def extra_super_categories(self):
....:                 return [As().B(), As().C()]
```

We can still construct, say:

```
sage: A3s().B()
Category of b a3s
sage: A3s().C()
Category of c a3s
```

However,

```
sage: A3s().B().C()            # not tested
```

runs into an infinite recursion loop, as `A3s().B().C()` wants to have `A3s().E().F()` as super category and reciprocally.

**Todo:** The above example violates the specifications (a category should be modelled by at most one class), so it's appropriate that it fails. Yet, the error message could be usefully complemented by some hint at what the source of the problem is (a category implemented in two distinct classes). Leaving a large enough piece of the backtrace would be useful though, so that one can explore where the issue comes from (e.g. with post mortem debugging).

### 1.3.2 Specifications

After fixing some vocabulary, we summarize here some specifications about categories and axioms.

#### The lattice of constructible categories

A mathematical category $C$ is *implemented* if there is a class in Sage modelling it; it is *constructible* if it is either implemented, or is the intersection of *implemented* categories; in the latter case it is modelled by a `JoinCategory`. The comparison of two constructible categories with the `Category.is_subcategory()` method is supposed to model the comparison of the corresponding mathematical categories for inclusion of the objects (see *On the category hierarchy: subcategories and super categories* for details). For example:

```
sage: Fields().is_subcategory(Rings())
True
```

However this modelling may be incomplete. It can happen that a mathematical fact implying that a category $A$ is a subcategory of a category $B$ is not implemented. Still, the comparison should endow the set of constructible categories with a poset structure and in fact a lattice structure.

In this lattice, the join of two categories (*Category.join()*) is supposed to model their intersection. Given that we compare categories for inclusion, it would be more natural to call this operation the *meet*; blames go to me (Nicolas) for originally comparing categories by *amount of structure* rather than by *inclusion*. In practice, the join of two categories may be a strict super category of their intersection; first because this intersection might not be constructible; second because Sage might miss some mathematical information to recover the smallest constructible super category of the intersection.

### Axioms

We say that an axiom `A` is *defined by* a category `Cs()` if `Cs` defines an appropriate method `Cs.SubcategoryMethods.A`, with the semantic of the axiom specified in the documentation; for any subcategory `Ds()`, `Ds().A()` models the subcategory of the objects of `Ds()` satisfying `A`. In this case, we say that the axiom `A` is *defined for* the category `Ds()`. Furthermore, `Ds` *implements the axiom* `A` if `Ds` has a category with axiom as nested class `Ds.A`. The category `Ds()` *satisfies* the axiom if `Ds()` is a subcategory of `Cs().A()` (meaning that all the objects of `Ds()` are known to satisfy the axiom `A`).

### A digression on the structure of fibers when adding an axiom

Consider the application $\phi_A$ which maps a category to its category of objects satisfying $A$. Equivalently, $\phi_A$ is computing the intersection with the defining category with axiom of $A$. It follows immediately from the latter that $\phi_A$ is a regressive endomorphism of the lattice of categories. It restricts to a regressive endomorphism `Cs() |-> Cs().A()` on the lattice of constructible categories.

This endomorphism may have non trivial fibers, as in our favorite example: `DivisionRings()` and `Fields()` are in the same fiber for the axiom `Finite`:

```
sage: DivisionRings().Finite() is Fields().Finite()
True
```

Consider the intersection $S$ of such a fiber of $\phi_A$ with the upper set $I_A$ of categories that do not satisfy `A`. The fiber itself is a sublattice. However $I_A$ is not guaranteed to be stable under intersection (though exceptions should be rare). Therefore, there is a priori no guarantee that $S$ would be stable under intersection. Also it's presumably finite, in fact small, but this is not guaranteed either.

### Specifications

- Any constructible category `C` should admit a finite number of larger constructible categories.
- The methods `super_categories`, `extra_super_categories`, and friends should always return strict supercategories.

  For example, to specify that a finite division ring is a finite field, `DivisionRings.Finite_extra_super_categories` should not return `Fields().Finite()`! It could possibly return `Fields()`; but it's preferable to return the largest category that contains the relevant information, in this case `Magmas().Commutative()`, and to let the infrastructure apply the derivations.
- The base category of a *CategoryWithAxiom* should be an implemented category (i.e. not a *JoinCategory*). This is checked by *CategoryWithAxiom._test_category_with_axiom()*.
- Arborescent structure: Let `Cs()` be a category, and $S$ be some set of axioms defined in some super categories of `Cs()` but not satisfied by `Cs()`. Suppose we want to provide a category with axiom for the elements of `Cs()`

satisfying the axioms in $S$. Then, there should be a single enumeration `A1, A2, ..., Ak` without repetition of axioms in $S$ such that `Cs.A1.A2....Ak` is an implemented category. Furthermore, every intermediate step `Cs.A1.A2....Ai` with $i \leq k$ should be a category with axiom having `Ai` as axiom and `Cs.A1.A2....Ai-1` as base category class; this base category class should not satisfy `Ai`. In particular, when some axioms of $S$ can be deduced from previous ones by deduction rules, they should not appear in the enumeration `A1, A2, ..., Ak`.

- In particular, if `Cs()` is a category that satisfies some axiom `A` (e.g. from one of its super categories), then it should not implement that axiom. For example, a category class `Cs` can never have a nested class `Cs.A.A`. Similarly, applying the specification recursively, a category satisfying `A` cannot have a nested class `Cs.A1.A2.A3.A` where `A1`, `A2`, `A3` are axioms.
- A category can only implement an axiom if this axiom is defined by some super category. The code has not been systematically checked to support having two super categories defining the same axiom (which should of course have the same semantic). You are welcome to try, at your own risk. :-)
- When a category defines an axiom or functorial construction `A`, this fixes the semantic of `A` for all the subcategories. In particular, if two categories define `A`, then these categories should be independent, and either the semantic of `A` should be the same, or there should be no natural intersection between the two hierarchies of subcategories.
- Any super category of a *CategoryWithParameters* should either be a *CategoryWithParameters* or a `Category_singleton`.
- A *CategoryWithAxiom* having a *Category_singleton* as base category should be a *CategoryWithAxiom_singleton*. This is handled automatically by *CategoryWithAxiom.__init__()* and checked in *CategoryWithAxiom._test_category_with_axiom()*.
- A *CategoryWithAxiom* having a `Category_over_base_ring` as base category should be a `Category_over_base_ring`. This currently has to be handled by hand, using *CategoryWithAxiom_over_base_ring*. This is checked in *CategoryWithAxiom._test_category_with_axiom()*.

---

**Todo:** The following specifications would be desirable but are not yet implemented:

- A functorial construction category (Graded, CartesianProducts, …) having a `Category_singleton` as base category should be a *CategoryWithAxiom_singleton*.

  Nothing difficult to implement, but this will need to rework the current “no subclass of a concrete class” assertion test of `Category_singleton.__classcall__()`.

- Similarly, a covariant functorial construction category having a `Category_over_base_ring` as base category should be a `Category_over_base_ring`.

The following specification might be desirable, or not:

- A join category involving a `Category_over_base_ring` should be a `Category_over_base_ring`. In the mean time, a `base_ring` method is automatically provided for most of those by *Modules.SubcategoryMethods.base_ring()*.

---

### 1.3.3 Design goals

As pointed out in the primer, the main design goal of the axioms infrastructure is to subdue the potential combinatorial explosion of the category hierarchy by letting the developer focus on implementing a few bookshelves for which there is actual code or mathematical information, and let Sage *compose dynamically and lazily* these building blocks to construct the minimal hierarchy of classes needed for the computation at hand. This allows for the infrastructure to scale smoothly as bookshelves are added, extended, or reorganized.

Other design goals include:

- Flexibility in the code layout: the category of, say, finite sets can be implemented either within the Sets category (in a nested class `Sets.Finite`), or in a separate file (typically in a class `FiniteSets` in a lazily imported module sage.categories.finite_sets).
- Single point of truth: a theorem, like Wedderburn's, should be implemented in a single spot.
- Single entry point: for example, from the entry *Rings*, one can explore a whole range of related categories just by applying axioms and constructions:

```
sage: Rings().Commutative().Finite().NoZeroDivisors()
Category of finite integral domains
sage: Rings().Finite().Division()
Category of finite enumerated fields
```

  This will allow for progressively getting rid of all the entries like *GradedHopfAlgebrasWithBasis* which are polluting the global name space.

  Note that this is not about precluding the existence of multiple natural ways to construct the same category:

```
sage: Groups().Finite()
Category of finite groups
sage: Monoids().Finite().Inverse()
Category of finite groups
sage: Sets().Finite() & Monoids().Inverse()
Category of finite groups
```

- Concise idioms for the users (adding axioms, . . . )
- Concise idioms and well highlighted hierarchy of bookshelves for the developer (especially with code folding)
- Introspection friendly (listing the axioms, recovering the mixins)

**Note:** The constructor for instances of this class takes as input the base category. Hence, they should in principle be constructed as:

```
sage: FiniteSets(Sets())
Category of finite sets

sage: Sets.Finite(Sets())
Category of finite sets
```

None of these idioms are really practical for the user. So instead, this object is to be constructed using any of the following idioms:

```
sage: Sets()._with_axiom('Finite')
Category of finite sets
sage: FiniteSets()
Category of finite sets
sage: Sets().Finite()
Category of finite sets
```

The later two are implemented using respectively *CategoryWithAxiom.__classcall__()* and *CategoryWithAxiom.__classget__()*.

### 1.3.4 Upcoming features

**Todo:**

- Implement compatibility axiom / functorial constructions. For example, one would want to have:

```
A.CartesianProducts() & B.CartesianProducts() = (A&B).CartesianProducts()
```

- Once full subcategories are implemented (see trac ticket #10668), make the relevant categories with axioms be such. This can be done systematically for, e.g., the axioms `Associative` or `Commutative`, but not for the axiom `Unital`: a semigroup morphism between two monoids need not preserve the unit.

  Should all full subcategories be implemented in term of axioms?

### 1.3.5 Algorithms

#### Computing joins

The workhorse of the axiom infrastructure is the algorithm for computing the join $J$ of a set $C_1, \ldots, C_k$ of categories (see `Category.join()`). Formally, $J$ is defined as the largest constructible category such that $J \subset C_i$ for all $i$, and $J \subset C.A()$ for every constructible category $C \supset J$ and any axiom $A$ satisfied by $J$.

The join $J$ is naturally computed as a closure in the lattice of constructible categories: it starts with the $C_i$'s, gathers the set $S$ of all the axioms satisfied by them, and repeatedly adds each axiom $A$ to those categories that do not yet satisfy $A$ using `Category._with_axiom()`. Due to deduction rules or (extra) super categories, new categories or new axioms may appear in the process. The process stops when each remaining category has been combined with each axiom. In practice, only the smallest categories are kept along the way; this is correct because adding an axiom is covariant: `C.A()` is a subcategory of `D.A()` whenever `C` is a subcategory of `D`.

As usual in such closure computations, the result does not depend on the order of execution. Futhermore, given that adding an axiom is an idempotent and regressive operation, the process is guaranteed to stop in a number of steps which is bounded by the number of super categories of $J$. In particular, it is a finite process.

**Todo:** Detail this a bit. What could typically go wrong is a situation where, for some category `C1`, `C1.A()` specifies a category `C2` as super category such that `C2.A()` specifies `C3` as super category such that ...; this would clearly cause an infinite execution. Note that this situation violates the specifications since `C1.A()` is supposed to be a subcategory of `C2.A()`, ... so we would have an infinite increasing chain of constructible categories.

It's reasonable to assume that there is a finite number of axioms defined in the code. There remains to use this assumption to argue that any infinite execution of the algorithm would give rise to such an infinite sequence.

#### Adding an axiom

Let `Cs` be a category and `A` an axiom defined for this category. To compute `Cs().A()`, there are two cases.

#### Adding an axiom `A` to a category `Cs()` not implementing it

In this case, `Cs().A()` returns the join of:

- `Cs()`

- `Bs().A()` for every direct super category `Bs()` of `Cs()`
- the categories appearing in `Cs().A_extra_super_categories()`

This is a highly recursive process. In fact, as such, it would run right away into an infinite loop! Indeed, the join of `Cs()` with `Bs().A()` would trigger the construction of `Cs().A()` and reciprocally. To avoid this, the `Category.join()` method itself does not use `Category._with_axiom()` to add axioms, but its sister `Category._with_axiom_as_tuple()`; the latter builds a tuple of categories that should be joined together but leaves the computation of the join to its caller, the master join calculation.

### Adding an axiom `A` to a category `Cs()` implementing it

In this case `Cs().A()` simply constructs an instance $D$ of `Cs.A` which models the desired category. The non trivial part is the construction of the super categories of $D$. Very much like above, this includes:

- `Cs()`
- `Bs().A()` for every super category `Bs()` of `Cs()`
- the categories appearing in `D.extra_super_categories()`

This by itself may not be sufficient, due in particular to deduction rules. On may for example discover a new axiom `A1` satisfied by $D$, imposing to add `A1` to all of the above categories. Therefore the super categories are computed as the join of the above categories. Up to one twist: as is, the computation of this join would trigger recursively a recalculation of `Cs().A()`! To avoid this, `Category.join()` is given an optional argument to specify that the axiom `A` should *not* be applied to `Cs()`.

### Sketch of proof of correctness and evaluation of complexity

As we have seen, this is a highly recursive process! In particular, one needs to argue that, as long as the specifications are satisfied, the algorithm won't run in an infinite recursion, in particular in case of deduction rule.

**Theorem**

Consider the construction of a category $C$ by adding an axiom to a category (or computing of a join). Let $H$ be the hierarchy of implemented categories above $C$. Let $n$ and $m$ be respectively the number of categories and the number of inheritance edges in $H$.

Assuming that the specifications are satisfied, the construction of $C$ involves constructing the categories in $H$ exactly once (and no other category), and at most $n$ join calculations. In particular, the time complexity should be, roughly speaking, bounded by $n^2$. In particular, it's finite.

**Remark**

It's actually to be expected that the complexity is more of the order of magnitude of $na + m$, where $a$ is the number of axioms satisfied by $C$. But this is to be checked in detail, in particular due to the many category inclusion tests involved.

The key argument is that `Category.join` cannot call itself recursively without going through the construction of some implemented category. In turn, the construction of some implemented category $C$ only involves constructing strictly smaller categories, and possibly a direct join calculation whose result is strictly smaller than $C$. This statement is obvious if $C$ implements the `super_categories` method directly, and easy to check for functorial construction categories. It requires a proof for categories with axioms since there is a recursive join involved.

**Lemma**

Let $C$ be a category implementing an axiom $A$. Recall that the construction of `C.A()` involves a single direct join calculation for computing the super categories. No other direct join calculation occur, and the calculation involves only implemented categories that are strictly smaller than `C.A()`.

**Proof**

Let $D$ be a category involved in the join calculation for the super categories of `C.A()`, and assume by induction that $D$ is strictly smaller than `C.A()`. A category $E$ newly constructed from $D$ can come from:

- `D.(extra_)super_categories()`

  In this case, the specifications impose that $E$ should be strictly smaller than $D$ and therefore strictly smaller than $C$.

- `D.with_axiom_as_tuple('B')` or `D.B_extra_super_categories()` for some axiom $B$

  In this case, the axiom $B$ is satisfied by some subcategory of `C.A()`, and therefore must be satisfied by `C.A()` itself. Since adding an axiom is a regressive construction, $E$ must be a subcategory of `C.A()`. If there is equality, then $E$ and `C.A()` must have the same class, and therefore, $E$ must be directly constructed as `C.A()`. However the join construction explicitly prevents this call.

Note that a call to `D.with_axiom_as_tuple('B')` does not trigger a direct join calculation; but of course, if $D$ implements $B$, the construction of the implemented category `E = D.B()` will involve a strictly smaller join calculation.

### 1.3.6 Conclusion

This is the end of the axioms documentation. Congratulations on having read that far!

### 1.3.7 Tests

**Note:** Quite a few categories with axioms are constructed early on during Sage's startup. Therefore, when playing around with the implementation of the axiom infrastructure, it is easy to break Sage. The following sequence of tests is designed to test the infrastructure from the ground up even in a partially broken Sage. Please don't remove the imports!

```
sage: Magmas()
Category of magmas
sage: Magmas().Finite()
Category of finite magmas

sage: Magmas().Unital()
Category of unital magmas
sage: Magmas().Commutative().Unital()
Category of commutative unital magmas
sage: Magmas().Associative()
Category of semigroups
sage: Magmas().Associative() & Magmas().Unital().Inverse() & Sets().Finite()
```

```
Category of finite groups
sage: _ is Groups().Finite()
True

sage: from sage.categories.semigroups import Semigroups
sage: Semigroups()
Category of semigroups
sage: Semigroups().Finite()
Category of finite semigroups

sage: from sage.categories.modules_with_basis import ModulesWithBasis
sage: ModulesWithBasis(QQ) is Modules(QQ).WithBasis()
True
sage: ModulesWithBasis(ZZ) is Modules(ZZ).WithBasis()
True

sage: Semigroups().Unital()
Category of monoids
sage: Semigroups().Unital().Commutative()
Category of commutative monoids
sage: Semigroups().Commutative()
Category of commutative semigroups
sage: Semigroups().Commutative().Unital()
Category of commutative monoids
sage: Semigroups().Commutative().Unital().super_categories()
[Category of monoids, Category of commutative magmas]

sage: AdditiveMagmas().AdditiveAssociative().AdditiveCommutative()
Category of commutative additive semigroups

sage: from sage.categories.magmas_and_additive_magmas import MagmasAndAdditiveMagmas
sage: C = CommutativeAdditiveMonoids() & Monoids() & MagmasAndAdditiveMagmas().
↪Distributive(); C
Category of semirings
sage: C is (CommutativeAdditiveMonoids() & Monoids()).Distributive()
True
sage: C.AdditiveInverse()
Category of rings
sage: Rings().axioms()
frozenset({'AdditiveAssociative',
           'AdditiveCommutative',
           'AdditiveInverse',
           'AdditiveUnital',
           'Associative',
           'Distributive',
           'Unital'})
sage: sorted(Rings().axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive', 'Unital']

sage: Domains().Commutative()
Category of integral domains

sage: DivisionRings().Finite() # Wedderburn's theorem
Category of finite enumerated fields

sage: FiniteMonoids().Algebras(QQ)
Join of Category of monoid algebras over Rational Field
```

```
    and Category of finite dimensional algebras with basis over Rational Field
    and Category of finite set algebras over Rational Field
sage: FiniteGroups().Algebras(QQ)
Category of finite group algebras over Rational Field
```

**class** sage.categories.category_with_axiom.**Bars**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

A toy singleton category, for testing purposes.

**See also:**

*Blahs*

**Unital_extra_super_categories**()

Return extraneous super categories for the unital objects of self.

This method specifies that a unital bar is a test object. Thus, the categories of unital bars and of unital test objects coincide.

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import Bars, TestObjects
sage: Bars().Unital_extra_super_categories()
[Category of test objects]
sage: Bars().Unital()
Category of unital test objects
sage: TestObjects().Unital().all_super_categories()
[Category of unital test objects,
 Category of unital blahs,
 Category of test objects,
 Category of bars,
 Category of blahs,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

**super_categories**()

**class** sage.categories.category_with_axiom.**Blahs**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

A toy singleton category, for testing purposes.

This is the root of a hierarchy of mathematically meaningless categories, used for testing Sage's category framework:

- *Bars*
- *TestObjects*
- *TestObjectsOverBaseRing*

**Blue_extra_super_categories**()

Illustrates a current limitation in the way to have an axiom imply another one.

Here, we would want Blue to imply Unital, and to put the class for the category of unital blue blahs in Blahs.Unital.Blue rather than Blahs.Blue.

This currently fails because Blahs is the category where the axiom Blue is defined, and the specifications currently impose that a category defining an axiom should also implement it (here in an category with axiom Blahs.Blue). In practice, due to this violation of the specifications, the axiom is lost during the join calculation.

**Todo:** Decide whether we care about this feature. In such a situation, we are not really defining a new axiom, but just defining an axiom as an alias for a couple others, which might not be that useful.

**Todo:** Improve the infrastructure to detect and report this violation of the specifications, if this is easy. Otherwise, it's not so bad: when defining an axiom A in a category `Cs` the first thing one is supposed to doctest is that `Cs().A()` works. So the problem should not go unnoticed.

**class Commutative**(*base_category*)
Bases: `sage.categories.category_with_axiom.CategoryWithAxiom`

**class Connected**(*base_category*)
Bases: `sage.categories.category_with_axiom.CategoryWithAxiom`

**class FiniteDimensional**(*base_category*)
Bases: `sage.categories.category_with_axiom.CategoryWithAxiom`

**class Flying**(*base_category*)
Bases: `sage.categories.category_with_axiom.CategoryWithAxiom`

**extra_super_categories**()
This illustrates a way to have an axiom imply another one.

Here, we want `Flying` to imply `Unital`, and to put the class for the category of unital flying blahs in `Blahs.Flying` rather than `Blahs.Unital.Flying`.

**class SubcategoryMethods**

**Blue**()

**Commutative**()

**Connected**()

**FiniteDimensional**()

**Flying**()

**Unital**()

**class Unital**(*base_category*)
Bases: `sage.categories.category_with_axiom.CategoryWithAxiom`

**class Blue**(*base_category*)
Bases: `sage.categories.category_with_axiom.CategoryWithAxiom`

**super_categories**()

**class** sage.categories.category_with_axiom.**CategoryWithAxiom**(*base_category*)
Bases: `sage.categories.category.Category`

An abstract class for categories obtained by adding an axiom to a base category.

See the *category primer*, and in particular its *section about axioms* for an introduction to axioms, and `CategoryWithAxiom` for how to implement axioms and the documentation of the axiom infrastructure.

**static __classcall__**(**args*, ***options*)
Make `FoosBar(**)` an alias for `Foos(**)._with_axiom("Bar")`.

EXAMPLES:

```
sage: FiniteGroups()
Category of finite groups
sage: ModulesWithBasis(ZZ)
Category of modules with basis over Integer Ring
sage: AlgebrasWithBasis(QQ)
Category of algebras with basis over Rational Field
```

This is relevant when e.g. `Foos(**)` does some non trivial transformations:

```
sage: Modules(QQ) is VectorSpaces(QQ)
True
sage: type(Modules(QQ))
<class 'sage.categories.vector_spaces.VectorSpaces_with_category'>

sage: ModulesWithBasis(QQ) is VectorSpaces(QQ).WithBasis()
True
sage: type(ModulesWithBasis(QQ))
<class 'sage.categories.vector_spaces.VectorSpaces.WithBasis_with_category'>
```

**static __classget__**(*base_category*, *base_category_class*)

Implement the binding behavior for categories with axioms.

This method implements a binding behavior on category with axioms so that, when a category `Cs` implements an axiom `A` with a nested class `Cs.A`, the expression `Cs().A` evaluates to the method defining the axiom `A` and not the nested class. See those design notes for the rationale behind this behavior.

EXAMPLES:

```
sage: Sets().Infinite()
Category of infinite sets
sage: Sets().Infinite
Cached version of <function Infinite at ...>
sage: Sets().Infinite.f == Sets.SubcategoryMethods.Infinite.f
True
```

We check that this also works when the class is implemented in a separate file, and lazy imported:

```
sage: Sets().Finite
Cached version of <function Finite at ...>
```

There is no binding behavior when accessing `Finite` or `Infinite` from the class of the category instead of the category itself:

```
sage: Sets.Finite
<class 'sage.categories.finite_sets.FiniteSets'>
sage: Sets.Infinite
<class 'sage.categories.sets_cat.Sets.Infinite'>
```

This method also initializes the attribute `_base_category_class_and_axiom` if not already set:

```
sage: Sets.Infinite._base_category_class_and_axiom
(<class 'sage.categories.sets_cat.Sets'>, 'Infinite')
sage: Sets.Infinite._base_category_class_and_axiom_origin
'set by __classget__'
```

**__init__**(*base_category*)

**_repr_object_names**()

The names of the objects of this category, as used by `_repr_`.

**See also:**

*Category._repr_object_names()*

EXAMPLES:

```
sage: FiniteSets()._repr_object_names()
'finite sets'
sage: AlgebrasWithBasis(QQ).FiniteDimensional()._repr_object_names()
'finite dimensional algebras with basis over Rational Field'
sage: Monoids()._repr_object_names()
'monoids'
sage: Semigroups().Unital().Finite()._repr_object_names()
'finite monoids'
sage: Algebras(QQ).Commutative()._repr_object_names()
'commutative algebras over Rational Field'
```

**Note:** This is implemented by taking _repr_object_names from self._without_axioms(named=True), and adding the names of the relevant axioms in appropriate order.

**static _repr_object_names_static**(*category*, *axioms*)

INPUT:

- `base_category` – a category
- `axioms` – a list or iterable of strings

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import CategoryWithAxiom
sage: CategoryWithAxiom._repr_object_names_static(Semigroups(), ["Flying",
↪"Blue"])
'flying blue semigroups'
sage: CategoryWithAxiom._repr_object_names_static(Algebras(QQ), ["Flying",
↪"WithBasis", "Blue"])
'flying blue algebras with basis over Rational Field'
sage: CategoryWithAxiom._repr_object_names_static(Algebras(QQ), ["WithBasis"])
'algebras with basis over Rational Field'
sage: CategoryWithAxiom._repr_object_names_static(Sets().Finite().
↪Subquotients(), ["Finite"])
'subquotients of finite sets'
sage: CategoryWithAxiom._repr_object_names_static(Monoids(), ["Unital"])
'monoids'
sage: CategoryWithAxiom._repr_object_names_static(Algebras(QQ['x']['y']), [
↪"Flying", "WithBasis", "Blue"])
'flying blue algebras with basis over Univariate Polynomial Ring in y over␣
↪Univariate Polynomial Ring in x over Rational Field'
```

If the axioms is a set or frozen set, then they are first sorted using `canonicalize_axioms()`:

```
sage: CategoryWithAxiom._repr_object_names_static(Semigroups(), set(["Finite",
↪ "Commutative", "Facade"]))
'facade finite commutative semigroups'
```

**See also:**

*_repr_object_names()*

**Note:** The logic here is shared between *_repr_object_names()* and *category.JoinCategory._repr_object_names()*

**_test_category_with_axiom**(***options*)

Run generic tests on this category with axioms.

**See also:**

TestSuite.

This check that an axiom category of a `Category_singleton` is a singleton category, and similarwise for `Category_over_base_ring`.

EXAMPLES:

```
sage: Sets().Finite()._test_category_with_axiom()
sage: Modules(ZZ).FiniteDimensional()._test_category_with_axiom()
```

**_without_axioms**(*named=False*)

Return the category without the axioms that have been added to create it.

EXAMPLES:

```
sage: Sets().Finite()._without_axioms()
Category of sets
sage: Monoids().Finite()._without_axioms()
Category of magmas
```

This is because:

```
sage: Semigroups().Unital() is Monoids()
True
```

If `named` is `True`, then `_without_axioms` stops at the first category that has an explicit name of its own:

```
sage: Sets().Finite()._without_axioms(named=True)
Category of sets
sage: Monoids().Finite()._without_axioms(named=True)
Category of monoids
```

Technically we test this by checking if the class specifies explicitly the attribute `_base_category_class_and_axiom` by looking up `_base_category_class_and_axiom_origin`.

Some more examples:

```
sage: Algebras(QQ).Commutative()._without_axioms()
Category of magmatic algebras over Rational Field
sage: Algebras(QQ).Commutative()._without_axioms(named=True)
Category of algebras over Rational Field
```

**additional_structure**()

Return the additional structure defined by `self`.

OUTPUT: `None`

By default, a category with axiom defines no additional structure.

**See also:**

*Category.additional_structure()*.

EXAMPLES:

sage: Sets().Finite().additional_structure() sage: Monoids().additional_structure()

**axioms**()

Return the axioms known to be satisfied by all the objects of `self`.

**See also:**

*Category.axioms()*

EXAMPLES:

```
sage: C = Sets.Finite(); C
Category of finite sets
sage: C.axioms()
frozenset({'Finite'})

sage: C = Modules(GF(5)).FiniteDimensional(); C
Category of finite dimensional vector spaces over Finite Field of size 5
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Finite', 'FiniteDimensional']

sage: sorted(FiniteMonoids().Algebras(QQ).axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive',
 'FiniteDimensional', 'Unital', 'WithBasis']
sage: sorted(FiniteMonoids().Algebras(GF(3)).axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive', 'Finite',
 'FiniteDimensional', 'Unital', 'WithBasis']

sage: from sage.categories.magmas_and_additive_magmas import␣
↪MagmasAndAdditiveMagmas
sage: MagmasAndAdditiveMagmas().Distributive().Unital().axioms()
frozenset({'Distributive', 'Unital'})

sage: D = MagmasAndAdditiveMagmas().Distributive()
sage: X = D.AdditiveAssociative().AdditiveCommutative().Associative()
sage: X.Unital().super_categories()[1]
Category of monoids
sage: X.Unital().super_categories()[1] is Monoids()
True
```

**base_category**()

Return the base category of `self`.

EXAMPLES:

```
sage: C = Sets.Finite(); C
Category of finite sets
sage: C.base_category()
Category of sets
sage: C._without_axioms()
Category of sets
```

**extra_super_categories**()

Return the extra super categories of a category with axiom.

Default implementation which returns `[]`.

EXAMPLES:

```
sage: FiniteSets().extra_super_categories()
[]
```

**super_categories**()

Return a list of the (immediate) super categories of `self`, as per *Category.super_categories()*.

This implements the property that if `As` is a subcategory of `Bs`, then the intersection of `As` with `FiniteSets()` is a subcategory of `As` and of the intersection of `Bs` with `FiniteSets()`.

EXAMPLES:

A finite magma is both a magma and a finite set:

```
sage: Magmas().Finite().super_categories()
[Category of magmas, Category of finite sets]
```

Variants:

```
sage: Sets().Finite().super_categories()
[Category of sets]

sage: Monoids().Finite().super_categories()
[Category of monoids, Category of finite semigroups]
```

EXAMPLES:

class sage.categories.category_with_axiom.**CategoryWithAxiom_over_base_ring**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*, *sage.categories.category_types.Category_over_base_ring*

class sage.categories.category_with_axiom.**CategoryWithAxiom_singleton**(*base_category*)

Bases: *sage.categories.category_singleton.Category_singleton*, *sage.categories.category_with_axiom.CategoryWithAxiom*

class sage.categories.category_with_axiom.**TestObjects**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

A toy singleton category, for testing purposes.

**See also:**

*Blahs*

class **Commutative**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

class **Facade**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

class **Finite**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

class **FiniteDimensional**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**class FiniteDimensional**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**class Finite**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**class Unital**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**class Commutative**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**class Unital**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**super_categories**()

**class** sage.categories.category_with_axiom.**TestObjectsOverBaseRing**(*base*, *name=None*)
Bases: *sage.categories.category_types.Category_over_base_ring*

A toy singleton category, for testing purposes.

**See also:**

*Blahs*

**class Commutative**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class Facade**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class Finite**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class FiniteDimensional**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class FiniteDimensional**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class Finite**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class Unital**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class Commutative**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class Unital**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**super_categories**()

sage.categories.category_with_axiom.**axiom**(*axiom*)
Return a function/method `self -> self._with_axiom(axiom)`.

This can used as a shorthand to define axioms, in particular in the tests below. Usually one will want to attach documentation to an axiom, so the need for such a shorthand in real life might not be that clear, unless we start creating lots of axioms.

In the long run maybe this could evolve into an `@axiom` decorator.

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import axiom
sage: axiom("Finite")(Semigroups())
Category of finite semigroups
```

Upon assigning the result to a class this becomes a method:

```
sage: class As:
....:     def _with_axiom(self, axiom): return self, axiom
....:     Finite = axiom("Finite")
sage: As().Finite()
(<__main__.As instance at ...>, 'Finite')
```

`sage.categories.category_with_axiom.`**`axiom_of_nested_class`**`(`*cls*, *nested_cls*`)`

Given a class and a nested axiom class, return the axiom.

EXAMPLES:

This uses some heuristics like checking if the nested_cls carries the name of the axiom, or is built by appending or prepending the name of the axiom to that of the class:

```
sage: from sage.categories.category_with_axiom import TestObjects, axiom_of_
↪nested_class
sage: axiom_of_nested_class(TestObjects, TestObjects.FiniteDimensional)
'FiniteDimensional'
sage: axiom_of_nested_class(TestObjects.FiniteDimensional, TestObjects.
↪FiniteDimensional.Finite)
'Finite'
sage: axiom_of_nested_class(Sets, FiniteSets)
'Finite'
sage: axiom_of_nested_class(Algebras, AlgebrasWithBasis)
'WithBasis'
```

In all other cases, the nested class should provide an attribute `_base_category_class_and_axiom`:

```
sage: Semigroups._base_category_class_and_axiom
(<class 'sage.categories.magmas.Magmas'>, 'Associative')
sage: axiom_of_nested_class(Magmas, Semigroups)
'Associative'
```

`sage.categories.category_with_axiom.`**`base_category_class_and_axiom`**`(`*cls*`)`

Try to deduce the base category and the axiom from the name of `cls`.

The heuristic is to try to decompose the name as the concatenation of the name of a category and the name of an axiom, and looking up that category in the standard location (i.e. in *`sage.categories.hopf_algebras`* for *`HopfAlgebras`*, and in *`sage.categories.sets_cat`* as a special case for *`Sets`*).

If the heuristic succeeds, the result is guaranteed to be correct. Otherwise, an error is raised.

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import base_category_class_and_
↪axiom, CategoryWithAxiom
sage: base_category_class_and_axiom(FiniteSets)
```

```
(<class 'sage.categories.sets_cat.Sets'>, 'Finite')
sage: Sets.Finite
<class 'sage.categories.finite_sets.FiniteSets'>
sage: base_category_class_and_axiom(Sets.Finite)
(<class 'sage.categories.sets_cat.Sets'>, 'Finite')

sage: base_category_class_and_axiom(FiniteDimensionalHopfAlgebrasWithBasis)
(<class 'sage.categories.hopf_algebras_with_basis.HopfAlgebrasWithBasis'>,
↪'FiniteDimensional')

sage: base_category_class_and_axiom(HopfAlgebrasWithBasis)
(<class 'sage.categories.hopf_algebras.HopfAlgebras'>, 'WithBasis')
```

Along the way, this does some sanity checks:

```
sage: class FacadeSemigroups(CategoryWithAxiom):
....:     pass
sage: base_category_class_and_axiom(FacadeSemigroups)
Traceback (most recent call last):
...
AssertionError: Missing (lazy import) link for <class 'sage.categories.semigroups.
↪Semigroups'> to <class '__main__.FacadeSemigroups'> for axiom Facade?

sage: Semigroups.Facade = FacadeSemigroups
sage: base_category_class_and_axiom(FacadeSemigroups)
(<class 'sage.categories.semigroups.Semigroups'>, 'Facade')
```

**Note:** In the following example, we could possibly retrieve `Sets` from the class name. However this cannot be implemented robustly until trac ticket #9107 is fixed. Anyway this feature has not been needed so far:

```
sage: Sets.Infinite
<class 'sage.categories.sets_cat.Sets.Infinite'>
sage: base_category_class_and_axiom(Sets.Infinite)
Traceback (most recent call last):
...
TypeError: Could not retrieve the base category class and axiom for <class 'sage.
↪categories.sets_cat.Sets.Infinite'>.
...
```

`sage.categories.category_with_axiom.`**`uncamelcase`**(*s*, *separator*=' ')

EXAMPLES:

```
sage: sage.categories.category_with_axiom.uncamelcase("FiniteDimensionalAlgebras")
'finite dimensional algebras'
sage: sage.categories.category_with_axiom.uncamelcase("JTrivialMonoids")
'j trivial monoids'
sage: sage.categories.category_with_axiom.uncamelcase("FiniteDimensionalAlgebras",
↪ "_")
'finite_dimensional_algebras'
```

## 1.4 Functors

AUTHORS:

- David Kohel and William Stein
- David Joyner (2005-12-17): examples
- Robert Bradshaw (2007-06-23): Pyrexify
- Simon King (2010-04-30): more examples, several bug fixes, re-implementation of the default call method, making functors applicable to morphisms (not only to objects)
- Simon King (2010-12): Pickling of functors without loosing domain and codomain

`sage.categories.functor.`**`ForgetfulFunctor`**(*domain*, *codomain*)

Construct the forgetful function from one category to another.

INPUT:

`C`, `D` - two categories

OUTPUT:

A functor that returns the corresponding object of `D` for any element of `C`, by forgetting the extra structure.

ASSUMPTION:

The category `C` must be a sub-category of `D`.

EXAMPLES:

```
sage: rings = Rings()
sage: abgrps = CommutativeAdditiveGroups()
sage: F = ForgetfulFunctor(rings, abgrps)
sage: F
The forgetful functor from Category of rings to Category of commutative additive␣
↪groups
```

It would be a mistake to call it in opposite order:

```
sage: F = ForgetfulFunctor(abgrps, rings)
Traceback (most recent call last):
...
ValueError: Forgetful functor not supported for domain Category of commutative␣
↪additive groups
```

If both categories are equal, the forgetful functor is the same as the identity functor:

```
sage: ForgetfulFunctor(abgrps, abgrps) == IdentityFunctor(abgrps)
True
```

**class** `sage.categories.functor.`**`ForgetfulFunctor_generic`**

Bases: *`sage.categories.functor.Functor`*

The forgetful functor, i.e., embedding of a subcategory.

NOTE:

Forgetful functors should be created using *`ForgetfulFunctor()`*, since the init method of this class does not check whether the domain is a subcategory of the codomain.

EXAMPLES:

```
sage: F = ForgetfulFunctor(FiniteFields(),Fields()) #indirect doctest
sage: F
The forgetful functor from Category of finite enumerated fields to Category of␣
↪fields
```

```
sage: F(GF(3))
Finite Field of size 3
```

**class** sage.categories.functor.**Functor**

Bases: sage.structure.sage_object.SageObject

A class for functors between two categories

NOTE:

- In the first place, a functor is given by its domain and codomain, which are both categories.
- When defining a sub-class, the user should not implement a call method. Instead, one should implement three methods, which are composed in the default call method:
  - `_coerce_into_domain(self, x)`: Return an object of `self`'s domain, corresponding to `x`, or raise a `TypeError`.
    * Default: Raise `TypeError` if `x` is not in `self`'s domain.
  - `_apply_functor(self, x)`: Apply `self` to an object `x` of `self`'s domain.
    * Default: Conversion into `self`'s codomain.
  - `_apply_functor_to_morphism(self, f)`: Apply `self` to a morphism `f` in `self`'s domain. - Default: Return `self(f.domain()).hom(f,self(f.codomain()))`.

EXAMPLES:

```
sage: rings  = Rings()
sage: abgrps = CommutativeAdditiveGroups()
sage: F = ForgetfulFunctor(rings, abgrps)
sage: F.domain()
Category of rings
sage: F.codomain()
Category of commutative additive groups
sage: from sage.categories.functor import is_Functor
sage: is_Functor(F)
True
sage: I = IdentityFunctor(abgrps)
sage: I
The identity functor on Category of commutative additive groups
sage: I.domain()
Category of commutative additive groups
sage: is_Functor(I)
True
```

Note that by default, an instance of the class Functor is coercion from the domain into the codomain. The above subclasses overloaded this behaviour. Here we illustrate the default:

```
sage: from sage.categories.functor import Functor
sage: F = Functor(Rings(),Fields())
sage: F
Functor from Category of rings to Category of fields
sage: F(ZZ)
Rational Field
sage: F(GF(2))
Finite Field of size 2
```

Functors are not only about the objects of a category, but also about their morphisms. We illustrate it, again, with the coercion functor from rings to fields.

```
sage: R1.<x> = ZZ[]
sage: R2.<a,b> = QQ[]
sage: f = R1.hom([a+b],R2)
sage: f
Ring morphism:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Multivariate Polynomial Ring in a, b over Rational Field
  Defn: x |--> a + b
sage: F(f)
Ring morphism:
  From: Fraction Field of Univariate Polynomial Ring in x over Integer Ring
  To:   Fraction Field of Multivariate Polynomial Ring in a, b over Rational Field
  Defn: x |--> a + b
sage: F(f)(1/x)
1/(a + b)
```

We can also apply a polynomial ring construction functor to our homomorphism. The result is a homomorphism that is defined on the base ring:

```
sage: F = QQ['t'].construction()[0]
sage: F
Poly[t]
sage: F(f)
Ring morphism:
  From: Univariate Polynomial Ring in t over Univariate Polynomial Ring in x over␣
↪Integer Ring
  To:   Univariate Polynomial Ring in t over Multivariate Polynomial Ring in a, b␣
↪over Rational Field
  Defn: Induced from base ring by
        Ring morphism:
          From: Univariate Polynomial Ring in x over Integer Ring
          To:   Multivariate Polynomial Ring in a, b over Rational Field
          Defn: x |--> a + b
sage: p = R1['t']('(-x^2 + x)*t^2 + (x^2 - x)*t - 4*x^2 - x + 1')
sage: F(f)(p)
(-a^2 - 2*a*b - b^2 + a + b)*t^2 + (a^2 + 2*a*b + b^2 - a - b)*t - 4*a^2 - 8*a*b -
↪ 4*b^2 - a - b + 1
```

**codomain**()

The codomain of self

EXAMPLES:

```
sage: F = ForgetfulFunctor(FiniteFields(),Fields())
sage: F.codomain()
Category of fields
```

**domain**()

The domain of self

EXAMPLES:

```
sage: F = ForgetfulFunctor(FiniteFields(),Fields())
sage: F.domain()
Category of finite enumerated fields
```

sage.categories.functor.**IdentityFunctor**(*C*)

Construct the identity functor of the given category.

INPUT:

A category, C.

OUTPUT:

The identity functor in C.

EXAMPLES:

```
sage: rings = Rings()
sage: F = IdentityFunctor(rings)
sage: F(ZZ['x','y']) is ZZ['x','y']
True
```

**class** sage.categories.functor.**IdentityFunctor_generic**(*C*)

Bases: *sage.categories.functor.ForgetfulFunctor_generic*

Generic identity functor on any category

NOTE:

This usually is created using *IdentityFunctor()*.

EXAMPLES:

```
sage: F = IdentityFunctor(Fields()) #indirect doctest
sage: F
The identity functor on Category of fields
sage: F(RR) is RR
True
sage: F(ZZ)
Traceback (most recent call last):
...
TypeError: x (=Integer Ring) is not in Category of fields
```

sage.categories.functor.**is_Functor**(*x*)

Test whether the argument is a functor

NOTE:

There is a deprecation warning when using it from top level. Therefore we import it in our doc test.

EXAMPLES:

```
sage: from sage.categories.functor import is_Functor
sage: F1 = QQ.construction()[0]
sage: F1
FractionField
sage: is_Functor(F1)
True
sage: is_Functor(FractionField)
False
sage: F2 = ForgetfulFunctor(Fields(), Rings())
sage: F2
The forgetful functor from Category of fields to Category of rings
sage: is_Functor(F2)
True
```

## 1.5 Implementing a new parent: a (draft of) tutorial

The easiest approach for implementing a new parent is to start from a close example in sage.categories.examples. Here, we will get through the process of implementing a new finite semigroup, taking as starting point the provided example:

```
sage: S = FiniteSemigroups().example()
sage: S
An example of a finite semigroup: the left regular band generated by ('a', 'b', 'c',
↪'d')
```

You may lookup the implementation of this example with:

```
sage: S??                               # not tested
```

Or by browsing the source code of *sage.categories.examples.finite_semigroups.LeftRegularBand*.

Copy-paste this code into, say, a cell of the notebook, and replace every occurrence of `FiniteSemigroups().example(...)` in the documentation by `LeftRegularBand`. This will be equivalent to:

```
sage: from sage.categories.examples.finite_semigroups import LeftRegularBand
```

Now, try:

```
sage: S = LeftRegularBand(); S
An example of a finite semigroup: the left regular band generated by ('a', 'b', 'c',
↪'d')
```

and play around with the examples in the documentation of `S` and of *FiniteSemigroups*.

Rename the class to `ShiftSemigroup`, and modify the product to implement the semigroup generated by the given alphabet such that $au = u$ for any $u$ of length 3.

Use `TestSuite` to test the newly implemented semigroup; draw its Cayley graph.

Add another option to the constructor to generalize the construction to any u of length $k$.

Lookup the Sloane for the sequence of the sizes of those semigroups.

Now implement the commutative monoid of subsets of $\{1, \ldots, n\}$ endowed with union as product. What is its category? What are the extra functionalities available there? Implement iteration and cardinality.

TODO: the tutorial should explain there how to reuse the enumerated set of subsets, and endow it with more structure.

CHAPTER

TWO

# MAPS AND MORPHISMS

## 2.1 Base class for maps

AUTHORS:

- Robert Bradshaw: initial implementation
- Sebastien Besnier (2014-05-5): *FormalCompositeMap* contains a list of Map instead of only two Map. See trac ticket #16291.

**class** sage.categories.map.**FormalCompositeMap**

Bases: *sage.categories.map.Map*

Formal composite maps.

A formal composite map is formed by two maps, so that the codomain of the first map is contained in the domain of the second map.

**Note:** When calling a composite with additional arguments, these arguments are *only* passed to the second underlying map.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: S.<a> = QQ[]
sage: from sage.categories.morphism import SetMorphism
sage: f = SetMorphism(Hom(R, S, Rings()), lambda p: p[0]*a^p.degree())
sage: g = S.hom([2*x])
sage: f*g
Composite map:
  From: Univariate Polynomial Ring in a over Rational Field
  To:   Univariate Polynomial Ring in a over Rational Field
  Defn:   Ring morphism:
          From: Univariate Polynomial Ring in a over Rational Field
          To:   Univariate Polynomial Ring in x over Rational Field
          Defn: a |--> 2*x
        then
          Generic morphism:
          From: Univariate Polynomial Ring in x over Rational Field
          To:   Univariate Polynomial Ring in a over Rational Field
sage: g*f
Composite map:
  From: Univariate Polynomial Ring in x over Rational Field
  To:   Univariate Polynomial Ring in x over Rational Field
```

```
  Defn:   Generic morphism:
          From: Univariate Polynomial Ring in x over Rational Field
          To:   Univariate Polynomial Ring in a over Rational Field
        then
          Ring morphism:
          From: Univariate Polynomial Ring in a over Rational Field
          To:   Univariate Polynomial Ring in x over Rational Field
          Defn: a |--> 2*x
sage: (f*g)(2*a^2+5)
5*a^2
sage: (g*f)(2*x^2+5)
20*x^2
```

**domains**()

Iterate over the domains of the factors of this map.

(This is useful in particular to check for loops in coercion maps.)

**See also:**

*Map.domains()*

EXAMPLES:

```
sage: f = QQ.coerce_map_from(ZZ)
sage: g = MatrixSpace(QQ, 2, 2).coerce_map_from(QQ)
sage: list((g*f).domains())
[Integer Ring, Rational Field]
```

**first**()

Return the first map in the formal composition.

If `self` represents $f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0$, then `self.first()` returns $f_0$. We have `self == self.then() * self.first()`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: S.<a> = QQ[]
sage: from sage.categories.morphism import SetMorphism
sage: f = SetMorphism(Hom(R, S, Rings()), lambda p: p[0]*a^p.degree())
sage: g = S.hom([2*x])
sage: fg = f * g
sage: fg.first() == g
True
sage: fg == fg.then() * fg.first()
True
```

**is_injective**()

Tell whether `self` is injective.

It raises `NotImplementedError` if it can't be determined.

EXAMPLES:

```
sage: V1 = QQ^2
sage: V2 = QQ^3
sage: phi1 = (QQ^1).hom(Matrix([[1, 1]]), V1)
sage: phi2 = V1.hom(Matrix([[1, 2, 3], [4, 5, 6]]), V2)
```

If both constituents are injective, the composition is injective:

```
sage: from sage.categories.map import FormalCompositeMap
sage: c1 = FormalCompositeMap(Hom(QQ^1, V2, phi1.category_for()), phi1, phi2)
sage: c1.is_injective()
True
```

If it cannot be determined whether the composition is injective, an error is raised:

```
sage: psi1 = V2.hom(Matrix([[1, 2], [3, 4], [5, 6]]), V1)
sage: c2 = FormalCompositeMap(Hom(V1, V1, phi2.category_for()), phi2, psi1)
sage: c2.is_injective()
Traceback (most recent call last):
...
NotImplementedError: Not enough information to deduce injectivity.
```

If the first map is surjective and the second map is not injective, then the composition is not injective:

```
sage: psi2 = V1.hom([[1], [1]], QQ^1)
sage: c3 = FormalCompositeMap(Hom(V2, QQ^1, phi2.category_for()), psi2, psi1)
sage: c3.is_injective()
False
```

**is_surjective**()

Tell whether `self` is surjective.

It raises `NotImplementedError` if it can't be determined.

EXAMPLES:

```
sage: from sage.categories.map import FormalCompositeMap
sage: V3 = QQ^3
sage: V2 = QQ^2
sage: V1 = QQ^1
```

If both maps are surjective, the composition is surjective:

```
sage: phi32 = V3.hom(Matrix([[1, 2], [3, 4], [5, 6]]), V2)
sage: phi21 = V2.hom(Matrix([[1], [1]]), V1)
sage: c_phi = FormalCompositeMap(Hom(V3, V1, phi32.category_for()), phi32,␣
↪phi21)
sage: c_phi.is_surjective()
True
```

If the second map is not surjective, the composition is not surjective:

```
sage: FormalCompositeMap(Hom(V3, V1, phi32.category_for()), phi32, V2.
↪hom(Matrix([[0], [0]]), V1)).is_surjective()
False
```

If the second map is an isomorphism and the first map is not surjective, then the composition is not surjective:

```
sage: FormalCompositeMap(Hom(V2, V1, phi32.category_for()), V2.
↪hom(Matrix([[0], [0]]), V1), V1.hom(Matrix([[1]]), V1)).is_surjective()
False
```

Otherwise, surjectivity of the composition cannot be determined:

```
sage: FormalCompositeMap(Hom(V2, V1, phi32.category_for()),
....:     V2.hom(Matrix([[1, 1], [1, 1]]), V2),
....:     V2.hom(Matrix([[1], [1]]), V1)).is_surjective()
Traceback (most recent call last):
...
NotImplementedError: Not enough information to deduce surjectivity.
```

**second**(**args*, ***kwds*)

Deprecated: Use *then()* instead. See trac ticket #16291 for details.

**then**()

Return the tail of the list of maps.

If `self` represents $f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0$, then `self.first()` returns $f_n \circ f_{n-1} \circ \cdots \circ f_1$. We have `self == self.then() * self.first()`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: S.<a> = QQ[]
sage: from sage.categories.morphism import SetMorphism
sage: f = SetMorphism(Hom(R, S, Rings()), lambda p: p[0]*a^p.degree())
sage: g = S.hom([2*x])
sage: (f*g).then() == f
True
```

**class** sage.categories.map.**Map**

Bases: sage.structure.element.Element

Basic class for all maps.

**Note:** The call method is of course not implemented in this base class. This must be done in the sub classes, by overloading `_call_` and possibly also `_call_with_args`.

EXAMPLES:

Usually, instances of this class will not be constructed directly, but for example like this:

```
sage: from sage.categories.morphism import SetMorphism
sage: X.<x> = ZZ[]
sage: Y = ZZ
sage: phi = SetMorphism(Hom(X, Y, Rings()), lambda p: p[0])
sage: phi(x^2+2*x-1)
-1
sage: R.<x,y> = QQ[]
sage: f = R.hom([x+y, x-y], R)
sage: f(x^2+2*x-1)
x^2 + 2*x*y + y^2 + 2*x + 2*y - 1
```

**category_for**()

Returns the category self is a morphism for.

**Note:** This is different from the category of maps to which this map belongs *as an object*.

EXAMPLES:

```
sage: from sage.categories.morphism import SetMorphism
sage: X.<x> = ZZ[]
sage: Y = ZZ
sage: phi = SetMorphism(Hom(X, Y, Rings()), lambda p: p[0])
sage: phi.category_for()
Category of rings
sage: phi.category()
Category of homsets of unital magmas and additive unital additive magmas
sage: R.<x,y> = QQ[]
sage: f = R.hom([x+y, x-y], R)
sage: f.category_for()
Join of Category of unique factorization domains
and Category of commutative algebras
over (number fields and quotient fields and metric spaces)
and Category of infinite sets
sage: f.category()
Category of endsets of unital magmas
 and right modules over (number fields and quotient fields and metric spaces)
 and left modules over (number fields and quotient fields and metric spaces)
```

FIXME: find a better name for this method

**codomain**

**domain**

**domains**()

Iterate over the domains of the factors of a (composite) map.

This default implementation simply yields the domain of this map.

**See also:**

*FormalCompositeMap.domains()*

EXAMPLES:

```
sage: list(QQ.coerce_map_from(ZZ).domains())
[Integer Ring]
```

**extend_codomain**(*new_codomain*)

INPUT:

- `self` – a member of Hom(X, Y)
- `new_codomain` – an object Z such that there is a canonical coercion $\phi$ in Hom(Y, Z)

OUTPUT:

An element of Hom(X, Z) obtained by composing self with $\phi$. If no canonical $\phi$ exists, a TypeError is raised.

EXAMPLES:

```
sage: mor = QQ.coerce_map_from(ZZ)
sage: mor.extend_codomain(RDF)
Composite map:
  From: Integer Ring
  To:   Real Double Field
  Defn:   Natural morphism:
          From: Integer Ring
          To:   Rational Field
```

```
        then
          Native morphism:
          From: Rational Field
          To:   Real Double Field
sage: mor.extend_codomain(GF(7))
Traceback (most recent call last):
...
TypeError: No coercion from Rational Field to Finite Field of size 7
```

**extend_domain** (*new_domain*)

INPUT:

- `self` – a member of Hom(Y, Z)
- `new_codomain` – an object X such that there is a canonical coercion $\phi$ in Hom(X, Y)

OUTPUT:

An element of Hom(X, Z) obtained by composing self with $\phi$. If no canonical $\phi$ exists, a TypeError is raised.

EXAMPLES:

```
sage: mor = CDF.coerce_map_from(RDF)
sage: mor.extend_domain(QQ)
Composite map:
  From: Rational Field
  To:   Complex Double Field
  Defn:   Native morphism:
          From: Rational Field
          To:   Real Double Field
        then
          Native morphism:
          From: Real Double Field
          To:   Complex Double Field
sage: mor.extend_domain(ZZ['x'])
Traceback (most recent call last):
...
TypeError: No coercion from Univariate Polynomial Ring in x over Integer Ring␣
↪to Real Double Field
```

**is_surjective** ()

Tells whether the map is surjective (not implemented in the base class).

**parent** ()

Return the homset containing this map.

**Note:** The method `_make_weak_references()`, that is used for the maps found by the coercion system, needs to remove the usual strong reference from the coercion map to the homset containing it. As long as the user keeps strong references to domain and codomain of the map, we will be able to reconstruct the homset. However, a strong reference to the coercion map does not prevent the domain from garbage collection!

EXAMPLES:

```
sage: Q = QuadraticField(-5)
sage: phi = CDF._internal_convert_map_from(Q)
```

```
sage: print(phi.parent())
Set of field embeddings from Number Field in a with defining polynomial x^2 +
↪5 to Complex Double Field
```

We now demonstrate that the reference to the coercion map $\phi$ does not prevent $Q$ from being garbage collected:

```
sage: import gc
sage: del Q
sage: _ = gc.collect()
sage: phi.parent()
Traceback (most recent call last):
...
ValueError: This map is in an invalid state, the domain has been garbage
↪collected
```

You can still obtain copies of the maps used by the coercion system with strong references:

```
sage: Q = QuadraticField(-5)
sage: phi = CDF.convert_map_from(Q)
sage: print(phi.parent())
Set of field embeddings from Number Field in a with defining polynomial x^2 +
↪5 to Complex Double Field
sage: import gc
sage: del Q
sage: _ = gc.collect()
sage: phi.parent()
Set of field embeddings from Number Field in a with defining polynomial x^2 +
↪5 to Complex Double Field
```

**post_compose**(*left*)

INPUT:

- `self` – a Map in some `Hom(X, Y, category_right)`
- `left` – a Map in some `Hom(Y, Z, category_left)`

Returns the composition of `self` followed by `right` as a morphism in `Hom(X, Z, category)` where `category` is the meet of `category_left` and `category_right`.

Caveat: see the current restrictions on `Category.meet()`

EXAMPLES:

```
sage: from sage.categories.morphism import SetMorphism
sage: X.<x> = ZZ[]
sage: Y = ZZ
sage: Z = QQ
sage: phi_xy = SetMorphism(Hom(X, Y, Rings()), lambda p: p[0])
sage: phi_yz = SetMorphism(Hom(Y, Z, Monoids()), lambda y: QQ(y**2))
sage: phi_xz = phi_xy.post_compose(phi_yz); phi_xz
Composite map:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Rational Field
  Defn:   Generic morphism:
          From: Univariate Polynomial Ring in x over Integer Ring
          To:   Integer Ring
        then
          Generic morphism:
```

```
          From: Integer Ring
          To:   Rational Field
sage: phi_xz.category_for()
Category of monoids
```

**pre_compose** (*right*)

INPUT:

- `self` – a Map in some `Hom(Y, Z, category_left)`
- `left` – a Map in some `Hom(X, Y, category_right)`

Returns the composition of `right` followed by `self` as a morphism in `Hom(X, Z, category)` where `category` is the meet of `category_left` and `category_right`.

EXAMPLES:

```
sage: from sage.categories.morphism import SetMorphism
sage: X.<x> = ZZ[]
sage: Y = ZZ
sage: Z = QQ
sage: phi_xy = SetMorphism(Hom(X, Y, Rings()), lambda p: p[0])
sage: phi_yz = SetMorphism(Hom(Y, Z, Monoids()), lambda y: QQ(y**2))
sage: phi_xz = phi_yz.pre_compose(phi_xy); phi_xz
Composite map:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Rational Field
  Defn:   Generic morphism:
          From: Univariate Polynomial Ring in x over Integer Ring
          To:   Integer Ring
        then
          Generic morphism:
          From: Integer Ring
          To:   Rational Field
sage: phi_xz.category_for()
Category of monoids
```

**section** ()

Return a section of self.

NOTE:

By default, it returns `None`. You may override it in subclasses.

**class** sage.categories.map.**Section**

Bases: *sage.categories.map.Map*

A formal section of a map.

NOTE:

Call methods are not implemented for the base class `Section`.

EXAMPLES:

```
sage: from sage.categories.map import Section
sage: R.<x,y> = ZZ[]
sage: S.<a,b> = QQ[]
sage: f = R.hom([a+b, a-b])
sage: sf = Section(f); sf
Section map:
```

```
  From: Multivariate Polynomial Ring in a, b over Rational Field
  To:   Multivariate Polynomial Ring in x, y over Integer Ring
sage: sf(a)
Traceback (most recent call last):
...
NotImplementedError: <type 'sage.categories.map.Section'>
```

**inverse**()

Return inverse of `self`.

`sage.categories.map.`**is_Map**(*x*)

Auxiliary function: Is the argument a map?

EXAMPLES:

```
sage: R.<x,y> = QQ[]
sage: f = R.hom([x+y, x-y], R)
sage: from sage.categories.map import is_Map
sage: is_Map(f)
True
```

`sage.categories.map.`**unpickle_map**(*_class*, *parent*, *_dict*, *_slots*)

Auxiliary function for unpickling a map.

## 2.2 Homsets

The class *Hom* is the base class used to represent sets of morphisms between objects of a given category. *Hom* objects are usually "weakly" cached upon creation so that they don't have to be generated over and over but can be garbage collected together with the corresponding objects when these are not strongly ref'ed anymore.

EXAMPLES:

In the following, the *Hom* object is indeed cached:

```
sage: K = GF(17)
sage: H = Hom(ZZ, K)
sage: H
Set of Homomorphisms from Integer Ring to Finite Field of size 17
sage: H is Hom(ZZ, K)
True
```

Nonetheless, garbage collection occurs when the original references are overwritten:

```
sage: for p in prime_range(200):
....:     K = GF(p)
....:     H = Hom(ZZ, K)
sage: import gc
sage: _ = gc.collect()
sage: from sage.rings.finite_rings.finite_field_prime_modn import FiniteField_prime_
↪modn as FF
sage: L = [x for x in gc.get_objects() if isinstance(x, FF)]
sage: len(L)
1
sage: L
[Finite Field of size 199]
```

AUTHORS:

- David Kohel and William Stein
- David Joyner (2005-12-17): added examples
- William Stein (2006-01-14): Changed from Homspace to Homset.
- Nicolas M. Thiery (2008-12-): Updated for the new category framework
- Simon King (2011-12): Use a weak cache for homsets
- Simon King (2013-02): added examples

sage.categories.homset.**End**(*X*, *category=None*)

Create the set of endomorphisms of X in the category category.

INPUT:

- X – anything
- category – (optional) category in which to coerce X

OUTPUT:

A set of endomorphisms in category

EXAMPLES:

```
sage: V = VectorSpace(QQ, 3)
sage: End(V)
Set of Morphisms (Linear Transformations) from
Vector space of dimension 3 over Rational Field to
Vector space of dimension 3 over Rational Field
```

```
sage: G = AlternatingGroup(3)
sage: S = End(G); S
Set of Morphisms from Alternating group of order 3!/2 as a permutation group to
↪Alternating group of order 3!/2 as a permutation group in Category of finite
↪enumerated permutation groups
sage: from sage.categories.homset import is_Endset
sage: is_Endset(S)
True
sage: S.domain()
Alternating group of order 3!/2 as a permutation group
```

To avoid creating superfluous categories, a homset in a category Cs() is in the homset category of the lowest full super category Bs() of Cs() that implements Bs.Homsets (or the join thereof if there are several). For example, finite groups form a full subcategory of unital magmas: any unital magma morphism between two finite groups is a finite group morphism. Since finite groups currently implement nothing more than unital magmas about their homsets, we have:

```
sage: G = GL(3,3)
sage: G.category()
Category of finite groups
sage: H = Hom(G,G)
sage: H.homset_category()
Category of finite groups
sage: H.category()
Category of endsets of unital magmas
```

Similarly, a ring morphism just needs to preserve addition, multiplication, zero, and one. Accordingly, and since the category of rings implements nothing specific about its homsets, a ring homset is currently constructed in the category of homsets of unital magmas and unital additive magmas:

```
sage: H = Hom(ZZ,ZZ,Rings())
sage: H.category()
Category of endsets of unital magmas and additive unital additive magmas
```

`sage.categories.homset.Hom(X, Y, category=None, check=True)`

Create the space of homomorphisms from X to Y in the category `category`.

INPUT:

- `X` – an object of a category
- `Y` – an object of a category
- `category` – a category in which the morphisms must be. (default: the meet of the categories of `X` and `Y`) Both `X` and `Y` must belong to that category.
- `check` – a boolean (default: `True`): whether to check the input, and in particular that `X` and `Y` belong to `category`.

OUTPUT: a homset in category

EXAMPLES:

```
sage: V = VectorSpace(QQ,3)
sage: Hom(V, V)
Set of Morphisms (Linear Transformations) from
Vector space of dimension 3 over Rational Field to
Vector space of dimension 3 over Rational Field
sage: G = AlternatingGroup(3)
sage: Hom(G, G)
Set of Morphisms from Alternating group of order 3!/2 as a permutation group to
↪Alternating group of order 3!/2 as a permutation group in Category of finite
↪enumerated permutation groups
sage: Hom(ZZ, QQ, Sets())
Set of Morphisms from Integer Ring to Rational Field in Category of sets

sage: Hom(FreeModule(ZZ,1), FreeModule(QQ,1))
Set of Morphisms from Ambient free module of rank 1 over the principal ideal
↪domain Integer Ring to Vector space of dimension 1 over Rational Field in
↪Category of commutative additive groups
sage: Hom(FreeModule(QQ,1), FreeModule(ZZ,1))
Set of Morphisms from Vector space of dimension 1 over Rational Field to Ambient
↪free module of rank 1 over the principal ideal domain Integer Ring in Category
↪of commutative additive groups
```

Here, we test against a memory leak that has been fixed at trac ticket #11521 by using a weak cache:

```
sage: for p in prime_range(10^3):
....:  K = GF(p)
....:  a = K(0)
sage: import gc
sage: gc.collect()       # random
624
sage: from sage.rings.finite_rings.finite_field_prime_modn import FiniteField_
↪prime_modn as FF
sage: L = [x for x in gc.get_objects() if isinstance(x, FF)]
sage: len(L), L[0]
(1, Finite Field of size 997)
```

To illustrate the choice of the category, we consider the following parents as running examples:

```
sage: X = ZZ; X
Integer Ring
sage: Y = SymmetricGroup(3); Y
Symmetric group of order 3! as a permutation group
```

By default, the smallest category containing both X and Y, is used:

```
sage: Hom(X, Y)
Set of Morphisms from Integer Ring
 to Symmetric group of order 3! as a permutation group
 in Category of enumerated monoids
```

Otherwise, if category is specified, then category is used, after checking that X and Y are indeed in category:

```
sage: Hom(X, Y, Magmas())
Set of Morphisms from Integer Ring to Symmetric group of order 3! as a␣
↪permutation group in Category of magmas

sage: Hom(X, Y, Groups())
Traceback (most recent call last):
...
ValueError: Integer Ring is not in Category of groups
```

A parent (or a parent class of a category) may specify how to construct certain homsets by implementing a method _Hom_(self, codomain, category). This method should either construct the requested homset or raise a TypeError. This hook is currently mostly used to create homsets in some specific subclass of *Homset* (e.g. sage.rings.homset.RingHomset):

```
sage: Hom(QQ,QQ).__class__
<class 'sage.rings.homset.RingHomset_generic_with_category'>
```

Do not call this hook directly to create homsets, as it does not handle unique representation:

```
sage: Hom(QQ,QQ) == QQ._Hom_(QQ, category=QQ.category())
True
sage: Hom(QQ,QQ) is QQ._Hom_(QQ, category=QQ.category())
False
```

**Todo:**

- Design decision: how much of the homset comes from the category of X and Y, and how much from the specific X and Y. In particular, do we need several parent classes depending on X and Y, or does the difference only lie in the elements (i.e. the morphism), and of course how the parent calls their constructors.
- Specify the protocol for the _Hom_ hook in case of ambiguity (e.g. if both a parent and some category thereof provide one).

**class** sage.categories.homset.**Homset**(*X*, *Y*, *category=None*, *base=None*, *check=True*)

Bases: sage.structure.parent.Set_generic

The class for collections of morphisms in a category.

EXAMPLES:

```
sage: H = Hom(QQ^2, QQ^3)
sage: loads(H.dumps()) is H
True
```

Homsets of unique parents are unique as well:

```
sage: H = End(AffineSpace(2, names='x,y'))
sage: loads(dumps(AffineSpace(2, names='x,y'))) is AffineSpace(2, names='x,y')
True
sage: loads(dumps(H)) is H
True
```

Conversely, homsets of non-unique parents are non-unique:

sage: H = End(ProjectiveSpace(2, names='x,y,z')) sage: loads(dumps(ProjectiveSpace(2, names='x,y,z'))) is ProjectiveSpace(2, names='x,y,z') False sage: loads(dumps(ProjectiveSpace(2, names='x,y,z'))) == ProjectiveSpace(2, names='x,y,z') True sage: loads(dumps(H)) is H False sage: loads(dumps(H)) == H True

**codomain**()

Return the codomain of this homset.

EXAMPLES:

```
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
sage: f.parent().codomain()
Univariate Polynomial Ring in t over Rational Field
sage: f.codomain() is f.parent().codomain()
True
```

**domain**()

Return the domain of this homset.

EXAMPLES:

```
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
sage: f.parent().domain()
Univariate Polynomial Ring in t over Integer Ring
sage: f.domain() is f.parent().domain()
True
```

**element_class_set_morphism**()

A base class for elements of this homset which are also `SetMorphism`, i.e. implemented by mean of a Python function.

This is currently plain `SetMorphism`, without inheritance from categories.

**Todo:** Refactor during the upcoming homset cleanup.

EXAMPLES:

```
sage: H = Hom(ZZ, ZZ)
sage: H.element_class_set_morphism
<type 'sage.categories.morphism.SetMorphism'>
```

**homset_category**()

Return the category that this is a Hom in, i.e., this is typically the category of the domain or codomain object.

EXAMPLES:

```
sage: H = Hom(AlternatingGroup(4), AlternatingGroup(7))
sage: H.homset_category()
Category of finite enumerated permutation groups
```

**identity**()

The identity map of this homset.

**Note:** Of course, this only exists for sets of endomorphisms.

EXAMPLES:

```
sage: H = Hom(QQ,QQ)
sage: H.identity()
Identity endomorphism of Rational Field
sage: H = Hom(ZZ,QQ)
sage: H.identity()
Traceback (most recent call last):
...
TypeError: Identity map only defined for endomorphisms. Try natural_map()␣
↪instead.
sage: H.natural_map()
Natural morphism:
  From: Integer Ring
  To:   Rational Field
```

**natural_map**()

Return the "natural map" of this homset.

**Note:** By default, a formal coercion morphism is returned.

EXAMPLES:

```
sage: H = Hom(ZZ['t'],QQ['t'], CommutativeAdditiveGroups())
sage: H.natural_map()
Coercion morphism:
  From: Univariate Polynomial Ring in t over Integer Ring
  To:   Univariate Polynomial Ring in t over Rational Field
sage: H = Hom(QQ['t'],GF(3)['t'])
sage: H.natural_map()
Traceback (most recent call last):
...
TypeError: natural coercion morphism from Univariate Polynomial Ring in t␣
↪over Rational Field to Univariate Polynomial Ring in t over Finite Field of␣
↪size 3 not defined
```

**one**()

The identity map of this homset.

**Note:** Of course, this only exists for sets of endomorphisms.

EXAMPLES:

```
sage: K = GaussianIntegers()
sage: End(K).one()
Identity endomorphism of Gaussian Integers in Number Field in I with defining␣
↪polynomial x^2 + 1
```

**reversed**()

Return the corresponding homset, but with the domain and codomain reversed.

EXAMPLES:

```
sage: H = Hom(ZZ^2, ZZ^3); H
Set of Morphisms from Ambient free module of rank 2 over
 the principal ideal domain Integer Ring to Ambient free module
 of rank 3 over the principal ideal domain Integer Ring in
 Category of finite dimensional modules with basis over (euclidean
 domains and infinite enumerated sets and metric spaces)
sage: type(H)
<class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
sage: H.reversed()
Set of Morphisms from Ambient free module of rank 3 over
 the principal ideal domain Integer Ring to Ambient free module
 of rank 2 over the principal ideal domain Integer Ring in
 Category of finite dimensional modules with basis over (euclidean
 domains and infinite enumerated sets and metric spaces)
sage: type(H.reversed())
<class 'sage.modules.free_module_homspace.FreeModuleHomspace_with_category'>
```

**class** sage.categories.homset.**HomsetWithBase**(*X*, *Y*, *category=None*, *check=True*, *base=None*)

Bases: *sage.categories.homset.Homset*

sage.categories.homset.**end**(*X*, *f*)

Return `End(X)(f)`, where `f` is data that defines an element of `End(X)`.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: phi = end(R, [x + 1])
sage: phi
Ring endomorphism of Univariate Polynomial Ring in x over Rational Field
  Defn: x |--> x + 1
sage: phi(x^2 + 5)
x^2 + 2*x + 6
```

sage.categories.homset.**hom**(*X*, *Y*, *f*)

Return `Hom(X,Y)(f)`, where `f` is data that defines an element of `Hom(X,Y)`.

EXAMPLES:

```
sage: phi = hom(QQ['x'], QQ, [2])
sage: phi(x^2 + 3)
7
```

sage.categories.homset.**is_Endset**(*x*)

Return True if x is a set of endomorphisms in a category.

EXAMPLES:

```
sage: from sage.categories.homset import is_Endset
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
sage: is_Endset(f.parent())
False
sage: g = P.hom([2*t])
sage: is_Endset(g.parent())
True
```

sage.categories.homset.**is_Homset**(*x*)

Return True if x is a set of homomorphisms in a category.

EXAMPLES:

```
sage: from sage.categories.homset import is_Homset
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
sage: is_Homset(f)
False
sage: is_Homset(f.category())
False
sage: is_Homset(f.parent())
True
```

## 2.3 Morphisms

AUTHORS:

- William Stein: initial version
- David Joyner (12-17-2005): added examples
- Robert Bradshaw (2007-06-25) Pyrexification

**class** sage.categories.morphism.**CallMorphism**

Bases: *sage.categories.morphism.Morphism*

**class** sage.categories.morphism.**FormalCoercionMorphism**

Bases: *sage.categories.morphism.Morphism*

**class** sage.categories.morphism.**IdentityMorphism**

Bases: *sage.categories.morphism.Morphism*

**is_identity**()

Return True if this morphism is the identity morphism.

EXAMPLES:

```
sage: E = End(Partitions(5))
sage: E.identity().is_identity()
True
```

Check that trac ticket #15478 is fixed:

```
sage: K.<z> = GF(4)
sage: phi = End(K)([z^2])
sage: R.<t> = K[]
sage: psi = End(R)(phi)
sage: psi.is_identity()
False
```

**is_injective**()

Return whether this morphism is injective.

EXAMPLES:

```
sage: Hom(ZZ, ZZ).identity().is_injective()
True
```

**is_surjective**()

Return whether this morphism is surjective.

EXAMPLES:

```
sage: Hom(ZZ, ZZ).identity().is_surjective()
True
```

**class** sage.categories.morphism.**Morphism**

Bases: *sage.categories.map.Map*

**category**()

Return the category of the parent of this morphism.

EXAMPLES:

```
sage: R.<t> = ZZ[]
sage: f = R.hom([t**2])
sage: f.category()
Category of endsets of unital magmas and right modules over
 (euclidean domains and infinite enumerated sets and metric spaces)
 and left modules over (euclidean domains
 and infinite enumerated sets and metric spaces)

sage: K = CyclotomicField(12)
sage: L = CyclotomicField(132)
sage: phi = L._internal_coerce_map_from(K)
sage: phi.category()
Category of homsets of number fields
```

**is_endomorphism**()

Return `True` if this morphism is an endomorphism.

EXAMPLES:

```
sage: R.<t> = ZZ[]
sage: f = R.hom([t])
sage: f.is_endomorphism()
True

sage: K = CyclotomicField(12)
sage: L = CyclotomicField(132)
sage: phi = L._internal_coerce_map_from(K)
```

```
sage: phi.is_endomorphism()
False
```

**is_identity()**

Return `True` if this morphism is the identity morphism.

**Note:** Implemented only when the domain has a method gens()

EXAMPLES:

```
sage: R.<t> = ZZ[]
sage: f = R.hom([t])
sage: f.is_identity()
True
sage: g = R.hom([t+1])
sage: g.is_identity()
False
```

A morphism between two different spaces cannot be the identity:

```
sage: R2.<t2> = QQ[]
sage: h = R.hom([t2])
sage: h.is_identity()
False
```

**pushforward**(*I*)

**register_as_coercion()**

Register this morphism as a coercion to Sage's coercion model (see `sage.structure.coerce`).

EXAMPLES:

By default, adding polynomials over different variables triggers an error:

```
sage: X.<x> = ZZ[]
sage: Y.<y> = ZZ[]
sage: x^2 + y
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +: 'Univariate Polynomial Ring
↪in x over Integer Ring' and 'Univariate Polynomial Ring in y over Integer
↪Ring'
```

Let us declare a coercion from $\mathbf{Z}[x]$ to $\mathbf{Z}[z]$:

```
sage: Z.<z> = ZZ[]
sage: phi = Hom(X, Z)(z)
sage: phi(x^2+1)
z^2 + 1
sage: phi.register_as_coercion()
```

Now we can add elements from $\mathbf{Z}[x]$ and $\mathbf{Z}[z]$, because the elements of the former are allowed to be implicitly coerced into the later:

```
sage: x^2 + z
z^2 + z
```

Caveat: the registration of the coercion must be done before any other coercion is registered or discovered:

```
sage: phi = Hom(X, Y)(y)
sage: phi.register_as_coercion()
Traceback (most recent call last):
...
AssertionError: coercion from Univariate Polynomial Ring in x over Integer␣
↪Ring to Univariate Polynomial Ring in y over Integer Ring already␣
↪registered or discovered
```

**register_as_conversion**()

Register this morphism as a conversion to Sage's coercion model

(see `sage.structure.coerce`).

EXAMPLES:

Let us declare a conversion from the symmetric group to $\mathbf{Z}$ through the sign map:

```
sage: S = SymmetricGroup(4)
sage: phi = Hom(S, ZZ)(lambda x: ZZ(x.sign()))
sage: x = S.an_element(); x
(1,2,3,4)
sage: phi(x)
-1
sage: phi.register_as_conversion()
sage: ZZ(x)
-1
```

**class** sage.categories.morphism.**SetMorphism**

Bases: *sage.categories.morphism.Morphism*

INPUT:

- `parent` – a Homset
- `function` – a Python function that takes elements of the domain as input and returns elements of the domain.

EXAMPLES:

```
sage: from sage.categories.morphism import SetMorphism
sage: f = SetMorphism(Hom(QQ, ZZ, Sets()), numerator)
sage: f.parent()
Set of Morphisms from Rational Field to Integer Ring in Category of sets
sage: f.domain()
Rational Field
sage: f.codomain()
Integer Ring
sage: TestSuite(f).run()
```

sage.categories.morphism.**is_Morphism**(*x*)

## 2.4 Coercion via Construction Functors

**class** sage.categories.pushout.**AlgebraicClosureFunctor**

Bases: *sage.categories.pushout.ConstructionFunctor*

Algebraic Closure.

EXAMPLES:

```
sage: F = CDF.construction()[0]
sage: F(QQ)
Algebraic Field
sage: F(RR)
Complex Field with 53 bits of precision
sage: F(F(QQ)) is F(QQ)
True
```

**merge** (*other*)

Mathematically, Algebraic Closure subsumes Algebraic Extension. However, it seems that people do want to work with algebraic extensions of RR. Therefore, we do not merge with algebraic extension.

**class** sage.categories.pushout.**AlgebraicExtensionFunctor** (*polys*, *names*, *embeddings=None*, *structures=None*, *cyclotomic=None*, ***kwds*)

Bases: *sage.categories.pushout.ConstructionFunctor*

Algebraic extension (univariate polynomial ring modulo principal ideal).

EXAMPLES:

```
sage: K.<a> = NumberField(x^3+x^2+1)
sage: F = K.construction()[0]
sage: F(ZZ['t'])
Univariate Quotient Polynomial Ring in a over Univariate Polynomial Ring in t␣
↪over Integer Ring with modulus a^3 + a^2 + 1
```

Note that, even if a field is algebraically closed, the algebraic extension will be constructed as the quotient of a univariate polynomial ring:

```
sage: F(CC)
Univariate Quotient Polynomial Ring in a over Complex Field with 53 bits of␣
↪precision with modulus a^3 + a^2 + 1.00000000000000
sage: F(RR)
Univariate Quotient Polynomial Ring in a over Real Field with 53 bits of␣
↪precision with modulus a^3 + a^2 + 1.00000000000000
```

Note that the construction functor of a number field applied to the integers returns an order (not necessarily maximal) of that field, similar to the behaviour of `ZZ.extension(...)`:

```
sage: F(ZZ)
Order in Number Field in a with defining polynomial x^3 + x^2 + 1
```

This also holds for non-absolute number fields:

```
sage: K.<a,b> = NumberField([x^3+x^2+1,x^2+x+1])
sage: F = K.construction()[0]
sage: O = F(ZZ); O
Relative Order in Number Field in a with defining polynomial x^3 + x^2 + 1 over␣
↪its base field
sage: O.ambient() is K
True
```

**expand** ()

Decompose the functor $F$ into sub-functors, whose product returns $F$.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: K.<a> = NumberField(x^3-5,embedding=0)
sage: L.<b> = K.extension(x^2+a)
sage: F,R = L.construction()
sage: prod(F.expand())(R) == L
True
sage: K = NumberField([x^2-2, x^2-3],'a')
sage: F, R = K.construction()
sage: F
AlgebraicExtensionFunctor
sage: L = F.expand(); L
[AlgebraicExtensionFunctor, AlgebraicExtensionFunctor]
sage: L[-1](QQ)
Number Field in a1 with defining polynomial x^2 - 3
```

**merge** (*other*)

Merging with another `AlgebraicExtensionFunctor`.

INPUT:

`other` – Construction Functor.

OUTPUT:

- If `self==other`, `self` is returned.
- If `self` and `other` are simple extensions and both provide an embedding, then it is tested whether one of the number fields provided by the functors coerces into the other; the functor associated with the target of the coercion is returned. Otherwise, the construction functor associated with the pushout of the codomains of the two embeddings is returned, provided that it is a number field.
- If these two extensions are defined by Conway polynomials over finite fields, merges them into a single extension of degree the lcm of the two degrees.
- Otherwise, None is returned.

REMARK:

Algebraic extension with embeddings currently only works when applied to the rational field. This is why we use the admittedly strange rule above for merging.

EXAMPLES:

The following demonstrate coercions for finite fields using Conway or pseudo-Conway polynomials:

```
sage: k = GF(3^2, prefix='z'); a = k.gen()
sage: l = GF(3^3, prefix='z'); b = l.gen()
sage: a + b # indirect doctest
z6^5 + 2*z6^4 + 2*z6^3 + z6^2 + 2*z6 + 1
```

Note that embeddings are compatible in lattices of such finite fields:

```
sage: m = GF(3^5, prefix='z'); c = m.gen()
sage: (a+b)+c == a+(b+c) # indirect doctest
True
sage: from sage.categories.pushout import pushout
sage: n = pushout(k, l)
sage: o = pushout(l, m)
sage: q = pushout(n, o)
sage: q(o(b)) == q(n(b)) # indirect doctest
True
```

Coercion is also available for number fields:

```
sage: P.<x> = QQ[]
sage: L.<b> = NumberField(x^8-x^4+1, embedding=CDF.0)
sage: M1.<c1> = NumberField(x^2+x+1, embedding=b^4-1)
sage: M2.<c2> = NumberField(x^2+1, embedding=-b^6)
sage: M1.coerce_map_from(M2)
sage: M2.coerce_map_from(M1)
sage: c1+c2; parent(c1+c2)    #indirect doctest
-b^6 + b^4 - 1
Number Field in b with defining polynomial x^8 - x^4 + 1
sage: pushout(M1['x'],M2['x'])
Univariate Polynomial Ring in x over Number Field in b with defining␣
↪polynomial x^8 - x^4 + 1
```

In the previous example, the number field `L` becomes the pushout of `M1` and `M2` since both are provided with an embedding into `L`, *and* since `L` is a number field. If two number fields are embedded into a field that is not a numberfield, no merging occurs:

```
sage: K.<a> = NumberField(x^3-2, embedding=CDF(1/2*I*2^(1/3)*sqrt(3) - 1/2*2^
↪(1/3)))
sage: L.<b> = NumberField(x^6-2, embedding=1.1)
sage: L.coerce_map_from(K)
sage: K.coerce_map_from(L)
sage: pushout(K,L)
Traceback (most recent call last):
...
CoercionException: ('Ambiguous Base Extension', Number Field in a with␣
↪defining polynomial x^3 - 2, Number Field in b with defining polynomial x^6␣
↪- 2)
```

**class** sage.categories.pushout.**BlackBoxConstructionFunctor**(*box*)

Bases: *sage.categories.pushout.ConstructionFunctor*

Construction functor obtained from any callable object.

EXAMPLES:

```
sage: from sage.categories.pushout import BlackBoxConstructionFunctor
sage: FG = BlackBoxConstructionFunctor(gap)
sage: FS = BlackBoxConstructionFunctor(singular)
sage: FG
BlackBoxConstructionFunctor
sage: FG(ZZ)
Integers
sage: FG(ZZ).parent()
Gap
sage: FS(QQ['t'])
polynomial ring, over a field, global ordering
//   coefficients: QQ
//   number of vars : 1
//        block   1 : ordering lp
//                  : names    t
//        block   2 : ordering C
sage: FG == FS
False
sage: FG == loads(dumps(FG))
True
```

**class** sage.categories.pushout.**CompletionFunctor**(*p*, *prec*, *extras=None*)

Bases: *sage.categories.pushout.ConstructionFunctor*

Completion of a ring with respect to a given prime (including infinity).

EXAMPLES:

```
sage: R = Zp(5)
sage: R
5-adic Ring with capped relative precision 20
sage: F1 = R.construction()[0]
sage: F1
Completion[5, prec=20]
sage: F1(ZZ) is R
True
sage: F1(QQ)
5-adic Field with capped relative precision 20
sage: F2 = RR.construction()[0]
sage: F2
Completion[+Infinity, prec=53]
sage: F2(QQ) is RR
True
sage: P.<x> = ZZ[]
sage: Px = P.completion(x) # currently the only implemented completion of P
sage: Px
Power Series Ring in x over Integer Ring
sage: F3 = Px.construction()[0]
sage: F3(GF(3)['x'])
Power Series Ring in x over Finite Field of size 3
```

**commutes**(*other*)

Completion commutes with fraction fields.

EXAMPLES:

```
sage: F1 = Qp(5).construction()[0]
sage: F2 = QQ.construction()[0]
sage: F1.commutes(F2)
True
```

**merge**(*other*)

Two Completion functors are merged, if they are equal. If the precisions of both functors coincide, then a Completion functor is returned that results from updating the `extras` dictionary of `self` by `other.extras`. Otherwise, if the completion is at infinity then merging does not increase the set precision, and if the completion is at a finite prime, merging does not decrease the capped precision.

EXAMPLES:

```
sage: R1.<a> = Zp(5,prec=20)[]
sage: R2 = Qp(5,prec=40)
sage: R2(1)+a          # indirect doctest
(1 + O(5^20))*a + (1 + O(5^40))
sage: R3 = RealField(30)
sage: R4 = RealField(50)
sage: R3(1) + R4(1)    # indirect doctest
2.0000000
sage: (R3(1) + R4(1)).parent()
Real Field with 30 bits of precision
```

```
sage: P0 = ZpFM(5, 10)
sage: P1 = ZpFM(5, 20)
sage: P2 = ZpCR(5, 10)
sage: P3 = ZpCR(5, 20)
sage: P4 = ZpCA(5, 10)
sage: P5 = ZpCA(5, 20)
sage: P6 = Qp(5, 10)
sage: P7 = Qp(5, 20)
sage: Plist = [P2,P3,P4,P5,P6,P7]
sage: from sage.categories.pushout import pushout
sage: pushouts = [P2,P3,P4,P5,P6,P7,P3,P3,P5,P5,P7,P7,P4,P5,P4,P5,P6,P7,P5,P5,
↪P5,P5,P7,P7,P6,P7,P6,P7,P6,P7,P7,P7,P7,P7,P7,P7]
sage: all([P is Q for P, Q in zip(pushouts, [pushout(a, b) for a in Plist for␣
↪b in Plist])])
True
```

**class** sage.categories.pushout.**CompositeConstructionFunctor**(**args*)

Bases: *sage.categories.pushout.ConstructionFunctor*

A Construction Functor composed by other Construction Functors.

INPUT:

`F1, F2,...`: A list of Construction Functors. The result is the composition `F1` followed by `F2` followed by `...`

EXAMPLES:

```
sage: from sage.categories.pushout import CompositeConstructionFunctor
sage: F = CompositeConstructionFunctor(QQ.construction()[0],ZZ['x'].
↪construction()[0],QQ.construction()[0],ZZ['y'].construction()[0])
sage: F
Poly[y](FractionField(Poly[x](FractionField(...))))
sage: F == loads(dumps(F))
True
sage: F == CompositeConstructionFunctor(*F.all)
True
sage: F(GF(2)['t'])
Univariate Polynomial Ring in y over Fraction Field of Univariate Polynomial Ring␣
↪in x over Fraction Field of Univariate Polynomial Ring in t over Finite Field␣
↪of size 2 (using GF2X)
```

**expand**()

Return expansion of a CompositeConstructionFunctor.

NOTE:

The product over the list of components, as returned by the `expand()` method, is equal to `self`.

EXAMPLES:

```
sage: from sage.categories.pushout import CompositeConstructionFunctor
sage: F = CompositeConstructionFunctor(QQ.construction()[0],ZZ['x'].
↪construction()[0],QQ.construction()[0],ZZ['y'].construction()[0])
sage: F
Poly[y](FractionField(Poly[x](FractionField(...))))
sage: prod(F.expand()) == F
True
```

**class** sage.categories.pushout.**ConstructionFunctor**

Bases: *sage.categories.functor.Functor*

Base class for construction functors.

A construction functor is a functorial algebraic construction, such as the construction of a matrix ring over a given ring or the fraction field of a given ring.

In addition to the class *Functor*, construction functors provide rules for combining and merging constructions. This is an important part of Sage's coercion model, namely the pushout of two constructions: When a polynomial `p` in a variable `x` with integer coefficients is added to a rational number `q`, then Sage finds that the parents `ZZ['x']` and `QQ` are obtained from `ZZ` by applying a polynomial ring construction respectively the fraction field construction. Each construction functor has an attribute `rank`, and the rank of the polynomial ring construction is higher than the rank of the fraction field construction. This means that the pushout of `QQ` and `ZZ['x']`, and thus a common parent in which `p` and `q` can be added, is `QQ['x']`, since the construction functor with a lower rank is applied first.

```
sage: F1, R = QQ.construction()
sage: F1
FractionField
sage: R
Integer Ring
sage: F2, R = (ZZ['x']).construction()
sage: F2
Poly[x]
sage: R
Integer Ring
sage: F3 = F2.pushout(F1)
sage: F3
Poly[x](FractionField(...))
sage: F3(R)
Univariate Polynomial Ring in x over Rational Field
sage: from sage.categories.pushout import pushout
sage: P.<x> = ZZ[]
sage: pushout(QQ,P)
Univariate Polynomial Ring in x over Rational Field
sage: ((x+1) + 1/2).parent()
Univariate Polynomial Ring in x over Rational Field
```

When composing two construction functors, they are sometimes merged into one, as is the case in the Quotient construction:

```
sage: Q15, R = (ZZ.quo(15*ZZ)).construction()
sage: Q15
QuotientFunctor
sage: Q35, R = (ZZ.quo(35*ZZ)).construction()
sage: Q35
QuotientFunctor
sage: Q15.merge(Q35)
QuotientFunctor
sage: Q15.merge(Q35)(ZZ)
Ring of integers modulo 5
```

Functors can not only be applied to objects, but also to morphisms in the respective categories. For example:

```
sage: P.<x,y> = ZZ[]
sage: F = P.construction()[0]; F
MPoly[x,y]
sage: A.<a,b> = GF(5)[]
```

```
sage: f = A.hom([a+b,a-b],A)
sage: F(A)
Multivariate Polynomial Ring in x, y over Multivariate Polynomial Ring in a, b␣
↪over Finite Field of size 5
sage: F(f)
Ring endomorphism of Multivariate Polynomial Ring in x, y over Multivariate␣
↪Polynomial Ring in a, b over Finite Field of size 5
  Defn: Induced from base ring by
        Ring endomorphism of Multivariate Polynomial Ring in a, b over Finite␣
↪Field of size 5
          Defn: a |--> a + b
                b |--> a - b
sage: F(f)(F(A)(x)*a)
(a + b)*x
```

**common_base**(*other_functor*, *self_bases*, *other_bases*)

This function is called by *pushout()* when no common parent is found in the construction tower.

---

**Note:** The main use is for multivariate construction functors, which use this function to implement recursion for *pushout()*.

---

INPUT:

- `other_functor` – a construction functor.
- `self_bases` – the arguments passed to this functor.
- `other_bases` – the arguments passed to the functor `other_functor`.

OUTPUT:

Nothing, since a `CoercionException` is raised.

---

**Note:** Overload this function in derived class, see e.e. *MultivariateConstructionFunctor*.

---

**commutes**(*other*)

Determine whether `self` commutes with another construction functor.

NOTE:

By default, `False` is returned in all cases (even if the two functors are the same, since in this case *merge()* will apply anyway). So far there is no construction functor that overloads this method. Anyway, this method only becomes relevant if two construction functors have the same rank.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: P = ZZ['t'].construction()[0]
sage: F.commutes(P)
False
sage: P.commutes(F)
False
sage: F.commutes(F)
False
```

**expand**()

Decompose `self` into a list of construction functors.

NOTE:

The default is to return the list only containing `self`.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: F.expand()
[FractionField]
sage: Q = ZZ.quo(2).construction()[0]
sage: Q.expand()
[QuotientFunctor]
sage: P = ZZ['t'].construction()[0]
sage: FP = F*P
sage: FP.expand()
[FractionField, Poly[t]]
```

**merge** (*other*)

Merge `self` with another construction functor, or return None.

NOTE:

The default is to merge only if the two functors coincide. But this may be overloaded for subclasses, such as the quotient functor.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: P = ZZ['t'].construction()[0]
sage: F.merge(F)
FractionField
sage: F.merge(P)
sage: P.merge(F)
sage: P.merge(P)
Poly[t]
```

**pushout** (*other*)

Composition of two construction functors, ordered by their ranks.

NOTE:

- This method seems not to be used in the coercion model.
- By default, the functor with smaller rank is applied first.

**class** sage.categories.pushout.**FractionField**

Bases: *sage.categories.pushout.ConstructionFunctor*

Construction functor for fraction fields.

EXAMPLES:

```
sage: F = QQ.construction()[0]
sage: F
FractionField
sage: F.domain()
Category of integral domains
sage: F.codomain()
Category of fields
sage: F(GF(5)) is GF(5)
True
sage: F(ZZ['t'])
```

```
Fraction Field of Univariate Polynomial Ring in t over Integer Ring
sage: P.<x,y> = QQ[]
sage: f = P.hom([x+2*y,3*x-y],P)
sage: F(f)
Ring endomorphism of Fraction Field of Multivariate Polynomial Ring in x, y over␣
↪Rational Field
  Defn: x |--> x + 2*y
        y |--> 3*x - y
sage: F(f)(1/x)
1/(x + 2*y)
sage: F == loads(dumps(F))
True
```

**class** sage.categories.pushout.**IdentityConstructionFunctor**

Bases: *sage.categories.pushout.ConstructionFunctor*

A construction functor that is the identity functor.

**class** sage.categories.pushout.**InfinitePolynomialFunctor**(*gens*, *order*, *implementation*)

Bases: *sage.categories.pushout.ConstructionFunctor*

A Construction Functor for Infinite Polynomial Rings (see `infinite_polynomial_ring`).

AUTHOR:

– Simon King

This construction functor is used to provide uniqueness of infinite polynomial rings as parent structures. As usual, the construction functor allows for constructing pushouts.

Another purpose is to avoid name conflicts of variables of the to-be-constructed infinite polynomial ring with variables of the base ring, and moreover to keep the internal structure of an Infinite Polynomial Ring as simple as possible: If variables $v_1, ..., v_n$ of the given base ring generate an *ordered* sub-monoid of the monomials of the ambient Infinite Polynomial Ring, then they are removed from the base ring and merged with the generators of the ambient ring. However, if the orders don't match, an error is raised, since there was a name conflict without merging.

EXAMPLES:

```
sage: A.<a,b> = InfinitePolynomialRing(ZZ['t'])
sage: A.construction()
[InfPoly{[a,b], "lex", "dense"},
 Univariate Polynomial Ring in t over Integer Ring]
sage: type(_[0])
<class 'sage.categories.pushout.InfinitePolynomialFunctor'>
sage: B.<x,y,a_3,a_1> = PolynomialRing(QQ, order='lex')
sage: B.construction()
(MPoly[x,y,a_3,a_1], Rational Field)
sage: A.construction()[0]*B.construction()[0]
InfPoly{[a,b], "lex", "dense"}(MPoly[x,y](...))
```

Apparently the variables $a_1, a_3$ of the polynomial ring are merged with the variables $a_0, a_1, a_2, ...$ of the infinite polynomial ring; indeed, they form an ordered sub-structure. However, if the polynomial ring was given a different ordering, merging would not be allowed, resulting in a name conflict:

```
sage: A.construction()[0]*PolynomialRing(QQ,names=['x','y','a_3','a_1']).
↪construction()[0]
Traceback (most recent call last):
```

```
...
CoercionException: Incompatible term orders lex, degrevlex
```

In an infinite polynomial ring with generator $a_*$, the variable $a_3$ will always be greater than the variable $a_1$. Hence, the orders are incompatible in the next example as well:

```
sage: A.construction()[0]*PolynomialRing(QQ,names=['x','y','a_1','a_3'], order=
↪'lex').construction()[0]
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('a', 'b'),['a_1', 'a_3']) are␣
↪incompatible
```

Another requirement is that after merging the order of the remaining variables must be unique. This is not the case in the following example, since it is not clear whether the variables $x, y$ should be greater or smaller than the variables $b_*$:

```
sage: A.construction()[0]*PolynomialRing(QQ,names=['a_3','a_1','x','y'], order=
↪'lex').construction()[0]
Traceback (most recent call last):
...
CoercionException: Overlapping variables (('a', 'b'),['a_3', 'a_1']) are␣
↪incompatible
```

Since the construction functors are actually used to construct infinite polynomial rings, the following result is no surprise:

```
sage: C.<a,b> = InfinitePolynomialRing(B); C
Infinite polynomial ring in a, b over Multivariate Polynomial Ring in x, y over␣
↪Rational Field
```

There is also an overlap in the next example:

```
sage: X.<w,x,y> = InfinitePolynomialRing(ZZ)
sage: Y.<x,y,z> = InfinitePolynomialRing(QQ)
```

$X$ and $Y$ have an overlapping generators $x_*, y_*$. Since the default lexicographic order is used in both rings, it gives rise to isomorphic sub-monoids in both $X$ and $Y$. They are merged in the pushout, which also yields a common parent for doing arithmetic:

```
sage: P = sage.categories.pushout.pushout(Y,X); P
Infinite polynomial ring in w, x, y, z over Rational Field
sage: w[2]+z[3]
w_2 + z_3
sage: _.parent() is P
True
```

**expand**()

Decompose the functor $F$ into sub-functors, whose product returns $F$.

EXAMPLES:

```
sage: F = InfinitePolynomialRing(QQ, ['x','y'],order='degrevlex').
↪construction()[0]; F
InfPoly{[x,y], "degrevlex", "dense"}
sage: F.expand()
[InfPoly{[y], "degrevlex", "dense"}, InfPoly{[x], "degrevlex", "dense"}]
```

```
sage: F = InfinitePolynomialRing(QQ, ['x','y','z'],order='degrevlex').
↪construction()[0]; F
InfPoly{[x,y,z], "degrevlex", "dense"}
sage: F.expand()
[InfPoly{[z], "degrevlex", "dense"},
 InfPoly{[y], "degrevlex", "dense"},
 InfPoly{[x], "degrevlex", "dense"}]
sage: prod(F.expand())==F
True
```

**merge** (*other*)

Merge two construction functors of infinite polynomial rings, regardless of monomial order and implementation.

The purpose is to have a pushout (and thus, arithmetic) even in cases when the parents are isomorphic as rings, but not as ordered rings.

EXAMPLES:

```
sage: X.<x,y> = InfinitePolynomialRing(QQ,implementation='sparse')
sage: Y.<x,y> = InfinitePolynomialRing(QQ,order='degrevlex')
sage: X.construction()
[InfPoly{[x,y], "lex", "sparse"}, Rational Field]
sage: Y.construction()
[InfPoly{[x,y], "degrevlex", "dense"}, Rational Field]
sage: Y.construction()[0].merge(Y.construction()[0])
InfPoly{[x,y], "degrevlex", "dense"}
sage: y[3] + X(x[2])
x_2 + y_3
sage: _.parent().construction()
[InfPoly{[x,y], "degrevlex", "dense"}, Rational Field]
```

**class** sage.categories.pushout.**LaurentPolynomialFunctor** (*var*, *multi_variate=False*)

Bases: *sage.categories.pushout.ConstructionFunctor*

Construction functor for Laurent polynomial rings.

EXAMPLES:

```
sage: L.<t> = LaurentPolynomialRing(ZZ)
sage: F = L.construction()[0]
sage: F
LaurentPolynomialFunctor
sage: F(QQ)
Univariate Laurent Polynomial Ring in t over Rational Field
sage: K.<x> = LaurentPolynomialRing(ZZ)
sage: F(K)
Univariate Laurent Polynomial Ring in t over Univariate Laurent Polynomial Ring␣
↪in x over Integer Ring
sage: P.<x,y> = ZZ[]
sage: f = P.hom([x+2*y,3*x-y],P)
sage: F(f)
Ring endomorphism of Univariate Laurent Polynomial Ring in t over Multivariate␣
↪Polynomial Ring in x, y over Integer Ring
  Defn: Induced from base ring by
        Ring endomorphism of Multivariate Polynomial Ring in x, y over Integer␣
↪Ring
          Defn: x |--> x + 2*y
                y |--> 3*x - y
```

```
sage: F(f)(x*F(P).gen()^-2+y*F(P).gen()^3)
(x + 2*y)*t^-2 + (3*x - y)*t^3
```

**merge**(*other*)

Two Laurent polynomial construction functors merge if the variable names coincide. The result is multivariate if one of the arguments is multivariate.

EXAMPLES:

```
sage: from sage.categories.pushout import LaurentPolynomialFunctor
sage: F1 = LaurentPolynomialFunctor('t')
sage: F2 = LaurentPolynomialFunctor('t', multi_variate=True)
sage: F1.merge(F2)
LaurentPolynomialFunctor
sage: F1.merge(F2)(LaurentPolynomialRing(GF(2),'a'))
Multivariate Laurent Polynomial Ring in a, t over Finite Field of size 2
sage: F1.merge(F1)(LaurentPolynomialRing(GF(2),'a'))
Univariate Laurent Polynomial Ring in t over Univariate Laurent Polynomial
↪Ring in a over Finite Field of size 2
```

**class** sage.categories.pushout.**MatrixFunctor**(*nrows*, *ncols*, *is_sparse=False*)

Bases: *sage.categories.pushout.ConstructionFunctor*

A construction functor for matrices over rings.

EXAMPLES:

```
sage: MS = MatrixSpace(ZZ,2, 3)
sage: F = MS.construction()[0]; F
MatrixFunctor
sage: MS = MatrixSpace(ZZ,2)
sage: F = MS.construction()[0]; F
MatrixFunctor
sage: P.<x,y> = QQ[]
sage: R = F(P); R
Full MatrixSpace of 2 by 2 dense matrices over Multivariate Polynomial Ring in x,
↪y over Rational Field
sage: f = P.hom([x+y,x-y],P); F(f)
Ring endomorphism of Full MatrixSpace of 2 by 2 dense matrices over Multivariate
↪Polynomial Ring in x, y over Rational Field
  Defn: Induced from base ring by
        Ring endomorphism of Multivariate Polynomial Ring in x, y over Rational
↪Field
          Defn: x |--> x + y
                y |--> x - y
sage: M = R([x,y,x*y,x+y])
sage: F(f)(M)
[    x + y     x - y]
[x^2 - y^2       2*x]
```

**merge**(*other*)

Merging is only happening if both functors are matrix functors of the same dimension. The result is sparse if and only if both given functors are sparse.

EXAMPLES:

```
sage: F1 = MatrixSpace(ZZ,2,2).construction()[0]
sage: F2 = MatrixSpace(ZZ,2,3).construction()[0]
sage: F3 = MatrixSpace(ZZ,2,2,sparse=True).construction()[0]
```

```
sage: F1.merge(F2)
sage: F1.merge(F3)
MatrixFunctor
sage: F13 = F1.merge(F3)
sage: F13.is_sparse
False
sage: F1.is_sparse
False
sage: F3.is_sparse
True
sage: F3.merge(F3).is_sparse
True
```

**class** sage.categories.pushout.**MultiPolynomialFunctor**(*vars*, *term_order*)

Bases: *sage.categories.pushout.ConstructionFunctor*

A constructor for multivariate polynomial rings.

EXAMPLES:

```
sage: P.<x,y> = ZZ[]
sage: F = P.construction()[0]; F
MPoly[x,y]
sage: A.<a,b> = GF(5)[]
sage: F(A)
Multivariate Polynomial Ring in x, y over Multivariate Polynomial Ring in a, b␣
↪over Finite Field of size 5
sage: f = A.hom([a+b,a-b],A)
sage: F(f)
Ring endomorphism of Multivariate Polynomial Ring in x, y over Multivariate␣
↪Polynomial Ring in a, b over Finite Field of size 5
  Defn: Induced from base ring by
        Ring endomorphism of Multivariate Polynomial Ring in a, b over Finite␣
↪Field of size 5
          Defn: a |--> a + b
                b |--> a - b
sage: F(f)(F(A)(x)*a)
(a + b)*x
```

**expand**()

Decompose `self` into a list of construction functors.

EXAMPLES:

```
sage: F = QQ['x,y,z,t'].construction()[0]; F
MPoly[x,y,z,t]
sage: F.expand()
[MPoly[t], MPoly[z], MPoly[y], MPoly[x]]
```

Now an actual use case:

```
sage: R.<x,y,z> = ZZ[]
sage: S.<z,t> = QQ[]
sage: x+t
x + t
sage: parent(x+t)
Multivariate Polynomial Ring in x, y, z, t over Rational Field
sage: T.<y,s> = QQ[]
sage: x + s
```

```
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +: 'Multivariate Polynomial Ring␣
↪in x, y, z over Integer Ring' and 'Multivariate Polynomial Ring in y, s␣
↪over Rational Field'
sage: R = PolynomialRing(ZZ, 'x', 500)
sage: S = PolynomialRing(GF(5), 'x', 200)
sage: R.gen(0) + S.gen(0)
2*x0
```

**merge** (*other*)

Merge `self` with another construction functor, or return None.

EXAMPLES:

```
sage: F = sage.categories.pushout.MultiPolynomialFunctor(['x','y'], None)
sage: G = sage.categories.pushout.MultiPolynomialFunctor(['t'], None)
sage: F.merge(G) is None
True
sage: F.merge(F)
MPoly[x,y]
```

**class** sage.categories.pushout.**MultivariateConstructionFunctor**

Bases: *sage.categories.pushout.ConstructionFunctor*

An abstract base class for functors that take multiple inputs (e.g. Cartesian products).

**common_base** (*other_functor*, *self_bases*, *other_bases*)

This function is called by *pushout()* when no common parent is found in the construction tower.

INPUT:

- `other_functor` – a construction functor.
- `self_bases` – the arguments passed to this functor.
- `other_bases` – the arguments passed to the functor `other_functor`.

OUTPUT:

A parent.

If no common base is found a `sage.structure.coerce_exceptions.CoercionException` is raised.

**Note:** Overload this function in derived class, see e.g. *MultivariateConstructionFunctor*.

**class** sage.categories.pushout.**PermutationGroupFunctor** (*gens*, *domain*)

Bases: *sage.categories.pushout.ConstructionFunctor*

EXAMPLES:

```
sage: from sage.categories.pushout import PermutationGroupFunctor
sage: PF = PermutationGroupFunctor([PermutationGroupElement([(1,2)])], [1,2]); PF
PermutationGroupFunctor[(1,2)]
```

**gens** ()

EXAMPLES:

```
sage: P1 = PermutationGroup([[(1,2)]])
sage: PF, P = P1.construction()
sage: PF.gens()
[(1,2)]
```

**merge**(*other*)

Merge `self` with another construction functor, or return None.

EXAMPLES:

```
sage: P1 = PermutationGroup([[(1,2)]])
sage: PF1, P = P1.construction()
sage: P2 = PermutationGroup([[(1,3)]])
sage: PF2, P = P2.construction()
sage: PF1.merge(PF2)
PermutationGroupFunctor[(1,2), (1,3)]
```

**class** sage.categories.pushout.**PolynomialFunctor**(*var*, *multi_variate=False*, *sparse=False*)

Bases: *sage.categories.pushout.ConstructionFunctor*

Construction functor for univariate polynomial rings.

EXAMPLES:

```
sage: P = ZZ['t'].construction()[0]
sage: P(GF(3))
Univariate Polynomial Ring in t over Finite Field of size 3
sage: P == loads(dumps(P))
True
sage: R.<x,y> = GF(5)[]
sage: f = R.hom([x+2*y,3*x-y],R)
sage: P(f)((x+y)*P(R).0)
(-x + y)*t
```

By trac ticket #9944, the construction functor distinguishes sparse and dense polynomial rings. Before, the following example failed:

```
sage: R.<x> = PolynomialRing(GF(5), sparse=True)
sage: F,B = R.construction()
sage: F(B) is R
True
sage: S.<x> = PolynomialRing(ZZ)
sage: R.has_coerce_map_from(S)
False
sage: S.has_coerce_map_from(R)
False
sage: S.0 + R.0
2*x
sage: (S.0 + R.0).parent()
Univariate Polynomial Ring in x over Finite Field of size 5
sage: (S.0 + R.0).parent().is_sparse()
False
```

**merge**(*other*)

Merge `self` with another construction functor, or return None.

NOTE:

Internally, the merging is delegated to the merging of multipolynomial construction functors. But in effect, this does the same as the default implementation, that returns `None` unless the to-be-merged functors coincide.

EXAMPLES:

```
sage: P = ZZ['x'].construction()[0]
sage: Q = ZZ['y','x'].construction()[0]
sage: P.merge(Q)
sage: P.merge(P) is P
True
```

**class** sage.categories.pushout.**QuotientFunctor**(*I*, *names=None*, *as_field=False*)

Bases: *sage.categories.pushout.ConstructionFunctor*

Construction functor for quotient rings.

NOTE:

The functor keeps track of variable names.

EXAMPLES:

```
sage: P.<x,y> = ZZ[]
sage: Q = P.quo([x^2+y^2]*P)
sage: F = Q.construction()[0]
sage: F(QQ['x','y'])
Quotient of Multivariate Polynomial Ring in x, y over Rational Field by the ideal
↪(x^2 + y^2)
sage: F(QQ['x','y']) == QQ['x','y'].quo([x^2+y^2]*QQ['x','y'])
True
sage: F(QQ['x','y','z'])
Traceback (most recent call last):
...
CoercionException: Can not apply this quotient functor to Multivariate Polynomial
↪Ring in x, y, z over Rational Field
sage: F(QQ['y','z'])
Traceback (most recent call last):
...
TypeError: Could not find a mapping of the passed element to this ring.
```

**merge**(*other*)

Two quotient functors with coinciding names are merged by taking the gcd of their moduli.

EXAMPLES:

```
sage: P.<x> = QQ[]
sage: Q1 = P.quo([(x^2+1)^2*(x^2-3)])
sage: Q2 = P.quo([(x^2+1)^2*(x^5+3)])
sage: from sage.categories.pushout import pushout
sage: pushout(Q1,Q2)     # indirect doctest
Univariate Quotient Polynomial Ring in xbar over Rational Field with modulus
↪x^4 + 2*x^2 + 1
```

The following was fixed in trac ticket #8800:

```
sage: pushout(GF(5), Integers(5))
Finite Field of size 5
```

**class** sage.categories.pushout.**SubspaceFunctor**(*basis*)

Bases: *sage.categories.pushout.ConstructionFunctor*

Constructing a subspace of an ambient free module, given by a basis.

NOTE:

This construction functor keeps track of the basis. It can only be applied to free modules into which this basis coerces.

EXAMPLES:

```
sage: M = ZZ^3
sage: S = M.submodule([(1,2,3),(4,5,6)]); S
Free module of degree 3 and rank 2 over Integer Ring
Echelon basis matrix:
[1 2 3]
[0 3 6]
sage: F = S.construction()[0]
sage: F(GF(2)^3)
Vector space of degree 3 and dimension 2 over Finite Field of size 2
User basis matrix:
[1 0 1]
[0 1 0]
```

**merge** (*other*)

Two Subspace Functors are merged into a construction functor of the sum of two subspaces.

EXAMPLES:

```
sage: M = GF(5)^3
sage: S1 = M.submodule([(1,2,3),(4,5,6)])
sage: S2 = M.submodule([(2,2,3)])
sage: F1 = S1.construction()[0]
sage: F2 = S2.construction()[0]
sage: F1.merge(F2)
SubspaceFunctor
sage: F1.merge(F2)(GF(5)^3) == S1+S2
True
sage: F1.merge(F2)(GF(5)['t']^3)
Free module of degree 3 and rank 3 over Univariate Polynomial Ring in t over
↪Finite Field of size 5
User basis matrix:
[1 0 0]
[0 1 0]
[0 0 1]
```

**class** sage.categories.pushout.**VectorFunctor** (*n*, *is_sparse=False*, *inner_product_matrix=None*)

Bases: *sage.categories.pushout.ConstructionFunctor*

A construction functor for free modules over commutative rings.

EXAMPLES:

```
sage: F = (ZZ^3).construction()[0]
sage: F
VectorFunctor
sage: F(GF(2)['t'])
Ambient free module of rank 3 over the principal ideal domain Univariate
↪Polynomial Ring in t over Finite Field of size 2 (using GF2X)
```

**merge** (*other*)

Two constructors of free modules merge, if the module ranks and the inner products coincide. If both have explicitly given inner product matrices, they must coincide as well.

EXAMPLES:

Two modules without explicitly given inner product allow coercion:

```
sage: M1 = QQ^3
sage: P.<t> = ZZ[]
sage: M2 = FreeModule(P,3)
sage: M1([1,1/2,1/3]) + M2([t,t^2+t,3])      # indirect doctest
(t + 1, t^2 + t + 1/2, 10/3)
```

If only one summand has an explicit inner product, the result will be provided with it:

```
sage: M3 = FreeModule(P,3, inner_product_matrix = Matrix(3,3,range(9)))
sage: M1([1,1/2,1/3]) + M3([t,t^2+t,3])
(t + 1, t^2 + t + 1/2, 10/3)
sage: (M1([1,1/2,1/3]) + M3([t,t^2+t,3])).parent().inner_product_matrix()
[0 1 2]
[3 4 5]
[6 7 8]
```

If both summands have an explicit inner product (even if it is the standard inner product), then the products must coincide. The only difference between `M1` and `M4` in the following example is the fact that the default inner product was *explicitly* requested for `M4`. It is therefore not possible to coerce with a different inner product:

```
sage: M4 = FreeModule(QQ,3, inner_product_matrix = Matrix(3,3,1))
sage: M4 == M1
True
sage: M4.inner_product_matrix() == M1.inner_product_matrix()
True
sage: M4([1,1/2,1/3]) + M3([t,t^2+t,3])      # indirect doctest
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for +: 'Ambient quadratic space of␣
↪dimension 3 over Rational Field
Inner product matrix:
[1 0 0]
[0 1 0]
[0 0 1]' and 'Ambient free quadratic module of rank 3 over the integral␣
↪domain Univariate Polynomial Ring in t over Integer Ring
Inner product matrix:
[0 1 2]
[3 4 5]
[6 7 8]'
```

`sage.categories.pushout.`**`construction_tower`**(*R*)

An auxiliary function that is used in *pushout()* and *pushout_lattice()*.

INPUT:

An object

OUTPUT:

A constructive description of the object from scratch, by a list of pairs of a construction functor and an object to which the construction functor is to be applied. The first pair is formed by `None` and the given object.

EXAMPLES:

```
sage: from sage.categories.pushout import construction_tower
sage: construction_tower(MatrixSpace(FractionField(QQ['t']),2))
[(None, Full MatrixSpace of 2 by 2 dense matrices over Fraction Field of␣
↪Univariate Polynomial Ring in t over Rational Field), (MatrixFunctor, Fraction␣
↪Field of Univariate Polynomial Ring in t over Rational Field), (FractionField,␣
↪Univariate Polynomial Ring in t over Rational Field), (Poly[t], Rational Field),
↪ (FractionField, Integer Ring)]
```

sage.categories.pushout.**expand_tower**(*tower*)

An auxiliary function that is used in *pushout()*.

INPUT:

A construction tower as returned by *construction_tower()*.

OUTPUT:

A new construction tower with all the construction functors expanded.

EXAMPLES:

```
sage: from sage.categories.pushout import construction_tower, expand_tower
sage: construction_tower(QQ['x,y,z'])
[(None, Multivariate Polynomial Ring in x, y, z over Rational Field),
 (MPoly[x,y,z], Rational Field),
 (FractionField, Integer Ring)]
sage: expand_tower(construction_tower(QQ['x,y,z']))
[(None, Multivariate Polynomial Ring in x, y, z over Rational Field),
 (MPoly[z], Univariate Polynomial Ring in y over Univariate Polynomial Ring in x␣
↪over Rational Field),
 (MPoly[y], Univariate Polynomial Ring in x over Rational Field),
 (MPoly[x], Rational Field),
 (FractionField, Integer Ring)]
```

sage.categories.pushout.**pushout**(*R*, *S*)

Given a pair of objects $R$ and $S$, try to construct a reasonable object $Y$ and return maps such that canonically $R \leftarrow Y \rightarrow S$.

ALGORITHM:

This incorporates the idea of functors discussed at Sage Days 4. Every object $R$ can be viewed as an initial object and a series of functors (e.g. polynomial, quotient, extension, completion, vector/matrix, etc.). Call the series of increasingly simple objects (with the associated functors) the "tower" of $R$. The construction method is used to create the tower.

Given two objects $R$ and $S$, try to find a common initial object $Z$. If the towers of $R$ and $S$ meet, let $Z$ be their join. Otherwise, see if the top of one coerces naturally into the other.

Now we have an initial object and two ordered lists of functors to apply. We wish to merge these in an unambiguous order, popping elements off the top of one or the other tower as we apply them to $Z$.

- If the functors are of distinct types, there is an absolute ordering given by the rank attribute. Use this.
- Otherwise:
  - If the tops are equal, we (try to) merge them.
  - If exactly one occurs lower in the other tower, we may unambiguously apply the other (hoping for a later merge).
  - If the tops commute, we can apply either first.
  - Otherwise fail due to ambiguity.

The algorithm assumes by default that when a construction $F$ is applied to an object $X$, the object $F(X)$ admits a coercion map from $X$. However, the algorithm can also handle the case where $F(X)$ has a coercion map *to* $X$ instead. In this case, the attribute `coercion_reversed` of the class implementing $F$ should be set to `True`.

EXAMPLES:

Here our "towers" are $R = Complete_7(Frac(\mathbf{Z}))$ and $Frac(Poly_x(\mathbf{Z}))$, which give us $Frac(Poly_x(Complete_7(Frac(\mathbf{Z}))))$:

```
sage: from sage.categories.pushout import pushout
sage: pushout(Qp(7), Frac(ZZ['x']))
Fraction Field of Univariate Polynomial Ring in x over 7-adic Field with capped␣
↪relative precision 20
```

Note we get the same thing with

```
sage: pushout(Zp(7), Frac(QQ['x']))
Fraction Field of Univariate Polynomial Ring in x over 7-adic Field with capped␣
↪relative precision 20
sage: pushout(Zp(7)['x'], Frac(QQ['x']))
Fraction Field of Univariate Polynomial Ring in x over 7-adic Field with capped␣
↪relative precision 20
```

Note that polynomial variable ordering must be unambiguously determined.

```
sage: pushout(ZZ['x,y,z'], QQ['w,z,t'])
Traceback (most recent call last):
...
CoercionException: ('Ambiguous Base Extension', Multivariate Polynomial Ring in x,
↪ y, z over Integer Ring, Multivariate Polynomial Ring in w, z, t over Rational␣
↪Field)
sage: pushout(ZZ['x,y,z'], QQ['w,x,z,t'])
Multivariate Polynomial Ring in w, x, y, z, t over Rational Field
```

Some other examples:

```
sage: pushout(Zp(7)['y'], Frac(QQ['t'])['x,y,z'])
Multivariate Polynomial Ring in x, y, z over Fraction Field of Univariate␣
↪Polynomial Ring in t over 7-adic Field with capped relative precision 20
sage: pushout(ZZ['x,y,z'], Frac(ZZ['x'])['y'])
Multivariate Polynomial Ring in y, z over Fraction Field of Univariate Polynomial␣
↪Ring in x over Integer Ring
sage: pushout(MatrixSpace(RDF, 2, 2), Frac(ZZ['x']))
Full MatrixSpace of 2 by 2 dense matrices over Fraction Field of Univariate␣
↪Polynomial Ring in x over Real Double Field
sage: pushout(ZZ, MatrixSpace(ZZ[['x']], 3, 3))
Full MatrixSpace of 3 by 3 dense matrices over Power Series Ring in x over␣
↪Integer Ring
sage: pushout(QQ['x,y'], ZZ[['x']])
Univariate Polynomial Ring in y over Power Series Ring in x over Rational Field
sage: pushout(Frac(ZZ['x']), QQ[['x']])
Laurent Series Ring in x over Rational Field
```

A construction with `coercion_reversed = True` (currently only the *SubspaceFunctor* construction) is only applied if it leads to a valid coercion:

```
sage: A = ZZ^2
sage: V = span([[1, 2]], QQ)
sage: P = sage.categories.pushout.pushout(A, V)
```

```
sage: P
Vector space of dimension 2 over Rational Field
sage: P.has_coerce_map_from(A)
True

sage: V = (QQ^3).span([[1, 2, 3/4]])
sage: A = ZZ^3
sage: pushout(A, V)
Vector space of dimension 3 over Rational Field
sage: B = A.span([[0, 0, 2/3]])
sage: pushout(B, V)
Vector space of degree 3 and dimension 2 over Rational Field
User basis matrix:
[1 2 0]
[0 0 1]
```

Some more tests with `coercion_reversed = True`:

```
sage: from sage.categories.pushout import ConstructionFunctor
sage: class EvenPolynomialRing(type(QQ['x'])):
....:     def __init__(self, base, var):
....:         super(EvenPolynomialRing, self).__init__(base, var)
....:         self.register_embedding(base[var])
....:     def __repr__(self):
....:         return "Even Power " + super(EvenPolynomialRing, self).__repr__()
....:     def construction(self):
....:         return EvenPolynomialFunctor(), self.base()[self.variable_name()]
....:     def _coerce_map_from_(self, R):
....:         return self.base().has_coerce_map_from(R)
sage: class EvenPolynomialFunctor(ConstructionFunctor):
....:     rank = 10
....:     coercion_reversed = True
....:     def __init__(self):
....:         ConstructionFunctor.__init__(self, Rings(), Rings())
....:     def _apply_functor(self, R):
....:         return EvenPolynomialRing(R.base(), R.variable_name())
sage: pushout(EvenPolynomialRing(QQ, 'x'), ZZ)
Even Power Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), QQ)
Even Power Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), RR)
Even Power Univariate Polynomial Ring in x over Real Field with 53 bits of␣
↪precision

sage: pushout(EvenPolynomialRing(QQ, 'x'), ZZ['x'])
Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), QQ['x'])
Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), RR['x'])
Univariate Polynomial Ring in x over Real Field with 53 bits of precision

sage: pushout(EvenPolynomialRing(QQ, 'x'), EvenPolynomialRing(QQ, 'x'))
Even Power Univariate Polynomial Ring in x over Rational Field
sage: pushout(EvenPolynomialRing(QQ, 'x'), EvenPolynomialRing(RR, 'x'))
Even Power Univariate Polynomial Ring in x over Real Field with 53 bits of␣
↪precision

sage: pushout(EvenPolynomialRing(QQ, 'x')^2, RR^2)
```

```
Ambient free module of rank 2 over the principal ideal domain Even Power␣
↪Univariate Polynomial Ring in x over Real Field with 53 bits of precision
sage: pushout(EvenPolynomialRing(QQ, 'x')^2, RR['x']^2)
Ambient free module of rank 2 over the principal ideal domain Univariate␣
↪Polynomial Ring in x over Real Field with 53 bits of precision
```

Some more tests related to univariate/multivariate constructions. We consider a generalization of polynomial rings, where in addition to the coefficient ring $C$ we also specify an additive monoid $E$ for the exponents of the indeterminate. In particular, the elements of such a parent are given by

$$\sum_{i=0}^{I} c_i X^{e_i}$$

with $c_i \in C$ and $e_i \in E$. We define

```
sage: class GPolynomialRing(Parent):
....:     def __init__(self, coefficients, var, exponents):
....:         self.coefficients = coefficients
....:         self.var = var
....:         self.exponents = exponents
....:         super(GPolynomialRing, self).__init__(category=Rings())
....:     def _repr_(self):
....:         return 'Generalized Polynomial Ring in %s^(%s) over %s' % (
....:                 self.var, self.exponents, self.coefficients)
....:     def construction(self):
....:         return GPolynomialFunctor(self.var, self.exponents), self.
↪coefficients
....:     def _coerce_map_from_(self, R):
....:         return self.coefficients.has_coerce_map_from(R)
```

and

```
sage: class GPolynomialFunctor(ConstructionFunctor):
....:     rank = 10
....:     def __init__(self, var, exponents):
....:         self.var = var
....:         self.exponents = exponents
....:         ConstructionFunctor.__init__(self, Rings(), Rings())
....:     def _repr_(self):
....:         return 'GPoly[%s^(%s)]' % (self.var, self.exponents)
....:     def _apply_functor(self, coefficients):
....:         return GPolynomialRing(coefficients, self.var, self.exponents)
....:     def merge(self, other):
....:         if isinstance(other, GPolynomialFunctor) and self.var == other.var:
....:             exponents = pushout(self.exponents, other.exponents)
....:             return GPolynomialFunctor(self.var, exponents)
```

We can construct a parent now in two different ways:

```
sage: GPolynomialRing(QQ, 'X', ZZ)
Generalized Polynomial Ring in X^(Integer Ring) over Rational Field
sage: GP_ZZ = GPolynomialFunctor('X', ZZ); GP_ZZ
GPoly[X^(Integer Ring)]
sage: GP_ZZ(QQ)
Generalized Polynomial Ring in X^(Integer Ring) over Rational Field
```

Since the construction

```
sage: GP_ZZ(QQ).construction()
(GPoly[X^(Integer Ring)], Rational Field)
```

uses the coefficient ring, we have the usual coercion with respect to this parameter:

```
sage: pushout(GP_ZZ(ZZ), GP_ZZ(QQ))
Generalized Polynomial Ring in X^(Integer Ring) over Rational Field
sage: pushout(GP_ZZ(ZZ['t']), GP_ZZ(QQ))
Generalized Polynomial Ring in X^(Integer Ring) over Univariate Polynomial Ring␣
↪in t over Rational Field
sage: pushout(GP_ZZ(ZZ['a,b']), GP_ZZ(ZZ['b,c']))
Generalized Polynomial Ring in X^(Integer Ring)
  over Multivariate Polynomial Ring in a, b, c over Integer Ring
sage: pushout(GP_ZZ(ZZ['a,b']), GP_ZZ(QQ['b,c']))
Generalized Polynomial Ring in X^(Integer Ring)
  over Multivariate Polynomial Ring in a, b, c over Rational Field
sage: pushout(GP_ZZ(ZZ['a,b']), GP_ZZ(ZZ['c,d']))
Traceback (most recent call last):
...
CoercionException: ('Ambiguous Base Extension', ...)
```

```
sage: GP_QQ = GPolynomialFunctor('X', QQ)
sage: pushout(GP_ZZ(ZZ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Rational Field) over Integer Ring
sage: pushout(GP_QQ(ZZ), GP_ZZ(ZZ))
Generalized Polynomial Ring in X^(Rational Field) over Integer Ring
```

```
sage: GP_ZZt = GPolynomialFunctor('X', ZZ['t'])
sage: pushout(GP_ZZt(ZZ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t
  over Rational Field) over Integer Ring
```

```
sage: pushout(GP_ZZ(ZZ), GP_QQ(QQ))
Generalized Polynomial Ring in X^(Rational Field) over Rational Field
sage: pushout(GP_ZZ(QQ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Rational Field) over Rational Field
sage: pushout(GP_ZZt(QQ), GP_QQ(ZZ))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t
  over Rational Field) over Rational Field
sage: pushout(GP_ZZt(ZZ), GP_QQ(QQ))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t
  over Rational Field) over Rational Field
sage: pushout(GP_ZZt(ZZ['a,b']), GP_QQ(ZZ['c,d']))
Traceback (most recent call last):
...
CoercionException: ('Ambiguous Base Extension', ...)
sage: pushout(GP_ZZt(ZZ['a,b']), GP_QQ(ZZ['b,c']))
Generalized Polynomial Ring in X^(Univariate Polynomial Ring in t over Rational␣
↪Field)
  over Multivariate Polynomial Ring in a, b, c over Integer Ring
```

Some tests with Cartesian products:

```
sage: from sage.sets.cartesian_product import CartesianProduct
sage: A = CartesianProduct((ZZ['x'], QQ['y'], QQ['z']), Sets().
↪CartesianProducts())
sage: B = CartesianProduct((ZZ['x'], ZZ['y'], ZZ['t']['z']), Sets().
↪CartesianProducts())
```

```
sage: A.construction()
(The cartesian_product functorial construction,
 (Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Rational Field,
  Univariate Polynomial Ring in z over Rational Field))
sage: pushout(A, B)
The Cartesian product of
 (Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Rational Field,
  Univariate Polynomial Ring in z over Univariate Polynomial Ring in t over␣
↪Rational Field)
sage: pushout(ZZ, cartesian_product([ZZ, QQ]))
Traceback (most recent call last):
...
CoercionException: 'NoneType' object is not iterable
```

```
sage: from sage.categories.pushout import PolynomialFunctor
sage: from sage.sets.cartesian_product import CartesianProduct
sage: class CartesianProductPoly(CartesianProduct):
....:     def __init__(self, polynomial_rings):
....:         sort = sorted(polynomial_rings, key=lambda P: P.variable_name())
....:         super(CartesianProductPoly, self).__init__(sort, Sets().
↪CartesianProducts())
....:     def vars(self):
....:         return tuple(P.variable_name() for P in self.cartesian_factors())
....:     def _pushout_(self, other):
....:         if isinstance(other, CartesianProductPoly):
....:             s_vars = self.vars()
....:             o_vars = other.vars()
....:             if s_vars == o_vars:
....:                 return
....:             return pushout(CartesianProductPoly(
....:                     self.cartesian_factors() +
....:                     tuple(f for f in other.cartesian_factors()
....:                           if f.variable_name() not in s_vars)),
....:                 CartesianProductPoly(
....:                     other.cartesian_factors() +
....:                     tuple(f for f in self.cartesian_factors()
....:                           if f.variable_name() not in o_vars)))
....:         C = other.construction()
....:         if C is None:
....:             return
....:         elif isinstance(C[0], PolynomialFunctor):
....:             return pushout(self, CartesianProductPoly((other,)))
```

```
sage: pushout(CartesianProductPoly((ZZ['x'],)),
....:         CartesianProductPoly((ZZ['y'],)))
The Cartesian product of
 (Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Integer Ring)
sage: pushout(CartesianProductPoly((ZZ['x'], ZZ['y'])),
....:         CartesianProductPoly((ZZ['x'], ZZ['z'])))
The Cartesian product of
 (Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Integer Ring,
  Univariate Polynomial Ring in z over Integer Ring)
sage: pushout(CartesianProductPoly((QQ['a,b']['x'], QQ['y'])),
```

```
....:         CartesianProductPoly((ZZ['b,c']['x'], SR['z'])))
The Cartesian product of
 (Univariate Polynomial Ring in x over
    Multivariate Polynomial Ring in a, b, c over Rational Field,
  Univariate Polynomial Ring in y over Rational Field,
  Univariate Polynomial Ring in z over Symbolic Ring)
```

```
sage: pushout(CartesianProductPoly((ZZ['x'],)), ZZ['y'])
The Cartesian product of
 (Univariate Polynomial Ring in x over Integer Ring,
  Univariate Polynomial Ring in y over Integer Ring)
sage: pushout(QQ['b,c']['y'], CartesianProductPoly((ZZ['a,b']['x'],)))
The Cartesian product of
 (Univariate Polynomial Ring in x over
    Multivariate Polynomial Ring in a, b over Integer Ring,
  Univariate Polynomial Ring in y over
    Multivariate Polynomial Ring in b, c over Rational Field)
```

```
sage: pushout(CartesianProductPoly((ZZ['x'],)), ZZ)
Traceback (most recent call last):
...
CoercionException: No common base ("join") found for
The cartesian_product functorial construction(...) and None(Integer Ring):
(Multivariate) functors are incompatible.
```

AUTHORS:

- Robert Bradshaw
- Peter Bruin
- Simon King
- Daniel Krenn
- David Roe

sage.categories.pushout.**pushout_lattice**(*R*, *S*)

Given a pair of objects $R$ and $S$, try to construct a reasonable object $Y$ and return maps such that canonically $R \leftarrow Y \rightarrow S$.

ALGORITHM:

This is based on the model that arose from much discussion at Sage Days 4. Going up the tower of constructions of $R$ and $S$ (e.g. the reals come from the rationals come from the integers), try to find a common parent, and then try to fill in a lattice with these two towers as sides with the top as the common ancestor and the bottom will be the desired ring.

See the code for a specific worked-out example.

EXAMPLES:

```
sage: from sage.categories.pushout import pushout_lattice
sage: A, B = pushout_lattice(Qp(7), Frac(ZZ['x']))
sage: A.codomain()
Fraction Field of Univariate Polynomial Ring in x over 7-adic Field with capped␣
↪relative precision 20
sage: A.codomain() is B.codomain()
True
sage: A, B = pushout_lattice(ZZ, MatrixSpace(ZZ[['x']], 3, 3))
```

```
sage: B
Identity endomorphism of Full MatrixSpace of 3 by 3 dense matrices over Power␣
↪Series Ring in x over Integer Ring
```

AUTHOR:

- Robert Bradshaw

sage.categories.pushout.**type_to_parent**(*P*)

An auxiliary function that is used in *pushout()*.

INPUT:

A type

OUTPUT:

A Sage parent structure corresponding to the given type

CHAPTER

THREE

# INDIVIDUAL CATEGORIES

## 3.1 Group, ring, etc. actions on objects.

The terminology and notation used is suggestive of groups acting on sets, but this framework can be used for modules, algebras, etc.

A group action $G \times S \to S$ is a functor from $G$ to Sets.

**Warning:** An *Action* object only keeps a weak reference to the underlying set which is acted upon. This decision was made in trac ticket #715 in order to allow garbage collection within the coercion framework (this is where actions are mainly used) and avoid memory leaks.

```
sage: from sage.categories.action import Action
sage: class P: pass
sage: A = Action(P(),P())
sage: import gc
sage: _ = gc.collect()
sage: A
<repr(<sage.categories.action.Action at 0x...>) failed: RuntimeError: This action␣
↪acted on a set that became garbage collected>
```

To avoid garbage collection of the underlying set, it is sufficient to create a strong reference to it before the action is created.

```
sage: _ = gc.collect()
sage: from sage.categories.action import Action
sage: class P: pass
sage: q = P()
sage: A = Action(P(),q)
sage: gc.collect()
0
sage: A
Left action by <__main__.P instance at ...> on <__main__.P instance at ...>
```

AUTHOR:

- Robert Bradshaw: initial version

**class** sage.categories.action.**Action**

Bases: *sage.categories.functor.Functor*

**G**

**act** (*g*, *a*)

This is a consistent interface for acting on a by g, regardless of whether it's a left or right action.

**actor**()

**codomain**()

**domain**()

**is_left**()

**left_domain**()

**op**

**operation**()

**right_domain**()

**class** sage.categories.action.**ActionEndomorphism**

Bases: *sage.categories.morphism.Morphism*

The endomorphism defined by the action of one element.

EXAMPLES:

```
sage: A = ZZ['x'].get_action(QQ, self_on_left=False, op=operator.mul)
sage: A
Left scalar multiplication by Rational Field on Univariate Polynomial
Ring in x over Integer Ring
sage: A(1/2)
Action of 1/2 on Univariate Polynomial Ring in x over Integer Ring
under Left scalar multiplication by Rational Field on Univariate
Polynomial Ring in x over Integer Ring.
```

**class** sage.categories.action.**InverseAction**

Bases: *sage.categories.action.Action*

An action that acts as the inverse of the given action.

EXAMPLES:

```
sage: V = QQ^3
sage: v = V((1, 2, 3))
sage: cm = get_coercion_model()

sage: a = cm.get_action(V, QQ, operator.mul)
sage: a
Right scalar multiplication by Rational Field on Vector space of dimension 3 over
↪Rational Field
sage: ~a
Right inverse action by Rational Field on Vector space of dimension 3 over
↪Rational Field
sage: (~a)(v, 1/3)
(3, 6, 9)

sage: b = cm.get_action(QQ, V, operator.mul)
sage: b
Left scalar multiplication by Rational Field on Vector space of dimension 3 over
↪Rational Field
sage: ~b
Left inverse action by Rational Field on Vector space of dimension 3 over
↪Rational Field
sage: (~b)(1/3, v)
(3, 6, 9)
```

```
sage: c = cm.get_action(ZZ, list, operator.mul)
sage: c
Left action by Integer Ring on <... 'list'>
sage: ~c
Traceback (most recent call last):
...
TypeError: no inverse defined for Left action by Integer Ring on <... 'list'>
```

**codomain**()

**class** sage.categories.action.**PrecomposedAction**

Bases: *sage.categories.action.Action*

A precomposed action first applies given maps, and then applying an action to the return values of the maps.

EXAMPLES:

We demonstrate that an example discussed on trac ticket #14711 did not become a problem:

```
sage: E = ModularSymbols(11).2
sage: s = E.modular_symbol_rep()
sage: del E,s
sage: import gc
sage: _ = gc.collect()
sage: E = ModularSymbols(11).2
sage: v = E.manin_symbol_rep()
sage: c,x = v[0]
sage: y = x.modular_symbol_rep()
sage: A = y.parent().get_action(QQ, self_on_left=False, op=operator.mul)
sage: A
Left scalar multiplication by Rational Field on Abelian Group of all Formal␣
↪Finite Sums over Rational Field
with precomposition on right by Coercion map:
  From: Abelian Group of all Formal Finite Sums over Integer Ring
  To:   Abelian Group of all Formal Finite Sums over Rational Field
```

**codomain**()

**domain**()

## 3.2 Additive groups

**class** sage.categories.additive_groups.**AdditiveGroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of additive groups.

An *additive group* is a set with an internal binary operation $+$ which is associative, admits a zero, and where every element can be negated.

EXAMPLES:

```
sage: from sage.categories.additive_groups import AdditiveGroups
sage: from sage.categories.additive_monoids import AdditiveMonoids
sage: AdditiveGroups()
Category of additive groups
sage: AdditiveGroups().super_categories()
[Category of additive inverse additive unital additive magmas,
```

```
 Category of additive monoids]
sage: AdditiveGroups().all_super_categories()
[Category of additive groups,
 Category of additive inverse additive unital additive magmas,
 Category of additive monoids,
 Category of additive unital additive magmas,
 Category of additive semigroups,
 Category of additive magmas,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]

sage: AdditiveGroups().axioms()
frozenset({'AdditiveAssociative', 'AdditiveInverse', 'AdditiveUnital'})
sage: AdditiveGroups() is AdditiveMonoids().AdditiveInverse()
True
```

**AdditiveCommutative**
alias of *CommutativeAdditiveGroups*

**class Algebras**(*category*, **args*)
Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**group**()
Return the underlying group of the group algebra.

EXAMPLES:

```
sage: GroupAlgebras(QQ).example(GL(3, GF(11))).group()
General Linear Group of degree 3 over Finite Field of size 11
sage: SymmetricGroup(10).algebra(QQ).group()
Symmetric group of order 10! as a permutation group
```

**class Finite**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class Algebras**(*category*, **args*)
Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**extra_super_categories**()
Implement Maschke's theorem.

In characteristic 0 all finite group algebras are semisimple.

EXAMPLES:

```
sage: FiniteGroups().Algebras(QQ).is_subcategory(Algebras(QQ).
↪Semisimple())
True
sage: FiniteGroups().Algebras(FiniteField(7)).is_
↪subcategory(Algebras(FiniteField(7)).Semisimple())
False
sage: FiniteGroups().Algebras(ZZ).is_subcategory(Algebras(ZZ).
↪Semisimple())
False
sage: FiniteGroups().Algebras(Fields()).is_
↪subcategory(Algebras(Fields()).Semisimple())
```

```
False

sage: Cat = CommutativeAdditiveGroups().Finite()
sage: Cat.Algebras(QQ).is_subcategory(Algebras(QQ).Semisimple())
True
sage: Cat.Algebras(GF(7)).is_subcategory(Algebras(GF(7)).Semisimple())
False
sage: Cat.Algebras(ZZ).is_subcategory(Algebras(ZZ).Semisimple())
False
sage: Cat.Algebras(Fields()).is_subcategory(Algebras(Fields()).
↪Semisimple())
False
```

## 3.3 Additive Magmas

**class** sage.categories.additive_magmas.**AdditiveMagmas**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of additive magmas.

An additive magma is a set endowed with a binary operation $+$.

EXAMPLES:

```
sage: AdditiveMagmas()
Category of additive magmas
sage: AdditiveMagmas().super_categories()
[Category of sets]
sage: AdditiveMagmas().all_super_categories()
[Category of additive magmas, Category of sets, Category of sets with partial␣
↪maps, Category of objects]
```

The following axioms are defined by this category:

```
sage: AdditiveMagmas().AdditiveAssociative()
Category of additive semigroups
sage: AdditiveMagmas().AdditiveUnital()
Category of additive unital additive magmas
sage: AdditiveMagmas().AdditiveCommutative()
Category of additive commutative additive magmas
sage: AdditiveMagmas().AdditiveUnital().AdditiveInverse()
Category of additive inverse additive unital additive magmas
sage: AdditiveMagmas().AdditiveAssociative().AdditiveCommutative()
Category of commutative additive semigroups
sage: AdditiveMagmas().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital()
Category of commutative additive monoids
sage: AdditiveMagmas().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital().AdditiveInverse()
Category of commutative additive groups
```

**AdditiveAssociative**

alias of AdditiveSemigroups

**class AdditiveCommutative**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class Algebras**(*category*, **args*)
Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**extra_super_categories**()
Implement the fact that the algebra of a commutative additive magmas is commutative.

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveCommutative().Algebras(QQ).extra_super_
↪categories()
[Category of commutative magmas]

sage: AdditiveMagmas().AdditiveCommutative().Algebras(QQ).super_
↪categories()
[Category of additive magma algebras over Rational Field,
 Category of commutative magmas]
```

**class CartesianProducts**(*category*, **args*)
Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**extra_super_categories**()
Implement the fact that a Cartesian product of commutative additive magmas is a commutative additive magma.

EXAMPLES:

```
sage: C = AdditiveMagmas().AdditiveCommutative().CartesianProducts()
sage: C.extra_super_categories();
[Category of additive commutative additive magmas]
sage: C.axioms()
frozenset({'AdditiveCommutative'})
```

**class AdditiveUnital**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class AdditiveInverse**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class CartesianProducts**(*category*, **args*)
Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**class ElementMethods**

**extra_super_categories**()
Implement the fact that a Cartesian product of additive magmas with inverses is an additive magma with inverse.

EXAMPLES:

```
sage: C = AdditiveMagmas().AdditiveUnital().AdditiveInverse().
↪CartesianProducts()
sage: C.extra_super_categories();
[Category of additive inverse additive unital additive magmas]
sage: sorted(C.axioms())
['AdditiveInverse', 'AdditiveUnital']
```

**class Algebras**(*category*, **args*)
Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**one_basis**()

Return the zero of this additive magma, which index the one of this algebra, as per `AlgebrasWithBasis.ParentMethods.one_basis()`.

EXAMPLES:

```
sage: S = CommutativeAdditiveMonoids().example(); S
An example of a commutative monoid: the free commutative monoid␣
↪generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(ZZ)
sage: A.one_basis()
0
sage: A.one()
B[0]
sage: A(3)
3*B[0]
```

**extra_super_categories**()

EXAMPLES:

```
sage: C = AdditiveMagmas().AdditiveUnital().Algebras(QQ)
sage: C.extra_super_categories()
[Category of unital magmas]

sage: C.super_categories()
[Category of unital algebras with basis over Rational Field, Category␣
↪of additive magma algebras over Rational Field]
```

**class CartesianProducts**(*category*, **args*)

Bases: `sage.categories.cartesian_product.CartesianProductsCategory`

**class ParentMethods**

**zero**()

Returns the zero of this group

EXAMPLES:

```
sage: GF(8,'x').cartesian_product(GF(5)).zero()
(0, 0)
```

**extra_super_categories**()

Implement the fact that a Cartesian product of unital additive magmas is a unital additive magma.

EXAMPLES:

```
sage: C = AdditiveMagmas().AdditiveUnital().CartesianProducts()
sage: C.extra_super_categories();
[Category of additive unital additive magmas]
sage: C.axioms()
frozenset({'AdditiveUnital'})
```

**class ElementMethods**

**class Homsets**(*category*, **args*)

Bases: `sage.categories.homsets.HomsetsCategory`

**class ParentMethods**

**zero**()

EXAMPLES:

```
sage: R = QQ['x']
sage: H = Hom(ZZ, R, AdditiveMagmas().AdditiveUnital())
sage: f = H.zero()
sage: f
Generic morphism:
  From: Integer Ring
  To:   Univariate Polynomial Ring in x over Rational Field
sage: f(3)
0
sage: f(3) is R.zero()
True
```

**extra_super_categories**()

Implement the fact that a homset between two unital additive magmas is a unital additive magma.

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveUnital().Homsets().extra_super_
↪categories()
[Category of additive unital additive magmas]
sage: AdditiveMagmas().AdditiveUnital().Homsets().super_categories()
[Category of additive unital additive magmas, Category of homsets]
```

**class ParentMethods**

**is_empty**()

Return whether this set is empty.

Since this set is an additive magma it has a zero element and hence is not empty. This method thus always returns `False`.

EXAMPLES:

```
sage: A = AdditiveAbelianGroup([3,3])
sage: A in AdditiveMagmas()
True
sage: A.is_empty()
False

sage: B = CommutativeAdditiveMonoids().example()
sage: B.is_empty()
False
```

**zero**()

Return the zero of this additive magma, that is the unique neutral element for $+$.

The default implementation is to coerce `0` into `self`.

It is recommended to override this method because the coercion from the integers:

- is not always meaningful (except for 0), and
- often uses `self.zero()` otherwise.

EXAMPLES:

```
sage: S = CommutativeAdditiveMonoids().example()
sage: S.zero()
0
```

**class SubcategoryMethods**

**AdditiveInverse**()

Return the full subcategory of the additive inverse objects of `self`.

An inverse *additive magma* is a `unital additive magma` such that every element admits both an additive inverse on the left and on the right. Such an additive magma is also called an *additive loop*.

**See also:**

Wikipedia article Inverse_element, Wikipedia article Quasigroup

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveUnital().AdditiveInverse()
Category of additive inverse additive unital additive magmas
sage: from sage.categories.additive_monoids import AdditiveMonoids
sage: AdditiveMonoids().AdditiveInverse()
Category of additive groups
```

**class WithRealizations**(*category*, **args*)

Bases: *sage.categories.with_realizations.WithRealizationsCategory*

**class ParentMethods**

**zero**()

Return the zero of this unital additive magma.

This default implementation returns the zero of the realization of `self` given by *a_realization()*.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.zero.__module__
'sage.categories.additive_magmas'
sage: A.zero()
0
```

**additional_structure**()

Return whether `self` is a structure category.

**See also:**

*Category.additional_structure()*

The category of unital additive magmas defines the zero as additional structure, and this zero shall be preserved by morphisms.

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveUnital().additional_structure()
Category of additive unital additive magmas
```

**class Algebras**(*category*, **args*)
Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**algebra_generators**()
The generators of this algebra, as per *MagmaticAlgebras.ParentMethods.algebra_generators()*.

They correspond to the generators of the additive semigroup.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative monoid: the free commutative monoid␣
↪generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)
sage: A.algebra_generators()
Finite family {0: B[a], 1: B[b], 2: B[c], 3: B[d]}
```

**product_on_basis**(*g1*, *g2*)
Product, on basis elements, as per *MagmaticAlgebras.WithBasis.ParentMethods.product_on_basis()*.

The product of two basis elements is induced by the addition of the corresponding elements of the group.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative monoid: the free commutative monoid␣
↪generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)
sage: a,b,c,d = A.algebra_generators()
sage: a * b + b * d * c
B[c + b + d] + B[a + b]
```

**extra_super_categories**()
EXAMPLES:

```
sage: AdditiveMagmas().Algebras(QQ).extra_super_categories()
[Category of magmatic algebras with basis over Rational Field]

sage: AdditiveMagmas().Algebras(QQ).super_categories()
[Category of magmatic algebras with basis over Rational Field, Category␣
↪of set algebras over Rational Field]
```

**class CartesianProducts**(*category*, **args*)
Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**class ElementMethods**

**extra_super_categories**()
Implement the fact that a Cartesian product of additive magmas is an additive magma.

EXAMPLES:

```
sage: C = AdditiveMagmas().CartesianProducts()
sage: C.extra_super_categories()
[Category of additive magmas]
```

```
sage: C.super_categories()
[Category of additive magmas, Category of Cartesian products of sets]
sage: C.axioms()
frozenset()
```

**class ElementMethods**

**class Homsets**(*category*, **args*)

Bases: *sage.categories.homsets.HomsetsCategory*

**extra_super_categories**()

Implement the fact that a homset between two magmas is a magma.

EXAMPLES:

```
sage: AdditiveMagmas().Homsets().extra_super_categories()
[Category of additive magmas]
sage: AdditiveMagmas().Homsets().super_categories()
[Category of additive magmas, Category of homsets]
```

**class ParentMethods**

**addition_table**(*names='letters'*, *elements=None*)

Return a table describing the addition operation.

---

**Note:** The order of the elements in the row and column headings is equal to the order given by the table's `column_keys()` method. The association can also be retrieved with the `translation()` method.

---

INPUT:

- `names` – the type of names used:
  - `'letters'` - lowercase ASCII letters are used for a base 26 representation of the elements' positions in the list given by `column_keys()`, padded to a common width with leading 'a's.
  - `'digits'` - base 10 representation of the elements' positions in the list given by `column_keys()`, padded to a common width with leading zeros.
  - `'elements'` - the string representations of the elements themselves.
  - a list - a list of strings, where the length of the list equals the number of elements.
- `elements` – (default: `None`) A list of elements of the additive magma, in forms that can be coerced into the structure, eg. their string representations. This may be used to impose an alternate ordering on the elements, perhaps when this is used in the context of a particular structure. The default is to use whatever ordering the `S.list` method returns. Or the `elements` can be a subset which is closed under the operation. In particular, this can be used when the base set is infinite.

OUTPUT:

The addition table as an object of the class `OperationTable` which defines several methods for manipulating and displaying the table. See the documentation there for full details to supplement the documentation here.

EXAMPLES:

All that is required is that an algebraic structure has an addition defined.The default is to represent elements as lowercase ASCII letters.

```
sage: R=IntegerModRing(5)
sage: R.addition_table()
```

```
+  a b c d e
 +----------
a| a b c d e
b| b c d e a
c| c d e a b
d| d e a b c
e| e a b c d
```

The `names` argument allows displaying the elements in different ways. Requesting `elements` will use the representation of the elements of the set. Requesting `digits` will include leading zeros as padding.

```
sage: R=IntegerModRing(11)
sage: P=R.addition_table(names='elements')
sage: P
 +   0  1  2  3  4  5  6  7  8  9 10
  +---------------------------------
 0|  0  1  2  3  4  5  6  7  8  9 10
 1|  1  2  3  4  5  6  7  8  9 10  0
 2|  2  3  4  5  6  7  8  9 10  0  1
 3|  3  4  5  6  7  8  9 10  0  1  2
 4|  4  5  6  7  8  9 10  0  1  2  3
 5|  5  6  7  8  9 10  0  1  2  3  4
 6|  6  7  8  9 10  0  1  2  3  4  5
 7|  7  8  9 10  0  1  2  3  4  5  6
 8|  8  9 10  0  1  2  3  4  5  6  7
 9|  9 10  0  1  2  3  4  5  6  7  8
10| 10  0  1  2  3  4  5  6  7  8  9

sage: T=R.addition_table(names='digits')
sage: T
+  00 01 02 03 04 05 06 07 08 09 10
  +---------------------------------
00| 00 01 02 03 04 05 06 07 08 09 10
01| 01 02 03 04 05 06 07 08 09 10 00
02| 02 03 04 05 06 07 08 09 10 00 01
03| 03 04 05 06 07 08 09 10 00 01 02
04| 04 05 06 07 08 09 10 00 01 02 03
05| 05 06 07 08 09 10 00 01 02 03 04
06| 06 07 08 09 10 00 01 02 03 04 05
07| 07 08 09 10 00 01 02 03 04 05 06
08| 08 09 10 00 01 02 03 04 05 06 07
09| 09 10 00 01 02 03 04 05 06 07 08
10| 10 00 01 02 03 04 05 06 07 08 09
```

Specifying the elements in an alternative order can provide more insight into how the operation behaves.

```
sage: S=IntegerModRing(7)
sage: elts = [0, 3, 6, 2, 5, 1, 4]
sage: S.addition_table(elements=elts)
+  a b c d e f g
 +--------------
a| a b c d e f g
b| b c d e f g a
c| c d e f g a b
d| d e f g a b c
e| e f g a b c d
```

```
f| f g a b c d e
g| g a b c d e f
```

The `elements` argument can be used to provide a subset of the elements of the structure. The subset must be closed under the operation. Elements need only be in a form that can be coerced into the set. The `names` argument can also be used to request that the elements be represented with their usual string representation.

```
sage: T=IntegerModRing(12)
sage: elts=[0, 3, 6, 9]
sage: T.addition_table(names='elements', elements=elts)
+  0 3 6 9
 +--------
0| 0 3 6 9
3| 3 6 9 0
6| 6 9 0 3
9| 9 0 3 6
```

The table returned can be manipulated in various ways. See the documentation for `OperationTable` for more comprehensive documentation.

```
sage: R=IntegerModRing(3)
sage: T=R.addition_table()
sage: T.column_keys()
(0, 1, 2)
sage: sorted(T.translation().items())
[('a', 0), ('b', 1), ('c', 2)]
sage: T.change_names(['x', 'y', 'z'])
sage: sorted(T.translation().items())
[('x', 0), ('y', 1), ('z', 2)]
sage: T
+  x y z
 +------
x| x y z
y| y z x
z| z x y
```

**summation** $(x, y)$

Return the sum of `x` and `y`.

The binary addition operator of this additive magma.

INPUT:

- `x`, `y` – elements of this additive magma

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example()
sage: (a,b,c,d) = S.additive_semigroup_generators()
sage: S.summation(a, b)
a + b
```

A parent in `AdditiveMagmas()` must either implement *`summation()`* in the parent class or `_add_` in the element class. By default, the addition method on elements `x._add_(y)` calls `S.summation(x,y)`, and reciprocally.

As a bonus effect, `S.summation` by itself models the binary function from `S` to `S`:

```
sage: bin = S.summation
sage: bin(a,b)
a + b
```

Here, `S.summation` is just a bound method. Whenever possible, it is recommended to enrich `S.summation` with extra mathematical structure. Lazy attributes can come handy for this.

**Todo:** Add an example.

**summation_from_element_class_add**(*x*, *y*)

Return the sum of `x` and `y`.

The binary addition operator of this additive magma.

INPUT:

- `x`, `y` – elements of this additive magma

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example()
sage: (a,b,c,d) = S.additive_semigroup_generators()
sage: S.summation(a, b)
a + b
```

A parent in `AdditiveMagmas()` must either implement *summation()* in the parent class or `_add_` in the element class. By default, the addition method on elements `x._add_(y)` calls `S.summation(x,y)`, and reciprocally.

As a bonus effect, `S.summation` by itself models the binary function from `S` to `S`:

```
sage: bin = S.summation
sage: bin(a,b)
a + b
```

Here, `S.summation` is just a bound method. Whenever possible, it is recommended to enrich `S.summation` with extra mathematical structure. Lazy attributes can come handy for this.

**Todo:** Add an example.

**class SubcategoryMethods**

**AdditiveAssociative**()

Return the full subcategory of the additive associative objects of `self`.

An *additive magma* $M$ is *associative* if, for all $x, y, z \in M$,

$$x + (y + z) = (x + y) + z$$

**See also:**

Wikipedia article Associative_property

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveAssociative()
Category of additive semigroups
```

**AdditiveCommutative**()

Return the full subcategory of the commutative objects of `self`.

An *additive magma* $M$ is *commutative* if, for all $x, y \in M$,

$$x + y = y + x$$

**See also:**

Wikipedia article Commutative_property

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveCommutative()
Category of additive commutative additive magmas
sage: AdditiveMagmas().AdditiveAssociative().AdditiveUnital().
↪AdditiveCommutative()
Category of commutative additive monoids
sage: _ is CommutativeAdditiveMonoids()
True
```

**AdditiveUnital**()

Return the subcategory of the unital objects of `self`.

An *additive magma* $M$ is *unital* if it admits an element $0$, called *neutral element*, such that for all $x \in M$,

$$0 + x = x + 0 = x$$

This element is necessarily unique, and should be provided as `M.zero()`.

**See also:**

Wikipedia article Unital_magma#unital

EXAMPLES:

```
sage: AdditiveMagmas().AdditiveUnital()
Category of additive unital additive magmas
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: AdditiveSemigroups().AdditiveUnital()
Category of additive monoids
sage: CommutativeAdditiveMonoids().AdditiveUnital()
Category of commutative additive monoids
```

**super_categories**()

EXAMPLES:

```
sage: AdditiveMagmas().super_categories()
[Category of sets]
```

## 3.4 Additive monoids

**class** sage.categories.additive_monoids.**AdditiveMonoids**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of additive monoids.

An *additive monoid* is a unital `additive semigroup`, that is a set endowed with a binary operation $+$ which is associative and admits a zero (see Wikipedia article Monoid).

EXAMPLES:

```
sage: from sage.categories.additive_monoids import AdditiveMonoids
sage: C = AdditiveMonoids(); C
Category of additive monoids
sage: C.super_categories()
[Category of additive unital additive magmas, Category of additive semigroups]
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveUnital']
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: C is AdditiveSemigroups().AdditiveUnital()
True
```

**AdditiveCommutative**
alias of `CommutativeAdditiveMonoids`

**AdditiveInverse**
alias of `AdditiveGroups`

**class Homsets**(*category*, **args*)
Bases: `sage.categories.homsets.HomsetsCategory`

**extra_super_categories**()
Implement the fact that a homset between two monoids is associative.

EXAMPLES:

```
sage: from sage.categories.additive_monoids import AdditiveMonoids
sage: AdditiveMonoids().Homsets().extra_super_categories()
[Category of additive semigroups]
sage: AdditiveMonoids().Homsets().super_categories()
[Category of homsets of additive unital additive magmas, Category of␣
↪additive monoids]
```

**Todo:** This could be deduced from `AdditiveSemigroups.Homsets.extra_super_categories()`. See comment in `Objects.SubcategoryMethods.Homsets()`.

**class ParentMethods**

**sum**(*args*)
Return the sum of the elements in `args`, as an element of `self`.

INPUT:

- `args` – a list (or iterable) of elements of `self`

EXAMPLES:

```
sage: S = CommutativeAdditiveMonoids().example()
sage: (a,b,c,d) = S.additive_semigroup_generators()
sage: S.sum((a,b,a,c,a,b))
3*a + c + 2*b
sage: S.sum(())
0
```

```
sage: S.sum(()).parent() == S
True
```

## 3.5 Additive semigroups

**class** sage.categories.additive_semigroups.**AdditiveSemigroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of additive semigroups.

An *additive semigroup* is an associative *additive magma*, that is a set endowed with an operation + which is associative.

EXAMPLES:

```
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: C = AdditiveSemigroups(); C
Category of additive semigroups
sage: C.super_categories()
[Category of additive magmas]
sage: C.all_super_categories()
[Category of additive semigroups,
 Category of additive magmas,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]

sage: C.axioms()
frozenset({'AdditiveAssociative'})
sage: C is AdditiveMagmas().AdditiveAssociative()
True
```

**AdditiveCommutative**

alias of *CommutativeAdditiveSemigroups*

**AdditiveUnital**

alias of AdditiveMonoids

**class Algebras**(*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**algebra_generators**()

Return the generators of this algebra, as per *MagmaticAlgebras.ParentMethods.algebra_generators()*.

They correspond to the generators of the additive semigroup.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative monoid: the free commutative monoid␣
↪generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)
sage: A.algebra_generators()
Finite family {0: B[a], 1: B[b], 2: B[c], 3: B[d]}
```

**product_on_basis**(*g1*, *g2*)

Product, on basis elements, as per *MagmaticAlgebras.WithBasis.ParentMethods.product_on_basis()*.

The product of two basis elements is induced by the addition of the corresponding elements of the group.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative monoid: the free commutative monoid␣
↪generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)
sage: a,b,c,d = A.algebra_generators()
sage: a * b + b * d * c
B[c + b + d] + B[a + b]
```

**extra_super_categories**()

EXAMPLES:

```
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: AdditiveSemigroups().Algebras(QQ).extra_super_categories()
[Category of semigroups]
sage: CommutativeAdditiveSemigroups().Algebras(QQ).super_categories()
[Category of additive semigroup algebras over Rational Field,
 Category of additive commutative additive magma algebras over Rational␣
↪Field]
```

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**extra_super_categories**()

Implement the fact that a Cartesian product of additive semigroups is an additive semigroup.

EXAMPLES:

```
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: C = AdditiveSemigroups().CartesianProducts()
sage: C.extra_super_categories()
[Category of additive semigroups]
sage: C.axioms()
frozenset({'AdditiveAssociative'})
```

**class Homsets**(*category*, **args*)

Bases: *sage.categories.homsets.HomsetsCategory*

**extra_super_categories**()

Implement the fact that a homset between two semigroups is a semigroup.

EXAMPLES:

```
sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: AdditiveSemigroups().Homsets().extra_super_categories()
[Category of additive semigroups]
sage: AdditiveSemigroups().Homsets().super_categories()
[Category of homsets of additive magmas, Category of additive semigroups]
```

**class ParentMethods**

## 3.6 Affine Weyl Groups

**class** sage.categories.affine_weyl_groups.**AffineWeylGroups**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of affine Weyl groups

**Todo:** add a description of this category

**See also:**

- Wikipedia article Affine_weyl_group
- *WeylGroups*, WeylGroup

EXAMPLES:

```
sage: C = AffineWeylGroups(); C
Category of affine weyl groups
sage: C.super_categories()
[Category of infinite weyl groups]

sage: C.example()
NotImplemented
sage: W = WeylGroup(["A",4,1]); W
Weyl Group of type ['A', 4, 1] (as a matrix group acting on the root space)
sage: W.category()
Category of irreducible affine weyl groups
```

**class ElementMethods**

**affine_grassmannian_to_core**()

Bijection between affine Grassmannian elements of type $A_k^{(1)}$ and $(k+1)$-cores.

INPUT:

- self – an affine Grassmannian element of some affine Weyl group of type $A_k^{(1)}$

Recall that an element $w$ of an affine Weyl group is affine Grassmannian if all its all reduced words end in 0, see *is_affine_grassmannian()*.

OUTPUT:

- a $(k+1)$-core

See also *affine_grassmannian_to_partition()*.

EXAMPLES:

```
sage: W = WeylGroup(['A',2,1])
sage: w = W.from_reduced_word([0,2,1,0])
sage: la = w.affine_grassmannian_to_core(); la
[4, 2]
sage: type(la)
<class 'sage.combinat.core.Cores_length_with_category.element_class'>
sage: la.to_grassmannian() == w
True

sage: w = W.from_reduced_word([0,2,1])
sage: w.affine_grassmannian_to_core()
```

```
Traceback (most recent call last):
...
ValueError: Error! this only works on type 'A' affine Grassmannian␣
↪elements
```

**affine_grassmannian_to_partition**()

Bijection between affine Grassmannian elements of type $A_k^{(1)}$ and $k$-bounded partitions.

INPUT:

- `self` is affine Grassmannian element of the affine Weyl group of type $A_k^{(1)}$ (i.e. all reduced words end in 0)

OUTPUT:

- $k$-bounded partition

See also *affine_grassmannian_to_core()*.

EXAMPLES:

```
sage: k = 2
sage: W = WeylGroup(['A',k,1])
sage: w = W.from_reduced_word([0,2,1,0])
sage: la = w.affine_grassmannian_to_partition(); la
[2, 2]
sage: la.from_kbounded_to_grassmannian(k) == w
True
```

**is_affine_grassmannian**()

Tests whether `self` is affine Grassmannian

An element of an affine Weyl group is *affine Grassmannian* if any of the following equivalent properties holds:

- all reduced words for self end with 0.
- self is the identity, or 0 is its single right descent.
- self is a mimimal coset representative for W / cl W.

EXAMPLES:

```
sage: W=WeylGroup(['A',3,1])
sage: w=W.from_reduced_word([2,1,0])
sage: w.is_affine_grassmannian()
True
sage: w=W.from_reduced_word([2,0])
sage: w.is_affine_grassmannian()
False
sage: W.one().is_affine_grassmannian()
True
```

**class ParentMethods**

**affine_grassmannian_elements_of_given_length**($k$)

Returns the affine Grassmannian elements of length $k$, as a list.

EXAMPLES:

```
sage: W=WeylGroup(['A',3,1])
sage: [x.reduced_word() for x in W.affine_grassmannian_elements_of_given_
↪length(3)]
[[2, 1, 0], [3, 1, 0], [2, 3, 0]]
```

**See also:**

*AffineWeylGroups.ElementMethods.is_affine_grassmannian()*

**Todo:** should return an enumerated set, with iterator, ...

**special_node()**

Returns the distinguished special node of the underlying Dynkin diagram

EXAMPLES:

```
sage: W=WeylGroup(['A',3,1])
sage: W.special_node()
0
```

**additional_structure()**

Return `None`.

Indeed, the category of affine Weyl groups defines no additional structure: affine Weyl groups are a special class of Weyl groups.

**See also:**

*Category.additional_structure()*

**Todo:** Should this category be a `CategoryWithAxiom`?

EXAMPLES:

```
sage: AffineWeylGroups().additional_structure()
```

**super_categories()**

EXAMPLES:

```
sage: AffineWeylGroups().super_categories()
[Category of infinite weyl groups]
```

## 3.7 AlgebraIdeals

**class** sage.categories.algebra_ideals.**AlgebraIdeals**($A$)

Bases: *sage.categories.category_types.Category_ideal*

The category of two-sided ideals in a fixed algebra $A$.

EXAMPLES:

```
sage: AlgebraIdeals(QQ['a'])
Category of algebra ideals in Univariate Polynomial Ring in a over Rational Field
```

**Todo:**

- Add support for non commutative rings (this is currently not supported by the subcategory *AlgebraModules*).
- Make `AlgebraIdeals(R)`, return `CommutativeAlgebraIdeals(R)` when `R` is commutative.

- If useful, implement `AlgebraLeftIdeals` and `AlgebraRightIdeals` of which `AlgebraIdeals` would be a subcategory.

---

**algebra**()

EXAMPLES:

```
sage: AlgebraIdeals(QQ['x']).algebra()
Univariate Polynomial Ring in x over Rational Field
```

**super_categories**()

The category of algebra modules should be a super category of this category.

However, since algebra modules are currently only available over commutative rings, we have to omit it if our ring is non-commutative.

EXAMPLES:

```
sage: AlgebraIdeals(QQ['x']).super_categories()
[Category of algebra modules over Univariate Polynomial Ring in x over␣
↪Rational Field]
sage: C = AlgebraIdeals(FreeAlgebra(QQ,2,'a,b'))
sage: C.super_categories()
[]
```

## 3.8 Algebra modules

**class** sage.categories.algebra_modules.**AlgebraModules**($A$)

Bases: *sage.categories.category_types.Category_module*

The category of modules over a fixed algebra $A$.

EXAMPLES:

```
sage: AlgebraModules(QQ['a'])
Category of algebra modules over Univariate Polynomial Ring in a over Rational␣
↪Field
sage: AlgebraModules(QQ['a']).super_categories()
[Category of modules over Univariate Polynomial Ring in a over Rational Field]
```

Note: as of now, $A$ is required to be commutative, ensuring that the categories of left and right modules are isomorphic. Feedback and use cases for potential generalizations to the non commutative case are welcome.

**algebra**()

EXAMPLES:

```
sage: AlgebraModules(QQ['x']).algebra()
Univariate Polynomial Ring in x over Rational Field
```

**classmethod an_instance**()

Returns an instance of this class

EXAMPLES:

```
sage: AlgebraModules.an_instance()
Category of algebra modules over Univariate Polynomial Ring in x over␣
↪Rational Field
```

**super_categories**()

EXAMPLES:

```
sage: AlgebraModules(QQ['x']).super_categories()
[Category of modules over Univariate Polynomial Ring in x over Rational Field]
```

## 3.9 Algebras

AUTHORS:

- David Kohel & William Stein (2005): initial revision
- Nicolas M. Thiery (2008-2011): rewrote for the category framework

**class** sage.categories.algebras.**Algebras**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of associative and unital algebras over a given base ring.

An associative and unital algebra over a ring $R$ is a module over $R$ which is itself a ring.

**Warning:** *Algebras* will be eventually be replaced by *magmatic_algebras.MagmaticAlgebras* for consistency with e.g. Wikipedia article Algebras which assumes neither associativity nor the existence of a unit (see trac ticket #15043).

**Todo:** Should $R$ be a commutative ring?

EXAMPLES:

```
sage: Algebras(ZZ)
Category of algebras over Integer Ring
sage: sorted(Algebras(ZZ).super_categories(), key=str)
[Category of associative algebras over Integer Ring,
 Category of rings,
 Category of unital algebras over Integer Ring]
```

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

The category of algebras constructed as Cartesian products of algebras

This construction gives the direct product of algebras. See discussion on:

- http://groups.google.fr/group/sage-devel/browse_thread/thread/35a72b1d0a2fc77a/348f42ae77a66d16#348f42ae77a66d16
- Wikipedia article Direct_product

**extra_super_categories**()

A Cartesian product of algebras is endowed with a natural algebra structure.

EXAMPLES:

```
sage: C = Algebras(QQ).CartesianProducts()
sage: C.extra_super_categories()
[Category of algebras over Rational Field]
```

```
sage: sorted(C.super_categories(), key=str)
[Category of Cartesian products of distributive magmas and additive␣
↪magmas,
 Category of Cartesian products of monoids,
 Category of Cartesian products of vector spaces over Rational Field,
 Category of algebras over Rational Field]
```

**Commutative**

alias of *CommutativeAlgebras*

**class DualObjects** (*category*, **args*)

Bases: *sage.categories.dual.DualObjectsCategory*

**extra_super_categories** ()

Returns the dual category

EXAMPLES:

The category of algebras over the Rational Field is dual to the category of coalgebras over the same field:

```
sage: C = Algebras(QQ)
sage: C.dual()
Category of duals of algebras over Rational Field
sage: C.dual().extra_super_categories()
[Category of coalgebras over Rational Field]
```

**Warning:** This is only correct in certain cases (finite dimension, . . . ). See trac ticket #15647.

**class ElementMethods**

**Filtered**

alias of `FilteredAlgebras`

**Graded**

alias of *GradedAlgebras*

**class Quotients** (*category*, **args*)

Bases: *sage.categories.quotients.QuotientsCategory*

**class ParentMethods**

**algebra_generators** ()

Return algebra generators for `self`.

This implementation retracts the algebra generators from the ambient algebra.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: S = A.semisimple_quotient()
sage: S.algebra_generators()
Finite family {'y': B['y'], 'x': B['x'], 'b': 0, 'a': 0}
```

**Todo:** this could possibly remove the elements that retract to zero.

**Semisimple**
alias of SemisimpleAlgebras

**class SubcategoryMethods**

**Semisimple**()
Return the subcategory of semisimple objects of self.

**Note:** This mimics the syntax of axioms for a smooth transition if Semisimple becomes one.

EXAMPLES:

```
sage: Algebras(QQ).Semisimple()
Category of semisimple algebras over Rational Field
sage: Algebras(QQ).WithBasis().FiniteDimensional().Semisimple()
Category of finite dimensional semisimple algebras with basis over␣
↪Rational Field
```

**Super**
alias of SuperAlgebras

**class TensorProducts**(*category*, **args*)
Bases: *sage.categories.tensor.TensorProductsCategory*

**class ElementMethods**

**class ParentMethods**

**extra_super_categories**()
EXAMPLES:

```
sage: Algebras(QQ).TensorProducts().extra_super_categories()
[Category of algebras over Rational Field]
sage: Algebras(QQ).TensorProducts().super_categories()
[Category of algebras over Rational Field,
 Category of tensor products of vector spaces over Rational Field]
```

Meaning: a tensor product of algebras is an algebra

**WithBasis**
alias of *AlgebrasWithBasis*

## 3.10 Algebras With Basis

**class** sage.categories.algebras_with_basis.**AlgebrasWithBasis**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of algebras with a distinguished basis.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ); C
Category of algebras with basis over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of algebras over Rational Field,
 Category of unital algebras with basis over Rational Field]
```

We construct a typical parent in this category, and do some computations with it:

```
sage: A = C.example(); A
An example of an algebra with basis: the free algebra on the generators ('a', 'b',
↪ 'c') over Rational Field

sage: A.category()
Category of algebras with basis over Rational Field

sage: A.one_basis()
word:
sage: A.one()
B[word: ]

sage: A.base_ring()
Rational Field
sage: A.basis().keys()
Finite words over {'a', 'b', 'c'}

sage: (a,b,c) = A.algebra_generators()
sage: a^3, b^2
(B[word: aaa], B[word: bb])
sage: a*c*b
B[word: acb]

sage: A.product
<bound method FreeAlgebra_with_category._product_from_product_on_basis_multiply of
 An example of an algebra with basis: the free algebra on the generators ('a', 'b
↪', 'c') over Rational Field>
sage: A.product(a*b,b)
B[word: abb]

sage: TestSuite(A).run(verbose=True)
running ._test_additive_associativity() . . . pass
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_characteristic() . . . pass
running ._test_distributivity() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
```

```
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
running ._test_zero() . . . pass
sage: A.__class__
<class 'sage.categories.examples.algebras_with_basis.FreeAlgebra_with_category'>
sage: A.element_class
<class 'sage.categories.examples.algebras_with_basis.FreeAlgebra_with_category.
↪element_class'>
```

Please see the source code of $A$ (with `A??`) for how to implement other algebras with basis.

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

The category of algebras with basis, constructed as Cartesian products of algebras with basis.

Note: this construction give the direct products of algebras with basis. See comment in *Algebras.CartesianProducts*

**class ParentMethods**

**one**()

**one_from_cartesian_product_of_one_basis**()

Returns the one of this Cartesian product of algebras, as per `Monoids.ParentMethods.one`

It is constructed as the Cartesian product of the ones of the summands, using their `one_basis()` methods.

This implementation does not require multiplication by scalars nor calling cartesian_product. This might help keeping things as lazy as possible upon initialization.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example(); A
An example of an algebra with basis: the free algebra on the␣
↪generators ('a', 'b', 'c') over Rational Field
sage: A.one_basis()
word:

sage: B = cartesian_product((A, A, A))
sage: B.one_from_cartesian_product_of_one_basis()
B[(0, word: )] + B[(1, word: )] + B[(2, word: )]
sage: B.one()
B[(0, word: )] + B[(1, word: )] + B[(2, word: )]

sage: cartesian_product([SymmetricGroupAlgebra(QQ, 3),␣
↪SymmetricGroupAlgebra(QQ, 4)]).one()
B[(0, [1, 2, 3])] + B[(1, [1, 2, 3, 4])]
```

**extra_super_categories**()

A Cartesian product of algebras with basis is endowed with a natural algebra with basis structure.

EXAMPLES:

```
sage: AlgebrasWithBasis(QQ).CartesianProducts().extra_super_categories()
[Category of algebras with basis over Rational Field]
sage: AlgebrasWithBasis(QQ).CartesianProducts().super_categories()
[Category of algebras with basis over Rational Field,
 Category of Cartesian products of algebras over Rational Field,
 Category of Cartesian products of vector spaces with basis over Rational␣
↪Field]
```

**class ElementMethods**

**Filtered**
alias of `FilteredAlgebrasWithBasis`

**FiniteDimensional**
alias of *`FiniteDimensionalAlgebrasWithBasis`*

**Graded**
alias of *`GradedAlgebrasWithBasis`*

**class ParentMethods**

**hochschild_complex**(*M*)
Return the Hochschild complex of `self` with coefficients in `M`.

**See also:**

`HochschildComplex`

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: A = algebras.DifferentialWeyl(R)
sage: H = A.hochschild_complex(A)

sage: SGA = SymmetricGroupAlgebra(QQ, 3)
sage: T = SGA.trivial_representation()
sage: H = SGA.hochschild_complex(T)
```

**one**()
Return the multiplicative unit element.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()
sage: A.one_basis()
word:
sage: A.one()
B[word: ]
```

**Super**
alias of `SuperAlgebrasWithBasis`

**class TensorProducts**(*category*, **args*)
Bases: *`sage.categories.tensor.TensorProductsCategory`*

The category of algebras with basis constructed by tensor product of algebras with basis

**class ElementMethods**
Implements operations on elements of tensor products of algebras with basis

**class ParentMethods**
implements operations on tensor products of algebras with basis

**one_basis**()
Returns the index of the one of this tensor product of algebras, as per `AlgebrasWithBasis.ParentMethods.one_basis`

It is the tuple whose operands are the indices of the ones of the operands, as returned by their *one_basis()* methods.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example(); A
An example of an algebra with basis: the free algebra on the␣
↪generators ('a', 'b', 'c') over Rational Field
sage: A.one_basis()
word:
sage: B = tensor((A, A, A))
sage: B.one_basis()
(word: , word: , word: )
sage: B.one()
B[word: ] # B[word: ] # B[word: ]
```

**product_on_basis**(*t1*, *t2*)
The product of the algebra on the basis, as per `AlgebrasWithBasis.ParentMethods.product_on_basis`.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example(); A
An example of an algebra with basis: the free algebra on the␣
↪generators ('a', 'b', 'c') over Rational Field
sage: (a,b,c) = A.algebra_generators()

sage: x = tensor( (a, b, c) ); x
B[word: a] # B[word: b] # B[word: c]
sage: y = tensor( (c, b, a) ); y
B[word: c] # B[word: b] # B[word: a]
sage: x*y
B[word: ac] # B[word: bb] # B[word: ca]

sage: x = tensor( ((a+2*b), c) )    ; x
B[word: a] # B[word: c] + 2*B[word: b] # B[word: c]
sage: y = tensor( (c,       a) ) + 1; y
B[word: ] # B[word: ] + B[word: c] # B[word: a]
sage: x*y
B[word: a] # B[word: c] + B[word: ac] # B[word: ca] + 2*B[word: b] #␣
↪B[word: c] + 2*B[word: bc] # B[word: ca]
```

TODO: optimize this implementation!

**extra_super_categories**()
EXAMPLES:

```
sage: AlgebrasWithBasis(QQ).TensorProducts().extra_super_categories()
[Category of algebras with basis over Rational Field]
sage: AlgebrasWithBasis(QQ).TensorProducts().super_categories()
[Category of algebras with basis over Rational Field,
 Category of tensor products of algebras over Rational Field,
 Category of tensor products of vector spaces with basis over Rational␣
↪Field]
```

**example** (*alphabet=('a', 'b', 'c')*)

Return an example of algebra with basis.

EXAMPLES:

```
sage: AlgebrasWithBasis(QQ).example()
An example of an algebra with basis: the free algebra on the generators ('a',
↪'b', 'c') over Rational Field
```

An other set of generators can be specified as optional argument:

```
sage: AlgebrasWithBasis(QQ).example((1,2,3))
An example of an algebra with basis: the free algebra on the generators (1, 2,
↪ 3) over Rational Field
```

## 3.11 Aperiodic semigroups

**class** sage.categories.aperiodic_semigroups.**AperiodicSemigroups** (*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**extra_super_categories** ()

Implement the fact that an aperiodic semigroup is $H$-trivial.

EXAMPLES:

```
sage: Semigroups().Aperiodic().extra_super_categories()
[Category of h trivial semigroups]
```

## 3.12 Associative algebras

**class** sage.categories.associative_algebras.**AssociativeAlgebras** (*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of associative algebras over a given base ring.

An associative algebra over a ring $R$ is a module over $R$ which is also a not necessarily unital ring.

> **Warning:** Until trac ticket #15043 is implemented, *Algebras* is the category of associative unital algebras; thus, unlike the name suggests, *AssociativeAlgebras* is not a subcategory of *Algebras* but of *MagmaticAlgebras*.

EXAMPLES:

```
sage: from sage.categories.associative_algebras import AssociativeAlgebras
sage: C = AssociativeAlgebras(ZZ); C
Category of associative algebras over Integer Ring
```

**Unital**

alias of *Algebras*

## 3.13 Bialgebras

**class** sage.categories.bialgebras.**Bialgebras**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of bialgebras

EXAMPLES:

```
sage: Bialgebras(ZZ)
Category of bialgebras over Integer Ring
sage: Bialgebras(ZZ).super_categories()
[Category of algebras over Integer Ring, Category of coalgebras over Integer Ring]
```

**class Super**(*base_category*)

Bases: *sage.categories.super_modules.SuperModulesCategory*

**WithBasis**

alias of *BialgebrasWithBasis*

**additional_structure**()

Return None.

Indeed, the category of bialgebras defines no additional structure: a morphism of coalgebras and of algebras between two bialgebras is a bialgebra morphism.

**See also:**

*Category.additional_structure()*

**Todo:** This category should be a CategoryWithAxiom.

EXAMPLES:

```
sage: Bialgebras(QQ).additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: Bialgebras(QQ).super_categories()
[Category of algebras over Rational Field, Category of coalgebras over Rational Field]
```

## 3.14 Bialgebras with basis

**class** sage.categories.bialgebras_with_basis.**BialgebrasWithBasis**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of bialgebras with a distinguished basis.

EXAMPLES:

```
sage: C = BialgebrasWithBasis(QQ); C
Category of bialgebras with basis over Rational Field

sage: sorted(C.super_categories(), key=str)
```

```
[Category of algebras with basis over Rational Field,
 Category of bialgebras over Rational Field,
 Category of coalgebras with basis over Rational Field]
```

**class ElementMethods**

**adams_operator**(*n*)

Compute the $n$-th convolution power of the identity morphism Id on `self`.

INPUT:

- `n` – a nonnegative integer

OUTPUT:

- the image of `self` under the convolution power $\mathrm{Id}^{*n}$

**Note:** In the literature, this is also called a Hopf power or Sweedler power, cf. [AL2015].

**See also:**

`sage.categories.bialgebras.ElementMethods.convolution_product()`

**Todo:** Remove dependency on `modules_with_basis` methods.

EXAMPLES:

```
sage: h = SymmetricFunctions(QQ).h()
sage: h[5].adams_operator(2)
2*h[3, 2] + 2*h[4, 1] + 2*h[5]
sage: h[5].plethysm(2*h[1])
2*h[3, 2] + 2*h[4, 1] + 2*h[5]
sage: h([]).adams_operator(0)
h[]
sage: h([]).adams_operator(1)
h[]
sage: h[3,2].adams_operator(0)
0
sage: h[3,2].adams_operator(1)
h[3, 2]
```

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: S[4].adams_operator(5)
5*S[1, 1, 1, 1] + 10*S[1, 1, 2] + 10*S[1, 2, 1] + 10*S[1, 3] + 10*S[2, 1,
↪1] + 10*S[2, 2] + 10*S[3, 1] + 5*S[4]
```

```
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: m[[1,3],[2]].adams_operator(-2)
3*m{{1}, {2, 3}} + 3*m{{1, 2}, {3}} + 6*m{{1, 2, 3}} - 2*m{{1, 3}, {2}}
```

**convolution_product**(**maps*)

Return the image of `self` under the convolution product (map) of the maps.

Let $A$ and $B$ be bialgebras over a commutative ring $R$. Given maps $f_i : A \to B$ for $1 \leq i < n$, define the convolution product

$$(f_1 * f_2 * \cdots * f_n) := \mu^{(n-1)} \circ (f_1 \otimes f_2 \otimes \cdots \otimes f_n) \circ \Delta^{(n-1)},$$

where $\Delta^{(k)} := \left(\Delta \otimes \mathrm{Id}^{\otimes(k-1)}\right) \circ \Delta^{(k-1)}$, with $\Delta^{(1)} = \Delta$ (the ordinary coproduct in $A$) and $\Delta^{(0)} = \mathrm{Id}$; and with $\mu^{(k)} := \mu \circ \left(\mu^{(k-1)} \otimes \mathrm{Id}\right)$ and $\mu^{(1)} = \mu$ (the ordinary product in $B$). See [Swe1969].

(In the literature, one finds, e.g., $\Delta^{(2)}$ for what we denote above as $\Delta^{(1)}$. See [KMN2012].)

INPUT:

- `maps` – any number $n \geq 0$ of linear maps $f_1, f_2, \ldots, f_n$ on `self.parent()`; or a single `list` or `tuple` of such maps

OUTPUT:

- the convolution product of `maps` applied to `self`

AUTHORS:

- Amy Pang - 12 June 2015 - Sage Days 65

**Todo:** Remove dependency on `modules_with_basis` methods.

EXAMPLES:

We compute convolution products of the identity and antipode maps on Schur functions:

```
sage: Id = lambda x: x
sage: Antipode = lambda x: x.antipode()
sage: s = SymmetricFunctions(QQ).schur()
sage: s[3].convolution_product(Id, Id)
2*s[2, 1] + 4*s[3]
sage: s[3,2].convolution_product(Id) == s[3,2]
True
```

The method accepts multiple arguments, or a single argument consisting of a list of maps:

```
sage: s[3,2].convolution_product(Id, Id)
2*s[2, 1, 1, 1] + 6*s[2, 2, 1] + 6*s[3, 1, 1] + 12*s[3, 2] + 6*s[4, 1] +
↪2*s[5]
sage: s[3,2].convolution_product([Id, Id])
2*s[2, 1, 1, 1] + 6*s[2, 2, 1] + 6*s[3, 1, 1] + 12*s[3, 2] + 6*s[4, 1] +
↪2*s[5]
```

We test the defining property of the antipode morphism; namely, that the antipode is the inverse of the identity map in the convolution algebra whose identity element is the composition of the counit and unit:

```
sage: s[3,2].convolution_product() == s[3,2].convolution_product(Antipode,
↪ Id) == s[3,2].convolution_product(Id, Antipode)
True
```

```
sage: Psi = NonCommutativeSymmetricFunctions(QQ).Psi()
sage: Psi[2,1].convolution_product(Id, Id, Id)
3*Psi[1, 2] + 6*Psi[2, 1]
sage: (Psi[5,1] - Psi[1,5]).convolution_product(Id, Id, Id)
-3*Psi[1, 5] + 3*Psi[5, 1]
```

```
sage: G = SymmetricGroup(3)
sage: QG = GroupAlgebra(G,QQ)
sage: x = QG.sum_of_terms([(p,p.length()) for p in Permutations(3)]); x
[1, 3, 2] + [2, 1, 3] + 2*[2, 3, 1] + 2*[3, 1, 2] + 3*[3, 2, 1]
sage: x.convolution_product(Id, Id)
5*[1, 2, 3] + 2*[2, 3, 1] + 2*[3, 1, 2]
sage: x.convolution_product(Id, Id, Id)
```

```
4*[1, 2, 3] + [1, 3, 2] + [2, 1, 3] + 3*[3, 2, 1]
sage: x.convolution_product([Id]*6)
9*[1, 2, 3]
```

```
sage: h = SymmetricFunctions(QQ).h()
sage: h[5].convolution_product([Id, Id])
2*h[3, 2] + 2*h[4, 1] + 2*h[5]
sage: h.one().convolution_product([Id, Antipode])
h[]
sage: h[3,2].convolution_product([Id, Antipode])
0
sage: h.one().convolution_product([Id, Antipode]) == h.one().convolution_
↪product()
True
```

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: S[4].convolution_product([Id]*5)
5*S[1, 1, 1, 1] + 10*S[1, 1, 2] + 10*S[1, 2, 1] + 10*S[1, 3]
 + 10*S[2, 1, 1] + 10*S[2, 2] + 10*S[3, 1] + 5*S[4]
```

```
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: m[[1,3],[2]].convolution_product([Antipode, Antipode])
3*m{{1}, {2, 3}} + 3*m{{1, 2}, {3}} + 6*m{{1, 2, 3}} - 2*m{{1, 3}, {2}}
sage: m[[]].convolution_product([])
m{}
sage: m[[1,3],[2]].convolution_product([])
0
```

```
sage: QS = SymmetricGroupAlgebra(QQ, 5)
sage: x = QS.sum_of_terms(zip(Permutations(5)[3:6],[1,2,3])); x
[1, 2, 4, 5, 3] + 2*[1, 2, 5, 3, 4] + 3*[1, 2, 5, 4, 3]
sage: x.convolution_product([Antipode, Id])
6*[1, 2, 3, 4, 5]
sage: x.convolution_product(Id, Antipode, Antipode, Antipode)
3*[1, 2, 3, 4, 5] + [1, 2, 4, 5, 3] + 2*[1, 2, 5, 3, 4]
```

```
sage: G = SymmetricGroup(3)
sage: QG = GroupAlgebra(G,QQ)
sage: x = QG.sum_of_terms([(p,p.length()) for p in Permutations(3)]); x
[1, 3, 2] + [2, 1, 3] + 2*[2, 3, 1] + 2*[3, 1, 2] + 3*[3, 2, 1]
sage: x.convolution_product(Antipode, Id)
9*[1, 2, 3]
sage: x.convolution_product([Id, Antipode, Antipode, Antipode])
5*[1, 2, 3] + 2*[2, 3, 1] + 2*[3, 1, 2]
```

```
sage: s[3,2].counit().parent() == s[3,2].convolution_product().parent()
False
```

**class ParentMethods**

**convolution_product**(**maps*)

Return the convolution product (a map) of the given maps.

Let $A$ and $B$ be bialgebras over a commutative ring $R$. Given maps $f_i : A \to B$ for $1 \leq i < n$, define the convolution product

$$(f_1 * f_2 * \cdots * f_n) := \mu^{(n-1)} \circ (f_1 \otimes f_2 \otimes \cdots \otimes f_n) \circ \Delta^{(n-1)},$$

where $\Delta^{(k)} := \left(\Delta \otimes \mathrm{Id}^{\otimes(k-1)}\right) \circ \Delta^{(k-1)}$, with $\Delta^{(1)} = \Delta$ (the ordinary coproduct in $A$) and $\Delta^{(0)} = \mathrm{Id}$; and with $\mu^{(k)} := \mu \circ \left(\mu^{(k-1)} \otimes \mathrm{Id}\right)$ and $\mu^{(1)} = \mu$ (the ordinary product in $B$). See [Swe1969].

(In the literature, one finds, e.g., $\Delta^{(2)}$ for what we denote above as $\Delta^{(1)}$. See [KMN2012].)

INPUT:

- `maps` – any number $n \geq 0$ of linear maps $f_1, f_2, \ldots, f_n$ on `self`; or a single `list` or `tuple` of such maps

OUTPUT:

- the new map $f_1 * f_2 * \cdots * f_2$ representing their convolution product

**See also:**

`sage.categories.bialgebras.ElementMethods.convolution_product()`

AUTHORS:

- Aaron Lauve - 12 June 2015 - Sage Days 65

**Todo:** Remove dependency on `modules_with_basis` methods.

EXAMPLES:

We construct some maps: the identity, the antipode and projection onto the homogeneous componente of degree 2:

```
sage: Id = lambda x: x
sage: Antipode = lambda x: x.antipode()
sage: Proj2 = lambda x: x.parent().sum_of_terms([(m, c) for (m, c) in x␣
↪if m.size() == 2])
```

Compute the convolution product of the identity with itself and with the projection `Proj2` on the Hopf algebra of non-commutative symmetric functions:

```
sage: R = NonCommutativeSymmetricFunctions(QQ).ribbon()
sage: T = R.convolution_product([Id, Id])
sage: [T(R(comp)) for comp in Compositions(3)]
[4*R[1, 1, 1] + R[1, 2] + R[2, 1],
 2*R[1, 1, 1] + 4*R[1, 2] + 2*R[2, 1] + 2*R[3],
 2*R[1, 1, 1] + 2*R[1, 2] + 4*R[2, 1] + 2*R[3],
 R[1, 2] + R[2, 1] + 4*R[3]]
sage: T = R.convolution_product(Proj2, Id)
sage: [T(R([i])) for i in range(1, 5)]
[0, R[2], R[2, 1] + R[3], R[2, 2] + R[4]]
```

Compute the convolution product of no maps on the Hopf algebra of symmetric functions in non-commuting variables. This is the composition of the counit with the unit:

```
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: T = m.convolution_product()
sage: [T(m(lam)) for lam in SetPartitions(0).list() + SetPartitions(2).
↪list()]
[m{}, 0, 0]
```

Compute the convolution product of the projection `Proj2` with the identity on the Hopf algebra of symmetric functions in non-commuting variables:

```
sage: T = m.convolution_product(Proj2, Id)
sage: [T(m(lam)) for lam in SetPartitions(3)]
[0,
 m{{1, 2}, {3}} + m{{1, 2, 3}},
 m{{1, 2}, {3}} + m{{1, 2, 3}},
 m{{1, 2}, {3}} + m{{1, 2, 3}},
 3*m{{1}, {2}, {3}} + 3*m{{1}, {2, 3}} + 3*m{{1, 3}, {2}}]
```

Compute the convolution product of the antipode with itself and the identity map on group algebra of the symmetric group:

```
sage: G = SymmetricGroup(3)
sage: QG = GroupAlgebra(G, QQ)
sage: x = QG.sum_of_terms([(p,p.number_of_peaks() + p.number_of_
↪inversions()) for p in Permutations(3)]); x
2*[1, 3, 2] + [2, 1, 3] + 3*[2, 3, 1] + 2*[3, 1, 2] + 3*[3, 2, 1]
sage: T = QG.convolution_product(Antipode, Antipode, Id)
sage: T(x)
2*[1, 3, 2] + [2, 1, 3] + 2*[2, 3, 1] + 3*[3, 1, 2] + 3*[3, 2, 1]
```

## 3.15 Bimodules

**class** sage.categories.bimodules.**Bimodules**(*left_base*, *right_base*, *name=None*)

Bases: *sage.categories.category.CategoryWithParameters*

The category of $(R, S)$-bimodules

For $R$ and $S$ rings, a $(R, S)$-bimodule $X$ is a left $R$-module and right $S$-module such that the left and right actions commute: $r * (x * s) = (r * x) * s$.

EXAMPLES:

```
sage: Bimodules(QQ, ZZ)
Category of bimodules over Rational Field on the left and Integer Ring on the␣
↪right
sage: Bimodules(QQ, ZZ).super_categories()
[Category of left modules over Rational Field, Category of right modules over␣
↪Integer Ring]
```

**class ElementMethods**

**class ParentMethods**

**additional_structure**()

Return None.

Indeed, the category of bimodules defines no additional structure: a left and right module morphism between two bimodules is a bimodule morphism.

**See also:**

*Category.additional_structure()*

**Todo:** Should this category be a CategoryWithAxiom?

EXAMPLES:

```
sage: Bimodules(QQ, ZZ).additional_structure()
```

**classmethod an_instance()**
Return an instance of this class.

EXAMPLES:

```
sage: Bimodules.an_instance()
Category of bimodules over Rational Field on the left and Real Field with 53␣
↪bits of precision on the right
```

**left_base_ring()**
Return the left base ring over which elements of this category are defined.

EXAMPLES:

```
sage: Bimodules(QQ, ZZ).left_base_ring()
Rational Field
```

**right_base_ring()**
Return the right base ring over which elements of this category are defined.

EXAMPLES:

```
sage: Bimodules(QQ, ZZ).right_base_ring()
Integer Ring
```

**super_categories()**
EXAMPLES:

```
sage: Bimodules(QQ, ZZ).super_categories()
[Category of left modules over Rational Field, Category of right modules over␣
↪Integer Ring]
```

## 3.16 Classical Crystals

**class** sage.categories.classical_crystals.**ClassicalCrystals**(*s=None*)
Bases: *sage.categories.category_singleton.Category_singleton*

The category of classical crystals, that is crystals of finite Cartan type.

EXAMPLES:

```
sage: C = ClassicalCrystals()
sage: C
Category of classical crystals
sage: C.super_categories()
[Category of regular crystals,
 Category of finite crystals,
 Category of highest weight crystals]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

**class ElementMethods**

**lusztig_involution**()

Return the Lusztig involution on the classical highest weight crystal `self`.

The Lusztig involution on a finite-dimensional highest weight crystal $B(\lambda)$ of highest weight $\lambda$ maps the highest weight vector to the lowest weight vector and the Kashiwara operator $f_i$ to $e_{i^*}$, where $i^*$ is defined as $\alpha_{i^*} = -w_0(\alpha_i)$. Here $w_0$ is the longest element of the Weyl group acting on the $i$-th simple root $\alpha_i$.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',3],shape=[2,1])
sage: b = B(rows=[[1,2],[4]])
sage: b.lusztig_involution()
[[1, 4], [3]]
sage: b.to_tableau().schuetzenberger_involution(n=4)
[[1, 4], [3]]

sage: all(b.lusztig_involution().to_tableau() == b.to_tableau().
↪schuetzenberger_involution(n=4) for b in B)
True

sage: B = crystals.Tableaux(['D',4],shape=[1])
sage: [[b,b.lusztig_involution()] for b in B]
[[[[1]], [[-1]]], [[[2]], [[-2]]], [[[3]], [[-3]]], [[[4]], [[-4]]], [[[-
↪4]],
[[4]]], [[[-3]], [[3]]], [[[-2]], [[2]]], [[[-1]], [[1]]]]

sage: B = crystals.Tableaux(['D',3],shape=[1])
sage: [[b,b.lusztig_involution()] for b in B]
[[[[1]], [[-1]]], [[[2]], [[-2]]], [[[3]], [[3]]], [[[-3]], [[-3]]],
[[[-2]], [[2]]], [[[-1]], [[1]]]]

sage: C = CartanType(['E',6])
sage: La = C.root_system().weight_lattice().fundamental_weights()
sage: T = crystals.HighestWeight(La[1])
sage: t = T[3]; t
[(-4, 2, 5)]
sage: t.lusztig_involution()
[(-2, -3, 4)]
```

**class ParentMethods**

**cardinality**()

Returns the number of elements of the crystal, using Weyl's dimension formula on each connected component.

EXAMPLES:

```
sage: C = ClassicalCrystals().example(5)
sage: C.cardinality()
6
```

**character**(*R=None*)

Returns the character of this crystal.

INPUT:

- `R` – a `WeylCharacterRing` (default: the default `WeylCharacterRing` for this Cartan type)

Returns the character of `self` as an element of `R`.

EXAMPLES:

```
sage: C = crystals.Tableaux("A2", shape=[2,1])
sage: chi = C.character(); chi
A2(2,1,0)

sage: T = crystals.TensorProduct(C,C)
sage: chiT = T.character(); chiT
A2(2,2,2) + 2*A2(3,2,1) + A2(3,3,0) + A2(4,1,1) + A2(4,2,0)
sage: chiT == chi^2
True
```

One may specify an alternate `WeylCharacterRing`:

```
sage: R = WeylCharacterRing("A2", style="coroots")
sage: chiT = T.character(R); chiT
A2(0,0) + 2*A2(1,1) + A2(0,3) + A2(3,0) + A2(2,2)
sage: chiT in R
True
```

It should have the same Cartan type and use the same realization of the weight lattice as `self`:

```
sage: R = WeylCharacterRing("A3", style="coroots")
sage: T.character(R)
Traceback (most recent call last):
...
ValueError: Weyl character ring does not have the right Cartan type
```

**demazure_character**(*w*, *f=None*)

Returns the Demazure character associated to `w`.

INPUT:

- `w` – an element of the ambient weight lattice realization of the crystal, or a reduced word, or an element in the associated Weyl group

OPTIONAL:

- `f` – a function from the crystal to a module

This is currently only supported for crystals whose underlying weight space is the ambient space.

The Demazure character is obtained by applying the Demazure operator $D_w$ (see `sage.categories.regular_crystals.RegularCrystals.ParentMethods.demazure_operator()`) to the highest weight element of the classical crystal. The simple Demazure operators $D_i$ (see `sage.categories.regular_crystals.RegularCrystals.ElementMethods.demazure_operator_simple()`) do not braid on the level of crystals, but on the level of characters they do. That is why it makes sense to input `w` either as a weight, a reduced word, or as an element of the underlying Weyl group.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape = [2,1])
sage: e = T.weight_lattice_realization().basis()
sage: weight = e[0] + 2*e[2]
sage: weight.reduced_word()
[2, 1]
sage: T.demazure_character(weight)
x1^2*x2 + x1*x2^2 + x1^2*x3 + x1*x2*x3 + x1*x3^2

sage: T = crystals.Tableaux(['A',3],shape=[2,1])
```

```
sage: T.demazure_character([1,2,3])
x1^2*x2 + x1*x2^2 + x1^2*x3 + x1*x2*x3 + x2^2*x3
sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([1,2,3])
sage: T.demazure_character(w)
x1^2*x2 + x1*x2^2 + x1^2*x3 + x1*x2*x3 + x2^2*x3

sage: T = crystals.Tableaux(['B',2], shape = [2])
sage: e = T.weight_lattice_realization().basis()
sage: weight = -2*e[1]
sage: T.demazure_character(weight)
x1^2 + x1*x2 + x2^2 + x1 + x2 + x1/x2 + 1/x2 + 1/x2^2 + 1

sage: T = crystals.Tableaux("B2",shape=[1/2,1/2])
sage: b2=WeylCharacterRing("B2",base_ring=QQ).ambient()
sage: T.demazure_character([1,2],f=lambda x:b2(x.weight()))
b2(-1/2,1/2) + b2(1/2,-1/2) + b2(1/2,1/2)
```

REFERENCES:

- [De1974]
- [Ma2009]

**opposition_automorphism**()

Deprecated in trac ticket #15560. Use the corresponding method in Cartan type.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',5],shape=[1])
sage: T.opposition_automorphism()
doctest:...: DeprecationWarning: opposition_automorphism is deprecated.
Use opposition_automorphism from the Cartan type instead.
See http://trac.sagemath.org/15560 for details.
Finite family {1: 5, 2: 4, 3: 3, 4: 2, 5: 1}
```

**class TensorProducts**(*category*, **args*)

Bases: *sage.categories.tensor.TensorProductsCategory*

The category of classical crystals constructed by tensor product of classical crystals.

**extra_super_categories**()

EXAMPLES:

```
sage: ClassicalCrystals().TensorProducts().extra_super_categories()
[Category of classical crystals]
```

**additional_structure**()

Return None.

Indeed, the category of classical crystals defines no additional structure: it only states that its objects are $U_q(\mathfrak{g})$-crystals, where $\mathfrak{g}$ is of finite type.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: ClassicalCrystals().additional_structure()
```

**example** (*n=3*)

Returns an example of highest weight crystals, as per `Category.example()`.

EXAMPLES:

```
sage: B = ClassicalCrystals().example(); B
Highest weight crystal of type A_3 of highest weight omega_1
```

**super_categories** ()

EXAMPLES:

```
sage: ClassicalCrystals().super_categories()
[Category of regular crystals,
 Category of finite crystals,
 Category of highest weight crystals]
```

## 3.17 Coalgebras

**class** sage.categories.coalgebras.**Coalgebras** (*base*, *name=None*)

Bases: `sage.categories.category_types.Category_over_base_ring`

The category of coalgebras

EXAMPLES:

```
sage: Coalgebras(QQ)
Category of coalgebras over Rational Field
sage: Coalgebras(QQ).super_categories()
[Category of vector spaces over Rational Field]
```

**class DualObjects** (*category*, **args*)

Bases: `sage.categories.dual.DualObjectsCategory`

**extra_super_categories** ()

Return the dual category.

EXAMPLES:

The category of coalgebras over the Rational Field is dual to the category of algebras over the same field:

```
sage: C = Coalgebras(QQ)
sage: C.dual()
Category of duals of coalgebras over Rational Field
sage: C.dual().super_categories() # indirect doctest
[Category of algebras over Rational Field, Category of duals of vector␣
↪spaces over Rational Field]
```

> **Warning:** This is only correct in certain cases (finite dimension, . . . ). See trac ticket #15647.

**class ElementMethods**

**coproduct** ()

Returns the coproduct of `self`

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, a.coproduct()
(B[(1,2,3)], B[(1,2,3)] # B[(1,2,3)])
sage: b, b.coproduct()
(B[(1,3)], B[(1,3)] # B[(1,3)])
```

**counit**()

Returns the counit of `self`

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, a.counit()
(B[(1,2,3)], 1)
sage: b, b.counit()
(B[(1,3)], 1)
```

**class ParentMethods**

**coproduct**(*x*)

Returns the coproduct of x.

Eventually, there will be a default implementation, delegating to the overloading mechanism and forcing the conversion back

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, A.coproduct(a)
(B[(1,2,3)], B[(1,2,3)] # B[(1,2,3)])
sage: b, A.coproduct(b)
(B[(1,3)], B[(1,3)] # B[(1,3)])
```

**counit**(*x*)

Returns the counit of x.

Eventually, there will be a default implementation, delegating to the overloading mechanism and forcing the conversion back

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, A.counit(a)
(B[(1,2,3)], 1)
```

```
sage: b, A.counit(b)
(B[(1,3)], 1)
```

TODO: implement some tests of the axioms of coalgebras, bialgebras and Hopf algebras using the counit.

**class Realizations**(*category*, **args*)

Bases: *sage.categories.realizations.RealizationsCategory*

**class ParentMethods**

**coproduct_by_coercion**(*x*)

Return the coproduct by coercion if `coproduct_by_basis` is not implemented.

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: m = Sym.monomial()
sage: f = m[2,1]
sage: f.coproduct.__module__
'sage.categories.coalgebras'
sage: m.coproduct_on_basis
NotImplemented
sage: m.coproduct == m.coproduct_by_coercion
True
sage: f.coproduct()
m[] # m[2, 1] + m[1] # m[2] + m[2] # m[1] + m[2, 1] # m[]
```

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: R = N.ribbon()
sage: R.coproduct_by_coercion.__module__
'sage.categories.coalgebras'
sage: R.coproduct_on_basis
NotImplemented
sage: R.coproduct == R.coproduct_by_coercion
True
sage: R[1].coproduct()
R[] # R[1] + R[1] # R[]
```

**counit_by_coercion**(*x*)

Return the counit of `x` if `counit_by_basis` is not implemented.

EXAMPLES:

```
sage: sp = SymmetricFunctions(QQ).sp()
sage: sp.an_element()
2*sp[] + 2*sp[1] + 3*sp[2]
sage: sp.counit(sp.an_element())
2

sage: o = SymmetricFunctions(QQ).o()
sage: o.an_element()
2*o[] + 2*o[1] + 3*o[2]
sage: o.counit(o.an_element())
-1
```

**class Super**(*base_category*)

Bases: *sage.categories.super_modules.SuperModulesCategory*

**extra_super_categories**()

EXAMPLES:

```
sage: Coalgebras(ZZ).Super().extra_super_categories()
[Join of Category of graded modules over Integer Ring
    and Category of coalgebras over Integer Ring]
sage: Coalgebras(ZZ).Super().super_categories()
[Category of super modules over Integer Ring,
 Category of coalgebras over Integer Ring]
```

Compare this with the situation for bialgebras:

```
sage: Bialgebras(ZZ).Super().extra_super_categories()
[]
sage: Bialgebras(ZZ).Super().super_categories()
[Category of super algebras over Integer Ring,
 Category of super coalgebras over Integer Ring]
```

The category of bialgebras does not occur in these results, since super bialgebras are not bialgebras.

**class TensorProducts**(*category*, **args*)

Bases: `sage.categories.tensor.TensorProductsCategory`

**class ElementMethods**

**class ParentMethods**

**extra_super_categories**()

EXAMPLES:

```
sage: Coalgebras(QQ).TensorProducts().extra_super_categories()
[Category of coalgebras over Rational Field]
sage: Coalgebras(QQ).TensorProducts().super_categories()
[Category of tensor products of vector spaces over Rational Field,
 Category of coalgebras over Rational Field]
```

Meaning: a tensor product of coalgebras is a coalgebra

**WithBasis**

alias of `CoalgebrasWithBasis`

**class WithRealizations**(*category*, **args*)

Bases: `sage.categories.with_realizations.WithRealizationsCategory`

**class ParentMethods**

**coproduct**(*x*)

Returns the coproduct of `x`.

EXAMPLES:

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: S = N.complete()
sage: N.coproduct.__module__
'sage.categories.coalgebras'
sage: N.coproduct(S[2])
S[] # S[2] + S[1] # S[1] + S[2] # S[]
```

**counit**(*x*)

Return the counit of `x`.

EXAMPLES:

```
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.schur()
sage: f = s[2,1]
sage: f.counit.__module__
'sage.categories.coalgebras'
sage: f.counit()
0
```

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: N.counit.__module__
'sage.categories.coalgebras'
sage: N.counit(N.one())
1
sage: x = N.an_element(); x
2*S[] + 2*S[1] + 3*S[1, 1]
sage: N.counit(x)
2
```

**super_categories**()

EXAMPLES:

```
sage: Coalgebras(QQ).super_categories()
[Category of vector spaces over Rational Field]
```

## 3.18 Coalgebras with basis

**class** sage.categories.coalgebras_with_basis.**CoalgebrasWithBasis**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of coalgebras with a distinguished basis.

EXAMPLES:

```
sage: CoalgebrasWithBasis(ZZ)
Category of coalgebras with basis over Integer Ring
sage: sorted(CoalgebrasWithBasis(ZZ).super_categories(), key=str)
[Category of coalgebras over Integer Ring,
 Category of modules with basis over Integer Ring]
```

**class ElementMethods**

**coproduct_iterated**(*n=1*)

Apply `n` coproducts to `self`.

**Todo:** Remove dependency on `modules_with_basis` methods.

EXAMPLES:

```
sage: Psi = NonCommutativeSymmetricFunctions(QQ).Psi()
sage: Psi[2,2].coproduct_iterated(0)
Psi[2, 2]
sage: Psi[2,2].coproduct_iterated(2)
```

```
Psi[] # Psi[] # Psi[2, 2] + 2*Psi[] # Psi[2] # Psi[2]
 + Psi[] # Psi[2, 2] # Psi[] + 2*Psi[2] # Psi[] # Psi[2]
 + 2*Psi[2] # Psi[2] # Psi[] + Psi[2, 2] # Psi[] # Psi[]
```

```
sage: Psi = NonCommutativeSymmetricFunctions(QQ).Psi()
sage: Psi[2,2].coproduct_iterated(0)
Psi[2, 2]
sage: Psi[2,2].coproduct_iterated(3)
Psi[] # Psi[] # Psi[] # Psi[2, 2] + 2*Psi[] # Psi[] # Psi[2] # Psi[2]
 + Psi[] # Psi[] # Psi[2, 2] # Psi[] + 2*Psi[] # Psi[2] # Psi[] # Psi[2]
 + 2*Psi[] # Psi[2] # Psi[2] # Psi[] + Psi[] # Psi[2, 2] # Psi[] # Psi[]
 + 2*Psi[2] # Psi[] # Psi[] # Psi[2] + 2*Psi[2] # Psi[] # Psi[2] # Psi[]
 + 2*Psi[2] # Psi[2] # Psi[] # Psi[] + Psi[2, 2] # Psi[] # Psi[] # Psi[]
```

```
sage: m = SymmetricFunctionsNonCommutingVariables(QQ).m()
sage: m[[1,3],[2]].coproduct_iterated(2)
m{} # m{} # m{{1, 3}, {2}} + m{} # m{{1}} # m{{1, 2}}
 + m{} # m{{1, 2}} # m{{1}} + m{} # m{{1, 3}, {2}} # m{}
 + m{{1}} # m{} # m{{1, 2}} + m{{1}} # m{{1, 2}} # m{}
 + m{{1, 2}} # m{} # m{{1}} + m{{1, 2}} # m{{1}} # m{}
 + m{{1, 3}, {2}} # m{} # m{}
sage: m[[]].coproduct_iterated(3), m[[1,3],[2]].coproduct_iterated(0)
(m{} # m{} # m{} # m{}, m{{1, 3}, {2}})
```

**class ParentMethods**

**coproduct**()

If *coproduct_on_basis()* is available, construct the coproduct morphism from `self` to `self` $\otimes$ `self` by extending it by linearity. Otherwise, use *coproduct_by_coercion()*, if available.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, A.coproduct(a)
(B[(1,2,3)], B[(1,2,3)] # B[(1,2,3)])
sage: b, A.coproduct(b)
(B[(1,3)], B[(1,3)] # B[(1,3)])
```

**coproduct_on_basis**(*i*)

The coproduct of the algebra on the basis (optional).

INPUT:

- `i` – the indices of an element of the basis of `self`

Returns the coproduct of the corresponding basis elements If implemented, the coproduct of the algebra is defined from it by linearity.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: (a, b) = A._group.gens()
sage: A.coproduct_on_basis(a)
B[(1,2,3)] # B[(1,2,3)]
```

**counit**()

If *counit_on_basis()* is available, construct the counit morphism from self to self ⊗ self by extending it by linearity

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, A.counit(a)
(B[(1,2,3)], 1)
sage: b, A.counit(b)
(B[(1,3)], 1)
```

**counit_on_basis**(*i*)

The counit of the algebra on the basis (optional).

INPUT:

- i – the indices of an element of the basis of self

Returns the counit of the corresponding basis elements If implemented, the counit of the algebra is defined from it by linearity.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: (a, b) = A._group.gens()
sage: A.counit_on_basis(a)
1
```

**class Super**(*base_category*)

Bases: *sage.categories.super_modules.SuperModulesCategory*

## 3.19 Commutative additive groups

**class** sage.categories.commutative_additive_groups.**CommutativeAdditiveGroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*, *sage.categories.category_types.AbelianCategory*

The category of abelian groups, i.e. additive abelian monoids where each element has an inverse.

EXAMPLES:

```
sage: C = CommutativeAdditiveGroups(); C
Category of commutative additive groups
sage: C.super_categories()
[Category of additive groups, Category of commutative additive monoids]
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse', 'AdditiveUnital
↪']
sage: C is CommutativeAdditiveMonoids().AdditiveInverse()
True
sage: from sage.categories.additive_groups import AdditiveGroups
sage: C is AdditiveGroups().AdditiveCommutative()
True
```

**Note:** This category is currently empty. It's left there for backward compatibility and because it is likely to grow in the future.

**class Algebras**(*category*, **args*)
Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class CartesianProducts**(*category*, **args*)
Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**class ElementMethods**

**additive_order**()
Return the additive order of this element.

EXAMPLES:

```
sage: G = cartesian_product([Zmod(3), Zmod(6), Zmod(5)])
sage: G((1,1,1)).additive_order()
30
sage: any((i * G((1,1,1))).is_zero() for i in range(1,30))
False
sage: 30 * G((1,1,1))
(0, 0, 0)

sage: G = cartesian_product([ZZ, ZZ])
sage: G((0,0)).additive_order()
1
sage: G((0,1)).additive_order()
+Infinity

sage: K = GF(9)
sage: H = cartesian_product([cartesian_product([Zmod(2),Zmod(9)]), K])
sage: z = H(((1,2), K.gen()))
sage: z.additive_order()
18
```

## 3.20 Commutative additive monoids

**class** sage.categories.commutative_additive_monoids.**CommutativeAdditiveMonoids**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of commutative additive monoids, that is abelian additive semigroups with a unit

EXAMPLES:

```
sage: C = CommutativeAdditiveMonoids(); C
Category of commutative additive monoids
sage: C.super_categories()
[Category of additive monoids, Category of commutative additive semigroups]
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveUnital']
sage: C is AdditiveMagmas().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital()
True
```

**Note:** This category is currently empty and only serves as a place holder to make `C.example()` work.

## 3.21 Commutative additive semigroups

**class** sage.categories.commutative_additive_semigroups.**CommutativeAdditiveSemigroups**(*base_cate*

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of additive abelian semigroups, i.e. sets with an associative and abelian operation +.

EXAMPLES:

```
sage: C = CommutativeAdditiveSemigroups(); C
Category of commutative additive semigroups
sage: C.example()
An example of a commutative monoid: the free commutative monoid generated by ('a',
↪ 'b', 'c', 'd')

sage: sorted(C.super_categories(), key=str)
[Category of additive commutative additive magmas,
 Category of additive semigroups]
sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative']
sage: C is AdditiveMagmas().AdditiveAssociative().AdditiveCommutative()
True
```

**Note:** This category is currently empty and only serves as a place holder to make `C.example()` work.

## 3.22 Commutative algebra ideals

**class** sage.categories.commutative_algebra_ideals.**CommutativeAlgebraIdeals**(*A*)

Bases: *sage.categories.category_types.Category_ideal*

The category of ideals in a fixed commutative algebra $A$.

EXAMPLES:

```
sage: C = CommutativeAlgebraIdeals(QQ['x'])
sage: C
Category of commutative algebra ideals in Univariate Polynomial Ring in x over␣
↪Rational Field
```

**algebra**()

EXAMPLES:

```
sage: CommutativeAlgebraIdeals(QQ['x']).algebra()
Univariate Polynomial Ring in x over Rational Field
```

**super_categories**()

EXAMPLES:

```
sage: CommutativeAlgebraIdeals(QQ['x']).super_categories()
[Category of algebra ideals in Univariate Polynomial Ring in x over Rational␣
↪Field]
```

## 3.23 Commutative algebras

**class** sage.categories.commutative_algebras.**CommutativeAlgebras**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of commutative algebras with unit over a given base ring.

EXAMPLES:

```
sage: M = CommutativeAlgebras(GF(19))
sage: M
Category of commutative algebras over Finite Field of size 19
sage: CommutativeAlgebras(QQ).super_categories()
[Category of algebras over Rational Field, Category of commutative rings]
```

This is just a shortcut for:

```
sage: Algebras(QQ).Commutative()
Category of commutative algebras over Rational Field
```

## 3.24 Commutative ring ideals

**class** sage.categories.commutative_ring_ideals.**CommutativeRingIdeals**(*R*)
Bases: *sage.categories.category_types.Category_ideal*

The category of ideals in a fixed commutative ring.

EXAMPLES:

```
sage: C = CommutativeRingIdeals(IntegerRing())
sage: C
Category of commutative ring ideals in Integer Ring
```

**super_categories**()
EXAMPLES:

```
sage: CommutativeRingIdeals(ZZ).super_categories()
[Category of ring ideals in Integer Ring]
```

## 3.25 Commutative rings

**class** sage.categories.commutative_rings.**CommutativeRings**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of commutative rings

commutative rings with unity, i.e. rings with commutative * and a multiplicative identity

EXAMPLES:

```
sage: C = CommutativeRings(); C
Category of commutative rings
sage: C.super_categories()
[Category of rings, Category of commutative monoids]
```

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**extra_super_categories**()

Let Sage knows that Cartesian products of commutative rings is a commutative ring.

EXAMPLES:

```
sage: CommutativeRings().Commutative().CartesianProducts().extra_super_
↪categories()
[Category of commutative rings]
sage: cartesian_product([ZZ, Zmod(34), QQ, GF(5)]) in CommutativeRings()
True
```

**class ElementMethods**

**class Finite**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

Check that Sage knows that Cartesian products of finite commutative rings is a finite commutative ring.

EXAMPLES:

```
sage: cartesian_product([Zmod(34), GF(5)]) in Rings().Commutative().Finite()
True
```

**class ParentMethods**

**cyclotomic_cosets**(*q*, *cosets=None*)

Return the (multiplicative) orbits of `q` in the ring.

Let $R$ be a finite commutative ring. The group of invertible elements $R^*$ in $R$ gives rise to a group action on $R$ by multiplication. An orbit of the subgroup generated by an invertible element $q$ is called a *q-cyclotomic coset* (since in a finite ring, each invertible element is a root of unity).

These cosets arise in the theory of minimal polynomials of finite fields, duadic codes and combinatorial designs. Fix a primitive element $z$ of $GF(q^k)$. The minimal polynomial of $z^s$ over $GF(q)$ is given by

$$M_s(x) = \prod_{i \in C_s} (x - z^i),$$

where $C_s$ is the $q$-cyclotomic coset mod $n$ containing $s$, $n = q^k - 1$.

**Note:** When $R = \mathbf{Z}/n\mathbf{Z}$ the smallest element of each coset is sometimes called a *coset leader*. This function returns sorted lists so that the coset leader will always be the first element of the coset.

INPUT:

- `q` – an invertible element of the ring
- `cosets` – an optional lists of elements of `self`. If provided, the function only return the list of cosets that contain some element from `cosets`.

OUTPUT:

A list of lists.

EXAMPLES:

```
sage: Zmod(11).cyclotomic_cosets(2)
[[0], [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]]
sage: Zmod(15).cyclotomic_cosets(2)
[[0], [1, 2, 4, 8], [3, 6, 9, 12], [5, 10], [7, 11, 13, 14]]
```

Since the group of invertible elements of a finite field is cyclic, the set of squares is a particular case of cyclotomic coset:

```
sage: K = GF(25,'z')
sage: a = K.multiplicative_generator()
sage: K.cyclotomic_cosets(a**2,cosets=[1])
[[1, 2, 3, 4, z + 1, z + 3,
  2*z + 1, 2*z + 2, 3*z + 3,
  3*z + 4, 4*z + 2, 4*z + 4]]
sage: sorted(b for b in K if not b.is_zero() and b.is_square())
[1, 2, 3, 4, z + 1, z + 3,
 2*z + 1, 2*z + 2, 3*z + 3,
 3*z + 4, 4*z + 2, 4*z + 4]
```

We compute some examples of minimal polynomials:

```
sage: K = GF(27,'z')
sage: a = K.multiplicative_generator()
sage: R.<X> = PolynomialRing(K, 'X')
sage: a.minimal_polynomial('X')
X^3 + 2*X + 1
sage: cyc3 = Zmod(26).cyclotomic_cosets(3,cosets=[1]); cyc3
[[1, 3, 9]]
sage: prod(X - a**i for i in cyc3[0])
X^3 + 2*X + 1

sage: (a**7).minimal_polynomial('X')
X^3 + X^2 + 2*X + 1
sage: cyc7 = Zmod(26).cyclotomic_cosets(3,cosets=[7]); cyc7
[[7, 11, 21]]
sage: prod(X - a**i for i in cyc7[0])
X^3 + X^2 + 2*X + 1
```

Cyclotomic cosets of fields are useful in combinatorial design theory to provide so called difference families (see Wikipedia article Difference_set and `difference_family`). This is illustrated on the following examples:

```
sage: K = GF(5)
sage: a = K.multiplicative_generator()
sage: H = K.cyclotomic_cosets(a**2, cosets=[1,2]); H
[[1, 4], [2, 3]]
sage: sorted(x-y for D in H for x in D for y in D if x != y)
[1, 2, 3, 4]

sage: K = GF(37)
sage: a = K.multiplicative_generator()
sage: H = K.cyclotomic_cosets(a**4, cosets=[1]); H
[[1, 7, 9, 10, 12, 16, 26, 33, 34]]
```

```
sage: sorted(x-y for D in H for x in D for y in D if x != y)
[1, 1, 2, 2, 3, 3, 4, 4, 5, 5, ..., 33, 34, 34, 35, 35, 36, 36]
```

The method `cyclotomic_cosets` works on any finite commutative ring:

```
sage: R = cartesian_product([GF(7), Zmod(14)])
sage: a = R((3,5))
sage: R.cyclotomic_cosets((3,5), [(1,1)])
[[(1, 1), (3, 5), (2, 11), (6, 13), (4, 9), (5, 3)]]
```

## 3.26 Complete Discrete Valuation Rings (CDVR) and Fields (CDVF)

**class** sage.categories.complete_discrete_valuation.**CompleteDiscreteValuationFields**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of complete discrete valuation fields

EXAMPLES:

```
sage: Zp(7) in CompleteDiscreteValuationFields()
False
sage: QQ in CompleteDiscreteValuationFields()
False
sage: LaurentSeriesRing(QQ,'u') in CompleteDiscreteValuationFields()
True
sage: Qp(7) in CompleteDiscreteValuationFields()
True
sage: TestSuite(CompleteDiscreteValuationFields()).run()
```

**class ElementMethods**

**denominator**()

Return the denominator of this element normalized as a power of the uniformizer

EXAMPLES:

```
sage: K = Qp(7)
sage: x = K(1/21)
sage: x.denominator()
7 + O(7^21)

sage: x = K(7)
sage: x.denominator()
1 + O(7^20)
```

Note that the denominator lives in the ring of integers:

```
sage: x.denominator().parent()
7-adic Ring with capped relative precision 20
```

An error is raised when the input is indistinguishable from 0:

```
sage: x = K(0,5); x
O(7^5)
sage: x.denominator()
```

```
Traceback (most recent call last):
...
ValueError: Cannot determine the denominator of an element␣
↪indistinguishable from 0
```

**valuation**()

Return the valuation of this element.

EXAMPLES:

```
sage: K = Qp(7)
sage: x = K(7); x
7 + O(7^21)
sage: x.valuation()
1
```

**super_categories**()

EXAMPLES:

```
sage: CompleteDiscreteValuationFields().super_categories()
[Category of discrete valuation fields]
```

**class** sage.categories.complete_discrete_valuation.**CompleteDiscreteValuationRings**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of complete discrete valuation rings

EXAMPLES:

```
sage: Zp(7) in CompleteDiscreteValuationRings()
True
sage: QQ in CompleteDiscreteValuationRings()
False
sage: QQ[['u']] in CompleteDiscreteValuationRings()
True
sage: Qp(7) in CompleteDiscreteValuationRings()
False
sage: TestSuite(CompleteDiscreteValuationRings()).run()
```

**class ElementMethods**

**denominator**()

Return the denominator of this element normalized as a power of the uniformizer

EXAMPLES:

```
sage: K = Qp(7)
sage: x = K(1/21)
sage: x.denominator()
7 + O(7^21)

sage: x = K(7)
sage: x.denominator()
1 + O(7^20)
```

Note that the denominator lives in the ring of integers:

```
sage: x.denominator().parent()
7-adic Ring with capped relative precision 20
```

An error is raised when the input is indistinguishable from 0:

```
sage: x = K(0,5); x
O(7^5)
sage: x.denominator()
Traceback (most recent call last):
...
ValueError: Cannot determine the denominator of an element␣
↪indistinguishable from 0
```

**valuation**()
Return the valuation of this element.

EXAMPLES:

```
sage: R = Zp(7)
sage: x = R(7); x
7 + O(7^21)
sage: x.valuation()
1
```

**super_categories**()
EXAMPLES:

```
sage: CompleteDiscreteValuationRings().super_categories()
[Category of discrete valuation rings]
```

## 3.27 Complex reflection groups

**class** sage.categories.complex_reflection_groups.**ComplexReflectionGroups**(*s=None*)
Bases: *sage.categories.category_singleton.Category_singleton*

The category of complex reflection groups.

Let $V$ be a complex vector space. A *complex reflection* is an element of $\mathrm{GL}(V)$ fixing an hyperplane pointwise and acting by multiplication by a root of unity on a complementary line.

A *complex reflection group* is a group $W$ that is (isomorphic to) a subgroup of some general linear group $\mathrm{GL}(V)$ generated by a distinguished set of complex reflections.

The dimension of $V$ is the *rank* of $W$.

For a comprehensive treatment of complex reflection groups and many definitions and theorems used here, we refer to [LT2009]. See also Wikipedia article Reflection_group.

**See also:**

ReflectionGroup() for usage examples of this category.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups()
Category of complex reflection groups
```

```
sage: ComplexReflectionGroups().super_categories()
[Category of complex reflection or generalized coxeter groups]
sage: ComplexReflectionGroups().all_super_categories()
[Category of complex reflection groups,
 Category of complex reflection or generalized coxeter groups,
 Category of groups,
 Category of monoids,
 Category of finitely generated semigroups,
 Category of semigroups,
 Category of finitely generated magmas,
 Category of inverse unital magmas,
 Category of unital magmas,
 Category of magmas,
 Category of enumerated sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

An example of a reflection group:

```
sage: W = ComplexReflectionGroups().example(); W
5-colored permutations of size 3
```

`W` is in the category of complex reflection groups:

```
sage: W in ComplexReflectionGroups()
True
```

**Finite**
alias of `FiniteComplexReflectionGroups`

**class ParentMethods**

**rank**()
Return the rank of `self`.

The rank of `self` is the dimension of the smallest faithfull reflection representation of `self`.

EXAMPLES:

```
sage: W = CoxeterGroups().example(); W
The symmetric group on {0, ..., 3}
sage: W.rank()
3
```

**additional_structure**()
Return `None`.

Indeed, all the structure complex reflection groups have in addition to groups (simple reflections, ...) is already defined in the super category.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().additional_structure()
```

**example**()

Return an example of a complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().example()
5-colored permutations of size 3
```

**super_categories**()

Return the super categories of `self`.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().super_categories()
[Category of complex reflection or generalized coxeter groups]
```

## 3.28 Common category for Generalized Coxeter Groups or Complex Reflection Groups

**class** sage.categories.complex_reflection_or_generalized_coxeter_groups.**ComplexReflectionOrG**

Bases: *sage.categories.category_singleton.Category_singleton*

The category of complex reflection groups or generalized Coxeter groups.

Finite Coxeter groups can be defined equivalently as groups generated by reflections, or by presentations. Over the last decades, the theory has been generalized in both directions, leading to the study of (finite) complex reflection groups on the one hand, and (finite) generalized Coxeter groups on the other hand. Many of the features remain similar, yet, in the current state of the art, there is no general theory covering both directions.

This is reflected by the name of this category which is about factoring out the common code, tests, and declarations.

A group in this category has:

- A distinguished finite set of generators $(s_i)_I$, called *simple reflections*. The set $I$ is called the *index set*. The name "reflection" is somewhat of an abuse as they can have higher order; still, they are all of finite order: $s_i^k = 1$ for some $k$.
- A collection of *distinguished reflections* which are the conjugates of the simple reflections. For complex reflection groups, they are in one-to-one correspondence with the reflection hyperplanes and share the same index set.
- A collection of *reflections* which are the conjugates of all the non trivial powers of the simple reflections.

The usual notions of reduced words, length, irreducibility, etc can be canonically defined from the above.

The following methods must be implemented:

- *ComplexReflectionOrGeneralizedCoxeterGroups.ParentMethods.index_set()*

- *ComplexReflectionOrGeneralizedCoxeterGroups.ParentMethods.simple_reflection()*

Optionally one can define analog methods for distinguished reflections and reflections (see below).

At least one of the following methods must be implemented:

- *ComplexReflectionOrGeneralizedCoxeterGroups.ElementMethods.apply_simple_reflection()*
- *ComplexReflectionOrGeneralizedCoxeterGroups.ElementMethods.apply_simple_reflection_left()*
- *ComplexReflectionOrGeneralizedCoxeterGroups.ElementMethods.apply_simple_reflection_right()*
- ComplexReflectionOrGeneralizedCoxeterGroups.ElementMethods._mul_()

It's recommended to implement either _mul_ or both apply_simple_reflection_left and apply_simple_reflection_right.

**See also:**

- complex_reflection_groups.ComplexReflectionGroups
- generalized_coxeter_groups.GeneralizedCoxeterGroups

EXAMPLES:

```
sage: from sage.categories.complex_reflection_or_generalized_coxeter_groups␣
↪import ComplexReflectionOrGeneralizedCoxeterGroups
sage: C = ComplexReflectionOrGeneralizedCoxeterGroups(); C
Category of complex reflection or generalized coxeter groups
sage: C.super_categories()
[Category of finitely generated enumerated groups]

sage: C.required_methods()
{'element': {'optional': ['reflection_length'],
             'required': []},
  'parent': {'optional': ['distinguished_reflection', 'hyperplane_index_set',
                           'irreducible_components',
                           'reflection', 'reflection_index_set'],
            'required':  ['__contains__', 'index_set']}}
```

**class ElementMethods**

**apply_conjugation_by_simple_reflection**(*i*)

Conjugate self by the i-th simple reflection.

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.apply_conjugation_by_simple_reflection(1).reduced_word()
[3, 2]
```

**apply_reflections**(*word*, *side='right'*, *word_type='all'*)

Return the result of the (left/right) multiplication of self by word.

INPUT:

- word – a sequence of indices of reflections

- side – (default: 'right') indicates multiplying from left or right
- word_type – (optional, default: 'all'): either 'simple', 'distinguished', or 'all'

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))           # optional - gap3
sage: W.one().apply_reflections([1])         # optional - gap3
(1,4)(2,3)(5,6)
sage: W.one().apply_reflections([2])         # optional - gap3
(1,3)(2,5)(4,6)
sage: W.one().apply_reflections([2,1])       # optional - gap3
(1,2,6)(3,4,5)


sage: W = CoxeterGroups().example()
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.apply_reflections([0,1], word_type='simple')
(2, 3, 1, 0)
sage: w
(1, 2, 3, 0)
sage: w.apply_reflections([0,1], side='left', word_type='simple')
(0, 1, 3, 2)


sage: W = ReflectionGroup((1,1,3))           # optional - gap3
sage: W.one().apply_reflections([1], word_type='distinguished')    #␣
↪optional - gap3
(1,4)(2,3)(5,6)
sage: W.one().apply_reflections([2],   word_type='distinguished')    #␣
↪optional - gap3
(1,3)(2,5)(4,6)
sage: W.one().apply_reflections([3],   word_type='distinguished')    #␣
↪optional - gap3
(1,5)(2,4)(3,6)
sage: W.one().apply_reflections([2,1], word_type='distinguished')    #␣
↪optional - gap3
(1,2,6)(3,4,5)

sage: W = ReflectionGroup((1,1,3), hyperplane_index_set=['A','B','C']); W␣
↪   # optional - gap3
Irreducible real reflection group of rank 2 and type A2
sage: W.one().apply_reflections(['A'], word_type='distinguished')    #␣
↪optional - gap3
(1,4)(2,3)(5,6)
```

**apply_simple_reflection**(*i*, *side='right'*)

Return self multiplied by the simple reflection s[i].

INPUT:

- i – an element of the index set
- side – (default: "right") "left" or "right"

This default implementation simply calls *apply_simple_reflection_left()* or *apply_simple_reflection_right()*.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.an_element(); w
```

```
(1, 2, 3, 0)
sage: w.apply_simple_reflection(0, side = "left")
(0, 2, 3, 1)
sage: w.apply_simple_reflection(1, side = "left")
(2, 1, 3, 0)
sage: w.apply_simple_reflection(2, side = "left")
(1, 3, 2, 0)

sage: w.apply_simple_reflection(0, side = "right")
(2, 1, 3, 0)
sage: w.apply_simple_reflection(1, side = "right")
(1, 3, 2, 0)
sage: w.apply_simple_reflection(2, side = "right")
(1, 2, 0, 3)
```

By default, `side` is `"right"`:

```
sage: w.apply_simple_reflection(0)
(2, 1, 3, 0)
```

Some tests with a complex reflection group:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: W = ComplexReflectionGroups().example(); W
5-colored permutations of size 3
sage: w = W.an_element(); w
[[1, 0, 0], [3, 1, 2]]
sage: w.apply_simple_reflection(1, side="left")
[[0, 1, 0], [1, 3, 2]]
sage: w.apply_simple_reflection(2, side="left")
[[1, 0, 0], [3, 2, 1]]
sage: w.apply_simple_reflection(3, side="left")
[[1, 0, 1], [3, 1, 2]]

sage: w.apply_simple_reflection(1, side="right")
[[1, 0, 0], [3, 2, 1]]
sage: w.apply_simple_reflection(2, side="right")
[[1, 0, 0], [2, 1, 3]]
sage: w.apply_simple_reflection(3, side="right")
[[2, 0, 0], [3, 1, 2]]
```

**apply_simple_reflection_left**(*i*)

Return `self` multiplied by the simple reflection `s[i]` on the left.

This low level method is used intensively. Coxeter groups are encouraged to override this straightforward implementation whenever a faster approach exists.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.apply_simple_reflection_left(0)
(0, 2, 3, 1)
sage: w.apply_simple_reflection_left(1)
(2, 1, 3, 0)
sage: w.apply_simple_reflection_left(2)
```

```
(1, 3, 2, 0)
```

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: W = ComplexReflectionGroups().example()
sage: w = W.an_element(); w
[[1, 0, 0], [3, 1, 2]]
sage: w.apply_simple_reflection_left(1)
[[0, 1, 0], [1, 3, 2]]
sage: w.apply_simple_reflection_left(2)
[[1, 0, 0], [3, 2, 1]]
sage: w.apply_simple_reflection_left(3)
[[1, 0, 1], [3, 1, 2]]
```

**apply_simple_reflection_right**(*i*)

Return `self` multiplied by the simple reflection `s[i]` on the right.

This low level method is used intensively. Coxeter groups are encouraged to override this straightforward implementation whenever a faster approach exists.

EXAMPLES:

```
sage: W=CoxeterGroups().example()
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.apply_simple_reflection_right(0)
(2, 1, 3, 0)
sage: w.apply_simple_reflection_right(1)
(1, 3, 2, 0)
sage: w.apply_simple_reflection_right(2)
(1, 2, 0, 3)

sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: W = ComplexReflectionGroups().example()
sage: w = W.an_element(); w
[[1, 0, 0], [3, 1, 2]]
sage: w.apply_simple_reflection_right(1)
[[1, 0, 0], [3, 2, 1]]
sage: w.apply_simple_reflection_right(2)
[[1, 0, 0], [2, 1, 3]]
sage: w.apply_simple_reflection_right(3)
[[2, 0, 0], [3, 1, 2]]
```

**apply_simple_reflections**(*word*, *side='right'*, *type='simple'*)

Return the result of the (left/right) multiplication of `self` by `word`.

INPUT:

- `word` – a sequence of indices of simple reflections
- `side` – (default: `'right'`) indicates multiplying from left or right

This is a specialized implementation of *`apply_reflections()`* for the simple reflections. The rationale for its existence are:

- It can take advantage of `apply_simple_reflection`, which often is less expensive than computing a product.
- It reduced burden on implementations that would want to provide an optimized version of this method.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.apply_simple_reflections([0,1])
(2, 3, 1, 0)
sage: w
(1, 2, 3, 0)
sage: w.apply_simple_reflections([0,1],side='left')
(0, 1, 3, 2)
```

**inverse**()

Return the inverse of `self`.

EXAMPLES:

```
sage: W = WeylGroup(['B',7])
sage: w = W.an_element()
sage: u = w.inverse()
sage: u == ~w
True
sage: u * w == w * u
True
sage: u * w
[1 0 0 0 0 0 0]
[0 1 0 0 0 0 0]
[0 0 1 0 0 0 0]
[0 0 0 1 0 0 0]
[0 0 0 0 1 0 0]
[0 0 0 0 0 1 0]
[0 0 0 0 0 0 1]
```

**is_reflection**()

Return whether `self` is a reflection.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,4))                    # optional - gap3
sage: [t.is_reflection() for t in W.reflections()]      # optional - gap3
[True, True, True, True, True, True]
sage: len([t for t in W.reflections() if t.is_reflection()])     #␣
↪optional - gap3
6

sage: W = ReflectionGroup((2,1,3))                    # optional - gap3
sage: [t.is_reflection() for t in W.reflections()]      # optional - gap3
[True, True, True, True, True, True, True, True, True]
sage: len([t for t in W.reflections() if t.is_reflection()])     #␣
↪optional - gap3
9
```

**reflection_length**()

Return the reflection length of `self`.

This is the minimal length of a factorization of `self` into reflections.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,2))                         # optional - gap3
sage: sorted([t.reflection_length() for t in W])          # optional - gap3
[0, 1]

sage: W = ReflectionGroup((2,1,2))                         # optional - gap3
sage: sorted([t.reflection_length() for t in W])          # optional - gap3
[0, 1, 1, 1, 1, 2, 2, 2]

sage: W = ReflectionGroup((3,1,2))                         # optional - gap3
sage: sorted([t.reflection_length() for t in W])          # optional - gap3
[0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]

sage: W = ReflectionGroup((2,2,2))                         # optional - gap3
sage: sorted([t.reflection_length() for t in W])          # optional - gap3
[0, 1, 1, 2]
```

**class Irreducible**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class ParentMethods**

**irreducible_components**()

Return a list containing all irreducible components of `self` as finite reflection groups.

EXAMPLES:

```
sage: W = ColoredPermutations(4, 3)
sage: W.irreducible_components()
[4-colored permutations of size 3]
```

**class ParentMethods**

**distinguished_reflection**(*i*)

Return the $i$-th distinguished reflection of `self`.

INPUT:

- `i` – an element of the index set of the distinguished reflections.

**See also:**

- *distinguished_reflections()*
- *hyperplane_index_set()*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,4), hyperplane_index_set=('a','b','c','d',
↪'e','f'))  # optional - gap3
sage: for i in W.hyperplane_index_set():                         # optional -␣
↪gap3
....:     print('%s %s'%(i, W.distinguished_reflection(i))) # optional -␣
↪gap3
a (1,7)(2,4)(5,6)(8,10)(11,12)
b (1,4)(2,8)(3,5)(7,10)(9,11)
c (2,5)(3,9)(4,6)(8,11)(10,12)
d (1,8)(2,7)(3,6)(4,10)(9,12)
e (1,6)(2,9)(3,8)(5,11)(7,12)
f (1,11)(3,10)(4,9)(5,7)(6,12)
```

**distinguished_reflections**()

Return a finite family containing the distinguished reflections of `self`, indexed by *hyperplane_index_set()*.

A *distinguished reflection* is a conjugate of a simple reflection. For a Coxeter group, reflections and distinguished reflections coincide. For a Complex reflection groups this is a reflection acting on the complement of the fixed hyperplane $H$ as $\exp(2\pi i/n)$, where $n$ is the order of the reflection subgroup fixing $H$.

**See also:**

- *distinguished_reflection()*
- *hyperplane_index_set()*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                      # optional - gap3
sage: distinguished_reflections = W.distinguished_reflections() #␣
↪optional - gap3
sage: for index in sorted(distinguished_reflections.keys()):        #␣
↪optional - gap3
....:     print('%s %s'%(index, distinguished_reflections[index]))  #␣
↪optional - gap3
1 (1,4)(2,3)(5,6)
2 (1,3)(2,5)(4,6)
3 (1,5)(2,4)(3,6)

sage: W = ReflectionGroup((1,1,3),hyperplane_index_set=['a','b','c'])    #␣
↪optional - gap3
sage: distinguished_reflections = W.distinguished_reflections() #␣
↪optional - gap3
sage: for index in sorted(distinguished_reflections.keys()):        #␣
↪optional - gap3
....:     print('%s %s'%(index, distinguished_reflections[index]))  #␣
↪optional - gap3
a (1,4)(2,3)(5,6)
b (1,3)(2,5)(4,6)
c (1,5)(2,4)(3,6)

sage: W = ReflectionGroup((3,1,1))                      # optional - gap3
sage: distinguished_reflections = W.distinguished_reflections() #␣
↪optional - gap3
sage: for index in sorted(distinguished_reflections.keys()):        #␣
↪optional - gap3
....:     print('%s %s'%(index, distinguished_reflections[index]))  #␣
↪optional - gap3
1 (1,2,3)

sage: W = ReflectionGroup((1,1,3), (3,1,2))             # optional - gap3
sage: distinguished_reflections = W.distinguished_reflections() #␣
↪optional - gap3
sage: for index in sorted(distinguished_reflections.keys()):    #␣
↪optional - gap3
....:     print('%s %s'%(index, distinguished_reflections[index]))  #␣
↪optional - gap3
1 (1,6)(2,5)(7,8)
2 (1,5)(2,7)(6,8)
3 (3,9,15)(4,10,16)(12,17,23)(14,18,24)(20,25,29)(21,22,26)(27,28,30)
4 (3,11)(4,12)(9,13)(10,14)(15,19)(16,20)(17,21)(18,22)(23,27)(24,28)(25,
↪26)(29,30)
```

```
5  (1,7)(2,6)(5,8)
6  (3,19)(4,25)(9,11)(10,17)(12,28)(13,15)(14,30)(16,18)(20,27)(21,29)(22,
↪23)(24,26)
7  (4,21,27)(10,22,28)(11,13,19)(12,14,20)(16,26,30)(17,18,25)(23,24,29)
8  (3,13)(4,24)(9,19)(10,29)(11,15)(12,26)(14,21)(16,23)(17,30)(18,27)(20,
↪22)(25,28)
```

**from_reduced_word**(*word*, *word_type='simple'*)

Return an element of `self` from its (reduced) word.

INPUT:

- `word` – a list (or iterable) of elements of the index set of `self` (resp. of the distinguished or of all reflections)
- `word_type` – (optional, default: `'simple'`): either `'simple'`, `'distinguished'`, or `'all'`

If `word` is $[i_1, i_2, \ldots, i_k]$, then this returns the corresponding product of simple reflections $s_{i_1} s_{i_2} \cdots s_{i_k}$.

If `word_type` is `'distinguished'` (resp. `'all'`), then the product of the distinguished reflections (resp. all reflections) is returned.

**Note:** The main use case is for constructing elements from reduced words, hence the name of this method. However, the input word need *not* be reduced.

**See also:**

- *index_set()*
- *reflection_index_set()*
- *hyperplane_index_set()*
- *apply_simple_reflections()*
- `reduced_word()`
- `_test_reduced_word()`

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: W
The symmetric group on {0, ..., 3}
sage: s = W.simple_reflections()
sage: W.from_reduced_word([0,2,0,1])
(0, 3, 1, 2)
sage: W.from_reduced_word((0,2,0,1))
(0, 3, 1, 2)
sage: s[0]*s[2]*s[0]*s[1]
(0, 3, 1, 2)
```

We now experiment with the different values for `word_type` for the colored symmetric group:

```
sage: W = ColoredPermutations(1,4)
sage: W.from_reduced_word([1,2,1,2,1,2])
[[0, 0, 0, 0], [1, 2, 3, 4]]

sage: W.from_reduced_word([1, 2, 3]).reduced_word()
[1, 2, 3]

sage: W = ReflectionGroup((1,1,4))           # optional - gap3
sage: W.from_reduced_word([1,2,3], word_type='all').reduced_word()   #␣
↪optional - gap3
```

```
[1, 2, 3]

sage: W.from_reduced_word([1,2,3], word_type='all').reduced_word_in_
↪reflections()    # optional - gap3
[1, 2, 3]

sage: W.from_reduced_word([1,2,3]).reduced_word_in_reflections()     #␣
↪optional - gap3
[1, 2, 3]
```

**group_generators**()

Return the simple reflections of `self`, as distinguished group generators.

**See also:**

- *simple_reflections()*
- *Groups.ParentMethods.group_generators()*
- *Semigroups.ParentMethods.semigroup_generators()*

EXAMPLES:

```
sage: D10 = FiniteCoxeterGroups().example(10)
sage: D10.group_generators()
Finite family {1: (1,), 2: (2,)}
sage: SymmetricGroup(5).group_generators()
Finite family {1: (1,2), 2: (2,3), 3: (3,4), 4: (4,5)}

sage: W = ColoredPermutations(3,2)
sage: W.group_generators()
Finite family {1: [[0, 0],
                   [2, 1]],
               2: [[0, 1],
                   [1, 2]]}
```

The simple reflections are also semigroup generators, even for an infinite group:

```
sage: W = WeylGroup(["A",2,1])
sage: W.semigroup_generators()
Finite family {0: [-1  1  1]
                  [ 0  1  0]
                  [ 0  0  1],
               1: [ 1  0  0]
                  [ 1 -1  1]
                  [ 0  0  1],
               2: [ 1  0  0]
                  [ 0  1  0]
                  [ 1  1 -1]}
```

**hyperplane_index_set**()

Return the index set of the distinguished reflections of `self`.

This is also the index set of the reflection hyperplanes of `self`, hence the name. This name is slightly abusive since the concept of reflection hyperplanes is not defined for all generalized Coxeter groups. However for all practical purposes this is only used for complex reflection groups, and there this is the desirable name.

**See also:**

- *distinguished_reflection()*

- *distinguished_reflections()*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,4))                          # optional - gap3
sage: W.hyperplane_index_set()                              # optional - gap3
(1, 2, 3, 4, 5, 6)
sage: W = ReflectionGroup((1,1,4), hyperplane_index_set=[1,3,'asdf',7,9,
↪11])     # optional - gap3
sage: W.hyperplane_index_set()                              # optional - gap3
(1, 3, 'asdf', 7, 9, 11)
sage: W = ReflectionGroup((1,1,4), hyperplane_index_set=('a','b','c','d',
↪'e','f'))  # optional - gap3
sage: W.hyperplane_index_set()                              # optional - gap3
('a', 'b', 'c', 'd', 'e', 'f')
```

**index_set**()

Return the index set of (the simple reflections of) `self`, as a list (or iterable).

**See also:**

- *simple_reflection()*
- *simple_reflections()*

EXAMPLES:

```
sage: W = CoxeterGroups().Finite().example(); W
The 5-th dihedral group of order 10
sage: W.index_set()
(1, 2)

sage: W = ColoredPermutations(1, 4)
sage: W.index_set()
(1, 2, 3)
sage: W = ReflectionGroup((1,1,4), index_set=[1,3,'asdf'])  # optional -␣
↪gap3
sage: W.index_set()                                         # optional - gap3
(1, 3, 'asdf')
sage: W = ReflectionGroup((1,1,4), index_set=('a','b','c')) # optional -␣
↪gap3
sage: W.index_set()                                         # optional - gap3
('a', 'b', 'c')
```

**irreducible_component_index_sets**()

Return a list containing the index sets of the irreducible components of `self` as finite reflection groups.

EXAMPLES:

```
sage: W = ReflectionGroup([1,1,3], [3,1,3], 4); W       # optional - gap3
Reducible complex reflection group of rank 7 and type A2 x G(3,1,3) x ST4
sage: sorted(W.irreducible_component_index_sets())      # optional - gap3
[[1, 2], [3, 4, 5], [6, 7]]
```

ALGORITHM:

Take the connected components of the graph on the index set with edges `(i,j)`, where `s[i]` and `s[j]` do not commute.

**irreducible_components**()

Return the irreducible components of `self` as finite reflection groups.

EXAMPLES:

```
sage: W = ReflectionGroup([1,1,3], [3,1,3], 4)          # optional - gap3
sage: W.irreducible_components()                         # optional - gap3
[Irreducible real reflection group of rank 2 and type A2,
 Irreducible complex reflection group of rank 3 and type G(3,1,3),
 Irreducible complex reflection group of rank 2 and type ST4]
```

**is_irreducible**()

Return True if self is irreducible.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3); W
1-colored permutations of size 3
sage: W.is_irreducible()
True

sage: W = ReflectionGroup((1,1,3),(2,1,3)); W           # optional - gap3
Reducible real reflection group of rank 5 and type A2 x B3
sage: W.is_irreducible()                                 # optional - gap3
False
```

**is_reducible**()

Return True if self is not irreducible.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3); W
1-colored permutations of size 3
sage: W.is_reducible()
False

sage: W = ReflectionGroup((1,1,3), (2,1,3)); W          # optional - gap3
Reducible real reflection group of rank 5 and type A2 x B3
sage: W.is_reducible()                                   # optional - gap3
True
```

**number_of_irreducible_components**()

Return the number of irreducible components of self.

EXAMPLES:

```
sage: SymmetricGroup(3).number_of_irreducible_components()
1

sage: ColoredPermutations(1,3).number_of_irreducible_components()
1

sage: ReflectionGroup((1,1,3),(2,1,3)).number_of_irreducible_components()␣
↪  # optional - gap3
2
```

**number_of_simple_reflections**()

Return the number of simple reflections of self.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: W.number_of_simple_reflections()
```

```
2
sage: W = ColoredPermutations(2,3)
sage: W.number_of_simple_reflections()
3
sage: W = ColoredPermutations(4,3)
sage: W.number_of_simple_reflections()
3
sage: W = ReflectionGroup((4,2,3))                          # optional - gap3
sage: W.number_of_simple_reflections()                      # optional - gap3
4
```

**reflection**($i$)

Return the $i$-th reflection of self.

For $i$ in $1, \dots, N$, this gives the $i$-th reflection of self.

**See also:**

- reflections_index_set()
- *reflections()*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,4))                          # optional - gap3
sage: for i in W.reflection_index_set():                    # optional - gap3
....:     print('%s %s'%(i, W.reflection(i)))               # optional - gap3
1 (1,7)(2,4)(5,6)(8,10)(11,12)
2 (1,4)(2,8)(3,5)(7,10)(9,11)
3 (2,5)(3,9)(4,6)(8,11)(10,12)
4 (1,8)(2,7)(3,6)(4,10)(9,12)
5 (1,6)(2,9)(3,8)(5,11)(7,12)
6 (1,11)(3,10)(4,9)(5,7)(6,12)
```

**reflection_index_set**()

Return the index set of the reflections of self.

**See also:**

- *reflection()*
- *reflections()*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,4))                          # optional - gap3
sage: W.reflection_index_set()                              # optional - gap3
(1, 2, 3, 4, 5, 6)
sage: W = ReflectionGroup((1,1,4), reflection_index_set=[1,3,'asdf',7,9,
↪11])     # optional - gap3
sage: W.reflection_index_set()                              # optional - gap3
(1, 3, 'asdf', 7, 9, 11)
sage: W = ReflectionGroup((1,1,4), reflection_index_set=('a','b','c','d',
↪'e','f'))   # optional - gap3
sage: W.reflection_index_set()                              # optional - gap3
('a', 'b', 'c', 'd', 'e', 'f')
```

**reflections**()

Return a finite family containing the reflections of self, indexed by *reflection_index_set()*.

**See also:**

- *reflection()*
- *reflection_index_set()*

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                          # optional - gap3
sage: reflections = W.reflections()                         # optional - gap3
sage: for index in sorted(reflections.keys()):              # optional - gap3
....:     print('%s %s'%(index, reflections[index]))        # optional - gap3
1 (1,4)(2,3)(5,6)
2 (1,3)(2,5)(4,6)
3 (1,5)(2,4)(3,6)

sage: W = ReflectionGroup((1,1,3),reflection_index_set=['a','b','c'])    #␣
↪optional - gap3
sage: reflections = W.reflections()                         # optional - gap3
sage: for index in sorted(reflections.keys()):              # optional - gap3
....:     print('%s %s'%(index, reflections[index]))        # optional - gap3
a (1,4)(2,3)(5,6)
b (1,3)(2,5)(4,6)
c (1,5)(2,4)(3,6)

sage: W = ReflectionGroup((3,1,1))                          # optional - gap3
sage: reflections = W.reflections()                         # optional - gap3
sage: for index in sorted(reflections.keys()):              # optional - gap3
....:     print('%s %s'%(index, reflections[index]))        # optional - gap3
1 (1,2,3)
2 (1,3,2)

sage: W = ReflectionGroup((1,1,3), (3,1,2))                 # optional - gap3
sage: reflections = W.reflections()                         # optional - gap3
sage: for index in sorted(reflections.keys()):              # optional - gap3
....:     print('%s %s'%(index, reflections[index]))        # optional - gap3
1 (1,6)(2,5)(7,8)
2 (1,5)(2,7)(6,8)
3 (3,9,15)(4,10,16)(12,17,23)(14,18,24)(20,25,29)(21,22,26)(27,28,30)
4 (3,11)(4,12)(9,13)(10,14)(15,19)(16,20)(17,21)(18,22)(23,27)(24,28)(25,
↪26)(29,30)
5 (1,7)(2,6)(5,8)
6 (3,19)(4,25)(9,11)(10,17)(12,28)(13,15)(14,30)(16,18)(20,27)(21,29)(22,
↪23)(24,26)
7 (4,21,27)(10,22,28)(11,13,19)(12,14,20)(16,26,30)(17,18,25)(23,24,29)
8 (3,13)(4,24)(9,19)(10,29)(11,15)(12,26)(14,21)(16,23)(17,30)(18,27)(20,
↪22)(25,28)
9 (3,15,9)(4,16,10)(12,23,17)(14,24,18)(20,29,25)(21,26,22)(27,30,28)
10 (4,27,21)(10,28,22)(11,19,13)(12,20,14)(16,30,26)(17,25,18)(23,29,24)
```

**semigroup_generators**()

Return the simple reflections of `self`, as distinguished group generators.

**See also:**

- *simple_reflections()*
- *Groups.ParentMethods.group_generators()*
- *Semigroups.ParentMethods.semigroup_generators()*

EXAMPLES:

```
sage: D10 = FiniteCoxeterGroups().example(10)
sage: D10.group_generators()
```

```
Finite family {1: (1,), 2: (2,)}
sage: SymmetricGroup(5).group_generators()
Finite family {1: (1,2), 2: (2,3), 3: (3,4), 4: (4,5)}

sage: W = ColoredPermutations(3,2)
sage: W.group_generators()
Finite family {1: [[0, 0],
                   [2, 1]],
               2: [[0, 1],
                   [1, 2]]}
```

The simple reflections are also semigroup generators, even for an infinite group:

```
sage: W = WeylGroup(["A",2,1])
sage: W.semigroup_generators()
Finite family {0: [-1  1  1]
                  [ 0  1  0]
                  [ 0  0  1],
               1: [ 1  0  0]
                  [ 1 -1  1]
                  [ 0  0  1],
               2: [ 1  0  0]
                  [ 0  1  0]
                  [ 1  1 -1]}
```

**simple_reflection**(*i*)
Return the $i$-th simple reflection $s_i$ of `self`.

INPUT:

- `i` – an element from the index set

**See also:**

- *index_set()*
- *simple_reflections()*

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: W
The symmetric group on {0, ..., 3}
sage: W.simple_reflection(1)
(0, 2, 1, 3)
sage: s = W.simple_reflections()
sage: s[1]
(0, 2, 1, 3)

sage: W = ReflectionGroup((1,1,4), index_set=[1,3,'asdf'])  # optional -
↪gap3
sage: for i in W.index_set():                             # optional - gap3
....:     print('%s %s'%(i, W.simple_reflection(i)))     # optional - gap3
1 (1,7)(2,4)(5,6)(8,10)(11,12)
3 (1,4)(2,8)(3,5)(7,10)(9,11)
asdf (2,5)(3,9)(4,6)(8,11)(10,12)
```

**simple_reflection_orders**()
Return the orders of the simple reflections.

EXAMPLES:

```
sage: W = WeylGroup(['B',3])
sage: W.simple_reflection_orders()
[2, 2, 2]
sage: W = CoxeterGroup(['C',4])
sage: W.simple_reflection_orders()
[2, 2, 2, 2]
sage: SymmetricGroup(5).simple_reflection_orders()
[2, 2, 2, 2]
sage: C = ColoredPermutations(4, 3)
sage: C.simple_reflection_orders()
[2, 2, 4]
```

**simple_reflections**()

Return the simple reflections $(s_i)_{i \in I}$ of `self` as a family indexed by *index_set()*.

**See also:**

- *simple_reflection()*
- *index_set()*

EXAMPLES:

For the symmetric group, we recognize the simple transpositions:

```
sage: W = SymmetricGroup(4); W
Symmetric group of order 4! as a permutation group
sage: s = W.simple_reflections()
sage: s
Finite family {1: (1,2), 2: (2,3), 3: (3,4)}
sage: s[1]
(1,2)
sage: s[2]
(2,3)
sage: s[3]
(3,4)
```

Here are the simple reflections for a colored symmetric group and a reflection group:

```
sage: W = ColoredPermutations(1,3)
sage: W.simple_reflections()
Finite family {1: [[0, 0, 0], [2, 1, 3]], 2: [[0, 0, 0], [1, 3, 2]]}

sage: W = ReflectionGroup((1,1,3), index_set=['a','b']) # optional - gap3
sage: W.simple_reflections()                            # optional - gap3
Finite family {'a': (1,4)(2,3)(5,6), 'b': (1,3)(2,5)(4,6)}
```

This default implementation uses *index_set()* and *simple_reflection()*.

**some_elements**()

Implement *Sets.ParentMethods.some_elements()* by returning some typical elements of `self`.

The result is currently composed of the simple reflections together with the unit and the result of `an_element()`.

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: W.some_elements()
[
```

```
[0 1 0 0]  [1 0 0 0]  [1 0 0 0]  [1 0 0 0]  [0 0 0 1]
[1 0 0 0]  [0 0 1 0]  [0 1 0 0]  [0 1 0 0]  [1 0 0 0]
[0 0 1 0]  [0 1 0 0]  [0 0 0 1]  [0 0 1 0]  [0 1 0 0]
[0 0 0 1], [0 0 0 1], [0 0 1 0], [0 0 0 1], [0 0 1 0]
]

sage: W = ColoredPermutations(1,4)
sage: W.some_elements()
[[[0, 0, 0, 0], [2, 1, 3, 4]],
 [[0, 0, 0, 0], [1, 3, 2, 4]],
 [[0, 0, 0, 0], [1, 2, 4, 3]],
 [[0, 0, 0, 0], [1, 2, 3, 4]],
 [[0, 0, 0, 0], [4, 1, 2, 3]]]
```

**class SubcategoryMethods**

**Irreducible**()

Return the full subcategory of irreducible objects of `self`.

A complex reflection group, or generalized coxeter group is *reducible* if its simple reflections can be split in two sets $X$ and $Y$ such that the elements of $X$ commute with that of $Y$. In particular, the group is then direct product of $\langle X \rangle$ and $\langle Y \rangle$. It's *irreducible* otherwise.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().Irreducible()
Category of irreducible complex reflection groups
sage: CoxeterGroups().Irreducible()
Category of irreducible coxeter groups
```

**super_categories**()

Return the super categories of `self`.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().super_categories()
[Category of complex reflection or generalized coxeter groups]
```

## 3.29 Coxeter Group Algebras

**class** sage.categories.coxeter_group_algebras.**CoxeterGroupAlgebras**(*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**demazure_lusztig_eigenvectors**(*q1*, *q2*)

Return the family of eigenvectors for the Cherednik operators.

INPUT:

- `self` – a finite Coxeter group $W$

- q1, q2 – two elements of the ground ring $K$

The affine Hecke algebra $H_{q_1,q_2}(\tilde{W})$ acts on the group algebra of $W$ through the Demazure-Lusztig operators $T_i$. Its Cherednik operators $Y^\lambda$ can be simultaneously diagonalized as long as $q_1/q_2$ is not a small root of unity [HST2008].

This method returns the family of joint eigenvectors, indexed by $W$.

**See also:**

- *demazure_lusztig_operators()*
- sage.combinat.root_system.hecke_algebra_representation.CherednikOperatorsEigenvectors

EXAMPLES:

```
sage: W = WeylGroup(["B",2])
sage: W.element_class._repr_=lambda x: "".join(str(i) for i in x.reduced_
→word())
sage: K = QQ['q1,q2'].fraction_field()
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: E = KW.demazure_lusztig_eigenvectors(q1,q2)
sage: E.keys()
Weyl Group of type ['B', 2] (as a matrix group acting on the ambient␣
→space)
sage: w = W.an_element()
sage: E[w]
(q2/(-q1+q2))*2121 + ((-q2)/(-q1+q2))*121 - 212 + 12
```

**demazure_lusztig_operator_on_basis**(*w, i, q1, q2, side='right'*)

Return the result of applying the $i$-th Demazure Lusztig operator on w.

INPUT:

- w – an element of the Coxeter group
- i – an element of the index set
- q1, q2 – two elements of the ground ring
- bar – a boolean (default False)

See *demazure_lusztig_operators()* for details.

EXAMPLES:

```
sage: W = WeylGroup(["B",3])
sage: W.element_class._repr_=lambda x: "".join(str(i) for i in x.reduced_
→word())
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: w = W.an_element()
sage: KW.demazure_lusztig_operator_on_basis(w, 0, q1, q2)
(-q2)*323123 + (q1+q2)*123
sage: KW.demazure_lusztig_operator_on_basis(w, 1, q1, q2)
q1*1231
sage: KW.demazure_lusztig_operator_on_basis(w, 2, q1, q2)
q1*1232
sage: KW.demazure_lusztig_operator_on_basis(w, 3, q1, q2)
(q1+q2)*123 + (-q2)*12
```

At $q_1 = 1$ and $q_2 = 0$ we recover the action of the isobaric divided differences $\pi_i$:

```
sage: KW.demazure_lusztig_operator_on_basis(w, 0, 1, 0)
123
sage: KW.demazure_lusztig_operator_on_basis(w, 1, 1, 0)
1231
sage: KW.demazure_lusztig_operator_on_basis(w, 2, 1, 0)
1232
sage: KW.demazure_lusztig_operator_on_basis(w, 3, 1, 0)
123
```

At $q_1 = 1$ and $q_2 = -1$ we recover the action of the simple reflection $s_i$:

```
sage: KW.demazure_lusztig_operator_on_basis(w, 0, 1, -1)
323123
sage: KW.demazure_lusztig_operator_on_basis(w, 1, 1, -1)
1231
sage: KW.demazure_lusztig_operator_on_basis(w, 2, 1, -1)
1232
sage: KW.demazure_lusztig_operator_on_basis(w, 3, 1, -1)
12
```

**demazure_lusztig_operators**(*q1*, *q2*, *side='right'*, *affine=True*)

Return the Demazure Lusztig operators acting on `self`.

INPUT:

- `q1,q2` – two elements of the ground ring $K$
- `side` – `"left"` or `"right"` (default: `"right"`); which side to act upon
- `affine` – a boolean (default: `True`)

The Demazure-Lusztig operator $T_i$ is the linear map $R \to R$ obtained by interpolating between the simple projection $\pi_i$ (see `CoxeterGroups.ElementMethods.simple_projection()`) and the simple reflection $s_i$ so that $T_i$ has eigenvalues $q_1$ and $q_2$:

$$(q_1 + q_2)\pi_i - q_2 s_i.$$

The Demazure-Lusztig operators give the usual representation of the operators $T_i$ of the $q_1, q_2$ Hecke algebra associated to the Coxeter group.

For a finite Coxeter group, and if `affine=True`, the Demazure-Lusztig operators $T_1, \ldots, T_n$ are completed by $T_0$ to implement the level 0 action of the affine Hecke algebra.

EXAMPLES:

```
sage: W = WeylGroup(["B",3])
sage: W.element_class._repr_=lambda x: "".join(str(i) for i in x.reduced_
↪word())
sage: K = QQ['q1,q2']
sage: q1, q2 = K.gens()
sage: KW = W.algebra(K)
sage: T = KW.demazure_lusztig_operators(q1, q2, affine=True)
sage: x = KW.monomial(W.an_element()); x
123
sage: T[0](x)
(-q2)*323123 + (q1+q2)*123
sage: T[1](x)
q1*1231
sage: T[2](x)
q1*1232
sage: T[3](x)
(q1+q2)*123 + (-q2)*12
```

```
sage: T._test_relations()
```

**Note:** For a finite Weyl group $W$, the level 0 action of the affine Weyl group $\tilde{W}$ only depends on the Coxeter diagram of the affinization, not its Dynkin diagram. Hence it is possible to explore all cases using only untwisted affinizations.

## 3.30 Coxeter Groups

**class** sage.categories.coxeter_groups.**CoxeterGroups**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of Coxeter groups.

A *Coxeter group* is a group $W$ with a distinguished (finite) family of involutions $(s_i)_{i\in I}$, called the *simple reflections*, subject to relations of the form $(s_is_j)^{m_{i,j}} = 1$.

$I$ is the *index set* of $W$ and $|I|$ is the *rank* of $W$.

See Wikipedia article Coxeter_group for details.

EXAMPLES:

```
sage: C = CoxeterGroups(); C
Category of coxeter groups
sage: C.super_categories()
[Category of generalized coxeter groups]

sage: W = C.example(); W
The symmetric group on {0, ..., 3}

sage: W.simple_reflections()
Finite family {0: (1, 0, 2, 3), 1: (0, 2, 1, 3), 2: (0, 1, 3, 2)}
```

Here are some further examples:

```
sage: FiniteCoxeterGroups().example()
The 5-th dihedral group of order 10
sage: FiniteWeylGroups().example()
The symmetric group on {0, ..., 3}
sage: WeylGroup(["B", 3])
Weyl Group of type ['B', 3] (as a matrix group acting on the ambient space)
```

Those will eventually be also in this category:

```
sage: SymmetricGroup(4)
Symmetric group of order 4! as a permutation group
sage: DihedralGroup(5)
Dihedral group of order 10 as a permutation group
```

**Todo:** add a demo of usual computations on Coxeter groups.

**See also:**

- sage.combinat.root_system
- *WeylGroups*
- GeneralizedCoxeterGroups

**Warning:** It is assumed that morphisms in this category preserve the distinguished choice of simple reflections. In particular, subobjects in this category are parabolic subgroups. In this sense, this category might be better named Coxeter Systems. In the long run we might want to have two distinct categories, one for Coxeter groups (with morphisms being just group morphisms) and one for Coxeter systems:

```
sage: CoxeterGroups().is_full_subcategory(Groups())
False
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: CoxeterGroups().is_full_subcategory(GeneralizedCoxeterGroups())
True
```

**Algebras**
alias of CoxeterGroupAlgebras

**class ElementMethods**

**absolute_covers**()
Return the list of covers of self in absolute order.

**See also:**

*absolute_length()*

EXAMPLES:

```
sage: W = WeylGroup(["A", 3])
sage: s = W.simple_reflections()
sage: w0 = s[1]
sage: w1 = s[1]*s[2]*s[3]
sage: w0.absolute_covers()
[
[0 0 1 0]  [0 1 0 0]  [0 0 0 1]  [0 1 0 0]  [0 1 0 0]
[1 0 0 0]  [1 0 0 0]  [1 0 0 0]  [0 0 1 0]  [0 0 0 1]
[0 1 0 0]  [0 0 0 1]  [0 0 1 0]  [1 0 0 0]  [0 0 1 0]
[0 0 0 1], [0 0 1 0], [0 1 0 0], [0 0 0 1], [1 0 0 0]
]
```

**absolute_le**(*other*)
Return whether self is smaller than other in the absolute order.

A general reflection is an element of the form $ws_iw^{-1}$, where $s_i$ is a simple reflection. The absolute order is defined analogously to the weak order but using general reflections rather than just simple reflections.

This partial order can be used to define noncrossing partitions associated with this Coxeter group.

**See also:**

*absolute_length()*

EXAMPLES:

```
sage: W = WeylGroup(["A", 3])
sage: s = W.simple_reflections()
sage: w0 = s[1]
sage: w1 = s[1]*s[2]*s[3]
sage: w0.absolute_le(w1)
True
sage: w1.absolute_le(w0)
False
sage: w1.absolute_le(w1)
True
```

**absolute_length**()

Return the absolute length of `self`.

The absolute length is the length of the shortest expression of the element as a product of reflections.

For permutations in the symmetric groups, the absolute length is the size minus the number of its disjoint cycles.

**See also:**

*absolute_le()*

EXAMPLES:

```
sage: W = WeylGroup(["A", 3])
sage: s = W.simple_reflections()
sage: (s[1]*s[2]*s[3]).absolute_length()
3

sage: W = SymmetricGroup(4)
sage: s = W.simple_reflections()
sage: (s[3]*s[2]*s[1]).absolute_length()
3
```

**apply_demazure_product**(*element*, *side='right'*, *length_increasing=True*)

Returns the Demazure or 0-Hecke product of `self` with another Coxeter group element.

See *CoxeterGroups.ParentMethods.simple_projections()*.

INPUT:

- **element – either an element of the same Coxeter** group as `self` or a tuple or a list (such as a reduced word) of elements from the index set of the Coxeter group.
- **side – 'left' or 'right' (default: 'right'); the** side of `self` on which the element should be applied. If `side` is 'left' then the operation is applied on the left.
- **length_increasing – a boolean (default True)** whether to act length increasingly or decreasingly

EXAMPLES:

```
sage: W = WeylGroup(['C',4],prefix="s")
sage: v = W.from_reduced_word([1,2,3,4,3,1])
sage: v.apply_demazure_product([1,3,4,3,3])
s4*s1*s2*s3*s4*s3*s1
sage: v.apply_demazure_product([1,3,4,3],side='left')
s3*s4*s1*s2*s3*s4*s2*s3*s1
sage: v.apply_demazure_product((1,3,4,3),side='left')
s3*s4*s1*s2*s3*s4*s2*s3*s1
sage: v.apply_demazure_product(v)
s2*s3*s4*s1*s2*s3*s4*s2*s3*s2*s1
```

**apply_simple_projection**(*i*, *side='right'*, *length_increasing=True*)

INPUT:

- `i` - an element of the index set of the Coxeter group
- `side` - 'left' or 'right' (default: 'right')
- `length_increasing` - a boolean (default: True) specifying the direction of the projection

Returns the result of the application of the simple projection $\pi_i$ (resp. $\overline{\pi}_i$) on `self`.

See *CoxeterGroups.ParentMethods.simple_projections()* for the definition of the simple projections.

EXAMPLES:

```
sage: W=CoxeterGroups().example()
sage: w=W.an_element()
sage: w
(1, 2, 3, 0)
sage: w.apply_simple_projection(2)
(1, 2, 3, 0)
sage: w.apply_simple_projection(2, length_increasing=False)
(1, 2, 0, 3)
sage: W = WeylGroup(['C',4],prefix="s")
sage: v = W.from_reduced_word([1,2,3,4,3,1])
sage: v
s1*s2*s3*s4*s3*s1
sage: v.apply_simple_projection(2)
s1*s2*s3*s4*s3*s1*s2
sage: v.apply_simple_projection(2, side='left')
s1*s2*s3*s4*s3*s1
sage: v.apply_simple_projection(1, length_increasing = False)
s1*s2*s3*s4*s3
```

**binary_factorizations**(*predicate=The constant function (...) -> True*)

Returns the set of all the factorizations $self = uv$ such that $l(self) = l(u) + l(v)$.

Iterating through this set is Constant Amortized Time (counting arithmetic operations in the Coxeter group as constant time) complexity, and memory linear in the length of $self$.

One can pass as optional argument a predicate p such that $p(u)$ implies $p(u')$ for any $u$ left factor of $self$ and $u'$ left factor of $u$. Then this returns only the factorizations $self = uv$ such $p(u)$ holds.

EXAMPLES:

We construct the set of all factorizations of the maximal element of the group:

```
sage: W = WeylGroup(['A',3])
sage: s = W.simple_reflections()
sage: w0 = W.from_reduced_word([1,2,3,1,2,1])
sage: w0.binary_factorizations().cardinality()
24
```

The same number of factorizations, by bounded length:

```
sage: [w0.binary_factorizations(lambda u: u.length() <= l).cardinality()␣
↪for l in [-1,0,1,2,3,4,5,6]]
[0, 1, 4, 9, 15, 20, 23, 24]
```

The number of factorizations of the elements just below the maximal element:

```
sage: [(s[i]*w0).binary_factorizations().cardinality() for i in [1,2,3]]
[12, 12, 12]
```

```
sage: w0.binary_factorizations(lambda u: False).cardinality()
0
```

**bruhat_le**(*other*)

Bruhat comparison

INPUT:

- other – an element of the same Coxeter group

OUTPUT: a boolean

Returns whether `self` <= `other` in the Bruhat order.

EXAMPLES:

```
sage: W = WeylGroup(["A",3])
sage: u = W.from_reduced_word([1,2,1])
sage: v = W.from_reduced_word([1,2,3,2,1])
sage: u.bruhat_le(u)
True
sage: u.bruhat_le(v)
True
sage: v.bruhat_le(u)
False
sage: v.bruhat_le(v)
True
sage: s = W.simple_reflections()
sage: s[1].bruhat_le(W.one())
False
```

The implementation uses the equivalent condition that any reduced word for `other` contains a reduced word for `self` as subword. See Stembridge, A short derivation of the Möbius function for the Bruhat order. J. Algebraic Combin. 25 (2007), no. 2, 141–148, Proposition 1.1.

Complexity: $O(l * c)$, where $l$ is the minimum of the lengths of $u$ and of $v$, and $c$ is the cost of the low level methods `first_descent()`, `has_descent()`, `apply_simple_reflection()`), etc. Those are typically $O(n)$, where $n$ is the rank of the Coxeter group.

**bruhat_lower_covers**()

Returns all elements that `self` covers in (strong) Bruhat order.

If `w = self` has a descent at $i$, then the elements that $w$ covers are exactly $\{ws_i, u_1s_i, u_2s_i, ..., u_js_i\}$, where the $u_k$ are elements that $ws_i$ covers that also do not have a descent at $i$.

EXAMPLES:

```
sage: W = WeylGroup(["A",3])
sage: w = W.from_reduced_word([3,2,3])
sage: print([v.reduced_word() for v in w.bruhat_lower_covers()])
[[3, 2], [2, 3]]

sage: W = WeylGroup(["A",3])
sage: print([v.reduced_word() for v in W.simple_reflection(1).bruhat_
↪lower_covers()])
[[]]
sage: print([v.reduced_word() for v in W.one().bruhat_lower_covers()])
[]
sage: W = WeylGroup(["B",4,1])
sage: w = W.from_reduced_word([0,2])
```

```
sage: print([v.reduced_word() for v in w.bruhat_lower_covers()])
[[2], [0]]
```

We now show how to construct the Bruhat poset:

```
sage: W = WeylGroup(["A",3])
sage: covers = tuple([u, v] for v in W for u in v.bruhat_lower_covers() )
sage: P = Poset((W, covers), cover_relations = True)
sage: P.show()
```

Alternatively, one can just use:

```
sage: P = W.bruhat_poset()
```

The algorithm is taken from Stembridge's 'coxeter/weyl' package for Maple.

**bruhat_lower_covers_reflections**()

Returns all 2-tuples of lower_covers and reflections (v, r) where v is covered by self and r is the reflection such that self = v r.

ALGORITHM:

See *bruhat_lower_covers()*

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.bruhat_lower_covers_reflections()
[(s1*s2*s1, s1*s2*s3*s2*s1), (s3*s2*s1, s2), (s3*s1*s2, s1)]
```

**bruhat_upper_covers**()

Returns all elements that cover self in (strong) Bruhat order.

The algorithm works recursively, using the 'inverse' of the method described for lower covers *bruhat_lower_covers()*. Namely, it runs through all $i$ in the index set. Let $w$ equal self. If $w$ has no right descent $i$, then $ws_i$ is a cover; if $w$ has a decent at $i$, then $u_j s_i$ is a cover of $w$ where $u_j$ is a cover of $ws_i$.

EXAMPLES:

```
sage: W = WeylGroup(['A',3,1], prefix="s")
sage: w = W.from_reduced_word([1,2,1])
sage: w.bruhat_upper_covers()
[s1*s2*s1*s0, s1*s2*s0*s1, s0*s1*s2*s1, s3*s1*s2*s1, s2*s3*s1*s2,␣
↪s1*s2*s3*s1]

sage: W = WeylGroup(['A',3])
sage: w = W.long_element()
sage: w.bruhat_upper_covers()
[]

sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([1,2,1])
sage: S = [v for v in W if w in v.bruhat_lower_covers()]
sage: C = w.bruhat_upper_covers()
sage: set(S) == set(C)
True
```

**bruhat_upper_covers_reflections**()

Returns all 2-tuples of covers and reflections (v, r) where v covers self and r is the reflection such that self = v r.

ALGORITHM:

See *bruhat_upper_covers()*

EXAMPLES:

```
sage: W = WeylGroup(['A',4], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.bruhat_upper_covers_reflections()
[(s1*s2*s3*s2*s1, s3), (s2*s3*s1*s2*s1, s2*s3*s2), (s3*s4*s1*s2*s1, s4),
↪(s4*s3*s1*s2*s1, s1*s2*s3*s4*s3*s2*s1)]
```

**canonical_matrix**()

Return the matrix of self in the canonical faithful representation.

This is an $n$-dimension real faithful essential representation, where $n$ is the number of generators of the Coxeter group. Note that this is not always the most natural matrix representation, for instance in type $A_n$.

EXAMPLES:

```
sage: W = WeylGroup(["A", 3])
sage: s = W.simple_reflections()
sage: (s[1]*s[2]*s[3]).canonical_matrix()
[ 0  0 -1]
[ 1  0 -1]
[ 0  1 -1]
```

**coset_representative**(*index_set*, *side='right'*)

INPUT:

- index_set - a subset (or iterable) of the nodes of the Dynkin diagram
- side - 'left' or 'right'

Returns the unique shortest element of the Coxeter group $W$ which is in the same left (resp. right) coset as self, with respect to the parabolic subgroup $W_I$.

EXAMPLES:

```
sage: W = CoxeterGroups().example(5)
sage: s = W.simple_reflections()
sage: w = s[2]*s[1]*s[3]
sage: w.coset_representative([]).reduced_word()
[2, 3, 1]
sage: w.coset_representative([1]).reduced_word()
[2, 3]
sage: w.coset_representative([1,2]).reduced_word()
[2, 3]
sage: w.coset_representative([1,3]                 ).reduced_word()
[2]
sage: w.coset_representative([2,3]                 ).reduced_word()
[2, 1]
sage: w.coset_representative([1,2,3]               ).reduced_word()
[]
sage: w.coset_representative([],      side = 'left').reduced_word()
[2, 3, 1]
sage: w.coset_representative([1],     side = 'left').reduced_word()
[2, 3, 1]
```

```
sage: w.coset_representative([1,2],   side = 'left').reduced_word()
[3]
sage: w.coset_representative([1,3],   side = 'left').reduced_word()
[2, 3, 1]
sage: w.coset_representative([2,3],   side = 'left').reduced_word()
[1]
sage: w.coset_representative([1,2,3], side = 'left').reduced_word()
[]
```

**cover_reflections** (*side='right'*)

Returns the set of reflections `t` such that `self t` covers `self`.

If `side` is 'left', `t self` covers `self`.

EXAMPLES:

```
sage: W = WeylGroup(['A',4], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.cover_reflections()
[s3, s2*s3*s2, s4, s1*s2*s3*s4*s3*s2*s1]
sage: w.cover_reflections(side = 'left')
[s4, s2, s1*s2*s1, s3*s4*s3]
```

**coxeter_sorting_word** (*c*)

Return the `c`-sorting word of `self`.

For a Coxeter element $c$ and an element $w$, the $c$-sorting word of $w$ is the lexicographic minimal reduced expression of $w$ in the infinite word $c^\infty$.

INPUT:

- `c`– a Coxeter element.

OUTPUT:

the `c`-sorting word of `self` as a list of integers.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: c = W.from_reduced_word([0,2,1])
sage: w = W.from_reduced_word([1,2,1,0,1])
sage: w.coxeter_sorting_word(c)
[2, 1, 2, 0, 1]
```

**deodhar_factor_element** (*w, index_set*)

Returns Deodhar's Bruhat order factoring element.

INPUT:

- `w` is an element of the same Coxeter group `W` as `self`
- `index_set` is a subset of Dynkin nodes defining a parabolic subgroup `W'` of `W`

It is assumed that `v = self` and `w` are minimum length coset representatives for `W/W'` such that `v` $\leq$ `w` in Bruhat order.

OUTPUT:

Deodhar's element `f(v,w)` is the unique element of `W'` such that, for all `v'` and `w'` in `W'`, `vv'` $\leq$ `ww'` in `W` if and only if `v'` $\leq$ `f(v,w) * w'` in `W'` where $\star$ is the Demazure product.

EXAMPLES:

```
sage: W = WeylGroup(['A',5],prefix="s")
sage: v = W.from_reduced_word([5])
sage: w = W.from_reduced_word([4,5,2,3,1,2])
sage: v.deodhar_factor_element(w,[1,3,4])
s3*s1
sage: W=WeylGroup(['C',2])
sage: w=W.from_reduced_word([2,1])
sage: w.deodhar_factor_element(W.from_reduced_word([2]),[1])
Traceback (most recent call last):
...
ValueError: [2, 1] is not of minimum length in its coset for the␣
↪parabolic subgroup with index set [1]
```

REFERENCES:

- [Deo1987a]

**deodhar_lift_down**(*w*, *index_set*)

Letting `v = self`, given a Bruhat relation `v W'` $\geq$ `w W'` among cosets with respect to the subgroup `W'` given by the Dynkin node subset `index_set`, returns the Bruhat-maximum lift `x` of `wW'` such that `v` $\geq$ `x`.

INPUT:

- `w` is an element of the same Coxeter group `W` as `self`.
- `index_set` is a subset of Dynkin nodes defining a parabolic subgroup `W'`.

OUTPUT:

The unique Bruhat-maximum element `x` in `W` such that `x W' = w W'` and `v `\ge` ``x`.

**See also:**

*sage.categories.coxeter_groups.CoxeterGroups.ElementMethods.deodhar_lift_up()*

EXAMPLES:

```
sage: W = WeylGroup(['A',3],prefix="s")
sage: v = W.from_reduced_word([1,2,3,2])
sage: w = W.from_reduced_word([3,2])
sage: v.deodhar_lift_down(w, [3])
s2*s3*s2
```

**deodhar_lift_up**(*w*, *index_set*)

Letting `v = self`, given a Bruhat relation `v W'` $\leq$ `w W'` among cosets with respect to the subgroup `W'` given by the Dynkin node subset `index_set`, returns the Bruhat-minimum lift `x` of `wW'` such that `v` $\leq$ `x`.

INPUT:

- `w` is an element of the same Coxeter group `W` as `self`.
- `index_set` is a subset of Dynkin nodes defining a parabolic subgroup `W'`.

OUTPUT:

The unique Bruhat-minimum element `x` in `W` such that `x W' = w W'` and `v` $\leq$ `x`.

**See also:**

*sage.categories.coxeter_groups.CoxeterGroups.ElementMethods.deodhar_lift_down()*

EXAMPLES:

```
sage: W = WeylGroup(['A',3],prefix="s")
sage: v = W.from_reduced_word([1,2,3])
sage: w = W.from_reduced_word([1,3,2])
sage: v.deodhar_lift_up(w, [3])
s1*s2*s3*s2
```

**descents** (*side='right'*, *index_set=None*, *positive=False*)

INPUT:

- `index_set` - a subset (as a list or iterable) of the nodes of the Dynkin diagram; (default: all of them)
- `side` - 'left' or 'right' (default: 'right')
- `positive` - a boolean (default: `False`)

Returns the descents of self, as a list of elements of the index_set.

The `index_set` option can be used to restrict to the parabolic subgroup indexed by `index_set`.

If positive is `True`, then returns the non-descents instead

TODO: find a better name for `positive`: complement? non_descent?

Caveat: the return type may change to some other iterable (tuple, ...) in the future. Please use keyword arguments also, as the order of the arguments may change as well.

EXAMPLES:

```
sage: W=CoxeterGroups().example()
sage: s=W.simple_reflections()
sage: w=s[0]*s[1]
sage: w.descents()
[1]
sage: w=s[0]*s[2]
sage: w.descents()
[0, 2]

TODO: side, index_set, positive
```

**first_descent** (*side='right'*, *index_set=None*, *positive=False*)

Returns the first left (resp. right) descent of self, as ane element of `index_set`, or `None` if there is none.

See *descents()* for a description of the options.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[2]*s[0]
sage: w.first_descent()
0
sage: w = s[0]*s[2]
sage: w.first_descent()
0
sage: w = s[0]*s[1]
sage: w.first_descent()
1
```

**has_descent** (*i*, *side='right'*, *positive=False*)

Returns whether i is a (left/right) descent of self.

See *descents()* for a description of the options.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[0] * s[1] * s[2]
sage: w.has_descent(2)
True
sage: [ w.has_descent(i)                  for i in [0,1,2] ]
[False, False, True]
sage: [ w.has_descent(i, side = 'left')   for i in [0,1,2] ]
[True, False, False]
sage: [ w.has_descent(i, positive = True) for i in [0,1,2] ]
[True, True, False]
```

This default implementation delegates the work to *has_left_descent()* and *has_right_descent()*.

**has_full_support**()

Return whether `self` has full support.

An element is said to have full support if its support contains all simple reflections.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.from_reduced_word([1,2,1])
sage: w.has_full_support()
False
sage: w = W.from_reduced_word([1,2,1,0,1])
sage: w.has_full_support()
True
```

**has_left_descent**(*i*)

Returns whether $i$ is a left descent of self.

This default implementation uses that a left descent of $w$ is a right descent of $w^{-1}$.

EXAMPLES:

```
sage: W = CoxeterGroups().example(); W
The symmetric group on {0, ..., 3}
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.has_left_descent(0)
True
sage: w.has_left_descent(1)
False
sage: w.has_left_descent(2)
False
```

**has_right_descent**(*i*)

Returns whether `i` is a right descent of self.

EXAMPLES:

```
sage: W = CoxeterGroups().example(); W
The symmetric group on {0, ..., 3}
sage: w = W.an_element(); w
(1, 2, 3, 0)
sage: w.has_right_descent(0)
```

```
False
sage: w.has_right_descent(1)
False
sage: w.has_right_descent(2)
True
```

**inversions_as_reflections**()

Returns the set of reflections `r` such that `self r < self`.

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.inversions_as_reflections()
[s1, s1*s2*s1, s2, s1*s2*s3*s2*s1]
```

**is_coxeter_sortable**(*c*, *sorting_word=None*)

Return whether `self` is `c`-sortable.

Given a Coxeter element $c$, an element $w$ is $c$-sortable if its $c$-sorting word decomposes into a sequence of weakly decreasing subwords of $c$.

INPUT:

- `c` – a Coxeter element.
- `sorting_word` – sorting word (default: None) used to not recompute the `c`-sorting word if already computed.

OUTPUT:

is `self` `c`-sortable

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: c = W.from_reduced_word([0,2,1])
sage: w = W.from_reduced_word([1,2,1,0,1])
sage: w.coxeter_sorting_word(c)
[2, 1, 2, 0, 1]
sage: w.is_coxeter_sortable(c)
False
sage: w = W.from_reduced_word([0,2,1,0,2])
sage: w.coxeter_sorting_word(c)
[2, 0, 1, 2, 0]
sage: w.is_coxeter_sortable(c)
True
sage: W = CoxeterGroup(['A',3])
sage: c = W.from_reduced_word([1,2,3])
sage: len([w for w in W if w.is_coxeter_sortable(c)]) # number of c-
↪sortable elements in A_3 (Catalan number)
14
```

**is_grassmannian**(*side='right'*)

INPUT:

- `side` - "left" or "right" (default: "right")

Tests whether `self` is Grassmannian, i.e. it has at most one descent on the right (resp. on the left).

EXAMPLES:

```
sage: W = CoxeterGroups().example(); W
The symmetric group on {0, ..., 3}
```

```
sage: s = W.simple_reflections()
sage: W.one().is_grassmannian()
True
sage: s[1].is_grassmannian()
True
sage: (s[1]*s[2]).is_grassmannian()
True
sage: (s[0]*s[1]).is_grassmannian()
True
sage: (s[1]*s[2]*s[1]).is_grassmannian()
False

sage: (s[0]*s[2]*s[1]).is_grassmannian(side = "left")
False
sage: (s[0]*s[2]*s[1]).is_grassmannian(side = "right")
True
sage: (s[0]*s[2]*s[1]).is_grassmannian()
True
```

**left_inversions_as_reflections**()

Returns the set of reflections r such that r self < self.

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.left_inversions_as_reflections()
[s1, s3, s1*s2*s3*s2*s1, s2*s3*s2]
```

**length**()

Return the length of self.

This is the minimal length of a product of simple reflections giving self.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: s1 = W.simple_reflection(1)
sage: s2 = W.simple_reflection(2)
sage: s1.length()
1
sage: (s1*s2).length()
2
sage: W = CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: w = s[0]*s[1]*s[0]
sage: w.length()
3
sage: W = CoxeterGroups().example()
sage: sum((x^w.length()) for w in W) - expand(prod(sum(x^i for i in range(j+1)) for j in range(4))) # This is scandalously slow!!!
0
```

**See also:**

*reduced_word()*

**Todo:** Should use reduced_word_iterator (or reverse_iterator)

**lower_cover_reflections**(*side='right'*)

Returns the reflections t such that self covers self t.

If side is 'left', self covers t self.

EXAMPLES:

```
sage: W = WeylGroup(['A',3],prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.lower_cover_reflections()
[s1*s2*s3*s2*s1, s2, s1]
sage: w.lower_cover_reflections(side = 'left')
[s2*s3*s2, s3, s1]
```

**lower_covers**(*side='right'*, *index_set=None*)

Returns all elements that self covers in weak order.

INPUT:

- side – 'left' or 'right' (default: 'right')
- index_set – a list of indices or None

OUTPUT: a list

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([3,2,1])
sage: [x.reduced_word() for x in w.lower_covers()]
[[3, 2]]
```

To obtain covers for left weak order, set the option side to 'left':

```
sage: [x.reduced_word() for x in w.lower_covers(side='left')]
[[2, 1]]
sage: w = W.from_reduced_word([3,2,3,1])
sage: [x.reduced_word() for x in w.lower_covers()]
[[2, 3, 2], [3, 2, 1]]
```

Covers w.r.t. a parabolic subgroup are obtained with the option index_set:

```
sage: [x.reduced_word() for x in w.lower_covers(index_set = [1,2])]
[[2, 3, 2]]
sage: [x.reduced_word() for x in w.lower_covers(side='left')]
[[3, 2, 1], [2, 3, 1]]
```

**min_demazure_product_greater**(*element*)

Finds the unique Bruhat-minimum element u such that $v \le w * u$ where v is self, w is element and $*$ is the Demazure product.

INPUT:

- element is either an element of the same Coxeter group as self or a list (such as a reduced word) of elements from the index set of the Coxeter group.

EXAMPLES:

```
sage: W = WeylGroup(['A',4],prefix="s")
sage: v = W.from_reduced_word([2,3,4,1,2])
sage: u = W.from_reduced_word([2,3,2,1])
sage: v.min_demazure_product_greater(u)
s4*s2
sage: v.min_demazure_product_greater([2,3,2,1])
```

```
s4*s2
sage: v.min_demazure_product_greater((2,3,2,1))
s4*s2
```

**reduced_word**()

Return a reduced word for `self`.

This is a word $[i_1, i_2, \ldots, i_k]$ of minimal length such that $s_{i_1} s_{i_2} \cdots s_{i_k} =$ self, where the $s_i$ are the simple reflections.

EXAMPLES:

```
sage: W=CoxeterGroups().example()
sage: s=W.simple_reflections()
sage: w=s[0]*s[1]*s[2]
sage: w.reduced_word()
[0, 1, 2]
sage: w=s[0]*s[2]
sage: w.reduced_word()
[2, 0]
```

**See also:**

- *reduced_words()*, *reduced_word_reverse_iterator()*,
- *length()*, *reduced_word_graph()*

**reduced_word_graph**()

Return the reduced word graph of `self`.

The reduced word graph of an element $w$ in a Coxeter group is the graph whose vertices are the reduced words for $w$ (see *reduced_word()* for a definition of this term), and which has an $m$-colored edge between two reduced words $x$ and $y$ whenever $x$ and $y$ differ by exactly one length-$m$ braid move (with $m \geq 2$).

This graph is always connected (a theorem due to Tits) and has no multiple edges.

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix='s')
sage: w0 = W.long_element()
sage: G = w0.reduced_word_graph()
sage: G.num_verts()
16
sage: len(w0.reduced_words())
16
sage: G.num_edges()
18
sage: len([e for e in G.edges() if e[2] == 2])
10
sage: len([e for e in G.edges() if e[2] == 3])
8
```

**See also:**

*reduced_words()*, *reduced_word_reverse_iterator()*, *length()*, *reduced_word()*

**reduced_word_reverse_iterator**()

Return a reverse iterator on a reduced word for `self`.

EXAMPLES:

```
sage: W=CoxeterGroups().example()
sage: s = W.simple_reflections()
sage: sigma = s[0]*s[1]*s[2]
sage: rI=sigma.reduced_word_reverse_iterator()
sage: [i for i in rI]
[2, 1, 0]
sage: s[0]*s[1]*s[2]==sigma
True
sage: sigma.length()
3
```

**See also:**

*reduced_word()*

Default implementation: recursively remove the first right descent until the identity is reached (see *first_descent()* and *apply_simple_reflection()*).

**reduced_words()**

Return all reduced words for `self`.

See *reduced_word()* for the definition of a reduced word.

EXAMPLES:

```
sage: W=CoxeterGroups().example()
sage: s=W.simple_reflections()
sage: w=s[0]*s[2]
sage: w.reduced_words()
[[2, 0], [0, 2]]
sage: W=WeylGroup(['E',6])
sage: w=W.from_reduced_word([2,3,4,2])
sage: w.reduced_words()
[[3, 2, 4, 2], [2, 3, 4, 2], [3, 4, 2, 4]]
```

TODO: the result should be full featured finite enumerated set (e.g. counting can be done much faster than iterating).

**See also:**

*reduced_word()*, *reduced_word_reverse_iterator()*, *length()*, *reduced_word_graph()*

**reflection_length()**

Return the reflection length of `self`.

The reflection length is the length of the shortest expression of the element as a product of reflections.

**See also:**

*absolute_length()*

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: s = W.simple_reflections()
sage: (s[1]*s[2]*s[3]).reflection_length()
3

sage: W = SymmetricGroup(4)
sage: s = W.simple_reflections()
```

```
sage: (s[3]*s[2]*s[3]).reflection_length()
1
```

**support**()

Return the support of `self`, that is the simple reflections that appear in the reduced expressions of `self`.

OUTPUT:

The support of `self` as a set of integers

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: w = W.from_reduced_word([1,2,1])
sage: w.support()
{1, 2}
```

**upper_covers**(*side='right'*, *index_set=None*)

Returns all elements that cover `self` in weak order.

INPUT:

- side – 'left' or 'right' (default: 'right')
- index_set – a list of indices or None

OUTPUT: a list

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([2,3])
sage: [x.reduced_word() for x in w.upper_covers()]
[[2, 3, 1], [2, 3, 2]]
```

To obtain covers for left weak order, set the option `side` to 'left':

```
sage: [x.reduced_word() for x in w.upper_covers(side = 'left')]
[[1, 2, 3], [2, 3, 2]]
```

Covers w.r.t. a parabolic subgroup are obtained with the option `index_set`:

```
sage: [x.reduced_word() for x in w.upper_covers(index_set = [1])]
[[2, 3, 1]]
sage: [x.reduced_word() for x in w.upper_covers(side = 'left', index_set = [1])]
[[1, 2, 3]]
```

**weak_covers**(*side='right'*, *index_set=None*, *positive=False*)

Returns all elements that `self` covers in weak order.

INPUT:

- side – 'left' or 'right' (default: 'right')
- positive – a boolean (default: False)
- index_set – a list of indices or None

OUTPUT: a list

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([3,2,1])
```

```
sage: [x.reduced_word() for x in w.weak_covers()]
[[3, 2]]
```

To obtain instead elements that cover self, set `positive = True`:

```
sage: [x.reduced_word() for x in w.weak_covers(positive = True)]
[[3, 1, 2, 1], [2, 3, 2, 1]]
```

To obtain covers for left weak order, set the option side to 'left':

```
sage: [x.reduced_word() for x in w.weak_covers(side='left')]
[[2, 1]]
sage: w = W.from_reduced_word([3,2,3,1])
sage: [x.reduced_word() for x in w.weak_covers()]
[[2, 3, 2], [3, 2, 1]]
sage: [x.reduced_word() for x in w.weak_covers(side='left')]
[[3, 2, 1], [2, 3, 1]]
```

Covers w.r.t. a parabolic subgroup are obtained with the option `index_set`:

```
sage: [x.reduced_word() for x in w.weak_covers(index_set = [1,2])]
[[2, 3, 2]]
```

**weak_le**(*other*, *side='right'*)

comparison in weak order

INPUT:

- other – an element of the same Coxeter group
- side – 'left' or 'right' (default: 'right')

OUTPUT: a boolean

Returns whether `self` <= `other` in left (resp. right) weak order, that is if 'v' can be obtained from 'v' by length increasing multiplication by simple reflections on the left (resp. right).

EXAMPLES:

```
sage: W = WeylGroup(["A",3])
sage: u = W.from_reduced_word([1,2])
sage: v = W.from_reduced_word([1,2,3,2])
sage: u.weak_le(u)
True
sage: u.weak_le(v)
True
sage: v.weak_le(u)
False
sage: v.weak_le(v)
True
```

Comparison for left weak order is achieved with the option `side`:

```
sage: u.weak_le(v, side = 'left')
False
```

The implementation uses the equivalent condition that any reduced word for $u$ is a right (resp. left) prefix of some reduced word for $v$.

Complexity: $O(l * c)$, where $l$ is the minimum of the lengths of $u$ and of $v$, and $c$ is the cost of the low level methods *first_descent()*, *has_descent()*, *apply_simple_reflection()*), etc. Those are typically $O(n)$, where $n$ is the rank of the Coxeter group.

We now run consistency tests with permutations:

```
sage: W = WeylGroup(["A",3])
sage: P4 = Permutations(4)
sage: def P4toW(w): return W.from_reduced_word(w.reduced_word())
sage: for u in P4:  # long time (5s on sage.math, 2011)
....:     for v in P4:
....:         assert u.permutohedron_lequal(v) == P4toW(u).weak_
↪le(P4toW(v))
....:         assert u.permutohedron_lequal(v, side='left') == P4toW(u).
↪weak_le(P4toW(v), side='left')
```

**Finite**
alias of *FiniteCoxeterGroups*

**class ParentMethods**

**bruhat_graph** (*x=None*, *y=None*, *edge_labels=False*)
Return the Bruhat graph as a directed graph, with an edge $u \to v$ if and only if $u < v$ in the Bruhat order, and $u = r \cdot v$.

The Bruhat graph $\Gamma(x, y)$, defined if $x \leq y$ in the Bruhat order, has as its vertices the Bruhat interval $\{t | x \leq t \leq y\}$, and as its edges are the pairs $(u, v)$ such that $u = r \cdot v$ where $r$ is a reflection, that is, a conjugate of a simple reflection.

REFERENCES:

Carrell, The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties. Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), 53–61, Proc. Sympos. Pure Math., 56, Part 1, Amer. Math. Soc., Providence, RI, 1994.

EXAMPLES:

```
sage: W = CoxeterGroup(['H',3])
sage: G = W.bruhat_graph(); G
Digraph on 120 vertices

sage: W = CoxeterGroup(['A',2,1])
sage: s1, s2, s3 = W.simple_reflections()
sage: W.bruhat_graph(s1, s1*s3*s2*s3)
Digraph on 6 vertices

sage: W.bruhat_graph(s1, s3*s2*s3)
Digraph on 0 vertices

sage: W = WeylGroup("A3", prefix="s")
sage: s1, s2, s3 = W.simple_reflections()
sage: G = W.bruhat_graph(s1*s3, s1*s2*s3*s2*s1); G
Digraph on 10 vertices
```

Check that the graph has the correct number of edges (see trac ticket #17744):

```
sage: len(G.edges())
16
```

**bruhat_interval** (*x*, *y*)
Return the list of `t` such that `x <= t <= y`.

EXAMPLES:

```
sage: W = WeylGroup("A3", prefix="s")
sage: [s1,s2,s3]=W.simple_reflections()
sage: W.bruhat_interval(s2,s1*s3*s2*s1*s3)
[s1*s2*s3*s2*s1, s2*s3*s2*s1, s3*s1*s2*s1, s1*s2*s3*s1,
 s1*s2*s3*s2, s3*s2*s1, s2*s3*s1, s2*s3*s2, s1*s2*s1,
 s3*s1*s2, s1*s2*s3, s2*s1, s3*s2, s2*s3, s1*s2, s2]

sage: W = WeylGroup(['A',2,1], prefix="s")
sage: [s0,s1,s2]=W.simple_reflections()
sage: W.bruhat_interval(1,s0*s1*s2)
[s0*s1*s2, s1*s2, s0*s2, s0*s1, s2, s1, s0, 1]
```

**bruhat_interval_poset** (*x*, *y*, *facade=False*)

Return the poset of the Bruhat interval between `x` and `y` in Bruhat order.

EXAMPLES:

```
sage: W = WeylGroup("A3", prefix="s")
sage: s1,s2,s3 = W.simple_reflections()
sage: W.bruhat_interval_poset(s2, s1*s3*s2*s1*s3)
Finite poset containing 16 elements

sage: W = WeylGroup(['A',2,1], prefix="s")
sage: s0,s1,s2 = W.simple_reflections()
sage: W.bruhat_interval_poset(1, s0*s1*s2)
Finite poset containing 8 elements
```

**canonical_representation** ()

Return the canonical faithful representation of `self`.

EXAMPLES:

```
sage: W = WeylGroup("A3")
sage: W.canonical_representation()
Finite Coxeter group over Integer Ring with Coxeter matrix:
[1 3 2]
[3 1 3]
[2 3 1]
```

**coxeter_diagram** ()

Return the Coxeter diagram of `self`.

EXAMPLES:

```
sage: W = CoxeterGroup(['H',3], implementation="reflection")
sage: G = W.coxeter_diagram(); G
Graph on 3 vertices
sage: G.edges()
[(1, 2, 3), (2, 3, 5)]
sage: CoxeterGroup(G) is W
True
sage: G = Graph([(0, 1, 3), (1, 2, oo)])
sage: W = CoxeterGroup(G)
sage: W.coxeter_diagram() == G
True
sage: CoxeterGroup(W.coxeter_diagram()) is W
True
```

**coxeter_element**()

Return a Coxeter element.

The result is the product of the simple reflections, in some order.

---

**Note:** This implementation is shared with well generated complex reflection groups. It would be nicer to put it in some joint super category; however, in the current state of the art, there is none where it's clear that this is the right construction for obtaining a Coxeter element.

In this context, this is an element having a regular eigenvector (a vector not contained in any reflection hyperplane of `self`).

---

EXAMPLES:

```
sage: CoxeterGroup(['A', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['B', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['D', 4]).coxeter_element().reduced_word()
[1, 2, 4, 3]
sage: CoxeterGroup(['F', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['E', 8]).coxeter_element().reduced_word()
[1, 3, 2, 4, 5, 6, 7, 8]
sage: CoxeterGroup(['H', 3]).coxeter_element().reduced_word()
[1, 2, 3]
```

This method is also used for well generated finite complex reflection groups:

```
sage: W = ReflectionGroup((1,1,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3]

sage: W = ReflectionGroup((2,1,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3, 4]

sage: W = ReflectionGroup((4,1,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3, 4]

sage: W = ReflectionGroup((4,4,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3, 4]
```

**coxeter_matrix**()

Return the Coxeter matrix associated to `self`.

EXAMPLES:

```
sage: G = WeylGroup(['A',3])
sage: G.coxeter_matrix()
[1 3 2]
[3 1 3]
[2 3 1]
```

**coxeter_type**()

Return the Coxeter type of `self`.

EXAMPLES:

```
sage: W = CoxeterGroup(['H',3])
sage: W.coxeter_type()
Coxeter type of ['H', 3]
```

**demazure_product** (*Q*)
Returns the Demazure product of the list `Q` in `self`.

INPUT:
- `Q` is a list of elements from the index set of `self`.

This returns the Coxeter group element that represents the composition of 0-Hecke or Demazure operators. See *CoxeterGroups.ParentMethods.simple_projections()*.

EXAMPLES:

```
sage: W = WeylGroup(['A',2])
sage: w = W.demazure_product([2,2,1])
sage: w.reduced_word()
[2, 1]

sage: w = W.demazure_product([2,1,2,1,2])
sage: w.reduced_word()
[1, 2, 1]

sage: W = WeylGroup(['B',2])
sage: w = W.demazure_product([2,1,2,1,2])
sage: w.reduced_word()
[2, 1, 2, 1]
```

**elements_of_length** (*n*)
Return all elements of length $n$.

EXAMPLES:

```
sage: A = AffinePermutationGroup(['A',2,1])
sage: [len(list(A.elements_of_length(i))) for i in [0..5]]
[1, 3, 6, 9, 12, 15]

sage: W = CoxeterGroup(['H',3])
sage: [len(list(W.elements_of_length(i))) for i in range(4)]
[1, 3, 5, 7]

sage: W = CoxeterGroup(['A',2])
sage: [len(list(W.elements_of_length(i))) for i in range(6)]
[1, 2, 2, 1, 0, 0]
```

**grassmannian_elements** (*side='right'*)
Return the left or right Grassmannian elements of `self` as an enumerated set.

INPUT:
- `side` – (default: `"right"`) `"left"` or `"right"`

EXAMPLES:

```
sage: S = CoxeterGroups().example()
sage: G = S.grassmannian_elements()
sage: G.cardinality()
12
sage: G.list()
```

```
[(0, 1, 2, 3), (1, 0, 2, 3), (0, 2, 1, 3), (0, 1, 3, 2),
 (2, 0, 1, 3), (1, 2, 0, 3), (0, 3, 1, 2), (0, 2, 3, 1),
 (3, 0, 1, 2), (1, 3, 0, 2), (1, 2, 3, 0), (2, 3, 0, 1)]
sage: sorted(tuple(w.descents()) for w in G)
[(), (0,), (0,), (0,), (1,), (1,), (1,), (1,), (1,), (2,), (2,), (2,)]
sage: G = S.grassmannian_elements(side = "left")
sage: G.cardinality()
12
sage: sorted(tuple(w.descents(side = "left")) for w in G)
[(), (0,), (0,), (0,), (1,), (1,), (1,), (1,), (1,), (2,), (2,), (2,)]
```

**index_set**()

Return the index set of `self`.

EXAMPLES:

```
sage: W = CoxeterGroup([[1,3],[3,1]])
sage: W.index_set()
(1, 2)
sage: W = CoxeterGroup([[1,3],[3,1]], index_set=['x', 'y'])
sage: W.index_set()
('x', 'y')
sage: W = CoxeterGroup(['H',3])
sage: W.index_set()
(1, 2, 3)
```

**random_element_of_length**(*n*)

Return a random element of length `n` in `self`.

Starts at the identity, then chooses an upper cover at random.

Not very uniform: actually constructs a uniformly random reduced word of length $n$. Thus we most likely get elements with lots of reduced words!

EXAMPLES:

```
sage: A = AffinePermutationGroup(['A', 7, 1])
sage: p = A.random_element_of_length(10)
sage: p in A
True
sage: p.length() == 10
True

sage: W = CoxeterGroup(['A', 4])
sage: p = W.random_element_of_length(5)
sage: p in W
True
sage: p.length() == 5
True
```

**simple_projection**(*i*, *side='right'*, *length_increasing=True*)

INPUT:

- `i` - an element of the index set of `self`

Returns the simple projection $\pi_i$ (or $\overline{\pi}_i$ if $length_increasing$ is False).

See *simple_projections()* for the options and for the definition of the simple projections.

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: W
The symmetric group on {0, ..., 3}
sage: s = W.simple_reflections()
sage: sigma=W.an_element()
sage: sigma
(1, 2, 3, 0)
sage: u0=W.simple_projection(0)
sage: d0=W.simple_projection(0,length_increasing=False)
sage: sigma.length()
3
sage: pi=sigma*s[0]
sage: pi.length()
4
sage: u0(sigma)
(2, 1, 3, 0)
sage: pi
(2, 1, 3, 0)
sage: u0(pi)
(2, 1, 3, 0)
sage: d0(sigma)
(1, 2, 3, 0)
sage: d0(pi)
(1, 2, 3, 0)
```

**simple_projections**(*side='right'*, *length_increasing=True*)

Returns the family of simple projections, also known as 0-Hecke or Demazure operators.

INPUT:

- `self` - a Coxeter group $W$
- `side` - 'left' or 'right' (default: 'right')
- `length_increasing` - a boolean (default: True) specifying whether the operator increases or decreases length

Returns the simple projections of $W$, as a family.

To each simple reflection $s_i$ of $W$, corresponds a *simple projection* $\pi_i$ from $W$ to $W$ defined by:

$\pi_i(w) = ws_i$ if $i$ is not a descent of $w$ $\pi_i(w) = w$ otherwise.

The simple projections $(\pi_i)_{i\in I}$ move elements down the right permutohedron, toward the maximal element. They satisfy the same braid relations as the simple reflections, but are idempotents $\pi_i^2 = \pi$ not involutions $s_i^2 = 1$. As such, the simple projections generate the $0$-Hecke monoid.

By symmetry, one can also define the projections $(\overline{\pi}_i)_{i\in I}$ (when the option `length_increasing` is False):

$\overline{\pi}_i(w) = ws_i$ if $i$ is a descent of $w$ $\overline{\pi}_i(w) = w$ otherwise.

as well as the analogues acting on the left (when the option `side` is 'left').

EXAMPLES:

```
sage: W = CoxeterGroups().example()
sage: W
The symmetric group on {0, ..., 3}
sage: s = W.simple_reflections()
sage: sigma=W.an_element()
sage: sigma
(1, 2, 3, 0)
sage: pi=W.simple_projections()
sage: pi
Finite family {0: <function <lambda> at ...>, 1: <function <lambda> at ...
↪>, 2: <function <lambda> ...>}
```

```
sage: pi[1](sigma)
(1, 3, 2, 0)
sage: W.simple_projection(1)(sigma)
(1, 3, 2, 0)
```

**standard_coxeter_elements**()

Return all standard Coxeter elements in `self`.

This is the set of all elements in self obtained from any product of the simple reflections in `self`.

**Note:**

- `self` is assumed to be well-generated.
- This works even beyond real reflection groups, but the conjugacy class is not unique and we only obtain one such class.

EXAMPLES:

```
sage: W = ReflectionGroup(4)                     # optional - gap3
sage: sorted(W.standard_coxeter_elements())   # optional - gap3
[(1,7,6,12,23,20)(2,8,17,24,9,5)(3,16,10,19,15,21)(4,14,11,22,18,13),
 (1,10,4,12,21,22)(2,11,19,24,13,3)(5,15,7,17,16,23)(6,18,8,20,14,9)]
```

**weak_order_ideal**(*predicate*, *side='right'*, *category=None*)

Returns a weak order ideal defined by a predicate

INPUT:

- `predicate`: a predicate on the elements of `self` defining an weak order ideal in `self`
- `side`: "left" or "right" (default: "right")

OUTPUT: an enumerated set

EXAMPLES:

```
sage: D6 = FiniteCoxeterGroups().example(5)
sage: I = D6.weak_order_ideal(predicate = lambda w: w.length() <= 3)
sage: I.cardinality()
7
sage: list(I)
[(), (1,), (2,), (1, 2), (2, 1), (1, 2, 1), (2, 1, 2)]
```

We now consider an infinite Coxeter group:

```
sage: W = WeylGroup(["A",1,1])
sage: I = W.weak_order_ideal(predicate = lambda w: w.length() <= 2)
sage: list(iter(I))
[
[1 0]  [-1  2]  [ 1  0]  [ 3 -2]  [-1  2]
[0 1], [ 0  1], [ 2 -1], [ 2 -1], [-2  3]
]
```

Even when the result is finite, some features of *FiniteEnumeratedSets* are not available:

```
sage: I.cardinality() # todo: not implemented
5
sage: list(I)          # todo: not implemented
```

unless this finiteness is explicitly specified:

```
sage: I = W.weak_order_ideal(predicate = lambda w: w.length() <= 2,
....:                         category = FiniteEnumeratedSets())
sage: I.cardinality()
5
sage: list(I)
[
[1 0]  [-1  2]  [ 1  0]  [ 3 -2]  [-1  2]
[0 1], [ 0  1], [ 2 -1], [ 2 -1], [-2  3]
]
```

#### Background

The weak order is returned as a SearchForest. This is achieved by assigning to each element $u1$ of the ideal a single ancestor $u = u1s_i$, where $i$ is the smallest descent of $u$.

This allows for iterating through the elements in roughly Constant Amortized Time and constant memory (taking the operations and size of the generated objects as constants).

**additional_structure**()

Return None.

Indeed, all the structure Coxeter groups have in addition to groups (simple reflections, ...) is already defined in the super category.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: CoxeterGroups().additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: CoxeterGroups().super_categories()
[Category of generalized coxeter groups]
```

## 3.31 Crystals

**class** sage.categories.crystals.**CrystalHomset**(*X*, *Y*, *category=None*)

Bases: *sage.categories.homset.Homset*

The set of crystal morphisms from one crystal to another.

An $U_q(\mathfrak{g})$ $I$-crystal morphism $\Psi : B \to C$ is a map $\Psi : B \cup \{0\} \to C \cup \{0\}$ such that:

- $\Psi(0) = 0$.
- If $b \in B$ and $\Psi(b) \in C$, then $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in I$.
- If $b, b' \in B$, $\Psi(b), \Psi(b') \in C$ and $f_i b = b'$, then $f_i \Psi(b) = \Psi(b')$ and $\Psi(b) = e_i \Psi(b')$ for all $i \in I$.

If the Cartan type is unambiguous, it is surpressed from the notation.

We can also generalize the definition of a crystal morphism by considering a map of $\sigma$ of the (now possibly different) Dynkin diagrams corresponding to $B$ and $C$ along with scaling factors $\gamma_i \in \mathbf{Z}$ for $i \in I$. Let $\sigma_i$

denote the orbit of $i$ under $\sigma$. We write objects for $B$ as $X$ with corresponding objects of $C$ as $\widehat{X}$. Then a *virtual* crystal morphism $\Psi$ is a map such that the following holds:

- $\Psi(0) = 0$.
- If $b \in B$ and $\Psi(b) \in C$, then for all $j \in \sigma_i$:

$$\varepsilon_i(b) = \frac{1}{\gamma_j}\widehat{\varepsilon}_j(\Psi(b)), \quad \varphi_i(b) = \frac{1}{\gamma_j}\widehat{\varphi}_j(\Psi(b)), \quad \mathrm{wt}(\Psi(b)) = \sum_i c_i \sum_{j \in \sigma_i} \gamma_j \widehat{\Lambda}_j,$$

where $\mathrm{wt}(b) = \sum_i c_i \Lambda_i$.

- If $b, b' \in B$, $\Psi(b), \Psi(b') \in C$ and $f_i b = b'$, then independent of the ordering of $\sigma_i$ we have:

$$\Psi(b') = e_i \Psi(b) = \prod_{j \in \sigma_i} \widehat{e}_j^{\gamma_i} \Psi(b), \quad \Psi(b') = f_i \Psi(b) = \prod_{j \in \sigma_i} \widehat{f}_j^{\gamma_i} \Psi(b).$$

If $\gamma_i = 1$ for all $i \in I$ and the Dynkin diagrams are the same, then we call $\Psi$ a *twisted* crystal morphism.

INPUT:

- `X` – the domain
- `Y` – the codomain
- `category` – (optional) the category of the crystal morphisms

**See also:**

For the construction of an element of the homset, see `CrystalMorphismByGenerators` and `crystal_morphism()`.

EXAMPLES:

We begin with the natural embedding of $B(2\Lambda_1)$ into $B(\Lambda_1) \otimes B(\Lambda_1)$ in type $A_1$:

```
sage: B = crystals.Tableaux(['A',1], shape=[2])
sage: F = crystals.Tableaux(['A',1], shape=[1])
sage: T = crystals.TensorProduct(F, F)
sage: v = T.highest_weight_vectors()[0]; v
[[[1]], [[1]]]
sage: H = Hom(B, T)
sage: psi = H([v])
sage: b = B.highest_weight_vector(); b
[[1, 1]]
sage: psi(b)
[[[1]], [[1]]]
sage: b.f(1)
[[1, 2]]
sage: psi(b.f(1))
[[[1]], [[2]]]
```

We now look at the decomposition of $B(\Lambda_1) \otimes B(\Lambda_1)$ into $B(2\Lambda_1) \oplus B(0)$:

```
sage: B0 = crystals.Tableaux(['A',1], shape=[])
sage: D = crystals.DirectSum([B, B0])
sage: H = Hom(T, D)
sage: psi = H(D.module_generators)
sage: psi
['A', 1] Crystal morphism:
  From: Full tensor product of the crystals
   [The crystal of tableaux of type ['A', 1] and shape(s) [[1]],
```

```
    The crystal of tableaux of type ['A', 1] and shape(s) [[1]]]
  To:   Direct sum of the crystals Family
    (The crystal of tableaux of type ['A', 1] and shape(s) [[2]],
    The crystal of tableaux of type ['A', 1] and shape(s) [[]])
  Defn: [[[1]], [[1]]] |--> [[1, 1]]
        [[[2]], [[1]]] |--> []
sage: psi.is_isomorphism()
True
```

We can always construct the trivial morphism which sends everything to $0$:

```
sage: Binf = crystals.infinity.Tableaux(['B', 2])
sage: B = crystals.Tableaux(['B',2], shape=[1])
sage: H = Hom(Binf, B)
sage: psi = H(lambda x: None)
sage: psi(Binf.highest_weight_vector())
```

For Kirillov-Reshetikhin crystals, we consider the map to the corresponding classical crystal:

```
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,1)
sage: B = K.classical_decomposition()
sage: H = Hom(K, B)
sage: psi = H(lambda x: x.lift(), cartan_type=['D',4])
sage: L = [psi(mg) for mg in K.module_generators]; L
[[], [[1], [2]]]
sage: all(x.parent() == B for x in L)
True
```

Next we consider a type $D_4$ crystal morphism where we twist by $3 \leftrightarrow 4$:

```
sage: B = crystals.Tableaux(['D',4], shape=[1])
sage: H = Hom(B, B)
sage: d = {1:1, 2:2, 3:4, 4:3}
sage: psi = H(B.module_generators, automorphism=d)
sage: b = B.highest_weight_vector()
sage: b.f_string([1,2,3])
[[4]]
sage: b.f_string([1,2,4])
[[-4]]
sage: psi(b.f_string([1,2,3]))
[[-4]]
sage: psi(b.f_string([1,2,4]))
[[4]]
```

We construct the natural virtual embedding of a type $B_3$ into a type $D_4$ crystal:

```
sage: B = crystals.Tableaux(['B',3], shape=[1])
sage: C = crystals.Tableaux(['D',4], shape=[2])
sage: H = Hom(B, C)
sage: psi = H(C.module_generators)
sage: psi
['B', 3] -> ['D', 4] Virtual Crystal morphism:
  From: The crystal of tableaux of type ['B', 3] and shape(s) [[1]]
  To:   The crystal of tableaux of type ['D', 4] and shape(s) [[2]]
  Defn: [[1]] |--> [[1, 1]]
sage: for b in B: print("{} |--> {}".format(b, psi(b)))
[[1]] |--> [[1, 1]]
[[2]] |--> [[2, 2]]
```

```
[[3]] |--> [[3, 3]]
[[0]] |--> [[3, -3]]
[[-3]] |--> [[-3, -3]]
[[-2]] |--> [[-2, -2]]
[[-1]] |--> [[-1, -1]]
```

**Element**
alias of *CrystalMorphismByGenerators*

**class** sage.categories.crystals.**CrystalMorphism**(*parent*, *cartan_type=None*, *virtualization=None*, *scaling_factors=None*)
Bases: *sage.categories.morphism.Morphism*

A crystal morphism.

INPUT:

- parent – a homset
- cartan_type – (optional) a Cartan type; the default is the Cartan type of the domain
- virtualization – (optional) a dictionary whose keys are in the index set of the domain and whose values are lists of entries in the index set of the codomain
- scaling_factors – (optional) a dictionary whose keys are in the index set of the domain and whose values are scaling factors for the weight, $\varepsilon$ and $\varphi$

**cartan_type**()
Return the Cartan type of self.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: psi = Hom(B, B).an_element()
sage: psi.cartan_type()
['A', 2]
```

**is_injective**()
Return if self is an injective crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: psi = Hom(B, B).an_element()
sage: psi.is_injective()
False
```

**is_surjective**()
Check if self is a surjective crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['C',2], shape=[1,1])
sage: C = crystals.Tableaux(['C',2], ([2,1], [1,1]))
sage: psi = B.crystal_morphism(C.module_generators[1:], codomain=C)
sage: psi.is_surjective()
False
sage: im_gens = [None, B.module_generators[0]]
sage: psi = C.crystal_morphism(im_gens, codomain=B)
sage: psi.is_surjective()
True
```

```
sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: B = crystals.infinity.Tableaux(['A',2])
sage: La = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: W = crystals.elementary.T(['A',2], La[1]+La[2])
sage: T = W.tensor(B)
sage: mg = T(W.module_generators[0], B.module_generators[0])
sage: psi = Hom(C,T)([mg])
sage: psi.is_surjective()
False
```

**scaling_factors**()

Return the scaling factors $\gamma_i$.

EXAMPLES:

```
sage: B = crystals.Tableaux(['B',3], shape=[1])
sage: C = crystals.Tableaux(['D',4], shape=[2])
sage: psi = B.crystal_morphism(C.module_generators)
sage: psi.scaling_factors()
Finite family {1: 2, 2: 2, 3: 1}
```

**virtualization**()

Return the virtualization sets $\sigma_i$.

EXAMPLES:

```
sage: B = crystals.Tableaux(['B',3], shape=[1])
sage: C = crystals.Tableaux(['D',4], shape=[2])
sage: psi = B.crystal_morphism(C.module_generators)
sage: psi.virtualization()
Finite family {1: (1,), 2: (2,), 3: (3, 4)}
```

**class** sage.categories.crystals.**CrystalMorphismByGenerators**(*parent*, *on_gens*, *cartan_type=None*, *virtualization=None*, *scaling_factors=None*, *gens=None*, *check=True*)

Bases: *sage.categories.crystals.CrystalMorphism*

A crystal morphism defined by a set of generators which create a virtual crystal inside the codomain.

INPUT:

- parent – a homset
- on_gens – a function or list that determines the image of the generators (if given a list, then this uses the order of the generators of the domain) of the domain under self
- cartan_type – (optional) a Cartan type; the default is the Cartan type of the domain
- virtualization – (optional) a dictionary whose keys are in the index set of the domain and whose values are lists of entries in the index set of the codomain
- scaling_factors – (optional) a dictionary whose keys are in the index set of the domain and whose values are scaling factors for the weight, $\varepsilon$ and $\varphi$
- gens – (optional) a finite list of generators to define the morphism; the default is to use the highest weight vectors of the crystal

- check – (default: True) check if the crystal morphism is valid

**See also:**

*sage.categories.crystals.Crystals.ParentMethods.crystal_morphism()*

**im_gens**()
Return the image of the generators of self as a tuple.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: F = crystals.Tableaux(['A',2], shape=[1])
sage: T = crystals.TensorProduct(F, F, F)
sage: H = Hom(T, B)
sage: b = B.highest_weight_vector()
sage: psi = H((None, b, b, None), generators=T.highest_weight_vectors())
sage: psi.im_gens()
(None, [[1, 1], [2]], [[1, 1], [2]], None)
```

**image**()
Return the image of self in the codomain as a Subcrystal.

**Warning:** This assumes that self is a strict crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['B',3], shape=[1])
sage: C = crystals.Tableaux(['D',4], shape=[2])
sage: H = Hom(B, C)
sage: psi = H(C.module_generators)
sage: psi.image()
Virtual crystal of The crystal of tableaux of type ['D', 4] and shape(s)␣
↪[[2]] of type ['B', 3]
```

**to_module_generator**($x$)
Return a generator mg and a path of $e_i$ and $f_i$ operations to mg.

OUTPUT:

A tuple consisting of:

- a module generator,
- a list of 'e' and 'f' to denote which operation, and
- a list of matching indices.

EXAMPLES:

```
sage: B = crystals.elementary.Elementary(['A',2], 2)
sage: psi = B.crystal_morphism(B.module_generators)
sage: psi.to_module_generator(B(4))
(0, ['f', 'f', 'f', 'f'], [2, 2, 2, 2])
sage: psi.to_module_generator(B(-2))
(0, ['e', 'e'], [2, 2])
```

**class** sage.categories.crystals.**Crystals**(*s=None*)
Bases: *sage.categories.category_singleton.Category_singleton*

The category of crystals.

See `sage.combinat.crystals.crystals` for an introduction to crystals.

EXAMPLES:

```
sage: C = Crystals()
sage: C
Category of crystals
sage: C.super_categories()
[Category of... enumerated sets]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

Parents in this category should implement the following methods:

- either an attribute `_cartan_type` or a method `cartan_type`
- `module_generators`: a list (or container) of distinct elements which generate the crystal using $f_i$

Furthermore, their elements `x` should implement the following methods:

- `x.e(i)` (returning $e_i(x)$)
- `x.f(i)` (returning $f_i(x)$)
- `x.epsilon(i)` (returning $\varepsilon_i(x)$)
- `x.phi(i)` (returning $\varphi_i(x)$)

EXAMPLES:

```
sage: from sage.misc.abstract_method import abstract_methods_of_class
sage: abstract_methods_of_class(Crystals().element_class)
{'optional': [], 'required': ['e', 'epsilon', 'f', 'phi', 'weight']}
```

**class ElementMethods**

**Epsilon**()

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(0).Epsilon()
(0, 0, 0, 0, 0, 0)
sage: C(1).Epsilon()
(0, 0, 0, 0, 0, 0)
sage: C(2).Epsilon()
(1, 0, 0, 0, 0, 0)
```

**Phi**()

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(0).Phi()
(0, 0, 0, 0, 0, 0)
sage: C(1).Phi()
(1, 0, 0, 0, 0, 0)
sage: C(2).Phi()
(1, 1, 0, 0, 0, 0)
```

**all_paths_to_highest_weight** (*index_set=None*)

Iterate over all paths to the highest weight from `self` with respect to $index_set$.

INPUT:

- `index_set` – (optional) a subset of the index set of `self`

EXAMPLES:

```
sage: B = crystals.infinity.Tableaux("A2")
sage: b0 = B.highest_weight_vector()
sage: b = b0.f_string([1, 2, 1, 2])
sage: L = b.all_paths_to_highest_weight()
sage: list(L)
[[2, 1, 2, 1], [2, 2, 1, 1]]

sage: Y = crystals.infinity.GeneralizedYoungWalls(3)
sage: y0 = Y.highest_weight_vector()
sage: y = y0.f_string([0, 1, 2, 3, 2, 1, 0])
sage: list(y.all_paths_to_highest_weight())
[[0, 1, 2, 3, 2, 1, 0],
 [0, 1, 3, 2, 2, 1, 0],
 [0, 3, 1, 2, 2, 1, 0],
 [0, 3, 2, 1, 1, 0, 2],
 [0, 3, 2, 1, 1, 2, 0]]

sage: B = crystals.Tableaux("A3", shape=[4,2,1])
sage: b0 = B.highest_weight_vector()
sage: b = b0.f_string([1, 1, 2, 3])
sage: list(b.all_paths_to_highest_weight())
[[1, 3, 2, 1], [3, 1, 2, 1], [3, 2, 1, 1]]
```

**cartan_type** ()

Returns the Cartan type associated to `self`

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C(1).cartan_type()
['A', 5]
```

**e** (*i*)

Return $e_i$ of `self` if it exists or `None` otherwise.

This method should be implemented by the element class of the crystal.

EXAMPLES:

```
sage: C = Crystals().example(5)
sage: x = C[2]; x
3
sage: x.e(1), x.e(2), x.e(3)
(None, 2, None)
```

**e_string** (*list*)

Applies $e_{i_r} \cdots e_{i_1}$ to self for `list` as $[i_1, ..., i_r]$

EXAMPLES:

```
sage: C = crystals.Letters(['A',3])
sage: b = C(3)
sage: b.e_string([2,1])
```

```
1
sage: b.e_string([1,2])
```

**epsilon**(*i*)

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).epsilon(1)
0
sage: C(2).epsilon(1)
1
```

**f**(*i*)

Return $f_i$ of `self` if it exists or `None` otherwise.

This method should be implemented by the element class of the crystal.

EXAMPLES:

```
sage: C = Crystals().example(5)
sage: x = C[1]; x
2
sage: x.f(1), x.f(2), x.f(3)
(None, 3, None)
```

**f_string**(*list*)

Applies $f_{i_r} \cdots f_{i_1}$ to self for `list` as $[i_1, ..., i_r]$

EXAMPLES:

```
sage: C = crystals.Letters(['A',3])
sage: b = C(1)
sage: b.f_string([1,2])
3
sage: b.f_string([2,1])
```

**index_set**()

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).index_set()
(1, 2, 3, 4, 5)
```

**is_highest_weight**(*index_set=None*)

Returns `True` if `self` is a highest weight. Specifying the option `index_set` to be a subset $I$ of the index set of the underlying crystal, finds all highest weight vectors for arrows in $I$.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).is_highest_weight()
True
sage: C(2).is_highest_weight()
False
sage: C(2).is_highest_weight(index_set = [2,3,4,5])
True
```

**is_lowest_weight**(*index_set=None*)

Returns `True` if `self` is a lowest weight. Specifying the option `index_set` to be a subset $I$ of the index set of the underlying crystal, finds all lowest weight vectors for arrows in $I$.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).is_lowest_weight()
False
sage: C(6).is_lowest_weight()
True
sage: C(4).is_lowest_weight(index_set = [1,3])
True
```

**phi** (*i*)

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).phi(1)
1
sage: C(2).phi(1)
0
```

**phi_minus_epsilon** (*i*)

Return $\varphi_i - \varepsilon_i$ of `self`.

There are sometimes better implementations using the weight for this. It is used for reflections along a string.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).phi_minus_epsilon(1)
1
```

**s** (*i*)

Return the reflection of `self` along its $i$-string.

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: b = C(rows=[[1,1],[3]])
sage: b.s(1)
[[2, 2], [3]]
sage: b = C(rows=[[1,2],[3]])
sage: b.s(2)
[[1, 2], [3]]
sage: T = crystals.Tableaux(['A',2],shape=[4])
sage: t = T(rows=[[1,2,2,2]])
sage: t.s(1)
[[1, 1, 1, 2]]
```

**subcrystal** (*index_set=None, max_depth=inf, direction='both', contained=None, cartan_type=None, category=None*)

Construct the subcrystal generated by `self` using $e_i$ and/or $f_i$ for all $i$ in `index_set`.

INPUT:

- `index_set` – (default: `None`) the index set; if `None` then use the index set of the crystal
- `max_depth` – (default: infinity) the maximum depth to build
- `direction` – (default: `'both'`) the direction to build the subcrystal; it can be one of the following:
  - `'both'` - using both $e_i$ and $f_i$

  - 'upper' - using $e_i$
  - 'lower' - using $f_i$
- contained – (optional) a set (or function) defining the containment in the subcrystal
- cartan_type – (optional) specify the Cartan type of the subcrystal
- category – (optional) specify the category of the subcrystal

**See also:**

- *Crystals.ParentMethods.subcrystal()*

EXAMPLES:

```
sage: C = crystals.KirillovReshetikhin(['A',3,1], 1, 2)
sage: elt = C(1,4)
sage: list(elt.subcrystal(index_set=[1,3]))
[[[1, 4]], [[1, 3]], [[2, 4]], [[2, 3]]]
sage: list(elt.subcrystal(index_set=[1,3], max_depth=1))
[[[1, 4]], [[1, 3]], [[2, 4]]]
sage: list(elt.subcrystal(index_set=[1,3], direction='upper'))
[[[1, 4]], [[1, 3]]]
sage: list(elt.subcrystal(index_set=[1,3], direction='lower'))
[[[1, 4]], [[2, 4]]]
```

**to_highest_weight**(*index_set=None*)

Return the highest weight element $u$ and a list $[i_1, ..., i_k]$ such that $self = f_{i_1} ... f_{i_k} u$, where $i_1, ..., i_k$ are elements in $index_set$. By default the index set is assumed to be the full index set of self.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',3], shape = [1])
sage: t = T(rows = [[3]])
sage: t.to_highest_weight()
[[[1]], [2, 1]]
sage: T = crystals.Tableaux(['A',3], shape = [2,1])
sage: t = T(rows = [[1,2],[4]])
sage: t.to_highest_weight()
[[[1, 1], [2]], [1, 3, 2]]
sage: t.to_highest_weight(index_set = [3])
[[[1, 2], [3]], [3]]
sage: K = crystals.KirillovReshetikhin(['A',3,1],2,1)
sage: t = K(rows=[[2],[3]]); t.to_highest_weight(index_set=[1])
[[[1], [3]], [1]]
sage: t.to_highest_weight()
Traceback (most recent call last):
...
ValueError: This is not a highest weight crystals!
```

**to_lowest_weight**(*index_set=None*)

Return the lowest weight element $u$ and a list $[i_1, ..., i_k]$ such that $self = e_{i_1} ... e_{i_k} u$, where $i_1, ..., i_k$ are elements in $index_set$. By default the index set is assumed to be the full index set of self.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',3], shape = [1])
sage: t = T(rows = [[3]])
sage: t.to_lowest_weight()
[[[4]], [3]]
sage: T = crystals.Tableaux(['A',3], shape = [2,1])
sage: t = T(rows = [[1,2],[4]])
sage: t.to_lowest_weight()
```

```
[[[3, 4], [4]], [1, 2, 2, 3]]
sage: t.to_lowest_weight(index_set = [3])
[[[1, 2], [4]], []]
sage: K = crystals.KirillovReshetikhin(['A',3,1],2,1)
sage: t = K.module_generator(); t
[[1], [2]]
sage: t.to_lowest_weight(index_set=[1,2,3])
[[[3], [4]], [2, 1, 3, 2]]
sage: t.to_lowest_weight()
Traceback (most recent call last):
...
ValueError: This is not a highest weight crystals!
```

**weight**()

Return the weight of this crystal element.

This method should be implemented by the element class of the crystal.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).weight()
(1, 0, 0, 0, 0, 0)
```

**Finite**

alias of *FiniteCrystals*

**class MorphismMethods**

**is_embedding**()

Check if `self` is an injective crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['C',2], shape=[1,1])
sage: C = crystals.Tableaux(['C',2], ([2,1], [1,1]))
sage: psi = B.crystal_morphism(C.module_generators[1:], codomain=C)
sage: psi.is_embedding()
True

sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: B = crystals.infinity.Tableaux(['A',2])
sage: La = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: W = crystals.elementary.T(['A',2], La[1]+La[2])
sage: T = W.tensor(B)
sage: mg = T(W.module_generators[0], B.module_generators[0])
sage: psi = Hom(C,T)([mg])
sage: psi.is_embedding()
True
```

**is_isomorphism**()

Check if `self` is a crystal isomorphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['C',2], shape=[1,1])
sage: C = crystals.Tableaux(['C',2], ([2,1], [1,1]))
sage: psi = B.crystal_morphism(C.module_generators[1:], codomain=C)
```

```
sage: psi.is_isomorphism()
False
```

**is_strict()**

Check if `self` is a strict crystal morphism.

EXAMPLES:

```
sage: B = crystals.Tableaux(['C',2], shape=[1,1])
sage: C = crystals.Tableaux(['C',2], ([2,1], [1,1]))
sage: psi = B.crystal_morphism(C.module_generators[1:], codomain=C)
sage: psi.is_strict()
True
```

**class ParentMethods**

**Lambda()**

Returns the fundamental weights in the weight lattice realization for the root system associated with the crystal

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C.Lambda()
Finite family {1: (1, 0, 0, 0, 0, 0), 2: (1, 1, 0, 0, 0, 0), 3: (1, 1, 1,␣
↪0, 0, 0), 4: (1, 1, 1, 1, 0, 0), 5: (1, 1, 1, 1, 1, 0)}
```

**an_element()**

Returns an element of `self`

sage: C = crystals.Letters([‘A’, 5]) sage: C.an_element() 1

**cartan_type()**

Returns the Cartan type of the crystal

EXAMPLES:

```
sage: C = crystals.Letters(['A',2])
sage: C.cartan_type()
['A', 2]
```

**connected_components()**

Return the connected components of `self` as subcrystals.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(B,C)
sage: T.connected_components()
[Subcrystal of Full tensor product of the crystals
 [The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
  The crystal of letters for type ['A', 2]],
 Subcrystal of Full tensor product of the crystals
 [The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
  The crystal of letters for type ['A', 2]],
 Subcrystal of Full tensor product of the crystals
 [The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
  The crystal of letters for type ['A', 2]]]
```

**connected_components_generators**()

Return a tuple of generators for each of the connected components of `self`.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(B,C)
sage: T.connected_components_generators()
([[[1, 1], [2]], 1], [[[1, 2], [2]], 1], [[[1, 2], [3]], 1])
```

**crystal_morphism**(*on_gens*, *codomain=None*, *cartan_type=None*, *index_set=None*, *generators=None*, *automorphism=None*, *virtualization=None*, *scaling_factors=None*, *category=None*, *check=True*)

Construct a crystal morphism from `self` to another crystal `codomain`.

INPUT:

- `on_gens` – a function or list that determines the image of the generators (if given a list, then this uses the order of the generators of the domain) of `self` under the crystal morphism
- `codomain` – (default: `self`) the codomain of the morphism
- `cartan_type` – (optional) the Cartan type of the morphism; the default is the Cartan type of `self`
- `index_set` – (optional) the index set of the morphism; the default is the index set of the Cartan type
- `generators` – (optional) the generators to define the morphism; the default is the generators of `self`
- `automorphism` – (optional) the automorphism to perform the twisting
- `virtualization` – (optional) a dictionary whose keys are in the index set of the domain and whose values are lists of entries in the index set of the codomain; the default is the identity dictionary
- `scaling_factors` – (optional) a dictionary whose keys are in the index set of the domain and whose values are scaling factors for the weight, $\varepsilon$ and $\varphi$; the default are all scaling factors to be one
- `category` – (optional) the category for the crystal morphism; the default is the category of *`Crystals`*.
- `check` – (default: `True`) check if the crystal morphism is valid

**See also:**

For more examples, see *`sage.categories.crystals.CrystalHomset`*.

EXAMPLES:

We construct the natural embedding of a crystal using tableaux into the tensor product of single boxes via the reading word:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: F = crystals.Tableaux(['A',2], shape=[1])
sage: T = crystals.TensorProduct(F, F, F)
sage: mg = T.highest_weight_vectors()[2]; mg
[[[1]], [[2]], [[1]]]
sage: psi = B.crystal_morphism([mg], codomain=T); psi
['A', 2] Crystal morphism:
  From: The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]]
  To:   Full tensor product of the crystals
         [The crystal of tableaux of type ['A', 2] and shape(s) [[1]],
          The crystal of tableaux of type ['A', 2] and shape(s) [[1]],
          The crystal of tableaux of type ['A', 2] and shape(s) [[1]]]
  Defn: [[1, 1], [2]] |--> [[[1]], [[2]], [[1]]]
```

```
sage: b = B.module_generators[0]
sage: b.pp()
  1  1
  2
sage: psi(b)
[[[1]], [[2]], [[1]]]
sage: psi(b.f(2))
[[[1]], [[3]], [[1]]]
sage: psi(b.f_string([2,1,1]))
[[[2]], [[3]], [[2]]]
sage: lw = b.to_lowest_weight()[0]
sage: lw.pp()
  2  3
  3
sage: psi(lw)
[[[3]], [[3]], [[2]]]
sage: psi(lw) == mg.to_lowest_weight()[0]
True
```

We now take the other isomorphic highest weight component in the tensor product:

```
sage: mg = T.highest_weight_vectors()[1]; mg
[[[2]], [[1]], [[1]]]
sage: psi = B.crystal_morphism([mg], codomain=T)
sage: psi(lw)
[[[3]], [[2]], [[3]]]
```

We construct a crystal morphism of classical crystals using a Kirillov-Reshetikhin crystal:

```
sage: B = crystals.Tableaux(['D', 4], shape=[1,1])
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,2)
sage: K.module_generators
[[], [[1], [2]], [[1, 1], [2, 2]]]
sage: v = K.module_generators[1]
sage: psi = B.crystal_morphism([v], codomain=K, category=FiniteCrystals())
sage: psi
['D', 4] -> ['D', 4, 1] Virtual Crystal morphism:
  From: The crystal of tableaux of type ['D', 4] and shape(s) [[1, 1]]
  To:   Kirillov-Reshetikhin crystal of type ['D', 4, 1] with (r,s)=(2,2)
  Defn: [[1], [2]] |--> [[1], [2]]
sage: b = B.module_generators[0]
sage: psi(b)
[[1], [2]]
sage: psi(b.to_lowest_weight()[0])
[[-2], [-1]]
```

We can define crystal morphisms using a different set of generators. For example, we construct an example using the lowest weight vector:

```
sage: B = crystals.Tableaux(['A',2], shape=[1])
sage: La = RootSystem(['A',2]).weight_lattice().fundamental_weights()
sage: T = crystals.elementary.T(['A',2], La[2])
sage: Bp = T.tensor(B)
sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: x = C.module_generators[0].f_string([1,2])
sage: psi = Bp.crystal_morphism([x], generators=Bp.lowest_weight_
↪vectors())
sage: psi(Bp.highest_weight_vector())
```

```
[[1, 1], [2]]
```

We can also use a dictionary to specify the generators and their images:

```
sage: psi = Bp.crystal_morphism({Bp.lowest_weight_vectors()[0]: x})
sage: psi(Bp.highest_weight_vector())
[[1, 1], [2]]
```

We construct a twisted crystal morphism induced from the diagram automorphism of type $A_3^{(1)}$:

```
sage: La = RootSystem(['A',3,1]).weight_lattice(extended=True).
↪fundamental_weights()
sage: B0 = crystals.GeneralizedYoungWalls(3, La[0])
sage: B1 = crystals.GeneralizedYoungWalls(3, La[1])
sage: phi = B0.crystal_morphism(B1.module_generators, automorphism={0:1,␣
↪1:2, 2:3, 3:0})
sage: phi
['A', 3, 1] Twisted Crystal morphism:
  From: Highest weight crystal of generalized Young walls of Cartan type [
↪'A', 3, 1] and highest weight Lambda[0]
  To:   Highest weight crystal of generalized Young walls of Cartan type [
↪'A', 3, 1] and highest weight Lambda[1]
  Defn: [] |--> []
sage: x = B0.module_generators[0].f_string([0,1,2,3]); x
[[0, 3], [1], [2]]
sage: phi(x)
[[], [1, 0], [2], [3]]
```

We construct a virtual crystal morphism from type $G_2$ into type $D_4$:

```
sage: D = crystals.Tableaux(['D',4], shape=[1,1])
sage: G = crystals.Tableaux(['G',2], shape=[1])
sage: psi = G.crystal_morphism(D.module_generators,
....:                          virtualization={1:[2],2:[1,3,4]},
....:                          scaling_factors={1:1, 2:1})
sage: for x in G:
....:     ascii_art(x, psi(x), sep='  |-->  ')
....:     print("")
          1
  1  |-->  2

          1
  2  |-->  3

          2
  3  |--> -3

          3
  0  |--> -3

          3
 -3  |--> -2

         -3
 -2  |--> -1

         -2
 -1  |--> -1
```

**digraph** (*subset=None*, *index_set=None*)

Return the DiGraph associated to self.

INPUT:

- subset – (optional) a subset of vertices for which the digraph should be constructed
- index_set – (optional) the index set to draw arrows

EXAMPLES:

```
sage: C = Crystals().example(5)
sage: C.digraph()
Digraph on 6 vertices
```

The edges of the crystal graph are by default colored using blue for edge 1, red for edge 2, and green for edge 3:

```
sage: C = Crystals().example(3)
sage: G = C.digraph()
sage: view(G)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

One may also overwrite the colors:

```
sage: C = Crystals().example(3)
sage: G = C.digraph()
sage: G.set_latex_options(color_by_label = {1:"red", 2:"purple", 3:"blue"}
↪)
sage: view(G)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

Or one may add colors to yet unspecified edges:

```
sage: C = Crystals().example(4)
sage: G = C.digraph()
sage: C.cartan_type()._index_set_coloring[4]="purple"
sage: view(G)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

Here is an example of how to take the top part up to a given depth of an infinite dimensional crystal:

```
sage: C = CartanType(['C',2,1])
sage: La = C.root_system().weight_lattice().fundamental_weights()
sage: T = crystals.HighestWeight(La[0])
sage: S = T.subcrystal(max_depth=3)
sage: G = T.digraph(subset=S); G
Digraph on 5 vertices
sage: sorted(G.vertices(), key=str)
[(-Lambda[0] + 2*Lambda[1] - delta,),
 (1/2*Lambda[0] + Lambda[1] - Lambda[2] - 1/2*delta, -1/2*Lambda[0] +␣
↪Lambda[1] - 1/2*delta),
 (1/2*Lambda[0] - Lambda[1] + Lambda[2] - 1/2*delta, -1/2*Lambda[0] +␣
↪Lambda[1] - 1/2*delta),
 (Lambda[0] - 2*Lambda[1] + 2*Lambda[2] - delta,),
 (Lambda[0],)]
```

Here is a way to construct a picture of a Demazure crystal using the subset option:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = B.highest_weight_vector()
```

```
sage: D = B.demazure_subcrystal(t, [2,1])
sage: list(D)
[[[1, 1], [2]], [[1, 1], [3]], [[1, 2], [2]],
 [[1, 3], [2]], [[1, 3], [3]]]
sage: view(D)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

We can also choose to display particular arrows using the `index_set` option:

```
sage: C = crystals.KirillovReshetikhin(['D',4,1], 2, 1)
sage: G = C.digraph(index_set=[1,3])
sage: len(G.edges())
20
sage: view(G)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

---

**Todo:** Add more tests.

---

**direct_sum**(*X*)

Return the direct sum of `self` with `X`.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: B.direct_sum(C)
Direct sum of the crystals Family
(The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
 The crystal of letters for type ['A', 2])
```

As a shorthand, we can use +:

```
sage: B + C
Direct sum of the crystals Family
(The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]],
 The crystal of letters for type ['A', 2])
```

**dot_tex**()

Return a dot_tex string representation of `self`.

EXAMPLES:

```
sage: C = crystals.Letters(['A',2])
sage: C.dot_tex()
'digraph G { \n  node [ shape=plaintext ];\n  N_0 [ label = " ", texlbl =
↪"$1$" ];\n  N_1 [ label = " ", texlbl = "$2$" ];\n  N_2 [ label = " ",␣
↪texlbl = "$3$" ];\n  N_0 -> N_1 [ label = " ", texlbl = "1" ];\n  N_1 ->
↪ N_2 [ label = " ", texlbl = "2" ];\n}'
```

**index_set**()

Returns the index set of the Dynkin diagram underlying the crystal

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C.index_set()
(1, 2, 3, 4, 5)
```

**is_connected**()

Return `True` if `self` is a connected crystal.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(B,C)
sage: B.is_connected()
True
sage: T.is_connected()
False
```

**latex**(***options*)

Returns the crystal graph as a latex string. This can be exported to a file with self.latex_file('filename').

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2],shape=[1])
sage: T._latex_()
'...tikzpicture...'
sage: view(T) # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

One can for example also color the edges using the following options:

```
sage: T = crystals.Tableaux(['A',2],shape=[1])
sage: T._latex_(color_by_label={0:"black", 1:"red", 2:"blue"})
'...tikzpicture...'
```

**latex_file**(*filename*)

Export a file, suitable for pdflatex, to 'filename'.

This requires a proper installation of `dot2tex` in sage-python. For more information see the documentation for `self.latex()`.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: fn = tmp_filename(ext='.tex')
sage: C.latex_file(fn)
```

**metapost**(*filename*, *thicklines=False*, *labels=True*, *scaling_factor=1.0*, *tallness=1.0*)

Use C.metapost("filename.mp",[options]), where options can be:

thicklines = True (for thicker edges) labels = False (to suppress labeling of the vertices) scaling_factor=value, where value is a floating point number, 1.0 by default. Increasing or decreasing the scaling factor changes the size of the image. tallness=1.0. Increasing makes the image taller without increasing the width.

Root operators e(1) or f(1) move along red lines, e(2) or f(2) along green. The highest weight is in the lower left. Vertices with the same weight are kept close together. The concise labels on the nodes are strings introduced by Berenstein and Zelevinsky and Littelmann; see Littelmann's paper Cones, Crystals, Patterns, sections 5 and 6.

For Cartan types B2 or C2, the pattern has the form

a2 a3 a4 a1

where c*a2 = a3 = 2*a4 =0 and a1=0, with c=2 for B2, c=1 for C2. Applying e(2) a1 times, e(1) a2 times, e(2) a3 times, e(1) a4 times returns to the highest weight. (Observe that Littelmann writes the roots in opposite of the usual order, so our e(1) is his e(2) for these Cartan types.) For type A2, the pattern has the form

a3 a2 a1

where applying e(1) a1 times, e(2) a2 times then e(3) a1 times returns to the highest weight. These data determine the vertex and may be translated into a Gelfand-Tsetlin pattern or tableau.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 2])
sage: C.metapost(tmp_filename())
```

```
sage: C = crystals.Letters(['A', 5])
sage: C.metapost(tmp_filename())
Traceback (most recent call last):
...
NotImplementedError
```

**number_of_connected_components**()

Return the number of connected components of `self`.

EXAMPLES:

```
sage: B = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(B,C)
sage: T.number_of_connected_components()
3
```

**plot**(***options*)

Return the plot of `self` as a directed graph.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: print(C.plot())
Graphics object consisting of 17 graphics primitives
```

**plot3d**(***options*)

Return the 3-dimensional plot of `self` as a directed graph.

EXAMPLES:

```
sage: C = crystals.KirillovReshetikhin(['A',3,1],2,1)
sage: print(C.plot3d())
Graphics3d Object
```

**subcrystal**(*index_set=None, generators=None, max_depth=inf, direction='both', contained=None, virtualization=None, scaling_factors=None, cartan_type=None, category=None*)

Construct the subcrystal from `generators` using $e_i$ and/or $f_i$ for all $i$ in `index_set`.

INPUT:

- `index_set` – (default: `None`) the index set; if `None` then use the index set of the crystal
- `generators` – (default: `None`) the list of generators; if `None` then use the module generators of the crystal
- `max_depth` – (default: infinity) the maximum depth to build

- direction – (default: 'both') the direction to build the subcrystal; it can be one of the following:
  - 'both' - using both $e_i$ and $f_i$
  - 'upper' - using $e_i$
  - 'lower' - using $f_i$
- contained – (optional) a set or function defining the containment in the subcrystal
- virtualization, scaling_factors – (optional) dictionaries whose key $i$ corresponds to the sets $\sigma_i$ and $\gamma_i$ respectively used to define virtual crystals; see VirtualCrystal
- cartan_type – (optional) specify the Cartan type of the subcrystal
- category – (optional) specify the category of the subcrystal

EXAMPLES:

```
sage: C = crystals.KirillovReshetikhin(['A',3,1], 1, 2)
sage: S = list(C.subcrystal(index_set=[1,2])); S
[[[1, 1]], [[1, 2]], [[1, 3]], [[2, 2]], [[2, 3]], [[3, 3]]]
sage: C.cardinality()
10
sage: len(S)
6
sage: list(C.subcrystal(index_set=[1,3], generators=[C(1,4)]))
[[[1, 4]], [[1, 3]], [[2, 4]], [[2, 3]]]
sage: list(C.subcrystal(index_set=[1,3], generators=[C(1,4)], max_
↪depth=1))
[[[1, 4]], [[1, 3]], [[2, 4]]]
sage: list(C.subcrystal(index_set=[1,3], generators=[C(1,4)], direction=
↪'upper'))
[[[1, 4]], [[1, 3]]]
sage: list(C.subcrystal(index_set=[1,3], generators=[C(1,4)], direction=
↪'lower'))
[[[1, 4]], [[2, 4]]]

sage: G = C.subcrystal(index_set=[1,2,3]).digraph()
sage: GA = crystals.Tableaux('A3', shape=[2]).digraph()
sage: G.is_isomorphic(GA, edge_labels=True)
True
```

We construct the subcrystal which contains the necessary data to construct the corresponding dual equivalence graph:

```
sage: C = crystals.Tableaux(['A',5], shape=[3,3])
sage: is_wt0 = lambda x: all(x.epsilon(i) == x.phi(i) for i in x.parent().
↪index_set())
sage: def check(x):
....:     if is_wt0(x):
....:         return True
....:     for i in x.parent().index_set()[:-1]:
....:         L = [x.e(i), x.e_string([i,i+1]), x.f(i), x.f_string([i,
↪i+1])]
....:         if any(y is not None and is_wt0(y) for y in L):
....:             return True
....:     return False
sage: wt0 = [x for x in C if is_wt0(x)]
sage: S = C.subcrystal(contained=check, generators=wt0)
sage: S.module_generators[0]
[[1, 3, 5], [2, 4, 6]]
sage: S.module_generators[0].e(2).e(3).f(2).f(3)
[[1, 2, 5], [3, 4, 6]]
```

An example of a type $B_2$ virtual crystal inside of a type $A_3$ ambient crystal:

```
sage: A = crystals.Tableaux(['A',3], shape=[2,1,1])
sage: S = A.subcrystal(virtualization={1:[1,3], 2:[2]},
....:                  scaling_factors={1:1,2:1}, cartan_type=['B',2])
sage: B = crystals.Tableaux(['B',2], shape=[1])
sage: S.digraph().is_isomorphic(B.digraph(), edge_labels=True)
True
```

**tensor** (**crystals*, ***options*)

Return the tensor product of `self` with the crystals `B`.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 3])
sage: B = crystals.infinity.Tableaux(['A', 3])
sage: T = C.tensor(C, B); T
Full tensor product of the crystals
 [The crystal of letters for type ['A', 3],
  The crystal of letters for type ['A', 3],
  The infinity crystal of tableaux of type ['A', 3]]
sage: tensor([C, C, B]) is T
True

sage: C = crystals.Letters(['A',2])
sage: T = C.tensor(C, C, generators=[[C(2),C(1),C(1)],[C(1),C(2),C(1)]]);
↪T
The tensor product of the crystals
 [The crystal of letters for type ['A', 2],
  The crystal of letters for type ['A', 2],
  The crystal of letters for type ['A', 2]]
sage: T.module_generators
([2, 1, 1], [1, 2, 1])
```

**weight_lattice_realization** ()

Return the weight lattice realization used to express weights in `self`.

This default implementation uses the ambient space of the root system for (non relabelled) finite types and the weight lattice otherwise. This is a legacy from when ambient spaces were partially implemented, and may be changed in the future.

For affine types, this returns the extended weight lattice by default.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C.weight_lattice_realization()
Ambient space of the Root system of type ['A', 5]
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.weight_lattice_realization()
Weight lattice of the Root system of type ['A', 2, 1]
```

**class SubcategoryMethods**

Methods for all subcategories.

**TensorProducts** ()

Return the full subcategory of objects of `self` constructed as tensor products.

**See also:**

- *tensor.TensorProductsCategory*

- `RegressiveCovariantFunctorialConstruction`.

EXAMPLES:

```
sage: HighestWeightCrystals().TensorProducts()
Category of tensor products of highest weight crystals
```

**class TensorProducts**(*category*, **args*)

Bases: `sage.categories.tensor.TensorProductsCategory`

The category of crystals constructed by tensor product of crystals.

**extra_super_categories**()

EXAMPLES:

```
sage: Crystals().TensorProducts().extra_super_categories()
[Category of crystals]
```

**example**(*choice='highwt'*, ***kwds*)

Returns an example of a crystal, as per `Category.example()`.

INPUT:

- `choice` – str [default: 'highwt']. Can be either 'highwt' for the highest weight crystal of type A, or 'naive' for an example of a broken crystal.
- `**kwds` – keyword arguments passed onto the constructor for the chosen crystal.

EXAMPLES:

```
sage: Crystals().example(choice='highwt', n=5)
Highest weight crystal of type A_5 of highest weight omega_1
sage: Crystals().example(choice='naive')
A broken crystal, defined by digraph, of dimension five.
```

**super_categories**()

EXAMPLES:

```
sage: Crystals().super_categories()
[Category of enumerated sets]
```

## 3.32 CW Complexes

**class** sage.categories.cw_complexes.**CWComplexes**(*s=None*)

Bases: `sage.categories.category_singleton.Category_singleton`

The category of CW complexes.

A CW complex is a Closure-finite cell complex in the Weak topology.

REFERENCES:

- Wikipedia article CW_complex

**Note:** The notion of "finite" is that the number of cells is finite.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: C = CWComplexes(); C
Category of CW complexes
```

**Compact_extra_super_categories()**

Return extraneous super categories for `CWComplexes().Compact()`.

A compact CW complex is finite, see Proposition A.1 in [Hat2002].

---

**Todo:** Fix the name of finite CW complexes.

---

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: CWComplexes().Compact() # indirect doctest
Category of finite finite dimensional CW complexes
sage: CWComplexes().Compact() is CWComplexes().Finite()
True
```

**class Connected**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

The category of connected CW complexes.

**class ElementMethods**

**dimension()**

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: X.an_element().dimension()
2
```

**class Finite**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

Category of finite CW complexes.

A finite CW complex is a CW complex with a finite number of cells.

**class ParentMethods**

**dimension()**

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: X.dimension()
2
```

**extra_super_categories()**

Return the extra super categories of `self`.

A finite CW complex is a compact finite-dimensional CW complex.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: C = CWComplexes().Finite()
sage: C.extra_super_categories()
[Category of finite dimensional CW complexes,
 Category of compact topological spaces]
```

**class FiniteDimensional**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

Category of finite dimensional CW complexes.

**class ParentMethods**

**cells**()

Return the cells of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: C = X.cells()
sage: sorted((d, C[d]) for d in C.keys())
[(0, (0-cell v,)),
 (1, (0-cell e1, 0-cell e2)),
 (2, (2-cell f,))]
```

**dimension**()

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: X.dimension()
2
```

**class SubcategoryMethods**

**Connected**()

Return the full subcategory of the connected objects of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: CWComplexes().Connected()
Category of connected CW complexes
```

**FiniteDimensional**()

Return the full subcategory of the finite dimensional objects of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: C = CWComplexes().FiniteDimensional(); C
Category of finite dimensional CW complexes
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: CWComplexes().super_categories()
[Category of topological spaces]
```

## 3.33 Discrete Valuation Rings (DVR) and Fields (DVF)

**class** sage.categories.discrete_valuation.**DiscreteValuationFields**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of discrete valuation fields

EXAMPLES:

```
sage: Qp(7) in DiscreteValuationFields()
True
sage: TestSuite(DiscreteValuationFields()).run()
```

**class ElementMethods**

**valuation**()

Return the valuation of this element.

EXAMPLES:

```
sage: x = Qp(5)(50)
sage: x.valuation()
2
```

**class ParentMethods**

**residue_field**()

Return the residue field of the ring of integers of this discrete valuation field.

EXAMPLES:

```
sage: Qp(5).residue_field()
Finite Field of size 5

sage: K.<u> = LaurentSeriesRing(QQ)
sage: K.residue_field()
Rational Field
```

**uniformizer**()

Return a uniformizer of this ring.

EXAMPLES:

```
sage: Qp(5).uniformizer()
5 + O(5^21)
```

**super_categories**()

EXAMPLES:

```
sage: DiscreteValuationFields().super_categories()
[Category of fields]
```

**class** sage.categories.discrete_valuation.**DiscreteValuationRings**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of discrete valuation rings

EXAMPLES:

```
sage: GF(7)[['x']] in DiscreteValuationRings()
True
sage: TestSuite(DiscreteValuationRings()).run()
```

**class ElementMethods**

**euclidean_degree**()

Return the Euclidean degree of this element.

**gcd**(*other*)

Return the greatest common divisor of self and other, normalized so that it is a power of the distinguished uniformizer.

**is_unit**()

Return True if self is invertible.

EXAMPLES:

```
sage: x = Zp(5)(50)
sage: x.is_unit()
False

sage: x = Zp(7)(50)
sage: x.is_unit()
True
```

**lcm**(*other*)

Return the least common multiple of self and other, normalized so that it is a power of the distinguished uniformizer.

**quo_rem**(*other*)

Return the quotient and remainder for Euclidean division of `self` by `other`.

**valuation**()

Return the valuation of this element.

EXAMPLES:

```
sage: x = Zp(5)(50)
sage: x.valuation()
2
```

**class ParentMethods**

**residue_field**()

Return the residue field of this ring.

EXAMPLES:

```
sage: Zp(5).residue_field()
Finite Field of size 5

sage: K.<u> = QQ[[]]
sage: K.residue_field()
Rational Field
```

**uniformizer**()

Return a uniformizer of this ring.

EXAMPLES:

```
sage: Zp(5).uniformizer()
5 + O(5^21)

sage: K.<u> = QQ[[]]
sage: K.uniformizer()
u
```

**super_categories**()

EXAMPLES:

```
sage: DiscreteValuationRings().super_categories()
[Category of euclidean domains]
```

## 3.34 Distributive Magmas and Additive Magmas

**class** sage.categories.distributive_magmas_and_additive_magmas.**DistributiveMagmasAndAdditive**

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of sets $(S, +, *)$ with $*$ distributing on $+$.

This is similar to a ring, but $+$ and $*$ are only required to be (additive) magmas.

EXAMPLES:

```
sage: from sage.categories.distributive_magmas_and_additive_magmas import␣
↪DistributiveMagmasAndAdditiveMagmas
sage: C = DistributiveMagmasAndAdditiveMagmas(); C
Category of distributive magmas and additive magmas
sage: C.super_categories()
[Category of magmas and additive magmas]
```

**class AdditiveAssociative**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class AdditiveCommutative**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class AdditiveUnital**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class Associative**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**AdditiveInverse**
alias of *Rngs*

**Unital**
alias of *Semirings*

**class CartesianProducts**(*category*, **args*)
Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**extra_super_categories**()
Implement the fact that a Cartesian product of magmas distributing over additive magmas is a magma distributing over an additive magma.

EXAMPLES:

```
sage: C = (Magmas() & AdditiveMagmas()).Distributive().CartesianProducts()
sage: C.extra_super_categories();
[Category of distributive magmas and additive magmas]
sage: C.axioms()
frozenset({'Distributive'})
```

**class ParentMethods**

## 3.35 Division rings

**class** sage.categories.division_rings.**DivisionRings**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of division rings

A division ring (or skew field) is a not necessarily commutative ring where all non-zero elements have multiplicative inverses

EXAMPLES:

```
sage: DivisionRings()
Category of division rings
sage: DivisionRings().super_categories()
[Category of domains]
```

**Commutative**
alias of *Fields*

**class ElementMethods**

**Finite_extra_super_categories**()
Return extraneous super categories for `DivisionRings().Finite()`.

EXAMPLES:

Any field is a division ring:

```
sage: Fields().is_subcategory(DivisionRings())
True
```

This methods specifies that, by Weddeburn theorem, the reciprocal holds in the finite case: a finite division ring is commutative and thus a field:

```
sage: DivisionRings().Finite_extra_super_categories()
(Category of commutative magmas,)
sage: DivisionRings().Finite()
Category of finite enumerated fields
```

> **Warning:** This is not implemented in `DivisionRings.Finite.extra_super_categories` because the categories of finite division rings and of finite fields coincide. See the section *Deduction rules* in the documentation of axioms.

**class ParentMethods**

**extra_super_categories**()

Return the `Domains` category.

This method specifies that a division ring has no zero divisors, i.e. is a domain.

**See also:**

The *Deduction rules* section in the documentation of axioms

EXAMPLES:

sage: DivisionRings().extra_super_categories() (Category of domains,) sage: "NoZeroDivisors" in DivisionRings().axioms() True

## 3.36 Domains

**class** sage.categories.domains.**Domains**(*base_category*)

Bases: `sage.categories.category_with_axiom.CategoryWithAxiom_singleton`

The category of domains

A domain (or non-commutative integral domain), is a ring, not necessarily commutative, with no nonzero zero divisors.

EXAMPLES:

```
sage: C = Domains(); C
Category of domains
sage: C.super_categories()
[Category of rings]
sage: C is Rings().NoZeroDivisors()
True
```

**Commutative**

alias of `IntegralDomains`

**class ElementMethods**

**class ParentMethods**

**super_categories**()

EXAMPLES:

```
sage: Domains().super_categories()
[Category of rings]
```

## 3.37 Enumerated Sets

**class** sage.categories.enumerated_sets.**EnumeratedSets**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of enumerated sets

An *enumerated set* is a *finite* or *countable* set or multiset $S$ together with a canonical enumeration of its elements; conceptually, this is very similar to an immutable list. The main difference lies in the names and the return type of the methods, and of course the fact that the list of elements is not supposed to be expanded in memory. Whenever possible one should use one of the two sub-categories *FiniteEnumeratedSets* or *InfiniteEnumeratedSets*.

The purpose of this category is threefold:

- to fix a common interface for all these sets;
- to provide a bunch of default implementations;
- to provide consistency tests.

The standard methods for an enumerated set `S` are:

- `S.cardinality()`: the number of elements of the set. This is the equivalent for `len` on a list except that the return value is specified to be a Sage `Integer` or `infinity`, instead of a Python `int`.
- `iter(S)`: an iterator for the elements of the set;
- `S.list()`: the list of the elements of the set, when possible; raises a NotImplementedError if the list is predictably too large to be expanded in memory.
- `S.unrank(n)`: the `n-th` element of the set when `n` is a sage `Integer`. This is the equivalent for `l[n]` on a list.
- `S.rank(e)`: the position of the element `e` in the set; This is equivalent to `l.index(e)` for a list except that the return value is specified to be a Sage `Integer`, instead of a Python `int`.
- `S.first()`: the first object of the set; it is equivalent to `S.unrank(0)`.
- `S.next(e)`: the object of the set which follows `e`; It is equivalent to `S.unrank(S.rank(e)+1)`.
- `S.random_element()`: a random generator for an element of the set. Unless otherwise stated, and for finite enumerated sets, the probability is uniform.

For examples and tests see:

- `FiniteEnumeratedSets().example()`
- `InfiniteEnumeratedSets().example()`

EXAMPLES:

```
sage: EnumeratedSets()
Category of enumerated sets
sage: EnumeratedSets().super_categories()
[Category of sets]
sage: EnumeratedSets().all_super_categories()
[Category of enumerated sets, Category of sets, Category of sets with partial␣
↪maps, Category of objects]
```

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**class ParentMethods**

**first**()

Return the first element.

EXAMPLES:

```
sage: cartesian_product([ZZ]*10).first()
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
```

**class ElementMethods**

**rank**()

Return the rank of `self` in its parent.

See also *EnumeratedSets.ElementMethods.rank()*

EXAMPLES:

```
sage: F = FiniteSemigroups().example(('a','b','c'))
sage: L = list(F); L
['a', 'b', 'c', 'ac', 'ab', 'ba', 'bc', 'cb', 'ca',
 'acb', 'abc', 'bca', 'cba', 'bac', 'cab']
sage: L[7].rank()
7
```

**Finite**

alias of *FiniteEnumeratedSets*

**Infinite**

alias of *InfiniteEnumeratedSets*

**class ParentMethods**

**first**()

The "first" element of `self`.

`self.first()` returns the first element of the set `self`. This is a generic implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.first() # indirect doctest
1
```

**is_empty**()

Return whether this set is empty.

EXAMPLES:

```
sage: F = FiniteEnumeratedSet([1,2,3])
sage: F.is_empty()
False
sage: F = FiniteEnumeratedSet([])
sage: F.is_empty()
True
```

**iterator_range** (*start=None*, *stop=None*, *step=None*)

Iterate over the range of elements of `self` starting at `start`, ending at `stop`, and stepping by `step`.

**See also:**

`unrank()`, `unrank_range()`

EXAMPLES:

```
sage: P = Partitions()
sage: list(P.iterator_range(stop=5))
[[], [1], [2], [1, 1], [3]]
sage: list(P.iterator_range(0, 5))
[[], [1], [2], [1, 1], [3]]
sage: list(P.iterator_range(3, 5))
[[1, 1], [3]]
sage: list(P.iterator_range(3, 10))
[[1, 1], [3], [2, 1], [1, 1, 1], [4], [3, 1], [2, 2]]
sage: list(P.iterator_range(3, 10, 2))
[[1, 1], [2, 1], [4], [2, 2]]
sage: it = P.iterator_range(3)
sage: [next(it) for x in range(10)]
[[1, 1],
 [3], [2, 1], [1, 1, 1],
 [4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1],
 [5]]
sage: it = P.iterator_range(3, step=2)
sage: [next(it) for x in range(5)]
[[1, 1],
 [2, 1],
 [4], [2, 2], [1, 1, 1, 1]]
sage: next(P.iterator_range(stop=-3))
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: next(P.iterator_range(start=-3))
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
```

**list** ()

Return a list of the elements of `self`.

The elements of set `x` are created and cached on the fist call of `x.list()`. Then each call of `x.list()` returns a new list from the cached result. Thus in looping, it may be better to do `for e in x:`, not `for e in x.list():`.

If `x` is not known to be finite, then an exception is raised.

EXAMPLES:

```
sage: (GF(3)^2).list()
[(0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (2, 2)]
sage: R = Integers(11)
sage: R.list()
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
sage: l = R.list(); l
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
sage: l.remove(0); l
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

```
sage: R.list()
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

**map** (*f*, *name=None*)

Return the image $\{f(x)|x \in \text{self}\}$ of this enumerated set by $f$, as an enumerated set.

$f$ is supposed to be injective.

EXAMPLES:

```
sage: R = SymmetricGroup(3).map(attrcall('reduced_word')); R
Image of Symmetric group of order 3! as a permutation group by *.reduced_
↪word()
sage: R.cardinality()
6
sage: R.list()
[[], [1], [2, 1], [1, 2], [2], [1, 2, 1]]
sage: [ r for r in R]
[[], [1], [2, 1], [1, 2], [2], [1, 2, 1]]
```

**Warning:** If the function is not injective, then there may be repeated elements:

```
sage: P = SymmetricGroup(3)
sage: P.list()
[(), (1,2), (1,2,3), (1,3,2), (2,3), (1,3)]
sage: P.map(attrcall('length')).list()
[0, 1, 2, 2, 1, 3]
```

**Warning:** `MapCombinatorialClass` needs to be refactored to use categories:

```
sage: R.category()             # todo: not implemented
Category of enumerated sets
sage: TestSuite(R).run(skip=['_test_an_element', '_test_category', '_
↪test_some_elements'])
```

**next** (*obj*)

The "next" element after `obj` in `self`.

`self.next(e)` returns the element of the set `self` which follows `e`. This is a generic implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided.

Remark: this is the default (brute force) implementation of the category `EnumeratedSets()`. Its complexity is $O(r)$, where $r$ is the rank of `obj`.

EXAMPLES:

```
sage: C = InfiniteEnumeratedSets().example()
sage: C._next_from_iterator(10) # indirect doctest
11
```

TODO: specify the behavior when `obj` is not in `self`.

**random_element** ()

Return a random element in `self`.

Unless otherwise stated, and for finite enumerated sets, the probability is uniform.

This is a generic implementation from the category `EnumeratedSets()`. It raise a `NotImplementedError` since one does not know whether the set is finite.

EXAMPLES:

```
sage: class broken(UniqueRepresentation, Parent):
....:  def __init__(self):
....:      Parent.__init__(self, category = EnumeratedSets())
sage: broken().random_element()
Traceback (most recent call last):
...
NotImplementedError: unknown cardinality
```

**rank** ($x$)

The rank of an element of `self`

`self.rank(x)` returns the rank of $x$, that is its position in the enumeration of `self`. This is an integer between `0` and `n-1` where `n` is the cardinality of `self`, or None if $x$ is not in $self$.

This is the default (brute force) implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided. Its complexity is $O(r)$, where $r$ is the rank of `obj`. For infinite enumerated sets, this won't terminate when $x$ is not in `self`

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: list(C)
[1, 2, 3]
sage: C.rank(3) # indirect doctest
2
sage: C.rank(5) # indirect doctest
```

**some_elements** ()

Return some elements in `self`.

See `TestSuite` for a typical use case.

This is a generic implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided. It returns an iterator for up to the first 100 elements of `self`

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: list(C.some_elements()) # indirect doctest
[1, 2, 3]
```

**unrank** ($r$)

The `r`-th element of `self`

`self.unrank(r)` returns the `r`-th element of `self`, where `r` is an integer between `0` and `n-1` where `n` is the cardinality of `self`.

This is the default (brute force) implementation from the category `EnumeratedSets()` which can be used when the method `__iter__` is provided. Its complexity is $O(r)$, where $r$ is the rank of `obj`.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.unrank(2) # indirect doctest
3
```

```
sage: C._unrank_from_iterator(5)
Traceback (most recent call last):
...
ValueError: the value must be between 0 and 2 inclusive
```

**unrank_range** (*start=None*, *stop=None*, *step=None*)

Return the range of elements of `self` starting at `start`, ending at `stop`, and stepping by `step`.

**See also:**

`unrank()`, `iterator_range()`

EXAMPLES:

```
sage: P = Partitions()
sage: P.unrank_range(stop=5)
[[], [1], [2], [1, 1], [3]]
sage: P.unrank_range(0, 5)
[[], [1], [2], [1, 1], [3]]
sage: P.unrank_range(3, 5)
[[1, 1], [3]]
sage: P.unrank_range(3, 10)
[[1, 1], [3], [2, 1], [1, 1, 1], [4], [3, 1], [2, 2]]
sage: P.unrank_range(3, 10, 2)
[[1, 1], [2, 1], [4], [2, 2]]
sage: P.unrank_range(3)
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: P.unrank_range(stop=-3)
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: P.unrank_range(start=-3)
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
```

**additional_structure** ()

Return `None`.

Indeed, morphisms of enumerated sets are not required to preserve the enumeration.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: EnumeratedSets().additional_structure()
```

**super_categories** ()

EXAMPLES:

```
sage: EnumeratedSets().super_categories()
[Category of sets]
```

## 3.38 Euclidean domains

AUTHORS:

- Teresa Gomez-Diaz (2008): initial version
- Julian Rueth (2013-09-13): added euclidean degree, quotient remainder, and their tests

**class** sage.categories.euclidean_domains.**EuclideanDomains**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of constructive euclidean domains, i.e., one can divide producing a quotient and a remainder where the remainder is either zero or its *ElementMethods.euclidean_degree()* is smaller than the divisor.

EXAMPLES:

```
sage: EuclideanDomains()
Category of euclidean domains
sage: EuclideanDomains().super_categories()
[Category of principal ideal domains]
```

**class ElementMethods**

**euclidean_degree**()

Return the degree of this element as an element of an Euclidean domain, i.e., for elements $a$, $b$ the euclidean degree $f$ satisfies the usual properties:

1. if $b$ is not zero, then there are elements $q$ and $r$ such that $a = bq + r$ with $r = 0$ or $f(r) < f(b)$
2. if $a, b$ are not zero, then $f(a) \leq f(ab)$

**Note:** The name euclidean_degree was chosen because the euclidean function has different names in different contexts, e.g., absolute value for integers, degree for polynomials.

OUTPUT:

For non-zero elements, a natural number. For the zero element, this might raise an exception or produce some other output, depending on the implementation.

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.euclidean_degree()
1
sage: ZZ.one().euclidean_degree()
1
```

**gcd**(*other*)

Return the greatest common divisor of this element and other.

INPUT:

- other – an element in the same ring as self

ALGORITHM:

Algorithm 3.2.1 in [Coh1993].

EXAMPLES:

```
sage: R.<x> = PolynomialRing(QQ, sparse=True)
sage: EuclideanDomains().element_class.gcd(x,x+1)
-1
```

**quo_rem**(*other*)

Return the quotient and remainder of the division of this element by the non-zero element `other`.

INPUT:

- `other` – an element in the same euclidean domain

OUTPUT:

a pair of elements

EXAMPLES:

```
sage: R.<x> = QQ[]
sage: x.quo_rem(x)
(1, 0)
```

**class ParentMethods**

**gcd_free_basis**(*elts*)

Compute a set of coprime elements that can be used to express the elements of `elts`.

INPUT:

- `elts` - A sequence of elements of `self`.

OUTPUT:

A GCD-free basis (also called a coprime base) of `elts`; that is, a set of pairwise relatively prime elements of `self` such that any element of `elts` can be written as a product of elements of the set.

ALGORITHM:

Naive implementation of the algorithm described in Section 4.8 of Bach & Shallit [BS1996].

EXAMPLES:

```
sage: ZZ.gcd_free_basis([1])
[]
sage: ZZ.gcd_free_basis([4, 30, 14, 49])
[2, 15, 7]

sage: Pol.<x> = QQ[]
sage: Pol.gcd_free_basis([
....:     (x+1)^3*(x+2)^3*(x+3), (x+1)*(x+2)*(x+3),
....:     (x+1)*(x+2)*(x+4)])
[x + 3, x + 4, x^2 + 3*x + 2]
```

**is_euclidean_domain**()

Return True, since this in an object of the category of Euclidean domains.

EXAMPLES:

```
sage: Parent(QQ,category=EuclideanDomains()).is_euclidean_domain()
True
```

**super_categories**()

EXAMPLES:

```
sage: EuclideanDomains().super_categories()
[Category of principal ideal domains]
```

## 3.39 Fields

**class** sage.categories.fields.**Fields**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of (commutative) fields, i.e. commutative rings where all non-zero elements have multiplicative inverses

EXAMPLES:

```
sage: K = Fields()
sage: K
Category of fields
sage: Fields().super_categories()
[Category of euclidean domains, Category of division rings]

sage: K(IntegerRing())
Rational Field
sage: K(PolynomialRing(GF(3), 'x'))
Fraction Field of Univariate Polynomial Ring in x over
Finite Field of size 3
sage: K(RealField())
Real Field with 53 bits of precision
```

**class ElementMethods**

**euclidean_degree**()

Return the degree of this element as an element of an Euclidean domain.

In a field, this returns 0 for all but the zero element (for which it is undefined).

EXAMPLES:

```
sage: QQ.one().euclidean_degree()
0
```

**factor**()

Return a factorization of `self`.

Since `self` is either a unit or zero, this function is trivial.

EXAMPLES:

```
sage: x = GF(7)(5)
sage: x.factor()
5
sage: RR(0).factor()
Traceback (most recent call last):
...
ArithmeticError: factorization of 0.000000000000000 is not defined
```

**gcd**(*other*)

Greatest common divisor.

**Note:** Since we are in a field and the greatest common divisor is only determined up to a unit, it is correct to either return zero or one. Note that fraction fields of unique factorization domains provide a more sophisticated gcd.

EXAMPLES:

```
sage: K = GF(5)
sage: K(2).gcd(K(1))
1
sage: K(0).gcd(K(0))
0
sage: all(x.gcd(y) == (0 if x == 0 and y == 0 else 1) for x in K for y in␣
↪K)
True
```

For field of characteristic zero, the gcd of integers is considered as if they were elements of the integer ring:

```
sage: gcd(15.0,12.0)
3.00000000000000
```

But for others floating point numbers, the gcd is just 0.0 or 1.0:

```
sage: gcd(3.2, 2.18)
1.00000000000000

sage: gcd(0.0, 0.0)
0.000000000000000
```

AUTHOR:

- Simon King (2011-02) – trac ticket #10771
- Vincent Delecroix (2015) – trac ticket #17671

**is_unit**()

Returns True if `self` has a multiplicative inverse.

EXAMPLES:

```
sage: QQ(2).is_unit()
True
sage: QQ(0).is_unit()
False
```

**lcm**(*other*)

Least common multiple.

**Note:** Since we are in a field and the least common multiple is only determined up to a unit, it is correct to either return zero or one. Note that fraction fields of unique factorization domains provide a more sophisticated lcm.

EXAMPLES:

```
sage: GF(2)(1).lcm(GF(2)(0))
0
sage: GF(2)(1).lcm(GF(2)(1))
1
```

For field of characteristic zero, the lcm of integers is considered as if they were elements of the integer ring:

```
sage: lcm(15.0,12.0)
60.0000000000000
```

But for others floating point numbers, it is just 0.0 or 1.0:

```
sage: lcm(3.2, 2.18)
1.00000000000000

sage: lcm(0.0, 0.0)
0.000000000000000
```

AUTHOR:

- Simon King (2011-02) – trac ticket #10771
- Vincent Delecroix (2015) – trac ticket #17671

**quo_rem**(*other*)

Return the quotient with remainder of the division of this element by `other`.

INPUT:

- `other` – an element of the field

EXAMPLES:

```
sage: f,g = QQ(1), QQ(2)
sage: f.quo_rem(g)
(1/2, 0)
```

**xgcd**(*other*)

Compute the extended gcd of `self` and `other`.

INPUT:

- `other` – an element with the same parent as `self`

OUTPUT:

A tuple `(r, s, t)` of elements in the parent of `self` such that `r = s * self + t * other`. Since the computations are done over a field, `r` is zero if `self` and `other` are zero, and one otherwise.

AUTHORS:

- Julian Rueth (2012-10-19): moved here from `sage.structure.element.FieldElement`

EXAMPLES:

```
sage: K = GF(5)
sage: K(2).xgcd(K(1))
(1, 3, 0)
sage: K(0).xgcd(K(4))
(1, 0, 4)
sage: K(1).xgcd(K(1))
(1, 1, 0)
sage: GF(5)(0).xgcd(GF(5)(0))
(0, 0, 0)
```

The xgcd of non-zero floating point numbers will be a triple of floating points. But if the input are two integral floating points the result is a floating point version of the standard gcd on $\mathbf{Z}$:

```
sage: xgcd(12.0, 8.0)
(4.00000000000000, 1.00000000000000, -1.00000000000000)
```

```
sage: xgcd(3.1, 2.98714)
(1.00000000000000, 0.322580645161290, 0.000000000000000)

sage: xgcd(0.0, 1.1)
(1.00000000000000, 0.000000000000000, 0.909090909090909)
```

**Finite**
alias of *FiniteFields*

**class ParentMethods**

**fraction_field**()
Returns the *fraction field* of `self`, which is `self`.

EXAMPLES:

```
sage: QQ.fraction_field() is QQ
True
```

**is_field**(*proof=True*)
Returns True as `self` is a field.

EXAMPLES:

```
sage: QQ.is_field()
True
sage: Parent(QQ,category=Fields()).is_field()
True
```

**is_integrally_closed**()
Return `True`, as per `IntegralDomain.is_integrally_closed()`: for every field $F$, $F$ is its own field of fractions, hence every element of $F$ is integral over $F$.

EXAMPLES:

```
sage: QQ.is_integrally_closed()
True
sage: QQbar.is_integrally_closed()
True
sage: Z5 = GF(5); Z5
Finite Field of size 5
sage: Z5.is_integrally_closed()
True
```

**is_perfect**()
Return whether this field is perfect, i.e., its characteristic is $p = 0$ or every element has a $p$-th root.

EXAMPLES:

```
sage: QQ.is_perfect()
True
sage: GF(2).is_perfect()
True
sage: FunctionField(GF(2), 'x').is_perfect()
False
```

**extra_super_categories**()
EXAMPLES:

```
sage: Fields().extra_super_categories()
[Category of euclidean domains]
```

## 3.40 Filtered Algebras

**class** sage.categories.filtered_algebras.**FilteredAlgebras**(*base_category*)
Bases: *sage.categories.filtered_modules.FilteredModulesCategory*

The category of filtered algebras.

An algebra $A$ over a commutative ring $R$ is *filtered* if $A$ is endowed with a structure of a filtered $R$-module (whose underlying $R$-module structure is identical with that of the $R$-algebra $A$) such that the indexing set $I$ (typically $I = \mathbf{N}$) is also an additive abelian monoid, the unity 1 of $A$ belongs to $F_0$, and we have $F_i \cdot F_j \subseteq F_{i+j}$ for all $i, j \in I$.

EXAMPLES:

```
sage: Algebras(ZZ).Filtered()
Category of filtered algebras over Integer Ring
sage: Algebras(ZZ).Filtered().super_categories()
[Category of algebras over Integer Ring,
 Category of filtered modules over Integer Ring]
```

REFERENCES:

- Wikipedia article Filtered_algebra

**class ParentMethods**

**graded_algebra**()
Return the associated graded algebra to `self`.

**Todo:** Implement a version of the associated graded algebra which does not require `self` to have a distinguished basis.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: A.graded_algebra()
Graded Algebra of An example of a filtered algebra with basis:
 the universal enveloping algebra of
 Lie algebra of RR^3 with cross product over Integer Ring
```

## 3.41 Filtered Algebras With Basis

A filtered algebra with basis over a commutative ring $R$ is a filtered algebra over $R$ endowed with the structure of a filtered module with basis (with the same underlying filtered-module structure). See *FilteredAlgebras* and *FilteredModulesWithBasis* for these two notions.

**class** sage.categories.filtered_algebras_with_basis.**FilteredAlgebrasWithBasis**(*base_category*)
Bases: *sage.categories.filtered_modules.FilteredModulesCategory*

The category of filtered algebras with a distinguished homogeneous basis.

A filtered algebra with basis over a commutative ring $R$ is a filtered algebra over $R$ endowed with the structure of a filtered module with basis (with the same underlying filtered-module structure). See *FilteredAlgebras* and *FilteredModulesWithBasis* for these two notions.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(ZZ).Filtered(); C
Category of filtered algebras with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of algebras with basis over Integer Ring,
 Category of filtered algebras over Integer Ring,
 Category of filtered modules with basis over Integer Ring]
```

**class ElementMethods**

**class ParentMethods**

**from_graded_conversion**()

Return the inverse of the canonical $R$-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of $A$ (where $A =$ "'$self$'). This inverse is an isomorphism $\operatorname{gr} A \to A$.

This is an isomorphism of $R$-modules, not of algebras. See the class documentation `AssociatedGradedAlgebra`.

**See also:**

*to_graded_conversion()*

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: p = A.an_element() + A.algebra_generators()['x'] + 2; p
U['x']^2*U['y']^2*U['z']^3 + 3*U['x'] + 3*U['y'] + 3
sage: q = A.to_graded_conversion()(p)
sage: A.from_graded_conversion()(q) == p
True
sage: q.parent() is A.graded_algebra()
True
```

**graded_algebra**()

Return the associated graded algebra to `self`.

See `AssociatedGradedAlgebra` for the definition and the properties of this.

If the filtered algebra `self` with basis is called $A$, then this method returns $\operatorname{gr} A$. The method *to_graded_conversion()* returns the canonical $R$-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of $A$, and the method *from_graded_conversion()* returns the inverse of this isomorphism. The method *projection()* projects elements of $A$ onto $\operatorname{gr} A$ according to their place in the filtration on $A$.

> **Warning:** When not overridden, this method returns the default implementation of an associated graded algebra – namely, `AssociatedGradedAlgebra(self)`, where `AssociatedGradedAlgebra` is `AssociatedGradedAlgebra`. But many instances of *FilteredAlgebrasWithBasis* override this method, as the associated graded algebra often is (isomorphic) to a simpler object (for instance, the associated graded algebra of a

graded algebra can be identified with the graded algebra itself). Generic code that uses associated graded algebras (such as the code of the `induced_graded_map()` method below) should make sure to only communicate with them via the `to_graded_conversion()`, `from_graded_conversion()`, and `projection()` methods (in particular, do not expect there to be a conversion from `self` to `self.graded_algebra()`; this currently does not work for Clifford algebras). Similarly, when overriding `graded_algebra()`, make sure to accordingly redefine these three methods, unless their definitions below still apply to your case (this will happen whenever the basis of your `graded_algebra()` has the same indexing set as `self`, and the partition of this indexing set according to degree is the same as for `self`).

**Todo:** Maybe the thing about the conversion from `self` to `self.graded_algebra()` on the Clifford at least could be made to work? (I would still warn the user against ASSUMING that it must work – as there is probably no way to guarantee it in all cases, and we shouldn't require users to mess with element constructors.)

EXAMPLES:

```
sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: A.graded_algebra()
Graded Algebra of An example of a filtered algebra with basis:
 the universal enveloping algebra of
 Lie algebra of RR^3 with cross product over Integer Ring
```

**induced_graded_map**(*other*, *f*)

Return the graded linear map between the associated graded algebras of `self` and `other` canonically induced by the filtration-preserving map `f : self -> other`.

Let $A$ and $B$ be two filtered algebras with basis, and let $(F_i)_{i\in I}$ and $(G_i)_{i\in I}$ be their filtrations. Let $f : A \to B$ be a linear map which preserves the filtration (i.e., satisfies $f(F_i) \subseteq G_i$ for all $i \in I$). Then, there is a canonically defined graded linear map $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ which satisfies

$$(\operatorname{gr} f)(p_i(a)) = p_i(f(a)) \qquad \text{for all } i \in I \text{ and } a \in F_i,$$

where the $p_i$ on the left hand side is the canonical projection from $F_i$ onto the $i$-th graded component of $\operatorname{gr} A$, while the $p_i$ on the right hand side is the canonical projection from $G_i$ onto the $i$-th graded component of $\operatorname{gr} B$.

INPUT:

- `other` – a filtered algebra with basis
- `f` – a filtration-preserving linear map from `self` to `other` (can be given as a morphism or as a function)

OUTPUT:

The graded linear map $\operatorname{gr} f$.

EXAMPLES:

**Example 1.**

We start with the universal enveloping algebra of the Lie algebra $\mathbf{R}^3$ (with the cross product serving as Lie bracket):

```
sage: A = AlgebrasWithBasis(QQ).Filtered().example(); A
An example of a filtered algebra with basis: the
 universal enveloping algebra of Lie algebra of RR^3
```

```
 with cross product over Rational Field
sage: M = A.indices(); M
Free abelian monoid indexed by {'x', 'y', 'z'}
sage: x,y,z = [A.basis()[M.gens()[i]] for i in "xyz"]
```

Let us define a stupid filtered map from A to itself:

```
sage: def map_on_basis(m):
....:     d = m.dict()
....:     i = d.get('x', 0); j = d.get('y', 0); k = d.get('z', 0)
....:     g = (y ** (i+j)) * (z ** k)
....:     if i > 0:
....:         g += i * (x ** (i-1)) * (y ** j) * (z ** k)
....:     return g
sage: f = A.module_morphism(on_basis=map_on_basis,
....:                       codomain=A)
sage: f(x)
U['y'] + 1
sage: f(x*y*z)
U['y']^2*U['z'] + U['y']*U['z']
sage: f(x*x*y*z)
U['y']^3*U['z'] + 2*U['x']*U['y']*U['z']
sage: f(A.one())
1
sage: f(y*z)
U['y']*U['z']
```

(There is nothing here that is peculiar to this universal enveloping algebra; we are only using its module structure, and we could just as well be using a polynomial algebra in its stead.)

We now compute gr $f$

```
sage: grA = A.graded_algebra(); grA
Graded Algebra of An example of a filtered algebra with
 basis: the universal enveloping algebra of Lie algebra
 of RR^3 with cross product over Rational Field
sage: xx, yy, zz = [A.to_graded_conversion()(i) for i in [x, y, z]]
sage: xx+yy*zz
bar(U['y']*U['z']) + bar(U['x'])
sage: grf = A.induced_graded_map(A, f); grf
Generic endomorphism of Graded Algebra of An example
 of a filtered algebra with basis: the universal
 enveloping algebra of Lie algebra of RR^3 with cross
 product over Rational Field
sage: grf(xx)
bar(U['y'])
sage: grf(xx*yy*zz)
bar(U['y']^2*U['z'])
sage: grf(xx*xx*yy*zz)
bar(U['y']^3*U['z'])
sage: grf(grA.one())
bar(1)
sage: grf(yy*zz)
bar(U['y']*U['z'])
sage: grf(yy*zz-2*yy)
bar(U['y']*U['z']) - 2*bar(U['y'])
```

**Example 2.**

We shall now construct gr $f$ for a different map $f$ out of the same A; the new map $f$ will lead into a graded algebra already, namely into the algebra of symmetric functions:

```
sage: h = SymmetricFunctions(QQ).h()
sage: def map_on_basis(m):  # redefining map_on_basis
....:     d = m.dict()
....:     i = d.get('x', 0); j = d.get('y', 0); k = d.get('z', 0)
....:     g = (h[1] ** i) * (h[2] ** (floor(j/2))) * (h[3] ** (floor(k/
↪3)))
....:     g += i * (h[1] ** (i+j+k))
....:     return g
sage: f = A.module_morphism(on_basis=map_on_basis,
....:                       codomain=h)  # redefining f
sage: f(x)
2*h[1]
sage: f(y)
h[]
sage: f(z)
h[]
sage: f(y**2)
h[2]
sage: f(x**2)
3*h[1, 1]
sage: f(x*y*z)
h[1] + h[1, 1, 1]
sage: f(x*x*y*y*z)
2*h[1, 1, 1, 1, 1] + h[2, 1, 1]
sage: f(A.one())
h[]
```

The algebra h of symmetric functions in the $h$-basis is already graded, so its associated graded algebra is implemented as itself:

```
sage: grh = h.graded_algebra(); grh is h
True
sage: grf = A.induced_graded_map(h, f); grf
Generic morphism:
  From: Graded Algebra of An example of a filtered
   algebra with basis: the universal enveloping
   algebra of Lie algebra of RR^3 with cross
   product over Rational Field
  To:   Symmetric Functions over Rational Field
   in the homogeneous basis
sage: grf(xx)
2*h[1]
sage: grf(yy)
0
sage: grf(zz)
0
sage: grf(yy**2)
h[2]
sage: grf(xx**2)
3*h[1, 1]
sage: grf(xx*yy*zz)
h[1, 1, 1]
sage: grf(xx*xx*yy*yy*zz)
2*h[1, 1, 1, 1, 1]
sage: grf(grA.one())
```

```
h[]
```

**Example 3.**

After having had a graded algebra as the codomain, let us try to have one as the domain instead. Our new `f` will go from `h` to `A`:

```
sage: def map_on_basis(lam):  # redefining map_on_basis
....:     return x ** (sum(lam)) + y ** (len(lam))
sage: f = h.module_morphism(on_basis=map_on_basis,
....:                        codomain=A)  # redefining f
sage: f(h[1])
U['x'] + U['y']
sage: f(h[2])
U['x']^2 + U['y']
sage: f(h[1, 1])
U['x']^2 + U['y']^2
sage: f(h[2, 2])
U['x']^4 + U['y']^2
sage: f(h[3, 2, 1])
U['x']^6 + U['y']^3
sage: f(h.one())
2
sage: grf = h.induced_graded_map(A, f); grf
Generic morphism:
  From: Symmetric Functions over Rational Field
   in the homogeneous basis
  To:   Graded Algebra of An example of a filtered
   algebra with basis: the universal enveloping
   algebra of Lie algebra of RR^3 with cross
   product over Rational Field
sage: grf(h[1])
bar(U['x']) + bar(U['y'])
sage: grf(h[2])
bar(U['x']^2)
sage: grf(h[1, 1])
bar(U['x']^2) + bar(U['y']^2)
sage: grf(h[2, 2])
bar(U['x']^4)
sage: grf(h[3, 2, 1])
bar(U['x']^6)
sage: grf(h.one())
2*bar(1)
```

**Example 4.**

The construct gr $f$ also makes sense when $f$ is a filtration-preserving map between graded algebras.

```
sage: def map_on_basis(lam):  # redefining map_on_basis
....:     return h[lam] + h[len(lam)]
sage: f = h.module_morphism(on_basis=map_on_basis,
....:                        codomain=h)  # redefining f
sage: f(h[1])
2*h[1]
sage: f(h[2])
h[1] + h[2]
sage: f(h[1, 1])
h[1, 1] + h[2]
sage: f(h[2, 1])
```

```
h[2] + h[2, 1]
sage: f(h.one())
2*h[]
sage: grf = h.induced_graded_map(h, f); grf
Generic endomorphism of Symmetric Functions over Rational
 Field in the homogeneous basis
sage: grf(h[1])
2*h[1]
sage: grf(h[2])
h[2]
sage: grf(h[1, 1])
h[1, 1] + h[2]
sage: grf(h[2, 1])
h[2, 1]
sage: grf(h.one())
2*h[]
```

**Example 5.**

For another example, let us compute $\operatorname{gr} f$ for a map $f$ between two Clifford algebras:

```
sage: Q = QuadraticForm(ZZ, 2, [1,2,3])
sage: B = CliffordAlgebra(Q, names=['u','v']); B
The Clifford algebra of the Quadratic form in 2
 variables over Integer Ring with coefficients:
[ 1 2 ]
[ * 3 ]
sage: m = Matrix(ZZ, [[1, 2], [1, -1]])
sage: f = B.lift_module_morphism(m, names=['x','y'])
sage: A = f.domain(); A
The Clifford algebra of the Quadratic form in 2
 variables over Integer Ring with coefficients:
[ 6 0 ]
[ * 3 ]
sage: x, y = A.gens()
sage: f(x)
u + v
sage: f(y)
2*u - v
sage: f(x**2)
6
sage: f(x*y)
-3*u*v + 3
sage: grA = A.graded_algebra(); grA
The exterior algebra of rank 2 over Integer Ring
sage: A.to_graded_conversion()(x)
x
sage: A.to_graded_conversion()(y)
y
sage: A.to_graded_conversion()(x*y)
x^y
sage: u = A.to_graded_conversion()(x*y+1); u
x^y + 1
sage: A.from_graded_conversion()(u)
x*y + 1
sage: A.projection(2)(x*y+1)
x^y
sage: A.projection(1)(x+2*y-2)
```

```
x + 2*y
sage: grf = A.induced_graded_map(B, f); grf
Generic morphism:
  From: The exterior algebra of rank 2 over Integer Ring
  To:   The exterior algebra of rank 2 over Integer Ring
sage: grf(A.to_graded_conversion()(x))
u + v
sage: grf(A.to_graded_conversion()(y))
2*u - v
sage: grf(A.to_graded_conversion()(x**2))
6
sage: grf(A.to_graded_conversion()(x*y))
-3*u^v
sage: grf(grA.one())
1
```

**projection**($i$)

Return the $i$-th projection $p_i : F_i \to G_i$ (in the notations of the class documentation AssociatedGradedAlgebra, where $A =$ "$self$").

This method actually does not return the map $p_i$ itself, but an extension of $p_i$ to the whole $R$-module $A$. This extension is the composition of the $R$-module isomorphism $A \to \operatorname{gr} A$ with the canonical projection of the graded $R$-module $\operatorname{gr} A$ onto its $i$-th graded component $G_i$. The codomain of this map is $\operatorname{gr} A$, although its actual image is $G_i$. The map $p_i$ is obtained from this map by restricting its domain to $F_i$ and its image to $G_i$.

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: p = A.an_element() + A.algebra_generators()['x'] + 2; p
U['x']^2*U['y']^2*U['z']^3 + 3*U['x'] + 3*U['y'] + 3
sage: q = A.projection(7)(p); q
bar(U['x']^2*U['y']^2*U['z']^3)
sage: q.parent() is A.graded_algebra()
True
sage: A.projection(8)(p)
0
```

**to_graded_conversion**()

Return the canonical $R$-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of $A$ (where $A =$ "$self$").

This is an isomorphism of $R$-modules, not of algebras. See the class documentation AssociatedGradedAlgebra.

**See also:**

*from_graded_conversion()*

EXAMPLES:

```
sage: A = Algebras(QQ).WithBasis().Filtered().example()
sage: p = A.an_element() + A.algebra_generators()['x'] + 2; p
U['x']^2*U['y']^2*U['z']^3 + 3*U['x'] + 3*U['y'] + 3
sage: q = A.to_graded_conversion()(p); q
bar(U['x']^2*U['y']^2*U['z']^3) + 3*bar(U['x'])
 + 3*bar(U['y']) + 3*bar(1)
sage: q.parent() is A.graded_algebra()
True
```

## 3.42 Filtered Modules

A *filtered module* over a ring $R$ with a totally ordered indexing set $I$ (typically $I = \mathbf{N}$) is an $R$-module $M$ equipped with a family $(F_i)_{i \in I}$ of $R$-submodules satisfying $F_i \subseteq F_j$ for all $i, j \in I$ having $i \leq j$, and $M = \bigcup_{i \in I} F_i$. This family is called a *filtration* of the given module $M$.

**Todo:** Implement a notion for decreasing filtrations: where $F_j \subseteq F_i$ when $i \leq j$.

**Todo:** Implement filtrations for all concrete categories.

**Todo:** Implement gr as a functor.

**class** sage.categories.filtered_modules.**FilteredModules**(*base_category*)

Bases: *sage.categories.filtered_modules.FilteredModulesCategory*

The category of filtered modules over a given ring $R$.

A *filtered module* over a ring $R$ with a totally ordered indexing set $I$ (typically $I = \mathbf{N}$) is an $R$-module $M$ equipped with a family $(F_i)_{i \in I}$ of $R$-submodules satisfying $F_i \subseteq F_j$ for all $i, j \in I$ having $i \leq j$, and $M = \bigcup_{i \in I} F_i$. This family is called a *filtration* of the given module $M$.

EXAMPLES:

```
sage: Modules(ZZ).Filtered()
Category of filtered modules over Integer Ring
sage: Modules(ZZ).Filtered().super_categories()
[Category of modules over Integer Ring]
```

REFERENCES:

- Wikipedia article Filtration_(mathematics)

**class Connected**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class SubcategoryMethods**

**Connected**()

Return the full subcategory of the connected objects of `self`.

A filtered $R$-module $M$ with filtration $(F_0, F_1, F_2, \ldots)$ (indexed by $\mathbf{N}$) is said to be *connected* if $F_0$ is isomorphic to $R$.

EXAMPLES:

```
sage: Modules(ZZ).Filtered().Connected()
Category of filtered connected modules over Integer Ring
sage: Coalgebras(QQ).Filtered().Connected()
Join of Category of filtered connected modules over Rational Field
    and Category of coalgebras over Rational Field
sage: AlgebrasWithBasis(QQ).Filtered().Connected()
Category of filtered connected algebras with basis over Rational Field
```

**extra_super_categories**()

Add *VectorSpaces* to the super categories of self if the base ring is a field.

EXAMPLES:

```
sage: Modules(QQ).Filtered().extra_super_categories()
[Category of vector spaces over Rational Field]
sage: Modules(ZZ).Filtered().extra_super_categories()
[]
```

This makes sure that Modules(QQ).Filtered() returns an instance of *FilteredModules* and not a join category of an instance of this class and of VectorSpaces(QQ):

```
sage: type(Modules(QQ).Filtered())
<class 'sage.categories.filtered_modules.FilteredModules_with_category'>
```

**Todo:** Get rid of this workaround once there is a more systematic approach for the alias Modules(QQ) -> VectorSpaces(QQ). Probably the latter should be a category with axiom, and covariant constructions should play well with axioms.

**class** sage.categories.filtered_modules.**FilteredModulesCategory**(*base_category*)

Bases: *sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory*, *sage.categories.category_types.Category_over_base_ring*

EXAMPLES:

```
sage: C = Algebras(QQ).Filtered()
sage: C
Category of filtered algebras over Rational Field
sage: C.base_category()
Category of algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of algebras over Rational Field,
 Category of filtered modules over Rational Field]

sage: AlgebrasWithBasis(QQ).Filtered().base_ring()
Rational Field
sage: HopfAlgebrasWithBasis(QQ).Filtered().base_ring()
Rational Field
```

## 3.43 Filtered Modules With Basis

A *filtered module with basis* over a ring $R$ means (for the purpose of this code) a filtered $R$-module $M$ with filtration $(F_i)_{i\in I}$ (typically $I = \mathbf{N}$) endowed with a basis $(b_j)_{j\in J}$ of $M$ and a partition $J = \bigsqcup_{i\in I} J_i$ of the set $J$ (it is allowed that some $J_i$ are empty) such that for every $n \in I$, the subfamily $(b_j)_{j\in U_n}$, where $U_n = \bigcup_{i\leq n} J_i$, is a basis of the $R$-submodule $F_n$.

For every $i \in I$, the $R$-submodule of $M$ spanned by $(b_j)_{j\in J_i}$ is called the $i$*-th graded component* (aka the $i$*-th homogeneous component*) of the filtered module with basis $M$; the elements of this submodule are referred to as *homogeneous elements of degree* $i$.

See the class documentation *FilteredModulesWithBasis* for further details.

**class** sage.categories.filtered_modules_with_basis.**FilteredModulesWithBasis**(*base_category*)

Bases: *sage.categories.filtered_modules.FilteredModulesCategory*

The category of filtered modules with a distinguished basis.

A *filtered module with basis* over a ring $R$ means (for the purpose of this code) a filtered $R$-module $M$ with filtration $(F_i)_{i\in I}$ (typically $I = \mathbf{N}$) endowed with a basis $(b_j)_{j\in J}$ of $M$ and a partition $J = \bigsqcup_{i\in I} J_i$ of the set $J$ (it is allowed that some $J_i$ are empty) such that for every $n \in I$, the subfamily $(b_j)_{j\in U_n}$, where $U_n = \bigcup_{i\leq n} J_i$, is a basis of the $R$-submodule $F_n$.

For every $i \in I$, the $R$-submodule of $M$ spanned by $(b_j)_{j\in J_i}$ is called the $i$*-th graded component* (aka the $i$*-th homogeneous component*) of the filtered module with basis $M$; the elements of this submodule are referred to as *homogeneous elements of degree* $i$. The $R$-module $M$ is the direct sum of its $i$-th graded components over all $i \in I$, and thus becomes a graded $R$-module with basis. Conversely, any graded $R$-module with basis canonically becomes a filtered $R$-module with basis (by defining $F_n = \bigoplus_{i\leq n} G_i$ where $G_i$ is the $i$-th graded component, and defining $J_i$ as the indexing set of the basis of the $i$-th graded component). Hence, the notion of a filtered $R$-module with basis is equivalent to the notion of a graded $R$-module with basis.

However, the *category* of filtered $R$-modules with basis is not the category of graded $R$-modules with basis. Indeed, the *morphisms* of filtered $R$-modules with basis are defined to be morphisms of $R$-modules which send each $F_n$ of the domain to the corresponding $F_n$ of the target; in contrast, the morphisms of graded $R$-modules with basis must preserve each homogeneous component. Also, the notion of a filtered algebra with basis differs from that of a graded algebra with basis.

**Note:** Currently, to make use of the functionality of this class, an instance of `FilteredModulesWithBasis` should fulfill the contract of a `CombinatorialFreeModule` (most likely by inheriting from it). It should also have the indexing set $J$ encoded as its `_indices` attribute, and `_indices.subset(size=i)` should yield the subset $J_i$ (as an iterable). If the latter conditions are not satisfied, then `basis()` must be overridden.

**Note:** One should implement a `degree_on_basis` method in the parent class in order to fully utilize the methods of this category. This might become a required abstract method in the future.

EXAMPLES:

```
sage: C = ModulesWithBasis(ZZ).Filtered(); C
Category of filtered modules with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of filtered modules over Integer Ring,
 Category of modules with basis over Integer Ring]
sage: C is ModulesWithBasis(ZZ).Filtered()
True
```

**class ElementMethods**

**degree**()

The degree of a nonzero homogeneous element `self` in the filtered module.

**Note:** This raises an error if the element is not homogeneous. To compute the maximum of the degrees of the homogeneous summands of a (not necessarily homogeneous) element, use *maximal_degree()* instead.

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A(Partition((3,2,1)))
sage: y = A(Partition((4,4,1)))
sage: z = A(Partition((2,2,2)))
sage: x.degree()
6
sage: (x + 2*z).degree()
6
sage: (y - x).degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

An example in a graded algebra:

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: (x, y) = (S[2], S[3])
sage: x.homogeneous_degree()
2
sage: (x^3 + 4*y^2).homogeneous_degree()
6
sage: ((1 + x)^3).homogeneous_degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

Let us now test a filtered algebra (but remember that the notion of homogeneity now depends on the choice of a basis):

```
sage: A = AlgebrasWithBasis(QQ).Filtered().example()
sage: x,y,z = A.algebra_generators()
sage: (x*y).homogeneous_degree()
2
sage: (y*x).homogeneous_degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
sage: A.one().homogeneous_degree()
0
```

**degree_on_basis** (*m*)

Return the degree of the basis element indexed by `m` in `self`.

EXAMPLES:

```
sage: A = GradedModulesWithBasis(QQ).example()
sage: A.degree_on_basis(Partition((2,1)))
3
sage: A.degree_on_basis(Partition((4,2,1,1,1,1)))
10
```

**homogeneous_component** (*n*)

Return the homogeneous component of degree `n` of the element `self`.

Let $m$ be an element of a filtered $R$-module $M$ with basis. Then, $m$ can be uniquely written in the form $m = \sum_{i \in I} m_i$, where each $m_i$ is a homogeneous element of degree $i$. For $n \in I$, we define the homogeneous component of degree $n$ of the element $m$ to be $m_n$.

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A.an_element(); x
2*P[] + 2*P[1] + 3*P[2]
sage: x.homogeneous_component(-1)
0
sage: x.homogeneous_component(0)
2*P[]
sage: x.homogeneous_component(1)
2*P[1]
sage: x.homogeneous_component(2)
3*P[2]
sage: x.homogeneous_component(3)
0

sage: A = ModulesWithBasis(ZZ).Graded().example()
sage: x = A.an_element(); x
2*P[] + 2*P[1] + 3*P[2]
sage: x.homogeneous_component(-1)
0
sage: x.homogeneous_component(0)
2*P[]
sage: x.homogeneous_component(1)
2*P[1]
sage: x.homogeneous_component(2)
3*P[2]
sage: x.homogeneous_component(3)
0

sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: G = A.algebra_generators()
sage: g = A.an_element() - 2 * G['x'] * G['y']; g
U['x']^2*U['y']^2*U['z']^3 - 2*U['x']*U['y']
 + 2*U['x'] + 3*U['y'] + 1
sage: g.homogeneous_component(-1)
0
sage: g.homogeneous_component(0)
1
sage: g.homogeneous_component(2)
-2*U['x']*U['y']
sage: g.homogeneous_component(5)
0
sage: g.homogeneous_component(7)
U['x']^2*U['y']^2*U['z']^3
sage: g.homogeneous_component(8)
0
```

**homogeneous_degree**()

The degree of a nonzero homogeneous element `self` in the filtered module.

**Note:** This raises an error if the element is not homogeneous. To compute the maximum of the degrees of the homogeneous summands of a (not necessarily homogeneous) element, use *maximal_degree()* instead.

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A(Partition((3,2,1)))
sage: y = A(Partition((4,4,1)))
sage: z = A(Partition((2,2,2)))
sage: x.degree()
6
sage: (x + 2*z).degree()
6
sage: (y - x).degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

An example in a graded algebra:

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: (x, y) = (S[2], S[3])
sage: x.homogeneous_degree()
2
sage: (x^3 + 4*y^2).homogeneous_degree()
6
sage: ((1 + x)^3).homogeneous_degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
```

Let us now test a filtered algebra (but remember that the notion of homogeneity now depends on the choice of a basis):

```
sage: A = AlgebrasWithBasis(QQ).Filtered().example()
sage: x,y,z = A.algebra_generators()
sage: (x*y).homogeneous_degree()
2
sage: (y*x).homogeneous_degree()
Traceback (most recent call last):
...
ValueError: element is not homogeneous
sage: A.one().homogeneous_degree()
0
```

**is_homogeneous**()

Return whether the element `self` is homogeneous.

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x=A(Partition((3,2,1)))
sage: y=A(Partition((4,4,1)))
sage: z=A(Partition((2,2,2)))
sage: (3*x).is_homogeneous()
True
sage: (x - y).is_homogeneous()
False
sage: (x+2*z).is_homogeneous()
True
```

Here is an example with a graded algebra:

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: (x, y) = (S[2], S[3])
sage: (3*x).is_homogeneous()
True
sage: (x^3 - y^2).is_homogeneous()
True
sage: ((x + y)^2).is_homogeneous()
False
```

Let us now test a filtered algebra (but remember that the notion of homogeneity now depends on the choice of a basis, or at least on a definition of homogeneous components):

```
sage: A = AlgebrasWithBasis(QQ).Filtered().example()
sage: x,y,z = A.algebra_generators()
sage: (x*y).is_homogeneous()
True
sage: (y*x).is_homogeneous()
False
sage: A.one().is_homogeneous()
True
sage: A.zero().is_homogeneous()
True
sage: (A.one()+x).is_homogeneous()
False
```

**maximal_degree**()

The maximum of the degrees of the homogeneous components of `self`.

This is also the smallest $i$ such that `self` belongs to $F_i$. Hence, it does not depend on the basis of the parent of `self`.

**See also:**

*homogeneous_degree()*

EXAMPLES:

sage: A = ModulesWithBasis(ZZ).Filtered().example() sage: x = A(Partition((3,2,1))) sage: y = A(Partition((4,4,1))) sage: z = A(Partition((2,2,2))) sage: x.maximal_degree() 6 sage: (x + 2*z).maximal_degree() 6 sage: (y - x).maximal_degree() 9 sage: (3*z).maximal_degree() 6

Now, we test this on a graded algebra:

```
sage: S = NonCommutativeSymmetricFunctions(QQ).S()
sage: (x, y) = (S[2], S[3])
sage: x.maximal_degree()
2
sage: (x^3 + 4*y^2).maximal_degree()
6
sage: ((1 + x)^3).maximal_degree()
6
```

Let us now test a filtered algebra:

```
sage: A = AlgebrasWithBasis(QQ).Filtered().example()
sage: x,y,z = A.algebra_generators()
sage: (x*y).maximal_degree()
2
sage: (y*x).maximal_degree()
2
```

```
sage: A.one().maximal_degree()
0
sage: A.zero().maximal_degree()
Traceback (most recent call last):
...
ValueError: the zero element does not have a well-defined degree
sage: (A.one()+x).maximal_degree()
1
```

**truncate**(*n*)

Return the sum of the homogeneous components of degree strictly less than `n` of `self`.

See *homogeneous_component()* for the notion of a homogeneous component.

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: x = A.an_element(); x
2*P[] + 2*P[1] + 3*P[2]
sage: x.truncate(0)
0
sage: x.truncate(1)
2*P[]
sage: x.truncate(2)
2*P[] + 2*P[1]
sage: x.truncate(3)
2*P[] + 2*P[1] + 3*P[2]

sage: A = ModulesWithBasis(ZZ).Graded().example()
sage: x = A.an_element(); x
2*P[] + 2*P[1] + 3*P[2]
sage: x.truncate(0)
0
sage: x.truncate(1)
2*P[]
sage: x.truncate(2)
2*P[] + 2*P[1]
sage: x.truncate(3)
2*P[] + 2*P[1] + 3*P[2]

sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: G = A.algebra_generators()
sage: g = A.an_element() - 2 * G['x'] * G['y']; g
U['x']^2*U['y']^2*U['z']^3 - 2*U['x']*U['y']
 + 2*U['x'] + 3*U['y'] + 1
sage: g.truncate(-1)
0
sage: g.truncate(0)
0
sage: g.truncate(2)
2*U['x'] + 3*U['y'] + 1
sage: g.truncate(3)
-2*U['x']*U['y'] + 2*U['x'] + 3*U['y'] + 1
sage: g.truncate(5)
-2*U['x']*U['y'] + 2*U['x'] + 3*U['y'] + 1
sage: g.truncate(7)
-2*U['x']*U['y'] + 2*U['x'] + 3*U['y'] + 1
sage: g.truncate(8)
U['x']^2*U['y']^2*U['z']^3 - 2*U['x']*U['y']
```

```
+ 2*U['x'] + 3*U['y'] + 1
```

**class ParentMethods**

**basis**(*d=None*)

Return the basis for (the d-th homogeneous component of) self.

INPUT:

- d – (optional, default None) nonnegative integer or None

OUTPUT:

If d is None, returns the basis of the module. Otherwise, returns the basis of the homogeneous component of degree d (i.e., the subfamily of the basis of the whole module which consists only of the basis vectors lying in $F_d \setminus \bigcup_{i<d} F_i$).

The basis is always returned as a family.

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: A.basis(4)
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions of the integer 4}
```

Without arguments, the full basis is returned:

```
sage: A.basis()
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions}
sage: A.basis()
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions}
```

Checking this method on a filtered algebra. Note that this will typically raise a NotImplementedError when this feature is not implemented.

```
sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: A.basis(4)
Traceback (most recent call last):
...
NotImplementedError: infinite set
```

Without arguments, the full basis is returned:

```
sage: A.basis()
Lazy family (Term map from Free abelian monoid indexed by
 {'x', 'y', 'z'} to An example of a filtered algebra with
 basis: the universal enveloping algebra of Lie algebra
 of RR^3 with cross product over Integer Ring(i))_{i in
 Free abelian monoid indexed by {'x', 'y', 'z'}}
```

An example with a graded algebra:

```
sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: E.basis()
```

```
Lazy family (Term map from Subsets of {0, 1} to
 The exterior algebra of rank 2 over Rational Field(i))_{i in
 Subsets of {0, 1}}
```

**from_graded_conversion()**

Return the inverse of the canonical $R$-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of $A$ (where $A =$ "'$self$'). This inverse is an isomorphism $\operatorname{gr} A \to A$.

This is an isomorphism of $R$-modules. See the class documentation `AssociatedGradedAlgebra`.

**See also:**

*to_graded_conversion()*

EXAMPLES:

```
sage: A = Modules(QQ).WithBasis().Filtered().example()
sage: p = -2 * A.an_element(); p
-4*P[] - 4*P[1] - 6*P[2]
sage: q = A.to_graded_conversion()(p); q
-4*Bbar[[]] - 4*Bbar[[1]] - 6*Bbar[[2]]
sage: A.from_graded_conversion()(q) == p
True
sage: q.parent() is A.graded_algebra()
True
```

**graded_algebra()**

Return the associated graded module to `self`.

See `AssociatedGradedAlgebra` for the definition and the properties of this.

If the filtered module `self` with basis is called $A$, then this method returns $\operatorname{gr} A$. The method *to_graded_conversion()* returns the canonical $R$-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of $A$, and the method *from_graded_conversion()* returns the inverse of this isomorphism. The method *projection()* projects elements of $A$ onto $\operatorname{gr} A$ according to their place in the filtration on $A$.

> **Warning:** When not overridden, this method returns the default implementation of an associated graded module – namely, `AssociatedGradedAlgebra(self)`, where `AssociatedGradedAlgebra` is `AssociatedGradedAlgebra`. But some instances of *FilteredModulesWithBasis* override this method, as the associated graded module often is (isomorphic) to a simpler object (for instance, the associated graded module of a graded module can be identified with the graded module itself). Generic code that uses associated graded modules (such as the code of the *induced_graded_map()* method below) should make sure to only communicate with them via the *to_graded_conversion()*, *from_graded_conversion()* and *projection()* methods (in particular, do not expect there to be a conversion from `self` to `self.graded_algebra()`; this currently does not work for Clifford algebras). Similarly, when overriding *graded_algebra()*, make sure to accordingly redefine these three methods, unless their definitions below still apply to your case (this will happen whenever the basis of your *graded_algebra()* has the same indexing set as `self`, and the partition of this indexing set according to degree is the same as for `self`).

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: A.graded_algebra()
Graded Module of An example of a filtered module with basis:
 the free module on partitions over Integer Ring
```

**homogeneous_component** (*d*)

Return the d-th homogeneous component of self.

EXAMPLES:

```
sage: A = GradedModulesWithBasis(ZZ).example()
sage: A.homogeneous_component(4)
Degree 4 homogeneous component of An example of a graded module
 with basis: the free module on partitions over Integer Ring
```

**homogeneous_component_basis** (*d*)

Return a basis for the d-th homogeneous component of self.

EXAMPLES:

```
sage: A = GradedModulesWithBasis(ZZ).example()
sage: A.homogeneous_component_basis(4)
Lazy family (Term map from Partitions to An example of a graded module␣
↪with basis:
             the free module on partitions over Integer Ring(i))_{i in␣
↪Partitions of the integer 4}

sage: cat = GradedModulesWithBasis(ZZ)
sage: C = CombinatorialFreeModule(ZZ, ['a', 'b'], category=cat)
sage: C.degree_on_basis = lambda x: 1 if x == 'a' else 2
sage: C.homogeneous_component_basis(1)
Finite family {'a': B['a']}
sage: C.homogeneous_component_basis(2)
Finite family {'b': B['b']}
```

**induced_graded_map** (*other*, *f*)

Return the graded linear map between the associated graded modules of self and other canonically induced by the filtration-preserving map f : self -> other.

Let $A$ and $B$ be two filtered modules with basis, and let $(F_i)_{i\in I}$ and $(G_i)_{i\in I}$ be their filtrations. Let $f : A \to B$ be a linear map which preserves the filtration (i.e., satisfies $f(F_i) \subseteq G_i$ for all $i \in I$). Then, there is a canonically defined graded linear map $\operatorname{gr} f : \operatorname{gr} A \to \operatorname{gr} B$ which satisfies

$$(\operatorname{gr} f)(p_i(a)) = p_i(f(a)) \qquad \text{for all } i \in I \text{ and } a \in F_i,$$

where the $p_i$ on the left hand side is the canonical projection from $F_i$ onto the $i$-th graded component of $\operatorname{gr} A$, while the $p_i$ on the right hand side is the canonical projection from $G_i$ onto the $i$-th graded component of $\operatorname{gr} B$.

INPUT:

- other – a filtered algebra with basis
- f – a filtration-preserving linear map from self to other (can be given as a morphism or as a function)

OUTPUT:

The graded linear map $\operatorname{gr} f$.

EXAMPLES:

**Example 1.**

We start with the free $\mathbf{Q}$-module with basis the set of all partitions:

```
sage: A = Modules(QQ).WithBasis().Filtered().example(); A
An example of a filtered module with basis: the free module
 on partitions over Rational Field
sage: M = A.indices(); M
Partitions
sage: p1, p2, p21, p321 = [A.basis()[Partition(i)] for i in [[1], [2], [2,
↪1], [3,2,1]]]
```

Let us define a map from `A` to itself which acts on the basis by sending every partition $\lambda$ to the sum of the conjugates of all partitions $\mu$ for which $\lambda/\mu$ is a horizontal strip:

```
sage: def map_on_basis(lam):
....:     return A.sum_of_monomials([Partition(mu).conjugate() for k in␣
↪range(sum(lam) + 1)
....:                                for mu in lam.remove_horizontal_
↪border_strip(k)])
sage: f = A.module_morphism(on_basis=map_on_basis,
....:                       codomain=A)
sage: f(p1)
P[] + P[1]
sage: f(p2)
P[] + P[1] + P[1, 1]
sage: f(p21)
P[1] + P[1, 1] + P[2] + P[2, 1]
sage: f(p21 - p1)
-P[] + P[1, 1] + P[2] + P[2, 1]
sage: f(p321)
P[2, 1] + P[2, 1, 1] + P[2, 2] + P[2, 2, 1]
 + P[3, 1] + P[3, 1, 1] + P[3, 2] + P[3, 2, 1]
```

We now compute $\operatorname{gr} f$

```
sage: grA = A.graded_algebra(); grA
Graded Module of An example of a filtered module with basis:
 the free module on partitions over Rational Field
sage: pp1, pp2, pp21, pp321 = [A.to_graded_conversion()(i) for i in [p1,␣
↪p2, p21, p321]]
sage: pp2 + 4 * pp21
Bbar[[2]] + 4*Bbar[[2, 1]]
sage: grf = A.induced_graded_map(A, f); grf
Generic endomorphism of Graded Module of An example of a
 filtered module with basis:
 the free module on partitions over Rational Field
sage: grf(pp1)
Bbar[[1]]
sage: grf(pp2 + 4 * pp21)
Bbar[[1, 1]] + 4*Bbar[[2, 1]]
```

**Example 2.**

We shall now construct $\operatorname{gr} f$ for a different map $f$ out of the same `A`; the new map $f$ will lead into a graded algebra already, namely into the algebra of symmetric functions:

```
sage: h = SymmetricFunctions(QQ).h()
sage: def map_on_basis(lam):  # redefining map_on_basis
....:     return h.sum_of_monomials([Partition(mu).conjugate() for k in␣
↪range(sum(lam) + 1)
```

```
....:                                 for mu in lam.remove_horizontal_
↪border_strip(k)])
sage: f = A.module_morphism(on_basis=map_on_basis,
....:                         codomain=h)  # redefining f
sage: f(p1)
h[] + h[1]
sage: f(p2)
h[] + h[1] + h[1, 1]
sage: f(A.zero())
0
sage: f(p2 - 3*p1)
-2*h[] - 2*h[1] + h[1, 1]
```

The algebra h of symmetric functions in the $h$-basis is already graded, so its associated graded algebra is implemented as itself:

```
sage: grh = h.graded_algebra(); grh is h
True
sage: grf = A.induced_graded_map(h, f); grf
Generic morphism:
  From: Graded Module of An example of a filtered
   module with basis: the free module on partitions
   over Rational Field
  To:   Symmetric Functions over Rational Field
   in the homogeneous basis
sage: grf(pp1)
h[1]
sage: grf(pp2)
h[1, 1]
sage: grf(pp321)
h[3, 2, 1]
sage: grf(pp2 - 3*pp1)
-3*h[1] + h[1, 1]
sage: grf(pp21)
h[2, 1]
sage: grf(grA.zero())
0
```

**Example 3.**

After having had a graded module as the codomain, let us try to have one as the domain instead. Our new f will go from h to A:

```
sage: def map_on_basis(lam):  # redefining map_on_basis
....:     return A.sum_of_monomials([Partition(mu).conjugate() for k in␣
↪range(sum(lam) + 1)
....:                                 for mu in lam.remove_horizontal_
↪border_strip(k)])
sage: f = h.module_morphism(on_basis=map_on_basis,
....:                         codomain=A)  # redefining f
sage: f(h[1])
P[] + P[1]
sage: f(h[2])
P[] + P[1] + P[1, 1]
sage: f(h[1, 1])
P[1] + P[2]
sage: f(h[2, 2])
P[1, 1] + P[2, 1] + P[2, 2]
```

```
sage: f(h[3, 2, 1])
P[2, 1] + P[2, 1, 1] + P[2, 2] + P[2, 2, 1]
 + P[3, 1] + P[3, 1, 1] + P[3, 2] + P[3, 2, 1]
sage: f(h.one())
P[]
sage: grf = h.induced_graded_map(A, f); grf
Generic morphism:
  From: Symmetric Functions over Rational Field
   in the homogeneous basis
  To:   Graded Module of An example of a filtered
   module with basis: the free module on partitions
   over Rational Field
sage: grf(h[1])
Bbar[[1]]
sage: grf(h[2])
Bbar[[1, 1]]
sage: grf(h[1, 1])
Bbar[[2]]
sage: grf(h[2, 2])
Bbar[[2, 2]]
sage: grf(h[3, 2, 1])
Bbar[[3, 2, 1]]
sage: grf(h.one())
Bbar[[]]
```

**Example 4.**

The construct gr $f$ also makes sense when $f$ is a filtration-preserving map between graded modules.

```
sage: def map_on_basis(lam):  # redefining map_on_basis
....:     return h.sum_of_monomials([Partition(mu).conjugate() for k in␣
↪range(sum(lam) + 1)
....:                                for mu in lam.remove_horizontal_
↪border_strip(k)])
sage: f = h.module_morphism(on_basis=map_on_basis,
....:                       codomain=h)  # redefining f
sage: f(h[1])
h[] + h[1]
sage: f(h[2])
h[] + h[1] + h[1, 1]
sage: f(h[1, 1])
h[1] + h[2]
sage: f(h[2, 1])
h[1] + h[1, 1] + h[2] + h[2, 1]
sage: f(h.one())
h[]
sage: grf = h.induced_graded_map(h, f); grf
Generic endomorphism of Symmetric Functions over Rational
 Field in the homogeneous basis
sage: grf(h[1])
h[1]
sage: grf(h[2])
h[1, 1]
sage: grf(h[1, 1])
h[2]
sage: grf(h[2, 1])
h[2, 1]
sage: grf(h.one())
```

```
h[]
```

**projection**($i$)

Return the $i$-th projection $p_i : F_i \to G_i$ (in the notations of the class documentation AssociatedGradedAlgebra, where $A =$ "'$self$').

This method actually does not return the map $p_i$ itself, but an extension of $p_i$ to the whole $R$-module $A$. This extension is the composition of the $R$-module isomorphism $A \to \operatorname{gr} A$ with the canonical projection of the graded $R$-module $\operatorname{gr} A$ onto its $i$-th graded component $G_i$. The codomain of this map is $\operatorname{gr} A$, although its actual image is $G_i$. The map $p_i$ is obtained from this map by restricting its domain to $F_i$ and its image to $G_i$.

EXAMPLES:

```
sage: A = Modules(ZZ).WithBasis().Filtered().example()
sage: p = -2 * A.an_element(); p
-4*P[] - 4*P[1] - 6*P[2]
sage: q = A.projection(2)(p); q
-6*Bbar[[2]]
sage: q.parent() is A.graded_algebra()
True
sage: A.projection(3)(p)
0
```

**to_graded_conversion**()

Return the canonical $R$-module isomorphism $A \to \operatorname{gr} A$ induced by the basis of $A$ (where $A =$ "'$self$').

This is an isomorphism of $R$-modules. See the class documentation AssociatedGradedAlgebra.

**See also:**

*from_graded_conversion()*

EXAMPLES:

```
sage: A = Modules(QQ).WithBasis().Filtered().example()
sage: p = -2 * A.an_element(); p
-4*P[] - 4*P[1] - 6*P[2]
sage: q = A.to_graded_conversion()(p); q
-4*Bbar[[]] - 4*Bbar[[1]] - 6*Bbar[[2]]
sage: q.parent() is A.graded_algebra()
True
```

## 3.44 Finite Complex Reflection Groups

**class** sage.categories.finite_complex_reflection_groups.**FiniteComplexReflectionGroups**(*base_ca*

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite complex reflection groups.

See ComplexReflectionGroups for the definition of complex reflection group. In the finite case, most of the information about the group can be recovered from its *degrees* and *codegrees*, and to a lesser extent to the explicit realization as subgroup of $GL(V)$. Hence the most important optional methods to implement are:

- ComplexReflectionGroups.Finite.ParentMethods.degrees(),

- `ComplexReflectionGroups.Finite.ParentMethods.codegrees()`,
- `ComplexReflectionGroups.Finite.ElementMethods.to_matrix()`.

Finite complex reflection groups are completely classified. In particular, if the group is irreducible, then it's uniquely determined by its degrees and codegrees and whether it's reflection representation is *primitive* or not (see [LT2009] Chapter 2.1 for the definition of primitive).

**See also:**

Wikipedia article Complex_reflection_groups

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import ComplexReflectionGroups
sage: ComplexReflectionGroups().Finite()
Category of finite complex reflection groups
sage: ComplexReflectionGroups().Finite().super_categories()
[Category of complex reflection groups,
 Category of finite groups,
 Category of finite finitely generated semigroups]
```

An example of a finite reflection group:

```
sage: W = ComplexReflectionGroups().Finite().example(); W       # optional - gap3
Reducible real reflection group of rank 4 and type A2 x B2

sage: W.reflections()                               # optional - gap3
Finite family {1: (1,8)(2,5)(9,12), 2: (1,5)(2,9)(8,12),
               3: (3,10)(4,7)(11,14), 4: (3,6)(4,11)(10,13),
               5: (1,9)(2,8)(5,12), 6: (4,14)(6,13)(7,11),
               7: (3,13)(6,10)(7,14)}
```

`W` is in the category of complex reflection groups:

```
sage: W in ComplexReflectionGroups().Finite()       # optional - gap3
True
```

**class ElementMethods**

**character_value**()

Return the value at `self` of the character of the reflection representation given by *to_matrix()*.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3); W
1-colored permutations of size 3
sage: [t.character_value() for t in W]
[3, 1, 1, 0, 0, 1]
```

Note that this could be a different (faithful) representation than that given by the corresponding root system:

```
sage: W = ReflectionGroup((1,1,3)); W       # optional - gap3
Irreducible real reflection group of rank 2 and type A2
sage: [t.character_value() for t in W]      # optional - gap3
[2, 0, 0, -1, -1, 0]

sage: W = ColoredPermutations(2,2); W
```

```
2-colored permutations of size 2
sage: [t.character_value() for t in W]
[2, 0, 0, -2, 0, 0, 0, 0]

sage: W = ColoredPermutations(3,1); W
3-colored permutations of size 1
sage: [t.character_value() for t in W]
[1, zeta3, -zeta3 - 1]
```

**reflection_length**(*in_unitary_group=False*)

Return the reflection length of `self`.

This is the minimal numbers of reflections needed to obtain `self`.

INPUT:

- `in_unitary_group` – (default: `False`) if `True`, the reflection length is computed in the unitary group which is the dimension of the move space of `self`

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                        # optional - gap3
sage: sorted([t.reflection_length() for t in W])          # optional - gap3
[0, 1, 1, 1, 2, 2]

sage: W = ReflectionGroup((2,1,2))                        # optional - gap3
sage: sorted([t.reflection_length() for t in W])          # optional - gap3
[0, 1, 1, 1, 1, 2, 2, 2]

sage: W = ReflectionGroup((2,2,2))                        # optional - gap3
sage: sorted([t.reflection_length() for t in W])          # optional - gap3
[0, 1, 1, 2]

sage: W = ReflectionGroup((3,1,2))                        # optional - gap3
sage: sorted([t.reflection_length() for t in W])          # optional - gap3
[0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]
```

**to_matrix**()

Return the matrix presentation of `self` acting on a vector space $V$.

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))          # optional - gap3
sage: [t.to_matrix() for t in W]            # optional - gap3
[
[1 0]   [ 1  1]  [-1  0]  [-1 -1]  [ 0  1]  [ 0 -1]
[0 1], [ 0 -1], [ 1  1], [ 1  0], [-1 -1], [-1  0]
]

sage: W = ColoredPermutations(1,3)
sage: [t.to_matrix() for t in W]
[
[1 0 0]  [1 0 0]  [0 1 0]  [0 0 1]  [0 1 0]  [0 0 1]
[0 1 0]  [0 0 1]  [1 0 0]  [1 0 0]  [0 0 1]  [0 1 0]
[0 0 1], [0 1 0], [0 0 1], [0 1 0], [1 0 0], [1 0 0]
]
```

A different representation is given by the colored permutations:

```
sage: W = ColoredPermutations(3, 1)
sage: [t.to_matrix() for t in W]
[[1], [zeta3], [-zeta3 - 1]]
```

**class Irreducible**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class ParentMethods**

**absolute_poset**(*in_unitary_group=False*)

Return the poset induced by the absolute order of `self` as a finite lattice.

INPUT:

- `in_unitary_group` – (default: `False`) if `False`, the relation is given by `\sigma \leq \tau` if $l_R(\sigma) + l_R(\sigma^{-1}\tau) = l_R(\tau)$ If `True`, the relation is given by $\sigma \leq \tau$ if $\dim(\mathrm{Fix}(\sigma)) + \dim(\mathrm{Fix}(\sigma^{-1}\tau)) = \dim(\mathrm{Fix}(\tau))$

**See also:**

*noncrossing_partition_lattice()*

EXAMPLES:

```
sage: P = ReflectionGroup((1,1,3)).absolute_poset(); P      # optional␣
↪- gap3
Finite poset containing 6 elements

sage: sorted(w.reduced_word() for w in P)                    # optional␣
↪- gap3
[[], [1], [1, 2], [1, 2, 1], [2], [2, 1]]

sage: W = ReflectionGroup(4); W                              # optional␣
↪- gap3
Irreducible complex reflection group of rank 2 and type ST4
sage: W.absolute_poset()                                     # optional␣
↪- gap3
Finite poset containing 24 elements
```

**coxeter_number**()

Return the Coxeter number of an irreducible reflection group.

This is defined as $\frac{N+N^*}{n}$ where $N$ is the number of reflections, $N^*$ is the number of reflection hyperplanes, and $n$ is the rank of `self`.

EXAMPLES:

```
sage: W = ReflectionGroup(31)          # optional - gap3
sage: W.coxeter_number()               # optional - gap3
30
```

**elements_below_coxeter_element**(*c=None*)

Return all elements in `self` in the interval $[1, c]$ in the absolute order of `self`.

This order is defined by

$$\omega \leq_R \tau \Leftrightarrow \ell_R(\omega) + \ell_R(\omega^{-1}\tau) = \ell_R(\tau),$$

where $\ell_R$ denotes the reflection length.

**Note:** `self` is assumed to be well-generated.

INPUT:

- `c` – (default: `None`) if an element `c` is given, it is used as the maximal element in the interval; if a list is given, the union of the various maximal elements is computed

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                              # optional␣
↪- gap3

sage: sorted( w.reduced_word() for w in W.elements_below_coxeter_
↪element() )     # optional - gap3
[[], [1], [1, 2], [1, 2, 1], [2]]

sage: sorted( w.reduced_word() for w in W.elements_below_coxeter_
↪element(W.from_reduced_word([2,1])) )   # optional - gap3
[[], [1], [1, 2, 1], [2], [2, 1]]

sage: sorted( w.reduced_word() for w in W.elements_below_coxeter_
↪element(W.from_reduced_word([2])) )     # optional - gap3
[[], [2]]
```

**generalized_noncrossing_partitions**(*m*, *c=None*, *positive=False*)

Return the set of all chains of length `m` in the noncrossing partition lattice of `self`, see *noncrossing_partition_lattice()*.

**Note:** `self` is assumed to be well-generated.

INPUT:

- `c` – (default: `None`) if an element `c` in `self` is given, it is used as the maximal element in the interval
- `positive` – (default: `False`) if `True`, only those generalized noncrossing partitions of full support are returned

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                              # optional␣
↪- gap3

sage: sorted([w.reduced_word() for w in chain]                 # optional␣
↪- gap3
....:         for chain in W.generalized_noncrossing_partitions(2))   #␣
↪optional - gap3
[[[], [], [1, 2]],
 [[], [1], [2]],
 [[], [1, 2], []],
 [[], [1, 2, 1], [1]],
 [[], [2], [1, 2, 1]],
 [[1], [], [2]],
 [[1], [2], []],
 [[1, 2], [], []],
 [[1, 2, 1], [], [1]],
 [[1, 2, 1], [1], []],
 [[2], [], [1, 2, 1]],
 [[2], [1, 2, 1], []]]
```

```
sage: sorted([w.reduced_word() for w in chain]              # optional␣
↪- gap3
....:         for chain in W.generalized_noncrossing_partitions(2,␣
↪positive=True))    # optional - gap3
[[[], [1, 2], []],
 [[], [1, 2, 1], [1]],
 [[1], [2], []],
 [[1, 2], [], []],
 [[1, 2, 1], [], [1]],
 [[1, 2, 1], [1], []],
 [[2], [1, 2, 1], []]]
```

**noncrossing_partition_lattice**(*c=None*, *L=None*, *in_unitary_group=False*)

Return the interval $[1, c]$ in the absolute order of `self` as a finite lattice.

**See also:**

*elements_below_coxeter_element()*

INPUT:

- `c` – (default: `None`) if an element `c` in `self` is given, it is used as the maximal element in the interval
- `L` – (default: `None`) if a subset `L` (must be hashable!) of `self` is given, it is used as the underlying set (only cover relations are checked)
- `in_unitary_group` – (default: `False`) if `False`, the relation is given by $\sigma \leq \tau$ if $l_R(\sigma) + l_R(\sigma^{-1}\tau) = l_R(\tau)$; if `True`, the relation is given by $\sigma \leq \tau$ if $\dim(\mathrm{Fix}(\sigma)) + \dim(\mathrm{Fix}(\sigma^{-1}\tau)) = \dim(\mathrm{Fix}(\tau))$

EXAMPLES:

```
sage: W = WeylGroup(['G', 2])
sage: W.noncrossing_partition_lattice()
Finite lattice containing 8 elements

sage: W = ReflectionGroup((1,1,3))                          # optional␣
↪- gap3

sage: sorted( w.reduced_word() for w in W.noncrossing_partition_
↪lattice() ) # optional - gap3
[[], [1], [1, 2], [1, 2, 1], [2]]

sage: sorted( w.reduced_word() for w in W.noncrossing_partition_
↪lattice(W.from_reduced_word([2,1])) )    # optional - gap3
[[], [1], [1, 2, 1], [2], [2, 1]]

sage: sorted( w.reduced_word() for w in W.noncrossing_partition_
↪lattice(W.from_reduced_word([2])) ) # optional - gap3
[[], [2]]
```

**example**()

Return an example of an irreducible complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().Finite().Irreducible().example()     #␣
↪optional - gap3
```

```
Irreducible complex reflection group of rank 3 and type G(4,2,3)
```

**class ParentMethods**

**base_change_matrix**()

Return the base change from the standard basis of the vector space of `self` to the basis given by the independent roots of `self`.

---

**Todo:** For non-well-generated groups there is a conflict with construction of the matrix for an element.

---

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))                           # optional -␣
↪gap3
sage: W.base_change_matrix()                                 # optional -␣
↪gap3
[1 0]
[0 1]

sage: W = ReflectionGroup(23)                                # optional -␣
↪gap3
sage: W.base_change_matrix()                                 # optional -␣
↪gap3
[1 0 0]
[0 1 0]
[0 0 1]

sage: W = ReflectionGroup((3,1,2))                           # optional -␣
↪gap3
sage: W.base_change_matrix()                                 # optional -␣
↪gap3
[1 0]
[1 1]

sage: W = ReflectionGroup((4,2,2))                           # optional -␣
↪gap3
sage: W.base_change_matrix()                                 # optional -␣
↪gap3
[   1    0]
[E(4)    1]
```

**cardinality**()

Return the cardinality of `self`.

It is given by the product of the degrees of `self`.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: W.cardinality()
6
sage: W = ColoredPermutations(2,3)
sage: W.cardinality()
48
sage: W = ColoredPermutations(4,3)
```

```
sage: W.cardinality()
384
sage: W = ReflectionGroup((4,2,3))          # optional - gap3
sage: W.cardinality()                       # optional - gap3
192
```

**codegrees**()

Return the codegrees of `self`.

OUTPUT: a tuple of Sage integers

EXAMPLES:

```
sage: W = ColoredPermutations(1,4)
sage: W.codegrees()
(2, 1, 0)

sage: W = ColoredPermutations(3,3)
sage: W.codegrees()
(6, 3, 0)

sage: W = ReflectionGroup(31)               # optional - gap3
sage: W.codegrees()                         # optional - gap3
(28, 16, 12, 0)
```

**degrees**()

Return the degrees of `self`.

OUTPUT: a tuple of Sage integers

EXAMPLES:

```
sage: W = ColoredPermutations(1,4)
sage: W.degrees()
(2, 3, 4)

sage: W = ColoredPermutations(3,3)
sage: W.degrees()
(3, 6, 9)

sage: W = ReflectionGroup(31)               # optional - gap3
sage: W.degrees()                           # optional - gap3
(8, 12, 20, 24)
```

**is_real**()

Return whether `self` is real.

A complex reflection group is *real* if it is isomorphic to a reflection group in $GL(V)$ over a real vector space $V$. Equivalently its character table has real entries.

This implementation uses the following statement: an irreducible complex reflection group is real if and only if 2 is a degree of `self` with multiplicity one. Hence, in general we just need to compare the number of occurrences of 2 as degree of `self` and the number of irreducible components.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: W.is_real()
True
```

```
sage: W = ColoredPermutations(4,3)
sage: W.is_real()
False
```

**Todo:** Add an example of non real finite complex reflection group that is generated by order 2 reflections.

**is_well_generated**()

Return whether `self` is well-generated.

A finite complex reflection group is *well generated* if the number of its simple reflections coincides with its rank.

**See also:**

`ComplexReflectionGroups.Finite.WellGenerated()`

**Note:**

- All finite real reflection groups are well generated.
- The complex reflection groups of type $G(r, 1, n)$ and of type $G(r, r, n)$ are well generated.
- The complex reflection groups of type $G(r, p, n)$ with $1 < p < r$ are *not* well generated.
- The direct product of two well generated finite complex reflection group is still well generated.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: W.is_well_generated()
True

sage: W = ColoredPermutations(4,3)
sage: W.is_well_generated()
True

sage: W = ReflectionGroup((4,2,3))          # optional - gap3
sage: W.is_well_generated()                 # optional - gap3
False

sage: W = ReflectionGroup((4,4,3))          # optional - gap3
sage: W.is_well_generated()                 # optional - gap3
True
```

**number_of_reflection_hyperplanes**()

Return the number of reflection hyperplanes of `self`.

This is also the number of distinguished reflections. For real groups, this coincides with the number of reflections.

This implementation uses that it is given by the sum of the codegrees of `self` plus its rank.

**See also:**

*number_of_reflections()*

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: W.number_of_reflection_hyperplanes()
```

```
3
sage: W = ColoredPermutations(2,3)
sage: W.number_of_reflection_hyperplanes()
9
sage: W = ColoredPermutations(4,3)
sage: W.number_of_reflection_hyperplanes()
15
sage: W = ReflectionGroup((4,2,3))          # optional - gap3
sage: W.number_of_reflection_hyperplanes()  # optional - gap3
15
```

**number_of_reflections**()

Return the number of reflections of `self`.

For real groups, this coincides with the number of reflection hyperplanes.

This implementation uses that it is given by the sum of the degrees of `self` minus its rank.

**See also:**

*number_of_reflection_hyperplanes()*

EXAMPLES:

```
sage: [SymmetricGroup(i).number_of_reflections() for i in range(int(8))]
[0, 0, 1, 3, 6, 10, 15, 21]

sage: W = ColoredPermutations(1,3)
sage: W.number_of_reflections()
3
sage: W = ColoredPermutations(2,3)
sage: W.number_of_reflections()
9
sage: W = ColoredPermutations(4,3)
sage: W.number_of_reflections()
21
sage: W = ReflectionGroup((4,2,3))         # optional - gap3
sage: W.number_of_reflections()            # optional - gap3
15
```

**rank**()

Return the rank of `self`.

The rank of `self` is the dimension of the smallest faithfull reflection representation of `self`.

This default implementation uses that the rank is the number of *degrees()*.

**See also:**

`ComplexReflectionGroups.rank()`

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: W.rank()
2
sage: W = ColoredPermutations(2,3)
sage: W.rank()
3
sage: W = ColoredPermutations(4,3)
sage: W.rank()
```

```
3
sage: W = ReflectionGroup((4,2,3))          # optional - gap3
sage: W.rank()                              # optional - gap3
3
```

**class SubcategoryMethods**

**WellGenerated**()

Return the full subcategory of well-generated objects of `self`.

A finite complex generated group is *well generated* if it is isomorphic to a subgroup of the general linear group $GL_n$ generated by $n$ reflections.

**See also:**

`ComplexReflectionGroups.Finite.ParentMethods.is_well_generated()`

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: C = ComplexReflectionGroups().Finite().WellGenerated(); C
Category of well generated finite complex reflection groups
```

Here is an example of a finite well-generated complex reflection group:

```
sage: W = C.example(); W                 # optional - gap3
Reducible complex reflection group of rank 4 and type A2 x G(3,1,2)
```

All finite Coxeter groups are well generated:

```
sage: CoxeterGroups().Finite().is_subcategory(C)
True
sage: SymmetricGroup(3) in C
True
```

**Note:** The category of well generated finite complex reflection groups is currently implemented as an axiom. See discussion on trac ticket #11187. This may be a bit of overkill. Still it's nice to have a full subcategory.

**class WellGenerated**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class Irreducible**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite irreducible well-generated finite complex reflection groups.

**class ParentMethods**

**catalan_number**(*positive=False*, *polynomial=False*)

Return the Catalan number associated to `self`.

It is defined by

$$\prod_{i=1}^{n} \frac{d_i + h}{d_i},$$

where $d_1, \ldots, d_n$ are the degrees and where $h$ is the Coxeter number. See [Ar2006] for further information.

INPUT:

- `positive` – optional boolean (default `False`) if `True`, return instead the positive Catalan number
- `polynomial` – optional boolean (default `False`) if `True`, return instead the $q$-analogue as a polynomial in $q$

**Note:**

- For the symmetric group $S_n$, it reduces to the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.
- The Catalan numbers for $G(r, 1, n)$ all coincide for $r > 1$.

EXAMPLES:

```
sage: [ColoredPermutations(1,n).catalan_number() for n in [3,4,5]]
[5, 14, 42]

sage: [ColoredPermutations(2,n).catalan_number() for n in [3,4,5]]
[20, 70, 252]

sage: [ReflectionGroup((2,2,n)).catalan_number() for n in [3,4,5]]
↪# optional - gap3
[14, 50, 182]
```

**coxeter_number**()

Return the Coxeter number of a well-generated, irreducible reflection group. This is defined to be the order of a regular element in `self`, and is equal to the highest degree of `self`.

**See also:**

`ComplexReflectionGroups.Finite.Irreducible()`

**Note:** This method overwrites the more general method for complex reflection groups since the expression given here is quicker to compute.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: W.coxeter_number()
3

sage: W = ColoredPermutations(4,3)
sage: W.coxeter_number()
12

sage: W = ReflectionGroup((4,4,3))  # optional - gap3
sage: W.coxeter_number()            # optional - gap3
8
```

**fuss_catalan_number**(*m*, *positive=False*, *polynomial=False*)

Return the `m`-th Fuss-Catalan number associated to `self`.

This is defined by

$$\prod_{i=1}^{n} \frac{d_i + mh}{d_i},$$

where $d_1, \dots, d_n$ are the degrees and $h$ is the Coxeter number.

INPUT:

- `positive` – optional boolean (default `False`) if `True`, return instead the positive Fuss-Catalan number
- `polynomial` – optional boolean (default `False`) if `True`, return instead the $q$-analogue as a polynomial in $q$

See [Ar2006] for further information.

**Note:**

- For the symmetric group $S_n$, it reduces to the Fuss-Catalan number $\frac{1}{mn+1}\binom{(m+1)n}{n}$.
- The Fuss-Catalan numbers for $G(r, 1, n)$ all coincide for $r > 1$.

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]
[5, 12, 22]

sage: W = ColoredPermutations(1,4)
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]
[14, 55, 140]

sage: W = ColoredPermutations(1,5)
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]
[42, 273, 969]

sage: W = ColoredPermutations(2,2)
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]
[6, 15, 28]

sage: W = ColoredPermutations(2,3)
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]
[20, 84, 220]

sage: W = ColoredPermutations(2,4)
sage: [W.fuss_catalan_number(i) for i in [1,2,3]]
[70, 495, 1820]
```

**number_of_reflections_of_full_support**()

Return the number of reflections with full support.

EXAMPLES:

```
sage: W = ColoredPermutations(1,4)
sage: W.number_of_reflections_of_full_support()
1

sage: W = ColoredPermutations(3,3)
sage: W.number_of_reflections_of_full_support()
3
```

**rational_catalan_number**(*p*, *polynomial=False*)

Return the `p`-th rational Catalan number associated to `self`.

It is defined by

$$\prod_{i=1}^{n} \frac{p + (p(d_i - 1)) \mod h)}{d_i},$$

where $d_1, \ldots, d_n$ are the degrees and $h$ is the Coxeter number. See [STW2016] for this formula.

INPUT:

- `polynomial` – optional boolean (default `False`) if `True`, return instead the $q$-analogue as a polynomial in $q$

EXAMPLES:

```
sage: W = ColoredPermutations(1,3)
sage: [W.rational_catalan_number(p) for p in [5,7,8]]
[7, 12, 15]

sage: W = ColoredPermutations(2,2)
sage: [W.rational_catalan_number(p) for p in [7,9,11]]
[10, 15, 21]
```

**example()**

Return an example of an irreducible well-generated complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().Finite().WellGenerated().Irreducible().
↪example()
4-colored permutations of size 3
```

**class ParentMethods**

**coxeter_element()**

Return a Coxeter element.

The result is the product of the simple reflections, in some order.

**Note:** This implementation is shared with well generated complex reflection groups. It would be nicer to put it in some joint super category; however, in the current state of the art, there is none where it's clear that this is the right construction for obtaining a Coxeter element.

In this context, this is an element having a regular eigenvector (a vector not contained in any reflection hyperplane of `self`).

EXAMPLES:

```
sage: CoxeterGroup(['A', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['B', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['D', 4]).coxeter_element().reduced_word()
[1, 2, 4, 3]
sage: CoxeterGroup(['F', 4]).coxeter_element().reduced_word()
[1, 2, 3, 4]
sage: CoxeterGroup(['E', 8]).coxeter_element().reduced_word()
[1, 3, 2, 4, 5, 6, 7, 8]
```

```
sage: CoxeterGroup(['H', 3]).coxeter_element().reduced_word()
[1, 2, 3]
```

This method is also used for well generated finite complex reflection groups:

```
sage: W = ReflectionGroup((1,1,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3]

sage: W = ReflectionGroup((2,1,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3, 4]

sage: W = ReflectionGroup((4,1,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3, 4]

sage: W = ReflectionGroup((4,4,4))          # optional - gap3
sage: W.coxeter_element().reduced_word()    # optional - gap3
[1, 2, 3, 4]
```

**coxeter_elements**()

Return the (unique) conjugacy class in `self` containing all Coxeter elements.

A Coxeter element is an element that has an eigenvalue $e^{2\pi i/h}$ where $h$ is the Coxeter number.

In case of finite Coxeter groups, these are exactly the elements that are conjugate to one (or, equivalently, all) standard Coxeter element, this is, to an element that is the product of the simple generators in some order.

**See also:**

`standard_coxeter_elements()`

EXAMPLES:

```
sage: W = ReflectionGroup((1,1,3))      # optional - gap3
sage: sorted(c.reduced_word() for c in W.coxeter_elements())     #␣
↪optional - gap3
[[1, 2], [2, 1]]

sage: W = ReflectionGroup((1,1,4))      # optional - gap3
sage: sorted(c.reduced_word() for c in W.coxeter_elements())     #␣
↪optional - gap3
[[1, 2, 1, 3, 2], [1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 1, 3, 2, 1], [3,
↪ 2, 1]]
```

**is_well_generated**()

Return `True` as `self` is well-generated.

EXAMPLES:

```
sage: W = ReflectionGroup((3,1,2))      # optional - gap3
sage: W.is_well_generated()             # optional - gap3
True
```

**standard_coxeter_elements**()

Return all standard Coxeter elements in `self`.

This is the set of all elements in self obtained from any product of the simple reflections in `self`.

**Note:**
- self is assumed to be well-generated.
- This works even beyond real reflection groups, but the conjugacy class is not unique and we only obtain one such class.

EXAMPLES:

```
sage: W = ReflectionGroup(4)                       # optional - gap3
sage: sorted(W.standard_coxeter_elements())   # optional - gap3
[(1,7,6,12,23,20)(2,8,17,24,9,5)(3,16,10,19,15,21)(4,14,11,22,18,13),
 (1,10,4,12,21,22)(2,11,19,24,13,3)(5,15,7,17,16,23)(6,18,8,20,14,9)]
```

**example**()
Return an example of a well-generated complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().Finite().WellGenerated().example()   #␣
↪optional - gap3
Reducible complex reflection group of rank 4 and type A2 x G(3,1,2)
```

**example**()
Return an example of a complex reflection group.

EXAMPLES:

```
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups
sage: ComplexReflectionGroups().Finite().example()          # optional - gap3
Reducible real reflection group of rank 4 and type A2 x B2
```

## 3.45 Finite Coxeter Groups

**class** sage.categories.finite_coxeter_groups.**FiniteCoxeterGroups**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite Coxeter groups.

EXAMPLES:

```
sage: CoxeterGroups.Finite()
Category of finite coxeter groups
sage: FiniteCoxeterGroups().super_categories()
[Category of finite generalized coxeter groups,
 Category of coxeter groups]

sage: G = CoxeterGroups().Finite().example()
sage: G.cayley_graph(side = "right").plot()
Graphics object consisting of 40 graphics primitives
```

Here are some further examples:

```
sage: WeylGroups().Finite().example()
The symmetric group on {0, ..., 3}

sage: WeylGroup(["B", 3])
Weyl Group of type ['B', 3] (as a matrix group acting on the ambient space)
```

Those other examples will eventually be also in this category:

```
sage: SymmetricGroup(4)
Symmetric group of order 4! as a permutation group
sage: DihedralGroup(5)
Dihedral group of order 10 as a permutation group
```

**class ElementMethods**

**bruhat_upper_covers**()

Returns all the elements that cover `self` in Bruhat order.

EXAMPLES:

```
sage: W = WeylGroup(["A",4])
sage: w = W.from_reduced_word([3,2])
sage: print([v.reduced_word() for v in w.bruhat_upper_covers()])
[[4, 3, 2], [3, 4, 2], [2, 3, 2], [3, 1, 2], [3, 2, 1]]

sage: W = WeylGroup(["B",6])
sage: w = W.from_reduced_word([1,2,1,4,5])
sage: C = w.bruhat_upper_covers()
sage: len(C)
9
sage: print([v.reduced_word() for v in C])
[[6, 4, 5, 1, 2, 1], [4, 5, 6, 1, 2, 1], [3, 4, 5, 1, 2, 1], [2, 3, 4, 5,
↪1, 2],
[1, 2, 3, 4, 5, 1], [4, 5, 4, 1, 2, 1], [4, 5, 3, 1, 2, 1], [4, 5, 2, 3,
↪1, 2],
[4, 5, 1, 2, 3, 1]]
sage: ww = W.from_reduced_word([5,6,5])
sage: CC = ww.bruhat_upper_covers()
sage: print([v.reduced_word() for v in CC])
[[6, 5, 6, 5], [4, 5, 6, 5], [5, 6, 4, 5], [5, 6, 5, 4], [5, 6, 5, 3], [5,
↪ 6, 5, 2],
[5, 6, 5, 1]]
```

Recursive algorithm: write $w$ for `self`. If $i$ is a non-descent of $w$, then the covers of $w$ are exactly $\{ws_i, u_1s_i, u_2s_i, ..., u_js_i\}$, where the $u_k$ are those covers of $ws_i$ that have a descent at $i$.

**covered_reflections_subgroup**()

Return the subgroup of $W$ generated by the conjugates by $w$ of the simple reflections indexed by right descents of $w$.

This is used to compute the shard intersection order on $W$.

EXAMPLES:

```
sage: W = CoxeterGroup(['A',3], base_ring=ZZ)
sage: len(W.long_element().covered_reflections_subgroup())
24
sage: s = W.simple_reflection(1)
```

```
sage: Gs = s.covered_reflections_subgroup()
sage: len(Gs)
2
sage: s in [u.lift() for u in Gs]
True
sage: len(W.one().covered_reflections_subgroup())
1
```

**coxeter_knuth_graph**()

Return the Coxeter-Knuth graph of type $A$.

The Coxeter-Knuth graph of type $A$ is generated by the Coxeter-Knuth relations which are given by $aa+1a \sim a+1aa+1$, $abc \sim acb$ if $b<a<c$ and $abc \sim bac$ if $a<c<b$.

EXAMPLES:

```
sage: W = WeylGroup(['A',4], prefix='s')
sage: w = W.from_reduced_word([1,2,1,3,2])
sage: D = w.coxeter_knuth_graph()
sage: D.vertices()
[(1, 2, 1, 3, 2),
(1, 2, 3, 1, 2),
(2, 1, 2, 3, 2),
(2, 1, 3, 2, 3),
(2, 3, 1, 2, 3)]
sage: D.edges()
[((1, 2, 1, 3, 2), (1, 2, 3, 1, 2), None),
((1, 2, 1, 3, 2), (2, 1, 2, 3, 2), None),
((2, 1, 2, 3, 2), (2, 1, 3, 2, 3), None),
((2, 1, 3, 2, 3), (2, 3, 1, 2, 3), None)]

sage: w = W.from_reduced_word([1,3])
sage: D = w.coxeter_knuth_graph()
sage: D.vertices()
[(1, 3), (3, 1)]
sage: D.edges()
[]
```

**coxeter_knuth_neighbor**($w$)

Return the Coxeter-Knuth (oriented) neighbors of the reduced word $w$ of `self`.

INPUT:

- `w` – reduced word of `self`

The Coxeter-Knuth relations are given by $aa+1a \sim a+1aa+1$, $abc \sim acb$ if $b<a<c$ and $abc \sim bac$ if $a<c<b$. This method returns all neighbors of `w` under the Coxeter-Knuth relations oriented from left to right.

EXAMPLES:

```
sage: W = WeylGroup(['A',4], prefix='s')
sage: word = [1,2,1,3,2]
sage: w = W.from_reduced_word(word)
sage: w.coxeter_knuth_neighbor(word)
{(1, 2, 3, 1, 2), (2, 1, 2, 3, 2)}

sage: word = [1,2,1,3,2,4,3]
sage: w = W.from_reduced_word(word)
sage: w.coxeter_knuth_neighbor(word)
{(1, 2, 1, 3, 4, 2, 3), (1, 2, 3, 1, 2, 4, 3), (2, 1, 2, 3, 2, 4, 3)}
```

**is_coxeter_element**()

Return whether this is a Coxeter element.

This is, whether `self` has an eigenvalue $e^{2\pi i/h}$ where $h$ is the Coxeter number.

**See also:**

`coxeter_elements()`

EXAMPLES:

```
sage: W = CoxeterGroup(['A',2])
sage: c = prod(W.gens())
sage: c.is_coxeter_element()
True
sage: W.one().is_coxeter_element()
False

sage: W = WeylGroup(['G', 2])
sage: c = prod(W.gens())
sage: c.is_coxeter_element()
True
sage: W.one().is_coxeter_element()
False
```

**class ParentMethods**

Ambiguity resolution: the implementation of `some_elements` is preferable to that of *FiniteGroups*. The same holds for `__iter__`, although a breath first search would be more natural; at least this maintains backward compatibility after trac ticket #13589.

**bhz_poset**()

Return the Bergeron-Hohlweg-Zabrocki partial order on the Coxeter group.

This is a partial order on the elements of a finite Coxeter group $W$, which is distinct from the Bruhat order, the weak order and the shard intersection order. It was defined in [BHZ05].

This partial order is not a lattice, as there is no unique maximal element. It can be succintly defined as follows.

Let $u$ and $v$ be two elements of the Coxeter group $W$. Let $S(u)$ be the support of $u$. Then $u \leq v$ if and only if $v_{S(u)} = u$ (here $v = v^I v_I$ denotes the usual parabolic decomposition with respect to the standard parabolic subgroup $W_I$).

**See also:**

*bruhat_poset()*, *shard_poset()*, *weak_poset()*

EXAMPLES:

```
sage: W = CoxeterGroup(['A',3], base_ring=ZZ)
sage: P = W.bhz_poset(); P
Finite poset containing 24 elements
sage: P.relations_number()
103
sage: P.chain_polynomial()
34*q^4 + 90*q^3 + 79*q^2 + 24*q + 1
sage: len(P.maximal_elements())
13
```

REFERENCE:

**bruhat_poset** (*facade=False*)

Return the Bruhat poset of `self`.

**See also:**

*bhz_poset()*, *shard_poset()*, *weak_poset()*

EXAMPLES:

```
sage: W = WeylGroup(["A", 2])
sage: P = W.bruhat_poset()
sage: P
Finite poset containing 6 elements
sage: P.show()
```

Here are some typical operations on this poset:

```
sage: W = WeylGroup(["A", 3])
sage: P = W.bruhat_poset()
sage: u = W.from_reduced_word([3,1])
sage: v = W.from_reduced_word([3,2,1,2,3])
sage: P(u) <= P(v)
True
sage: len(P.interval(P(u), P(v)))
10
sage: P.is_join_semilattice()
False
```

By default, the elements of $P$ are aware that they belong to $P$:

```
sage: P.an_element().parent()
Finite poset containing 24 elements
```

If instead one wants the elements to be plain elements of the Coxeter group, one can use the `facade` option:

```
sage: P = W.bruhat_poset(facade = True)
sage: P.an_element().parent()
Weyl Group of type ['A', 3] (as a matrix group acting on the ambient␣
↪space)
```

**See also:**

`Poset()` for more on posets and facade posets.

---

**Todo:**

- Use the symmetric group in the examples (for nicer output), and print the edges for a stronger test.
- The constructed poset should be lazy, in order to handle large / infinite Coxeter groups.

---

**cambrian_lattice** (*c*, *on_roots=False*)

Return the $c$-Cambrian lattice on delta sequences.

See Arxiv 1503.00710 and Arxiv math/0611106.

Delta sequences are certain 2-colored minimal factorizations of `c` into reflections.

INPUT:

- `c` – a standard Coxeter element in `self` (as a tuple, or as an element of `self`)

- on_roots (optional, default False) – if on_roots is True, the lattice is realized on roots rather than on reflections. In order for this to work, the ElementMethod reflection_to_root must be available.

EXAMPLES:

```
sage: CoxeterGroup(["A", 2]).cambrian_lattice((1,2))
Finite lattice containing 5 elements

sage: CoxeterGroup(["B", 2]).cambrian_lattice((1,2))
Finite lattice containing 6 elements

sage: CoxeterGroup(["G", 2]).cambrian_lattice((1,2))
Finite lattice containing 8 elements
```

**codegrees**()

Return the codegrees of the Coxeter group.

These are just the degrees minus 2.

EXAMPLES:

```
sage: CoxeterGroup(['A', 4]).codegrees()
(0, 1, 2, 3)
sage: CoxeterGroup(['B', 4]).codegrees()
(0, 2, 4, 6)
sage: CoxeterGroup(['D', 4]).codegrees()
(0, 2, 2, 4)
sage: CoxeterGroup(['F', 4]).codegrees()
(0, 4, 6, 10)
sage: CoxeterGroup(['E', 8]).codegrees()
(0, 6, 10, 12, 16, 18, 22, 28)
sage: CoxeterGroup(['H', 3]).codegrees()
(0, 4, 8)

sage: WeylGroup([["A",3], ["A",3], ["B",2]]).codegrees()
(0, 1, 2, 0, 1, 2, 0, 2)
```

**degrees**()

Return the degrees of the Coxeter group.

The output is an increasing list of integers.

EXAMPLES:

```
sage: CoxeterGroup(['A', 4]).degrees()
(2, 3, 4, 5)
sage: CoxeterGroup(['B', 4]).degrees()
(2, 4, 6, 8)
sage: CoxeterGroup(['D', 4]).degrees()
(2, 4, 4, 6)
sage: CoxeterGroup(['F', 4]).degrees()
(2, 6, 8, 12)
sage: CoxeterGroup(['E', 8]).degrees()
(2, 8, 12, 14, 18, 20, 24, 30)
sage: CoxeterGroup(['H', 3]).degrees()
(2, 6, 10)

sage: WeylGroup([["A",3], ["A",3], ["B",2]]).degrees()
(2, 3, 4, 2, 3, 4, 2, 4)
```

**inversion_sequence**(*word*)

Return the inversion sequence corresponding to the word in indices of simple generators of self.

If word corresponds to $[w_0, w_1, ...w_k]$, the output is $[w_0, w_0w_1w_0, \ldots, w_0w_1\cdots w_k\cdots w_1w_0]$.

INPUT:

- word – a word in the indices of the simple generators of self.

EXAMPLES:

```
sage: CoxeterGroup(["A", 2]).inversion_sequence([1,2,1])
[
[-1  1]  [ 0 -1]  [ 1  0]
[ 0  1], [-1  0], [ 1 -1]
]

sage: [t.reduced_word() for t in CoxeterGroup(["A",3]).inversion_
↪sequence([2,1,3,2,1,3])]
[[2], [1, 2, 1], [2, 3, 2], [1, 2, 3, 2, 1], [3], [1]]
```

**is_real**()

Return True since self is a real reflection group.

EXAMPLES:

```
sage: CoxeterGroup(['F',4]).is_real()
True
sage: CoxeterGroup(['H',4]).is_real()
True
```

**long_element**(*index_set=None*, *as_word=False*)

Return the longest element of self, or of the parabolic subgroup corresponding to the given index_set.

INPUT:

- index_set – a subset (as a list or iterable) of the nodes of the Dynkin diagram; (default: all of them)
- as_word – boolean (default False). If True, then return instead a reduced decomposition of the longest element.

Should this method be called maximal_element? longest_element?

EXAMPLES:

```
sage: D10 = FiniteCoxeterGroups().example(10)
sage: D10.long_element()
(1, 2, 1, 2, 1, 2, 1, 2, 1, 2)
sage: D10.long_element([1])
(1,)
sage: D10.long_element([2])
(2,)
sage: D10.long_element([])
()

sage: D7 = FiniteCoxeterGroups().example(7)
sage: D7.long_element()
(1, 2, 1, 2, 1, 2, 1)
```

One can require instead a reduced word for w0:

```
sage: A3 = CoxeterGroup(['A', 3])
sage: A3.long_element(as_word=True)
[1, 2, 1, 3, 2, 1]
```

**m_cambrian_lattice** (*c*, *m=1*, *on_roots=False*)

Return the $m$-Cambrian lattice on $m$-delta sequences.

See Arxiv 1503.00710 and Arxiv math/0611106.

The $m$-delta sequences are certain $m$-colored minimal factorizations of $c$ into reflections.

INPUT:

- $c$ – a Coxeter element of `self` (as a tuple, or as an element of `self`)
- $m$ – a positive integer (optional, default 1)
- `on_roots` (optional, default `False`) – if `on_roots` is `True`, the lattice is realized on roots rather than on reflections. In order for this to work, the ElementMethod `reflection_to_root` must be available.

EXAMPLES:

```
sage: CoxeterGroup(["A",2]).m_cambrian_lattice((1,2))
Finite lattice containing 5 elements

sage: CoxeterGroup(["A",2]).m_cambrian_lattice((1,2),2)
Finite lattice containing 12 elements
```

**permutahedron** (*point=None*, *base_ring=None*)

Return the permutahedron of `self`,

This is the convex hull of the point `point` in the weight basis under the action of `self` on the underlying vector space $V$.

**See also:**

`permutahedron()`

INPUT:

- `point` – optional, a point given by its coordinates in the weight basis (default is $(1, 1, 1, \ldots)$)
- `base_ring` – optional, the base ring of the polytope

**Note:** The result is expressed in the root basis coordinates.

**Note:** If function is too slow, switching the base ring to `RDF` will almost certainly speed things up.

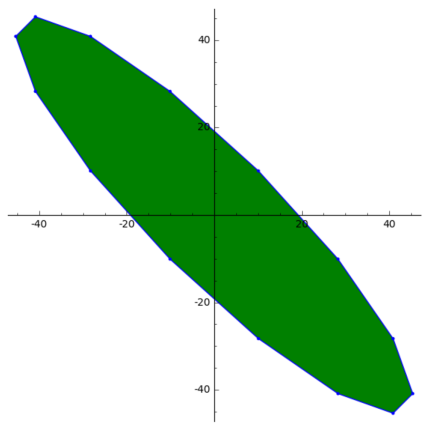
EXAMPLES:

```
sage: W = CoxeterGroup(['H',3], base_ring=RDF)
sage: W.permutahedron()
A 3-dimensional polyhedron in RDF^3 defined as the convex hull of 120␣
↪vertices

sage: W = CoxeterGroup(['I',7])
sage: W.permutahedron()
A 2-dimensional polyhedron in AA^2 defined as the convex hull of 14␣
↪vertices
sage: W.permutahedron(base_ring=RDF)
A 2-dimensional polyhedron in RDF^2 defined as the convex hull of 14␣
↪vertices
```

```
sage: W = ReflectionGroup(['A',3])                              # optional -
↪gap3
sage: W.permutahedron()                                         # optional -
↪gap3
A 3-dimensional polyhedron in QQ^3 defined as the convex hull
of 24 vertices

sage: W = ReflectionGroup(['A',3],['B',2])                      # optional -
↪gap3
sage: W.permutahedron()                                         # optional -
↪gap3
A 5-dimensional polyhedron in QQ^5 defined as the convex hull of 192
↪vertices
```




**reflections_from_w0**()

Return the reflections of `self` using the inversion set of `w_0`.

EXAMPLES:

```
sage: WeylGroup(['A',2]).reflections_from_w0()
[
[0 1 0]  [0 0 1]  [1 0 0]
[1 0 0]  [0 1 0]  [0 0 1]
[0 0 1], [1 0 0], [0 1 0]
]

sage: WeylGroup(['A',3]).reflections_from_w0()
[
[0 1 0 0]  [0 0 1 0]  [1 0 0 0]  [0 0 0 1]  [1 0 0 0]  [1 0 0 0]
[1 0 0 0]  [0 1 0 0]  [0 0 1 0]  [0 1 0 0]  [0 0 0 1]  [0 1 0 0]
[0 0 1 0]  [1 0 0 0]  [0 1 0 0]  [0 0 1 0]  [0 0 1 0]  [0 0 0 1]
[0 0 0 1], [0 0 0 1], [0 0 0 1], [1 0 0 0], [0 1 0 0], [0 0 1 0]
]
```

**shard_poset**(*side='right'*)

Return the shard intersection order attached to $W$.

This is a lattice structure on $W$, introduced in *[Reading]*. It contains the noncrossing partition lattice, as the induced lattice on the subset of $c$-sortable elements.

The partial order is given by simultaneous inclusion of inversion sets and subgroups attached to every element.

The precise description used here can be found in *[StThWi]*.

Another implementation for the symmetric groups is available as `shard_poset()`.

**See also:**

*bhz_poset()*, *bruhat_poset()*, *weak_poset()*

EXAMPLES:

```
sage: W = CoxeterGroup(['A',3], base_ring=ZZ)
sage: SH = W.shard_poset(); SH
Finite lattice containing 24 elements
sage: SH.is_graded()
True
sage: SH.characteristic_polynomial()
q^3 - 11*q^2 + 23*q - 13
sage: SH.f_polynomial()
34*q^3 + 22*q^2 + q
```

REFERENCES:

**w0**()

Return the longest element of `self`.

This attribute is deprecated, use *long_element()* instead.

EXAMPLES:

```
sage: D8 = FiniteCoxeterGroups().example(8)
sage: D8.w0
(1, 2, 1, 2, 1, 2, 1, 2)
sage: D3 = FiniteCoxeterGroups().example(3)
sage: D3.w0
(1, 2, 1)
```

**weak_lattice**(*side='right'*, *facade=False*)

INPUT:

- `side` – "left", "right", or "twosided" (default: "right")
- `facade` – a boolean (default: `False`)

Returns the left (resp. right) poset for weak order. In this poset, $u$ is smaller than $v$ if some reduced word of $u$ is a right (resp. left) factor of some reduced word of $v$.

**See also:**

*bhz_poset()*, *bruhat_poset()*, *shard_poset()*

EXAMPLES:

```
sage: W = WeylGroup(["A", 2])
sage: P = W.weak_poset()
sage: P
Finite lattice containing 6 elements
sage: P.show()
```

This poset is in fact a lattice:

```
sage: W = WeylGroup(["B", 3])
sage: P = W.weak_poset(side = "left")
```

```
sage: P.is_lattice()
True
```

so this method has an alias *weak_lattice()*:

```
sage: W.weak_lattice(side = "left") is W.weak_poset(side = "left")
True
```

As a bonus feature, one can create the left-right weak poset:

```
sage: W = WeylGroup(["A",2])
sage: P = W.weak_poset(side = "twosided")
sage: P.show()
sage: len(P.hasse_diagram().edges())
8
```

This is the transitive closure of the union of left and right order. In this poset, $u$ is smaller than $v$ if some reduced word of $u$ is a factor of some reduced word of $v$. Note that this is not a lattice:

```
sage: P.is_lattice()
False
```

By default, the elements of $P$ are aware of that they belong to $P$:

```
sage: P.an_element().parent()
Finite poset containing 6 elements
```

If instead one wants the elements to be plain elements of the Coxeter group, one can use the `facade` option:

```
sage: P = W.weak_poset(facade = True)
sage: P.an_element().parent()
Weyl Group of type ['A', 2] (as a matrix group acting on the ambient␣
↪space)
```

**See also:**

`Poset()` for more on posets and facade posets.

**Todo:**

- Use the symmetric group in the examples (for nicer output), and print the edges for a stronger test.
- The constructed poset should be lazy, in order to handle large / infinite Coxeter groups.

**weak_poset**(*side='right'*, *facade=False*)

INPUT:

- `side` – "left", "right", or "twosided" (default: "right")
- `facade` – a boolean (default: `False`)

Returns the left (resp. right) poset for weak order. In this poset, $u$ is smaller than $v$ if some reduced word of $u$ is a right (resp. left) factor of some reduced word of $v$.

**See also:**

*bhz_poset()*, *bruhat_poset()*, *shard_poset()*

EXAMPLES:

```
sage: W = WeylGroup(["A", 2])
sage: P = W.weak_poset()
sage: P
Finite lattice containing 6 elements
sage: P.show()
```

This poset is in fact a lattice:

```
sage: W = WeylGroup(["B", 3])
sage: P = W.weak_poset(side = "left")
sage: P.is_lattice()
True
```

so this method has an alias *weak_lattice()*:

```
sage: W.weak_lattice(side = "left") is W.weak_poset(side = "left")
True
```

As a bonus feature, one can create the left-right weak poset:

```
sage: W = WeylGroup(["A",2])
sage: P = W.weak_poset(side = "twosided")
sage: P.show()
sage: len(P.hasse_diagram().edges())
8
```

This is the transitive closure of the union of left and right order. In this poset, $u$ is smaller than $v$ if some reduced word of $u$ is a factor of some reduced word of $v$. Note that this is not a lattice:

```
sage: P.is_lattice()
False
```

By default, the elements of $P$ are aware of that they belong to $P$:

```
sage: P.an_element().parent()
Finite poset containing 6 elements
```

If instead one wants the elements to be plain elements of the Coxeter group, one can use the `facade` option:

```
sage: P = W.weak_poset(facade = True)
sage: P.an_element().parent()
Weyl Group of type ['A', 2] (as a matrix group acting on the ambient␣
↪space)
```

**See also:**

`Poset()` for more on posets and facade posets.

**Todo:**

- Use the symmetric group in the examples (for nicer output), and print the edges for a stronger test.
- The constructed poset should be lazy, in order to handle large / infinite Coxeter groups.

**extra_super_categories**()

EXAMPLES:

```
sage: CoxeterGroups().Finite().super_categories()
[Category of finite generalized coxeter groups,
 Category of coxeter groups]
```

## 3.46 Finite Crystals

**class** sage.categories.finite_crystals.**FiniteCrystals**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite crystals.

EXAMPLES:

```
sage: C = FiniteCrystals()
sage: C
Category of finite crystals
sage: C.super_categories()
[Category of crystals, Category of finite enumerated sets]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

**class TensorProducts**(*category*, **args*)
Bases: *sage.categories.tensor.TensorProductsCategory*

The category of finite crystals constructed by tensor product of finite crystals.

**extra_super_categories**()
EXAMPLES:

```
sage: FiniteCrystals().TensorProducts().extra_super_categories()
[Category of finite crystals]
```

**example**(*n=3*)
Returns an example of highest weight crystals, as per *Category.example()*.

EXAMPLES:

```
sage: B = FiniteCrystals().example(); B
Highest weight crystal of type A_3 of highest weight omega_1
```

**extra_super_categories**()
EXAMPLES:

```
sage: FiniteCrystals().extra_super_categories()
[Category of finite enumerated sets]
```

## 3.47 Finite dimensional algebras with basis

**Todo:** Quotients of polynomial rings.

Quotients in general.

Matrix rings.

REFERENCES:

- [CR1962]

**class** sage.categories.finite_dimensional_algebras_with_basis.**FiniteDimensionalAlgebrasWithB**

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of finite dimensional algebras with a distinguished basis.

EXAMPLES:

```
sage: C = FiniteDimensionalAlgebrasWithBasis(QQ); C
Category of finite dimensional algebras with basis over Rational Field
sage: C.super_categories()
[Category of algebras with basis over Rational Field,
 Category of finite dimensional modules with basis over Rational Field]
sage: C.example()
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
```

**class ElementMethods**

**on_left_matrix**(*base_ring=None*, *action=<built-in function mul>*, *side='left'*)

Return the matrix of the action of `self` on the algebra.

INPUT:

- `base_ring` – the base ring for the matrix to be constructed
- `action` – a bivariate function (default: `operator.mul()`)
- `side` – 'left' or 'right' (default: 'left')

EXAMPLES:

```
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: a = QS3([2,1,3])
sage: a.to_matrix(side='left')
[0 0 1 0 0 0]
[0 0 0 0 1 0]
[1 0 0 0 0 0]
[0 0 0 0 0 1]
[0 1 0 0 0 0]
[0 0 0 1 0 0]
sage: a.to_matrix(side='right')
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 0 0 0 1]
[0 0 0 0 1 0]
sage: a.to_matrix(base_ring=RDF, side="left")
[0.0 0.0 1.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 1.0 0.0]
[1.0 0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0 1.0]
[0.0 1.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 1.0 0.0 0.0]
```

AUTHORS: Mike Hansen, . . .

**to_matrix**(*base_ring=None*, *action=<built-in function mul>*, *side='left'*)

Return the matrix of the action of `self` on the algebra.

INPUT:

- base_ring – the base ring for the matrix to be constructed
- action – a bivariate function (default: operator.mul())
- side – 'left' or 'right' (default: 'left')

EXAMPLES:

```
sage: QS3 = SymmetricGroupAlgebra(QQ, 3)
sage: a = QS3([2,1,3])
sage: a.to_matrix(side='left')
[0 0 1 0 0 0]
[0 0 0 0 1 0]
[1 0 0 0 0 0]
[0 0 0 0 0 1]
[0 1 0 0 0 0]
[0 0 0 1 0 0]
sage: a.to_matrix(side='right')
[0 0 1 0 0 0]
[0 0 0 1 0 0]
[1 0 0 0 0 0]
[0 1 0 0 0 0]
[0 0 0 0 0 1]
[0 0 0 0 1 0]
sage: a.to_matrix(base_ring=RDF, side="left")
[0.0 0.0 1.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 1.0 0.0]
[1.0 0.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 0.0 0.0 1.0]
[0.0 1.0 0.0 0.0 0.0 0.0]
[0.0 0.0 0.0 1.0 0.0 0.0]
```

AUTHORS: Mike Hansen, ...

**class ParentMethods**

**cartan_invariants_matrix**()

Return the Cartan invariants matrix of the algebra.

OUTPUT: a matrix of non negative integers

Let $A$ be this finite dimensional algebra and $(S_i)_{i\in I}$ be representatives of the right simple modules of $A$. Note that their adjoints $S_i^*$ are representatives of the left simple modules.

Let $(P_i^L)_{i\in I}$ and $(P_i^R)_{i\in I}$ be respectively representatives of the corresponding indecomposable projective left and right modules of $A$. In particular, we assume that the indexing is consistent so that $S_i^* = \operatorname{top} P_i^L$ and $S_i = \operatorname{top} P_i^R$.

The *Cartan invariant matrix* $(C_{i,j})_{i,j\in I}$ is a matrix of non negative integers that encodes much of the representation theory of $A$; namely:

- $C_{i,j}$ counts how many times $S_i^* \otimes S_j$ appears as composition factor of $A$ seen as a bimodule over itself;
- $C_{i,j} = \dim Hom_A(P_j^R, P_i^R)$;
- $C_{i,j}$ counts how many times $S_j$ appears as composition factor of $P_i^R$;
- $C_{i,j} = \dim Hom_A(P_i^L, P_j^L)$;
- $C_{i,j}$ counts how many times $S_i^*$ appears as composition factor of $P_j^L$.

In the commutative case, the Cartan invariant matrix is diagonal. In the context of solving systems of multivariate polynomial equations of dimension zero, $A$ is the quotient of the polynomial ring by the ideal generated by the equations, the simple modules correspond to the roots, and the numbers $C_{i,i}$ give the multiplicities of those roots.

**Note:** For simplicity, the current implementation assumes that the index set $I$ is of the form $\{0, \ldots, n-1\}$. Better indexations will be possible in the future.

ALGORITHM:

The Cartan invariant matrix of $A$ is computed from the dimension of the summands of its Peirce decomposition.

**See also:**

- *peirce_decomposition()*
- *isotypic_projective_modules()*

EXAMPLES:

For a semisimple algebra, in particular for group algebras in characteristic zero, the Cartan invariants matrix is the identity:

```
sage: A3 = SymmetricGroup(3).algebra(QQ)
sage: A3.cartan_invariants_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

For the path algebra of a quiver, the Cartan invariants matrix counts the number of paths between two vertices:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example()
sage: A.cartan_invariants_matrix()
[1 2]
[0 1]
```

In the commutative case, the Cartan invariant matrix is diagonal:

```
sage: Z12 = Monoids().Finite().example(); Z12
An example of a finite multiplicative monoid: the integers modulo 12
sage: A = Z12.algebra(QQ)
sage: A.cartan_invariants_matrix()
[1 0 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0]
[0 0 2 0 0 0 0 0 0]
[0 0 0 1 0 0 0 0 0]
[0 0 0 0 2 0 0 0 0]
[0 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 2 0]
[0 0 0 0 0 0 0 0 1]
```

With the algebra of the 0-Hecke monoid:

```
sage: from sage.monoids.hecke_monoid import HeckeMonoid
sage: A = HeckeMonoid(SymmetricGroup(4)).algebra(QQ)
sage: A.cartan_invariants_matrix()
[1 0 0 0 0 0 0 0]
[0 2 1 0 1 1 0 0]
[0 1 1 0 1 0 0 0]
[0 0 0 1 0 1 1 0]
[0 1 1 0 1 0 0 0]
```

```
[0 1 0 1 0 2 1 0]
[0 0 0 1 0 1 1 0]
[0 0 0 0 0 0 0 1]
```

**center**()

Return the center of `self`.

**See also:**

*center_basis()*

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: center = A.center(); center
Center of An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: center in Algebras(QQ).WithBasis().FiniteDimensional().Commutative()
True
sage: center.dimension()
1
sage: center.basis()
Finite family {0: B[0]}
sage: center.ambient() is A
True
sage: [c.lift() for c in center.basis()]
[x + y]
```

The center of a semisimple algebra is semisimple:

```
sage: DihedralGroup(6).algebra(QQ).center() in Algebras(QQ).Semisimple()
True
```

**Todo:**

- Pickling by construction, as `A.center()`?
- Lazy evaluation of `_repr_`

**center_basis**()

Return a basis of the center of `self`.

OUTPUT:

- a list of elements of `self`.

**See also:**

*center()*

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.center_basis()
(x + y,)
```

```
```

**idempotent_lift**(*x*)

Lift an idempotent of the semisimple quotient into an idempotent of `self`.

Let $A$ be this finite dimensional algebra and $\pi$ be the projection $A \to \overline{A}$ on its semisimple quotient. Let $\overline{x}$ be an idempotent of $\overline{A}$, and $x$ any lift thereof in $A$. This returns an idempotent $e$ of $A$ such that $\pi(e) = \pi(x)$ and $e$ is a polynomial in $x$.

INPUT:

- $x$ – an element of $A$ that projects on an idempotent $\overline{x}$ of the semisimple quotient of $A$. Alternatively one may give as input the idempotent $\overline{x}$, in which case some lift thereof will be taken for $x$.

OUTPUT: the idempotent $e$ of `self`

ALGORITHM:

Iterate the formula $1 - (1 - x^2)^2$ until having an idempotent.

See [CR1962] for correctness and termination proofs.

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example()
sage: S = A.semisimple_quotient()
sage: A.idempotent_lift(S.basis()['x'])
x
sage: A.idempotent_lift(A.basis()['y'])
y
```

**Todo:** Add some non trivial example

**is_commutative**()

Return whether `self` is a commutative algebra.

EXAMPLES:

```
sage: S4 = SymmetricGroupAlgebra(QQ, 4)
sage: S4.is_commutative()
False
sage: S2 = SymmetricGroupAlgebra(QQ, 2)
sage: S2.is_commutative()
True
```

**is_identity_decomposition_into_orthogonal_idempotents**(*l*)

Return whether `l` is a decomposition of the identity into orthogonal idempotents.

INPUT:

- `l` – a list or iterable of elements of `self`

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field

sage: x,y,a,b = A.algebra_generators(); x,y,a,b
(x, y, a, b)
```

```
sage: A.is_identity_decomposition_into_orthogonal_idempotents([A.one()])
True
sage: A.is_identity_decomposition_into_orthogonal_idempotents([x,y])
True
sage: A.is_identity_decomposition_into_orthogonal_idempotents([x+a, y-a])
True
```

Here the idempotents do not sum up to 1:

```
sage: A.is_identity_decomposition_into_orthogonal_idempotents([x])
False
```

Here $1 + x$ and $-x$ are neither idempotent nor orthogonal:

```
sage: A.is_identity_decomposition_into_orthogonal_idempotents([1+x,-x])
False
```

With the algebra of the 0-Hecke monoid:

```
sage: from sage.monoids.hecke_monoid import HeckeMonoid
sage: A = HeckeMonoid(SymmetricGroup(4)).algebra(QQ)
sage: idempotents = A.orthogonal_idempotents_central_mod_radical()
sage: A.is_identity_decomposition_into_orthogonal_idempotents(idempotents)
True
```

Here are some more counterexamples:

1. Some orthogonal elements summing to 1 but not being idempotent:

```
sage: class PQAlgebra(CombinatorialFreeModule):
....:     def __init__(self, F, p):
....:         # Construct the quotient algebra F[x] / p,
....:         # where p is a univariate polynomial.
....:         R = parent(p); x = R.gen()
....:         I = R.ideal(p)
....:         self._xbar = R.quotient(I).gen()
....:         basis_keys = [self._xbar**i for i in range(p.degree())]
....:         CombinatorialFreeModule.__init__(self, F, basis_keys,
....:                 category=Algebras(F).FiniteDimensional().
↪WithBasis())
....:     def x(self):
....:         return self(self._xbar)
....:     def one(self):
....:         return self.basis()[self.base_ring().one()]
....:     def product_on_basis(self, w1, w2):
....:         return self.from_vector(vector(w1*w2))
sage: R.<x> = PolynomialRing(QQ)
sage: A = PQAlgebra(QQ, x**3 - x**2 + x + 1); y = A.x()
sage: a, b = y, 1-y
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, b))
False
```

For comparison:

```
sage: A = PQAlgebra(QQ, x**2 - x); y = A.x()
sage: a, b = y, 1-y
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, b))
True
```

```
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, A.
↪zero(), b))
True
sage: A = PQAlgebra(QQ, x**3 - x**2 + x - 1); y = A.x()
sage: a = (y**2 + 1) / 2
sage: b = 1 - a
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, b))
True
```

2. Some idempotents summing to 1 but not orthogonal:

```
sage: R.<x> = PolynomialRing(GF(2))
sage: A = PQAlgebra(GF(2), x)
sage: a = A.one()
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a,))
True
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a, a,␣
↪a))
False
```

3. Some orthogonal idempotents not summing to the identity:

```
sage: A.is_identity_decomposition_into_orthogonal_idempotents((a,a))
False
sage: A.is_identity_decomposition_into_orthogonal_idempotents(())
False
```

**isotypic_projective_modules**(*side='left'*)

Return the isotypic projective `side` `self`-modules.

Let $P_i$ be representatives of the indecomposable projective `side`-modules of this finite dimensional algebra $A$, and $S_i$ be the associated simple modules.

The regular `side` representation of $A$ can be decomposed as a direct sum $A = \bigoplus_i Q_i$ where each $Q_i$ is an isotypic projective module; namely $Q_i$ is the direct sum of $\dim S_i$ copies of the indecomposable projective module $P_i$. This decomposition is not unique.

The isotypic projective modules are constructed as $Q_i = e_i A$, where the $(e_i)_i$ is the decomposition of the identity into orthogonal idempotents obtained by lifting the central orthogonal idempotents of the semisimple quotient of $A$.

INPUT:

- `side` – 'left' or 'right' (default: 'left')

OUTPUT: a list of subspaces of `self`.

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: Q = A.isotypic_projective_modules(side="left"); Q
[Free module generated by {0} over Rational Field,
 Free module generated by {0, 1, 2} over Rational Field]
sage: [[x.lift() for x in Qi.basis()]
....:  for Qi in Q]
[[x],
 [y, a, b]]
```

We check that the sum of the dimensions of the isotypic projective modules is the dimension of `self`:

```
sage: sum([Qi.dimension() for Qi in Q]) == A.dimension()
True
```

**See also:**

- *orthogonal_idempotents_central_mod_radical()*
- *peirce_decomposition()*

**orthogonal_idempotents_central_mod_radical()**

Return a family of orthogonal idempotents of `self` that project on the central orthogonal idempotents of the semisimple quotient.

OUTPUT:

- a list of orthogonal idempotents obtained by lifting the central orthogonal idempotents of the semisimple quotient.

ALGORITHM:

The orthogonal idempotents of $A$ are obtained by lifting the central orthogonal idempotents of the semisimple quotient $\overline{A}$.

Namely, let $(\overline{f_i})$ be the central orthogonal idempotents of the semisimple quotient of $A$. We recursively construct orthogonal idempotents of $A$ by the following procedure: assuming $(f_i)_{i<n}$ is a set of already constructed orthogonal idempotent, we construct $f_k$ by idempotent lifting of $(1-f)g(1-f)$, where $g$ is any lift of $\overline{e_k}$ and $f = \sum_{i<k} f_i$.

See [CR1962] for correctness and termination proofs.

**See also:**

- `Algebras.SemiSimple.FiniteDimensional.WithBasis.ParentMethods.central_orthogonal_idempotents()`
- *idempotent_lift()*

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.orthogonal_idempotents_central_mod_radical()
(x, y)
```

```
sage: Z12 = Monoids().Finite().example(); Z12
An example of a finite multiplicative monoid: the integers modulo 12
sage: A = Z12.algebra(QQ)
sage: idempotents = A.orthogonal_idempotents_central_mod_radical()
sage: sorted(idempotents, key=str)
[-1/2*B[8] + 1/2*B[4],
 -B[0] + 1/2*B[8] + 1/2*B[4],
 -B[0] + 1/2*B[9] + 1/2*B[3],
 1/2*B[9] - 1/2*B[3],
 1/4*B[1] + 1/2*B[3] + 1/4*B[5] - 1/4*B[7] - 1/2*B[9] - 1/4*B[11],
 1/4*B[1] + 1/4*B[11] - 1/4*B[5] - 1/4*B[7],
 1/4*B[1] - 1/2*B[4] - 1/4*B[5] + 1/4*B[7] + 1/2*B[8] - 1/4*B[11],
 B[0],
 B[0] + 1/4*B[1] - 1/2*B[3] - 1/2*B[4] + 1/4*B[5] + 1/4*B[7] - 1/2*B[8] -
↪1/2*B[9] + 1/4*B[11]]
sage: sum(idempotents) == 1
True
sage: all(e*e == e for e in idempotents)
```

```
True
sage: all(e*f == 0 and f*e == 0 for e in idempotents for f in idempotents␣
↪if e != f)
True
```

This is best tested with:

```
sage: A.is_identity_decomposition_into_orthogonal_idempotents(idempotents)
True
```

We construct orthogonal idempotents for the algebra of the 0-Hecke monoid:

```
sage: from sage.monoids.hecke_monoid import HeckeMonoid
sage: A = HeckeMonoid(SymmetricGroup(4)).algebra(QQ)
sage: idempotents = A.orthogonal_idempotents_central_mod_radical()
sage: A.is_identity_decomposition_into_orthogonal_idempotents(idempotents)
True
```

**peirce_decomposition**(*idempotents=None*, *check=True*)

Return a Peirce decomposition of `self`.

Let $(e_i)_i$ be a collection of orthogonal idempotents of $A$ with sum $1$. The *Peirce decomposition* of $A$ is the decomposition of $A$ into the direct sum of the subspaces $e_i A e_j$.

With the default collection of orthogonal idempotents, one has

$$\dim e_i A e_j = C_{i,j} \dim S_i \dim S_j$$

where $(S_i)_i$ are the simple modules of $A$ and $(C_{i,j})_{i,j}$ is the Cartan invariants matrix.

INPUT:

- `idempotents` – a list of orthogonal idempotents $(e_i)_{i=0,\dots,n}$ of the algebra that sum to $1$ (default: the idempotents returned by *orthogonal_idempotents_central_mod_radical()*)
- `check` – (default: `True`) whether to check that the idempotents are indeed orthogonal and idempotent and sum to 1

OUTPUT:

A list of lists $l$ such that `l[i][j]` is the subspace $e_i A e_j$.

**See also:**

- *orthogonal_idempotents_central_mod_radical()*
- *cartan_invariants_matrix()*

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.orthogonal_idempotents_central_mod_radical()
(x, y)
sage: decomposition = A.peirce_decomposition(); decomposition
[[Free module generated by {0} over Rational Field,
  Free module generated by {0, 1} over Rational Field],
 [Free module generated by {} over Rational Field,
  Free module generated by {0} over Rational Field]]
sage: [ [[x.lift() for x in decomposition[i][j].basis()]
```

```
....:     for j in range(2)]
....:    for i in range(2)]
[[[x], [a, b]],
 [[], [y]]]
```

We recover that the group algebra of the symmetric group $S_4$ is a block matrix algebra:

```
sage: A = SymmetricGroup(4).algebra(QQ)
sage: decomposition = A.peirce_decomposition()    # long time
sage: [[decomposition[i][j].dimension()           # long time (4s)
....:    for j in range(len(decomposition))]
....:   for i in range(len(decomposition))]
[[1, 0, 0, 0, 0],
 [0, 9, 0, 0, 0],
 [0, 0, 4, 0, 0],
 [0, 0, 0, 9, 0],
 [0, 0, 0, 0, 1]]
```

The dimension of each block is $d^2$, where $d$ is the dimension of the corresponding simple module of $S_4$. The latter are given by:

```
sage: [p.standard_tableaux().cardinality() for p in Partitions(4)]
[1, 3, 2, 3, 1]
```

**peirce_summand** (*ei, ej*)

Return the Peirce decomposition summand $e_i A e_j$.

INPUT:

- `self` – an algebra $A$
- `ei`, `ej` – two idempotents of $A$

OUTPUT: $e_i A e_j$, as a subspace of $A$.

**See also:**

- *peirce_decomposition()*
- *principal_ideal()*

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example()
sage: idemp = A.orthogonal_idempotents_central_mod_radical()
sage: A.peirce_summand(idemp[0], idemp[1])
Free module generated by {0, 1} over Rational Field
sage: A.peirce_summand(idemp[1], idemp[0])
Free module generated by {} over Rational Field
```

We recover the $2 \times 2$ block of $\mathbf{Q}[S_4]$ corresponding to the unique simple module of dimension 2 of the symmetric group $S_4$:

```
sage: A4 = SymmetricGroup(4).algebra(QQ)
sage: e = A4.central_orthogonal_idempotents()[2]
sage: A4.peirce_summand(e, e)
Free module generated by {0, 1, 2, 3} over Rational Field
```

**principal_ideal** (*a, side='left'*)

Construct the `side` principal ideal generated by `a`.

EXAMPLES:

In order to highlight the difference between left and right principal ideals, our first example deals with a non commutative algebra:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: x, y, a, b = A.basis()
```

In this algebra, multiplication on the right by $x$ annihilates all basis elements but $x$:

```
sage: x*x, y*x, a*x, b*x
(x, 0, 0, 0)
```

so the left ideal generated by $x$ is one-dimensional:

```
sage: Ax = A.principal_ideal(x, side='left'); Ax
Free module generated by {0} over Rational Field
sage: [B.lift() for B in Ax.basis()]
[x]
```

Multiplication on the left by $x$ annihilates only $x$ and fixes the other basis elements:

```
sage: x*x, x*y, x*a, x*b
(x, 0, a, b)
```

so the right ideal generated by $x$ is 3-dimensional:

```
sage: xA = A.principal_ideal(x, side='right'); xA
Free module generated by {0, 1, 2} over Rational Field
sage: [B.lift() for B in xA.basis()]
[x, a, b]
```

**See also:**

- *peirce_summand()*

**radical**()

Return the Jacobson radical of `self`.

This uses *radical_basis()*, whose default implementation handles algebras over fields of characteristic zero or fields of characteristic $p$ in which we can compute $x^{1/p}$.

**See also:**

*radical_basis()*, *semisimple_quotient()*

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: radical = A.radical(); radical
Radical of An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
```

The radical is an ideal of $A$, and thus a finite dimensional non unital associative algebra:

```
sage: from sage.categories.associative_algebras import AssociativeAlgebras
sage: radical in AssociativeAlgebras(QQ).WithBasis().FiniteDimensional()
True
sage: radical in Algebras(QQ)
False

sage: radical.dimension()
2
sage: radical.basis()
Finite family {0: B[0], 1: B[1]}
sage: radical.ambient() is A
True
sage: [c.lift() for c in radical.basis()]
[a, b]
```

**Todo:**

- Tell Sage that the radical is in fact an ideal;
- Pickling by construction, as `A.center()`;
- Lazy evaluation of `_repr_`.

**radical_basis**()

Return a basis of the Jacobson radical of this algebra.

**Note:** This implementation handles algebras over fields of characteristic zero (using Dixon's lemma) or fields of characteristic $p$ in which we can compute $x^{1/p}$ [FR1985], [Eb1989].

OUTPUT:

- a list of elements of `self`.

**See also:**

*radical()*, *Algebras.Semisimple*

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.radical_basis()
(a, b)
```

We construct the group algebra of the Klein Four-Group over the rationals:

```
sage: A = KleinFourGroup().algebra(QQ)
```

This algebra belongs to the category of finite dimensional algebras over the rationals:

```
sage: A in Algebras(QQ).FiniteDimensional().WithBasis()
True
```

Since the field has characteristic $0$, Maschke's Theorem tells us that the group algebra is semisimple. So its radical is the zero ideal:

```
sage: A in Algebras(QQ).Semisimple()
True
```

```
sage: A.radical_basis()
()
```

Let's work instead over a field of characteristic 2:

```
sage: A = KleinFourGroup().algebra(GF(2))
sage: A in Algebras(GF(2)).Semisimple()
False
sage: A.radical_basis()
(() + (1,2)(3,4), (3,4) + (1,2)(3,4), (1,2) + (1,2)(3,4))
```

We now implement the algebra $A = K[x]/(x^p - 1)$, where $K$ is a finite field of characteristic $p$, and check its radical; alas, we currently need to wrap $A$ to make it a proper *ModulesWithBasis*:

```
sage: class AnAlgebra(CombinatorialFreeModule):
....:     def __init__(self, F):
....:         R.<x> = PolynomialRing(F)
....:         I = R.ideal(x**F.characteristic()-F.one())
....:         self._xbar = R.quotient(I).gen()
....:         basis_keys = [self._xbar**i for i in range(F.
↪characteristic())]
....:         CombinatorialFreeModule.__init__(self, F, basis_keys,
....:                 category=Algebras(F).FiniteDimensional().
↪WithBasis())
....:     def one(self):
....:         return self.basis()[self.base_ring().one()]
....:     def product_on_basis(self, w1, w2):
....:         return self.from_vector(vector(w1*w2))
sage: AnAlgebra(GF(3)).radical_basis()
(B[1] + 2*B[xbar^2], B[xbar] + 2*B[xbar^2])
sage: AnAlgebra(GF(16,'a')).radical_basis()
(B[1] + B[xbar],)
sage: AnAlgebra(GF(49,'a')).radical_basis()
(B[1] + 6*B[xbar^6], B[xbar] + 6*B[xbar^6], B[xbar^2] + 6*B[xbar^6],
 B[xbar^3] + 6*B[xbar^6], B[xbar^4] + 6*B[xbar^6], B[xbar^5] + 6*B[xbar^
↪6])
```

**semisimple_quotient**()

Return the semisimple quotient of `self`.

This is the quotient of `self` by its radical.

**See also:**

*radical()*

EXAMPLES:

```
sage: A = Algebras(QQ).FiniteDimensional().WithBasis().example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: a,b,x,y = sorted(A.basis())
sage: S = A.semisimple_quotient(); S
Semisimple quotient of An example of a finite dimensional algebra with␣
↪basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: S in Algebras(QQ).Semisimple()
```

```
True
sage: S.basis()
Finite family {'y': B['y'], 'x': B['x']}
sage: xs,ys = sorted(S.basis())
sage: (xs + ys) * xs
B['x']
```

Sanity check: the semisimple quotient of the $n$-th descent algebra of the symmetric group is of dimension the number of partitions of $n$:

```
sage: [ DescentAlgebra(QQ,n).B().semisimple_quotient().dimension()
....:   for n in range(6) ]
[1, 1, 2, 3, 5, 7]
sage: [Partitions(n).cardinality() for n in range(10)]
[1, 1, 2, 3, 5, 7, 11, 15, 22, 30]
```

**Todo:**

- Pickling by construction, as `A.semisimple_quotient()`?
- Lazy evaluation of `_repr_`

## 3.48 Finite dimensional bialgebras with basis

sage.categories.finite_dimensional_bialgebras_with_basis.**FiniteDimensionalBialgebrasWithBas**

The category of finite dimensional bialgebras with a distinguished basis

EXAMPLES:

```
sage: C = FiniteDimensionalBialgebrasWithBasis(QQ); C
Category of finite dimensional bialgebras with basis over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of bialgebras with basis over Rational Field,
 Category of finite dimensional algebras with basis over Rational Field]
sage: C is Bialgebras(QQ).WithBasis().FiniteDimensional()
True
```

## 3.49 Finite dimensional coalgebras with basis

sage.categories.finite_dimensional_coalgebras_with_basis.**FiniteDimensionalCoalgebrasWithBas**

The category of finite dimensional coalgebras with a distinguished basis

EXAMPLES:

```
sage: C = FiniteDimensionalCoalgebrasWithBasis(QQ); C
Category of finite dimensional coalgebras with basis over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of coalgebras with basis over Rational Field,
 Category of finite dimensional modules with basis over Rational Field]
sage: C is Coalgebras(QQ).WithBasis().FiniteDimensional()
True
```

## 3.50 Finite dimensional Hopf algebras with basis

**class** sage.categories.finite_dimensional_hopf_algebras_with_basis.**FiniteDimensionalHopfAlge**

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of finite dimensional Hopf algebras with a distinguished basis.

EXAMPLES:

```
sage: FiniteDimensionalHopfAlgebrasWithBasis(QQ) # fixme: Hopf should be␣
↪capitalized
Category of finite dimensional hopf algebras with basis over Rational Field
sage: FiniteDimensionalHopfAlgebrasWithBasis(QQ).super_categories()
[Category of hopf algebras with basis over Rational Field,
 Category of finite dimensional algebras with basis over Rational Field]
```

**class ElementMethods**

**class ParentMethods**

## 3.51 Finite Dimensional Lie Algebras With Basis

AUTHORS:

- Travis Scrimshaw (07-15-2013): Initial implementation

**class** sage.categories.finite_dimensional_lie_algebras_with_basis.**FiniteDimensionalLieAlgebr**

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

Category of finite dimensional Lie algebras with a basis.

**Todo:** Many of these tests should use non-abelian Lie algebras and need to be added after trac ticket #16820.

**class ElementMethods**

**adjoint_matrix**()

Return the matrix of the adjoint action of self.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.an_element().adjoint_matrix()
[0 0 0]
[0 0 0]
[0 0 0]
```

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'):{'x':1}})
sage: x.adjoint_matrix()
[0 0]
[1 0]
sage: y.adjoint_matrix()
[-1  0]
[ 0  0]
```

**to_vector**()

Return the vector in g.module() corresponding to the element self of g (where g is the parent of self).

Implement this if you implement g.module(). See sage.categories.lie_algebras.LieAlgebras.module() for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.an_element().to_vector()
(0, 0, 0)

sage: D = DescentAlgebra(QQ, 4).D()
sage: L = LieAlgebra(associative=D)
sage: L.an_element().to_vector()
(1, 1, 1, 1, 1, 1, 1, 1)
```

**class ParentMethods**

**as_finite_dimensional_algebra**()

Return self as a FiniteDimensionalAlgebra.

EXAMPLES:

```
sage: L = lie_algebras.cross_product(QQ)
sage: x,y,z = L.basis()
sage: F = L.as_finite_dimensional_algebra()
sage: X,Y,Z = F.basis()
sage: x.bracket(y)
Z
sage: X * Y
Z
```

**center**()

Return the center of self.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: Z = L.center(); Z
An example of a finite dimensional Lie algebra with basis: the
 3-dimensional abelian Lie algebra over Rational Field
sage: Z.basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

**centralizer**(*S*)

Return the centralizer of S in self.

INPUT:

- S – a subalgebra of self or a list of elements that represent generators for a subalgebra

**See also:**

*centralizer_basis()*

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a,b,c = L.lie_algebra_generators()
sage: S = L.centralizer([a + b, 2*a + c]); S
An example of a finite dimensional Lie algebra with basis:
 the 3-dimensional abelian Lie algebra over Rational Field
sage: S.basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

**centralizer_basis**($S$)

Return a basis of the centralizer of `S` in `self`.

INPUT:

- `S` – a subalgebra of `self` or a list of elements that represent generators for a subalgebra

**See also:**

*centralizer()*

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a,b,c = L.lie_algebra_generators()
sage: L.centralizer_basis([a + b, 2*a + c])
[(1, 0, 0), (0, 1, 0), (0, 0, 1)]

sage: H = lie_algebras.Heisenberg(QQ, 2)
sage: H.centralizer_basis(H)
[z]


sage: D = DescentAlgebra(QQ, 4).D()
sage: L = LieAlgebra(associative=D)
sage: L.centralizer_basis(L)
[D{},
 D{1} + D{1, 2} + D{2, 3} + D{3},
 D{1, 2, 3} + D{1, 3} + D{2}]
sage: D.center_basis()
(D{},
 D{1} + D{1, 2} + D{2, 3} + D{3},
 D{1, 2, 3} + D{1, 3} + D{2})
```

**derived_series**()

Return the derived series $(\mathfrak{g}^{(i)})_i$ of `self` where the rightmost $\mathfrak{g}^{(k)} = \mathfrak{g}^{(k+1)} = \cdots$.

We define the derived series of a Lie algebra $\mathfrak{g}$ recursively by $\mathfrak{g}^{(0)} := \mathfrak{g}$ and

$$\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$$

and recall that $\mathfrak{g}^{(k)} \supseteq \mathfrak{g}^{(k+1)}$. Alternatively we can express this as

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq \bigl[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]\bigr] \supseteq \Bigl[\bigl[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]\bigr], \bigl[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]\bigr]\Bigr] \supseteq \cdots .$$

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.derived_series()
(An example of a finite dimensional Lie algebra with basis:
```

```
    the 3-dimensional abelian Lie algebra over Rational Field,
 An example of a finite dimensional Lie algebra with basis:
    the 0-dimensional abelian Lie algebra over Rational Field
    with basis matrix:
    [])
```

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'):{'x':1}})
sage: L.derived_series() # todo: not implemented - #17416
(Lie algebra on 2 generators (x, y) over Rational Field,
 Subalgebra generated of Lie algebra on 2 generators (x, y) over Rational␣
↪Field with basis:
(x,),
 Subalgebra generated of Lie algebra on 2 generators (x, y) over Rational␣
↪Field with basis:
())
```

**derived_subalgebra**()

Return the derived subalgebra of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.derived_subalgebra()
An example of a finite dimensional Lie algebra with basis:
 the 0-dimensional abelian Lie algebra over Rational Field
 with basis matrix:
[]
```

**from_vector**(*v*)

Return the element of `self` corresponding to the vector `v` in `self.module()`.

Implement this if you implement *module()*; see the documentation of `sage.categories.lie_algebras.LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: u = L.from_vector(vector(QQ, (1, 0, 0))); u
(1, 0, 0)
sage: parent(u) is L
True
```

**is_abelian**()

Return if `self` is an abelian Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_abelian()
True
```

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'): {'x':1}})
sage: L.is_abelian()
False
```

**is_ideal**(*A*)

Return if `self` is an ideal of `A`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: I = L.ideal([2*a - c, b + c])
sage: I.is_ideal(L)
True

sage: L.<x,y> = LieAlgebra(QQ, {('x','y'):{'x':1}})
sage: L.is_ideal(L)
True

sage: F = LieAlgebra(QQ, 'F', representation='polynomial')
sage: L.is_ideal(F)
Traceback (most recent call last):
...
NotImplementedError: A must be a finite dimensional Lie algebra
 with basis
```

**is_nilpotent**()

Return if `self` is a nilpotent Lie algebra.

A Lie algebra is nilpotent if the lower central series eventually becomes 0.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_nilpotent()
True
```

**is_semisimple**()

Return if `self` if a semisimple Lie algebra.

A Lie algebra is semisimple if the solvable radical is zero. In characteristic 0, this is equivalent to saying the Killing form is non-degenerate.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_semisimple()
False
```

**is_solvable**()

Return if `self` is a solvable Lie algebra.

A Lie algebra is solvable if the derived series eventually becomes 0.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_solvable()
True
```

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'):{'x':1}})
sage: L.is_solvable() # todo: not implemented - #17416
False
```

**killing_form**($x$, $y$)

Return the Killing form on `x` and `y`, where `x` and `y` are two elements of `self`.

The Killing form is defined as

$$\langle x \mid y \rangle = \operatorname{tr}\left(\operatorname{ad}_x \circ \operatorname{ad}_y\right).$$

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a,b,c = L.lie_algebra_generators()
sage: L.killing_form(a, b)
0
```

**killing_form_matrix**()

Return the matrix of the Killing form of `self`.

The rows and the columns of this matrix are indexed by the elements of the basis of `self` (in the order provided by `basis()`).

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.killing_form_matrix()
[0 0 0]
[0 0 0]
[0 0 0]

sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example(0)
sage: m = L.killing_form_matrix(); m
[]
sage: parent(m)
Full MatrixSpace of 0 by 0 dense matrices over Rational Field
```

**killing_matrix**(*x*, *y*)

Return the Killing matrix of `x` and `y`, where `x` and `y` are two elements of `self`.

The Killing matrix is defined as the matrix corresponding to the action of $\operatorname{ad}_x \circ \operatorname{ad}_y$ in the basis of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a,b,c = L.lie_algebra_generators()
sage: L.killing_matrix(a, b)
[0 0 0]
[0 0 0]
[0 0 0]
```

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'):{'x':1}})
sage: L.killing_matrix(x, y)
[ 0  0]
[-1  0]
```

**lower_central_series**()

Return the lower central series $(\mathfrak{g}_i)_i$ of `self` where the rightmost $\mathfrak{g}_k = \mathfrak{g}_{k+1} = \cdots$.

We define the lower central series of a Lie algebra $\mathfrak{g}$ recursively by $\mathfrak{g}_0 := \mathfrak{g}$ and

$$\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k]$$

and recall that $\mathfrak{g}_k \supseteq \mathfrak{g}_{k+1}$. Alternatively we can express this as

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq \big[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}\big] \supseteq \Big[\big[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}\big], \mathfrak{g}\Big] \supseteq \cdots.$$

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.derived_series()
(An example of a finite dimensional Lie algebra with basis:
    the 3-dimensional abelian Lie algebra over Rational Field,
 An example of a finite dimensional Lie algebra with basis:
    the 0-dimensional abelian Lie algebra over Rational Field
    with basis matrix:
    [])
```

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'):{'x':1}})
sage: L.lower_central_series() # todo: not implemented - #17416
(Lie algebra on 2 generators (x, y) over Rational Field,
 Subalgebra generated of Lie algebra on 2 generators (x, y) over Rational␣
↪Field with basis:
(x,))
```

**module**(*R=None*)

Return a dense free module associated to `self` over `R`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L._dense_free_module()
Vector space of dimension 3 over Rational Field
```

**product_space**(*L, submodule=False*)

Return the product space `[self, L]`.

INPUT:

- `L` – a Lie subalgebra of `self`
- `submodule` – (default: `False`) if `True`, then the result is forced to be a submodule of `self`

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a,b,c = L.lie_algebra_generators()
sage: X = L.subalgebra([a, b+c])
sage: L.product_space(X)
An example of a finite dimensional Lie algebra with basis:
 the 0-dimensional abelian Lie algebra over Rational Field
 with basis matrix:
[]
sage: Y = L.subalgebra([a, 2*b-c])
sage: X.product_space(Y)
An example of a finite dimensional Lie algebra with basis:
 the 0-dimensional abelian Lie algebra over Rational
 Field with basis matrix:
[]
```

```
sage: H = lie_algebras.Heisenberg(ZZ, 4)
sage: Hp = H.product_space(H, submodule=True).basis()
sage: [H.from_vector(v) for v in Hp]
[z]
```

```
sage: L.<x,y> = LieAlgebra(QQ, {('x','y'):{'x':1}})
sage: Lp = L.product_space(L) # todo: not implemented - #17416
sage: Lp # todo: not implemented - #17416
```

```
Subalgebra generated of Lie algebra on 2 generators (x, y) over Rational␣
↪Field with basis:
(x,)
sage: Lp.product_space(L) # todo: not implemented - #17416
Subalgebra generated of Lie algebra on 2 generators (x, y) over Rational␣
↪Field with basis:
(x,)
sage: L.product_space(Lp) # todo: not implemented - #17416
Subalgebra generated of Lie algebra on 2 generators (x, y) over Rational␣
↪Field with basis:
(x,)
sage: Lp.product_space(Lp) # todo: not implemented - #17416
Subalgebra generated of Lie algebra on 2 generators (x, y) over Rational␣
↪Field with basis:
()
```

**structure_coefficients**(*include_zeros=False*)

Return the structure coefficients of `self`.

INPUT:

- `include_zeros` – (default: `False`) if `True`, then include the $[x, y] = 0$ pairs in the output

OUTPUT:

A dictionary whose keys are pairs of basis indices $(i, j)$ with $i < j$, and whose values are the corresponding *elements* $[b_i, b_j]$ in the Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.structure_coefficients()
Finite family {}
sage: L.structure_coefficients(True)
Finite family {(0, 1): (0, 0, 0), (1, 2): (0, 0, 0), (0, 2): (0, 0, 0)}
```

```
sage: G = SymmetricGroup(3)
sage: S = GroupAlgebra(G, QQ)
sage: L = LieAlgebra(associative=S)
sage: L.structure_coefficients()
Finite family {((1,3,2), (1,3)): (2,3) - (1,2),
               ((1,2), (1,2,3)): -(2,3) + (1,3),
               ((1,2,3), (1,3)): -(2,3) + (1,2),
               ((2,3), (1,3,2)): -(1,2) + (1,3),
               ((2,3), (1,3)): -(1,2,3) + (1,3,2),
               ((2,3), (1,2)): (1,2,3) - (1,3,2),
               ((2,3), (1,2,3)): (1,2) - (1,3),
               ((1,2), (1,3,2)): (2,3) - (1,3),
               ((1,2), (1,3)): (1,2,3) - (1,3,2)}
```

**class Subobjects**(*category*, **args*)

Bases: *sage.categories.subobjects.SubobjectsCategory*

A category for subalgebras of a finite dimensional Lie algebra with basis.

**class ParentMethods**

**ambient**()

Return the ambient Lie algebra of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: S = L.subalgebra([2*a+b, b + c])
sage: S.ambient() == L
True
```

**basis_matrix**()

Return the basis matrix of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: S = L.subalgebra([2*a+b, b + c])
sage: S.basis_matrix()
[   1    0 -1/2]
[   0    1    1]
```

**example**(*n=3*)

Return an example of a finite dimensional Lie algebra with basis as per *Category.example*.

EXAMPLES:

```
sage: C = LieAlgebras(QQ).FiniteDimensional().WithBasis()
sage: C.example()
An example of a finite dimensional Lie algebra with basis:
 the 3-dimensional abelian Lie algebra over Rational Field
```

Other dimensions can be specified as an optional argument:

```
sage: C.example(5)
An example of a finite dimensional Lie algebra with basis:
 the 5-dimensional abelian Lie algebra over Rational Field
```

## 3.52 Finite dimensional modules with basis

**class** sage.categories.finite_dimensional_modules_with_basis.**FiniteDimensionalModulesWithBas**

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of finite dimensional modules with a distinguished basis

EXAMPLES:

```
sage: C = FiniteDimensionalModulesWithBasis(ZZ); C
Category of finite dimensional modules with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of finite dimensional modules over Integer Ring,
 Category of modules with basis over Integer Ring]
sage: C is Modules(ZZ).WithBasis().FiniteDimensional()
True
```

**class ElementMethods**

**dense_coefficient_list**(*order=None*)

Return a list of *all* coefficients of `self`.

By default, this list is ordered in the same way as the indexing set of the basis of the parent of `self`.

INPUT:

- `order` – (optional) an ordering of the basis indexing set

EXAMPLES:

```
sage: v = vector([0, -1, -3])
sage: v.dense_coefficient_list()
[0, -1, -3]
sage: v.dense_coefficient_list([2,1,0])
[-3, -1, 0]
sage: sorted(v.coefficients())
[-3, -1]
```

**class MorphismMethods**

**image**()

Return the image of `self` as a submodule of the codomain.

EXAMPLES:

```
sage: SGA = SymmetricGroupAlgebra(QQ, 3)
sage: f = SGA.module_morphism(lambda x: SGA(x**2), codomain=SGA)
sage: f.image()
Free module generated by {0, 1, 2} over Rational Field
```

**image_basis**()

Return a basis for the image of `self` in echelon form.

EXAMPLES:

```
sage: SGA = SymmetricGroupAlgebra(QQ, 3)
sage: f = SGA.module_morphism(lambda x: SGA(x**2), codomain=SGA)
sage: f.image_basis()
([1, 2, 3], [2, 3, 1], [3, 1, 2])
```

**kernel**()

Return the kernel of `self` as a submodule of the domain.

EXAMPLES:

```
sage: SGA = SymmetricGroupAlgebra(QQ, 3)
sage: f = SGA.module_morphism(lambda x: SGA(x**2), codomain=SGA)
sage: K = f.kernel()
sage: K
Free module generated by {0, 1, 2} over Rational Field
sage: K.ambient()
Symmetric group algebra of order 3 over Rational Field
```

**kernel_basis**()

Return a basis of the kernel of `self` in echelon form.

EXAMPLES:

```
sage: SGA = SymmetricGroupAlgebra(QQ, 3)
sage: f = SGA.module_morphism(lambda x: SGA(x**2), codomain=SGA)
sage: f.kernel_basis()
([1, 2, 3] - [3, 2, 1], [1, 3, 2] - [3, 2, 1], [2, 1, 3] - [3, 2, 1])
```

**matrix** (*base_ring=None*, *side='left'*)
Return the matrix of this morphism in the distinguished bases of the domain and codomain.

INPUT:

- `base_ring` – a ring (default: `None`, meaning the base ring of the codomain)
- `side` – "left" or "right" (default: "left")

If `side` is "left", this morphism is considered as acting on the left; i.e. each column of the matrix represents the image of an element of the basis of the domain.

The order of the rows and columns matches with the order in which the bases are enumerated.

**See also:**

`Modules.WithBasis.ParentMethods.module_morphism()`

EXAMPLES:

```
sage: X = CombinatorialFreeModule(ZZ, [1,2]); x = X.basis()
sage: Y = CombinatorialFreeModule(ZZ, [3,4]); y = Y.basis()
sage: phi = X.module_morphism(on_basis = {1: y[3] + 3*y[4], 2: 2*y[3] +
↪5*y[4]}.__getitem__,
....:                         codomain = Y)
sage: phi.matrix()
[1 2]
[3 5]
sage: phi.matrix(side="right")
[1 3]
[2 5]

sage: phi.matrix().parent()
Full MatrixSpace of 2 by 2 dense matrices over Integer Ring
sage: phi.matrix(QQ).parent()
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

The resulting matrix is immutable:

```
sage: phi.matrix().is_mutable()
False
```

The zero morphism has a zero matrix:

```
sage: Hom(X,Y).zero().matrix()
[0 0]
[0 0]
```

**Todo:** Add support for morphisms where the codomain has a different base ring than the domain:

```
sage: Y = CombinatorialFreeModule(QQ, [3,4]); y = Y.basis()
sage: phi = X.module_morphism(on_basis = {1: y[3] + 3*y[4], 2: 2*y[3] + 5/
↪2*y[4]}.__getitem__,
....:                         codomain = Y)
sage: phi.matrix().parent()          # todo: not implemented
Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

This currently does not work because, in this case, the morphism is just in the category of commutative additive groups (i.e. the intersection of the categories of modules over $\mathbf{Z}$ and over $\mathbf{Q}$):

```
sage: phi.parent().homset_category()
Category of commutative additive semigroups
```

```
sage: phi.parent().homset_category() # todo: not implemented
Category of finite dimensional modules with basis over Integer Ring
```

**class ParentMethods**

**annihilator**(*S*, *action=<built-in function mul>*, *side='right'*, *category=None*)

Return the annihilator of a finite set.

INPUT:

- `S` – a finite set
- `action` – a function (default: `operator.mul`)
- `side` – 'left' or 'right' (default: 'right')
- `category` – a category

Assumptions:

- `action` takes elements of `self` as first argument and elements of `S` as second argument;
- The codomain is any vector space, and `action` is linear on its first argument; typically it is bilinear;
- If `side` is 'left', this is reversed.

OUTPUT:

The subspace of the elements $x$ of `self` such that `action(x,s) = 0` for all $s \in S$. If `side` is 'left' replace the above equation by `action(s,x) = 0`.

If `self` is a ring, `action` an action of `self` on a module $M$ and $S$ is a subset of $M$, we recover the Wikipedia article Annihilator_%28ring_theory%29. Similarly this can be used to compute torsion or orthogonals.

**See also:**

*annihilator_basis()* for lots of examples.

EXAMPLES:

```
sage: F = FiniteDimensionalAlgebrasWithBasis(QQ).example(); F
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: x,y,a,b = F.basis()
sage: A = F.annihilator([a + 3*b + 2*y]); A
Free module generated by {0} over Rational Field
sage: [b.lift() for b in A.basis()]
[-1/2*a - 3/2*b + x]
```

The category can be used to specify other properties of this subspace, like that this is a subalgebra:

```
sage: center = F.annihilator(F.basis(), F.bracket,
....:                        category=Algebras(QQ).Subobjects())
sage: (e,) = center.basis()
sage: e.lift()
x + y
sage: e * e == e
True
```

Taking annihilator is order reversing for inclusion:

```
sage: A   = F.annihilator([]);    A   .rename("A")
sage: Ax  = F.annihilator([x]);   Ax  .rename("Ax")
```

```
sage: Ay  = F.annihilator([y]);   Ay .rename("Ay")
sage: Axy = F.annihilator([x,y]); Axy.rename("Axy")
sage: P = Poset(([A, Ax, Ay, Axy], attrcall("is_submodule")))
sage: sorted(P.cover_relations(), key=str)
[[Ax, A], [Axy, Ax], [Axy, Ay], [Ay, A]]
```

**annihilator_basis** (*S*, *action=<built-in function mul>*, *side='right'*)

Return a basis of the annihilator of a finite set of elements.

INPUT:

- `S` – a finite set of objects
- `action` – a function (default: `operator.mul`)
- `side` – 'left' or 'right' (default: 'right'): on which side of `self` the elements of $S$ acts.

See `annihilator()` for the assumptions and definition of the annihilator.

EXAMPLES:

By default, the action is the standard $*$ operation. So our first example is about an algebra:

```
sage: F = FiniteDimensionalAlgebrasWithBasis(QQ).example(); F
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: x,y,a,b = F.basis()
```

In this algebra, multiplication on the right by $x$ annihilates all basis elements but $x$:

```
sage: x*x, y*x, a*x, b*x
(x, 0, 0, 0)
```

So the annihilator is the subspace spanned by $y$, $a$, and $b$:

```
sage: F.annihilator_basis([x])
(y, a, b)
```

The same holds for $a$ and $b$:

```
sage: x*a, y*a, a*a, b*a
(a, 0, 0, 0)
sage: F.annihilator_basis([a])
(y, a, b)
```

On the other hand, $y$ annihilates only $x$:

```
sage: F.annihilator_basis([y])
(x,)
```

Here is a non trivial annihilator:

```
sage: F.annihilator_basis([a + 3*b + 2*y])
(-1/2*a - 3/2*b + x,)
```

Let's check it:

```
sage: (-1/2*a - 3/2*b + x) * (a + 3*b + 2*y)
0
```

Doing the same calculations on the left exchanges the roles of $x$ and $y$:

```
sage: F.annihilator_basis([y], side="left")
(x, a, b)
sage: F.annihilator_basis([a], side="left")
(x, a, b)
sage: F.annihilator_basis([b], side="left")
(x, a, b)
sage: F.annihilator_basis([x], side="left")
(y,)
sage: F.annihilator_basis([a+3*b+2*x], side="left")
(-1/2*a - 3/2*b + y,)
```

By specifying an inner product, this method can be used to compute the orthogonal of a subspace:

```
sage: x,y,a,b = F.basis()
sage: def scalar(u,v): return vector([sum(u[i]*v[i] for i in F.basis().
↪keys())])
sage: F.annihilator_basis([x+y, a+b], scalar)
(x - y, a - b)
```

By specifying the standard Lie bracket as action, one can compute the commutator of a subspace of $F$:

```
sage: F.annihilator_basis([a+b], action=F.bracket)
(x + y, a, b)
```

In particular one can compute a basis of the center of the algebra. In our example, it is reduced to the identity:

```
sage: F.annihilator_basis(F.algebra_generators(), action=F.bracket)
(x + y,)
```

But see also `FiniteDimensionalAlgebrasWithBasis.ParentMethods.center_basis()`.

**from_vector**(*vector*, *order=None*)

Build an element of `self` from a vector.

EXAMPLES:

```
sage: p_mult = matrix([[0,0,0],[0,0,-1],[0,0,0]])
sage: q_mult = matrix([[0,0,1],[0,0,0],[0,0,0]])
sage: A = algebras.FiniteDimensional(QQ, [p_mult, q_mult, matrix(QQ,3,3)],
....:                                  'p,q,z')
sage: A.from_vector(vector([1,0,2]))
p + 2*z
```

**gens**()

Return the generators of `self`.

OUTPUT:

A tuple containing the basis elements of `self`.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(ZZ, ['a', 'b', 'c'])
sage: F.gens()
(B['a'], B['b'], B['c'])
```

**quotient_module** (*submodule*, *check=True*, *already_echelonized=False*, *category=None*)
Construct the quotient module `self/submodule`.

INPUT:

- `submodule` – a submodule with basis of `self`, or something that can be turned into one via `self.submodule(submodule)`.
- `check`, `already_echelonized` – passed down to *ModulesWithBasis.ParentMethods.submodule()*.

> **Warning:** At this point, this only supports quotients by free submodules admitting a basis in unitriangular echelon form. In this case, the quotient is also a free module, with a basis consisting of the retract of a subset of the basis of `self`.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
sage: Y = X.quotient_module([x[0]-x[1], x[1]-x[2]], already_
↪echelonized=True)
sage: Y.print_options(prefix='y'); Y
Free module generated by {2} over Rational Field
sage: y = Y.basis()
sage: y[2]
y[2]
sage: y[2].lift()
x[2]
sage: Y.retract(x[0]+2*x[1])
3*y[2]

sage: R.<a,b> = QQ[]
sage: C = CombinatorialFreeModule(R, range(3), prefix='x')
sage: x = C.basis()
sage: gens = [x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]]
sage: Y = X.quotient_module(gens)
```

**See also:**

- `Modules.WithBasis.ParentMethods.submodule()`
- *Rings.ParentMethods.quotient()*
- `sage.modules.with_basis.subquotient.QuotientModuleWithBasis`

## 3.53 Finite dimensional semisimple algebras with basis

**class** sage.categories.finite_dimensional_semisimple_algebras_with_basis.**FiniteDimensionalSe**
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of finite dimensional semisimple algebras with a distinguished basis

EXAMPLES:

```
sage: from sage.categories.finite_dimensional_semisimple_algebras_with_basis␣
↪import FiniteDimensionalSemisimpleAlgebrasWithBasis
sage: C = FiniteDimensionalSemisimpleAlgebrasWithBasis(QQ); C
Category of finite dimensional semisimple algebras with basis over Rational Field
```

This category is best constructed as:

```
sage: D = Algebras(QQ).Semisimple().FiniteDimensional().WithBasis(); D
Category of finite dimensional semisimple algebras with basis over Rational Field
sage: D is C
True
```

**class Commutative**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class ParentMethods**

**central_orthogonal_idempotents**()

Return the central orthogonal idempotents of this semisimple commutative algebra.

Those idempotents form a maximal decomposition of the identity into primitive orthogonal idempotents.

OUTPUT:

A list of orthogonal idempotents of `self`.

EXAMPLES:

```
sage: A4 = SymmetricGroup(4).algebra(QQ)
sage: Z4 = A4.center()
sage: idempotents = Z4.central_orthogonal_idempotents()
sage: idempotents
(1/24*B[0] + 1/24*B[1] + 1/24*B[2] + 1/24*B[3] + 1/24*B[4],
 3/8*B[0] + 1/8*B[1] - 1/8*B[2] - 1/8*B[4],
 1/6*B[0] + 1/6*B[2] - 1/12*B[3],
 3/8*B[0] - 1/8*B[1] - 1/8*B[2] + 1/8*B[4],
 1/24*B[0] - 1/24*B[1] + 1/24*B[2] + 1/24*B[3] - 1/24*B[4])
```

Lifting those idempotents from the center, we recognize among them the sum and alternating sum of all permutations:

```
sage: [e.lift() for e in idempotents]
[1/24*() + 1/24*(3,4) + 1/24*(2,3) + 1/24*(2,3,4) + 1/24*(2,4,3)
 + 1/24*(2,4) + 1/24*(1,2) + 1/24*(1,2)(3,4) + 1/24*(1,2,3)
 + 1/24*(1,2,3,4) + 1/24*(1,2,4,3) + 1/24*(1,2,4) + 1/24*(1,3,2)
 + 1/24*(1,3,4,2) + 1/24*(1,3) + 1/24*(1,3,4) + 1/24*(1,3)(2,4)
 + 1/24*(1,3,2,4) + 1/24*(1,4,3,2) + 1/24*(1,4,2) + 1/24*(1,4,3)
 + 1/24*(1,4) + 1/24*(1,4,2,3) + 1/24*(1,4)(2,3),
 ...,
 1/24*() - 1/24*(3,4) - 1/24*(2,3) + 1/24*(2,3,4) + 1/24*(2,4,3)
 - 1/24*(2,4) - 1/24*(1,2) + 1/24*(1,2)(3,4) + 1/24*(1,2,3)
 - 1/24*(1,2,3,4) - 1/24*(1,2,4,3) + 1/24*(1,2,4) + 1/24*(1,3,2)
 - 1/24*(1,3,4,2) - 1/24*(1,3) + 1/24*(1,3,4) + 1/24*(1,3)(2,4)
 - 1/24*(1,3,2,4) - 1/24*(1,4,3,2) + 1/24*(1,4,2) + 1/24*(1,4,3)
 - 1/24*(1,4) - 1/24*(1,4,2,3) + 1/24*(1,4)(2,3)]
```

We check that they indeed form a decomposition of the identity of $Z_4$ into orthogonal idempotents:

```
sage: Z4.is_identity_decomposition_into_orthogonal_
↪idempotents(idempotents)
True
```

**class ParentMethods**

**central_orthogonal_idempotents**()

Return a maximal list of central orthogonal idempotents of `self`.

*Central orthogonal idempotents* of an algebra $A$ are idempotents $(e_1, \ldots, e_n)$ in the center of $A$ such that $e_i e_j = 0$ whenever $i \neq j$.

With the maximality condition, they sum up to 1 and are uniquely determined (up to order).

INPUT:

- `self` – a semisimple algebra.

EXAMPLES:

For the algebra of the symmetric group $S_3$, we recover the sum and alternating sum of all permutations, together with a third idempotent:

```
sage: A3 = SymmetricGroup(3).algebra(QQ)
sage: idempotents = A3.central_orthogonal_idempotents()
sage: idempotents
(1/6*() + 1/6*(2,3) + 1/6*(1,2) + 1/6*(1,2,3) + 1/6*(1,3,2) + 1/6*(1,3),
 2/3*() - 1/3*(1,2,3) - 1/3*(1,3,2),
 1/6*() - 1/6*(2,3) - 1/6*(1,2) + 1/6*(1,2,3) + 1/6*(1,3,2) - 1/6*(1,3))
sage: A3.is_identity_decomposition_into_orthogonal_
↪idempotents(idempotents)
True
```

For the semisimple quotient of a quiver algebra, we recover the vertices of the quiver:

```
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver (containing
the arrows a:x->y and b:x->y) over Rational Field
sage: Aquo = A.semisimple_quotient()
sage: Aquo.central_orthogonal_idempotents()
(B['x'], B['y'])
```

**radical_basis**(***keywords*)

Return a basis of the Jacobson radical of this algebra.

- `keywords` – for compatibility; ignored.

OUTPUT: the empty list since this algebra is semisimple.

EXAMPLES:

```
sage: A = SymmetricGroup(4).algebra(QQ)
sage: A.radical_basis()
()
```

## 3.54 Finite Enumerated Sets

**class** sage.categories.finite_enumerated_sets.**FiniteEnumeratedSets**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite enumerated sets

EXAMPLES:

```
sage: FiniteEnumeratedSets()
Category of finite enumerated sets
sage: FiniteEnumeratedSets().super_categories()
[Category of enumerated sets, Category of finite sets]
sage: FiniteEnumeratedSets().all_super_categories()
[Category of finite enumerated sets,
 Category of enumerated sets,
 Category of finite sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

---

**Todo:** `sage.combinat.debruijn_sequence.DeBruijnSequences` should not inherit from this class. If that is solved, then *`FiniteEnumeratedSets`* shall be turned into a subclass of *`Category_singleton`*.

---

**class CartesianProducts**(*category*, **args*)

Bases: *`sage.categories.cartesian_product.CartesianProductsCategory`*

**class ParentMethods**

**cardinality**()

Return the cardinality of self.

EXAMPLES:

```
sage: E = FiniteEnumeratedSet([1,2,3])
sage: C = cartesian_product([E,SymmetricGroup(4)])
sage: C.cardinality()
72

sage: E = FiniteEnumeratedSet([])
sage: C = cartesian_product([E, ZZ, QQ])
sage: C.cardinality()
0

sage: C = cartesian_product([ZZ, QQ])
sage: C.cardinality()
+Infinity

sage: cartesian_product([GF(5), Permutations(10)]).cardinality()
18144000
sage: cartesian_product([GF(71)]*20).cardinality() == 71**20
True
```

**last**()

Return the last element

EXAMPLES:

```
sage: C = cartesian_product([Zmod(42), Partitions(10),
↪IntegerRange(5)])
sage: C.last()
(41, [1, 1, 1, 1, 1, 1, 1, 1, 1, 1], 4)
```

**random_element**(**args*)

Return a random element of this Cartesian product.

The extra arguments are passed down to each of the factors of the Cartesian product.

EXAMPLES:

```
sage: C = cartesian_product([Permutations(10)]*5)
sage: C.random_element()           # random
([2, 9, 4, 7, 1, 8, 6, 10, 5, 3],
 [8, 6, 5, 7, 1, 4, 9, 3, 10, 2],
 [5, 10, 3, 8, 2, 9, 1, 4, 7, 6],
 [9, 6, 10, 3, 2, 1, 5, 8, 7, 4],
 [8, 5, 2, 9, 10, 3, 7, 1, 4, 6])

sage: C = cartesian_product([ZZ]*10)
sage: c1 = C.random_element()
sage: c1                        # random
(3, 1, 4, 1, 1, -3, 0, -4, -17, 2)
sage: c2 = C.random_element(4,7)
sage: c2                        # random
(6, 5, 6, 4, 5, 6, 6, 4, 5, 5)
sage: all(4 <= i < 7 for i in c2)
True
```

**rank**(*x*)

Return the rank of an element of this Cartesian product.

The *rank* of `x` is its position in the enumeration. It is an integer between `0` and `n-1` where `n` is the cardinality of this set.

**See also:**

- *EnumeratedSets.ParentMethods.rank()*
- *unrank()*

EXAMPLES:

```
sage: C = cartesian_product([GF(2), GF(11), GF(7)])
sage: C.rank(C((1,2,5)))
96
sage: C.rank(C((0,0,0)))
0

sage: for c in C: print(C.rank(c))
0
1
2
3
4
5
...
150
151
152
153

sage: F1 = FiniteEnumeratedSet('abcdefgh')
sage: F2 = IntegerRange(250)
sage: F3 = Partitions(20)
```

```
sage: C = cartesian_product([F1, F2, F3])
sage: c = C(('a', 86, [7,5,4,4]))
sage: C.rank(c)
54213
sage: C.unrank(54213)
('a', 86, [7, 5, 4, 4])
```

**unrank** (*i*)

Return the i-th element of this Cartesian product.

INPUT:

- i – integer between 0 and n−1 where n is the cardinality of this set.

**See also:**

- *EnumeratedSets.ParentMethods.unrank()*
- *rank()*

EXAMPLES:

```
sage: C = cartesian_product([GF(3), GF(11), GF(7), GF(5)])
sage: c = C.unrank(123); c
(0, 3, 3, 3)
sage: C.rank(c)
123

sage: c = C.unrank(857); c
(2, 2, 3, 2)
sage: C.rank(c)
857

sage: C.unrank(2500)
Traceback (most recent call last):
...
IndexError: index i (=2) is greater than the cardinality
```

**extra_super_categories** ()

A Cartesian product of finite enumerated sets is a finite enumerated set.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().CartesianProducts()
sage: C.extra_super_categories()
[Category of finite enumerated sets]
```

**class IsomorphicObjects** (*category*, **args*)

Bases: *sage.categories.isomorphic_objects.IsomorphicObjectsCategory*

**class ParentMethods**

**cardinality** ()

Returns the cardinality of self which is the same as that of the ambient set self is isomorphic to.

EXAMPLES:

```
sage: A = FiniteEnumeratedSets().IsomorphicObjects().example(); A
The image by some isomorphism of An example of a finite enumerated␣
↪set: {1,2,3}
```

```
sage: A.cardinality()
3
```

**example**()

Returns an example of isomorphic object of a finite enumerated set, as per *Category.example*.

EXAMPLES:

```
sage: FiniteEnumeratedSets().IsomorphicObjects().example()
The image by some isomorphism of An example of a finite enumerated set:
↪{1,2,3}
```

**class ParentMethods**

**cardinality**(**ignored_args*, ***ignored_kwds*)

Return the cardinality of `self`.

This brute force implementation of *cardinality()* iterates through the elements of `self` to count them.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example(); C
An example of a finite enumerated set: {1,2,3}
sage: C._cardinality_from_iterator()
3
```

**iterator_range**(*start=None*, *stop=None*, *step=None*)

Iterate over the range of elements of `self` starting at `start`, ending at `stop`, and stepping by `step`.

**See also:**

`unrank()`, `unrank_range()`

EXAMPLES:

```
sage: F = FiniteEnumeratedSet([1,2,3])
sage: list(F.iterator_range(1))
[2, 3]
sage: list(F.iterator_range(stop=2))
[1, 2]
sage: list(F.iterator_range(stop=2, step=2))
[1]
sage: list(F.iterator_range(start=1, step=2))
[2]
sage: list(F.iterator_range(start=1, stop=2))
[2]
sage: list(F.iterator_range(start=0, stop=1))
[1]
sage: list(F.iterator_range(start=0, stop=3, step=2))
[1, 3]
sage: list(F.iterator_range(stop=-1))
[1, 2]

sage: F = FiniteEnumeratedSet([1,2,3,4])
sage: list(F.iterator_range(start=1, stop=3))
[2, 3]
sage: list(F.iterator_range(stop=10))
[1, 2, 3, 4]
```

**last**()

The last element of `self`.

`self.last()` returns the last element of `self`.

This is the default (brute force) implementation from the category `FiniteEnumeratedSet()` which can be used when the method `__iter__` is provided. Its complexity is $O(n)$ where $n$ is the size of `self`.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.last()
3
sage: C._last_from_iterator()
3
```

**list**()

Return a list of the elements of `self`.

The elements of set `x` is created and cashed on the fist call of `x.list()`. Then each call of `x.list()` returns a new list from the cashed result. Thus in looping, it may be better to do `for e in x:`, not `for e in x.list():`.

**See also:**

`_list_from_iterator()`, `_cardinality_from_list()`, `_iterator_from_list()`, and `_unrank_from_list()`

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.list()
[1, 2, 3]
```

**random_element**()

A random element in `self`.

`self.random_element()` returns a random element in `self` with uniform probability.

This is the default implementation from the category `EnumeratedSet()` which uses the method `unrank`.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.random_element()
1
sage: C._random_element_from_unrank()
2
```

TODO: implement _test_random which checks uniformness

**unrank_range**(*start=None*, *stop=None*, *step=None*)

Return the range of elements of `self` starting at `start`, ending at `stop`, and stepping by `step`.

See also `unrank()`.

EXAMPLES:

```
sage: F = FiniteEnumeratedSet([1,2,3])
sage: F.unrank_range(1)
```

```
[2, 3]
sage: F.unrank_range(stop=2)
[1, 2]
sage: F.unrank_range(stop=2, step=2)
[1]
sage: F.unrank_range(start=1, step=2)
[2]
sage: F.unrank_range(stop=-1)
[1, 2]

sage: F = FiniteEnumeratedSet([1,2,3,4])
sage: F.unrank_range(stop=10)
[1, 2, 3, 4]
```

## 3.55 Finite Fields

**class** sage.categories.finite_fields.**FiniteFields**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite fields.

EXAMPLES:

```
sage: K = FiniteFields(); K
Category of finite enumerated fields
```

A finite field is a finite monoid with the structure of a field; it is currently assumed to be enumerated:

```
sage: K.super_categories()
[Category of fields,
 Category of finite commutative rings,
 Category of finite enumerated sets]
```

Some examples of membership testing and coercion:

```
sage: FiniteField(17) in K
True
sage: RationalField() in K
False
sage: K(RationalField())
Traceback (most recent call last):
...
TypeError: unable to canonically associate a finite field to Rational Field
```

**class ElementMethods**

**class ParentMethods**

**extra_super_categories**()

Any finite field is assumed to be endowed with an enumeration.

## 3.56 FiniteGroups

**class** sage.categories.finite_groups.**FiniteGroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite (multiplicative) groups.

EXAMPLES:

```
sage: C = FiniteGroups(); C
Category of finite groups
sage: C.super_categories()
[Category of finite monoids, Category of groups]
sage: C.example()
General Linear Group of degree 2 over Finite Field of size 3
```

**class Algebras**(*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**extra_super_categories**()

Implement Maschke's theorem.

In characteristic 0 all finite group algebras are semisimple.

EXAMPLES:

```
sage: FiniteGroups().Algebras(QQ).is_subcategory(Algebras(QQ).
↪Semisimple())
True
sage: FiniteGroups().Algebras(FiniteField(7)).is_
↪subcategory(Algebras(FiniteField(7)).Semisimple())
False
sage: FiniteGroups().Algebras(ZZ).is_subcategory(Algebras(ZZ).
↪Semisimple())
False
sage: FiniteGroups().Algebras(Fields()).is_subcategory(Algebras(Fields()).
↪Semisimple())
False

sage: Cat = CommutativeAdditiveGroups().Finite()
sage: Cat.Algebras(QQ).is_subcategory(Algebras(QQ).Semisimple())
True
sage: Cat.Algebras(GF(7)).is_subcategory(Algebras(GF(7)).Semisimple())
False
sage: Cat.Algebras(ZZ).is_subcategory(Algebras(ZZ).Semisimple())
False
sage: Cat.Algebras(Fields()).is_subcategory(Algebras(Fields()).
↪Semisimple())
False
```

**class ElementMethods**

**class ParentMethods**

**cardinality**()

Returns the cardinality of `self`, as per `EnumeratedSets.ParentMethods.cardinality()`.

This default implementation calls *order()* if available, and otherwise resorts to `_cardinality_from_iterator()`. This is for backward compatibility only. Finite groups should override this method instead of *order()*.

EXAMPLES:

We need to use a finite group which uses this default implementation of cardinality:

```
sage: R.<x> = PolynomialRing(QQ)
sage: f = x^4 - 17*x^3 - 2*x + 1
sage: G = f.galois_group(pari_group=True); G
PARI group [24, -1, 5, "S4"] of degree 4
sage: G.cardinality.__module__
'sage.categories.finite_groups'
sage: G.cardinality()
24
```

**cayley_graph_disabled**(*connecting_set=None*)

AUTHORS:

- Bobby Moretti (2007-08-10)
- Robert Miller (2008-05-01): editing

**conjugacy_classes**()

Return a list with all the conjugacy classes of the group.

This will eventually be a fall-back method for groups not defined over GAP. Right now just raises a `NotImplementedError`, until we include a non-GAP way of listing the conjugacy classes representatives.

EXAMPLES:

```
sage: from sage.groups.group import FiniteGroup
sage: G = FiniteGroup()
sage: G.conjugacy_classes()
Traceback (most recent call last):
...
NotImplementedError: Listing the conjugacy classes for group <sage.groups.
↪group.FiniteGroup object at ...> is not implemented
```

**conjugacy_classes_representatives**()

Return a list of the conjugacy classes representatives of the group.

EXAMPLES:

```
sage: G = SymmetricGroup(3)
sage: G.conjugacy_classes_representatives()
[(), (1,2), (1,2,3)]
```

**monoid_generators**()

Return monoid generators for `self`.

For finite groups, the group generators are also monoid generators. Hence, this default implementation calls *group_generators()*.

EXAMPLES:

```
sage: A = AlternatingGroup(4)
sage: A.monoid_generators()
Family ((2,3,4), (1,2,3))
```

**semigroup_generators**()

Returns semigroup generators for self.

For finite groups, the group generators are also semigroup generators. Hence, this default implementation calls *group_generators()*.

EXAMPLES:

```
sage: A = AlternatingGroup(4)
sage: A.semigroup_generators()
Family ((2,3,4), (1,2,3))
```

**some_elements**()

Return some elements of self.

EXAMPLES:

```
sage: A = AlternatingGroup(4)
sage: A.some_elements()
Family ((2,3,4), (1,2,3))
```

**example**()

Return an example of finite group, as per *Category.example()*.

EXAMPLES:

```
sage: G = FiniteGroups().example(); G
General Linear Group of degree 2 over Finite Field of size 3
```

## 3.57 Finite lattice posets

**class** sage.categories.finite_lattice_posets.**FiniteLatticePosets**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

The category of finite lattices, i.e. finite partially ordered sets which are also lattices.

EXAMPLES:

```
sage: FiniteLatticePosets()
Category of finite lattice posets
sage: FiniteLatticePosets().super_categories()
[Category of lattice posets, Category of finite posets]
sage: FiniteLatticePosets().example()
NotImplemented
```

**See also:**

*FinitePosets*, *LatticePosets*, FiniteLatticePoset

**class ParentMethods**

**irreducibles_poset**()

Return the poset of meet- or join-irreducibles of the lattice.

A *join-irreducible* element of a lattice is an element with exactly one lower cover. Dually a *meet-irreducible* element has exactly one upper cover.

This is the smallest poset with completion by cuts being isomorphic to the lattice. As a special case this returns one-element poset from one-element lattice.

**See also:**

completion_by_cuts().

EXAMPLES:

```
sage: L = LatticePoset({1: [2, 3, 4], 2: [5, 6], 3: [5],
....:                   4: [6], 5: [9, 7], 6: [9, 8], 7: [10],
....:                   8: [10], 9: [10], 10: [11]})
sage: L_ = L.irreducibles_poset()
sage: sorted(L_)
[2, 3, 4, 7, 8, 9, 10, 11]
sage: L_.completion_by_cuts().is_isomorphic(L)
True
```

**is_lattice_morphism** (*f*, *codomain*)

Return whether `f` is a morphism of posets from `self` to `codomain`.

A map $f : P \to Q$ is a poset morphism if

$$x \leq y \Rightarrow f(x) \leq f(y)$$

for all $x, y \in P$.

INPUT:

- `f` – a function from `self` to `codomain`
- `codomain` – a lattice

EXAMPLES:

We build the boolean lattice of $\{2, 2, 3\}$ and the lattice of divisors of 60, and check that the map $b \mapsto 5 \prod_{x \in b} x$ is a morphism of lattices:

```
sage: D = LatticePoset((divisors(60), attrcall("divides")))
sage: B = LatticePoset((Subsets([2,2,3]), attrcall("issubset")))
sage: def f(b): return D(5*prod(b))
sage: B.is_lattice_morphism(f, D)
True
```

We construct the boolean lattice $B_2$:

```
sage: B = posets.BooleanLattice(2)
sage: B.cover_relations()
[[0, 1], [0, 2], [1, 3], [2, 3]]
```

And the same lattice with new top and bottom elements numbered respectively $-1$ and 3:

```
sage: L = LatticePoset(DiGraph({-1:[0], 0:[1,2], 1:[3], 2:[3],3:[4]}))
sage: L.cover_relations()
[[-1, 0], [0, 1], [0, 2], [1, 3], [2, 3], [3, 4]]

sage: f = { B(0): L(0), B(1): L(1), B(2): L(2), B(3): L(3) }.__getitem__
sage: B.is_lattice_morphism(f, L)
True

sage: f = { B(0): L(-1),B(1): L(1), B(2): L(2), B(3): L(3) }.__getitem__
sage: B.is_lattice_morphism(f, L)
False

sage: f = { B(0): L(0), B(1): L(1), B(2): L(2), B(3): L(4) }.__getitem__
sage: B.is_lattice_morphism(f, L)
False
```

**See also:**

*is_poset_morphism()*

**join_irreducibles**()

Return the join-irreducible elements of this finite lattice.

A *join-irreducible element* of `self` is an element $x$ that is not minimal and that can not be written as the join of two elements different from $x$.

EXAMPLES:

```
sage: L = LatticePoset({0:[1,2],1:[3],2:[3,4],3:[5],4:[5]})
sage: L.join_irreducibles()
[1, 2, 4]
```

**See also:**

- Dual function: *meet_irreducibles()*
- Other: `double_irreducibles()`, *join_irreducibles_poset()*

**join_irreducibles_poset**()

Return the poset of join-irreducible elements of this finite lattice.

A *join-irreducible element* of `self` is an element $x$ that is not minimal and can not be written as the join of two elements different from $x$.

EXAMPLES:

```
sage: L = LatticePoset({0:[1,2,3],1:[4],2:[4],3:[4]})
sage: L.join_irreducibles_poset()
Finite poset containing 3 elements
```

**See also:**

- Dual function: *meet_irreducibles_poset()*
- Other: *join_irreducibles()*

**meet_irreducibles**()

Return the meet-irreducible elements of this finite lattice.

A *meet-irreducible element* of `self` is an element $x$ that is not maximal and that can not be written as the meet of two elements different from $x$.

EXAMPLES:

```
sage: L = LatticePoset({0:[1,2],1:[3],2:[3,4],3:[5],4:[5]})
sage: L.meet_irreducibles()
[1, 3, 4]
```

**See also:**

- Dual function: *join_irreducibles()*
- Other: `double_irreducibles()`, *meet_irreducibles_poset()*

**meet_irreducibles_poset**()

Return the poset of join-irreducible elements of this finite lattice.

A *meet-irreducible element* of `self` is an element $x$ that is not maximal and can not be written as the meet of two elements different from $x$.

EXAMPLES:

```
sage: L = LatticePoset({0:[1,2,3],1:[4],2:[4],3:[4]})
sage: L.join_irreducibles_poset()
Finite poset containing 3 elements
```

**See also:**

- Dual function: `join_irreducibles_poset()`
- Other: `meet_irreducibles()`

## 3.58 Finite Monoids

**class** sage.categories.finite_monoids.**FiniteMonoids**(*base_category*)

Bases: `sage.categories.category_with_axiom.CategoryWithAxiom_singleton`

The category of finite (multiplicative) `monoids`.

A finite monoid is a `finite sets` endowed with an associative unital binary operation $*$.

EXAMPLES:

```
sage: FiniteMonoids()
Category of finite monoids
sage: FiniteMonoids().super_categories()
[Category of monoids, Category of finite semigroups]
```

**class ElementMethods**

**pseudo_order**()

Returns the pair $[k, j]$ with $k$ minimal and $0 \leq j < k$ such that `self^k == self^j`.

Note that $j$ is uniquely determined.

EXAMPLES:

```
sage: M = FiniteMonoids().example(); M
An example of a finite multiplicative monoid: the integers modulo 12

sage: x = M(2)
sage: [ x^i for i in range(7) ]
[1, 2, 4, 8, 4, 8, 4]
sage: x.pseudo_order()
[4, 2]

sage: x = M(3)
sage: [ x^i for i in range(7) ]
[1, 3, 9, 3, 9, 3, 9]
sage: x.pseudo_order()
[3, 1]

sage: x = M(4)
sage: [ x^i for i in range(7) ]
[1, 4, 4, 4, 4, 4, 4]
sage: x.pseudo_order()
[2, 1]

sage: x = M(5)
sage: [ x^i for i in range(7) ]
[1, 5, 1, 5, 1, 5, 1]
sage: x.pseudo_order()
[2, 0]
```

TODO: more appropriate name? see, for example, Jean-Eric Pin's lecture notes on semigroups.

**class ParentMethods**

**nerve**()

The nerve (classifying space) of this monoid.

OUTPUT: the nerve $BG$ (if $G$ denotes this monoid), as a simplicial set. The $k$-dimensional simplices of this object are indexed by products of $k$ elements in the monoid:

$$a_1 * a_2 * \cdots * a_k$$

The 0th face of this is obtained by deleting $a_1$, and the $k$-th face is obtained by deleting $a_k$. The other faces are obtained by multiplying elements: the 1st face is

$$(a1 * a_2) * \cdots * a_k$$

and so on. See Wikipedia article Nerve_(category_theory), which describes the construction of the nerve as a simplicial set.

A simplex in this simplicial set will be degenerate if in the corresponding product of $k$ elements, one of those elements is the identity. So we only need to keep track of the products of non-identity elements. Similarly, if a product $a_{i-1}a_i$ is the identity element, then the corresponding face of the simplex will be a degenerate simplex.

EXAMPLES:

The nerve (classifying space) of the cyclic group of order 2 is infinite-dimensional real projective space.

```
sage: Sigma2 = groups.permutation.Cyclic(2)
sage: BSigma2 = Sigma2.nerve()
sage: BSigma2.cohomology(4, base_ring=GF(2))
Vector space of dimension 1 over Finite Field of size 2
```

The $k$-simplices of the nerve are named after the chains of $k$ non-unit elements to be multiplied. The group $\Sigma_2$ has two elements, written `()` (the identity element) and `(1,2)` in Sage. So the 1-cells and 2-cells in $B\Sigma_2$ are:

```
sage: BSigma2.n_cells(1)
[(1,2)]
sage: BSigma2.n_cells(2)
[(1,2) * (1,2)]
```

Another construction of the group, with different names for its elements:

```
sage: C2 = groups.misc.MultiplicativeAbelian([2])
sage: BC2 = C2.nerve()
sage: BC2.n_cells(0)
[1]
sage: BC2.n_cells(1)
[f]
sage: BC2.n_cells(2)
[f * f]
```

With mod $p$ coefficients, $B\Sigma_p$ should have its first nonvanishing homology group in dimension $p$:

```
sage: Sigma3 = groups.permutation.Symmetric(3)
sage: BSigma3 = Sigma3.nerve()
sage: BSigma3.homology(range(4), base_ring=GF(3))
```

```
{0: Vector space of dimension 0 over Finite Field of size 3,
1: Vector space of dimension 0 over Finite Field of size 3,
2: Vector space of dimension 0 over Finite Field of size 3,
3: Vector space of dimension 1 over Finite Field of size 3}
```

Note that we can construct the $n$-skeleton for $B\Sigma_2$ for relatively large values of $n$, while for $B\Sigma_3$, the complexes get large pretty quickly:

```
sage: Sigma2.nerve().n_skeleton(14)
Simplicial set with 15 non-degenerate simplices

sage: BSigma3 = Sigma3.nerve()
sage: BSigma3.n_skeleton(3)
Simplicial set with 156 non-degenerate simplices
sage: BSigma3.n_skeleton(4)
Simplicial set with 781 non-degenerate simplices
```

Finally, note that the classifying space of the order $p$ cyclic group is smaller than that of the symmetric group on $p$ letters, and its first homology group appears earlier:

```
sage: C3 = groups.misc.MultiplicativeAbelian([3])
sage: list(C3)
[1, f, f^2]
sage: BC3 = C3.nerve()
sage: BC3.n_cells(1)
[f, f^2]
sage: BC3.n_cells(2)
[f * f, f * f^2, f^2 * f, f^2 * f^2]
sage: len(BSigma3.n_cells(2))
25
sage: len(BC3.n_cells(3))
8
sage: len(BSigma3.n_cells(3))
125

sage: BC3.homology(range(5), base_ring=GF(3))
{0: Vector space of dimension 0 over Finite Field of size 3,
 1: Vector space of dimension 1 over Finite Field of size 3,
 2: Vector space of dimension 1 over Finite Field of size 3,
 3: Vector space of dimension 1 over Finite Field of size 3,
 4: Vector space of dimension 1 over Finite Field of size 3}

sage: BC5 = groups.permutation.Cyclic(5).nerve()
sage: BC5.homology(range(5), base_ring=GF(5))
{0: Vector space of dimension 0 over Finite Field of size 5,
1: Vector space of dimension 1 over Finite Field of size 5,
2: Vector space of dimension 1 over Finite Field of size 5,
3: Vector space of dimension 1 over Finite Field of size 5,
4: Vector space of dimension 1 over Finite Field of size 5}
```

**rhodes_radical_congruence**(*base_ring=None*)

Return the Rhodes radical congruence of the semigroup.

The Rhodes radical congruence is the congruence induced on S by the map $S \to kS \to kS/radkS$ with k a field.

INPUT:

- `base_ring` (default: $\mathbf{Q}$) a field

OUTPUT:

- A list of couples (m, n) with $m \neq n$ in the lexicographic order for the enumeration of the monoid `self`.

EXAMPLES:

```
sage: M = Monoids().Finite().example()
sage: M.rhodes_radical_congruence()
[(0, 6), (2, 8), (4, 10)]
sage: from sage.monoids.hecke_monoid import HeckeMonoid
sage: H3 = HeckeMonoid(SymmetricGroup(3))
sage: H3.repr_element_method(style="reduced")
sage: H3.rhodes_radical_congruence()
[([1, 2], [2, 1]), ([1, 2], [1, 2, 1]), ([2, 1], [1, 2, 1])]
```

By Maschke's theorem, every group algebra over $\mathbf{Q}$ is semisimple hence the Rhodes radical of a group must be trivial:

```
sage: SymmetricGroup(3).rhodes_radical_congruence()
[]
sage: DihedralGroup(10).rhodes_radical_congruence()
[]
```

REFERENCES:

- [Rho69]

## 3.59 Finite Permutation Groups

**class** `sage.categories.finite_permutation_groups.`**`FinitePermutationGroups`**(*base_category*)

Bases: *`sage.categories.category_with_axiom.CategoryWithAxiom`*

The category of finite permutation groups, i.e. groups concretely represented as groups of permutations acting on a finite set.

It is currently assumed that any finite permutation group comes endowed with a distinguished finite set of generators (method `group_generators`); this is the case for all the existing implementations in Sage.

EXAMPLES:

```
sage: C = PermutationGroups().Finite(); C
Category of finite enumerated permutation groups
sage: C.super_categories()
[Category of permutation groups,
 Category of finite groups,
 Category of finite finitely generated semigroups]

sage: C.example()
Dihedral group of order 6 as a permutation group
```

**class ElementMethods**

**class ParentMethods**

**`cycle_index`**(*parent=None*)

Return the *cycle index* of `self`.

INPUT:

- `self` - a permutation group $G$

- `parent` – a free module with basis indexed by partitions, or behave as such, with a `term` and `sum` method (default: the symmetric functions over the rational field in the $p$ basis)

The *cycle index* of a permutation group $G$ (Wikipedia article Cycle_index) is a gadget counting the elements of $G$ by cycle type, averaged over the group:

$$P = \frac{1}{|G|} \sum_{g \in G} p_{\text{cycle type}(g)}$$

EXAMPLES:

Among the permutations of the symmetric group $S_4$, there is the identity, 6 cycles of length 2, 3 products of two cycles of length 2, 8 cycles of length 3, and 6 cycles of length 4:

```
sage: S4 = SymmetricGroup(4)
sage: P = S4.cycle_index()
sage: 24 * P
p[1, 1, 1, 1] + 6*p[2, 1, 1] + 3*p[2, 2] + 8*p[3, 1] + 6*p[4]
```

If $l = (l_1, \dots, l_k)$ is a partition, `|G| P[l]` is the number of elements of $G$ with cycles of length $(p_1, \dots, p_k)$:

```
sage: 24 * P[ Partition([3,1]) ]
8
```

The cycle index plays an important role in the enumeration of objects modulo the action of a group (Pólya enumeration), via the use of symmetric functions and plethysms. It is therefore encoded as a symmetric function, expressed in the powersum basis:

```
sage: P.parent()
Symmetric Functions over Rational Field in the powersum basis
```

This symmetric function can have some nice properties; for example, for the symmetric group $S_n$, we get the complete symmetric function $h_n$:

```
sage: S = SymmetricFunctions(QQ); h = S.h()
sage: h( P )
h[4]
```

**Todo:** Add some simple examples of Pólya enumeration, once it will be easy to expand symmetric functions on any alphabet.

Here are the cycle indices of some permutation groups:

```
sage: 6 * CyclicPermutationGroup(6).cycle_index()
p[1, 1, 1, 1, 1, 1] + p[2, 2, 2] + 2*p[3, 3] + 2*p[6]

sage: 60 * AlternatingGroup(5).cycle_index()
p[1, 1, 1, 1, 1] + 15*p[2, 2, 1] + 20*p[3, 1, 1] + 24*p[5]

sage: for G in TransitiveGroups(5):               # optional - database_
↪gap # long time
....:     G.cardinality() * G.cycle_index()
p[1, 1, 1, 1, 1] + 4*p[5]
p[1, 1, 1, 1, 1] + 5*p[2, 2, 1] + 4*p[5]
p[1, 1, 1, 1, 1] + 5*p[2, 2, 1] + 10*p[4, 1] + 4*p[5]
p[1, 1, 1, 1, 1] + 15*p[2, 2, 1] + 20*p[3, 1, 1] + 24*p[5]
p[1, 1, 1, 1, 1] + 10*p[2, 1, 1, 1] + 15*p[2, 2, 1] + 20*p[3, 1, 1] +␣
↪20*p[3, 2] + 30*p[4, 1] + 24*p[5]
```

Permutation groups with arbitrary domains are supported (see trac ticket #22765):

```
sage: G = PermutationGroup([['b','c','a']], domain=['a','b','c'])
sage: G.cycle_index()
1/3*p[1, 1, 1] + 2/3*p[3]
```

One may specify another parent for the result:

```
sage: F = CombinatorialFreeModule(QQ, Partitions())
sage: P = CyclicPermutationGroup(6).cycle_index(parent = F)
sage: 6 * P
B[[1, 1, 1, 1, 1, 1]] + B[[2, 2, 2]] + 2*B[[3, 3]] + 2*B[[6]]
sage: P.parent() is F
True
```

This parent should be a module with basis indexed by partitions:

```
sage: CyclicPermutationGroup(6).cycle_index(parent = QQ)
Traceback (most recent call last):
  ...
ValueError: `parent` should be a module with basis indexed by partitions
```

REFERENCES:
- [Ke1991]

AUTHORS:
- Nicolas Borie and Nicolas M. Thiéry

**example**()

Returns an example of finite permutation group, as per *Category.example()*.

EXAMPLES:

```
sage: G = FinitePermutationGroups().example(); G
Dihedral group of order 6 as a permutation group
```

**extra_super_categories**()

Any permutation group is assumed to be endowed with a finite set of generators.

## 3.60 Finite posets

Here is some terminology used in this file:

- An *order filter* (or *upper set*) of a poset $P$ is a subset $S$ of $P$ such that if $x \leq y$ and $x \in S$ then $y \in S$.
- An *order ideal* (or *lower set*) of a poset $P$ is a subset $S$ of $P$ such that if $x \leq y$ and $y \in S$ then $x \in S$.

**class** sage.categories.finite_posets.**FinitePosets**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

The category of finite posets i.e. finite sets with a partial order structure.

EXAMPLES:

```
sage: FinitePosets()
Category of finite posets
sage: FinitePosets().super_categories()
[Category of posets, Category of finite sets]
```

```
sage: FinitePosets().example()
NotImplemented
```

**See also:**

*Posets*, Poset()

**class ParentMethods**

**antichains**()

Return all antichains of self.

EXAMPLES:

```
sage: A = posets.PentagonPoset().antichains(); A
Set of antichains of Finite lattice containing 5 elements
sage: list(A)
[[], [0], [1], [1, 2], [1, 3], [2], [3], [4]]
```

**birational_free_labelling**(*linear_extension=None*, *prefix='x'*, *base_field=None*, *reduced=False*, *addvars=None*)

Return the birational free labelling of self.

Let us hold back defining this, and introduce birational toggles and birational rowmotion first. These notions have been introduced in [EP2013] as generalizations of the notions of toggles (order_ideal_toggle()) and *rowmotion* on order ideals of a finite poset. They have been studied further in [GR2013].

Let $\mathbf{K}$ be a field, and $P$ be a finite poset. Let $\widehat{P}$ denote the poset obtained from $P$ by adding a new element 1 which is greater than all existing elements of $P$, and a new element 0 which is smaller than all existing elements of $P$ and 1. Now, a $\mathbf{K}$-*labelling of* $P$ will mean any function from $\widehat{P}$ to $\mathbf{K}$. The image of an element $v$ of $\widehat{P}$ under this labelling will be called the *label* of this labelling at $v$. The set of all $\mathbf{K}$-labellings of $P$ is clearly $\mathbf{K}^{\widehat{P}}$.

For any $v \in P$, we now define a rational map $T_v : \mathbf{K}^{\widehat{P}} \dashrightarrow \mathbf{K}^{\widehat{P}}$ as follows: For every $f \in \mathbf{K}^{\widehat{P}}$, the image $T_v f$ should send every element $u \in \widehat{P}$ distinct from $v$ to $f(u)$ (so the labels at all $u \neq v$ don't change), while $v$ is sent to

$$\frac{1}{f(v)} \cdot \frac{\sum_{u \lessdot v} f(u)}{\sum_{u \gtrdot v} \frac{1}{f(u)}}$$

(both sums are over all $u \in \widehat{P}$ satisfying the respectively given conditions). Here, $\lessdot$ and $\gtrdot$ mean (respectively) "covered by" and "covers", interpreted with respect to the poset $\widehat{P}$. This rational map $T_v$ is an involution and is called the *(birational) $v$-toggle*; see *birational_toggle()* for its implementation.

Now, *birational rowmotion* is defined as the composition $T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n}$, where $(v_1, v_2, \ldots, v_n)$ is a linear extension of $P$ (written as a linear ordering of the elements of $P$). This is a rational map $\mathbf{K}^{\widehat{P}} \dashrightarrow \mathbf{K}^{\widehat{P}}$ which does not depend on the choice of the linear extension; it is denoted by $R$. See *birational_rowmotion()* for its implementation.

The definitions of birational toggles and birational rowmotion extend to the case of $\mathbf{K}$ being any semifield rather than necessarily a field (although it becomes less clear what constitutes a rational map in this generality). The most useful case is that of the tropical semiring, in which case birational rowmotion relates to classical constructions such as promotion of rectangular semistandard Young tableaux (page 5 of [EP2013b] and future work, via the related notion of birational *promotion*) and rowmotion on order ideals of the poset ([EP2013]).

The *birational free labelling* is a special labelling defined for every finite poset $P$ and every linear extension $(v_1, v_2, \ldots, v_n)$ of $P$. It is given by sending every element $v_i$ in $P$ to $x_i$, sending the element $0$ of $\widehat{P}$ to $a$, and sending the element $1$ of $\widehat{P}$ to $b$, where the ground field $\mathbf{K}$ is the field of rational functions in $n + 2$ indeterminates $a, x_1, x_2, \ldots, x_n, b$ over $\mathbb{Q}$.

In Sage, a labelling $f$ of a poset $P$ is encoded as a $4$-tuple $(\mathbf{K}, d, u, v)$, where $\mathbf{K}$ is the ground field of the labelling (i. e., its target), $d$ is the dictionary containing the values of $f$ at the elements of $P$ (the keys being the respective elements of $P$), $u$ is the label of $f$ at $0$, and $v$ is the label of $f$ at $1$.

> **Warning:** The dictionary $d$ is labelled by the elements of $P$. If $P$ is a poset with `facade` option set to `False`, these might not be what they seem to be! (For instance, if `P == Poset({1: [2, 3]}, facade=False)`, then the value of $d$ at $1$ has to be accessed by `d[P(1)]`, not by `d[1]`.)

> **Warning:** Dictionaries are mutable. They do compare correctly, but are not hashable and need to be cloned to avoid spooky action at a distance. Be careful!

INPUT:

- `linear_extension` – (default: the default linear extension of `self`) a linear extension of `self` (as a linear extension or as a list), or more generally a list of all elements of all elements of `self` each occurring exactly once
- `prefix` – (default: `'x'`) the prefix to name the indeterminates corresponding to the elements of `self` in the labelling (so, setting it to `'frog'` will result in these indeterminates being called `frog1, frog2, ..., frogn` rather than `x1, x2, ..., xn`).
- `base_field` – (default: `QQ`) the base field to be used instead of $\mathbf{Q}$ to define the rational function field over; this is not going to be the base field of the labelling, because the latter will have indeterminates adjoined!
- `reduced` – (default: `False`) if set to `True`, the result will be the *reduced* birational free labelling, which differs from the regular one by having $0$ and $1$ both sent to $1$ instead of $a$ and $b$ (the indeterminates $a$ and $b$ then also won't appear in the ground field)
- `addvars` – (default: `''`) a string containing names of extra variables to be adjoined to the ground field (these don't have an effect on the labels)

OUTPUT:

The birational free labelling of the poset `self` and the linear extension `linear_extension`. Or, if `reduced` is set to `True`, the reduced birational free labelling.

EXAMPLES:

We construct the birational free labelling on a simple poset:

```
sage: P = Poset({1: [2, 3]})
sage: l = P.birational_free_labelling(); l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(l[1].items())
[(1, x1), (2, x2), (3, x3)]

sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2]); l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
```

```
 {...},
 a,
 b)
sage: sorted(l[1].items())
[(1, x1), (2, x3), (3, x2)]

sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2],␣
↪reduced=True, addvars="spam, eggs"); l
(Fraction Field of Multivariate Polynomial Ring in x1, x2, x3, spam, eggs␣
↪over Rational Field,
 {...},
 1,
 1)
sage: sorted(l[1].items())
[(1, x1), (2, x3), (3, x2)]

sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2], prefix=
↪"wut", reduced=True, addvars="spam, eggs"); l
(Fraction Field of Multivariate Polynomial Ring in wut1, wut2, wut3, spam,
↪ eggs over Rational Field,
 {...},
 1,
 1)
sage: sorted(l[1].items())
[(1, wut1), (2, wut3), (3, wut2)]

sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2],␣
↪reduced=False, addvars="spam, eggs"); l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b, spam,
↪ eggs over Rational Field,
 {...},
 a,
 b)
sage: sorted(l[1].items())
[(1, x1), (2, x3), (3, x2)]
sage: l[1][2]
x3
```

Illustrating the warning about facade:

```
sage: P = Poset({1: [2, 3]}, facade=False)
sage: l = P.birational_free_labelling(linear_extension=[1, 3, 2],␣
↪reduced=False, addvars="spam, eggs"); l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b, spam,
↪ eggs over Rational Field,
 {...},
 a,
 b)
sage: l[1][2]
Traceback (most recent call last):
...
KeyError: 2
sage: l[1][P(2)]
x3
```

Another poset:

```
sage: P = posets.SSTPoset([2,1])
sage: lext = sorted(P)
sage: l = P.birational_free_labelling(linear_extension=lext, addvars="ohai
↪")
sage: l
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, x4, x5,␣
↪x6, x7, x8, b, ohai over Rational Field,
 {...},
 a,
 b)
sage: sorted(l[1].items())
[(([[1, 1], [2]], x1), ([[1, 1], [3]], x2), ([[1, 2], [2]], x3), ([[1, 2],␣
↪[3]], x4),
 ([[1, 3], [2]], x5), ([[1, 3], [3]], x6), ([[2, 2], [3]], x7), ([[2, 3],␣
↪[3]], x8)]
```

See `birational_rowmotion()`, `birational_toggle()` and `birational_toggles()` for more substantial examples of what one can do with the birational free labelling.

**birational_rowmotion**(*labelling*)

Return the result of applying birational rowmotion to the $\mathbf{K}$-labelling `labelling` of the poset `self`.

See the documentation of `birational_free_labelling()` for a definition of birational rowmotion and $\mathbf{K}$-labellings and for an explanation of how $\mathbf{K}$-labellings are to be encoded to be understood by Sage. This implementation allows $\mathbf{K}$ to be a semifield, not just a field. Birational rowmotion is only a rational map, so an exception (most likely, `ZeroDivisionError`) will be thrown if the denominator is zero.

INPUT:

- `labelling` – a $\mathbf{K}$-labelling of `self` in the sense as defined in the documentation of `birational_free_labelling()`

OUTPUT:

The image of the $\mathbf{K}$-labelling $f$ under birational rowmotion.

EXAMPLES:

```
sage: P = Poset({1: [2, 3], 2: [4], 3: [4]})
sage: lex = [1, 2, 3, 4]
sage: t = P.birational_free_labelling(linear_extension=lex); t
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, x4, b␣
↪over Rational Field,
 {...},
 a,
 b)
sage: sorted(t[1].items())
[(1, x1), (2, x2), (3, x3), (4, x4)]
sage: t = P.birational_rowmotion(t); t
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, x4, b␣
↪over Rational Field,
 {...},
 a,
 b)
sage: sorted(t[1].items())
[(1, a*b/x4), (2, (x1*x2*b + x1*x3*b)/(x2*x4)),
 (3, (x1*x2*b + x1*x3*b)/(x3*x4)), (4, (x2*b + x3*b)/x4)]
```

A result of [GR2013] states that applying birational rowmotion $n + m$ times to a $\mathbf{K}$-labelling $f$ of the poset $[n] \times [m]$ gives back $f$. Let us check this:

```
sage: def test_rectangle_periodicity(n, m, k):
....:     P = posets.ChainPoset(n).product(posets.ChainPoset(m))
....:     t0 = P.birational_free_labelling(P)
....:     t = t0
....:     for i in range(k):
....:         t = P.birational_rowmotion(t)
....:     return t == t0
sage: test_rectangle_periodicity(2, 2, 4)
True
sage: test_rectangle_periodicity(2, 2, 2)
False
sage: test_rectangle_periodicity(2, 3, 5)  # long time
True
```

While computations with the birational free labelling quickly run out of memory due to the complexity of the rational functions involved, it is computationally cheap to check properties of birational rowmotion on examples in the tropical semiring:

```
sage: def test_rectangle_periodicity_tropical(n, m, k):
....:     P = posets.ChainPoset(n).product(posets.ChainPoset(m))
....:     TT = TropicalSemiring(ZZ)
....:     t0 = (TT, {v: TT(floor(random()*100)) for v in P}, TT(0),␣
↪TT(124))
....:     t = t0
....:     for i in range(k):
....:         t = P.birational_rowmotion(t)
....:     return t == t0
sage: test_rectangle_periodicity_tropical(7, 6, 13)
True
```

Tropicalization is also what relates birational rowmotion to classical rowmotion on order ideals. In fact, if $T$ denotes the `tropical semiring` of $\mathbf{Z}$ and $P$ is a finite poset, then we can define an embedding $\phi$ from the set $J(P)$ of all order ideals of $P$ into the set $T^{\widehat{P}}$ of all $T$-labellings of $P$ by sending every $I \in J(P)$ to the indicator function of $I$ extended by the value 1 at the element 0 and the value 0 at the element 1. This map $\phi$ has the property that $R \circ \phi = \phi \circ r$, where $R$ denotes birational rowmotion, and $r$ denotes *`classical rowmotion`* on $J(P)$. An example:

```
sage: P = posets.IntegerPartitions(5)
sage: TT = TropicalSemiring(ZZ)
sage: def indicator_labelling(I):
....:     # send order ideal `I` to a `T`-labelling of `P`.
....:     dct = {v: TT(v in I) for v in P}
....:     return (TT, dct, TT(1), TT(0))
sage: all(indicator_labelling(P.rowmotion(I))
....:     == P.birational_rowmotion(indicator_labelling(I))
....:     for I in P.order_ideals_lattice(facade=True))
True
```

**birational_toggle**(*v*, *labelling*)

Return the result of applying the birational $v$-toggle $T_v$ to the $\mathbf{K}$-labelling `labelling` of the poset `self`.

See the documentation of *`birational_free_labelling()`* for a definition of this toggle and of $\mathbf{K}$-labellings as well as an explanation of how $\mathbf{K}$-labellings are to be encoded to be understood by Sage. This implementation allows $\mathbf{K}$ to be a semifield, not just a field. The birational $v$-toggle is only a rational map, so an exception (most likely, `ZeroDivisionError`) will be thrown if the denominator is zero.

INPUT:

- v – an element of self (must have self as parent if self is a facade=False poset)
- labelling – a $\mathbf{K}$-labelling of self in the sense as defined in the documentation of *birational_free_labelling()*

OUTPUT:

The $\mathbf{K}$-labelling $T_v f$ of self, where $f$ is labelling.

EXAMPLES:

Let us start with the birational free labelling of the "V"-poset (the three-element poset with Hasse diagram looking like a "V"):

```
sage: V = Poset({1: [2, 3]})
sage: s = V.birational_free_labelling(); s
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(s[1].items())
[(1, x1), (2, x2), (3, x3)]
```

The image of $s$ under the 1-toggle $T_1$ is:

```
sage: s1 = V.birational_toggle(1, s); s1
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(s1[1].items())
[(1, a*x2*x3/(x1*x2 + x1*x3)), (2, x2), (3, x3)]
```

Now let us apply the 2-toggle $T_2$ (to the old s):

```
sage: s2 = V.birational_toggle(2, s); s2
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(s2[1].items())
[(1, x1), (2, x1*b/x2), (3, x3)]
```

On the other hand, we can also apply $T_2$ to the image of $s$ under $T_1$:

```
sage: s12 = V.birational_toggle(2, s1); s12
(Fraction Field of Multivariate Polynomial Ring in a, x1, x2, x3, b over␣
↪Rational Field,
 {...},
 a,
 b)
sage: sorted(s12[1].items())
[(1, a*x2*x3/(x1*x2 + x1*x3)), (2, a*x3*b/(x1*x2 + x1*x3)), (3, x3)]
```

Each toggle is an involution:

```
sage: all( V.birational_toggle(i, V.birational_toggle(i, s)) == s
....:      for i in V )
True
```

We can also start with a less generic labelling:

```
sage: t = (QQ, {1: 3, 2: 6, 3: 7}, 2, 10)
sage: t1 = V.birational_toggle(1, t); t1
(Rational Field, {...}, 2, 10)
sage: sorted(t1[1].items())
[(1, 28/13), (2, 6), (3, 7)]
sage: t13 = V.birational_toggle(3, t1); t13
(Rational Field, {...}, 2, 10)
sage: sorted(t13[1].items())
[(1, 28/13), (2, 6), (3, 40/13)]
```

However, labellings have to be sufficiently generic, lest denominators vanish:

```
sage: t = (QQ, {1: 3, 2: 5, 3: -5}, 1, 15)
sage: t1 = V.birational_toggle(1, t)
Traceback (most recent call last):
...
ZeroDivisionError: rational division by zero
```

We don't get into zero-division issues in the tropical semiring (unless the zero of the tropical semiring appears in the labelling):

```
sage: TT = TropicalSemiring(QQ)
sage: t = (TT, {1: TT(2), 2: TT(4), 3: TT(1)}, TT(6), TT(0))
sage: t1 = V.birational_toggle(1, t); t1
(Tropical semiring over Rational Field, {...}, 6, 0)
sage: sorted(t1[1].items())
[(1, 8), (2, 4), (3, 1)]
sage: t12 = V.birational_toggle(2, t1); t12
(Tropical semiring over Rational Field, {...}, 6, 0)
sage: sorted(t12[1].items())
[(1, 8), (2, 4), (3, 1)]
sage: t123 = V.birational_toggle(3, t12); t123
(Tropical semiring over Rational Field, {...}, 6, 0)
sage: sorted(t123[1].items())
[(1, 8), (2, 4), (3, 7)]
```

We turn to more interesting posets. Here is the 6-element poset arising from the weak order on $S_3$:

```
sage: P = posets.SymmetricGroupWeakOrderPoset(3)
sage: sorted(list(P))
['123', '132', '213', '231', '312', '321']
sage: t = (TT, {'123': TT(4), '132': TT(2), '213': TT(3), '231': TT(1),
↪'321': TT(1), '312': TT(2)}, TT(7), TT(1))
sage: t1 = P.birational_toggle('123', t); t1
(Tropical semiring over Rational Field, {...}, 7, 1)
sage: sorted(t1[1].items())
[('123', 6), ('132', 2), ('213', 3), ('231', 1), ('312', 2), ('321', 1)]
sage: t13 = P.birational_toggle('213', t1); t13
(Tropical semiring over Rational Field, {...}, 7, 1)
sage: sorted(t13[1].items())
[('123', 6), ('132', 2), ('213', 4), ('231', 1), ('312', 2), ('321', 1)]
```

Let us verify on this example some basic properties of toggles. First of all, again let us check that $T_v$ is an involution for every $v$:

```
sage: all( P.birational_toggle(v, P.birational_toggle(v, t)) == t
....:      for v in P )
True
```

Furthermore, two toggles $T_v$ and $T_w$ commute unless one of $v$ or $w$ covers the other:

```
sage: all( P.covers(v, w) or P.covers(w, v)
....:      or P.birational_toggle(v, P.birational_toggle(w, t))
....:         == P.birational_toggle(w, P.birational_toggle(v, t))
....:      for v in P for w in P )
True
```

**birational_toggles**(*vs*, *labelling*)

Return the result of applying a sequence of birational toggles (specified by `vs`) to the $\mathbf{K}$-labelling `labelling` of the poset `self`.

See the documentation of *birational_free_labelling()* for a definition of birational toggles and $\mathbf{K}$-labellings and for an explanation of how $\mathbf{K}$-labellings are to be encoded to be understood by Sage. This implementation allows $\mathbf{K}$ to be a semifield, not just a field. The birational $v$-toggle is only a rational map, so an exception (most likely, `ZeroDivisionError`) will be thrown if the denominator is zero.

INPUT:

- `vs` – an iterable comprising elements of `self` (which must have `self` as parent if `self` is a `facade=False` poset)
- `labelling` – a $\mathbf{K}$-labelling of `self` in the sense as defined in the documentation of *birational_free_labelling()*

OUTPUT:

The $\mathbf{K}$-labelling $T_{v_n} T_{v_{n-1}} \cdots T_{v_1} f$ of `self`, where $f$ is `labelling` and $(v_1, v_2, \ldots, v_n)$ is `vs` (written as list).

EXAMPLES:

```
sage: P = posets.SymmetricGroupBruhatOrderPoset(3)
sage: sorted(list(P))
['123', '132', '213', '231', '312', '321']
sage: TT = TropicalSemiring(ZZ)
sage: t = (TT, {'123': TT(4), '132': TT(2), '213': TT(3), '231': TT(1),
↪'321': TT(1), '312': TT(2)}, TT(7), TT(1))
sage: tA = P.birational_toggles(['123', '231', '312'], t); tA
(Tropical semiring over Integer Ring, {...}, 7, 1)
sage: sorted(tA[1].items())
[('123', 6), ('132', 2), ('213', 3), ('231', 2), ('312', 1), ('321', 1)]
sage: tAB = P.birational_toggles(['132', '213', '321'], tA); tAB
(Tropical semiring over Integer Ring, {...}, 7, 1)
sage: sorted(tAB[1].items())
[('123', 6), ('132', 6), ('213', 5), ('231', 2), ('312', 1), ('321', 1)]

sage: P = Poset({1: [2, 3], 2: [4], 3: [4]})
sage: Qx = PolynomialRing(QQ, 'x').fraction_field()
sage: x = Qx.gen()
sage: t = (Qx, {1: 1, 2: x, 3: (x+1)/x, 4: x^2}, 1, 1)
sage: t1 = P.birational_toggles((i for i in range(1, 5)), t); t1
(Fraction Field of Univariate Polynomial Ring in x over Rational Field,
 {...},
```

```
 1,
 1)
sage: sorted(t1[1].items())
[(1, (x^2 + x)/(x^2 + x + 1)), (2, (x^3 + x^2)/(x^2 + x + 1)), (3, x^4/(x^
↪2 + x + 1)), (4, 1)]
sage: t2 = P.birational_toggles(reversed(range(1, 5)), t)
sage: sorted(t2[1].items())
[(1, 1/x^2), (2, (x^2 + x + 1)/x^4), (3, (x^2 + x + 1)/(x^3 + x^2)), (4,␣
↪(x^2 + x + 1)/x^3)]
```

Facade set to `False` works:

```
sage: P = Poset({'x': ['y', 'w'], 'y': ['z'], 'w': ['z']}, facade=False)
sage: lex = ['x', 'y', 'w', 'z']
sage: t = P.birational_free_labelling(linear_extension=lex)
sage: sorted(P.birational_toggles([P('x'), P('y')], t)[1].items())
[(x, a*x2*x3/(x1*x2 + x1*x3)), (y, a*x3*x4/(x1*x2 + x1*x3)), (w, x3), (z,␣
↪x4)]
```

**deprecated_function_alias**(*trac_number*, *func*)

Create an aliased version of a function or a method which raise a deprecation warning message.

If f is a function or a method, write `g = deprecated_function_alias(trac_number, f)` to make a deprecated aliased version of f.

INPUT:

- `trac_number` – integer. The trac ticket number where the deprecation is introduced.
- `func` – the function or method to be aliased

EXAMPLES:

```
sage: from sage.misc.superseded import deprecated_function_alias
sage: g = deprecated_function_alias(13109, number_of_partitions)
sage: g(5)
doctest:...: DeprecationWarning: g is deprecated. Please use sage.
↪combinat.partition.number_of_partitions instead.
See http://trac.sagemath.org/13109 for details.
7
```

This also works for methods:

```
sage: class cls(object):
....:    def new_meth(self): return 42
....:    old_meth = deprecated_function_alias(13109, new_meth)
sage: cls().old_meth()
doctest:...: DeprecationWarning: old_meth is deprecated. Please use new_
↪meth instead.
See http://trac.sagemath.org/13109 for details.
42
```

trac ticket #11585:

```
sage: def a(): pass
sage: b = deprecated_function_alias(13109, a)
sage: b()
doctest:...: DeprecationWarning: b is deprecated. Please use a instead.
See http://trac.sagemath.org/13109 for details.
```

AUTHORS:

- Florent Hivert (2009-11-23), with the help of Mike Hansen.
- Luca De Feo (2011-07-11), printing the full module path when different from old path

**directed_subsets** (*direction*)

Return the order filters (resp. order ideals) of `self`, as lists.

If `direction` is 'up', returns the order filters (upper sets).

If `direction` is 'down', returns the order ideals (lower sets).

INPUT:

- `direction` – 'up' or 'down'

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")), facade=True)
sage: A = P.directed_subsets('up')
sage: sorted(list(A))
[[], [1, 2, 4, 3, 6, 12], [2, 4, 3, 6, 12], [2, 4, 6, 12], [3, 6, 12], [4,
↪ 3, 6, 12], [4, 6, 12], [4, 12], [6, 12], [12]]
```

**is_lattice** ()

Return whether the poset is a lattice.

A poset is a lattice if all pairs of elements have both a least upper bound ("join") and a greatest lower bound ("meet") in the poset.

EXAMPLES:

```
sage: P = Poset([[1, 3, 2], [4], [4, 5, 6], [6], [7], [7], [7], []])
sage: P.is_lattice()
True

sage: P = Poset([[1, 2], [3], [3], []])
sage: P.is_lattice()
True

sage: P = Poset({0: [2, 3], 1: [2, 3]})
sage: P.is_lattice()
False

sage: P = Poset({1: [2, 3, 4], 2: [5, 6], 3: [5, 7], 4: [6, 7], 5: [8, 9],
....:            6: [8, 10], 7: [9, 10], 8: [11], 9: [11], 10: [11]})
sage: P.is_lattice()
False
```

**See also:**

- Weaker properties: `is_join_semilattice()`, `is_meet_semilattice()`

**is_poset_isomorphism** (*f*, *codomain*)

Return whether $f$ is an isomorphism of posets from `self` to `codomain`.

INPUT:

- `f` – a function from `self` to `codomain`
- `codomain` – a poset

EXAMPLES:

We build the poset $D$ of divisors of 30, and check that it is isomorphic to the boolean lattice $B$ of the subsets of $\{2, 3, 5\}$ ordered by inclusion, via the reverse function $f : B \to D, b \mapsto \prod_{x \in b} x$:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: B = Poset(([frozenset(s) for s in Subsets([2,3,5])], attrcall(
↪"issubset")))
sage: def f(b): return D(prod(b))
sage: B.is_poset_isomorphism(f, D)
True
```

On the other hand, $f$ is not an isomorphism to the chain of divisors of 30, ordered by usual comparison:

```
sage: P = Poset((divisors(30), operator.le))
sage: def f(b): return P(prod(b))
sage: B.is_poset_isomorphism(f, P)
False
```

A non surjective case:

```
sage: B = Poset(([frozenset(s) for s in Subsets([2,3])], attrcall(
↪"issubset")))
sage: def f(b): return D(prod(b))
sage: B.is_poset_isomorphism(f, D)
False
```

A non injective case:

```
sage: B = Poset(([frozenset(s) for s in Subsets([2,3,5,6])], attrcall(
↪"issubset")))
sage: def f(b): return D(gcd(prod(b), 30))
sage: B.is_poset_isomorphism(f, D)
False
```

**Note:** since `D` and `B` are not facade posets, `f` is responsible for the conversions between integers and subsets to elements of `D` and `B` and back.

**See also:**

*FiniteLatticePosets.ParentMethods.is_lattice_morphism()*

**is_poset_morphism**(*f*, *codomain*)

Return whether $f$ is a morphism of posets from `self` to `codomain`, that is

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

for all $x$ and $y$ in `self`.

INPUT:

- `f` – a function from `self` to `codomain`
- `codomain` – a poset

EXAMPLES:

We build the boolean lattice of the subsets of $\{2, 3, 5, 6\}$ and the lattice of divisors of 30, and check that the map $b \mapsto \gcd(\prod_{x \in b} x, 30)$ is a morphism of posets:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: B = Poset(([frozenset(s) for s in Subsets([2,3,5,6])], attrcall(
↪"issubset")))
sage: def f(b): return D(gcd(prod(b), 30))
```

```
sage: B.is_poset_morphism(f, D)
True
```

**Note:** since D and B are not facade posets, f is responsible for the conversions between integers and subsets to elements of D and B and back.

*f* is also a morphism of posets to the chain of divisors of 30, ordered by usual comparison:

```
sage: P = Poset((divisors(30), operator.le))
sage: def f(b): return P(gcd(prod(b), 30))
sage: B.is_poset_morphism(f, P)
True
```

FIXME: should this be `is_order_preserving_morphism`?

**See also:**

*is_poset_isomorphism()*

**is_self_dual**()

Return whether the poset is *self-dual*.

A poset is self-dual if it is isomorphic to its dual poset.

EXAMPLES:

```
sage: P = Poset({1: [3, 4], 2: [3, 4]})
sage: P.is_self_dual()
True

sage: P = Poset({1: [2, 3]})
sage: P.is_self_dual()
False
```

**See also:**

- Stronger properties: `is_orthocomplemented()` (for lattices)
- Other: `dual()`

**is_selfdual**(**args*, ***kwds*)

Deprecated: Use *is_self_dual()* instead. See trac ticket #24048 for details.

**order_filter_generators**(*filter*)

Generators for an order filter

INPUT:

- `filter` – an order filter of `self`, as a list (or iterable)

EXAMPLES:

```
sage: P = Poset((Subsets([1,2,3]), attrcall("issubset")))
sage: I = P.order_filter([Set([1,2]), Set([2,3]), Set([1])]); I
[{2, 3}, {1}, {1, 2}, {1, 3}, {1, 2, 3}]
sage: P.order_filter_generators(I)
{{2, 3}, {1}}
```

**See also:**

*order_ideal_generators()*

**order_ideal_complement_generators** (*antichain*, *direction='up'*)

Return the Panyushev complement of the antichain `antichain`.

Given an antichain $A$ of a poset $P$, the Panyushev complement of $A$ is defined to be the antichain consisting of the minimal elements of the order filter $B$, where $B$ is the (set-theoretic) complement of the order ideal of $P$ generated by $A$.

Setting the optional keyword variable `direction` to `'down'` leads to the inverse Panyushev complement being computed instead of the Panyushev complement. The inverse Panyushev complement of an antichain $A$ is the antichain whose Panyushev complement is $A$. It can be found as the antichain consisting of the maximal elements of the order ideal $C$, where $C$ is the (set-theoretic) complement of the order filter of $P$ generated by $A$.

*panyushev_complement()* is an alias for this method.

Panyushev complementation is related (actually, isomorphic) to rowmotion (*rowmotion()*).

INPUT:

- `antichain` – an antichain of `self`, as a list (or iterable), or, more generally, generators of an order ideal (resp. order filter)
- `direction` – 'up' or 'down' (default: 'up')

OUTPUT:

- the generating antichain of the complement order filter (resp. order ideal) of the order ideal (resp. order filter) generated by the antichain `antichain`

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: P.order_ideal_complement_generators([1])
{2}
sage: P.order_ideal_complement_generators([3])
set()
sage: P.order_ideal_complement_generators([1,2])
{3}
sage: P.order_ideal_complement_generators([1,2,3])
set()

sage: P.order_ideal_complement_generators([1], direction="down")
{2}
sage: P.order_ideal_complement_generators([3], direction="down")
{1, 2}
sage: P.order_ideal_complement_generators([1,2], direction="down")
set()
sage: P.order_ideal_complement_generators([1,2,3], direction="down")
set()
```

**Warning:** This is a brute force implementation, building the order ideal generated by the antichain, and searching for order filter generators of its complement

**order_ideal_generators** (*ideal*, *direction='down'*)

Return the antichain of (minimal) generators of the order ideal (resp. order filter) `ideal`.

INPUT:

- `ideal` – an order ideal $I$ (resp. order filter) of `self`, as a list (or iterable); this should be an order ideal if `direction` is set to `'down'`, and an order filter if `direction` is set to `'up'`.
- `direction` – `'up'` or `'down'` (default: `'down'`).

The antichain of (minimal) generators of an order ideal $I$ in a poset $P$ is the set of all minimal elements of $P$. In the case of an order filter, the definition is similar, but with "maximal" used instead of "minimal".

EXAMPLES:

We build the boolean lattice of all subsets of $\{1, 2, 3\}$ ordered by inclusion, and compute an order ideal there:

```
sage: P = Poset((Subsets([1,2,3]), attrcall("issubset")))
sage: I = P.order_ideal([Set([1,2]), Set([2,3]), Set([1])]); I
[{}, {3}, {2}, {2, 3}, {1}, {1, 2}]
```

Then, we retrieve the generators of this ideal:

```
sage: P.order_ideal_generators(I)
{{1, 2}, {2, 3}}
```

If `direction` is 'up', then this instead computes the minimal generators for an order filter:

```
sage: I = P.order_filter([Set([1,2]), Set([2,3]), Set([1])]); I
[{2, 3}, {1}, {1, 2}, {1, 3}, {1, 2, 3}]
sage: P.order_ideal_generators(I, direction='up')
{{2, 3}, {1}}
```

Complexity: $O(n+m)$ where $n$ is the cardinality of $I$, and $m$ the number of upper covers of elements of $I$.

**order_ideals_lattice**(*as_ideals=True, facade=None*)

Return the lattice of order ideals of a poset `self`, ordered by inclusion.

The lattice of order ideals of a poset $P$ is usually denoted by $J(P)$. Its underlying set is the set of order ideals of $P$, and its partial order is given by inclusion.

The order ideals of $P$ are in a canonical bijection with the antichains of $P$. The bijection maps every order ideal to the antichain formed by its maximal elements. By setting the `as_ideals` keyword variable to `False`, one can make this method apply this bijection before returning the lattice.

INPUT:

- `as_ideals` – Boolean, if `True` (default) returns a poset on the set of order ideals, otherwise on the set of antichains
- `facade` – Boolean or `None` (default). Whether to return a facade lattice or not. By default return facade lattice if the poset is a facade poset.

EXAMPLES:

```
sage: P = posets.PentagonPoset()
sage: P.cover_relations()
[[0, 1], [0, 2], [1, 4], [2, 3], [3, 4]]
sage: J = P.order_ideals_lattice(); J
Finite lattice containing 8 elements
sage: list(J)
[{}, {0}, {0, 2}, {0, 2, 3}, {0, 1}, {0, 1, 2}, {0, 1, 2, 3}, {0, 1, 2, 3,
↪ 4}]
```

As a lattice on antichains:

```
sage: J2 = P.order_ideals_lattice(False); J2
Finite lattice containing 8 elements
sage: list(J2)
[(0,), (1, 2), (1, 3), (1,), (2,), (3,), (4,), ()]
```

**panyushev_complement** (*antichain*, *direction='up'*)

Return the Panyushev complement of the antichain `antichain`.

Given an antichain $A$ of a poset $P$, the Panyushev complement of $A$ is defined to be the antichain consisting of the minimal elements of the order filter $B$, where $B$ is the (set-theoretic) complement of the order ideal of $P$ generated by $A$.

Setting the optional keyword variable `direction` to `'down'` leads to the inverse Panyushev complement being computed instead of the Panyushev complement. The inverse Panyushev complement of an antichain $A$ is the antichain whose Panyushev complement is $A$. It can be found as the antichain consisting of the maximal elements of the order ideal $C$, where $C$ is the (set-theoretic) complement of the order filter of $P$ generated by $A$.

*`panyushev_complement()`* is an alias for this method.

Panyushev complementation is related (actually, isomorphic) to rowmotion (*`rowmotion()`*).

INPUT:

- `antichain` – an antichain of `self`, as a list (or iterable), or, more generally, generators of an order ideal (resp. order filter)
- `direction` – 'up' or 'down' (default: 'up')

OUTPUT:

- the generating antichain of the complement order filter (resp. order ideal) of the order ideal (resp. order filter) generated by the antichain `antichain`

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: P.order_ideal_complement_generators([1])
{2}
sage: P.order_ideal_complement_generators([3])
set()
sage: P.order_ideal_complement_generators([1,2])
{3}
sage: P.order_ideal_complement_generators([1,2,3])
set()

sage: P.order_ideal_complement_generators([1], direction="down")
{2}
sage: P.order_ideal_complement_generators([3], direction="down")
{1, 2}
sage: P.order_ideal_complement_generators([1,2], direction="down")
set()
sage: P.order_ideal_complement_generators([1,2,3], direction="down")
set()
```

> **Warning:** This is a brute force implementation, building the order ideal generated by the antichain, and searching for order filter generators of its complement

**panyushev_orbit_iter** (*antichain*, *element_constructor=<type 'set'>*, *stop=True*, *check=True*)

Iterate over the Panyushev orbit of an antichain `antichain` of `self`.

The Panyushev orbit of an antichain is its orbit under Panyushev complementation (see *`panyushev_complement()`*).

INPUT:

- `antichain` – an antichain of `self`, given as an iterable.

- `element_constructor` (defaults to `set`) – a type constructor (`set`, `tuple`, `list`, `frozenset`, `iter`, etc.) which is to be applied to the antichains before they are yielded.
- `stop` – a Boolean (default: `True`) determining whether the iterator should stop once it completes its cycle (this happens when it is set to `True`) or go on forever (this happens when it is set to `False`).
- `check` – a Boolean (default: `True`) determining whether `antichain` should be checked for being an antichain.

OUTPUT:

- an iterator over the orbit of the antichain `antichain` under Panyushev complementation. This iterator $I$ has the property that `I[0] == antichain` and each $i$ satisfies `self.order_ideal_complement_generators(I[i]) == I[i+1]`, where `I[i+1]` has to be understood as `I[0]` if it is undefined. The entries `I[i]` are sets by default, but depending on the optional keyword variable `element_constructors` they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: list(P.panyushev_orbit_iter(set([1, 2])))
[{1, 2}, {3}, set()]
sage: list(P.panyushev_orbit_iter([1, 2]))
[{1, 2}, {3}, set()]
sage: list(P.panyushev_orbit_iter([2, 1]))
[{1, 2}, {3}, set()]
sage: list(P.panyushev_orbit_iter(set([1, 2]), element_constructor=list))
[[1, 2], [3], []]
sage: list(P.panyushev_orbit_iter(set([1, 2]), element_
↪constructor=frozenset))
[frozenset({1, 2}), frozenset({3}), frozenset()]
sage: list(P.panyushev_orbit_iter(set([1, 2]), element_constructor=tuple))
[(1, 2), (3,), ()]

sage: P = Poset( {} )
sage: list(P.panyushev_orbit_iter([]))
[set()]

sage: P = Poset({ 1: [2, 3], 2: [4], 3: [4], 4: [] })
sage: Piter = P.panyushev_orbit_iter([2], stop=False)
sage: next(Piter)
{2}
sage: next(Piter)
{3}
sage: next(Piter)
{2}
sage: next(Piter)
{3}
```

**panyushev_orbits**(*element_constructor=<type 'set'>*)

Return the Panyushev orbits of antichains in `self`.

The Panyushev orbit of an antichain is its orbit under Panyushev complementation (see *panyushev_complement()*).

INPUT:

- `element_constructor` (defaults to `set`) – a type constructor (`set`, `tuple`, `list`, `frozenset`, `iter`, etc.) which is to be applied to the antichains before they are returned.

OUTPUT:

- the partition of the set of all antichains of `self` into orbits under Panyushev complementation. This is returned as a list of lists `L` such that for each `L` and `i`, cyclically: `self.order_ideal_complement_generators(L[i]) == L[i+1]`. The entries `L[i]` are sets by default, but depending on the optional keyword variable `element_constructors` they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: P.panyushev_orbits()
[[{2}, {1}], [set(), {1, 2}, {3}]]
sage: P.panyushev_orbits(element_constructor=list)
[[[2], [1]], [[], [1, 2], [3]]]
sage: P.panyushev_orbits(element_constructor=frozenset)
[[frozenset({2}), frozenset({1})],
 [frozenset(), frozenset({1, 2}), frozenset({3})]]
sage: P.panyushev_orbits(element_constructor=tuple)
[[(2,), (1,)], [(), (1, 2), (3,)]]
sage: P = Poset( {} )
sage: P.panyushev_orbits()
[[set()]]
```

**rowmotion**(*order_ideal*)

The image of the order ideal `order_ideal` under rowmotion in `self`.

Rowmotion on a finite poset $P$ is an automorphism of the set $J(P)$ of all order ideals of $P$. One way to define it is as follows: Given an order ideal $I \in J(P)$, we let $F$ be the set-theoretic complement of $I$ in $P$. Furthermore we let $A$ be the antichain consisting of all minimal elements of $F$. Then, the rowmotion of $I$ is defined to be the order ideal of $P$ generated by the antichain $A$ (that is, the order ideal consisting of each element of $P$ which has some element of $A$ above it).

Rowmotion is related (actually, isomorphic) to Panyushev complementation (*panyushev_complement()*).

INPUT:

- `order_ideal` – an order ideal of `self`, as a set

OUTPUT:

- the image of `order_ideal` under rowmotion, as a set again

EXAMPLES:

```
sage: P = Poset( {1: [2, 3], 2: [], 3: [], 4: [8], 5: [], 6: [5], 7: [1, 
↪4], 8: []} )
sage: I = Set({2, 6, 1, 7})
sage: P.rowmotion(I)
{1, 3, 4, 5, 6, 7}

sage: P = Poset( {} )
sage: I = Set({})
sage: P.rowmotion(I)
{}
```

**rowmotion_orbit_iter**(*oideal*, *element_constructor=<type 'set'>*, *stop=True*, *check=True*)

Iterate over the rowmotion orbit of an order ideal `oideal` of `self`.

The rowmotion orbit of an order ideal is its orbit under rowmotion (see *rowmotion()*).

INPUT:

- `oideal` – an order ideal of `self`, given as an iterable.
- `element_constructor` (defaults to `set`) – a type constructor (`set`, `tuple`, `list`, `frozenset`, `iter`, etc.) which is to be applied to the order ideals before they are yielded.
- `stop` – a Boolean (default: `True`) determining whether the iterator should stop once it completes its cycle (this happens when it is set to `True`) or go on forever (this happens when it is set to `False`).

- check – a Boolean (default: True) determining whether oideal should be checked for being an order ideal.

OUTPUT:

- an iterator over the orbit of the order ideal oideal under rowmotion. This iterator $I$ has the property that I[0] == oideal and that every $i$ satisfies self.rowmotion(I[i]) == I[i+1], where I[i+1] has to be understood as I[0] if it is undefined. The entries I[i] are sets by default, but depending on the optional keyword variable element_constructors they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: list(P.rowmotion_orbit_iter(set([1, 2])))
[{1, 2}, {1, 2, 3}, set()]
sage: list(P.rowmotion_orbit_iter([1, 2]))
[{1, 2}, {1, 2, 3}, set()]
sage: list(P.rowmotion_orbit_iter([2, 1]))
[{1, 2}, {1, 2, 3}, set()]
sage: list(P.rowmotion_orbit_iter(set([1, 2]), element_constructor=list))
[[1, 2], [1, 2, 3], []]
sage: list(P.rowmotion_orbit_iter(set([1, 2]), element_
↪constructor=frozenset))
[frozenset({1, 2}), frozenset({1, 2, 3}), frozenset()]
sage: list(P.rowmotion_orbit_iter(set([1, 2]), element_constructor=tuple))
[(1, 2), (1, 2, 3), ()]

sage: P = Poset( {} )
sage: list(P.rowmotion_orbit_iter([]))
[set()]

sage: P = Poset({ 1: [2, 3], 2: [4], 3: [4], 4: [] })
sage: Piter = P.rowmotion_orbit_iter([1, 2, 3], stop=False)
sage: next(Piter)
{1, 2, 3}
sage: next(Piter)
{1, 2, 3, 4}
sage: next(Piter)
set()
sage: next(Piter)
{1}
sage: next(Piter)
{1, 2, 3}

sage: P = Poset({ 1: [4], 2: [4, 5], 3: [5] })
sage: list(P.rowmotion_orbit_iter([1, 2], element_constructor=list))
[[1, 2], [1, 2, 3, 4], [2, 3, 5], [1], [2, 3], [1, 2, 3, 5], [1, 2, 4],␣
↪[3]]
```

**rowmotion_orbits**(*element_constructor=<type 'set'>*)

Return the rowmotion orbits of order ideals in self.

The rowmotion orbit of an order ideal is its orbit under rowmotion (see *rowmotion()*).

INPUT:

- element_constructor (defaults to set) – a type constructor (set, tuple, list, frozenset, iter, etc.) which is to be applied to the antichains before they are returned.

OUTPUT:

- the partition of the set of all order ideals of self into orbits under rowmotion. This is returned as a list of lists L such that for each L and i, cyclically: self.rowmotion(L[i]) ==

L[i+1]. The entries L[i] are sets by default, but depending on the optional keyword variable element_constructors they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( {1: [2, 3], 2: [], 3: [], 4: [2]} )
sage: sorted(len(o) for o in P.rowmotion_orbits())
[3, 5]
sage: sorted(P.rowmotion_orbits(element_constructor=list))
[[[1, 3], [4], [1], [4, 1, 3], [4, 1, 2]], [[4, 1], [4, 1, 2, 3], []]]
sage: sorted(P.rowmotion_orbits(element_constructor=tuple))
[[(1, 3), (4,), (1,), (4, 1, 3), (4, 1, 2)], [(4, 1), (4, 1, 2, 3), ()]]
sage: P = Poset({})
sage: sorted(P.rowmotion_orbits(element_constructor=tuple))
[[()]]
```

**toggling_orbit_iter**(*vs*, *oideal*, *element_constructor=<type 'set'>*, *stop=True*, *check=True*)

Iterate over the orbit of an order ideal oideal of self under the operation of toggling the vertices vs[0], vs[1], ... in this order.

See order_ideal_toggle() for a definition of toggling.

> **Warning:** The orbit is that under the composition of toggles, *not* under the single toggles themselves. Thus, for example, if vs == [1,2], then the orbit has the form $(I, T_2T_1I, T_2T_1T_2T_1I, \ldots)$ (where $I$ denotes oideal and $T_i$ means toggling at $i$) rather than $(I, T_1I, T_2T_1I, T_1T_2T_1I, \ldots)$.

INPUT:

- vs: a list (or other iterable) of elements of self (but since the output depends on the order, sets should not be used as vs).
- oideal – an order ideal of self, given as an iterable.
- element_constructor (defaults to set) – a type constructor (set, tuple, list, frozenset, iter, etc.) which is to be applied to the order ideals before they are yielded.
- stop – a Boolean (default: True) determining whether the iterator should stop once it completes its cycle (this happens when it is set to True) or go on forever (this happens when it is set to False).
- check – a Boolean (default: True) determining whether oideal should be checked for being an order ideal.

OUTPUT:

- an iterator over the orbit of the order ideal oideal under toggling the vertices in the list vs in this order. This iterator $I$ has the property that I[0] == oideal and that every $i$ satisfies self.order_ideal_toggles(I[i], vs) == I[i+1], where I[i+1] has to be understood as I[0] if it is undefined. The entries I[i] are sets by default, but depending on the optional keyword variable element_constructors they can also be tuples, lists etc.

EXAMPLES:

```
sage: P = Poset( ( [1,2,3], [ [1,3], [2,3] ] ) )
sage: list(P.toggling_orbit_iter([1, 3, 1], set([1, 2])))
[{1, 2}]
sage: list(P.toggling_orbit_iter([1, 2, 3], set([1, 2])))
[{1, 2}, set(), {1, 2, 3}]
sage: list(P.toggling_orbit_iter([3, 2, 1], set([1, 2])))
[{1, 2}, {1, 2, 3}, set()]
sage: list(P.toggling_orbit_iter([3, 2, 1], set([1, 2]), element_
↪constructor=list))
```

```
[[1, 2], [1, 2, 3], []]
sage: list(P.toggling_orbit_iter([3, 2, 1], set([1, 2]), element_
↪constructor=frozenset))
[frozenset({1, 2}), frozenset({1, 2, 3}), frozenset()]
sage: list(P.toggling_orbit_iter([3, 2, 1], set([1, 2]), element_
↪constructor=tuple))
[(1, 2), (1, 2, 3), ()]
sage: list(P.toggling_orbit_iter([3, 2, 1], [2, 1], element_
↪constructor=tuple))
[(1, 2), (1, 2, 3), ()]

sage: P = Poset( {} )
sage: list(P.toggling_orbit_iter([], []))
[set()]

sage: P = Poset({ 1: [2, 3], 2: [4], 3: [4], 4: [] })
sage: Piter = P.toggling_orbit_iter([1, 2, 4, 3], [1, 2, 3], stop=False)
sage: next(Piter)
{1, 2, 3}
sage: next(Piter)
{1}
sage: next(Piter)
set()
sage: next(Piter)
{1, 2, 3}
sage: next(Piter)
{1}
```

**toggling_orbits**(*vs*, *element_constructor=<type 'set'>*)

Return the orbits of order ideals in `self` under the operation of toggling the vertices `vs[0]`, `vs[1]`, `...` in this order.

See `order_ideal_toggle()` for a definition of toggling.

> **Warning:** The orbits are those under the composition of toggles, *not* under the single toggles themselves. Thus, for example, if `vs == [1,2]`, then the orbits have the form $(I, T_2T_1I, T_2T_1T_2T_1I, \ldots)$ (where $I$ denotes an order ideal and $T_i$ means toggling at $i$) rather than $(I, T_1I, T_2T_1I, T_1T_2T_1I, \ldots)$.

INPUT:

- `vs`: a list (or other iterable) of elements of `self` (but since the output depends on the order, sets should not be used as `vs`).

OUTPUT:

- a partition of the order ideals of `self`, as a list of sets `L` such that for each `L` and `i`, cyclically: `self.order_ideal_toggles(L[i], vs) == L[i+1]`.

EXAMPLES:

```
sage: P = Poset( {1: [2, 4], 2: [], 3: [4], 4: []} )
sage: sorted(len(o) for o in P.toggling_orbits([1, 2]))
[2, 3, 3]
sage: P = Poset( {1: [3], 2: [1, 4], 3: [], 4: [3]} )
sage: sorted(len(o) for o in P.toggling_orbits((1, 2, 4, 3)))
[3, 3]
```

## 3.61 Finite semigroups

**class** sage.categories.finite_semigroups.**FiniteSemigroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite (multiplicative) semigroups.

A finite semigroup is a *finite set* endowed with an associative binary operation $*$.

**Warning:** Finite semigroups in Sage used to be automatically endowed with an *enumerated set* structure; the default enumeration is then obtained by iteratively multiplying the semigroup generators. This forced any finite semigroup to either implement an enumeration, or provide semigroup generators; this was often inconvenient.

Instead, finite semigroups that provide a distinguished finite set of generators with `semigroup_generators()` should now explicitly declare themselves in the category of `finitely generated semigroups`:

```
sage: Semigroups().FinitelyGenerated()
Category of finitely generated semigroups
```

This is a backward incompatible change.

EXAMPLES:

```
sage: C = FiniteSemigroups(); C
Category of finite semigroups
sage: C.super_categories()
[Category of semigroups, Category of finite sets]
sage: sorted(C.axioms())
['Associative', 'Finite']
sage: C.example()
An example of a finite semigroup: the left regular band generated by ('a', 'b', 'c
↪', 'd')
```

**class ParentMethods**

**idempotents**()

Returns the idempotents of the semigroup

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('x','y'))
sage: sorted(S.idempotents())
['x', 'xy', 'y', 'yx']
```

**j_classes**()

Returns the $J$-classes of the semigroup.

Two elements $u$ and $v$ of a monoid are in the same $J$-class if $u$ divides $v$ and $v$ divides $u$.

OUTPUT:

All the $J$-classes of self, as a list of lists.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('a','b', 'c'))
sage: sorted(map(sorted, S.j_classes()))
[['a'], ['ab', 'ba'], ['abc', 'acb', 'bac', 'bca', 'cab', 'cba'], ['ac',
↪'ca'], ['b'], ['bc', 'cb'], ['c']]
```

**j_classes_of_idempotents**()

Returns all the idempotents of self, grouped by J-class.

OUTPUT:

a list of lists.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('a','b', 'c'))
sage: sorted(map(sorted, S.j_classes_of_idempotents()))
[['a'], ['ab', 'ba'], ['abc', 'acb', 'bac', 'bca', 'cab', 'cba'], ['ac',
↪'ca'], ['b'], ['bc', 'cb'], ['c']]
```

**j_transversal_of_idempotents**()

Returns a list of one idempotent per regular J-class

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('a','b', 'c'))
sage: sorted(S.j_transversal_of_idempotents())
['a', 'ab', 'abc', 'ac', 'b', 'bc', 'c']
```

## 3.62 Finite sets

**class** sage.categories.finite_sets.**FiniteSets**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite sets.

EXAMPLES:

```
sage: C = FiniteSets(); C
Category of finite sets
sage: C.super_categories()
[Category of sets]
sage: C.all_super_categories()
[Category of finite sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
sage: C.example()
NotImplemented
```

**class Algebras**(*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**extra_super_categories**()

EXAMPLES:

```
sage: FiniteSets().Algebras(QQ).extra_super_categories()
[Category of finite dimensional vector spaces with basis over Rational␣
↪Field]
```

This implements the fact that the algebra of a finite set is finite dimensional:

```
sage: FiniteMonoids().Algebras(QQ).is_subcategory(AlgebrasWithBasis(QQ).
↪FiniteDimensional())
True
```

**class ParentMethods**

**is_finite**()

Return `True` since `self` is finite.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.is_finite()
True
```

**class Subquotients**(*category*, **args*)

Bases: *sage.categories.subquotients.SubquotientsCategory*

**extra_super_categories**()

EXAMPLES:

```
sage: FiniteSets().Subquotients().extra_super_categories()
[Category of finite sets]
```

This implements the fact that a subquotient (and therefore a quotient or subobject) of a finite set is finite:

```
sage: FiniteSets().Subquotients().is_subcategory(FiniteSets())
True
sage: FiniteSets().Quotients   ().is_subcategory(FiniteSets())
True
sage: FiniteSets().Subobjects  ().is_subcategory(FiniteSets())
True
```

## 3.63 Finite Weyl Groups

**class** sage.categories.finite_weyl_groups.**FiniteWeylGroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite Weyl groups.

EXAMPLES:

```
sage: C = FiniteWeylGroups()
sage: C
Category of finite weyl groups
sage: C.super_categories()
[Category of finite coxeter groups, Category of weyl groups]
sage: C.example()
The symmetric group on {0, ..., 3}
```

**class ElementMethods**

**class ParentMethods**

## 3.64 Finitely generated magmas

**class** sage.categories.finitely_generated_magmas.**FinitelyGeneratedMagmas**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finitely generated (multiplicative) magmas.

See *Magmas.SubcategoryMethods.FinitelyGeneratedAsMagma()* for details.

EXAMPLES:

```
sage: C = Magmas().FinitelyGeneratedAsMagma(); C
Category of finitely generated magmas
sage: C.super_categories()
[Category of magmas]
sage: sorted(C.axioms())
['FinitelyGeneratedAsMagma']
```

**class ParentMethods**

**magma_generators**()

Return distinguished magma generators for self.

OUTPUT: a finite family

This method should be implemented by all *finitely generated magmas*.

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S.magma_generators()
Family ('a', 'b', 'c', 'd')
```

## 3.65 Finitely generated semigroups

**class** sage.categories.finitely_generated_semigroups.**FinitelyGeneratedSemigroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finitely generated (multiplicative) semigroups.

A *finitely generated semigroup* is a *semigroup* endowed with a distinguished finite set of generators (see *FinitelyGeneratedSemigroups.ParentMethods.semigroup_generators()*). This makes it into an *enumerated set*.

EXAMPLES:

```
sage: C = Semigroups().FinitelyGenerated(); C
Category of finitely generated semigroups
sage: C.super_categories()
[Category of semigroups,
 Category of finitely generated magmas,
 Category of enumerated sets]
sage: sorted(C.axioms())
['Associative', 'Enumerated', 'FinitelyGeneratedAsMagma']
sage: C.example()
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c', 'd')
```

**class Finite**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class ParentMethods**

**some_elements**()

Return an iterable containing some elements of the semigroup.

OUTPUT: the ten first elements of the semigroup, if they exist.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('x','y'))
sage: S.some_elements()
['x', 'y', 'yx', 'xy']
sage: S = FiniteSemigroups().example(alphabet=('x','y','z'))
sage: S.some_elements()
['x', 'y', 'z', 'xz', 'yx', 'yz', 'zx', 'zy', 'xy', 'yxz']
```

**class ParentMethods**

**ideal**(*gens*, *side='twosided'*)

Return the `side`-sided ideal generated by `gens`.

This brute force implementation recursively multiplies the elements of `gens` by the distinguished generators of this semigroup.

**See also:**

*semigroup_generators()*

INPUT:

- `gens` – a list (or iterable) of elements of `self`
- `side` – [default: “twosided”] “left”, “right” or “twosided”

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: list(S.ideal([S('cab')], side="left"))
['cab', 'acb', 'dcab', 'bca', 'abc', 'adcb', 'bdca',
 'cba', 'cdab', 'bac', 'dacb', 'dbca', 'adbc', 'bcda',
 'dbac', 'dabc', 'cbda', 'cdba', 'abdc', 'bdac', 'dcba',
 'cadb', 'badc', 'acdb', 'abcd', 'cbad', 'bacd', 'acbd',
 'bcad', 'cabd']
sage: list(S.ideal([S('cab')], side="right"))
['cab', 'cabd']
sage: list(S.ideal([S('cab')], side="twosided"))
['cab', 'acb', 'dcab', 'bca', 'cabd', 'abc', 'adcb',
 'acbd', 'bdca', 'bcad', 'cba', 'cdab', 'bac', 'dacb',
 'dbca', 'abcd', 'cbad', 'bacd', 'bcda', 'dbac', 'dabc',
 'cbda', 'cdba', 'abdc', 'adbc', 'bdac', 'dcba', 'cadb',
 'badc', 'acdb']
sage: list(S.ideal([S('cab')]))
['cab', 'acb', 'dcab', 'bca', 'cabd', 'abc', 'adcb',
 'acbd', 'bdca', 'bcad', 'cba', 'cdab', 'bac', 'dacb',
 'dbca', 'abcd', 'cbad', 'bacd', 'bcda', 'dbac', 'dabc',
 'cbda', 'cdba', 'abdc', 'adbc', 'bdac', 'dcba', 'cadb',
 'badc', 'acdb']
```

**semigroup_generators**()

Return distinguished semigroup generators for `self`.

OUTPUT: a finite family

This method should be implemented by all semigroups in *FinitelyGeneratedSemigroups*.

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

**succ_generators**(*side='twosided'*)

Return the successor function of the `side`-sided Cayley graph of `self`.

This is a function that maps an element of `self` to all the products of `x` by a generator of this semigroup, where the product is taken on the left, right, or both sides.

INPUT:

- `side`: "left", "right", or "twosided"

**Todo:** Design choice:

- find a better name for this method
- should we return a set? a family?

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S.succ_generators("left" )(S('ca'))
('ac', 'bca', 'ca', 'dca')
sage: S.succ_generators("right")(S('ca'))
('ca', 'cab', 'ca', 'cad')
sage: S.succ_generators("twosided" )(S('ca'))
('ac', 'bca', 'ca', 'dca', 'ca', 'cab', 'ca', 'cad')
```

**example**()

EXAMPLES:

```
sage: Semigroups().FinitelyGenerated().example()
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c', 'd
↪')
```

**extra_super_categories**()

State that a finitely generated semigroup is endowed with a default enumeration.

EXAMPLES:

```
sage: Semigroups().FinitelyGenerated().extra_super_categories()
[Category of enumerated sets]
```

## 3.66 Function fields

**class** sage.categories.function_fields.**FunctionFields**(*s=None*)

Bases: *sage.categories.category.Category*

The category of function fields.

EXAMPLES:

We create the category of function fields:

```
sage: C = FunctionFields()
sage: C
Category of function fields
```

**class ElementMethods**

**class ParentMethods**

**super_categories**()

Returns the Category of which this is a direct sub-Category For a list off all super caategories see all_super_categories

EXAMPLES:

```
sage: FunctionFields().super_categories()
[Category of fields]
```

## 3.67 G-Sets

**class** sage.categories.g_sets.**GSets**($G$)

Bases: *sage.categories.category.Category*

The category of $G$-sets, for a group $G$.

EXAMPLES:

```
sage: S = SymmetricGroup(3)
sage: GSets(S)
Category of G-sets for Symmetric group of order 3! as a permutation group
```

TODO: should this derive from Category_over_base?

**classmethod an_instance**()

Returns an instance of this class.

EXAMPLES:

```
sage: GSets.an_instance() # indirect doctest
Category of G-sets for Symmetric group of order 8! as a permutation group
```

**super_categories**()

EXAMPLES:

```
sage: GSets(SymmetricGroup(8)).super_categories()
[Category of sets]
```

## 3.68 Gcd domains

**class** sage.categories.gcd_domains.**GcdDomains**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of gcd domains domains where gcd can be computed but where there is no guarantee of factorisation into irreducibles

EXAMPLES:

```
sage: GcdDomains()
Category of gcd domains
sage: GcdDomains().super_categories()
[Category of integral domains]
```

**class ElementMethods**

**class ParentMethods**

**additional_structure**()

Return None.

Indeed, the category of gcd domains defines no additional structure: a ring morphism between two gcd domains is a gcd domain morphism.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: GcdDomains().additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: GcdDomains().super_categories()
[Category of integral domains]
```

## 3.69 Generalized Coxeter Groups

**class** sage.categories.generalized_coxeter_groups.**GeneralizedCoxeterGroups**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of generalized Coxeter groups.

A generalized Coxeter group is a group with a presentation of the following form:

$$\langle s_i \mid s_i^{p_i}, s_i s_j \cdots = s_j s_i \cdots \rangle,$$

where $p_i > 1$, $i \in I$, and the factors in the braid relation occur $m_{ij} = m_{ji}$ times for all $i \neq j \in I$.

EXAMPLES:

```
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: C = GeneralizedCoxeterGroups(); C
Category of generalized coxeter groups
```

**class Finite**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of finite generalized Coxeter groups.

**extra_super_categories**()

Implement that a finite generalized Coxeter group is a well-generated complex reflection group.

EXAMPLES:

```
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: from sage.categories.complex_reflection_groups import␣
↪ComplexReflectionGroups

sage: Cat = GeneralizedCoxeterGroups().Finite()
sage: Cat.extra_super_categories()
[Category of well generated finite complex reflection groups]
sage: Cat.is_subcategory(ComplexReflectionGroups().Finite().
↪WellGenerated())
True
```

**additional_structure**()

Return `None`.

Indeed, all the structure generalized Coxeter groups have in addition to groups (simple reflections, . . . ) is already defined in the super category.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: GeneralizedCoxeterGroups().additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.generalized_coxeter_groups import␣
↪GeneralizedCoxeterGroups
sage: GeneralizedCoxeterGroups().super_categories()
[Category of complex reflection or generalized coxeter groups]
```

## 3.70 Graded Algebras

**class** sage.categories.graded_algebras.**GradedAlgebras**(*base_category*)

Bases: *sage.categories.graded_modules.GradedModulesCategory*

The category of graded algebras

EXAMPLES:

```
sage: GradedAlgebras(ZZ)
Category of graded algebras over Integer Ring
sage: GradedAlgebras(ZZ).super_categories()
[Category of filtered algebras over Integer Ring,
 Category of graded modules over Integer Ring]
```

**class ElementMethods**

**class ParentMethods**

**graded_algebra**()

Return the associated graded algebra to `self`.

Since `self` is already graded, this just returns `self`.

EXAMPLES:

```
sage: m = SymmetricFunctions(QQ).m()
sage: m.graded_algebra() is m
True
```

## 3.71 Graded algebras with basis

**class** sage.categories.graded_algebras_with_basis.**GradedAlgebrasWithBasis**(*base_category*)

Bases: *sage.categories.graded_modules.GradedModulesCategory*

The category of graded algebras with a distinguished basis

EXAMPLES:

```
sage: C = GradedAlgebrasWithBasis(ZZ); C
Category of graded algebras with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of filtered algebras with basis over Integer Ring,
 Category of graded algebras over Integer Ring,
 Category of graded modules with basis over Integer Ring]
```

**class ElementMethods**

**class ParentMethods**

**graded_algebra**()

Return the associated graded algebra to `self`.

This is `self`, because `self` is already graded. See `graded_algebra()` for the general behavior of this method, and see `AssociatedGradedAlgebra` for the definition and properties of associated graded algebras.

EXAMPLES:

```
sage: m = SymmetricFunctions(QQ).m()
sage: m.graded_algebra() is m
True
```

## 3.72 Graded bialgebras

sage.categories.graded_bialgebras.**GradedBialgebras**(*base_ring*)

The category of graded bialgebras

EXAMPLES:

```
sage: C = GradedBialgebras(QQ); C
Join of Category of graded algebras over Rational Field
    and Category of bialgebras over Rational Field
sage: C is Bialgebras(QQ).Graded()
True
```

## 3.73 Graded bialgebras with basis

sage.categories.graded_bialgebras_with_basis.**GradedBialgebrasWithBasis**(*base_ring*)

The category of graded bialgebras with a distinguished basis

EXAMPLES:

```
sage: C = GradedBialgebrasWithBasis(QQ); C
Join of Category of ...
sage: sorted(C.super_categories(), key=str)
[Category of bialgebras with basis over Rational Field,
 Category of graded algebras with basis over Rational Field]
```

## 3.74 Graded Coalgebras

sage.categories.graded_coalgebras.**GradedCoalgebras**(*base_ring*)

The category of graded coalgebras

EXAMPLES:

```
sage: C = GradedCoalgebras(QQ); C
Join of Category of graded modules over Rational Field
    and Category of coalgebras over Rational Field
sage: C is Coalgebras(QQ).Graded()
True
```

## 3.75 Graded coalgebras with basis

sage.categories.graded_coalgebras_with_basis.**GradedCoalgebrasWithBasis**(*base_ring*)

The category of graded coalgebras with a distinguished basis

EXAMPLES:

```
sage: C = GradedCoalgebrasWithBasis(QQ); C
Join of Category of graded modules with basis over Rational Field
    and Category of coalgebras with basis over Rational Field
sage: C is Coalgebras(QQ).WithBasis().Graded()
True
```

## 3.76 Graded Hopf algebras

sage.categories.graded_hopf_algebras.**GradedHopfAlgebras**(*base_ring*)

The category of graded Hopf algebras.

EXAMPLES:

```
sage: C = GradedHopfAlgebras(QQ); C
Join of Category of hopf algebras over Rational Field
    and Category of graded algebras over Rational Field
sage: C is HopfAlgebras(QQ).Graded()
True
```

## 3.77 Graded Hopf algebras with basis

**class** sage.categories.graded_hopf_algebras_with_basis.**GradedHopfAlgebrasWithBasis**(*base_catego*
Bases: *sage.categories.graded_modules.GradedModulesCategory*

The category of graded Hopf algebras with a distinguished basis.

EXAMPLES:

```
sage: C = GradedHopfAlgebrasWithBasis(ZZ); C
Category of graded hopf algebras with basis over Integer Ring
sage: C.super_categories()
[Category of filtered hopf algebras with basis over Integer Ring,
 Category of graded algebras with basis over Integer Ring]

sage: C is HopfAlgebras(ZZ).WithBasis().Graded()
True
sage: C is HopfAlgebras(ZZ).Graded().WithBasis()
False
```

**class Connected**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class ElementMethods**

**class ParentMethods**

**antipode_on_basis**(*index*)
The antipode on the basis element indexed by `index`.

INPUT:
- `index` – an element of the index set

For a graded connected Hopf algebra, we can define an antipode recursively by

$$S(x) := -\sum_{x^L \neq x} S(x^L) \times x^R$$

when $|x| > 0$, and by $S(x) = x$ when $|x| = 0$.

**counit_on_basis**(*i*)
The default counit of a graded connected Hopf algebra.

INPUT:
- `i` – an element of the index set

OUTPUT:
- an element of the base ring

$$c(i) := \begin{cases} 1 & \text{if } i \text{ indexes the 1 of the algebra} \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLES:

```
sage: H = GradedHopfAlgebrasWithBasis(QQ).Connected().example()
sage: H.monomial(4).counit() # indirect doctest
0
sage: H.monomial(0).counit() # indirect doctest
1
```

**example**()

Return an example of a graded connected Hopf algebra with a distinguished basis.

**class ElementMethods**

**class ParentMethods**

**class WithRealizations**(*category*, **args*)

Bases: *sage.categories.with_realizations.WithRealizationsCategory*

**super_categories**()

EXAMPLES:

```
sage: GradedHopfAlgebrasWithBasis(QQ).WithRealizations().super_
↪categories()
[Join of Category of hopf algebras over Rational Field
     and Category of graded algebras over Rational Field]
```

**example**()

Return an example of a graded Hopf algebra with a distinguished basis.

## 3.78 Graded modules

**class** sage.categories.graded_modules.**GradedModules**(*base_category*)

Bases: *sage.categories.graded_modules.GradedModulesCategory*

The category of graded modules.

We consider every graded module $M = \bigoplus_i M_i$ as a filtered module under the (natural) filtration given by

$$F_i = \bigoplus_{j<i} M_j.$$

EXAMPLES:

```
sage: GradedModules(ZZ)
Category of graded modules over Integer Ring
sage: GradedModules(ZZ).super_categories()
[Category of filtered modules over Integer Ring]
```

The category of graded modules defines the graded structure which shall be preserved by morphisms:

```
sage: Modules(ZZ).Graded().additional_structure()
Category of graded modules over Integer Ring
```

**class ElementMethods**

**class ParentMethods**

**class** sage.categories.graded_modules.**GradedModulesCategory**(*base_category*)
Bases: *sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory*, *sage.categories.category_types.Category_over_base_ring*

EXAMPLES:

```
sage: C = GradedAlgebras(QQ)
sage: C
Category of graded algebras over Rational Field
sage: C.base_category()
Category of algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of filtered algebras over Rational Field,
 Category of graded modules over Rational Field]

sage: AlgebrasWithBasis(QQ).Graded().base_ring()
Rational Field
sage: GradedHopfAlgebrasWithBasis(QQ).base_ring()
Rational Field
```

**classmethod default_super_categories**(*category*, **args*)
Return the default super categories of `category.Graded()`.

Mathematical meaning: every graded object (module, algebra, etc.) is a filtered object with the (implicit) filtration defined by $F_i = \bigoplus_{j \leq i} G_j$.

INPUT:

- `cls` – the class `GradedModulesCategory`
- `category` – a category

OUTPUT: a (join) category

In practice, this returns `category.Filtered()`, joined together with the result of the method *RegressiveCovariantConstructionCategory.default_super_categories()* (that is the join of `category.Filtered()` and `cat` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Algebras()`, which has `cat=Modules()` as super category. Then, a grading of an algebra $G$ is also a filtration of $G$:

```
sage: Algebras(QQ).Graded().super_categories()
[Category of filtered algebras over Rational Field,
 Category of graded modules over Rational Field]
```

This resulted from the following call:

```
sage: sage.categories.graded_modules.GradedModulesCategory.default_super_
↪categories(Algebras(QQ))
Join of Category of filtered algebras over Rational Field
 and Category of graded modules over Rational Field
```

## 3.79 Graded modules with basis

**class** sage.categories.graded_modules_with_basis.**GradedModulesWithBasis**(*base_category*)
Bases: *sage.categories.graded_modules.GradedModulesCategory*

The category of graded modules with a distinguished basis.

EXAMPLES:

```
sage: C = GradedModulesWithBasis(ZZ); C
Category of graded modules with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of filtered modules with basis over Integer Ring,
 Category of graded modules over Integer Ring]
sage: C is ModulesWithBasis(ZZ).Graded()
True
```

**class ElementMethods**

**degree_negation**()

Return the image of `self` under the degree negation automorphism of the graded module to which `self` belongs.

The degree negation is the module automorphism which scales every homogeneous element of degree $k$ by $(-1)^k$ (for all $k$). This assumes that the module to which `self` belongs (that is, the module `self.parent()`) is $\mathbf{Z}$-graded.

EXAMPLES:

```
sage: E.<a,b> = ExteriorAlgebra(QQ)
sage: ((1 + a) * (1 + b)).degree_negation()
a^b - a - b + 1
sage: E.zero().degree_negation()
0

sage: P = GradedModulesWithBasis(ZZ).example(); P
An example of a graded module with basis: the free module on partitions␣
↪over Integer Ring
sage: pbp = lambda x: P.basis()[Partition(list(x))]
sage: p = pbp([3,1]) - 2 * pbp([2]) + 4 * pbp([1])
sage: p.degree_negation()
-4*P[1] - 2*P[2] + P[3, 1]
```

**class ParentMethods**

**degree_negation**(*element*)

Return the image of `element` under the degree negation automorphism of the graded module `self`.

The degree negation is the module automorphism which scales every homogeneous element of degree $k$ by $(-1)^k$ (for all $k$). This assumes that the module `self` is $\mathbf{Z}$-graded.

INPUT:

- `element` – element of the module `self`

EXAMPLES:

```
sage: E.<a,b> = ExteriorAlgebra(QQ)
sage: E.degree_negation((1 + a) * (1 + b))
a^b - a - b + 1
sage: E.degree_negation(E.zero())
0

sage: P = GradedModulesWithBasis(ZZ).example(); P
An example of a graded module with basis: the free module on partitions␣
↪over Integer Ring
```

```
sage: pbp = lambda x: P.basis()[Partition(list(x))]
sage: p = pbp([3,1]) - 2 * pbp([2]) + 4 * pbp([1])
sage: P.degree_negation(p)
-4*P[1] - 2*P[2] + P[3, 1]
```

## 3.80 Graphs

**class** sage.categories.graphs.**Graphs**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of graphs.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs(); C
Category of graphs
```

**class ParentMethods**

**dimension**()

Return the dimension of self as a CW complex.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.dimension()
1
```

**edges**()

Return the edges of self.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.edges()
[(0, 1), (1, 2), (2, 3), (3, 4), (4, 0)]
```

**faces**()

Return the faces of self.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: sorted(C.faces(), key=lambda x: (x.dimension(), x.value))
[0, 1, 2, 3, 4, (0, 1), (1, 2), (2, 3), (3, 4), (4, 0)]
```

**facets**()

Return the facets of self.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.facets()
[(0, 1), (1, 2), (2, 3), (3, 4), (4, 0)]
```

**vertices**()

Return the vertices of `self`.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.vertices()
[0, 1, 2, 3, 4]
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: Graphs().super_categories()
[Category of simplicial complexes]
```

## 3.81 Group Algebras

This module implements the category of group algebras for arbitrary groups over arbitrary commutative rings. For details, see *sage.categories.algebra_functor*.

AUTHOR:

- David Loeffler (2008-08-24): initial version
- Martin Raum (2009-08): update to use new coercion model – see trac ticket #6670.
- John Palmieri (2011-07): more updates to coercion, categories, etc., group algebras constructed using CombinatorialFreeModule – see trac ticket #6670.
- Nicolas M. Thiéry (2010-2017), Travis Scrimshaw (2017): generalization to a covariant functorial construction for monoid algebras, and beyond – see e.g. trac ticket #18700.

**class** sage.categories.group_algebras.**GroupAlgebras**(*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

The category of group algebras over a given base ring.

EXAMPLES:

```
sage: C = Groups().Algebras(ZZ); C
Category of group algebras over Integer Ring
sage: C.super_categories()
[Category of hopf algebras with basis over Integer Ring,
 Category of monoid algebras over Integer Ring]
```

We can also construct this category with:

```
sage: C is GroupAlgebras(ZZ)
True
```

Here is how to create the group algebra of a group $G$:

```
sage: G = DihedralGroup(5)
sage: QG = G.algebra(QQ); QG
Algebra of Dihedral group of order 10 as a permutation group over Rational Field
```

and an example of computation:

```
sage: g = G.an_element(); g
(1,2,3,4,5)
sage: (QG.term(g) + 1)**3
() + 3*(1,2,3,4,5) + 3*(1,3,5,2,4) + (1,4,2,5,3)
```

**Todo:**

- Check which methods would be better located in `Monoid.Algebras` or `Groups.Finite.Algebras`.

**class ElementMethods**

**central_form()**

Return `self` expressed in the canonical basis of the center of the group algebra.

INPUT:

- `self` – an element of the center of the group algebra

OUTPUT:

- A formal linear combination of the conjugacy class representatives representing its coordinates in the canonical basis of the center. See `Groups.Algebras.ParentMethods.center_basis()` for details.

**Warning:**

- This method requires the underlying group to have a method `conjugacy_classes_representatives` (every permutation group has one, thanks GAP!).
- This method does not check that the element is indeed central. Use the method *`Monoids.Algebras.ElementMethods.is_central()`* for this purpose.
- This function has a complexity linear in the number of conjugacy classes of the group. One could easily implement a function whose complexity is linear in the size of the support of `self`.

EXAMPLES:

```
sage: QS3 = SymmetricGroup(3).algebra(QQ)
sage: A = QS3([2,3,1]) + QS3([3,1,2])
sage: A.central_form()
B[(1,2,3)]
sage: QS4 = SymmetricGroup(4).algebra(QQ)
sage: B = sum(len(s.cycle_type())*QS4(s) for s in Permutations(4))
sage: B.central_form()
4*B[()] + 3*B[(1,2)] + 2*B[(1,2)(3,4)] + 2*B[(1,2,3)] + B[(1,2,3,4)]

sage: QG = GroupAlgebras(QQ).example(PermutationGroup([[(1,2,3),(4,5)],
↪[(3,4)]]))
sage: sum(i for i in QG.basis()).central_form()
```

```
B[()] + B[(4,5)] + B[(3,4,5)] + B[(2,3)(4,5)] + B[(2,3,4,5)] + B[(1,2)(3,
↪4,5)] + B[(1,2,3,4,5)]
```

**See also:**

- `Groups.Algebras.ParentMethods.center_basis()`
- *`Monoids.Algebras.ElementMethods.is_central()`*

**class ParentMethods**

**antipode_on_basis**(*g*)

Return the antipode of the element `g` of the basis.

Each basis element `g` is group-like, and so has antipode $g^{-1}$. This method is used to compute the antipode of any element.

EXAMPLES:

```
sage: A = CyclicPermutationGroup(6).algebra(ZZ); A
Algebra of Cyclic group of order 6 as a permutation group over Integer␣
↪Ring
sage: g = CyclicPermutationGroup(6).an_element();g
(1,2,3,4,5,6)
sage: A.antipode_on_basis(g)
(1,6,5,4,3,2)
sage: a = A.an_element(); a
() + 3*(1,2,3,4,5,6) + 3*(1,3,5)(2,4,6)
sage: a.antipode()
() + 3*(1,5,3)(2,6,4) + 3*(1,6,5,4,3,2)
```

**center_basis**()

Return a basis of the center of the group algebra.

The canonical basis of the center of the group algebra is the family $(f_\sigma)_{\sigma\in C}$, where $C$ is any collection of representatives of the conjugacy classes of the group, and $f_\sigma$ is the sum of the elements in the conjugacy class of $\sigma$.

OUTPUT:

- `tuple` of elements of `self`

> **Warning:**
>
> - This method requires the underlying group to have a method `conjugacy_classes` (every permutation group has one, thanks GAP!).

EXAMPLES:

```
sage: SymmetricGroup(3).algebra(QQ).center_basis()
((), (2,3) + (1,2) + (1,3), (1,2,3) + (1,3,2))
```

**See also:**

- `Groups.Algebras.ElementMethods.central_form()`
- *`Monoids.Algebras.ElementMethods.is_central()`*

**coproduct_on_basis**(*g*)

Return the coproduct of the element `g` of the basis.

Each basis element `g` is group-like. This method is used to compute the coproduct of any element.

EXAMPLES:

```
sage: A = CyclicPermutationGroup(6).algebra(ZZ); A
Algebra of Cyclic group of order 6 as a permutation group over Integer␣
↪Ring
sage: g = CyclicPermutationGroup(6).an_element(); g
(1,2,3,4,5,6)
sage: A.coproduct_on_basis(g)
(1,2,3,4,5,6) # (1,2,3,4,5,6)
sage: a = A.an_element(); a
() + 3*(1,2,3,4,5,6) + 3*(1,3,5)(2,4,6)
sage: a.coproduct()
() # () + 3*(1,2,3,4,5,6) # (1,2,3,4,5,6) + 3*(1,3,5)(2,4,6) # (1,3,5)(2,
↪4,6)
```

**counit** (*x*)

Return the counit of the element x of the group algebra.

This is the sum of all coefficients of x with respect to the standard basis of the group algebra.

EXAMPLES:

```
sage: A = CyclicPermutationGroup(6).algebra(ZZ); A
Algebra of Cyclic group of order 6 as a permutation group over Integer␣
↪Ring
sage: a = A.an_element(); a
() + 3*(1,2,3,4,5,6) + 3*(1,3,5)(2,4,6)
sage: a.counit()
7
```

**counit_on_basis** (*g*)

Return the counit of the element g of the basis.

Each basis element g is group-like, and so has counit 1. This method is used to compute the counit of any element.

EXAMPLES:

```
sage: A=CyclicPermutationGroup(6).algebra(ZZ);A
Algebra of Cyclic group of order 6 as a permutation group over Integer␣
↪Ring
sage: g=CyclicPermutationGroup(6).an_element();g
(1,2,3,4,5,6)
sage: A.counit_on_basis(g)
1
```

**group** ()

Return the underlying group of the group algebra.

EXAMPLES:

```
sage: GroupAlgebras(QQ).example(GL(3, GF(11))).group()
General Linear Group of degree 3 over Finite Field of size 11
sage: SymmetricGroup(10).algebra(QQ).group()
Symmetric group of order 10! as a permutation group
```

**is_integral_domain** (*proof=True*)

Return True if self is an integral domain.

This is false unless `self.base_ring()` is an integral domain, and even then it is false unless `self.group()` has no nontrivial elements of finite order. I don't know if this condition suffices, but it obviously does if the group is abelian and finitely generated.

EXAMPLES:

```
sage: GroupAlgebra(SymmetricGroup(2)).is_integral_domain()
False
sage: GroupAlgebra(SymmetricGroup(1)).is_integral_domain()
True
sage: GroupAlgebra(SymmetricGroup(1), IntegerModRing(4)).is_integral_
↪domain()
False
sage: GroupAlgebra(AbelianGroup(1)).is_integral_domain()
True
sage: GroupAlgebra(AbelianGroup(2, [0,2])).is_integral_domain()
False
sage: GroupAlgebra(GL(2, ZZ)).is_integral_domain() # not implemented
False
```

**example** (*G=None*)

Return an example of group algebra.

EXAMPLES:

```
sage: GroupAlgebras(QQ['x']).example()
Algebra of Dihedral group of order 8 as a permutation group over Univariate␣
↪Polynomial Ring in x over Rational Field
```

An other group can be specified as optional argument:

```
sage: GroupAlgebras(QQ).example(AlternatingGroup(4))
Algebra of Alternating group of order 4!/2 as a permutation group over␣
↪Rational Field
```

**extra_super_categories** ()

Implement the fact that the algebra of a group is a Hopf algebra.

EXAMPLES:

```
sage: C = Groups().Algebras(QQ)
sage: C.extra_super_categories()
[Category of hopf algebras over Rational Field]
sage: sorted(C.super_categories(), key=str)
[Category of hopf algebras with basis over Rational Field,
 Category of monoid algebras over Rational Field]
```

## 3.82 Groupoid

**class** sage.categories.groupoid.**Groupoid** (*G=None*)

Bases: *sage.categories.category.CategoryWithParameters*

The category of groupoids, for a set (usually a group) $G$.

FIXME:

- Groupoid or Groupoids ?

- definition and link with Wikipedia article Groupoid
- Should Groupoid inherit from Category_over_base?

EXAMPLES:

```
sage: Groupoid(DihedralGroup(3))
Groupoid with underlying set Dihedral group of order 6 as a permutation group
```

**classmethod an_instance**()

Returns an instance of this class.

EXAMPLES:

```
sage: Groupoid.an_instance() # indirect doctest
Groupoid with underlying set Symmetric group of order 8! as a permutation␣
↪group
```

**super_categories**()

EXAMPLES:

```
sage: Groupoid(DihedralGroup(3)).super_categories()
[Category of sets]
```

## 3.83 Groups

**class** sage.categories.groups.**Groups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of (multiplicative) groups, i.e. monoids with inverses.

EXAMPLES:

```
sage: Groups()
Category of groups
sage: Groups().super_categories()
[Category of monoids, Category of inverse unital magmas]
```

**Algebras**

alias of *GroupAlgebras*

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

The category of groups constructed as Cartesian products of groups.

This construction gives the direct product of groups. See Wikipedia article Direct_product and Wikipedia article Direct_product_of_groups for more information.

**class ElementMethods**

**multiplicative_order**()

Return the multiplicative order of this element.

EXAMPLES:

```
sage: G1 = SymmetricGroup(3)
sage: G2 = SL(2,3)
sage: G = cartesian_product([G1,G2])
sage: G((G1.gen(0), G2.gen(1))).multiplicative_order()
12
```

**class ParentMethods**

**group_generators**()

Return the group generators of `self`.

EXAMPLES:

```
sage: C5 = CyclicPermutationGroup(5)
sage: C4 = CyclicPermutationGroup(4)
sage: S4 = SymmetricGroup(3)
sage: C = cartesian_product([C5, C4, S4])
sage: C.group_generators()
Family (((1,2,3,4,5), (), ()),
        ((), (1,2,3,4), ()),
        ((), (), (1,2)),
        ((), (), (2,3)))
```

We check the other portion of trac ticket #16718 is fixed:

```
sage: len(C.j_classes())
1
```

An example with an infinitely generated group (a better output is needed):

```
sage: G = Groups.free([1,2])
sage: H = Groups.free(ZZ)
sage: C = cartesian_product([G, H])
sage: C.monoid_generators()
Lazy family (gen(i))_{i in The Cartesian product of (...)}
```

**order**()

Return the cardinality of self.

EXAMPLES:

```
sage: C = cartesian_product([SymmetricGroup(10), SL(2,GF(3))])
sage: C.order()
87091200
```

**Todo:** this method is just here to prevent `FiniteGroups.ParentMethods` to call `_cardinality_from_iterator`.

**extra_super_categories**()

A Cartesian product of groups is endowed with a natural group structure.

EXAMPLES:

```
sage: C = Groups().CartesianProducts()
sage: C.extra_super_categories()
[Category of groups]
```

```
sage: sorted(C.super_categories(), key=str)
[Category of Cartesian products of inverse unital magmas,
 Category of Cartesian products of monoids,
 Category of groups]
```

**class Commutative**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

Category of commutative (abelian) groups.

A group $G$ is *commutative* if $xy = yx$ for all $x, y \in G$.

**static free**(*index_set=None*, *names=None*, ***kwds*)

Return the free commutative group.

INPUT:

- `index_set` – (optional) an index set for the generators; if an integer, then this represents $\{0, 1, \ldots, n-1\}$
- `names` – a string or list/tuple/iterable of strings (default: `'x'`); the generator names or name prefix

EXAMPLES:

```
sage: Groups.Commutative.free(index_set=ZZ)
Free abelian group indexed by Integer Ring
sage: Groups().Commutative().free(ZZ)
Free abelian group indexed by Integer Ring
sage: Groups().Commutative().free(5)
Multiplicative Abelian group isomorphic to Z x Z x Z x Z x Z
sage: F.<x,y,z> = Groups().Commutative().free(); F
Multiplicative Abelian group isomorphic to Z x Z x Z
```

**class ElementMethods**

**conjugacy_class**()

Return the conjugacy class of `self`.

EXAMPLES:

```
sage: D = DihedralGroup(5)
sage: g = D((1,3,5,2,4))
sage: g.conjugacy_class()
Conjugacy class of (1,3,5,2,4) in Dihedral group of order 10 as a␣
↪permutation group

sage: H = MatrixGroup([matrix(GF(5),2,[1,2, -1, 1]), matrix(GF(5),2, [1,1,
↪ 0,1])])
sage: h = H(matrix(GF(5),2,[1,2, -1, 1]))
sage: h.conjugacy_class()
Conjugacy class of [1 2]
[4 1] in Matrix group over Finite Field of size 5 with 2 generators (
[1 2]  [1 1]
[4 1], [0 1]
)

sage: G = SL(2, GF(2))
sage: g = G.gens()[0]
sage: g.conjugacy_class()
Conjugacy class of [1 1]
```

```
[0 1] in Special Linear Group of degree 2 over Finite Field of size 2

sage: G = SL(2, QQ)
sage: g = G([[1,1],[0,1]])
sage: g.conjugacy_class()
Conjugacy class of [1 1]
[0 1] in Special Linear Group of degree 2 over Rational Field
```

**Finite**
alias of *FiniteGroups*

**Lie**
alias of LieGroups

**class ParentMethods**

**cayley_table**(*names='letters'*, *elements=None*)
Returns the "multiplication" table of this multiplicative group, which is also known as the "Cayley table".

---

**Note:** The order of the elements in the row and column headings is equal to the order given by the table's column_keys() method. The association between the actual elements and the names/symbols used in the table can also be retrieved as a dictionary with the translation() method.

---

For groups, this routine should behave identically to the *multiplication_table()* method for magmas, which applies in greater generality.

INPUT:

- names - the type of names used, values are:
  - 'letters' - lowercase ASCII letters are used for a base 26 representation of the elements' positions in the list given by list(), padded to a common width with leading 'a's.
  - 'digits' - base 10 representation of the elements' positions in the list given by column_keys(), padded to a common width with leading zeros.
  - 'elements' - the string representations of the elements themselves.
  - a list - a list of strings, where the length of the list equals the number of elements.
- elements - default = None. A list of elements of the group, in forms that can be coerced into the structure, eg. their string representations. This may be used to impose an alternate ordering on the elements, perhaps when this is used in the context of a particular structure. The default is to use whatever ordering is provided by the the group, which is reported by the column_keys() method. Or the elements can be a subset which is closed under the operation. In particular, this can be used when the base set is infinite.

OUTPUT: An object representing the multiplication table. This is an OperationTable object and even more documentation can be found there.

EXAMPLES:

Permutation groups, matrix groups and abelian groups can all compute their multiplication tables.

```
sage: G = DiCyclicGroup(3)
sage: T = G.cayley_table()
sage: T.column_keys()
((), (1,3,2,4)(5,7), ..., (1,2)(3,4)(5,7,6))
sage: T
*  a b c d e f g h i j k l
```

```
 +------------------------
a| a b c d e f g h i j k l
b| b e f j i h d k a l c g
c| c g d e h b k l j f i a
d| d k e h l g i a f b j c
e| e i h l a k j c b g f d
f| f d j i k e c g l h a b
g| g h b f j l e i c a d k
h| h j l a c i f d g k b e
i| i a k g b c l f e d h j
j| j c i k g d a b h e l f
k| k l g b f a h j d c e i
l| l f a c d j b e k i g h
```

```
sage: M=SL(2,2)
sage: M.cayley_table()
*  a b c d e f
 +------------
a| a b c d e f
b| b a d c f e
c| c f e b a d
d| d e f a b c
e| e d a f c b
f| f c b e d a
```

```
sage: A=AbelianGroup([2,3])
sage: A.cayley_table()
*  a b c d e f
 +------------
a| a b c d e f
b| b c a e f d
c| c a b f d e
d| d e f a b c
e| e f d b c a
f| f d e c a b
```

Lowercase ASCII letters are the default symbols used for the table, but you can also specify the use of decimal digit strings, or provide your own strings (in the proper order if they have meaning). Also, if the elements themselves are not too complex, you can choose to just use the string representations of the elements themselves.

```
sage: C=CyclicPermutationGroup(11)
sage: C.cayley_table(names='digits')
 *   00 01 02 03 04 05 06 07 08 09 10
  +---------------------------------
00| 00 01 02 03 04 05 06 07 08 09 10
01| 01 02 03 04 05 06 07 08 09 10 00
02| 02 03 04 05 06 07 08 09 10 00 01
03| 03 04 05 06 07 08 09 10 00 01 02
04| 04 05 06 07 08 09 10 00 01 02 03
05| 05 06 07 08 09 10 00 01 02 03 04
06| 06 07 08 09 10 00 01 02 03 04 05
07| 07 08 09 10 00 01 02 03 04 05 06
08| 08 09 10 00 01 02 03 04 05 06 07
09| 09 10 00 01 02 03 04 05 06 07 08
10| 10 00 01 02 03 04 05 06 07 08 09
```

```
sage: G=QuaternionGroup()
sage: names=['1', 'I', 'J', '-1', '-K', 'K', '-I', '-J']
sage: G.cayley_table(names=names)
 *   1  I  J -1 -K  K -I -J
  +------------------------
 1|  1  I  J -1 -K  K -I -J
 I|  I -1  K -I  J -J  1 -K
 J|  J -K -1 -J -I  I  K  1
-1| -1 -I -J  1  K -K  I  J
-K| -K -J  I  K -1  1  J -I
 K|  K  J -I -K  1 -1 -J  I
-I| -I  1 -K  I -J  J -1  K
-J| -J  K  1  J  I -I -K -1
```

```
sage: A=AbelianGroup([2,2])
sage: A.cayley_table(names='elements')
    *       1    f1    f0 f0*f1
     +------------------------
    1|      1    f1    f0 f0*f1
   f1|     f1     1 f0*f1    f0
   f0|     f0 f0*f1     1    f1
f0*f1|  f0*f1    f0    f1     1
```

The `change_names()` routine behaves similarly, but changes an existing table "in-place."

```
sage: G=AlternatingGroup(3)
sage: T=G.cayley_table()
sage: T.change_names('digits')
sage: T
*  0 1 2
 +------
0| 0 1 2
1| 1 2 0
2| 2 0 1
```

For an infinite group, you can still work with finite sets of elements, provided the set is closed under multiplication. Elements will be coerced into the group as part of setting up the table.

```
sage: G=SL(2,ZZ)
sage: G
Special Linear Group of degree 2 over Integer Ring
sage: identity = matrix(ZZ, [[1,0], [0,1]])
sage: G.cayley_table(elements=[identity, -identity])
*  a b
 +----
a| a b
b| b a
```

The `OperationTable` class provides even greater flexibility, including changing the operation. Here is one such example, illustrating the computation of commutators. `commutator` is defined as a function of two variables, before being used to build the table. From this, the commutator subgroup seems obvious, and creating a Cayley table with just these three elements confirms that they form a closed subset in the group.

```
sage: from sage.matrix.operation_table import OperationTable
sage: G=DiCyclicGroup(3)
sage: commutator = lambda x, y: x*y*x^-1*y^-1
```

```
sage: T=OperationTable(G, commutator)
sage: T
.  a b c d e f g h i j k l
 +------------------------
a| a a a a a a a a a a a a
b| a a h d a d h h a h d d
c| a d a a a d d a d d d a
d| a h a a a h h a h h h a
e| a a a a a a a a a a a a
f| a h h d a a d h h d a d
g| a d h d a h a h d a h d
h| a d a a a d d a d d d a
i| a a h d a d h h a h d d
j| a d h d a h a h d a h d
k| a h h d a a d h h d a d
l| a h a a a h h a h h h a
sage: trans = T.translation()
sage: comm = [trans['a'], trans['d'],trans['h']]
sage: comm
[(), (5,7,6), (5,6,7)]
sage: P=G.cayley_table(elements=comm)
sage: P
*  a b c
 +------
a| a b c
b| b c a
c| c a b
```

**Todo:** Arrange an ordering of elements into cosets of a normal subgroup close to size $\sqrt{n}$. Then the quotient group structure is often apparent in the table. See comments on trac ticket #7555.

AUTHOR:

- Rob Beezer (2010-03-15)

**conjugacy_class** (*g*)

Return the conjugacy class of the element g.

This is a fall-back method for groups not defined over GAP.

EXAMPLES:

```
sage: A = AbelianGroup([2,2])
sage: c = A.conjugacy_class(A.an_element())
sage: type(c)
<class 'sage.groups.conjugacy_classes.ConjugacyClass_with_category'>
```

**group_generators** ()

Return group generators for `self`.

This default implementation calls `gens()`, for backward compatibility.

EXAMPLES:

```
sage: A = AlternatingGroup(4)
sage: A.group_generators()
Family ((2,3,4), (1,2,3))
```

**holomorph**()

The holomorph of a group

The holomorph of a group $G$ is the semidirect product $G \rtimes_{id} Aut(G)$, where $id$ is the identity function on $Aut(G)$, the automorphism group of $G$.

See Wikipedia article Holomorph (mathematics)

EXAMPLES:

```
sage: G = Groups().example()
sage: G.holomorph()
Traceback (most recent call last):
...
NotImplementedError: holomorph of General Linear Group of degree 4 over␣
↪Rational Field not yet implemented
```

**monoid_generators**()

Return the generators of `self` as a monoid.

Let $G$ be a group with generating set $X$. In general, the generating set of $G$ as a monoid is given by $X \cup X^{-1}$, where $X^{-1}$ is the set of inverses of $X$. If $G$ is a finite group, then the generating set as a monoid is $X$.

EXAMPLES:

```
sage: A = AlternatingGroup(4)
sage: A.monoid_generators()
Family ((2,3,4), (1,2,3))
sage: F.<x,y> = FreeGroup()
sage: F.monoid_generators()
Family (x, y, x^-1, y^-1)
```

**semidirect_product**(*N*, *mapping*, *check=True*)

The semi-direct product of two groups

EXAMPLES:

```
sage: G = Groups().example()
sage: G.semidirect_product(G,Morphism(G,G))
Traceback (most recent call last):
...
NotImplementedError: semidirect product of General Linear Group of degree␣
↪4 over Rational Field and General Linear Group of degree 4 over␣
↪Rational Field not yet implemented
```

**class Topological**(*category*, **args*)

Bases: *sage.categories.topological_spaces.TopologicalSpacesCategory*

Category of topological groups.

A topological group $G$ is a group which has a topology such that multiplication and taking inverses are continuous functions.

REFERENCES:

- Wikipedia article Topological_group

**example**()

EXAMPLES:

```
sage: Groups().example()
General Linear Group of degree 4 over Rational Field
```

**static free**(*index_set=None*, *names=None*, ***kwds*)

Return the free group.

INPUT:

- `index_set` – (optional) an index set for the generators; if an integer, then this represents $\{0, 1, \ldots, n-1\}$
- `names` – a string or list/tuple/iterable of strings (default: `'x'`); the generator names or name prefix

When the index set is an integer or only variable names are given, this returns `FreeGroup_class`, which currently has more features due to the interface with GAP than `IndexedFreeGroup`.

EXAMPLES:

```
sage: Groups.free(index_set=ZZ)
Free group indexed by Integer Ring
sage: Groups().free(ZZ)
Free group indexed by Integer Ring
sage: Groups().free(5)
Free Group on generators {x0, x1, x2, x3, x4}
sage: F.<x,y,z> = Groups().free(); F
Free Group on generators {x, y, z}
```

## 3.84 Hecke modules

**class** sage.categories.hecke_modules.**HeckeModules**(*R*)

Bases: *sage.categories.category_types.Category_module*

The category of Hecke modules.

A Hecke module is a module $M$ over the emph{anemic} Hecke algebra, i.e., the Hecke algebra generated by Hecke operators $T_n$ with $n$ coprime to the level of $M$. (Every Hecke module defines a level function, which is a positive integer.) The reason we require that $M$ only be a module over the anemic Hecke algebra is that many natural maps, e.g., degeneracy maps, Atkin-Lehner operators, etc., are $\mathbf{T}$-module homomorphisms; but they are homomorphisms over the anemic Hecke algebra.

EXAMPLES:

We create the category of Hecke modules over $\mathbf{Q}$:

```
sage: C = HeckeModules(RationalField()); C
Category of Hecke modules over Rational Field
```

TODO: check that this is what we want:

```
sage: C.super_categories()
[Category of vector spaces with basis over Rational Field]
```

# [Category of vector spaces over Rational Field]

Note that the base ring can be an arbitrary commutative ring:

```
sage: HeckeModules(IntegerRing())
Category of Hecke modules over Integer Ring
sage: HeckeModules(FiniteField(5))
Category of Hecke modules over Finite Field of size 5
```

The base ring doesn't have to be a principal ideal domain:

```
sage: HeckeModules(PolynomialRing(IntegerRing(), 'x'))
Category of Hecke modules over Univariate Polynomial Ring in x over Integer Ring
```

**class Homsets**(*category*, **args*)

Bases: *sage.categories.homsets.HomsetsCategory*

**class ParentMethods**

**base_ring**()

EXAMPLES:

```
sage: HeckeModules(QQ).Homsets().base_ring()
Rational Field
```

**extra_super_categories**()

**class ParentMethods**

**super_categories**()

EXAMPLES:

```
sage: HeckeModules(QQ).super_categories()
[Category of vector spaces with basis over Rational Field]
```

## 3.85 Highest Weight Crystals

**class** sage.categories.highest_weight_crystals.**HighestWeightCrystalHomset**(*X*, *Y*, *category=None*)

Bases: *sage.categories.crystals.CrystalHomset*

The set of crystal morphisms from a highest weight crystal to another crystal.

**See also:**

See *sage.categories.crystals.CrystalHomset* for more information.

**Element**

alias of *HighestWeightCrystalMorphism*

**class** sage.categories.highest_weight_crystals.**HighestWeightCrystalMorphism**(*parent*, *on_gens*, *cartan_type=None*, *virtualization=None*, *scaling_factors=None*, *gens=None*, *check=True*)

Bases: *sage.categories.crystals.CrystalMorphismByGenerators*

A virtual crystal morphism whose domain is a highest weight crystal.

INPUT:

- `parent` – a homset
- `on_gens` – a function or list that determines the image of the generators (if given a list, then this uses the order of the generators of the domain) of the domain under `self`
- `cartan_type` – (optional) a Cartan type; the default is the Cartan type of the domain
- `virtualization` – (optional) a dictionary whose keys are in the index set of the domain and whose values are lists of entries in the index set of the codomain
- `scaling_factors` – (optional) a dictionary whose keys are in the index set of the domain and whose values are scaling factors for the weight, $\varepsilon$ and $\varphi$
- `gens` – (optional) a list of generators to define the morphism; the default is to use the highest weight vectors of the crystal
- `check` – (default: `True`) check if the crystal morphism is valid

**class** sage.categories.highest_weight_crystals.**HighestWeightCrystals**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of highest weight crystals.

A crystal is highest weight if it is acyclic; in particular, every connected component has a unique highest weight element, and that element generate the component.

EXAMPLES:

```
sage: C = HighestWeightCrystals()
sage: C
Category of highest weight crystals
sage: C.super_categories()
[Category of crystals]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

**class ElementMethods**

**string_parameters**(*word=None*)

Return the string parameters of `self` corresponding to the reduced word `word`.

Given a reduced expression $w = s_{i_1} \cdots s_{i_k}$, the string parameters of $b \in B$ corresponding to $w$ are $(a_1, \ldots, a_k)$ such that

$$e_{i_m}^{a_m} \cdots e_{i_1}^{a_1} b \neq 0$$
$$e_{i_m}^{a_m+1} \cdots e_{i_1}^{a_1} b = 0$$

for all $1 \leq m \leq k$.

For connected components isomorphic to $B(\lambda)$ or $B(\infty)$, if $w = w_0$ is the longest element of the Weyl group, then the path determined by the string parametrization terminates at the highest weight vector.

INPUT:

- `word` – a word in the alphabet of the index set; if not specified and we are in finite type, then this will be some reduced expression for the long element determined by the Weyl group

EXAMPLES:

```
sage: B = crystals.infinity.NakajimaMonomials(['A',3])
sage: mg = B.highest_weight_vector()
sage: w0 = [1,2,1,3,2,1]
sage: mg.string_parameters(w0)
[0, 0, 0, 0, 0, 0]
sage: mg.f_string([1]).string_parameters(w0)
[1, 0, 0, 0, 0, 0]
sage: mg.f_string([1,1,1]).string_parameters(w0)
[3, 0, 0, 0, 0, 0]
sage: mg.f_string([1,1,1,2,2]).string_parameters(w0)
[1, 2, 2, 0, 0, 0]
sage: mg.f_string([1,1,1,2,2]) == mg.f_string([1,1,2,2,1])
True
sage: x = mg.f_string([1,1,1,2,2,1,3,3,2,1,1,1])
sage: x.string_parameters(w0)
[4, 1, 1, 2, 2, 2]
sage: x.string_parameters([3,2,1,3,2,3])
[2, 3, 7, 0, 0, 0]
sage: x == mg.f_string([1]*7 + [2]*3 + [3]*2)
True
```

```
sage: B = crystals.infinity.Tableaux("A5")
sage: b = B(rows=[[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,3,6,6,6,6,6,6],
....:             [2,2,2,2,2,2,2,2,2,4,5,5,5,6],
....:             [3,3,3,3,3,3,3,5],
....:             [4,4,4,6,6,6],
....:             [5,6]])
sage: b.string_parameters([1,2,1,3,2,1,4,3,2,1,5,4,3,2,1])
[0, 1, 1, 1, 1, 0, 4, 4, 3, 0, 11, 10, 7, 7, 6]

sage: B = crystals.infinity.Tableaux("G2")
sage: b = B(rows=[[1,1,1,1,1,3,3,0,-3,-3,-2,-2,-1,-1,-1,-1],[2,3,3,3]])
sage: b.string_parameters([2,1,2,1,2,1])
[5, 13, 11, 15, 4, 4]
sage: b.string_parameters([1,2,1,2,1,2])
[7, 12, 15, 8, 10, 0]
```

```
sage: C = crystals.Tableaux(['C',2], shape=[2,1])
sage: mg = C.highest_weight_vector()
sage: lw = C.lowest_weight_vectors()[0]
sage: lw.string_parameters([1,2,1,2])
```

```
[1, 2, 3, 1]
sage: lw.string_parameters([2,1,2,1])
[1, 3, 2, 1]
sage: lw.e_string([2,1,1,1,2,2,1]) == mg
True
sage: lw.e_string([1,2,2,1,1,1,2]) == mg
True
```

**class ParentMethods**

**connected_components_generators**()

Returns the highest weight vectors of `self`

This default implementation selects among the module generators those that are highest weight, and caches the result. A crystal element $b$ is highest weight if $e_i(b) = 0$ for all $i$ in the index set.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C.highest_weight_vectors()
(1,)
```

```
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C,C,C,generators=[[C(2),C(1),C(1)],[C(1),
↪C(2),C(1)]])
sage: T.highest_weight_vectors()
([2, 1, 1], [1, 2, 1])
```

**digraph**(*subset=None*, *index_set=None*, *depth=None*)

Return the DiGraph associated to `self`.

INPUT:

- `subset` – (optional) a subset of vertices for which the digraph should be constructed
- `index_set` – (optional) the index set to draw arrows
- `depth` – the depth to draw; optional only for finite crystals

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: T.digraph()
Digraph on 8 vertices
sage: S = T.subcrystal(max_depth=2)
sage: len(S)
5
sage: G = T.digraph(subset=list(S))
sage: G.is_isomorphic(T.digraph(depth=2), edge_labels=True)
True
```

**highest_weight_vector**()

Returns the highest weight vector if there is a single one; otherwise, raises an error.

Caveat: this assumes that *highest_weight_vectors()* returns a list or tuple.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C.highest_weight_vector()
1
```

**highest_weight_vectors**()

Returns the highest weight vectors of `self`

This default implementation selects among the module generators those that are highest weight, and caches the result. A crystal element $b$ is highest weight if $e_i(b) = 0$ for all $i$ in the index set.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C.highest_weight_vectors()
(1,)
```

```
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C,C,C,generators=[[C(2),C(1),C(1)],[C(1),
↪C(2),C(1)]])
sage: T.highest_weight_vectors()
([2, 1, 1], [1, 2, 1])
```

**lowest_weight_vectors**()

Return the lowest weight vectors of `self`.

This default implementation selects among all elements of the crystal those that are lowest weight, and cache the result. A crystal element $b$ is lowest weight if $f_i(b) = 0$ for all $i$ in the index set.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C.lowest_weight_vectors()
(6,)
```

```
sage: C = crystals.Letters(['A',2])
sage: T = crystals.TensorProduct(C,C,C,generators=[[C(2),C(1),C(1)],[C(1),
↪C(2),C(1)]])
sage: T.lowest_weight_vectors()
([3, 2, 3], [3, 3, 2])
```

**q_dimension**(*q=None*, *prec=None*, *use_product=False*)

Return the $q$-dimension of `self`.

Let $B(\lambda)$ denote a highest weight crystal. Recall that the degree of the $\mu$-weight space of $B(\lambda)$ (under the principal gradation) is equal to $\langle \rho^\vee, \lambda - \mu \rangle$ where $\langle \rho^\vee, \alpha_i \rangle = 1$ for all $i \in I$ (in particular, take $\rho^\vee = \sum_{i \in I} h_i$).

The $q$-dimension of a highest weight crystal $B(\lambda)$ is defined as

$$\dim_q B(\lambda) := \sum_{j \geq 0} \dim(B_j) q^j,$$

where $B_j$ denotes the degree $j$ portion of $B(\lambda)$. This can be expressed as the product

$$\dim_q B(\lambda) = \prod_{\alpha^\vee \in \Delta_+^\vee} \left( \frac{1 - q^{\langle \lambda + \rho, \alpha^\vee \rangle}}{1 - q^{\langle \rho, \alpha^\vee \rangle}} \right)^{\mathrm{mult}\, \alpha},$$

where $\Delta_+^\vee$ denotes the set of positive coroots. Taking the limit as $q \to 1$ gives the dimension of $B(\lambda)$. For more information, see [Ka1990] Section 10.10.

INPUT:

- `q` – the (generic) parameter $q$

- prec – (default: None) The precision of the power series ring to use if the crystal is not known to be finite (i.e. the number of terms returned). If None, then the result is returned as a lazy power series.
- use_product – (default: False) if we have a finite crystal and True, use the product formula

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',2], shape=[2,1])
sage: qdim = C.q_dimension(); qdim
q^4 + 2*q^3 + 2*q^2 + 2*q + 1
sage: qdim(1)
8
sage: len(C) == qdim(1)
True
sage: C.q_dimension(use_product=True) == qdim
True
sage: C.q_dimension(prec=20)
q^4 + 2*q^3 + 2*q^2 + 2*q + 1
sage: C.q_dimension(prec=2)
2*q + 1

sage: R.<t> = QQ[]
sage: C.q_dimension(q=t^2)
t^8 + 2*t^6 + 2*t^4 + 2*t^2 + 1

sage: C = crystals.Tableaux(['A',2], shape=[5,2])
sage: C.q_dimension()
q^10 + 2*q^9 + 4*q^8 + 5*q^7 + 6*q^6 + 6*q^5
 + 6*q^4 + 5*q^3 + 4*q^2 + 2*q + 1

sage: C = crystals.Tableaux(['B',2], shape=[2,1])
sage: qdim = C.q_dimension(); qdim
q^10 + 2*q^9 + 3*q^8 + 4*q^7 + 5*q^6 + 5*q^5
 + 5*q^4 + 4*q^3 + 3*q^2 + 2*q + 1
sage: qdim == C.q_dimension(use_product=True)
True

sage: C = crystals.Tableaux(['D',4], shape=[2,1])
sage: C.q_dimension()
q^16 + 2*q^15 + 4*q^14 + 7*q^13 + 10*q^12 + 13*q^11
 + 16*q^10 + 18*q^9 + 18*q^8 + 18*q^7 + 16*q^6 + 13*q^5
 + 10*q^4 + 7*q^3 + 4*q^2 + 2*q + 1
```

We check with a finite tensor product:

```
sage: TP = crystals.TensorProduct(C, C)
sage: TP.cardinality()
25600
sage: qdim = TP.q_dimension(use_product=True); qdim # long time
q^32 + 2*q^31 + 8*q^30 + 15*q^29 + 34*q^28 + 63*q^27 + 110*q^26
 + 175*q^25 + 276*q^24 + 389*q^23 + 550*q^22 + 725*q^21
 + 930*q^20 + 1131*q^19 + 1362*q^18 + 1548*q^17 + 1736*q^16
 + 1858*q^15 + 1947*q^14 + 1944*q^13 + 1918*q^12 + 1777*q^11
 + 1628*q^10 + 1407*q^9 + 1186*q^8 + 928*q^7 + 720*q^6
 + 498*q^5 + 342*q^4 + 201*q^3 + 117*q^2 + 48*q + 26
sage: qdim(1) # long time
25600
sage: TP.q_dimension() == qdim # long time
True
```

The $q$-dimensions of infinite crystals are returned as formal power series:

```
sage: C = crystals.LSPaths(['A',2,1], [1,0,0])
sage: C.q_dimension(prec=5)
1 + q + 2*q^2 + 2*q^3 + 4*q^4 + O(q^5)
sage: C.q_dimension(prec=10)
1 + q + 2*q^2 + 2*q^3 + 4*q^4 + 5*q^5 + 7*q^6
 + 9*q^7 + 13*q^8 + 16*q^9 + O(q^10)
sage: qdim = C.q_dimension(); qdim
1 + q + 2*q^2 + 2*q^3 + 4*q^4 + 5*q^5 + 7*q^6
 + 9*q^7 + 13*q^8 + 16*q^9 + 22*q^10 + O(x^11)
sage: qdim.compute_coefficients(15)
sage: qdim
1 + q + 2*q^2 + 2*q^3 + 4*q^4 + 5*q^5 + 7*q^6
 + 9*q^7 + 13*q^8 + 16*q^9 + 22*q^10 + 27*q^11
 + 36*q^12 + 44*q^13 + 57*q^14 + 70*q^15 + O(x^16)
```

**class TensorProducts**(*category*, **args*)

Bases: *sage.categories.tensor.TensorProductsCategory*

The category of highest weight crystals constructed by tensor product of highest weight crystals.

**class ParentMethods**

Implements operations on tensor products of crystals.

**highest_weight_vectors**()

Return the highest weight vectors of `self`.

This works by using a backtracing algorithm since if $b_2 \otimes b_1$ is highest weight then $b_1$ is highest weight.

EXAMPLES:

```
sage: C = crystals.Tableaux(['D',4], shape=[2,2])
sage: D = crystals.Tableaux(['D',4], shape=[1])
sage: T = crystals.TensorProduct(D, C)
sage: T.highest_weight_vectors()
([[[1]], [[1, 1], [2, 2]]],
 [[[3]], [[1, 1], [2, 2]]],
 [[[-2]], [[1, 1], [2, 2]]])
sage: L = filter(lambda x: x.is_highest_weight(), T)
sage: tuple(L) == T.highest_weight_vectors()
True
```

**extra_super_categories**()

EXAMPLES:

```
sage: HighestWeightCrystals().TensorProducts().extra_super_categories()
[Category of highest weight crystals]
```

**additional_structure**()

Return `None`.

Indeed, the category of highest weight crystals defines no additional structure: it only guarantees the existence of a unique highest weight element in each component.

**See also:**

*Category.additional_structure()*

**Todo:** Should this category be a CategoryWithAxiom?

EXAMPLES:

```
sage: HighestWeightCrystals().additional_structure()
```

**example**()

Returns an example of highest weight crystals, as per *Category.example()*.

EXAMPLES:

```
sage: B = HighestWeightCrystals().example(); B
Highest weight crystal of type A_3 of highest weight omega_1
```

**super_categories**()

EXAMPLES:

```
sage: HighestWeightCrystals().super_categories()
[Category of crystals]
```

## 3.86 Hopf algebras

**class** sage.categories.hopf_algebras.**HopfAlgebras**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of Hopf algebras

EXAMPLES:

```
sage: HopfAlgebras(QQ)
Category of hopf algebras over Rational Field
sage: HopfAlgebras(QQ).super_categories()
[Category of bialgebras over Rational Field]
```

**class DualCategory**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of Hopf algebras constructed as dual of a Hopf algebra

**class ParentMethods**

**class ElementMethods**

**antipode**()

Return the antipode of self

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: [a,b] = A.algebra_generators()
sage: a, a.antipode()
(B[(1,2,3)], B[(1,3,2)])
sage: b, b.antipode()
(B[(1,3)], B[(1,3)])
```

**class Morphism**(*s=None*)

Bases: *sage.categories.category.Category*

The category of Hopf algebra morphisms

**class ParentMethods**

**class Realizations**(*category*, **args*)

Bases: *sage.categories.realizations.RealizationsCategory*

**class ParentMethods**

**antipode_by_coercion**(*x*)

Returns the image of x by the antipode

This default implementation coerces to the default realization, computes the antipode there, and coerces the result back.

EXAMPLES:

```
sage: N = NonCommutativeSymmetricFunctions(QQ)
sage: R = N.ribbon()
sage: R.antipode_by_coercion.__module__
'sage.categories.hopf_algebras'
sage: R.antipode_by_coercion(R[1,3,1])
-R[2, 1, 2]
```

**class Super**(*base_category*)

Bases: *sage.categories.super_modules.SuperModulesCategory*

**class TensorProducts**(*category*, **args*)

Bases: *sage.categories.tensor.TensorProductsCategory*

The category of Hopf algebras constructed by tensor product of Hopf algebras

**class ElementMethods**

**class ParentMethods**

**extra_super_categories**()

EXAMPLES:

```
sage: C = HopfAlgebras(QQ).TensorProducts()
sage: C.extra_super_categories()
[Category of hopf algebras over Rational Field]
sage: sorted(C.super_categories(), key=str)
[Category of hopf algebras over Rational Field,
 Category of tensor products of algebras over Rational Field,
 Category of tensor products of coalgebras over Rational Field]
```

**WithBasis**

alias of *HopfAlgebrasWithBasis*

**dual**()

Return the dual category

EXAMPLES:

The category of Hopf algebras over any field is self dual:

```
sage: C = HopfAlgebras(QQ)
sage: C.dual()
Category of hopf algebras over Rational Field
```

**super_categories**()

EXAMPLES:

```
sage: HopfAlgebras(QQ).super_categories()
[Category of bialgebras over Rational Field]
```

## 3.87 Hopf algebras with basis

**class** sage.categories.hopf_algebras_with_basis.**HopfAlgebrasWithBasis**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of Hopf algebras with a distinguished basis

EXAMPLES:

```
sage: C = HopfAlgebrasWithBasis(QQ)
sage: C
Category of hopf algebras with basis over Rational Field
sage: C.super_categories()
[Category of hopf algebras over Rational Field,
 Category of bialgebras with basis over Rational Field]
```

We now show how to use a simple Hopf algebra, namely the group algebra of the dihedral group (see also AlgebrasWithBasis):

```
sage: A = C.example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral group of␣
↪order 6 as a permutation group over Rational Field
sage: A.__custom_name = "A"
sage: A.category()
Category of finite dimensional hopf algebras with basis over Rational Field

sage: A.one_basis()
()
sage: A.one()
B[()]

sage: A.base_ring()
Rational Field
sage: A.basis().keys()
Dihedral group of order 6 as a permutation group

sage: [a,b] = A.algebra_generators()
sage: a, b
(B[(1,2,3)], B[(1,3)])
sage: a^3, b^2
(B[()], B[()])
sage: a*b
B[(1,2)]

sage: A.product           # todo: not quite ...
<bound method MyGroupAlgebra_with_category._product_from_product_on_basis_
↪multiply of A>
```

```
sage: A.product(b,b)
B[()]

sage: A.zero().coproduct()
0
sage: A.zero().coproduct().parent()
A # A
sage: a.coproduct()
B[(1,2,3)] # B[(1,2,3)]

sage: TestSuite(A).run(verbose=True)
running ._test_additive_associativity() . . . pass
running ._test_an_element() . . . pass
running ._test_antipode() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_characteristic() . . . pass
running ._test_distributivity() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
running ._test_zero() . . . pass
sage: A.__class__
<class 'sage.categories.examples.hopf_algebras_with_basis.MyGroupAlgebra_with_
↪category'>
sage: A.element_class
<class 'sage.categories.examples.hopf_algebras_with_basis.MyGroupAlgebra_with_
↪category.element_class'>
```

Let us look at the code for implementing A:

```
sage: A??                          # todo: not implemented
```

**class ElementMethods**

**Filtered**

alias of FilteredHopfAlgebrasWithBasis

**FiniteDimensional**

alias of *FiniteDimensionalHopfAlgebrasWithBasis*

**Graded**
alias of *GradedHopfAlgebrasWithBasis*

**class ParentMethods**

**antipode**()
The antipode of this Hopf algebra.

If `antipode_basis()` is available, this constructs the antipode morphism from `self` to `self` by extending it by linearity. Otherwise, `self.antipode_by_coercion()` is used, if available.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(ZZ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Integer Ring
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: [a,b] = A.algebra_generators()
sage: a, A.antipode(a)
(B[(1,2,3)], B[(1,3,2)])
sage: b, A.antipode(b)
(B[(1,3)], B[(1,3)])
```

**antipode_on_basis**(*x*)
The antipode of the Hopf algebra on the basis (optional)

INPUT:

- `x` – an index of an element of the basis of `self`

Returns the antipode of the basis element indexed by `x`.

If this method is implemented, then *antipode()* is defined from this by linearity.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: W = A.basis().keys(); W
Dihedral group of order 6 as a permutation group
sage: w = W.an_element(); w
(1,2,3)
sage: A.antipode_on_basis(w)
B[(1,3,2)]
```

**Super**
alias of `SuperHopfAlgebrasWithBasis`

**class TensorProducts**(*category*, **args*)
Bases: *sage.categories.tensor.TensorProductsCategory*

The category of hopf algebras with basis constructed by tensor product of hopf algebras with basis

**class ElementMethods**

**class ParentMethods**

**extra_super_categories**()
EXAMPLES:

```
sage: C = HopfAlgebrasWithBasis(QQ).TensorProducts()
sage: C.extra_super_categories()
[Category of hopf algebras with basis over Rational Field]
sage: sorted(C.super_categories(), key=str)
```

```
[Category of hopf algebras with basis over Rational Field,
 Category of tensor products of algebras with basis over Rational Field,
 Category of tensor products of hopf algebras over Rational Field]
```

**example**(*G=None*)

Returns an example of algebra with basis:

```
sage: HopfAlgebrasWithBasis(QQ['x']).example()
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Univariate Polynomial Ring in␣
↪x over Rational Field
```

An other group can be specified as optional argument:

```
sage: HopfAlgebrasWithBasis(QQ).example(SymmetricGroup(4))
An example of Hopf algebra with basis: the group algebra of the Symmetric␣
↪group of order 4! as a permutation group over Rational Field
```

## 3.88 H-trivial semigroups

**class** sage.categories.h_trivial_semigroups.**HTrivialSemigroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**Finite_extra_super_categories**()

Implement the fact that a finite $H$-trivial is aperiodic

EXAMPLES:

```
sage: Semigroups().HTrivial().Finite_extra_super_categories()
[Category of aperiodic semigroups]
sage: Semigroups().HTrivial().Finite() is Semigroups().Aperiodic().Finite()
True
```

**Inverse_extra_super_categories**()

Implement the fact that an $H$-trivial inverse semigroup is $J$-trivial.

---

**Todo:** Generalization for inverse semigroups.

Recall that there are two invertibility axioms for a semigroup $S$:

- One stating the existence, for all $x$, of a local inverse $y$ satisfying $x = xyx$ and $y = yxy$;
- One stating the existence, for all $x$, of a global inverse $y$ satisfying $xy = yx = 1$, where 1 is the unit of $S$ (which must of course exist).

It is sufficient to have local inverses for $H$-triviality to imply $J$-triviality. However, at this stage, only the second axiom is implemented in Sage (see *Magmas.Unital.SubcategoryMethods.Inverse()*). Therefore this fact is only implemented for semigroups with global inverses, that is groups. However the trivial group is the unique $H$-trivial group, so this is rather boring.

---

EXAMPLES:

```
sage: Semigroups().HTrivial().Inverse_extra_super_categories()
[Category of j trivial semigroups]
```

```
sage: Monoids().HTrivial().Inverse()
Category of h trivial groups
```

## 3.89 Infinite Enumerated Sets

AUTHORS:

- Florent Hivert (2009-11): initial revision.

**class** sage.categories.infinite_enumerated_sets.**InfiniteEnumeratedSets**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of infinite enumerated sets

An infinite enumerated sets is a countable set together with a canonical enumeration of its elements.

EXAMPLES:

```
sage: InfiniteEnumeratedSets()
Category of infinite enumerated sets
sage: InfiniteEnumeratedSets().super_categories()
[Category of enumerated sets, Category of infinite sets]
sage: InfiniteEnumeratedSets().all_super_categories()
[Category of infinite enumerated sets,
 Category of enumerated sets,
 Category of infinite sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

**class ParentMethods**

**list**()

Returns an error since self is an infinite enumerated set.

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
```

**random_element**()

Returns an error since self is an infinite enumerated set.

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN.random_element()
Traceback (most recent call last):
...
NotImplementedError: infinite set
```

TODO: should this be an optional abstract_method instead?

## 3.90 Integral domains

**class** sage.categories.integral_domains.**IntegralDomains**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of integral domains

An integral domain is commutative ring with no zero divisors, or equivalently a commutative domain.

EXAMPLES:

```
sage: C = IntegralDomains(); C
Category of integral domains
sage: sorted(C.super_categories(), key=str)
[Category of commutative rings, Category of domains]
sage: C is Domains().Commutative()
True
sage: C is Rings().Commutative().NoZeroDivisors()
True
```

**class ElementMethods**

**class ParentMethods**

**is_integral_domain**()

Return True, since this in an object of the category of integral domains.

EXAMPLES:

```
sage: QQ.is_integral_domain()
True
sage: Parent(QQ,category=IntegralDomains()).is_integral_domain()
True
```

## 3.91 J-trivial semigroups

**class** sage.categories.j_trivial_semigroups.**JTrivialSemigroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**extra_super_categories**()

Implement the fact that a $J$-trivial semigroup is $L$ and $R$-trivial.

EXAMPLES:

```
sage: Semigroups().JTrivial().extra_super_categories()
[Category of l trivial semigroups, Category of r trivial semigroups]
```

## 3.92 Lattice posets

**class** sage.categories.lattice_posets.**LatticePosets**(*s=None*)

Bases: *sage.categories.category.Category*

The category of lattices, i.e. partially ordered sets in which any two elements have a unique supremum (the elements' least upper bound; called their *join*) and a unique infimum (greatest lower bound; called their *meet*).

EXAMPLES:

```
sage: LatticePosets()
Category of lattice posets
sage: LatticePosets().super_categories()
[Category of posets]
sage: LatticePosets().example()
NotImplemented
```

**See also:**

*Posets*, *FiniteLatticePosets*, LatticePoset()

**Finite**
alias of *FiniteLatticePosets*

**class ParentMethods**

**join** (*x*, *y*)
Returns the join of $x$ and $y$ in this lattice

INPUT:
- x, y – elements of self

EXAMPLES:

```
sage: D = LatticePoset((divisors(60), attrcall("divides")))
sage: D.join( D(6), D(10) )
30
```

**meet** (*x*, *y*)
Returns the meet of $x$ and $y$ in this lattice

INPUT:
- x, y – elements of self

EXAMPLES:

```
sage: D = LatticePoset((divisors(30), attrcall("divides")))
sage: D.meet( D(6), D(15) )
3
```

**super_categories** ()
Returns a list of the (immediate) super categories of self, as per *Category.super_categories()*.

EXAMPLES:

```
sage: LatticePosets().super_categories()
[Category of posets]
```

## 3.93 Left modules

**class** sage.categories.left_modules.**LeftModules** (*base*, *name=None*)
Bases: *sage.categories.category_types.Category_over_base_ring*

The category of left modules left modules over an rng (ring not necessarily with unit), i.e. an abelian group with left multiplication by elements of the rng

EXAMPLES:

```
sage: LeftModules(ZZ)
Category of left modules over Integer Ring
sage: LeftModules(ZZ).super_categories()
[Category of commutative additive groups]
```

**class ElementMethods**

**class ParentMethods**

**super_categories()**

EXAMPLES:

```
sage: LeftModules(QQ).super_categories()
[Category of commutative additive groups]
```

## 3.94 Lie Algebras

AUTHORS:

- Travis Scrimshaw (07-15-2013): Initial implementation

**class** sage.categories.lie_algebras.**LieAlgebras**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of Lie algebras.

EXAMPLES:

```
sage: C = LieAlgebras(QQ); C
Category of Lie algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of vector spaces over Rational Field]
```

We construct a typical parent in this category, and do some computations with it:

```
sage: A = C.example(); A
An example of a Lie algebra: the Lie algebra from the associative
 algebra Symmetric group algebra of order 3 over Rational Field
 generated by ([2, 1, 3], [2, 3, 1])

sage: A.category()
Category of Lie algebras over Rational Field

sage: A.base_ring()
Rational Field

sage: a,b = A.lie_algebra_generators()
sage: a.bracket(b)
-[1, 3, 2] + [3, 2, 1]
sage: b.bracket(2*a + b)
2*[1, 3, 2] - 2*[3, 2, 1]

sage: A.bracket(a, b)
-[1, 3, 2] + [3, 2, 1]
```

Please see the source code of $A$ (with `A??`) for how to implement other Lie algebras.

**Todo:** Many of these tests should use Lie algebras that are not the minimal example and need to be added after trac ticket #16820 (and trac ticket #16823).

**class ElementMethods**

**bracket**(*rhs*)

Return the Lie bracket `[self, rhs]`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()
sage: x,y = L.lie_algebra_generators()
sage: x.bracket(y)
-[1, 3, 2] + [3, 2, 1]
sage: x.bracket(0)
0
```

**killing_form**(*x*)

Return the Killing form of `self` and `x`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: a.killing_form(b)
0
```

**lift**()

Return the image of `self` under the canonical lift from the Lie algebra to its universal enveloping algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: elt = 3*a + b - c
sage: elt.lift()
3*b0 + b1 - b2
```

```
sage: L.<x,y> = LieAlgebra(QQ, abelian=True)
sage: x.lift()
x
```

**to_vector**()

Return the vector in `g.module()` corresponding to the element `self` of `g` (where `g` is the parent of `self`).

Implement this if you implement `g.module()`. See `LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: u = L((1, 0, 0)).to_vector(); u
(1, 0, 0)
sage: parent(u)
Vector space of dimension 3 over Rational Field
```

**class FiniteDimensional**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**WithBasis**
alias of `FiniteDimensionalLieAlgebrasWithBasis`

**extra_super_categories**()
Implements the fact that a finite dimensional Lie algebra over a finite ring is finite.

EXAMPLES:

```
sage: LieAlgebras(IntegerModRing(4)).FiniteDimensional().extra_super_
↪categories()
[Category of finite sets]
sage: LieAlgebras(ZZ).FiniteDimensional().extra_super_categories()
[]
sage: LieAlgebras(GF(5)).FiniteDimensional().is_subcategory(Sets().
↪Finite())
True
sage: LieAlgebras(ZZ).FiniteDimensional().is_subcategory(Sets().Finite())
False
sage: LieAlgebras(GF(5)).WithBasis().FiniteDimensional().is_
↪subcategory(Sets().Finite())
True
```

**class ParentMethods**

**bracket**(*lhs, rhs*)
Return the Lie bracket `[lhs, rhs]` after coercing `lhs` and `rhs` into elements of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()
sage: x,y = L.lie_algebra_generators()
sage: L.bracket(x, x + y)
-[1, 3, 2] + [3, 2, 1]
sage: L.bracket(x, 0)
0
sage: L.bracket(0, x)
0
```

**from_vector**(*v*)
Return the element of `self` corresponding to the vector `v` in `self.module()`.

Implement this if you implement *module()*; see the documentation of the latter for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: u = L.from_vector(vector(QQ, (1, 0, 0))); u
(1, 0, 0)
sage: parent(u) is L
True
```

**ideal**(*gens, names=None, index_set=None, category=None*)
Return the ideal of `self` generated by `gens`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.ideal([2*a - c, b + c])
An example of a finite dimensional Lie algebra with basis:
 the 2-dimensional abelian Lie algebra over Rational Field
 with basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

```
sage: L = LieAlgebras(QQ).example()
sage: x,y = L.lie_algebra_generators()
sage: L.ideal([x + y])
Traceback (most recent call last):
...
NotImplementedError: ideals not yet implemented: see #16824
```

**is_abelian**()

Return True if this Lie algebra is abelian.

A Lie algebra $\mathfrak{g}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()
sage: L.is_abelian()
False
sage: R = QQ['x,y']
sage: L = LieAlgebras(QQ).example(R.gens())
sage: L.is_abelian()
True
```

```
sage: L.<x> = LieAlgebra(QQ,1)  # todo: not implemented - #16823
sage: L.is_abelian()  # todo: not implemented - #16823
True
sage: L.<x,y> = LieAlgebra(QQ,2)  # todo: not implemented - #16823
sage: L.is_abelian()  # todo: not implemented - #16823
False
```

**is_commutative**()

Return if self is commutative. This is equivalent to self being abelian.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()
sage: L.is_commutative()
False
```

```
sage: L.<x> = LieAlgebra(QQ, 1) # todo: not implemented - #16823
sage: L.is_commutative() # todo: not implemented - #16823
True
```

**is_ideal**(*A*)

Return if self is an ideal of A.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()
sage: L.is_ideal(L)
True
```

**is_nilpotent**()

Return if `self` is a nilpotent Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_nilpotent()
True
```

**is_solvable**()

Return if `self` is a solvable Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.is_solvable()
True
```

**killing_form**(*x*, *y*)

Return the Killing form of `x` and `y`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.killing_form(a, b+c)
0
```

**lift**()

Construct the lift morphism from `self` to the universal enveloping algebra of `self` (the latter is implemented as `universal_enveloping_algebra()`).

This is a Lie algebra homomorphism. It is injective if `self` is a free module over its base ring, or if the base ring is a $\mathbf{Q}$-algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: lifted = L.lift(2*a + b - c); lifted
2*b0 + b1 - b2
sage: lifted.parent() is L.universal_enveloping_algebra()
True
```

**module**()

Return an $R$-module which is isomorphic to the underlying $R$-module of `self`.

The rationale behind this method is to enable linear algebraic functionality on `self` (such as computing the span of a list of vectors in `self`) via an isomorphism from `self` to an $R$-module (typically, although not always, an $R$-module of the form $R^n$ for an $n \in \mathbf{N}$) on which such functionality already exists. For this method to be of any use, it should return an $R$-module which has linear algebraic functionality that `self` does not have.

For instance, if `self` has ordered basis $(e, f, h)$, then `self.module()` will be the $R$-module $R^3$, and the elements $e$, $f$ and $h$ of `self` will correspond to the basis vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$

of `self.module()`.

This method `module()` needs to be set whenever a finite-dimensional Lie algebra with basis is intended to support linear algebra (which is, e.g., used in the computation of centralizers and lower central series). One then needs to also implement the $R$-module isomorphism from `self` to `self.module()` in both directions; that is, implement:

- a `to_vector` ElementMethod which sends every element of `self` to the corresponding element of `self.module()`;
- a `from_vector` ParentMethod which sends every element of `self.module()` to an element of `self`.

The `from_vector` method will automatically serve as an element constructor of `self` (that is, `self(v)` for any `v` in `self.module()` will return `self.from_vector(v)`).

---

**Todo:** Ensure that this is actually so.

---

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.module()
Vector space of dimension 3 over Rational Field
```

**subalgebra**(*gens*, *names=None*, *index_set=None*, *category=None*)

Return the subalgebra of `self` generated by `gens`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.subalgebra([2*a - c, b + c])
An example of a finite dimensional Lie algebra with basis:
 the 2-dimensional abelian Lie algebra over Rational Field
 with basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

```
sage: L = LieAlgebras(QQ).example()
sage: x,y = L.lie_algebra_generators()
sage: L.subalgebra([x + y])
Traceback (most recent call last):
...
NotImplementedError: subalgebras not yet implemented: see #17416
```

**universal_enveloping_algebra**()

Return the universal enveloping algebra of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.universal_enveloping_algebra()
Noncommutative Multivariate Polynomial Ring in b0, b1, b2
 over Rational Field, nc-relations: {}
```

```
sage: L = LieAlgebra(QQ, 3, 'x', abelian=True)
sage: L.universal_enveloping_algebra()
Multivariate Polynomial Ring in x0, x1, x2 over Rational Field
```

**See also:**

*lift()*

**WithBasis**
alias of LieAlgebrasWithBasis

**example** (*gens=None*)
Return an example of a Lie algebra as per *Category.example*.

EXAMPLES:

```
sage: LieAlgebras(QQ).example()
An example of a Lie algebra: the Lie algebra from the associative algebra
 Symmetric group algebra of order 3 over Rational Field
 generated by ([2, 1, 3], [2, 3, 1])
```

Another set of generators can be specified as an optional argument:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: LieAlgebras(QQ).example(F.gens())
An example of a Lie algebra: the Lie algebra from the associative algebra
 Free Algebra on 3 generators (x, y, z) over Rational Field
 generated by (x, y, z)
```

**super_categories** ()
EXAMPLES:

```
sage: LieAlgebras(QQ).super_categories()
[Category of vector spaces over Rational Field]
```

**class** sage.categories.lie_algebras.**LiftMorphism** (*domain*, *codomain*)
Bases: *sage.categories.morphism.Morphism*

The natural lifting morphism from a Lie algebra to its enveloping algebra.

## 3.95 Lie Algebras With Basis

AUTHORS:

- Travis Scrimshaw (07-15-2013): Initial implementation

**class** sage.categories.lie_algebras_with_basis.**LieAlgebrasWithBasis** (*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

Category of Lie algebras with a basis.

**class ElementMethods**

**lift** ()
Lift self to the universal enveloping algebra.

EXAMPLES:

```
sage: S = SymmetricGroup(3).algebra(QQ)
sage: L = LieAlgebra(associative=S)
sage: x = L.gen(2)
sage: y = L.gen(1)
sage: x.lift()
```

```
b2
sage: y.lift()
b1
sage: x * y
b1*b2 + b4 - b5
```

**to_vector**()

Return the vector in `g.module()` corresponding to the element `self` of `g` (where `g` is the parent of `self`).

Implement this if you implement `g.module()`. See `sage.categories.lie_algebras.LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.an_element().to_vector()
(0, 0, 0)
```

**Todo:** Doctest this implementation on an example not overshadowed.

**class ParentMethods**

**bracket_on_basis**(*x*, *y*)

Return the bracket of basis elements indexed by `x` and `y` where `x < y`. If this is not implemented, then the method `_bracket_()` for the elements must be overwritten.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: L.bracket_on_basis(Partition([3,1]), Partition([2,2,1,1]))
0
```

**dimension**()

Return the dimension of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.dimension()
3
```

```
sage: L = LieAlgebra(QQ, 'x,y', {('x','y'): {'x':1}})
sage: L.dimension()
2
```

**from_vector**(*v*)

Return the element of `self` corresponding to the vector `v` in `self.module()`.

Implement this if you implement *module()*; see the documentation of `sage.categories.lie_algebras.LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: u = L.from_vector(vector(QQ, (1, 0, 0))); u
```

```
(1, 0, 0)
sage: parent(u) is L
True
```

**module**()

Return an $R$-module which is isomorphic to the underlying $R$-module of `self`.

See `sage.categories.lie_algebras.LieAlgebras.module()` for an explanation.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: L.module()
Free module generated by Partitions over Rational Field
```

**pbw_basis**(*basis_key=None*, ***kwds*)

Return the Poincare-Birkhoff-Witt basis of the universal enveloping algebra corresponding to `self`.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 2)
sage: PBW = L.pbw_basis()
```

**poincare_birkhoff_witt_basis**(*basis_key=None*, ***kwds*)

Return the Poincare-Birkhoff-Witt basis of the universal enveloping algebra corresponding to `self`.

EXAMPLES:

```
sage: L = lie_algebras.sl(QQ, 2)
sage: PBW = L.pbw_basis()
```

**example**(*gens=None*)

Return an example of a Lie algebra as per `Category.example`.

EXAMPLES:

```
sage: LieAlgebras(QQ).WithBasis().example()
An example of a Lie algebra: the abelian Lie algebra on the
 generators indexed by Partitions over Rational Field
```

Another set of generators can be specified as an optional argument:

```
sage: LieAlgebras(QQ).WithBasis().example(Compositions())
An example of a Lie algebra: the abelian Lie algebra on the
 generators indexed by Compositions of non-negative integers
 over Rational Field
```

## 3.96 Lie Groups

**class** sage.categories.lie_groups.**LieGroups**(*base*, *name=None*)

Bases: `sage.categories.category_types.Category_over_base_ring`

The category of Lie groups.

A Lie group is a topological group with a smooth manifold structure.

EXAMPLES:

```
sage: from sage.categories.lie_groups import LieGroups
sage: C = LieGroups(QQ); C
Category of Lie groups over Rational Field
```

**additional_structure**()

Return `None`.

Indeed, the category of Lie groups defines no new structure: a morphism of topological spaces and of smooth manifolds is a morphism as Lie groups.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: from sage.categories.lie_groups import LieGroups
sage: LieGroups(QQ).additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.lie_groups import LieGroups
sage: LieGroups(QQ).super_categories()
[Category of topological groups,
 Category of smooth manifolds over Rational Field]
```

## 3.97 Loop Crystals

**class** sage.categories.loop_crystals.**KirillovReshetikhinCrystals**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

Category of Kirillov-Reshetikhin crystals.

**class ElementMethods**

**energy_function**()

Return the energy function of `self`.

Let $B$ be a KR crystal. Let $b^\sharp$ denote the unique element such that $\varphi(b^\sharp) = \ell\Lambda_0$ with $\ell = \min\{\langle c, \varphi(b) \mid b \in B\}$. Let $u_B$ denote the maximal element of $B$. The *energy* of $b \in B$ is given by

$$D(b) = H(b \otimes b^\sharp) - H(u_B \otimes b^\sharp),$$

where $H$ is the *local energy function*.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['D',4,1], 2,1)
sage: for x in K.classically_highest_weight_vectors():
....:     x, x.energy_function()
([], 1)
([[1], [2]], 0)

sage: K = crystals.KirillovReshetikhin(['D',4,3], 1,2)
sage: for x in K.classically_highest_weight_vectors():
....:     x, x.energy_function()
```

```
([], 2)
([[1]], 1)
([[1, 1]], 0)
```

**lusztig_involution**()

Return the result of the classical Lusztig involution on `self`.

EXAMPLES:

```
sage: KRT = crystals.KirillovReshetikhin(['D',4,1], 2, 3, model='KR')
sage: mg = KRT.module_generators[1]
sage: mg.lusztig_involution()
[[-2, -2, 1], [-1, -1, 2]]
sage: elt = mg.f_string([2,1,3,2]); elt
[[3, -2, 1], [4, -1, 2]]
sage: elt.lusztig_involution()
[[-4, -2, 1], [-3, -1, 2]]
```

**class ParentMethods**

**R_matrix**($K$)

Return the combinatorial $R$-matrix of `self` to `K`.

The *combinatorial* $R$*-matrix* is the affine crystal isomorphism $R : L \otimes K \to K \otimes L$ which maps $u_L \otimes u_K$ to $u_K \otimes u_L$, where $u_K$ is the unique element in $K = B^{r,s}$ of weight $s\Lambda_r - sc\Lambda_0$ (see `maximal_vector()`).

INPUT:

- `self` – a crystal $L$
- `K` – a Kirillov-Reshetikhin crystal of the same type as $L$

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: L = crystals.KirillovReshetikhin(['A',2,1],1,2)
sage: f = K.R_matrix(L)
sage: [[b,f(b)] for b in crystals.TensorProduct(K,L)]
[[[[[1]], [[1, 1]]], [[[1, 1]], [[1]]]],
 [[[[1]], [[1, 2]]], [[[1, 1]], [[2]]]],
 [[[[1]], [[2, 2]]], [[[1, 2]], [[2]]]],
 [[[[1]], [[1, 3]]], [[[1, 1]], [[3]]]],
 [[[[1]], [[2, 3]]], [[[1, 2]], [[3]]]],
 [[[[1]], [[3, 3]]], [[[1, 3]], [[3]]]],
 [[[[2]], [[1, 1]]], [[[1, 2]], [[1]]]],
 [[[[2]], [[1, 2]]], [[[2, 2]], [[1]]]],
 [[[[2]], [[2, 2]]], [[[2, 2]], [[2]]]],
 [[[[2]], [[1, 3]]], [[[2, 3]], [[1]]]],
 [[[[2]], [[2, 3]]], [[[2, 2]], [[3]]]],
 [[[[2]], [[3, 3]]], [[[2, 3]], [[3]]]],
 [[[[3]], [[1, 1]]], [[[1, 3]], [[1]]]],
 [[[[3]], [[1, 2]]], [[[1, 3]], [[2]]]],
 [[[[3]], [[2, 2]]], [[[2, 3]], [[2]]]],
 [[[[3]], [[1, 3]]], [[[3, 3]], [[1]]]],
 [[[[3]], [[2, 3]]], [[[3, 3]], [[2]]]],
 [[[[3]], [[3, 3]]], [[[3, 3]], [[3]]]]]

sage: K = crystals.KirillovReshetikhin(['D',4,1],1,1)
sage: L = crystals.KirillovReshetikhin(['D',4,1],2,1)
sage: f = K.R_matrix(L)
```

```
sage: T = crystals.TensorProduct(K,L)
sage: b = T( K(rows=[[1]]), L(rows=[]) )
sage: f(b)
[[[2], [-2]], [[1]]]
```

Alternatively, one can compute the combinatorial $R$-matrix using the isomorphism method of digraphs:

```
sage: K1 = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: K2 = crystals.KirillovReshetikhin(['A',2,1],2,1)
sage: T1 = crystals.TensorProduct(K1,K2)
sage: T2 = crystals.TensorProduct(K2,K1)
sage: T1.digraph().is_isomorphic(T2.digraph(), edge_labels=True,␣
↪certificate=True) #todo: not implemented (see #10904 and #10549)
(True, {[[[1]], [[2], [3]]]: [[[1], [3]], [[2]]], [[[3]], [[2], [3]]]:␣
↪[[[2], [3]], [[3]]],
[[[3]], [[1], [3]]]: [[[1], [3]], [[3]]], [[[1]], [[1], [3]]]: [[[1],␣
↪[3]], [[1]]], [[[1]],
[[1], [2]]]: [[[1], [2]], [[1]]], [[[2]], [[1], [2]]]: [[[1], [2]],␣
↪[[2]]], [[[3]],
[[1], [2]]]: [[[2], [3]], [[1]]], [[[2]], [[1], [3]]]: [[[1], [2]],␣
↪[[3]]], [[[2]], [[2], [3]]]: [[[2], [3]], [[2]]]})
```

**affinization**()

Return the corresponding affinization crystal of `self`.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.affinization()
Affinization of Kirillov-Reshetikhin crystal of type ['A', 2, 1] with (r,
↪s)=(1,1)

sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1, model='KR')
sage: K.affinization()
Affinization of Kirillov-Reshetikhin tableaux of type ['A', 2, 1] and␣
↪shape (1, 1)
```

**b_sharp**()

Return the element $b^\sharp$ of `self`.

Let $B$ be a KR crystal. The element $b^\sharp$ is the unique element such that $\varphi(b^\sharp) = \ell\Lambda_0$ with $\ell = \min\{\langle c, \varphi(b)\rangle \mid b \in B\}$.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',6,2], 2,1)
sage: K.b_sharp()
[]
sage: K.b_sharp().Phi()
Lambda[0]

sage: K = crystals.KirillovReshetikhin(['C',3,1], 1,3)
sage: K.b_sharp()
[[-1]]
sage: K.b_sharp().Phi()
2*Lambda[0]

sage: K = crystals.KirillovReshetikhin(['D',6,2], 2,2)
```

```
sage: K.b_sharp() # long time
[]
sage: K.b_sharp().Phi() # long time
2*Lambda[0]
```

**cardinality**()

Return the cardinality of self.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['E',6,1], 1,1)
sage: K.cardinality()
27
sage: K = crystals.KirillovReshetikhin(['C',6,1], 4,3)
sage: K.cardinality()
4736732
```

**classical_decomposition**()

Return the classical decomposition of self.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,2)
sage: K.classical_decomposition()
The crystal of tableaux of type ['A', 3] and shape(s) [[2, 2]]
```

**classically_highest_weight_vectors**()

Return the classically highest weight elements of self.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['E',6,1],1,1)
sage: K.classically_highest_weight_vectors()
([(1,)],)
```

**is_perfect**(*ell=None*)

Check if self is a perfect crystal of level ell.

A crystal $\mathcal{B}$ is perfect of level $\ell$ if:

1. $\mathcal{B}$ is isomorphic to the crystal graph of a finite-dimensional $U_q'(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. There exists a $\lambda \in X$, such that $\mathrm{wt}(\mathcal{B}) \subset \lambda + \sum_{i\in I} \mathbf{Z}_{\leq 0}\alpha_i$ and there is a unique element in $\mathcal{B}$ of classical weight $\lambda$.
4. For all $b \in \mathcal{B}$, $\mathrm{level}(\varepsilon(b)) \geq \ell$.
5. For all $\Lambda$ dominant weights of level $\ell$, there exist unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that $\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda)$.

Points (1)-(3) are known to hold. This method checks points (4) and (5).

If self is the Kirillov-Reshetikhin crystal $B^{r,s}$, then it was proven for non-exceptional types in [FOS2010] that it is perfect if and only if $s/c_r$ is an integer (where $c_r$ is a constant related to the type of the crystal).

It is conjectured this is true for all affine types.

INPUT:

- ell – (default: $s/c_r$) integer; the level

REFERENCES:

[FOS2010]

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.is_perfect()
True

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 1)
sage: K.is_perfect()
False

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 2)
sage: K.is_perfect()
True

sage: K = crystals.KirillovReshetikhin(['E',6,1], 1,3)
sage: K.is_perfect()
True
```

**Todo:** Implement a version for tensor products of KR crystals.

**level**()

Return the level of `self` when `self` is a perfect crystal.

**See also:**

*is_perfect()*

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.level()
1
sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 2)
sage: K.level()
1
sage: K = crystals.KirillovReshetikhin(['D',4,1], 1, 3)
sage: K.level()
3

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 1)
sage: K.level()
Traceback (most recent call last):
...
ValueError: this crystal is not perfect
```

**local_energy_function**(*B*)

Return the local energy function of `self` and `B`.

See *LocalEnergyFunction* for a definition.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',6,2], 2,1)
sage: Kp = crystals.KirillovReshetikhin(['A',6,2], 1,1)
sage: H = K.local_energy_function(Kp); H
Local energy function of
 Kirillov-Reshetikhin crystal of type ['BC', 3, 2] with (r,s)=(2,1)
```

```
tensor
 Kirillov-Reshetikhin crystal of type ['BC', 3, 2] with (r,s)=(1,1)
```

**maximal_vector**()

Return the unique element of classical weight $s\Lambda_r$ in `self`.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['C',2,1],1,2)
sage: K.maximal_vector()
[[1, 1]]
sage: K = crystals.KirillovReshetikhin(['E',6,1],1,1)
sage: K.maximal_vector()
[(1,)]

sage: K = crystals.KirillovReshetikhin(['D',4,1],2,1)
sage: K.maximal_vector()
[[1], [2]]
```

**module_generator**()

Return the unique module generator of classical weight $s\Lambda_r$ of the Kirillov-Reshetikhin crystal $B^{r,s}$.

EXAMPLES:

```
sage: La = RootSystem(['G',2,1]).weight_space().fundamental_weights()
sage: K = crystals.ProjectedLevelZeroLSPaths(La[1])
sage: K.module_generator()
(-Lambda[0] + Lambda[1],)
```

**q_dimension**(*q=None*, *prec=None*, *use_product=False*)

Return the $q$-dimension of `self`.

The $q$-dimension of a KR crystal is defined as the $q$-dimension of the underlying classical crystal.

EXAMPLES:

```
sage: KRC = crystals.KirillovReshetikhin(['A',2,1], 2,2)
sage: KRC.q_dimension()
q^4 + q^3 + 2*q^2 + q + 1
sage: KRC = crystals.KirillovReshetikhin(['D',4,1], 2,1)
sage: KRC.q_dimension()
q^10 + q^9 + 3*q^8 + 3*q^7 + 4*q^6 + 4*q^5 + 4*q^4 + 3*q^3 + 3*q^2 + q + 2
```

**r**()

Return the value $r$ in `self` written as $B^{r,s}$.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,4)
sage: K.r()
2
```

**s**()

Return the value $s$ in `self` written as $B^{r,s}$.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',3,1], 2,4)
sage: K.s()
4
```

**class TensorProducts**(*category*, **args*)

Bases: *sage.categories.tensor.TensorProductsCategory*

The category of tensor products of Kirillov-Reshetikhin crystals.

**class ElementMethods**

**affine_grading**()

Return the affine grading of `self`.

The affine grading is calculated by finding a path from `self` to a ground state path (using the helper method *e_string_to_ground_state()*) and counting the number of affine Kashiwara operators $e_0$ applied on the way.

OUTPUT: an integer

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K)
sage: t = T.module_generators[0]
sage: t.affine_grading()
1

sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K,K)
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     print("{} {}".format(b, b.affine_grading()))
[[[1]], [[1]], [[1]]] 3
[[[2]], [[1]], [[1]]] 2
[[[1]], [[2]], [[1]]] 1
[[[3]], [[2]], [[1]]] 0

sage: K = crystals.KirillovReshetikhin(['C',2,1],1,1)
sage: T = crystals.TensorProduct(K,K,K)
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     print("{} {}".format(b, b.affine_grading()))
[[[1]], [[1]], [[1]]] 2
[[[2]], [[1]], [[1]]] 1
[[[-1]], [[1]], [[1]]] 1
[[[1]], [[2]], [[1]]] 1
[[[-2]], [[2]], [[1]]] 0
[[[1]], [[-1]], [[1]]] 0
```

**e_string_to_ground_state**()

Return a string of integers in the index set $(i_1, \ldots, i_k)$ such that $e_{i_k} \cdots e_{i_1}$ of `self` is the ground state.

This method calculates a path from `self` to a ground state path using Demazure arrows as defined in Lemma 7.3 in [ST2011].

OUTPUT: a tuple of integers $(i_1, \ldots, i_k)$

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K)
sage: t = T.module_generators[0]
sage: t.e_string_to_ground_state()
(0, 2)

sage: K = crystals.KirillovReshetikhin(['C',2,1],1,1)
sage: T = crystals.TensorProduct(K,K)
sage: t = T.module_generators[0]; t
[[[1]], [[1]]]
sage: t.e_string_to_ground_state()
(0,)
sage: x = t.e(0)
sage: x.e_string_to_ground_state()
()
sage: y = t.f_string([1,2,1,1,0]); y
[[[2]], [[1]]]
sage: y.e_string_to_ground_state()
()
```

**energy_function**(*algorithm=None*)

Return the energy function of `self`.

ALGORITHM:

### definition

Let $T$ be a tensor product of Kirillov-Reshetikhin crystals. Let $R_i$ and $H_i$ be the combinatorial $R$-matrix and local energy functions, respectively, acting on the $i$ and $i+1$ factors. Let $D_B$ be the energy function of a single Kirillov-Reshetikhin crystal. The *energy function* is given by

$$D = \sum_{j>i} H_i R_{i+1} R_{i+2} \cdots R_{j-1} + \sum_{j} D_B R_1 R_2 \cdots R_{j-1},$$

where $D_B$ acts on the rightmost factor.

### grading

If `self` is an element of $T$, a tensor product of perfect crystals of the same level, then use the affine grading to determine the energy. Specifically, let $g$ denote the affine grading of `self` and $d$ the affine grading of the maximal vector in $T$. Then the energy of `self` is given by $d - g$.

For more details, see Theorem 7.5 in [ST2011].

INPUT:

- `algorithm` – (default: `None`) use one of the following algorithms to determine the energy function:
  - `'definition'` - use the definition of the energy function;
  - `'grading'` - use the affine grading;

  if not specified, then this uses `'grading'` if all factors are perfect of the same level and otherwise this uses `'definition'`

OUTPUT: an integer

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: T = crystals.TensorProduct(K,K,K)
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     print("{} {}".format(b, b.energy_function()))
[[[1]], [[1]], [[1]]] 0
[[[2]], [[1]], [[1]]] 1
[[[1]], [[2]], [[1]]] 2
[[[3]], [[2]], [[1]]] 3

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 2)
sage: T = crystals.TensorProduct(K,K)
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     print("{} {}".format(b, b.energy_function()))
[[], []] 4
[[[1, 1]], []] 3
[[], [[1, 1]]] 1
[[[1, 1]], [[1, 1]]] 0
[[[1, 2]], [[1, 1]]] 1
[[[2, 2]], [[1, 1]]] 2
[[[-1, -1]], [[1, 1]]] 2
[[[1, -1]], [[1, 1]]] 2
[[[2, -1]], [[1, 1]]] 2

sage: K = crystals.KirillovReshetikhin(['C',2,1], 1, 1)
sage: T = crystals.TensorProduct(K)
sage: t = T.module_generators[0]
sage: t.energy_function('grading')
Traceback (most recent call last):
...
NotImplementedError: all crystals in the tensor product
 need to be perfect of the same level
```

**class ParentMethods**

**cardinality**()

Return the cardinality of `self`.

EXAMPLES:

```
sage: RC = RiggedConfigurations(['A', 3, 1], [[3, 2], [1, 2]])
sage: RC.cardinality()
100
sage: len(RC.list())
100

sage: RC = RiggedConfigurations(['E', 7, 1], [[1,1]])
sage: RC.cardinality()
134
sage: len(RC.list())
134

sage: RC = RiggedConfigurations(['B', 3, 1], [[2,2],[1,2]])
sage: RC.cardinality()
5130
```

**classically_highest_weight_vectors**()

Return the classically highest weight elements of `self`.

This works by using a backtracking algorithm since if $b_2 \otimes b_1$ is classically highest weight then $b_1$ is classically highest weight.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K,K)
sage: T.classically_highest_weight_vectors()
([[[1]], [[1]], [[1]]],
 [[[2]], [[1]], [[1]]],
 [[[1]], [[2]], [[1]]],
 [[[3]], [[2]], [[1]]])
```

**maximal_vector**()

Return the maximal vector of `self`.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K,K)
sage: T.maximal_vector()
[[[1]], [[1]], [[1]]]
```

**one_dimensional_configuration_sum**(*q=None*, *group_components=True*)

Compute the one-dimensional configuration sum of `self`.

INPUT:

- `q` – (default: `None`) a variable or `None`; if `None`, a variable $q$ is set in the code
- `group_components` – (default: `True`) boolean; if `True`, then the terms are grouped by classical component

The one-dimensional configuration sum is the sum of the weights of all elements in the crystal weighted by the energy function.

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['A',2,1],1,1)
sage: T = crystals.TensorProduct(K,K)
sage: T.one_dimensional_configuration_sum()
B[-2*Lambda[1] + 2*Lambda[2]] + (q+1)*B[-Lambda[1]]
 + (q+1)*B[Lambda[1] - Lambda[2]] + B[2*Lambda[1]]
 + B[-2*Lambda[2]] + (q+1)*B[Lambda[2]]
sage: R.<t> = ZZ[]
sage: T.one_dimensional_configuration_sum(t, False)
B[-2*Lambda[1] + 2*Lambda[2]] + (t+1)*B[-Lambda[1]]
 + (t+1)*B[Lambda[1] - Lambda[2]] + B[2*Lambda[1]]
 + B[-2*Lambda[2]] + (t+1)*B[Lambda[2]]

sage: R = RootSystem(['A',2,1])
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])
sage: LS.one_dimensional_configuration_sum() == T.one_dimensional_
↪configuration_sum() # long time
True
```

**extra_super_categories**()

EXAMPLES:

```
sage: from sage.categories.loop_crystals import␣
↪KirillovReshetikhinCrystals
sage: KirillovReshetikhinCrystals().TensorProducts().extra_super_
↪categories()
[Category of finite regular loop crystals]
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.loop_crystals import KirillovReshetikhinCrystals
sage: KirillovReshetikhinCrystals().super_categories()
[Category of finite regular loop crystals]
```

**class** sage.categories.loop_crystals.**LocalEnergyFunction**(*B*, *Bp*, *normalization=0*)

Bases: *sage.categories.map.Map*

The local energy function.

Let $B$ and $B'$ be Kirillov-Reshetikhin crystals with maximal vectors $u_B$ and $u_{B'}$ respectively. The *local energy function* $H : B \otimes B' \to \mathbf{Z}$ is the function which satisfies

$$H(e_0(b \otimes b')) = H(b \otimes b') + \begin{cases} 1 & \text{if } i = 0 \text{ and LL,} \\ -1 & \text{if } i = 0 \text{ and RR,} \\ 0 & \text{otherwise,} \end{cases}$$

where LL (resp. RR) denote $e_0$ acts on the left (resp. right) on both $b \otimes b'$ and $R(b \otimes b')$, and normalized by $H(u_B \otimes u_{B'}) = 0$.

INPUT:

- B – a Kirillov-Reshetikhin crystal
- Bp – a Kirillov-Reshetikhin crystal
- normalization – (default: 0) the normalization value

EXAMPLES:

```
sage: K = crystals.KirillovReshetikhin(['C',2,1], 1,2)
sage: K2 = crystals.KirillovReshetikhin(['C',2,1], 2,1)
sage: H = K.local_energy_function(K2)
sage: T = tensor([K, K2])
sage: hw = T.classically_highest_weight_vectors()
sage: for b in hw:
....:     b, H(b)
([[], [[1], [2]]], 1)
([[[1, 1]], [[1], [2]]], 0)
([[[2, -2]], [[1], [2]]], 1)
([[[1, -2]], [[1], [2]]], 1)
```

REFERENCES:

[KKMMNN1992]

**class** sage.categories.loop_crystals.**LoopCrystals**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of $U_q'(\mathfrak{g})$-crystals, where $\mathfrak{g}$ is of affine type.

The category is called loop crystals as we can also consider them as crystals corresponding to the loop algebra $\mathfrak{g}_0[t]$, where $\mathfrak{g}_0$ is the corresponding classical type.

EXAMPLES:

```
sage: from sage.categories.loop_crystals import LoopCrystals
sage: C = LoopCrystals()
sage: C
Category of loop crystals
sage: C.super_categories()
[Category of crystals]
sage: C.example()
Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(1,1)
```

**class ParentMethods**

**digraph** (*subset=None, index_set=None*)

Return the `DiGraph` associated to `self`.

INPUT:

- `subset` – (optional) a subset of vertices for which the digraph should be constructed
- `index_set` – (optional) the index set to draw arrows

**See also:**

*sage.categories.crystals.Crystals.ParentMethods.digraph()*

EXAMPLES:

```
sage: C = crystals.KirillovReshetikhin(['D',4,1], 2, 1)
sage: G = C.digraph()
sage: G.latex_options()  # optional - dot2tex
LaTeX options for Digraph on 29 vertices:
{...'edge_options': <function <lambda> at 0x...>,...}
sage: view(G, tightpage=True)  # optional - dot2tex graphviz, not tested
↪(opens external window)
```

**weight_lattice_realization** ()

Return the weight lattice realization used to express weights of elements in `self`.

The default is to use the non-extended affine weight lattice.

EXAMPLES:

```
sage: C = crystals.Letters(['A', 5])
sage: C.weight_lattice_realization()
Ambient space of the Root system of type ['A', 5]
sage: K = crystals.KirillovReshetikhin(['A',2,1], 1, 1)
sage: K.weight_lattice_realization()
Weight lattice of the Root system of type ['A', 2, 1]
```

**example** (*n=3*)

Return an example of Kirillov-Reshetikhin crystals, as per *Category.example()*.

EXAMPLES:

```
sage: from sage.categories.loop_crystals import LoopCrystals
sage: B = LoopCrystals().example(); B
Kirillov-Reshetikhin crystal of type ['A', 3, 1] with (r,s)=(1,1)
```

**super_categories** ()

EXAMPLES:

```
sage: from sage.categories.loop_crystals import LoopCrystals
sage: LoopCrystals().super_categories()
[Category of crystals]
```

**class** sage.categories.loop_crystals.**RegularLoopCrystals**(*s=None*)
Bases: *sage.categories.category_singleton.Category_singleton*

The category of regular $U_q'(\mathfrak{g})$-crystals, where $\mathfrak{g}$ is of affine type.

**class ElementMethods**

**classical_weight**()
Return the classical weight of `self`.

EXAMPLES:

```
sage: R = RootSystem(['A',2,1])
sage: La = R.weight_space().basis()
sage: LS = crystals.ProjectedLevelZeroLSPaths(2*La[1])
sage: hw = LS.classically_highest_weight_vectors()
sage: [(v.weight(), v.classical_weight()) for v in hw]
[(-2*Lambda[0] + 2*Lambda[1], (2, 0, 0)),
 (-Lambda[0] + Lambda[2], (1, 1, 0))]
```

**super_categories**()
EXAMPLES:

```
sage: from sage.categories.loop_crystals import RegularLoopCrystals
sage: RegularLoopCrystals().super_categories()
[Category of regular crystals,
 Category of loop crystals]
```

## 3.98 L-trivial semigroups

**class** sage.categories.l_trivial_semigroups.**LTrivialSemigroups**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**Commutative_extra_super_categories**()
Implement the fact that a commutative $R$-trivial semigroup is $J$-trivial.

EXAMPLES:

```
sage: Semigroups().LTrivial().Commutative_extra_super_categories()
[Category of j trivial semigroups]
```

**RTrivial_extra_super_categories**()
Implement the fact that an $L$-trivial and $R$-trivial semigroup is $J$-trivial.

EXAMPLES:

```
sage: Semigroups().LTrivial().RTrivial_extra_super_categories()
[Category of j trivial magmas]
```

**extra_super_categories**()
Implement the fact that a $L$-trivial semigroup is $H$-trivial.

EXAMPLES:

```
sage: Semigroups().LTrivial().extra_super_categories()
[Category of h trivial semigroups]
```

## 3.99 Magmas

**class** sage.categories.magmas.**Magmas**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of (multiplicative) magmas.

A magma is a set with a binary operation $*$.

EXAMPLES:

```
sage: Magmas()
Category of magmas
sage: Magmas().super_categories()
[Category of sets]
sage: Magmas().all_super_categories()
[Category of magmas, Category of sets,
 Category of sets with partial maps, Category of objects]
```

The following axioms are defined by this category:

```
sage: Magmas().Associative()
Category of semigroups
sage: Magmas().Unital()
Category of unital magmas
sage: Magmas().Commutative()
Category of commutative magmas
sage: Magmas().Unital().Inverse()
Category of inverse unital magmas
sage: Magmas().Associative()
Category of semigroups
sage: Magmas().Associative().Unital()
Category of monoids
sage: Magmas().Associative().Unital().Inverse()
Category of groups
```

**class Algebras**(*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**is_field**(*proof=True*)

Return `True` if `self` is a field.

For a magma algebra $RS$ this is always false unless $S$ is trivial and the base ring $R$` is a field.

EXAMPLES:

```
sage: SymmetricGroup(1).algebra(QQ).is_field()
True
sage: SymmetricGroup(1).algebra(ZZ).is_field()
False
sage: SymmetricGroup(2).algebra(QQ).is_field()
False
```

**extra_super_categories**()

EXAMPLES:

sage: Magmas().Commutative().Algebras(QQ).extra_super_categories() [Category of commutative magmas]

This implements the fact that the algebra of a commutative magma is commutative:

```
sage: Magmas().Commutative().Algebras(QQ).super_categories()
[Category of magma algebras over Rational Field, Category of commutative␣
↪magmas]
```

In particular, commutative monoid algebras are commutative algebras:

```
sage: Monoids().Commutative().Algebras(QQ).is_subcategory(Algebras(QQ).
↪Commutative())
True
```

**Associative**

alias of *Semigroups*

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**class ParentMethods**

**product**(*left*, *right*)

EXAMPLES:

```
sage: C = Magmas().CartesianProducts().example(); C
The Cartesian product of (Rational Field, Integer Ring, Integer Ring)
sage: x = C.an_element(); x
(1/2, 1, 1)
sage: x * x
(1/4, 1, 1)

sage: A = SymmetricGroupAlgebra(QQ, 3);
sage: x = cartesian_product([A([1,3,2]), A([2,3,1])])
sage: y = cartesian_product([A([1,3,2]), A([2,3,1])])
sage: cartesian_product([A,A]).product(x,y)
B[(0, [1, 2, 3])] + B[(1, [3, 1, 2])]
sage: x*y
B[(0, [1, 2, 3])] + B[(1, [3, 1, 2])]
```

**example**()

Return an example of Cartesian product of magmas.

EXAMPLES:

```
sage: C = Magmas().CartesianProducts().example(); C
The Cartesian product of (Rational Field, Integer Ring, Integer Ring)
sage: C.category()
Category of Cartesian products of commutative rings
sage: sorted(C.category().axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Commutative',
 'Distributive', 'Unital']

sage: TestSuite(C).run()
```

**extra_super_categories**()

This implements the fact that a subquotient (and therefore a quotient or subobject) of a finite set is finite.

EXAMPLES:

```
sage: Semigroups().CartesianProducts().extra_super_categories()
[Category of semigroups]
sage: Semigroups().CartesianProducts().super_categories()
[Category of semigroups, Category of Cartesian products of magmas]
```

**class Commutative**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class Algebras**(*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**extra_super_categories**()

EXAMPLES:

sage: Magmas().Commutative().Algebras(QQ).extra_super_categories() [Category of commutative magmas]

This implements the fact that the algebra of a commutative magma is commutative:

```
sage: Magmas().Commutative().Algebras(QQ).super_categories()
[Category of magma algebras over Rational Field,
 Category of commutative magmas]
```

In particular, commutative monoid algebras are commutative algebras:

```
sage: Monoids().Commutative().Algebras(QQ).is_subcategory(Algebras(QQ).
↪Commutative())
True
```

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**extra_super_categories**()

Implement the fact that a Cartesian product of commutative additive magmas is still an commutative additive magmas.

EXAMPLES:

```
sage: C = Magmas().Commutative().CartesianProducts()
sage: C.extra_super_categories()
[Category of commutative magmas]
sage: C.axioms()
frozenset({'Commutative'})
```

**class ParentMethods**

**is_commutative**()

Return `True`, since commutative magmas are commutative.

EXAMPLES:

```
sage: Parent(QQ,category=CommutativeRings()).is_commutative()
True
```

**class ElementMethods**

**is_idempotent**()

Test whether `self` is idempotent.

EXAMPLES:

```
sage: S = Semigroups().example("free"); S
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c',
↪ 'd')
sage: a = S('a')
sage: a^2
'aa'
sage: a.is_idempotent()
False
```

```
sage: L = Semigroups().example("leftzero"); L
An example of a semigroup: the left zero semigroup
sage: x = L('x')
sage: x^2
'x'
sage: x.is_idempotent()
True
```

**FinitelyGeneratedAsMagma**

alias of `FinitelyGeneratedMagmas`

**class JTrivial**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**class ParentMethods**

**multiplication_table**(*names='letters'*, *elements=None*)

Returns a table describing the multiplication operation.

**Note:** The order of the elements in the row and column headings is equal to the order given by the table's `list()` method. The association can also be retrieved with the `dict()` method.

INPUT:

- `names` - the type of names used
  - `'letters'` - lowercase ASCII letters are used for a base 26 representation of the elements' positions in the list given by `column_keys()`, padded to a common width with leading 'a's.
  - `'digits'` - base 10 representation of the elements' positions in the list given by `column_keys()`, padded to a common width with leading zeros.
  - `'elements'` - the string representations of the elements themselves.
  - a list - a list of strings, where the length of the list equals the number of elements.
- `elements` - default = `None`. A list of elements of the magma, in forms that can be coerced into the structure, eg. their string representations. This may be used to impose an alternate ordering on the elements, perhaps when this is used in the context of a particular structure. The default is to use whatever ordering the `S.list` method returns. Or the `elements` can be a subset which is closed under the operation. In particular, this can be used when the base set is infinite.

OUTPUT: The multiplication table as an object of the class `OperationTable` which defines several methods for manipulating and displaying the table. See the documentation there for full details to supplement the documentation here.

EXAMPLES:

The default is to represent elements as lowercase ASCII letters.

```
sage: G=CyclicPermutationGroup(5)
sage: G.multiplication_table()
*  a b c d e
 +----------
a| a b c d e
b| b c d e a
c| c d e a b
d| d e a b c
e| e a b c d
```

All that is required is that an algebraic structure has a multiplication defined. A `LeftRegularBand` is an example of a finite semigroup. The `names` argument allows displaying the elements in different ways.

```
sage: from sage.categories.examples.finite_semigroups import␣
↪LeftRegularBand
sage: L=LeftRegularBand(('a','b'))
sage: T=L.multiplication_table(names='digits')
sage: T.column_keys()
('a', 'b', 'ab', 'ba')
sage: T
*  0 1 2 3
 +--------
0| 0 2 2 2
1| 3 1 3 3
2| 2 2 2 2
3| 3 3 3 3
```

Specifying the elements in an alternative order can provide more insight into how the operation behaves.

```
sage: L=LeftRegularBand(('a','b','c'))
sage: elts = sorted(L.list())
sage: L.multiplication_table(elements=elts)
*  a b c d e f g h i j k l m n o
 +------------------------------
a| a b c d e b b c c c d d e e e
b| b b c c c b b c c c c c c c c
c| c c c c c c c c c c c c c c c
d| d e e d e e e e e d d e e e e
e| e e e e e e e e e e e e e e e
f| g g h h h f g h i j i j j i j
g| g g h h h g g h h h h h h h h
h| h h h h h h h h h h h h h h h
i| j j j j j i j j i j i j j i j
j| j j j j j j j j j j j j j j j
k| l m m l m n o o n o k l m n o
l| l m m l m m m m m m l l m m m
m| m m m m m m m m m m m m m m m
n| o o o o o n o o n o n o o n o
o| o o o o o o o o o o o o o o o
```

The `elements` argument can be used to provide a subset of the elements of the structure. The subset must be closed under the operation. Elements need only be in a form that can be coerced into the set.

The `names` argument can also be used to request that the elements be represented with their usual string representation.

```
sage: L=LeftRegularBand(('a','b','c'))
sage: elts=['a', 'c', 'ac', 'ca']
sage: L.multiplication_table(names='elements', elements=elts)
   *   'a'  'c' 'ac' 'ca'
    +--------------------
 'a'|  'a' 'ac' 'ac' 'ac'
 'c'| 'ca'  'c' 'ca' 'ca'
'ac'| 'ac' 'ac' 'ac' 'ac'
'ca'| 'ca' 'ca' 'ca' 'ca'
```

The table returned can be manipulated in various ways. See the documentation for `OperationTable` for more comprehensive documentation.

```
sage: G=AlternatingGroup(3)
sage: T=G.multiplication_table()
sage: T.column_keys()
((), (1,2,3), (1,3,2))
sage: sorted(T.translation().items())
[('a', ()), ('b', (1,2,3)), ('c', (1,3,2))]
sage: T.change_names(['x', 'y', 'z'])
sage: sorted(T.translation().items())
[('x', ()), ('y', (1,2,3)), ('z', (1,3,2))]
sage: T
*  x y z
 +------
x| x y z
y| y z x
z| z x y
```

**product** $(x, y)$

The binary multiplication of the magma.

INPUT:

- `x`, `y` – elements of this magma

OUTPUT:

- an element of the magma (the product of `x` and `y`)

EXAMPLES:

```
sage: S = Semigroups().example("free")
sage: x = S('a'); y = S('b')
sage: S.product(x, y)
'ab'
```

A parent in `Magmas()` must either implement *product()* in the parent class or `_mul_` in the element class. By default, the addition method on elements `x._mul_(y)` calls `S.product(x, y)`, and reciprocally.

As a bonus, `S.product` models the binary function from `S` to `S`:

```
sage: bin = S.product
sage: bin(x,y)
'ab'
```

Currently, `S.product` is just a bound method:

```
sage: bin  # py2
<bound method FreeSemigroup_with_category.product of An example of a␣
↪semigroup: the free semigroup generated by ('a', 'b', 'c', 'd')>
sage: bin  # py3, due to difference in how bound methods are repr'd
<bound method FreeSemigroup.product of An example of a semigroup: the␣
↪free semigroup generated by ('a', 'b', 'c', 'd')>
```

When Sage will support multivariate morphisms, it will be possible, and in fact recommended, to enrich `S.product` with extra mathematical structure. This will typically be implemented using lazy attributes.:

```
sage: bin                 # todo: not implemented
Generic binary morphism:
From: (S x S)
To:   S
```

**product_from_element_class_mul** (*x*, *y*)

The binary multiplication of the magma.

INPUT:

- `x`, `y` – elements of this magma

OUTPUT:

- an element of the magma (the product of `x` and `y`)

EXAMPLES:

```
sage: S = Semigroups().example("free")
sage: x = S('a'); y = S('b')
sage: S.product(x, y)
'ab'
```

A parent in `Magmas()` must either implement *product()* in the parent class or `_mul_` in the element class. By default, the addition method on elements `x._mul_(y)` calls `S.product(x, y)`, and reciprocally.

As a bonus, `S.product` models the binary function from `S` to `S`:

```
sage: bin = S.product
sage: bin(x,y)
'ab'
```

Currently, `S.product` is just a bound method:

```
sage: bin  # py2
<bound method FreeSemigroup_with_category.product of An example of a␣
↪semigroup: the free semigroup generated by ('a', 'b', 'c', 'd')>
sage: bin  # py3, due to difference in how bound methods are repr'd
<bound method FreeSemigroup.product of An example of a semigroup: the␣
↪free semigroup generated by ('a', 'b', 'c', 'd')>
```

When Sage will support multivariate morphisms, it will be possible, and in fact recommended, to enrich `S.product` with extra mathematical structure. This will typically be implemented using lazy attributes.:

```
sage: bin                 # todo: not implemented
Generic binary morphism:
From: (S x S)
To:   S
```

**class Realizations**(*category*, **args*)
Bases: `sage.categories.realizations.RealizationsCategory`

**class ParentMethods**

**product_by_coercion**(*left*, *right*)
Default implementation of product for realizations.

This method coerces to the realization specified by `self.realization_of().a_realization()`, computes the product in that realization, and then coerces back.

EXAMPLES:

```
sage: Out = Sets().WithRealizations().example().Out(); Out
The subset algebra of {1, 2, 3} over Rational Field in the Out basis
sage: Out.product
<bound method SubsetAlgebra.Out_with_category.product_by_coercion of␣
↪The subset algebra of {1, 2, 3} over Rational Field in the Out basis>
sage: Out.product.__module__
'sage.categories.magmas'
sage: x = Out.an_element()
sage: y = Out.an_element()
sage: Out.product(x, y)
Out[{}] + 4*Out[{1}] + 9*Out[{2}] + Out[{1, 2}]
```

**class SubcategoryMethods**

**Associative**()
Return the full subcategory of the associative objects of `self`.

A (multiplicative) `magma Magmas` $M$ is *associative* if, for all $x, y, z \in M$,

$$x * (y * z) = (x * y) * z$$

**See also:**

Wikipedia article Associative_property

EXAMPLES:

```
sage: Magmas().Associative()
Category of semigroups
```

**Commutative**()
Return the full subcategory of the commutative objects of `self`.

A (multiplicative) `magma Magmas` $M$ is *commutative* if, for all $x, y \in M$,

$$x * y = y * x$$

**See also:**

Wikipedia article Commutative_property

EXAMPLES:

```
sage: Magmas().Commutative()
Category of commutative magmas
sage: Monoids().Commutative()
Category of commutative monoids
```

**Distributive**()

Return the full subcategory of the objects of `self` where $*$ is distributive on $+$.

INPUT:

- `self` – a subcategory of *Magmas* and *AdditiveMagmas*

Given that Sage does not yet know that the category `MagmasAndAdditiveMagmas` is the intersection of the categories *Magmas* and *AdditiveMagmas*, the method `MagmasAndAdditiveMagmas.SubcategoryMethods.Distributive()` is not available, as would be desirable, for this intersection.

This method is a workaround. It checks that `self` is a subcategory of both *Magmas* and *AdditiveMagmas* and upgrades it to a subcategory of `MagmasAndAdditiveMagmas` before applying the axiom. It complains overwise, since the `Distributive` axiom does not make sense for a plain magma.

EXAMPLES:

```
sage: (Magmas() & AdditiveMagmas()).Distributive()
Category of distributive magmas and additive magmas
sage: (Monoids() & CommutativeAdditiveGroups()).Distributive()
Category of rings

sage: Magmas().Distributive()
Traceback (most recent call last):
...
ValueError: The distributive axiom only makes sense on a magma which is␣
↪simultaneously an additive magma
sage: Semigroups().Distributive()
Traceback (most recent call last):
...
ValueError: The distributive axiom only makes sense on a magma which is␣
↪simultaneously an additive magma
```

**FinitelyGenerated**()

Return the subcategory of the objects of `self` that are endowed with a distinguished finite set of (multiplicative) magma generators.

EXAMPLES:

This is a shorthand for *FinitelyGeneratedAsMagma()*, which see:

```
sage: Magmas().FinitelyGenerated()
Category of finitely generated magmas
sage: Semigroups().FinitelyGenerated()
Category of finitely generated semigroups
sage: Groups().FinitelyGenerated()
Category of finitely generated enumerated groups
```

An error is raised if this is ambiguous:

```
sage: (Magmas() & AdditiveMagmas()).FinitelyGenerated()
Traceback (most recent call last):
...
ValueError: FinitelyGenerated is ambiguous for
Join of Category of magmas and Category of additive magmas.
Please use explicitly one of the FinitelyGeneratedAsXXX methods
```

**Note:** Checking that there is no ambiguity currently assumes that all the other "finitely generated" axioms involve an additive structure. As of Sage 6.4, this is correct.

The use of this shorthand should be reserved for casual interactive use or when there is no risk of ambiguity.

**FinitelyGeneratedAsMagma**()

Return the subcategory of the objects of `self` that are endowed with a distinguished finite set of (multiplicative) magma generators.

A set $S$ of elements of a multiplicative magma form a *set of generators* if any element of the magma can be expressed recursively from elements of $S$ and products thereof.

It is not imposed that morphisms shall preserve the distinguished set of generators; hence this is a full subcategory.

**See also:**

Wikipedia article Unital_magma#unital

EXAMPLES:

```
sage: Magmas().FinitelyGeneratedAsMagma()
Category of finitely generated magmas
```

Being finitely generated does depend on the structure: for a ring, being finitely generated as a magma, as an additive magma, or as a ring are different concepts. Hence the name of this axiom is explicit:

```
sage: Rings().FinitelyGeneratedAsMagma()
Category of finitely generated as magma enumerated rings
```

On the other hand, it does not depend on the multiplicative structure: for example a group is finitely generated if and only if it is finitely generated as a magma. A short hand is provided when there is no ambiguity, and the output tries to reflect that:

```
sage: Semigroups().FinitelyGenerated()
Category of finitely generated semigroups
sage: Groups().FinitelyGenerated()
Category of finitely generated enumerated groups

sage: Semigroups().FinitelyGenerated().axioms()
frozenset({'Associative', 'Enumerated', 'FinitelyGeneratedAsMagma'})
```

Note that the set of generators may depend on the actual category; for example, in a group, one can often use less generators since it is allowed to take inverses.

**JTrivial**()

Return the full subcategory of the $J$-trivial objects of `self`.

This axiom is in fact only meaningful for *semigroups*. This stub definition is here as a workaround for trac ticket #20515, in order to define the $J$-trivial axiom as the intersection of the $L$ and $R$-trivial axioms.

**See also:**

*Semigroups.SubcategoryMethods.JTrivial()*

**Unital**()

Return the subcategory of the unital objects of `self`.

A (multiplicative) `magma Magmas` $M$ is *unital* if it admits an element 1, called *unit*, such that for all $x \in M$,

$$1 * x = x * 1 = x$$

This element is necessarily unique, and should be provided as `M.one()`.

**See also:**

Wikipedia article Unital_magma#unital

EXAMPLES:

```
sage: Magmas().Unital()
Category of unital magmas
sage: Semigroups().Unital()
Category of monoids
sage: Monoids().Unital()
Category of monoids
sage: from sage.categories.associative_algebras import AssociativeAlgebras
sage: AssociativeAlgebras(QQ).Unital()
Category of algebras over Rational Field
```

**class Subquotients**(*category*, **args*)

Bases: *sage.categories.subquotients.SubquotientsCategory*

The category of subquotient magmas.

See *Sets.SubcategoryMethods.Subquotients()* for the general setup for subquotients. In the case of a subquotient magma $S$ of a magma $G$, the condition that $r$ be a morphism in `As` can be rewritten as follows:

- for any two $a, b \in S$ the identity $a \times_S b = r(l(a) \times_G l(b))$ holds.

This is used by this category to implement the product $\times_S$ of $S$ from $l$ and $r$ and the product of $G$.

EXAMPLES:

```
sage: Semigroups().Subquotients().all_super_categories()
[Category of subquotients of semigroups, Category of semigroups,
 Category of subquotients of magmas, Category of magmas,
 Category of subquotients of sets, Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

**class ParentMethods**

**product**(*x*, *y*)

Return the product of two elements of `self`.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S
An example of a (sub)quotient semigroup:
a quotient of the left zero semigroup
sage: S.product(S(19), S(3))
19
```

Here is a more elaborate example involving a sub algebra:

```
sage: Z = SymmetricGroup(5).algebra(QQ).center()
sage: B = Z.basis()
sage: B[3] * B[2]
4*B[2] + 6*B[3] + 5*B[6]
```

**class Unital**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class Algebras**(*category*, **args*)
Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**extra_super_categories**()
EXAMPLES:
sage: Magmas().Commutative().Algebras(QQ).extra_super_categories() [Category of commutative magmas]
This implements the fact that the algebra of a commutative magma is commutative:

```
sage: Magmas().Commutative().Algebras(QQ).super_categories()
[Category of magma algebras over Rational Field,
 Category of commutative magmas]
```

In particular, commutative monoid algebras are commutative algebras:

```
sage: Monoids().Commutative().Algebras(QQ).is_subcategory(Algebras(QQ).
↪Commutative())
True
```

**class CartesianProducts**(*category*, **args*)
Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**class ElementMethods**

**class ParentMethods**

**one**()
Return the unit of this Cartesian product.

It is built from the units for the Cartesian factors of `self`.

EXAMPLES:

```
sage: cartesian_product([QQ, ZZ, RR]).one()
(1, 1, 1.00000000000000)
```

**extra_super_categories**()
Implement the fact that a Cartesian product of unital magmas is a unital magma

EXAMPLES:

```
sage: C = Magmas().Unital().CartesianProducts()
sage: C.extra_super_categories();
[Category of unital magmas]
sage: C.axioms()
frozenset({'Unital'})

sage: Monoids().CartesianProducts().is_subcategory(Monoids())
True
```

**class ElementMethods**

**class Inverse**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class CartesianProducts**(*category*, **args*)
Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**extra_super_categories**()
Implement the fact that a Cartesian product of magmas with inverses is a magma with inverse.

EXAMPLES:

```
sage: C = Magmas().Unital().Inverse().CartesianProducts()
sage: C.extra_super_categories();
[Category of inverse unital magmas]
sage: sorted(C.axioms())
['Inverse', 'Unital']
```

**class ParentMethods**

**is_empty**()
Return whether `self` is empty.

Since this set is a unital magma it is not empty and this method always return `False`.

EXAMPLES:

```
sage: S = SymmetricGroup(2)
sage: S.is_empty()
False

sage: M = Monoids().example()
sage: M.is_empty()
False
```

**one**()
Return the unit of the monoid, that is the unique neutral element for $*$.

---

**Note:** The default implementation is to coerce 1 into `self`. It is recommended to override this method because the coercion from the integers:
- is not always meaningful (except for 1);
- often uses `self.one()`.

---

EXAMPLES:

```
sage: M = Monoids().example(); M
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd
↪')
sage: M.one()
''
```

**class Realizations**(*category*, **args*)
Bases: *sage.categories.realizations.RealizationsCategory*

**class ParentMethods**

**one**()
Return the unit element of `self`.

sage: from sage.combinat.root_system.extended_affine_weyl_group import ExtendedAffineWeylGroup sage: PvW0 = ExtendedAffineWeylGroup(['A',2,1]).PvW0() sage: PvW0 in Magmas().Unital().Realizations() True sage: PvW0.one() 1

**class SubcategoryMethods**

**Inverse**()
Return the full subcategory of the inverse objects of `self`.

An inverse :class:' (multiplicative) magma <Magmas>' is a *unital magma* such that every element admits both an inverse on the left and on the right. Such a magma is also called a *loop*.

**See also:**

Wikipedia article Inverse_element, Wikipedia article Quasigroup

EXAMPLES:

```
sage: Magmas().Unital().Inverse()
Category of inverse unital magmas
sage: Monoids().Inverse()
Category of groups
```

**additional_structure**()
Return `self`.

Indeed, the category of unital magmas defines an additional structure, namely the unit of the magma which shall be preserved by morphisms.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: Magmas().Unital().additional_structure()
Category of unital magmas
```

**super_categories**()
EXAMPLES:

```
sage: Magmas().super_categories()
[Category of sets]
```

## 3.100 Magmas and Additive Magmas

**class** sage.categories.magmas_and_additive_magmas.**MagmasAndAdditiveMagmas**(*s=None*)
Bases: *sage.categories.category_singleton.Category_singleton*

The category of sets $(S, +, *)$ with an additive operation '+' and a multiplicative operation $*$

EXAMPLES:

```
sage: from sage.categories.magmas_and_additive_magmas import
↪MagmasAndAdditiveMagmas
sage: C = MagmasAndAdditiveMagmas(); C
Category of magmas and additive magmas
```

This is the base category for the categories of rings and their variants:

```
sage: C.Distributive()
Category of distributive magmas and additive magmas
sage: C.Distributive().Associative().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital().AdditiveInverse()
Category of rngs
sage: C.Distributive().Associative().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital().Unital()
Category of semirings
sage: C.Distributive().Associative().AdditiveAssociative().AdditiveCommutative().
↪AdditiveUnital().AdditiveInverse().Unital()
Category of rings
```

This category is really meant to represent the intersection of the categories of `Magmas` and `AdditiveMagmas`; however Sage's infrastructure does not allow yet to model this:

```
sage: Magmas() & AdditiveMagmas()
Join of Category of magmas and Category of additive magmas

sage: Magmas() & AdditiveMagmas()        # todo: not implemented
Category of magmas and additive magmas
```

**class CartesianProducts**(*category*, **args*)

Bases: `sage.categories.cartesian_product.CartesianProductsCategory`

**extra_super_categories**()

Implement the fact that this structure is stable under Cartesian products.

**Distributive**

alias of `DistributiveMagmasAndAdditiveMagmas`

**class SubcategoryMethods**

**Distributive**()

Return the full subcategory of the objects of `self` where $*$ is distributive on $+$.

A `magma` and `additive magma` $M$ is *distributive* if, for all $x, y, z \in M$,

$$x * (y + z) = x * y + x * z \text{ and } (x + y) * z = x * z + y * z$$

EXAMPLES:

```
sage: from sage.categories.magmas_and_additive_magmas import␣
↪MagmasAndAdditiveMagmas
sage: C = MagmasAndAdditiveMagmas().Distributive(); C
Category of distributive magmas and additive magmas
```

**Note:** Given that Sage does not know that `MagmasAndAdditiveMagmas` is the intersection of `Magmas` and `AdditiveMagmas`, this method is not available for:

```
sage: Magmas() & AdditiveMagmas()
Join of Category of magmas and Category of additive magmas
```

Still, the natural syntax works:

```
sage: (Magmas() & AdditiveMagmas()).Distributive()
Category of distributive magmas and additive magmas
```

thanks to a workaround implemented in *Magmas.SubcategoryMethods.Distributive()*:

```
sage: (Magmas() & AdditiveMagmas()).Distributive.__module__
'sage.categories.magmas'
```

---

**additional_structure**()

Return None.

Indeed, this category is meant to represent the join of *AdditiveMagmas* and *Magmas*. As such, it defines no additional structure.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: from sage.categories.magmas_and_additive_magmas import␣
↪MagmasAndAdditiveMagmas
sage: MagmasAndAdditiveMagmas().additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.magmas_and_additive_magmas import␣
↪MagmasAndAdditiveMagmas
sage: MagmasAndAdditiveMagmas().super_categories()
[Category of magmas, Category of additive magmas]
```

## 3.101 Non-unital non-associative algebras

**class** sage.categories.magmatic_algebras.**MagmaticAlgebras**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of algebras over a given base ring.

An algebra over a ring $R$ is a module over $R$ endowed with a bilinear multiplication.

**Warning:** *MagmaticAlgebras* will eventually replace the current *Algebras* for consistency with e.g. Wikipedia article Algebras which assumes neither associativity nor the existence of a unit (see trac ticket #15043).

EXAMPLES:

```
sage: from sage.categories.magmatic_algebras import MagmaticAlgebras
sage: C = MagmaticAlgebras(ZZ); C
Category of magmatic algebras over Integer Ring
sage: C.super_categories()
[Category of additive commutative additive associative additive unital␣
↪distributive magmas and additive magmas,
 Category of modules over Integer Ring]
```

**Associative**
alias of `AssociativeAlgebras`

**class ParentMethods**

**algebra_generators**()
Return a family of generators of this algebra.

EXAMPLES:

```
sage: F = AlgebrasWithBasis(QQ).example(); F
An example of an algebra with basis: the free algebra on the generators (
→'a', 'b', 'c') over Rational Field
sage: F.algebra_generators()
Family (B[word: a], B[word: b], B[word: c])
```

**Unital**
alias of `UnitalAlgebras`

**class WithBasis**(*base_category*)
Bases: `sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring`

**class ParentMethods**

**algebra_generators**()
Return generators for this algebra.

This default implementation returns the basis of this algebra.

OUTPUT: a family

**See also:**

- `basis()`
- `MagmaticAlgebras.ParentMethods.algebra_generators()`

EXAMPLES:

```
sage: D4 = DescentAlgebra(QQ, 4).B()
sage: D4.algebra_generators()
Lazy family (...)_{i in Compositions of 4}

sage: R.<x> = ZZ[]
sage: P = PartitionAlgebra(1, x, R)
sage: P.algebra_generators()
Lazy family (Term map from Partition diagrams of order 1 to
 Partition Algebra of rank 1 with parameter x over Univariate
→Polynomial Ring in x
 over Integer Ring(i))_{i in Partition diagrams of order 1}
```

**product**()
The product of the algebra, as per `Magmas.ParentMethods.product()`

By default, this is implemented using one of the following methods, in the specified order:

- `product_on_basis()`
- `_multiply()` or `_multiply_basis()`
- `product_by_coercion()`

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()
sage: a, b, c = A.algebra_generators()
sage: A.product(a + 2*b, 3*c)
3*B[word: ac] + 6*B[word: bc]
```

**product_on_basis** (*i*, *j*)

The product of the algebra on the basis (optional).

INPUT:

- `i, j` – the indices of two elements of the basis of `self`

Return the product of the two corresponding basis elements indexed by `i` and `j`.

If implemented, *product()* is defined from it by bilinearity.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()
sage: Word = A.basis().keys()
sage: A.product_on_basis(Word("abc"),Word("cba"))
B[word: abccba]
```

**additional_structure** ()

Return `None`.

Indeed, the category of (magmatic) algebras defines no new structure: a morphism of modules and of magmas between two (magmatic) algebras is a (magmatic) algebra morphism.

**See also:**

*Category.additional_structure()*

---

**Todo:** This category should be a *CategoryWithAxiom*, the axiom specifying the compatibility between the magma and module structure.

---

EXAMPLES:

```
sage: from sage.categories.magmatic_algebras import MagmaticAlgebras
sage: MagmaticAlgebras(ZZ).additional_structure()
```

**super_categories** ()

EXAMPLES:

```
sage: from sage.categories.magmatic_algebras import MagmaticAlgebras
sage: MagmaticAlgebras(ZZ).super_categories()
[Category of additive commutative additive associative additive unital␣
↪distributive magmas and additive magmas, Category of modules over Integer␣
↪Ring]

sage: from sage.categories.additive_semigroups import AdditiveSemigroups
sage: MagmaticAlgebras(ZZ).is_subcategory((AdditiveSemigroups() & Magmas()).
↪Distributive())
True
```

## 3.102 Manifolds

**class** sage.categories.manifolds.**ComplexManifolds**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of complex manifolds.

A $d$-dimensional complex manifold is a manifold whose underlying vector space is $\mathbf{C}^d$ and has a holomorphic atlas.

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).super_categories()
[Category of topological spaces]
```

**class** sage.categories.manifolds.**Manifolds**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of manifolds over any topological field.

Let $k$ be a topological field. A $d$-dimensional $k$-*manifold* $M$ is a second countable Hausdorff space such that the neighborhood of any point $x \in M$ is homeomorphic to $k^d$.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: C = Manifolds(RR); C
Category of manifolds over Real Field with 53 bits of precision
sage: C.super_categories()
[Category of topological spaces]
```

**class AlmostComplex**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of almost complex manifolds.

An *almost complex manifold* $M$ is a manifold with a smooth tensor field $J$ of rank $(1,1)$ such that $J^2 = -1$ when regarded as a vector bundle isomorphism $J : TM \to TM$ on the tangent bundle. The tensor field $J$ is called the *almost complex structure* of $M$.

**extra_super_categories**()

Return the extra super categories of `self`.

An almost complex manifold is smooth.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).AlmostComplex().super_categories() # indirect doctest
[Category of smooth manifolds
 over Real Field with 53 bits of precision]
```

**class Analytic**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of complex manifolds.

An analytic manifold is a manifold with an analytic atlas.

**extra_super_categories**()

Return the extra super categories of `self`.

An analytic manifold is smooth.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Analytic().super_categories() # indirect doctest
[Category of smooth manifolds
 over Real Field with 53 bits of precision]
```

**class Connected**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of connected manifolds.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: C = Manifolds(RR).Connected()
sage: TestSuite(C).run(skip="_test_category_over_bases")
```

**class Differentiable**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of differentiable manifolds.

A differentiable manifold is a manifold with a differentiable atlas.

**class FiniteDimensional**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

Category of finite dimensional manifolds.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: C = Manifolds(RR).FiniteDimensional()
sage: TestSuite(C).run(skip="_test_category_over_bases")
```

**class ParentMethods**

**dimension**()

Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: M = Manifolds(RR).example()
sage: M.dimension()
3
```

**class Smooth**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of smooth manifolds.

A smooth manifold is a manifold with a smooth atlas.

**extra_super_categories**()

Return the extra super categories of `self`.

A smooth manifold is differentiable.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Smooth().super_categories() # indirect doctest
[Category of differentiable manifolds
 over Real Field with 53 bits of precision]
```

**class SubcategoryMethods**

**AlmostComplex()**

Return the subcategory of the almost complex objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).AlmostComplex()
Category of almost complex manifolds
 over Real Field with 53 bits of precision
```

**Analytic()**

Return the subcategory of the analytic objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Analytic()
Category of analytic manifolds
 over Real Field with 53 bits of precision
```

**Complex()**

Return the subcategory of manifolds over $\mathbf{C}$ of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(CC).Complex()
Category of complex manifolds over
 Complex Field with 53 bits of precision
```

**Connected()**

Return the full subcategory of the connected objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Connected()
Category of connected manifolds
 over Real Field with 53 bits of precision
```

**Differentiable()**

Return the subcategory of the differentiable objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Differentiable()
Category of differentiable manifolds
 over Real Field with 53 bits of precision
```

**FiniteDimensional**()

Return the full subcategory of the finite dimensional objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: C = Manifolds(RR).Connected().FiniteDimensional(); C
Category of finite dimensional connected manifolds
 over Real Field with 53 bits of precision
```

**Smooth**()

Return the subcategory of the smooth objects of `self`.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).Smooth()
Category of smooth manifolds
 over Real Field with 53 bits of precision
```

**additional_structure**()

Return `None`.

Indeed, the category of manifolds defines no new structure: a morphism of topological spaces between manifolds is a manifold morphism.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: Manifolds(RR).super_categories()
[Category of topological spaces]
```

## 3.103 Matrix algebras

**class** sage.categories.matrix_algebras.**MatrixAlgebras**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of matrix algebras over a field.

EXAMPLES:

```
sage: MatrixAlgebras(RationalField())
Category of matrix algebras over Rational Field
```

**super_categories**()

EXAMPLES:

```
sage: MatrixAlgebras(QQ).super_categories()
[Category of algebras over Rational Field]
```

## 3.104 Metric Spaces

**class** sage.categories.metric_spaces.**MetricSpaces**(*category*, **args*)

Bases: *sage.categories.metric_spaces.MetricSpacesCategory*

The category of metric spaces.

A *metric* on a set $S$ is a function $d : S \times S \to \mathbf{R}$ such that:

- $d(a, b) \geq 0$,
- $d(a, b) = 0$ if and only if $a = b$.

A metric space is a set $S$ with a distinguished metric.

### Implementation

Objects in this category must implement either a `dist` on the parent or the elements or `metric` on the parent; otherwise this will cause an infinite recursion.

**Todo:**

- Implement a general geodesics class.
- Implement a category for metric additive groups and move the generic distance $d(a, b) = |a - b|$ there.
- Incorporate the length of a geodesic as part of the default distance cycle.

EXAMPLES:

```
sage: from sage.categories.metric_spaces import MetricSpaces
sage: C = MetricSpaces()
sage: C
Category of metric spaces
sage: TestSuite(C).run()
```

**class Complete**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

The category of complete metric spaces.

**class ElementMethods**

**abs**()

Return the absolute value of `self`.

EXAMPLES:

```
sage: CC(I).abs()
1.00000000000000
```

**dist** (*b*)

Return the distance between `self` and `other`.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: p1 = UHP.get_point(5 + 7*I)
sage: p2 = UHP.get_point(1 + I)
sage: p1.dist(p2)
arccosh(33/7)
```

**class ParentMethods**

**dist** (*a*, *b*)

Return the distance between `a` and `b` in `self`.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: p1 = UHP.get_point(5 + 7*I)
sage: p2 = UHP.get_point(1.0 + I)
sage: UHP.dist(p1, p2)
2.23230104635820

sage: PD = HyperbolicPlane().PD()
sage: PD.dist(PD.get_point(0), PD.get_point(I/2))
arccosh(5/3)
```

**metric** ()

Return the metric of `self`.

EXAMPLES:

```
sage: UHP = HyperbolicPlane().UHP()
sage: m = UHP.metric()
sage: p1 = UHP.get_point(5 + 7*I)
sage: p2 = UHP.get_point(1.0 + I)
sage: m(p1, p2)
2.23230104635820
```

**class SubcategoryMethods**

**Complete** ()

Return the full subcategory of the complete objects of `self`.

EXAMPLES:

```
sage: Sets().Metric().Complete()
Category of complete metric spaces
```

**class WithRealizations** (*category*, **args*)

Bases: `sage.categories.with_realizations.WithRealizationsCategory`

**class ParentMethods**

**dist** (*a*, *b*)

Return the distance between `a` and `b` by converting them to a realization of `self` and doing the computation.

EXAMPLES:

```
sage: H = HyperbolicPlane()
sage: PD = H.PD()
sage: p1 = PD.get_point(0)
sage: p2 = PD.get_point(I/2)
sage: H.dist(p1, p2)
arccosh(5/3)
```

**class** sage.categories.metric_spaces.**MetricSpacesCategory**(*category*, **args*)

Bases: *sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory*

**classmethod default_super_categories**(*category*)

Return the default super categories of `category.Metric()`.

Mathematical meaning: if $A$ is a metric space in the category $C$, then $A$ is also a topological space.

INPUT:

- `cls` – the class `MetricSpaces`
- `category` – a category $Cat$

OUTPUT:

A (join) category

In practice, this returns `category.Metric()`, joined together with the result of the method *RegressiveCovariantConstructionCategory.default_super_categories()* (that is the join of `category` and `cat.Metric()` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Groups()`. Then, a group $G$ with a metric is simultaneously a topological group by itself, and a metric space:

```
sage: Groups().Metric().super_categories()
[Category of topological groups, Category of metric spaces]
```

This resulted from the following call:

```
sage: sage.categories.metric_spaces.MetricSpacesCategory.default_super_
↪categories(Groups())
Join of Category of topological groups and Category of metric spaces
```

## 3.105 Modular abelian varieties

**class** sage.categories.modular_abelian_varieties.**ModularAbelianVarieties**(*Y*)

Bases: *sage.categories.category_types.Category_over_base*

The category of modular abelian varieties over a given field.

EXAMPLES:

```
sage: ModularAbelianVarieties(QQ)
Category of modular abelian varieties over Rational Field
```

**class Homsets**(*category*, **args*)

Bases: *sage.categories.homsets.HomsetsCategory*

**class Endset** (*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**extra_super_categories** ()

Implement the fact that an endset of modular abelian variety is a ring.

EXAMPLES:

```
sage: ModularAbelianVarieties(QQ).Endsets().extra_super_categories()
[Category of rings]
```

**base_field** ()

EXAMPLES:

```
sage: ModularAbelianVarieties(QQ).base_field()
Rational Field
```

**super_categories** ()

EXAMPLES:

```
sage: ModularAbelianVarieties(QQ).super_categories()
[Category of sets]
```

## 3.106 Modules

**class** sage.categories.modules.**Modules** (*base*, *name=None*)

Bases: *sage.categories.category_types.Category_module*

The category of all modules over a base ring $R$.

An $R$-module $M$ is a left and right $R$-module over a commutative ring $R$ such that:

$$r * (x * s) = (r * x) * s \qquad \forall r, s \in R \text{ and } x \in M$$

INPUT:

- `base_ring` – a ring $R$ or subcategory of `Rings()`
- `dispatch` – a boolean (for internal use; default: `True`)

When the base ring is a field, the category of vector spaces is returned instead (unless `dispatch == False`).

> **Warning:** Outside of the context of symmetric modules over a commutative ring, the specifications of this category are fuzzy and not yet set in stone (see below). The code in this category and its subcategories is therefore prone to bugs or arbitrary limitations in this case.

EXAMPLES:

```
sage: Modules(ZZ)
Category of modules over Integer Ring
sage: Modules(QQ)
Category of vector spaces over Rational Field

sage: Modules(Rings())
Category of modules over rings
sage: Modules(FiniteFields())
```

```
Category of vector spaces over finite enumerated fields

sage: Modules(Integers(9))
Category of modules over Ring of integers modulo 9

sage: Modules(Integers(9)).super_categories()
[Category of bimodules over Ring of integers modulo 9 on the left and Ring of␣
↪integers modulo 9 on the right]

sage: Modules(ZZ).super_categories()
[Category of bimodules over Integer Ring on the left and Integer Ring on the␣
↪right]

sage: Modules == RingModules
True

sage: Modules(ZZ['x']).is_abelian()    # see #6081
True
```

**Todo:**

- Clarify the distinction, if any, with `BiModules(R, R)`. In particular, if $R$ is a commutative ring (e.g. a field), some pieces of the code possibly assume that $M$ is a *symmetric 'R'-'R'-bimodule*:

$$r * x = x * r \qquad \forall r \in R \text{ and } x \in M$$

- Make sure that non symmetric modules are properly supported by all the code, and advertise it.
- Make sure that non commutative rings are properly supported by all the code, and advertise it.
- Add support for base semirings.
- Implement a `FreeModules(R)` category, when so prompted by a concrete use case: e.g. modeling a free module with several bases (using `Sets.SubcategoryMethods.Realizations()`) or with an atlas of local maps (see e.g. trac ticket #15916).

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

The category of modules constructed as Cartesian products of modules

This construction gives the direct product of modules. The implementation is based on the following resources:

- http://groups.google.fr/group/sage-devel/browse_thread/thread/35a72b1d0a2fc77a/348f42ae77a66d16#348f42ae77a66d16
- Wikipedia article Direct_product

**class ParentMethods**

**base_ring**()

Return the base ring of this Cartesian product.

EXAMPLES:

```
sage: E = CombinatorialFreeModule(ZZ, [1,2,3])
sage: F = CombinatorialFreeModule(ZZ, [2,3,4])
sage: C = cartesian_product([E, F]); C
Free module generated by {1, 2, 3} over Integer Ring (+)
Free module generated by {2, 3, 4} over Integer Ring
sage: C.base_ring()
Integer Ring
```

**extra_super_categories**()

A Cartesian product of modules is endowed with a natural module structure.

EXAMPLES:

```
sage: Modules(ZZ).CartesianProducts().extra_super_categories()
[Category of modules over Integer Ring]
sage: Modules(ZZ).CartesianProducts().super_categories()
[Category of Cartesian products of commutative additive groups,
 Category of modules over Integer Ring]
```

**class ElementMethods**

**Filtered**

alias of FilteredModules

**class FiniteDimensional**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**extra_super_categories**()

Implement the fact that a finite dimensional module over a finite ring is finite.

EXAMPLES:

```
sage: Modules(IntegerModRing(4)).FiniteDimensional().extra_super_
↪categories()
[Category of finite sets]
sage: Modules(ZZ).FiniteDimensional().extra_super_categories()
[]
sage: Modules(GF(5)).FiniteDimensional().is_subcategory(Sets().Finite())
True
sage: Modules(ZZ).FiniteDimensional().is_subcategory(Sets().Finite())
False

sage: Modules(Rings().Finite()).FiniteDimensional().is_subcategory(Sets().
↪Finite())
True
sage: Modules(Rings()).FiniteDimensional().is_subcategory(Sets().Finite())
False
```

**Graded**

alias of *GradedModules*

**class Homsets**(*category*, **args*)

Bases: *sage.categories.homsets.HomsetsCategory*

The category of homomorphism sets $\hom(X, Y)$ for $X$, $Y$ modules.

**class Endset**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of endomorphism sets $End(X)$ for $X$ a module (this is not used yet)

**extra_super_categories**()

Implement the fact that the endomorphism set of a module is an algebra.

**See also:**

`CategoryWithAxiom.extra_super_categories()`

EXAMPLES:

```
sage: Modules(ZZ).Endsets().extra_super_categories()
[Category of magmatic algebras over Integer Ring]

sage: End(ZZ^3) in Algebras(ZZ)
True
```

**class ParentMethods**

**base_ring**()

Return the base ring of `self`.

EXAMPLES:

```
sage: E = CombinatorialFreeModule(ZZ, [1,2,3])
sage: F = CombinatorialFreeModule(ZZ, [2,3,4])
sage: H = Hom(E, F)
sage: H.base_ring()
Integer Ring
```

This `base_ring` method is actually overridden by `sage.structure.category_object.CategoryObject.base_ring()`:

```
sage: H.base_ring.__module__
```

Here we call it directly:

```
sage: method = H.category().parent_class.base_ring
sage: method.__get__(H)()
Integer Ring
```

**zero**()

EXAMPLES:

```
sage: E = CombinatorialFreeModule(ZZ, [1,2,3])
sage: F = CombinatorialFreeModule(ZZ, [2,3,4])
sage: H = Hom(E, F)
sage: f = H.zero()
sage: f
Generic morphism:
  From: Free module generated by {1, 2, 3} over Integer Ring
  To:   Free module generated by {2, 3, 4} over Integer Ring
sage: f(E.monomial(2))
0
sage: f(E.monomial(3)) == F.zero()
True
```

**base_ring**()

EXAMPLES:

```
sage: Modules(ZZ).Homsets().base_ring()
Integer Ring
```

---

**Todo:** Generalize this so that any homset category of a full subcategory of modules over a base ring is a category over this base ring.

---

**extra_super_categories()**
EXAMPLES:

```
sage: Modules(ZZ).Homsets().extra_super_categories()
[Category of modules over Integer Ring]
```

**class ParentMethods**

**tensor_square()**
Returns the tensor square of `self`

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: A.tensor_square()
An example of Hopf algebra with basis:
 the group algebra of the Dihedral group of order 6
 as a permutation group over Rational Field # An example
 of Hopf algebra with basis: the group algebra of the Dihedral
 group of order 6 as a permutation group over Rational Field
```

**class SubcategoryMethods**

**DualObjects()**
Return the category of spaces constructed as duals of spaces of `self`.

The *dual* of a vector space $V$ is the space consisting of all linear functionals on $V$ (see Wikipedia article Dual_space). Additional structure on $V$ can endow its dual with additional structure; for example, if $V$ is a finite dimensional algebra, then its dual is a coalgebra.

This returns the category of spaces constructed as dual of spaces in `self`, endowed with the appropriate additional structure.

**Warning:**

- This semantic of `dual` and `DualObject` is imposed on all subcategories, in particular to make `dual` a covariant functorial construction.

  A subcategory that defines a different notion of dual needs to use a different name.

- Typically, the category of graded modules should define a separate `graded_dual` construction (see trac ticket #15647). For now the two constructions are not distinguished which is an oversimplified model.

**See also:**

- *dual.DualObjectsCategory*
- *CovariantFunctorialConstruction*.

EXAMPLES:

```
sage: VectorSpaces(QQ).DualObjects()
Category of duals of vector spaces over Rational Field
```

The dual of a vector space is a vector space:

```
sage: VectorSpaces(QQ).DualObjects().super_categories()
[Category of vector spaces over Rational Field]
```

The dual of an algebra is a coalgebra:

```
sage: sorted(Algebras(QQ).DualObjects().super_categories(), key=str)
[Category of coalgebras over Rational Field,
 Category of duals of vector spaces over Rational Field]
```

The dual of a coalgebra is an algebra:

```
sage: sorted(Coalgebras(QQ).DualObjects().super_categories(), key=str)
[Category of algebras over Rational Field,
 Category of duals of vector spaces over Rational Field]
```

As a shorthand, this category can be accessed with the `dual()` method:

```
sage: VectorSpaces(QQ).dual()
Category of duals of vector spaces over Rational Field
```

**Filtered**(*base_ring=None*)

Return the subcategory of the filtered objects of `self`.

INPUT:

- `base_ring` – this is ignored

EXAMPLES:

```
sage: Modules(ZZ).Filtered()
Category of filtered modules over Integer Ring

sage: Coalgebras(QQ).Filtered()
Join of Category of filtered modules over Rational Field
 and Category of coalgebras over Rational Field

sage: AlgebrasWithBasis(QQ).Filtered()
Category of filtered algebras with basis over Rational Field
```

**Todo:**

- Explain why this does not commute with `WithBasis()`
- Improve the support for covariant functorial constructions categories over a base ring so as to get rid of the `base_ring` argument.

**FiniteDimensional**()

Return the full subcategory of the finite dimensional objects of `self`.

EXAMPLES:

```
sage: Modules(ZZ).FiniteDimensional()
Category of finite dimensional modules over Integer Ring
sage: Coalgebras(QQ).FiniteDimensional()
```

```
Category of finite dimensional coalgebras over Rational Field
sage: AlgebrasWithBasis(QQ).FiniteDimensional()
Category of finite dimensional algebras with basis over Rational Field
```

**Graded** (*base_ring=None*)

Return the subcategory of the graded objects of `self`.

INPUT:

- `base_ring` – this is ignored

EXAMPLES:

```
sage: Modules(ZZ).Graded()
Category of graded modules over Integer Ring

sage: Coalgebras(QQ).Graded()
Join of Category of graded modules over Rational Field and Category of␣
↪coalgebras over Rational Field

sage: AlgebrasWithBasis(QQ).Graded()
Category of graded algebras with basis over Rational Field
```

**Todo:**

- Explain why this does not commute with *WithBasis()*
- Improve the support for covariant functorial constructions categories over a base ring so as to get rid of the `base_ring` argument.

**Super** (*base_ring=None*)

Return the super-analogue category of `self`.

INPUT:

- `base_ring` – this is ignored

EXAMPLES:

```
sage: Modules(ZZ).Super()
Category of super modules over Integer Ring

sage: Coalgebras(QQ).Super()
Category of super coalgebras over Rational Field

sage: AlgebrasWithBasis(QQ).Super()
Category of super algebras with basis over Rational Field
```

**Todo:**

- Explain why this does not commute with *WithBasis()*
- Improve the support for covariant functorial constructions categories over a base ring so as to get rid of the `base_ring` argument.

**TensorProducts** ()

Return the full subcategory of objects of `self` constructed as tensor products.

**See also:**

- *tensor.TensorProductsCategory*
- `RegressiveCovariantFunctorialConstruction`.

EXAMPLES:

```
sage: ModulesWithBasis(QQ).TensorProducts()
Category of tensor products of vector spaces with basis over Rational␣
↪Field
```

**WithBasis**()

Return the full subcategory of the objects of `self` with a distinguished basis.

EXAMPLES:

```
sage: Modules(ZZ).WithBasis()
Category of modules with basis over Integer Ring
sage: Coalgebras(QQ).WithBasis()
Category of coalgebras with basis over Rational Field
sage: AlgebrasWithBasis(QQ).WithBasis()
Category of algebras with basis over Rational Field
```

**base_ring**()

Return the base ring (category) for `self`.

This implements a `base_ring` method for all subcategories of `Modules(K)`.

EXAMPLES:

```
sage: C = Modules(QQ) & Semigroups(); C
Join of Category of semigroups and Category of vector spaces over␣
↪Rational Field
sage: C.base_ring()
Rational Field
sage: C.base_ring.__module__
'sage.categories.modules'

sage: C = Modules(Rings()) & Semigroups(); C
Join of Category of semigroups and Category of modules over rings
sage: C.base_ring()
Category of rings
sage: C.base_ring.__module__
'sage.categories.modules'

sage: C = DescentAlgebra(QQ,3).B().category()
sage: C.base_ring.__module__
'sage.categories.modules'
sage: C.base_ring()
Rational Field

sage: C = QuasiSymmetricFunctions(QQ).F().category()
sage: C.base_ring.__module__
'sage.categories.modules'
sage: C.base_ring()
Rational Field
```

**dual**()

Return the category of spaces constructed as duals of spaces of `self`.

The *dual* of a vector space $V$ is the space consisting of all linear functionals on $V$ (see Wikipedia article Dual_space). Additional structure on $V$ can endow its dual with additional structure; for example, if $V$ is a finite dimensional algebra, then its dual is a coalgebra.

This returns the category of spaces constructed as dual of spaces in `self`, endowed with the appropriate additional structure.

**Warning:**

- This semantic of `dual` and `DualObject` is imposed on all subcategories, in particular to make `dual` a covariant functorial construction.

  A subcategory that defines a different notion of dual needs to use a different name.

- Typically, the category of graded modules should define a separate `graded_dual` construction (see trac ticket #15647). For now the two constructions are not distinguished which is an oversimplified model.

**See also:**

- *dual.DualObjectsCategory*
- *CovariantFunctorialConstruction*.

EXAMPLES:

```
sage: VectorSpaces(QQ).DualObjects()
Category of duals of vector spaces over Rational Field
```

The dual of a vector space is a vector space:

```
sage: VectorSpaces(QQ).DualObjects().super_categories()
[Category of vector spaces over Rational Field]
```

The dual of an algebra is a coalgebra:

```
sage: sorted(Algebras(QQ).DualObjects().super_categories(), key=str)
[Category of coalgebras over Rational Field,
 Category of duals of vector spaces over Rational Field]
```

The dual of a coalgebra is an algebra:

```
sage: sorted(Coalgebras(QQ).DualObjects().super_categories(), key=str)
[Category of algebras over Rational Field,
 Category of duals of vector spaces over Rational Field]
```

As a shorthand, this category can be accessed with the *dual()* method:

```
sage: VectorSpaces(QQ).dual()
Category of duals of vector spaces over Rational Field
```

**Super**
alias of `SuperModules`

**class TensorProducts**(*category*, **args*)
Bases: *sage.categories.tensor.TensorProductsCategory*

The category of modules constructed by tensor product of modules.

**extra_super_categories**()
EXAMPLES:

```
sage: Modules(ZZ).TensorProducts().extra_super_categories()
[Category of modules over Integer Ring]
sage: Modules(ZZ).TensorProducts().super_categories()
[Category of modules over Integer Ring]
```

**WithBasis**

alias of *ModulesWithBasis*

**additional_structure**()

Return None.

Indeed, the category of modules defines no additional structure: a bimodule morphism between two modules is a module morphism.

**See also:**

*Category.additional_structure()*

**Todo:** Should this category be a *CategoryWithAxiom*?

EXAMPLES:

```
sage: Modules(ZZ).additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: Modules(ZZ).super_categories()
[Category of bimodules over Integer Ring on the left and Integer Ring on the␣
↪right]
```

Nota bene:

```
sage: Modules(QQ)
Category of vector spaces over Rational Field
sage: Modules(QQ).super_categories()
[Category of modules over Rational Field]
```

## 3.107 Modules With Basis

AUTHORS:

- Nicolas M. Thiery (2008-2014): initial revision, axiomatization
- Jason Bandlow and Florent Hivert (2010): Triangular Morphisms
- Christian Stump (2010): trac ticket #9648 module_morphism's to a wider class of codomains

**class** sage.categories.modules_with_basis.**ModulesWithBasis**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of modules with a distinguished basis.

The elements are represented by expanding them in the distinguished basis. The morphisms are not required to respect the distinguished basis.

EXAMPLES:

```
sage: ModulesWithBasis(ZZ)
Category of modules with basis over Integer Ring
sage: ModulesWithBasis(ZZ).super_categories()
[Category of modules over Integer Ring]
```

If the base ring is actually a field, this constructs instead the category of vector spaces with basis:

```
sage: ModulesWithBasis(QQ)
Category of vector spaces with basis over Rational Field

sage: ModulesWithBasis(QQ).super_categories()
[Category of modules with basis over Rational Field,
 Category of vector spaces over Rational Field]
```

Let $X$ and $Y$ be two modules with basis. We can build $Hom(X,Y)$:

```
sage: X = CombinatorialFreeModule(QQ, [1,2]); X.__custom_name = "X"
sage: Y = CombinatorialFreeModule(QQ, [3,4]); Y.__custom_name = "Y"
sage: H = Hom(X, Y); H
Set of Morphisms from X to Y in Category of finite dimensional vector spaces with
→basis over Rational Field
```

The simplest morphism is the zero map:

```
sage: H.zero()         # todo: move this test into module once we have an example
Generic morphism:
  From: X
  To:   Y
```

which we can apply to elements of $X$:

```
sage: x = X.monomial(1) + 3 * X.monomial(2)
sage: H.zero()(x)
0
```

EXAMPLES:

We now construct a more interesting morphism by extending a function by linearity:

```
sage: phi = H(on_basis = lambda i: Y.monomial(i+2)); phi
Generic morphism:
  From: X
  To:   Y
sage: phi(x)
B[3] + 3*B[4]
```

We can retrieve the function acting on indices of the basis:

```
sage: f = phi.on_basis()
sage: f(1), f(2)
(B[3], B[4])
```

$Hom(X,Y)$ has a natural module structure (except for the zero, the operations are not yet implemented though). However since the dimension is not necessarily finite, it is not a module with basis; but see `FiniteDimensionalModulesWithBasis` and `GradedModulesWithBasis`:

```
sage: H in ModulesWithBasis(QQ), H in Modules(QQ)
(False, True)
```

Some more playing around with categories and higher order homsets:

```
sage: H.category()
Category of homsets of modules with basis over Rational Field
sage: Hom(H, H).category()
Category of endsets of homsets of modules with basis over Rational Field
```

**Todo:** `End(X)` is an algebra.

**Note:** This category currently requires an implementation of an element method `support`. Once trac ticket #18066 is merged, an implementation of an `items` method will be required.

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

The category of modules with basis constructed by Cartesian products of modules with basis.

**class ParentMethods**

**extra_super_categories**()

EXAMPLES:

```
sage: ModulesWithBasis(QQ).CartesianProducts().extra_super_categories()
[Category of vector spaces with basis over Rational Field]
sage: ModulesWithBasis(QQ).CartesianProducts().super_categories()
[Category of Cartesian products of modules with basis over Rational Field,
 Category of vector spaces with basis over Rational Field,
 Category of Cartesian products of vector spaces over Rational Field]
```

**class DualObjects**(*category*, **args*)

Bases: *sage.categories.dual.DualObjectsCategory*

**extra_super_categories**()

EXAMPLES:

```
sage: ModulesWithBasis(ZZ).DualObjects().extra_super_categories()
[Category of modules over Integer Ring]
sage: ModulesWithBasis(QQ).DualObjects().super_categories()
[Category of duals of vector spaces over Rational Field, Category of␣
↪duals of modules with basis over Rational Field]
```

**class ElementMethods**

**coefficient**(*m*)

Return the coefficient of `m` in `self` and raise an error if `m` is not in the basis indexing set.

INPUT:

- `m` – a basis index of the parent of `self`

OUTPUT:

The `B[m]`-coordinate of `self` with respect to the basis `B`. Here, `B` denotes the given basis of the parent of `self`.

EXAMPLES:

```
sage: s = CombinatorialFreeModule(QQ, Partitions())
sage: z = s([4]) - 2*s([2,1]) + s([1,1,1]) + s([1])
sage: z.coefficient([4])
1
sage: z.coefficient([2,1])
-2
sage: z.coefficient(Partition([2,1]))
-2
sage: z.coefficient([1,2])
Traceback (most recent call last):
...
AssertionError: [1, 2] should be an element of Partitions
sage: z.coefficient(Composition([2,1]))
Traceback (most recent call last):
...
AssertionError: [2, 1] should be an element of Partitions
```

Test that `coefficient` also works for those parents that do not yet have an element_class:

```
sage: G = DihedralGroup(3)
sage: F = CombinatorialFreeModule(QQ, G)
sage: hasattr(G, "element_class")
False
sage: g = G.an_element()
sage: (2*F.monomial(g)).coefficient(g)
2
```

**coefficients**(*sort=True*)

Return a list of the (non-zero) coefficients appearing on the basis elements in `self` (in an arbitrary order).

INPUT:

- `sort` – (default: `True`) to sort the coefficients based upon the default ordering of the indexing set

**See also:**

*dense_coefficient_list()*

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: f.coefficients()
[1, -3]
sage: f = B['c'] - 3*B['a']
sage: f.coefficients()
[-3, 1]
```

```
sage: s = SymmetricFunctions(QQ).schur()
sage: z = s([4]) + s([2,1]) + s([1,1,1]) + s([1])
sage: z.coefficients()
[1, 1, 1, 1]
```

**is_zero**()

Return `True` if and only if `self == 0`.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: f.is_zero()
False
sage: F.zero().is_zero()
True
```

```
sage: s = SymmetricFunctions(QQ).schur()
sage: s([2,1]).is_zero()
False
sage: s(0).is_zero()
True
sage: (s([2,1]) - s([2,1])).is_zero()
True
```

**leading_coefficient**(**args*, ***kwds*)

Return the leading coefficient of `self`.

This is the coefficient of the term whose corresponding basis element is maximal. Note that this may not be the term which actually appears first when `self` is printed.

If the default term ordering is not what is desired, a comparison key, `key(x,y)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X")
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.leading_coefficient()
1
sage: def key(x): return -x
sage: x.leading_coefficient(key=key)
3

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
sage: f.leading_coefficient()
-5
```

**leading_item**(**args*, ***kwds*)

Return the pair `(k, c)` where

$$c \cdot (\text{the basis element indexed by } k)$$

is the leading term of `self`.

Here 'leading term' means that the corresponding basis element is maximal. Note that this may not be the term which actually appears first when `self` is printed.

If the default term ordering is not what is desired, a comparison function, `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.
↪basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + 4*X.monomial(3)
sage: x.leading_item()
(3, 4)
sage: def key(x): return -x
```

```
sage: x.leading_item(key=key)
(1, 3)

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
sage: f.leading_item()
([3], -5)
```

**leading_monomial**(**args*, ***kwds*)

Return the leading monomial of `self`.

This is the monomial whose corresponding basis element is maximal. Note that this may not be the term which actually appears first when `self` is printed.

If the default term ordering is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.
↪basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.leading_monomial()
B[3]
sage: def key(x): return -x
sage: x.leading_monomial(key=key)
B[1]

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
sage: f.leading_monomial()
s[3]
```

**leading_support**(**args*, ***kwds*)

Return the maximal element of the support of `self`.

Note that this may not be the term which actually appears first when `self` is printed.

If the default ordering of the basis elements is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3])
sage: X.rename("X"); x = X.basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + 4*X.monomial(3)
sage: x.leading_support()
3
sage: def key(x): return -x
sage: x.leading_support(key=key)
1

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
sage: f.leading_support()
[3]
```

**leading_term**(**args*, ***kwds*)

Return the leading term of `self`.

This is the term whose corresponding basis element is maximal. Note that this may not be the term which actually appears first when `self` is printed.

If the default term ordering is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.
↪basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.leading_term()
B[3]
sage: def key(x): return -x
sage: x.leading_term(key=key)
3*B[1]

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
sage: f.leading_term()
-5*s[3]
```

**length**()

Return the number of basis elements whose coefficients in `self` are nonzero.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: f.length()
2
```

```
sage: s = SymmetricFunctions(QQ).schur()
sage: z = s([4]) + s([2,1]) + s([1,1,1]) + s([1])
sage: z.length()
4
```

**map_coefficients**(*f*)

Mapping a function on coefficients.

INPUT:

- `f` – an endofunction on the coefficient ring of the free module

Return a new element of `self.parent()` obtained by applying the function `f` to all of the coefficients of `self`.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: f.map_coefficients(lambda x: x+5)
6*B['a'] + 2*B['c']
```

Killed coefficients are handled properly:

```
sage: f.map_coefficients(lambda x: 0)
0
sage: list(f.map_coefficients(lambda x: 0))
[]
```

```
sage: s = SymmetricFunctions(QQ).schur()
sage: a = s([2,1])+2*s([3,2])
sage: a.map_coefficients(lambda x: x*2)
2*s[2, 1] + 4*s[3, 2]
```

**map_item**(*f*)

Mapping a function on items.

INPUT:

- `f` – a function mapping pairs `(index, coeff)` to other such pairs

Return a new element of `self.parent()` obtained by applying the function $f$ to all items `(index, coeff)` of `self`.

EXAMPLES:

```
sage: B = CombinatorialFreeModule(ZZ, [-1, 0, 1])
sage: x = B.an_element(); x
2*B[-1] + 2*B[0] + 3*B[1]
sage: x.map_item(lambda i, c: (-i, 2*c))
6*B[-1] + 4*B[0] + 4*B[1]
```

`f` needs not be injective:

```
sage: x.map_item(lambda i, c: (1, 2*c))
14*B[1]

sage: s = SymmetricFunctions(QQ).schur()
sage: f = lambda m,c: (m.conjugate(), 2*c)
sage: a = s([2,1]) + s([1,1,1])
sage: a.map_item(f)
2*s[2, 1] + 2*s[3]
```

**map_support**(*f*)

Mapping a function on the support.

INPUT:

- `f` – an endofunction on the indices of the free module

Return a new element of `self.parent()` obtained by applying the function `f` to all of the objects indexing the basis elements.

EXAMPLES:

```
sage: B = CombinatorialFreeModule(ZZ, [-1, 0, 1])
sage: x = B.an_element(); x
2*B[-1] + 2*B[0] + 3*B[1]
sage: x.map_support(lambda i: -i)
3*B[-1] + 2*B[0] + 2*B[1]
```

`f` needs not be injective:

```
sage: x.map_support(lambda i: 1)
7*B[1]

sage: s = SymmetricFunctions(QQ).schur()
sage: a = s([2,1])+2*s([3,2])
sage: a.map_support(lambda x: x.conjugate())
s[2, 1] + 2*s[2, 2, 1]
```

**map_support_skip_none** (*f*)
Mapping a function on the support.

INPUT:

- f – an endofunction on the indices of the free module

Returns a new element of self.parent() obtained by applying the function $f$ to all of the objects indexing the basis elements.

EXAMPLES:

```
sage: B = CombinatorialFreeModule(ZZ, [-1, 0, 1])
sage: x = B.an_element(); x
2*B[-1] + 2*B[0] + 3*B[1]
sage: x.map_support_skip_none(lambda i: -i if i else None)
3*B[-1] + 2*B[1]
```

f needs not be injective:

```
sage: x.map_support_skip_none(lambda i: 1 if i else None)
5*B[1]
```

**monomial_coefficients** (*copy=True*)
Return a dictionary whose keys are indices of basis elements in the support of self and whose values are the corresponding coefficients.

INPUT:

- copy – (default: True) if self is internally represented by a dictionary d, then make a copy of d; if False, then this can cause undesired behavior by mutating d

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] + 3*B['c']
sage: d = f.monomial_coefficients()
sage: d['a']
1
sage: d['c']
3
```

**monomials** ()
Return a list of the monomials of self (in an arbitrary order).

The monomials of an element $a$ are defined to be the basis elements whose corresponding coefficients of $a$ are non-zero.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] + 2*B['c']
sage: f.monomials()
[B['a'], B['c']]

sage: (F.zero()).monomials()
[]
```

**support** ()
Return a list of the objects indexing the basis of self.parent() whose corresponding coefficients of self are non-zero.

This method returns these objects in an arbitrary order.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] - 3*B['c']
sage: sorted(f.support())
['a', 'c']
```

```
sage: s = SymmetricFunctions(QQ).schur()
sage: z = s([4]) + s([2,1]) + s([1,1,1]) + s([1])
sage: sorted(z.support())
[[1], [1, 1, 1], [2, 1], [4]]
```

**support_of_term**()

Return the support of `self`, where `self` is a monomial (possibly with coefficient).

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1,2,3,4]); X.rename("X")
sage: X.monomial(2).support_of_term()
2
sage: X.term(3, 2).support_of_term()
3
```

An exception is raised if `self` has more than one term:

```
sage: (X.monomial(2) + X.monomial(3)).support_of_term()
Traceback (most recent call last):
...
ValueError: B[2] + B[3] is not a single term
```

**tensor**(**elements*)

Return the tensor product of its arguments, as an element of the tensor product of the parents of those elements.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ)
sage: A = C.example()
sage: (a,b,c) = A.algebra_generators()
sage: a.tensor(b, c)
B[word: a] # B[word: b] # B[word: c]
```

FIXME: is this a policy that we want to enforce on all parents?

**terms**()

Return a list of the (non-zero) terms of `self` (in an arbitrary order).

**See also:**

*monomials()*

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: B = F.basis()
sage: f = B['a'] + 2*B['c']
```

```
sage: f.terms()
[B['a'], 2*B['c']]
```

**trailing_coefficient**(**args*, ***kwds*)

Return the trailing coefficient of `self`.

This is the coefficient of the monomial whose corresponding basis element is minimal. Note that this may not be the term which actually appears last when `self` is printed.

If the default term ordering is not what is desired, a comparison key `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.
↪basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.trailing_coefficient()
3
sage: def key(x): return -x
sage: x.trailing_coefficient(key=key)
1

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
sage: f.trailing_coefficient()
2
```

**trailing_item**(**args*, ***kwds*)

Return the pair `(c, k)` where `c*self.parent().monomial(k)` is the trailing term of `self`.

This is the monomial whose corresponding basis element is minimal. Note that this may not be the term which actually appears last when `self` is printed.

If the default term ordering is not what is desired, a comparison key `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.
↪basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.trailing_item()
(1, 3)
sage: def key(x): return -x
sage: x.trailing_item(key=key)
(3, 1)

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
sage: f.trailing_item()
([1], 2)
```

**trailing_monomial**(**args*, ***kwds*)

Return the trailing monomial of `self`.

This is the monomial whose corresponding basis element is minimal. Note that this may not be the term which actually appears last when `self` is printed.

If the default term ordering is not what is desired, a comparison key `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.
↪basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.trailing_monomial()
B[1]
sage: def key(x): return -x
sage: x.trailing_monomial(key=key)
B[3]

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
sage: f.trailing_monomial()
s[1]
```

**trailing_support**(**args*, ***kwds*)

Return the minimal element of the support of `self`. Note that this may not be the term which actually appears last when `self` is printed.

If the default ordering of the basis elements is not what is desired, a comparison key, `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.
↪basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + 4*X.monomial(3)
sage: x.trailing_support()
1

sage: def key(x): return -x
sage: x.trailing_support(key=key)
3

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
sage: f.trailing_support()
[1]
```

**trailing_term**(**args*, ***kwds*)

Return the trailing term of `self`.

This is the term whose corresponding basis element is minimal. Note that this may not be the term which actually appears last when `self` is printed.

If the default term ordering is not what is desired, a comparison key `key(x)`, can be provided.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1, 2, 3]); X.rename("X"); x = X.
↪basis()
sage: x = 3*X.monomial(1) + 2*X.monomial(2) + X.monomial(3)
sage: x.trailing_term()
3*B[1]
sage: def key(x): return -x
sage: x.trailing_term(key=key)
B[3]

sage: s = SymmetricFunctions(QQ).schur()
sage: f = 2*s[1] + 3*s[2,1] - 5*s[3]
```

```
sage: f.trailing_term()
2*s[1]
```

**Filtered**
alias of FilteredModulesWithBasis

**FiniteDimensional**
alias of *FiniteDimensionalModulesWithBasis*

**Graded**
alias of *GradedModulesWithBasis*

**class Homsets**(*category*, **args*)
Bases: *sage.categories.homsets.HomsetsCategory*

**class ParentMethods**

**class MorphismMethods**

**on_basis**()
Return the action of this morphism on basis elements.

OUTPUT:

- a function from the indices of the basis of the domain to the codomain

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, [1,2,3]);   X.rename("X")
sage: Y = CombinatorialFreeModule(QQ, [1,2,3,4]); Y.rename("Y")
sage: H = Hom(X, Y)
sage: x = X.basis()

sage: f = H(lambda x: Y.zero()).on_basis()
sage: f(2)
0

sage: f = lambda i: Y.monomial(i) + 2*Y.monomial(i+1)
sage: g = H(on_basis = f).on_basis()
sage: g(2)
B[2] + 2*B[3]
sage: g == f
True
```

**class ParentMethods**

**basis**()
Return the basis of self.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: F.basis()
Finite family {'a': B['a'], 'c': B['c'], 'b': B['b']}
```

```
sage: QS3 = SymmetricGroupAlgebra(QQ,3)
sage: list(QS3.basis())
[[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]
```

**cardinality**()

Return the cardinality of `self`.

EXAMPLES:

```
sage: S = SymmetricGroupAlgebra(QQ, 4)
sage: S.cardinality()
+Infinity
sage: S = SymmetricGroupAlgebra(GF(2), 4) # not tested -- MRO bug trac
↪#15475
sage: S.cardinality() # not tested -- MRO bug trac #15475
16777216
sage: S.cardinality().factor() # not tested -- MRO bug trac #15475
2^24

sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: E.cardinality()
+Infinity
sage: E.<x,y> = ExteriorAlgebra(GF(3))
sage: E.cardinality()
81

sage: s = SymmetricFunctions(GF(2)).s()
sage: s.cardinality()
+Infinity
```

**dimension**()

Return the dimension of `self`.

EXAMPLES:

```
sage: A.<x,y> = algebras.DifferentialWeyl(QQ)
sage: A.dimension()
+Infinity
```

**echelon_form**(*elements*, *row_reduced=False*)

Return a basis in echelon form of the subspace spanned by a finite set of elements.

INPUT:

- `elements` – a list or finite iterable of elements of `self`
- `row_reduced` – (default: `False`) whether to compute the basis for the row reduced echelon form

OUTPUT:

A list of elements of `self` whose expressions as vectors form a matrix in echelon form. If `base_ring` is specified, then the calculation is achieved in this base ring.

EXAMPLES:

```
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
sage: V = X.echelon_form([x[0]-x[1], x[0]-x[2],x[1]-x[2]]); V
[x[0] - x[2], x[1] - x[2]]
sage: matrix(list(map(vector, V)))
[ 1  0 -1]
[ 0  1 -1]
```

```
sage: F = CombinatorialFreeModule(ZZ, [1,2,3,4])
sage: B = F.basis()
```

```
sage: elements = [B[1]-17*B[2]+6*B[3], B[1]-17*B[2]+B[4]]
sage: F.echelon_form(elements)
[B[1] - 17*B[2] + B[4], 6*B[3] - B[4]]
```

```
sage: F = CombinatorialFreeModule(QQ, ['a','b','c'])
sage: a,b,c = F.basis()
sage: F.echelon_form([8*a+b+10*c, -3*a+b-c, a-b-c])
[B['a'] + B['c'], B['b'] + 2*B['c']]
```

```
sage: R.<x,y> = QQ[]
sage: C = CombinatorialFreeModule(R, range(3), prefix='x')
sage: x = C.basis()
sage: C.echelon_form([x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]])
[x[0] - x[2], x[1] - x[2]]
```

**is_finite**()

Return whether `self` is finite.

This is true if and only if `self.basis().keys()` and `self.base_ring()` are both finite.

EXAMPLES:

```
sage: GroupAlgebra(SymmetricGroup(2), IntegerModRing(10)).is_finite()
True
sage: GroupAlgebra(SymmetricGroup(2)).is_finite()
False
sage: GroupAlgebra(AbelianGroup(1), IntegerModRing(10)).is_finite()
False
```

**linear_combination**(*iter_of_elements_coeff*, *factor_on_left=True*)

Return the linear combination $\lambda_1 v_1 + \cdots + \lambda_k v_k$ (resp. the linear combination $v_1 \lambda_1 + \cdots + v_k \lambda_k$) where `iter_of_elements_coeff` iterates through the sequence $((\lambda_1, v_1), ..., (\lambda_k, v_k))$.

INPUT:

- `iter_of_elements_coeff` – iterator of pairs `(element, coeff)` with `element` in `self` and `coeff` in `self.base_ring()`
- `factor_on_left` – (optional) if `True`, the coefficients are multiplied from the left; if `False`, the coefficients are multiplied from the right

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J.<a,b,c> = JordanAlgebra(m)
sage: J.linear_combination(((a+b, 1), (-2*b + c, -1)))
1 + (3, -1)
```

**module_morphism**(*on_basis=None*, *matrix=None*, *function=None*, *diagonal=None*, *triangular=None*, *unitriangular=False*, ***keywords*)

Construct a module morphism from `self` to `codomain`.

Let `self` be a module $X$ with a basis indexed by $I$. This constructs a morphism $f : X \to Y$ by linearity from a map $I \to Y$ which is to be its restriction to the basis $(x_i)_{i \in I}$ of $X$. Some variants are possible too.

INPUT:

- `self` – a parent $X$ in `ModulesWithBasis(R)` with basis $x = (x_i)_{i \in I}$.

Exactly one of the four following options must be specified in order to define the morphism:

- `on_basis` – a function $f$ from $I$ to $Y$
- `diagonal` – a function $d$ from $I$ to $R$

- `function` – a function $f$ from $X$ to $Y$
- `matrix` – a matrix of size $\dim Y \times \dim X$ (if the keyword `side` is set to `'left'`) or $\dim Y \times \dim X$ (if this keyword is `'right'`)

Further options include:

- `codomain` – the codomain $Y$ of the morphism (default: `f.codomain()` if it's defined; otherwise it must be specified)
- `category` – a category or `None` (default: $None$‘)
- `zero` – the zero of the codomain (default: `codomain.zero()`); can be used (with care) to define affine maps. Only meaningful with `on_basis`.
- `position` – a non-negative integer specifying which positional argument in used as the input of the function $f$ (default: 0); this is currently only used with `on_basis`.
- `triangular` – (default: `None`) `"upper"` or `"lower"` or `None`:
  - `"upper"` - if the *`leading_support()`* of the image of the basis vector $x_i$ is $i$, or
  - `"lower"` - if the *`trailing_support()`* of the image of the basis vector $x_i$ is $i$.
- `unitriangular` – (default: `False`) a boolean. Only meaningful for a triangular morphism. As a shorthand, one may use `unitriangular="lower"` for `triangular="lower"`, `unitriangular=True`.
- `side` – "left" or "right" (default: "left") Only meaningful for a morphism built from a matrix.

EXAMPLES:

With the `on_basis` option, this returns a function $g$ obtained by extending $f$ by linearity on the `position`-th positional argument. For example, for `position == 1` and a ternary function $f$, one has:

$$g\left(a,\ \sum_i \lambda_i x_i,\ c\right) = \sum_i \lambda_i f(a, i, c).$$

```
sage: X = CombinatorialFreeModule(QQ, [1,2,3]);   X.rename("X")
sage: Y = CombinatorialFreeModule(QQ, [1,2,3,4]); Y.rename("Y")
sage: phi = X.module_morphism(lambda i: Y.monomial(i) + 2*Y.monomial(i+1),
↪ codomain = Y)
sage: x = X.basis(); y = Y.basis()
sage: phi(x[1] + x[3])
B[1] + 2*B[2] + B[3] + 2*B[4]

sage: phi
Generic morphism:
From: X
To:   Y
```

By default, the category is the first of `Modules(R).WithBasis().FiniteDimensional()`, `Modules(R).WithBasis()`, `Modules(R)`, and `CommutativeAdditiveMonoids()` that contains both the domain and the codomain:

```
sage: phi.category_for()
Category of finite dimensional vector spaces with basis over Rational␣
↪Field
```

With the `zero` argument, one can define affine morphisms:

```
sage: phi = X.module_morphism(lambda i: Y.monomial(i) + 2*Y.monomial(i+1),
....:                         codomain = Y, zero = 10*y[1])
sage: phi(x[1] + x[3])
11*B[1] + 2*B[2] + B[3] + 2*B[4]
```

In this special case, the default category is `Sets()`:

```
sage: phi.category_for()
Category of sets
```

One can construct morphisms with the base ring as codomain:

```
sage: X = CombinatorialFreeModule(ZZ,[1,-1])
sage: phi = X.module_morphism( on_basis=lambda i: i, codomain=ZZ )
sage: phi( 2 * X.monomial(1) + 3 * X.monomial(-1) )
-1
sage: phi.category_for()
Category of commutative additive semigroups
sage: phi.category_for() # todo: not implemented (ZZ is currently not in␣
↪Modules(ZZ))
Category of modules over Integer Ring
```

Or more generally any ring admitting a coercion map from the base ring:

```
sage: phi = X.module_morphism(on_basis=lambda i: i, codomain=RR )
sage: phi( 2 * X.monomial(1) + 3 * X.monomial(-1) )
-1.00000000000000
sage: phi.category_for()
Category of commutative additive semigroups
sage: phi.category_for() # todo: not implemented (RR is currently not in␣
↪Modules(ZZ))
Category of modules over Integer Ring

sage: phi = X.module_morphism(on_basis=lambda i: i, codomain=Zmod(4) )
sage: phi( 2 * X.monomial(1) + 3 * X.monomial(-1) )
3

sage: phi = Y.module_morphism(on_basis=lambda i: i, codomain=Zmod(4) )
Traceback (most recent call last):
...
ValueError: codomain(=Ring of integers modulo 4) should be a module over␣
↪the base ring of the domain(=Y)
```

On can also define module morphisms between free modules over different base rings; here we implement the natural map from $X = \mathbf{R}^2$ to $Y = \mathbf{C}$:

```
sage: X = CombinatorialFreeModule(RR,['x','y'])
sage: Y = CombinatorialFreeModule(CC,['z'])
sage: x = X.monomial('x')
sage: y = X.monomial('y')
sage: z = Y.monomial('z')
sage: def on_basis( a ):
....:     if a == 'x':
....:         return CC(1) * z
....:     elif a == 'y':
....:         return CC(I) * z
sage: phi = X.module_morphism( on_basis=on_basis, codomain=Y )
sage: v = 3 * x + 2 * y; v
3.00000000000000*B['x'] + 2.00000000000000*B['y']
sage: phi(v)
(3.00000000000000+2.00000000000000*I)*B['z']
sage: phi.category_for()
Category of commutative additive semigroups
sage: phi.category_for() # todo: not implemented (CC is currently not in␣
↪Modules(RR)!)
```

```
Category of vector spaces over Real Field with 53 bits of precision

sage: Y = CombinatorialFreeModule(CC['q'],['z'])
sage: z = Y.monomial('z')
sage: phi = X.module_morphism( on_basis=on_basis, codomain=Y )
sage: phi(v)
(3.00000000000000+2.00000000000000*I)*B['z']
```

Of course, there should be a coercion between the respective base rings of the domain and the codomain for this to be meaningful:

```
sage: Y = CombinatorialFreeModule(QQ,['z'])
sage: phi = X.module_morphism( on_basis=on_basis, codomain=Y )
Traceback (most recent call last):
...
ValueError: codomain(=Free module generated by {'z'} over Rational Field)
 should be a module over the base ring of the
 domain(=Free module generated by {'x', 'y'} over Real Field with 53 bits␣
↪of precision)

sage: Y = CombinatorialFreeModule(RR['q'],['z'])
sage: phi = Y.module_morphism( on_basis=on_basis, codomain=X )
Traceback (most recent call last):
...
ValueError: codomain(=Free module generated by {'x', 'y'} over Real Field␣
↪with 53 bits of precision)
 should be a module over the base ring of the
 domain(=Free module generated by {'z'} over Univariate Polynomial Ring␣
↪in q over Real Field with 53 bits of precision)
```

With the `diagonal=d` argument, this constructs the module morphism $g$ such that

$$`g(x_i) = d(i)y_i`.$$

This assumes that the respective bases $x$ and $y$ of $X$ and $Y$ have the same index set $I$:

```
sage: X = CombinatorialFreeModule(ZZ, [1,2,3]); X.rename("X")
sage: phi = X.module_morphism(diagonal=factorial, codomain=X)
sage: x = X.basis()
sage: phi(x[1]), phi(x[2]), phi(x[3])
(B[1], 2*B[2], 6*B[3])
```

See also: `sage.modules.with_basis.morphism.DiagonalModuleMorphism`.

With the `matrix=m` argument, this constructs the module morphism whose matrix in the distinguished basis of $X$ and $Y$ is $m$:

```
sage: X = CombinatorialFreeModule(ZZ, [1,2,3]); X.rename("X"); x = X.
↪basis()
sage: Y = CombinatorialFreeModule(ZZ, [3,4]); Y.rename("Y"); y = Y.basis()
sage: m = matrix([[0,1,2],[3,5,0]])
sage: phi = X.module_morphism(matrix=m, codomain=Y)
sage: phi(x[1])
3*B[4]
sage: phi(x[2])
B[3] + 5*B[4]
```

See also: `sage.modules.with_basis.morphism.ModuleMorphismFromMatrix`.

With `triangular="upper"`, the constructed module morphism is assumed to be upper triangular; that is its matrix in the distinguished basis of $X$ and $Y$ would be upper triangular with invertible elements on its diagonal. This is used to compute preimages and to invert the morphism:

```
sage: I = list(range(1, 200))
sage: X = CombinatorialFreeModule(QQ, I); X.rename("X"); x = X.basis()
sage: Y = CombinatorialFreeModule(QQ, I); Y.rename("Y"); y = Y.basis()
sage: f = Y.sum_of_monomials * divisors
sage: phi = X.module_morphism(f, triangular="upper", codomain = Y)
sage: phi(x[2])
B[1] + B[2]
sage: phi(x[6])
B[1] + B[2] + B[3] + B[6]
sage: phi(x[30])
B[1] + B[2] + B[3] + B[5] + B[6] + B[10] + B[15] + B[30]
sage: phi.preimage(y[2])
-B[1] + B[2]
sage: phi.preimage(y[6])
B[1] - B[2] - B[3] + B[6]
sage: phi.preimage(y[30])
-B[1] + B[2] + B[3] + B[5] - B[6] - B[10] - B[15] + B[30]
sage: (phi^-1)(y[30])
-B[1] + B[2] + B[3] + B[5] - B[6] - B[10] - B[15] + B[30]
```

Since trac ticket #8678, one can also define a triangular morphism from a function:

```
sage: X = CombinatorialFreeModule(QQ, [0,1,2,3,4]); x = X.basis()
sage: from sage.modules.with_basis.morphism import␣
↪TriangularModuleMorphismFromFunction
sage: def f(x): return x + X.term(0, sum(x.coefficients()))
sage: phi = X.module_morphism(function=f, codomain=X, triangular="upper")
sage: phi(x[2] + 3*x[4])
4*B[0] + B[2] + 3*B[4]
sage: phi.preimage(_)
B[2] + 3*B[4]
```

For details and further optional arguments, see `sage.modules.with_basis.morphism.TriangularModuleMorphism`.

**Warning:** As a temporary measure, until multivariate morphisms are implemented, the constructed morphism is in `Hom(codomain, domain, category)`. This is only correct for unary functions.

**Todo:**

- Should codomain be `self` by default in the diagonal, triangular, and matrix cases?
- Support for diagonal morphisms between modules not sharing the same index set

```
sage: X = CombinatorialFreeModule(ZZ, [1,2,3]); X.rename("X")
sage: phi = X.module_morphism(diagonal=factorial, matrix=matrix(),␣
↪codomain=X)
Traceback (most recent call last):
...
ValueError: module_morphism() takes exactly one option
out of `matrix`, `on_basis`, `function`, `diagonal`
```

```
sage: X = CombinatorialFreeModule(ZZ, [1,2,3]); X.rename("X")
sage: phi = X.module_morphism(matrix=factorial, codomain=X)
Traceback (most recent call last):
...
ValueError: matrix (=factorial) should be a matrix
```

```
sage: X = CombinatorialFreeModule(ZZ, [1,2,3]); X.rename("X")
sage: phi = X.module_morphism(diagonal=3, codomain=X)
Traceback (most recent call last):
...
ValueError: diagonal (=3) should be a function
```

**monomial** (*i*)

Return the basis element indexed by `i`.

INPUT:

- `i` – an element of the index set

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.monomial('a')
B['a']
```

`F.monomial` is in fact (almost) a map:

```
sage: F.monomial
Term map from {'a', 'b', 'c'} to Free module generated by {'a', 'b', 'c'}␣
↪over Rational Field
```

**monomial_or_zero_if_none** (*i*)

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.monomial_or_zero_if_none('a')
B['a']
sage: F.monomial_or_zero_if_none(None)
0
```

**random_element** (*n=2*)

Return a 'random' element of `self`.

INPUT:

- `n` – integer (default: 2); number of summands

ALGORITHM:

Return a sum of `n` terms, each of which is formed by multiplying a random element of the base ring by a random element of the group.

EXAMPLES:

```
sage: DihedralGroup(6).algebra(QQ).random_element()
-1/95*() - 1/2*(1,4)(2,5)(3,6)
sage: SU(2, 13).algebra(QQ).random_element(1)
1/2*[       3        0]
[11*a + 1        9]
sage: CombinatorialFreeModule(ZZ, Partitions(4)).random_element() # random
2*B[[2, 1, 1]] + B[[2, 2]]
```

**submodule**(*gens*, *check=True*, *already_echelonized=False*, *unitriangular=False*, *category=None*)

The submodule spanned by a finite set of elements.

INPUT:

- `gens` – a list or family of elements of `self`
- **`check` – (default: `True`) whether to verify that the** elements of `gens` are in `self`
- **`already_echelonized` – (default: `False`) whether** the elements of `gens` are already in (not necessarily reduced) echelon form
- `unitriangular` – (default: `False`) whether the lift morphism is unitriangular

If `already_echelonized` is `False`, then the generators are put in reduced echelon form using `echelonize()`, and reindexed by $0, 1, ...$.

> **Warning:** At this point, this method only works for finite dimensional submodules and if matrices can be echelonized over the base ring.

If in addition `unitriangular` is `True`, then the generators are made such that the coefficients of the pivots are 1, so that lifting map is unitriangular.

The basis of the submodule uses the same index set as the generators, and the lifting map sends $y_i$ to $gens[i]$.

**See also:**

- `ModulesWithBasis.FiniteDimensional.ParentMethods.quotient_module()`
- `sage.modules.with_basis.subquotient.SubmoduleWithBasis`

EXAMPLES:

We construct a submodule of the free $\mathbf{Q}$-module generated by $x_0, x_1, x_2$. The submodule is spanned by $y_0 = x_0 - x_1$ and $y_1 - x_1 - x_2$, and its basis elements are indexed by 0 and 1:

```
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
sage: gens = [x[0] - x[1], x[1] - x[2]]; gens
[x[0] - x[1], x[1] - x[2]]
sage: Y = X.submodule(gens, already_echelonized=True)
sage: Y.print_options(prefix='y'); Y
Free module generated by {0, 1} over Rational Field
sage: y = Y.basis()
sage: y[1]
y[1]
sage: y[1].lift()
x[1] - x[2]
sage: Y.retract(x[0]-x[2])
y[0] + y[1]
sage: Y.retract(x[0])
Traceback (most recent call last):
...
ValueError: x[0] is not in the image
```

By using a family to specify a basis of the submodule, we obtain a submodule whose index set coincides with the index set of the family:

```
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
sage: gens = Family({1 : x[0] - x[1], 3: x[1] - x[2]}); gens
```

```
Finite family {1: x[0] - x[1], 3: x[1] - x[2]}
sage: Y = X.submodule(gens, already_echelonized=True)
sage: Y.print_options(prefix='y'); Y
Free module generated by {1, 3} over Rational Field
sage: y = Y.basis()
sage: y[1]
y[1]
sage: y[1].lift()
x[0] - x[1]
sage: y[3].lift()
x[1] - x[2]
sage: Y.retract(x[0]-x[2])
y[1] + y[3]
sage: Y.retract(x[0])
Traceback (most recent call last):
...
ValueError: x[0] is not in the image
```

It is not necessary that the generators of the submodule form a basis (an explicit basis will be computed):

```
sage: X = CombinatorialFreeModule(QQ, range(3), prefix="x")
sage: x = X.basis()
sage: gens = [x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]]; gens
[x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]]
sage: Y = X.submodule(gens, already_echelonized=False)
sage: Y.print_options(prefix='y')
sage: Y
Free module generated by {0, 1} over Rational Field
sage: [b.lift() for b in Y.basis()]
[x[0] - x[2], x[1] - x[2]]
```

We now implement by hand the center of the algebra of the symmetric group $S_3$:

```
sage: S3 = SymmetricGroup(3)
sage: S3A = S3.algebra(QQ)
sage: basis = S3A.annihilator_basis(S3A.algebra_generators(), S3A.bracket)
sage: basis
((), (2,3) + (1,2) + (1,3), (1,2,3) + (1,3,2))
sage: center = S3A.submodule(basis,
....:                        category=AlgebrasWithBasis(QQ).Subobjects(),
....:                        already_echelonized=True)
sage: center
Free module generated by {0, 1, 2} over Rational Field
sage: center in Algebras
True
sage: center.print_options(prefix='c')
sage: c = center.basis()
sage: c[1].lift()
(2,3) + (1,2) + (1,3)
sage: c[0]^2
c[0]
sage: e = 1/6*(c[0]+c[1]+c[2])
sage: e.is_idempotent()
True
```

Of course, this center is best constructed using:

```
sage: center = S3A.center()
```

We can also automatically construct a basis such that the lift morphism is (lower) unitriangular:

```
sage: R.<a,b> = QQ[]
sage: C = CombinatorialFreeModule(R, range(3), prefix='x')
sage: x = C.basis()
sage: gens = [x[0] - x[1], 2*x[1] - 2*x[2], x[0] - x[2]]
sage: Y = C.submodule(gens, unitriangular=True)
sage: Y.lift.matrix()
[ 1  0]
[ 0  1]
[-1 -1]
```

**sum_of_monomials**()

Return the sum of the basis elements with indices in `indices`.

INPUT:

- `indices` – an list (or iterable) of indices of basis elements

EXAMPLES:

```
sage: F = CombinatorialFreeModule(QQ, ['a', 'b', 'c'])
sage: F.sum_of_monomials(['a', 'b'])
B['a'] + B['b']

sage: F.sum_of_monomials(['a', 'b', 'a'])
2*B['a'] + B['b']
```

`F.sum_of_monomials` is in fact (almost) a map:

```
sage: F.sum_of_monomials
A map to Free module generated by {'a', 'b', 'c'} over Rational Field
```

**sum_of_terms**(*terms*)

Construct a sum of terms of `self`.

INPUT:

- `terms` – a list (or iterable) of pairs `(index, coeff)`

OUTPUT:

Sum of `coeff * B[index]` over all `(index, coeff)` in `terms`, where `B` is the basis of `self`.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J.<a,b,c> = JordanAlgebra(m)
sage: J.sum_of_terms([(0, 2), (2, -3)])
2 + (0, -3)
```

**tensor**(**parents*)

Return the tensor product of the parents.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ)
sage: A = C.example(); A.rename("A")
sage: A.tensor(A,A)
```

```
A # A # A
sage: A.rename(None)
```

**term** (*index*, *coeff=None*)

Construct a term in `self`.

INPUT:

- `index` – the index of a basis element
- `coeff` – an element of the coefficient ring (default: one)

OUTPUT:

`coeff * B[index]`, where `B` is the basis of `self`.

EXAMPLES:

```
sage: m = matrix([[0,1],[1,1]])
sage: J.<a,b,c> = JordanAlgebra(m)
sage: J.term(1, -2)
0 + (-2, 0)
```

Design: should this do coercion on the coefficient ring?

**Super**

alias of `SuperModulesWithBasis`

**class TensorProducts** (*category*, **args*)

Bases: *sage.categories.tensor.TensorProductsCategory*

The category of modules with basis constructed by tensor product of modules with basis.

**class ElementMethods**

Implements operations on elements of tensor products of modules with basis.

**apply_multilinear_morphism** (*f*, *codomain=None*)

Return the result of applying the morphism induced by `f` to `self`.

INPUT:

- `f` – a multilinear morphism from the component modules of the parent tensor product to any module
- `codomain` – the codomain of `f` (optional)

By the universal property of the tensor product, `f` induces a linear morphism from $self.parent()$ to the target module. Returns the result of applying that morphism to `self`.

The codomain is used for optimizations purposes only. If it's not provided, it's recovered by calling `f` on the zero input.

EXAMPLES:

We start with simple (admittedly not so interesting) examples, with two modules $A$ and $B$:

```
sage: A = CombinatorialFreeModule(ZZ, [1,2], prefix="A"); A.rename("A")
sage: B = CombinatorialFreeModule(ZZ, [3,4], prefix="B"); B.rename("B")
```

and $f$ the bilinear morphism $(a, b) \mapsto b \otimes a$ from $A \times B$ to $B \otimes A$:

```
sage: def f(a,b):
....:     return tensor([b,a])
```

Now, calling applying $f$ on $a \otimes b$ returns the same as $f(a, b)$:

```
sage: a = A.monomial(1) + 2 * A.monomial(2); a
A[1] + 2*A[2]
sage: b = B.monomial(3) - 2 * B.monomial(4); b
B[3] - 2*B[4]
sage: f(a,b)
B[3] # A[1] + 2*B[3] # A[2] - 2*B[4] # A[1] - 4*B[4] # A[2]
sage: tensor([a,b]).apply_multilinear_morphism(f)
B[3] # A[1] + 2*B[3] # A[2] - 2*B[4] # A[1] - 4*B[4] # A[2]
```

$f$ may be a bilinear morphism to any module over the base ring of $A$ and $B$. Here the codomain is $\mathbf{Z}$:

```
sage: def f(a,b):
....:     return sum(a.coefficients(), 0) * sum(b.coefficients(), 0)
sage: f(a,b)
-3
sage: tensor([a,b]).apply_multilinear_morphism(f)
-3
```

Mind the $0$ in the sums above; otherwise $f$ would not return $0$ in $\mathbf{Z}$:

```
sage: def f(a,b):
....:     return sum(a.coefficients()) * sum(b.coefficients())
sage: type(f(A.zero(), B.zero()))
<... 'int'>
```

Which would be wrong and break this method:

```
sage: tensor([a,b]).apply_multilinear_morphism(f)
Traceback (most recent call last):
...
AttributeError: 'int' object has no attribute 'parent'
```

Here we consider an example where the codomain is a module with basis with a different base ring:

```
   sage: C = CombinatorialFreeModule(QQ, [(1,3),(2,4)], prefix="C"); C.
↪rename("C")
   sage: def f(a,b):
   ....:     return C.sum_of_terms( [((1,3), QQ(a[1]*b[3])), ((2,4),␣
↪QQ(a[2]*b[4]))] )
   sage: f(a,b)
   C[(1, 3)] - 4*C[(2, 4)]
   sage: tensor([a,b]).apply_multilinear_morphism(f)
   C[(1, 3)] - 4*C[(2, 4)]

We conclude with a real life application, where we
check that the antipode of the Hopf algebra of
Symmetric functions on the Schur basis satisfies its
defining formula::

   sage: Sym = SymmetricFunctions(QQ)
   sage: s = Sym.schur()
   sage: def f(a,b): return a*b.antipode()
   sage: x = 4*s.an_element(); x
   8*s[] + 8*s[1] + 12*s[2]
   sage: x.coproduct().apply_multilinear_morphism(f)
   8*s[]
```

```
sage: x.coproduct().apply_multilinear_morphism(f) == x.counit()
True
```

We recover the constant term of $x$, as desired.

**Todo:** Extract a method to linearize a multilinear morphism, and delegate the work there.

**class ParentMethods**
Implements operations on tensor products of modules with basis.

**extra_super_categories**()
EXAMPLES:

```
sage: ModulesWithBasis(QQ).TensorProducts().extra_super_categories()
[Category of vector spaces with basis over Rational Field]
sage: ModulesWithBasis(QQ).TensorProducts().super_categories()
[Category of tensor products of modules with basis over Rational Field,
 Category of vector spaces with basis over Rational Field,
 Category of tensor products of vector spaces over Rational Field]
```

**is_abelian**()
Return whether this category is abelian.

This is the case if and only if the base ring is a field.

EXAMPLES:

```
sage: ModulesWithBasis(QQ).is_abelian()
True
sage: ModulesWithBasis(ZZ).is_abelian()
False
```

## 3.108 Monoid algebras

sage.categories.monoid_algebras.**MonoidAlgebras**(*base_ring*)
The category of monoid algebras over `base_ring`

EXAMPLES:

```
sage: C = MonoidAlgebras(QQ); C
Category of monoid algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of algebras with basis over Rational Field,
 Category of semigroup algebras over Rational Field,
 Category of unital magma algebras over Rational Field]
```

This is just an alias for:

```
sage: C is Monoids().Algebras(QQ)
True
```

## 3.109 Monoids

**class** sage.categories.monoids.**Monoids**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of (multiplicative) monoids.

A *monoid* is a unital *semigroup*, that is a set endowed with a multiplicative binary operation $*$ which is associative and admits a unit (see Wikipedia article Monoid).

EXAMPLES:

```
sage: Monoids()
Category of monoids
sage: Monoids().super_categories()
[Category of semigroups, Category of unital magmas]
sage: Monoids().all_super_categories()
[Category of monoids,
 Category of semigroups,
 Category of unital magmas, Category of magmas,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]

sage: Monoids().axioms()
frozenset({'Associative', 'Unital'})
sage: Semigroups().Unital()
Category of monoids

sage: Monoids().example()
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
```

```
sage: S = Monoids().example()
sage: x = S("aa")
sage: x^0, x^1, x^2, x^3, x^4, x^5
('', 'aa', 'aaaa', 'aaaaaa', 'aaaaaaaa', 'aaaaaaaaaa')
```

**class Algebras**(*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ElementMethods**

**is_central**()

Return whether the element self is central.

EXAMPLES:

```
sage: SG4=SymmetricGroupAlgebra(ZZ,4)
sage: SG4(1).is_central()
True
sage: SG4(Permutation([1,3,2,4])).is_central()
False
sage: A=GroupAlgebras(QQ).example(); A
Algebra of Dihedral group of order 8 as a permutation group over␣
↪Rational Field
sage: sum(i for i in A.basis()).is_central()
True
```

**class ParentMethods**

**algebra_generators**()

Return generators for this algebra.

For a monoid algebra, the algebra generators are built from the monoid generators if available and from the semigroup generators otherwise.

**See also:**

- *Semigroups.Algebras.ParentMethods.algebra_generators()*
- *MagmaticAlgebras.ParentMethods.algebra_generators()*.

EXAMPLES:

```
sage: M = Monoids().example(); M
An example of a monoid:
the free monoid generated by ('a', 'b', 'c', 'd')
sage: M.monoid_generators()
Finite family {'a': 'a', 'c': 'c', 'b': 'b', 'd': 'd'}
sage: M.algebra(ZZ).algebra_generators()
Finite family {'a': B['a'], 'c': B['c'], 'b': B['b'], 'd': B['d']}

sage: Z12 = Monoids().Finite().example(); Z12
An example of a finite multiplicative monoid:
the integers modulo 12
sage: Z12.monoid_generators()
Traceback (most recent call last):
...
AttributeError: 'IntegerModMonoid_with_category' object
has no attribute 'monoid_generators'
sage: Z12.semigroup_generators()
Family (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
sage: Z12.algebra(QQ).algebra_generators()
Finite family {0: B[0], 1: B[1], 2: B[2], 3: B[3],  4: B[4],   5: B[5],
               6: B[6], 7: B[7], 8: B[8], 9: B[9], 10: B[10], 11:␣
↪B[11]}


sage: GroupAlgebras(QQ).example(AlternatingGroup(10)).algebra_
↪generators()
Finite family {0: (8,9,10), 1: (1,2,3,4,5,6,7,8,9)}

sage: A = DihedralGroup(3).algebra(QQ); A
Algebra of Dihedral group of order 6 as a permutation group
 over Rational Field
sage: A.algebra_generators()
Finite family {0: (1,2,3), 1: (1,3)}
```

**one_basis**()

Return the unit of the monoid, which indexes the unit of this algebra, as per `AlgebrasWithBasis.ParentMethods.one_basis()`.

EXAMPLES:

```
sage: A = Monoids().example().algebra(ZZ)
sage: A.one_basis()
''
sage: A.one()
```

```
B['']
sage: A(3)
3*B['']
```

**extra_super_categories**()

EXAMPLES:

```
sage: Monoids().Algebras(QQ).extra_super_categories()
[Category of monoids]
sage: Monoids().Algebras(QQ).super_categories()
[Category of algebras with basis over Rational Field,
 Category of semigroup algebras over Rational Field,
 Category of unital magma algebras over Rational Field]
```

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

The category of monoids constructed as Cartesian products of monoids.

This construction gives the direct product of monoids. See Wikipedia article Direct_product for more information.

**class ParentMethods**

**monoid_generators**()

Return the generators of `self`.

EXAMPLES:

```
sage: M = Monoids.free([1,2,3])
sage: N = Monoids.free(['a','b'])
sage: C = cartesian_product([M, N])
sage: C.monoid_generators()
Family ((F[1], 1), (F[2], 1), (F[3], 1),
        (1, F['a']), (1, F['b']))
```

An example with an infinitely generated group (a better output is needed):

```
sage: N = Monoids.free(ZZ)
sage: C = cartesian_product([M, N])
sage: C.monoid_generators()
Lazy family (gen(i))_{i in The Cartesian product of (...)}
```

**extra_super_categories**()

A Cartesian product of monoids is endowed with a natural group structure.

EXAMPLES:

```
sage: C = Monoids().CartesianProducts()
sage: C.extra_super_categories()
[Category of monoids]
sage: sorted(C.super_categories(), key=str)
[Category of Cartesian products of semigroups,
 Category of Cartesian products of unital magmas,
 Category of monoids]
```

**class Commutative**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

Category of commutative (abelian) monoids.

A monoid $M$ is *commutative* if $xy = yx$ for all $x, y \in M$.

**static free** (*index_set=None*, *names=None*, ***kwds*)

Return a free abelian monoid on $n$ generators or with the generators indexed by a set $I$.

A free monoid is constructed by specifing either:

- the number of generators and/or the names of the generators, or
- the indexing set for the generators.

INPUT:

- `index_set` – (optional) an index set for the generators; if an integer, then this represents $\{0, 1, \ldots, n-1\}$
- `names` – a string or list/tuple/iterable of strings (default: `'x'`); the generator names or name prefix

EXAMPLES:

```
sage: Monoids.Commutative.free(index_set=ZZ)
Free abelian monoid indexed by Integer Ring
sage: Monoids().Commutative().free(ZZ)
Free abelian monoid indexed by Integer Ring
sage: F.<x,y,z> = Monoids().Commutative().free(); F
Free abelian monoid indexed by {'x', 'y', 'z'}
```

**class ElementMethods**

**is_one** ()

Return whether `self` is the one of the monoid.

The default implementation is to compare with `self.one()`.

**powers** (*n*)

Return the list $[x^0, x^1, \ldots, x^{n-1}]$.

EXAMPLES:

```
sage: A = Matrix([[1, 1], [-1, 0]])
sage: A.powers(6)
[
[1 0]  [ 1  1]  [ 0  1]  [-1  0]  [-1 -1]  [ 0 -1]
[0 1], [-1  0], [-1 -1], [ 0 -1], [ 1  0], [ 1  1]
]
```

**Finite**

alias of `FiniteMonoids`

**Inverse**

alias of `Groups`

**class ParentMethods**

**one_element** ()

Backward compatibility alias for `one()`.

**prod** (*args*)

n-ary product of elements of `self`.

INPUT:

- `args` – a list (or iterable) of elements of `self`

Returns the product of the elements in `args`, as an element of `self`.

EXAMPLES:

```
sage: S = Monoids().example()
sage: S.prod([S('a'), S('b')])
'ab'
```

**semigroup_generators**()

Return the generators of `self` as a semigroup.

The generators of a monoid $M$ as a semigroup are the generators of $M$ as a monoid and the unit.

EXAMPLES:

```
sage: M = Monoids().free([1,2,3])
sage: M.semigroup_generators()
Family (1, F[1], F[2], F[3])
```

**submonoid**(*generators*, *category=None*)

Return the multiplicative submonoid generated by `generators`.

INPUT:

- `generators` – a finite family of elements of `self`, or a list, iterable, … that can be converted into one (see `Family`).
- `category` – a category

This is a shorthand for *Semigroups.ParentMethods.subsemigroup()* that specifies that this is a submonoid, and in particular that the unit is `self.one()`.

EXAMPLES:

```
sage: R = IntegerModRing(15)
sage: M = R.submonoid([R(3),R(5)]); M
A submonoid of (Ring of integers modulo 15) with 2 generators
sage: M.list()
[1, 3, 5, 9, 0, 10, 12, 6]
```

Not the presence of the unit, unlike in:

```
sage: S = R.subsemigroup([R(3),R(5)]); S
A subsemigroup of (Ring of integers modulo 15) with 2 generators
sage: S.list()
[3, 5, 9, 0, 10, 12, 6]
```

This method is really a shorthand for subsemigroup:

```
sage: M2 = R.subsemigroup([R(3),R(5)], one=R.one())
sage: M2 is M
True
```

**class Subquotients**(*category*, *args*)

Bases: *sage.categories.subquotients.SubquotientsCategory*

**class ParentMethods**

**one**()

Returns the multiplicative unit of this monoid, obtained by retracting that of the ambient monoid.

EXAMPLES:

```
sage: S = Monoids().Subquotients().example() # todo: not implemented
sage: S.one()                                # todo: not implemented
```

**class WithRealizations**(*category*, **args*)

Bases: *sage.categories.with_realizations.WithRealizationsCategory*

**class ParentMethods**

**one**()

Return the unit of this monoid.

This default implementation returns the unit of the realization of `self` given by *a_realization()*.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.one.__module__
'sage.categories.monoids'
sage: A.one()
F[{}]
```

**static free**(*index_set=None*, *names=None*, ***kwds*)

Return a free monoid on $n$ generators or with the generators indexed by a set $I$.

A free monoid is constructed by specifing either:

- the number of generators and/or the names of the generators
- the indexing set for the generators

INPUT:

- `index_set` – (optional) an index set for the generators; if an integer, then this represents $\{0, 1, \ldots, n-1\}$
- `names` – a string or list/tuple/iterable of strings (default: `'x'`); the generator names or name prefix

EXAMPLES:

```
sage: Monoids.free(index_set=ZZ)
Free monoid indexed by Integer Ring
sage: Monoids().free(ZZ)
Free monoid indexed by Integer Ring
sage: F.<x,y,z> = Monoids().free(); F
Free monoid indexed by {'x', 'y', 'z'}
```

## 3.110 Number fields

**class** sage.categories.number_fields.**NumberFields**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of number fields.

EXAMPLES:

We create the category of number fields:

```
sage: C = NumberFields()
sage: C
Category of number fields
```

By definition, it is infinite:

```
sage: NumberFields().Infinite() is NumberFields()
True
```

Notice that the rational numbers $\mathbf{Q}$ *are* considered as an object in this category:

```
sage: RationalField() in C
True
```

However, we can define a degree 1 extension of $\mathbf{Q}$, which is of course also in this category:

```
sage: x = PolynomialRing(RationalField(), 'x').gen()
sage: K = NumberField(x - 1, 'a'); K
Number Field in a with defining polynomial x - 1
sage: K in C
True
```

Number fields all lie in this category, regardless of the name of the variable:

```
sage: K = NumberField(x^2 + 1, 'a')
sage: K in C
True
```

**class ElementMethods**

**class ParentMethods**

**super_categories**()

EXAMPLES:

```
sage: NumberFields().super_categories()
[Category of infinite fields]
```

## 3.111 Objects

**class** sage.categories.objects.**Objects**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of all objects the basic category

EXAMPLES:

```
sage: Objects()
Category of objects
sage: Objects().super_categories()
[]
```

**class ParentMethods**

Methods for all category objects

**class SubcategoryMethods**

**Endsets**()

Return the category of endsets between objects of this category.

EXAMPLES:

```
sage: Sets().Endsets()
Category of endsets of sets

sage: Rings().Endsets()
Category of endsets of unital magmas and additive unital additive magmas
```

**See also:**

- *Homsets()*

**Homsets**()

Return the category of homsets between objects of this category.

EXAMPLES:

```
sage: Sets().Homsets()
Category of homsets of sets

sage: Rings().Homsets()
Category of homsets of unital magmas and additive unital additive magmas
```

This used to be called `hom_category`:

```
sage: Sets().hom_category()
doctest:...: DeprecationWarning: hom_category is deprecated. Please use␣
↪Homsets instead.
See http://trac.sagemath.org/10668 for details.
Category of homsets of sets
```

**Note:** Background

Information, code, documentation, and tests about the category of homsets of a category `Cs` should go in the nested class `Cs.Homsets`. They will then be made available to homsets of any subcategory of `Cs`.

Assume, for example, that homsets of `Cs` are `Cs` themselves. This information can be implemented in the method `Cs.Homsets.extra_super_categories` to make `Cs.Homsets()` a subcategory of `Cs()`.

Methods about the homsets themselves should go in the nested class `Cs.Homsets.ParentMethods`.

Methods about the morphisms can go in the nested class `Cs.Homsets.ElementMethods`. However it's generally preferable to put them in the nested class `Cs.MorphimMethods`; indeed they will then apply to morphisms of all subcategories of `Cs`, and not only full subcategories.

**See also:**

`FunctorialConstruction`

**Todo:**

- Design a mechanism to specify that an axiom is compatible with taking subsets. Examples: `Finite`, `Associative`, `Commutative` (when meaningful), but not `Infinite` nor `Unital`.
- Design a mechanism to specify that, when $B$ is a subcategory of $A$, a $B$-homset is a subset of the corresponding $A$ homset. And use it to recover all the relevant axioms from homsets in super categories.
- For instances of redundant code due to this missing feature, see:
  - `AdditiveMonoids.Homsets.extra_super_categories()`
  - `HomsetsCategory.extra_super_categories()` (slightly different nature)
  - plus plenty of spots where this is not implemented.

**hom_category**(**args*, ***kwds*)

Deprecated: Use *Homsets()* instead. See trac ticket #10668 for details.

**additional_structure**()

Return `None`

Indeed, by convention, the category of objects defines no additional structure.

**See also:**

*Category.additional_structure()*

EXAMPLES:

```
sage: Objects().additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: Objects().super_categories()
[]
```

## 3.112 Partially ordered monoids

**class** sage.categories.partially_ordered_monoids.**PartiallyOrderedMonoids**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of partially ordered monoids, that is partially ordered sets which are also monoids, and such that multiplication preserves the ordering: $x \leq y$ implies $x * z < y * z$ and $z * x < z * y$.

See Wikipedia article Ordered_monoid

EXAMPLES:

```
sage: PartiallyOrderedMonoids()
Category of partially ordered monoids
sage: PartiallyOrderedMonoids().super_categories()
[Category of posets, Category of monoids]
```

**class ElementMethods**

**class ParentMethods**

**super_categories**()

EXAMPLES:

```
sage: PartiallyOrderedMonoids().super_categories()
[Category of posets, Category of monoids]
```

## 3.113 Permutation groups

**class** sage.categories.permutation_groups.**PermutationGroups**(*s=None*)

Bases: *sage.categories.category.Category*

The category of permutation groups.

A *permutation group* is a group whose elements are concretely represented by permutations of some set. In other words, the group comes endowed with a distinguished action on some set.

This distinguished action should be preserved by permutation group morphisms. For details, see Wikipedia article Permutation_group#Permutation_isomorphic_groups.

**Todo:** shall we accept only permutations with finite support or not?

EXAMPLES:

```
sage: PermutationGroups()
Category of permutation groups
sage: PermutationGroups().super_categories()
[Category of groups]
```

The category of permutation groups defines additional structure that should be preserved by morphisms, namely the distinguished action:

```
sage: PermutationGroups().additional_structure()
Category of permutation groups
```

**Finite**

alias of *FinitePermutationGroups*

**super_categories**()

Return a list of the immediate super categories of `self`.

EXAMPLES:

```
sage: PermutationGroups().super_categories()
[Category of groups]
```

## 3.114 Pointed sets

**class** sage.categories.pointed_sets.**PointedSets**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of pointed sets.

EXAMPLES:

```
sage: PointedSets()
Category of pointed sets
```

**super_categories**()

EXAMPLES:

```
sage: PointedSets().super_categories()
[Category of sets]
```

## 3.115 Polyhedral subsets of free ZZ, QQ or RR-modules.

**class** sage.categories.polyhedra.**PolyhedralSets**(*R*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of polyhedra over a ring.

EXAMPLES:

We create the category of polyhedra over **Q**:

```
sage: PolyhedralSets(QQ)
Category of polyhedral sets over Rational Field
```

**super_categories**()

EXAMPLES:

```
sage: PolyhedralSets(QQ).super_categories()
[Category of commutative magmas, Category of additive monoids]
```

## 3.116 Posets

**class** sage.categories.posets.**Posets**(*s=None*)

Bases: *sage.categories.category.Category*

The category of posets i.e. sets with a partial order structure.

EXAMPLES:

```
sage: Posets()
Category of posets
sage: Posets().super_categories()
[Category of sets]
sage: P = Posets().example(); P
An example of a poset: sets ordered by inclusion
```

The partial order is implemented by the mandatory method `le()`:

```
sage: x = P(Set([1,3])); y = P(Set([1,2,3]))
sage: x, y
({1, 3}, {1, 2, 3})
sage: P.le(x, y)
True
sage: P.le(x, x)
True
sage: P.le(y, x)
False
```

The other comparison methods are called `lt()`, `ge()`, `gt()`, following Python's naming convention in `operator`. Default implementations are provided:

```
sage: P.lt(x, x)
False
sage: P.ge(y, x)
True
```

Unless the poset is a facade (see *Sets.Facade*), one can compare directly its elements using the usual Python operators:

```
sage: D = Poset((divisors(30), attrcall("divides")), facade = False)
sage: D(3) <= D(6)
True
sage: D(3) <= D(3)
True
sage: D(3) <= D(5)
False
sage: D(3) < D(3)
False
sage: D(10) >= D(5)
True
```

At this point, this has to be implemented by hand. Once trac ticket #10130 will be resolved, this will be automatically provided by this category:

```
sage: x < y      # todo: not implemented
True
sage: x < x      # todo: not implemented
False
sage: x <= x     # todo: not implemented
True
sage: y >= x     # todo: not implemented
True
```

**See also:**

`Poset()`, *FinitePosets*, *LatticePosets*

**class ElementMethods**

**Finite**
alias of *FinitePosets*

**class ParentMethods**

**CartesianProduct**
alias of `CartesianProductPoset`

**directed_subset** (*elements*, *direction*)
Return the order filter or the order ideal generated by a list of elements.

If `direction` is 'up', the order filter (upper set) is being returned.

If `direction` is 'down', the order ideal (lower set) is being returned.

INPUT:

- elements – a list of elements.
- direction – 'up' or 'down'.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.directed_subset([3, 8], 'up')
[3, 7, 8, 9, 10, 11, 12, 13, 14, 15]
sage: B.directed_subset([7, 10], 'down')
[0, 1, 2, 3, 4, 5, 6, 7, 8, 10]
```

**ge** (*x*, *y*)

Return whether $x \geq y$ in the poset `self`.

INPUT:

- `x`, `y` – elements of `self`.

This default implementation delegates the work to `le()`.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.ge( 6, 3 )
True
sage: D.ge( 3, 3 )
True
sage: D.ge( 3, 5 )
False
```

**gt** (*x*, *y*)

Return whether $x > y$ in the poset `self`.

INPUT:

- `x`, `y` – elements of `self`.

This default implementation delegates the work to `lt()`.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.gt( 3, 6 )
False
sage: D.gt( 3, 3 )
False
sage: D.gt( 3, 5 )
False
```

**is_antichain_of_poset** (*o*)

Return whether an iterable `o` is an antichain of `self`.

INPUT:

- `o` – an iterable (e. g., list, set, or tuple) containing some elements of `self`

OUTPUT:

`True` if the subset of `self` consisting of the entries of `o` is an antichain of `self`, and `False` otherwise.

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")), facade=True, linear_
↪extension=True)
sage: sorted(P.list())
[1, 2, 3, 4, 6, 12]
sage: P.is_antichain_of_poset([1, 3])
False
sage: P.is_antichain_of_poset([3, 1])
```

```
False
sage: P.is_antichain_of_poset([1, 1, 3])
False
sage: P.is_antichain_of_poset([])
True
sage: P.is_antichain_of_poset([1])
True
sage: P.is_antichain_of_poset([1, 1])
True
sage: P.is_antichain_of_poset([3, 4])
True
sage: P.is_antichain_of_poset([3, 4, 12])
False
sage: P.is_antichain_of_poset([6, 4])
True
sage: P.is_antichain_of_poset(i for i in divisors(12) if (2 < i and i <␣
↪6))
True
sage: P.is_antichain_of_poset(i for i in divisors(12) if (2 <= i and i <␣
↪6))
False

sage: Q = Poset({2: [3, 1], 3: [4], 1: [4]})
sage: Q.is_antichain_of_poset((1, 2))
False
sage: Q.is_antichain_of_poset((2, 4))
False
sage: Q.is_antichain_of_poset((4, 2))
False
sage: Q.is_antichain_of_poset((2, 2))
True
sage: Q.is_antichain_of_poset((3, 4))
False
sage: Q.is_antichain_of_poset((3, 1))
True
sage: Q.is_antichain_of_poset((1, ))
True
sage: Q.is_antichain_of_poset(())
True
```

An infinite poset:

```
sage: from sage.categories.examples.posets import␣
↪FiniteSetsOrderedByInclusion
sage: R = FiniteSetsOrderedByInclusion()
sage: R.is_antichain_of_poset([R(set([3, 1, 2])), R(set([1, 4])),␣
↪R(set([4, 5]))])
True
sage: R.is_antichain_of_poset([R(set([3, 1, 2, 4])), R(set([1, 4])),␣
↪R(set([4, 5]))])
False
```

**is_chain_of_poset**(*o*, *ordered=False*)

Return whether an iterable `o` is a chain of `self`, including a check for `o` being ordered from smallest to largest element if the keyword `ordered` is set to `True`.

INPUT:

- `o` – an iterable (e. g., list, set, or tuple) containing some elements of `self`

- ordered – a Boolean (default: False) which decides whether the notion of a chain includes being ordered

OUTPUT:

If ordered is set to False, the truth value of the following assertion is returned: The subset of self formed by the elements of o is a chain in self.

If ordered is set to True, the truth value of the following assertion is returned: Every element of the list o is (strictly!) smaller than its successor in self. (This makes no sense if ordered is a set.)

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")), facade=True, linear_
↪extension=True)
sage: sorted(P.list())
[1, 2, 3, 4, 6, 12]
sage: P.is_chain_of_poset([1, 3])
True
sage: P.is_chain_of_poset([3, 1])
True
sage: P.is_chain_of_poset([1, 3], ordered=True)
True
sage: P.is_chain_of_poset([3, 1], ordered=True)
False
sage: P.is_chain_of_poset([])
True
sage: P.is_chain_of_poset([], ordered=True)
True
sage: P.is_chain_of_poset((2, 12, 6))
True
sage: P.is_chain_of_poset((2, 6, 12), ordered=True)
True
sage: P.is_chain_of_poset((2, 12, 6), ordered=True)
False
sage: P.is_chain_of_poset((2, 12, 6, 3))
False
sage: P.is_chain_of_poset((2, 3))
False

sage: Q = Poset({2: [3, 1], 3: [4], 1: [4]})
sage: Q.is_chain_of_poset([1, 2], ordered=True)
False
sage: Q.is_chain_of_poset([1, 2])
True
sage: Q.is_chain_of_poset([2, 1], ordered=True)
True
sage: Q.is_chain_of_poset([2, 1, 1], ordered=True)
False
sage: Q.is_chain_of_poset([3])
True
sage: Q.is_chain_of_poset([4, 2, 3])
True
sage: Q.is_chain_of_poset([4, 2, 3], ordered=True)
False
sage: Q.is_chain_of_poset([2, 3, 4], ordered=True)
True
```

Examples with infinite posets:

```
sage: from sage.categories.examples.posets import FiniteSetsOrderedByInclusion
sage: R = FiniteSetsOrderedByInclusion()
sage: R.is_chain_of_poset([R(set([3, 1, 2])), R(set([1, 4])), R(set([4, 5]))])
False
sage: R.is_chain_of_poset([R(set([3, 1, 2])), R(set([1, 2])), R(set([1]))], ordered=True)
False
sage: R.is_chain_of_poset([R(set([3, 1, 2])), R(set([1, 2])), R(set([1]))])
True

sage: from sage.categories.examples.posets import PositiveIntegersOrderedByDivisibilityFacade
sage: T = PositiveIntegersOrderedByDivisibilityFacade()
sage: T.is_chain_of_poset((T(3), T(4), T(7)))
False
sage: T.is_chain_of_poset((T(3), T(6), T(3)))
True
sage: T.is_chain_of_poset((T(3), T(6), T(3)), ordered=True)
False
sage: T.is_chain_of_poset((T(3), T(3), T(6)))
True
sage: T.is_chain_of_poset((T(3), T(3), T(6)), ordered=True)
False
sage: T.is_chain_of_poset((T(3), T(6)), ordered=True)
True
sage: T.is_chain_of_poset((), ordered=True)
True
sage: T.is_chain_of_poset((T(3),), ordered=True)
True
sage: T.is_chain_of_poset((T(q) for q in divisors(27)))
True
sage: T.is_chain_of_poset((T(q) for q in divisors(18)))
False
```

**is_order_filter**(*o*)

Return whether `o` is an order filter of `self`, assuming `self` has no infinite ascending path.

INPUT:

- `o` – a list (or set, or tuple) containing some elements of `self`

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")), facade=True, linear_extension=True)
sage: sorted(P.list())
[1, 2, 3, 4, 6, 12]
sage: P.is_order_filter([4, 12])
True
sage: P.is_order_filter([])
True
sage: P.is_order_filter({3, 4, 12})
False
sage: P.is_order_filter({3, 6, 12})
True
```

**is_order_ideal**(*o*)

Return whether o is an order ideal of self, assuming self has no infinite descending path.

INPUT:

- o – a list (or set, or tuple) containing some elements of self

EXAMPLES:

```
sage: P = Poset((divisors(12), attrcall("divides")), facade=True, linear_
↪extension=True)
sage: sorted(P.list())
[1, 2, 3, 4, 6, 12]
sage: P.is_order_ideal([1, 3])
True
sage: P.is_order_ideal([])
True
sage: P.is_order_ideal({1, 3})
True
sage: P.is_order_ideal([1, 3, 4])
False
```

**le** (*x*, *y*)

Return whether $x \leq y$ in the poset self.

INPUT:

- x, y – elements of self.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.le( 3, 6 )
True
sage: D.le( 3, 3 )
True
sage: D.le( 3, 5 )
False
```

**lower_covers** (*x*)

Return the lower covers of $x$, that is, the elements $y$ such that $y < x$ and there exists no $z$ such that $y < z < x$.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.lower_covers(15)
[3, 5]
```

**lt** (*x*, *y*)

Return whether $x < y$ in the poset self.

INPUT:

- x, y – elements of self.

This default implementation delegates the work to *le()*.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.lt( 3, 6 )
True
sage: D.lt( 3, 3 )
False
sage: D.lt( 3, 5 )
False
```

**order_filter**(*elements*)

Return the order filter generated by a list of elements.

A subset $I$ of a poset is said to be an order filter if, for any $x$ in $I$ and $y$ such that $y \geq x$, then $y$ is in $I$.

This is also called the upper set generated by these elements.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.order_filter([3,8])
[3, 7, 8, 9, 10, 11, 12, 13, 14, 15]
```

**order_ideal**(*elements*)

Return the order ideal in `self` generated by the elements of an iterable `elements`.

A subset $I$ of a poset is said to be an order ideal if, for any $x$ in $I$ and $y$ such that $y \leq x$, then $y$ is in $I$.

This is also called the lower set generated by these elements.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.order_ideal([7,10])
[0, 1, 2, 3, 4, 5, 6, 7, 8, 10]
```

**order_ideal_toggle**(*I*, *v*)

Return the result of toggling the element `v` in the order ideal `I`.

If $v$ is an element of a poset $P$, then toggling the element $v$ is an automorphism of the set $J(P)$ of all order ideals of $P$. It is defined as follows: If $I$ is an order ideal of $P$, then the image of $I$ under toggling the element $v$ is

- the set $I \cup \{v\}$, if $v \notin I$ but every element of $P$ smaller than $v$ is in $I$;
- the set $I \setminus \{v\}$, if $v \in I$ but no element of $P$ greater than $v$ is in $I$;
- $I$ otherwise.

This image always is an order ideal of $P$.

EXAMPLES:

```
sage: P = Poset({1: [2,3], 2: [4], 3: []})
sage: I = Set({1, 2})
sage: I in P.order_ideals_lattice()
True
sage: P.order_ideal_toggle(I, 1)
{1, 2}
sage: P.order_ideal_toggle(I, 2)
{1}
sage: P.order_ideal_toggle(I, 3)
{1, 2, 3}
sage: P.order_ideal_toggle(I, 4)
{1, 2, 4}
sage: P4 = Posets(4)
sage: all(all(all(P.order_ideal_toggle(P.order_ideal_toggle(I, i), i) == I
....:                 for i in range(4))
....:             for I in P.order_ideals_lattice(facade=True))
....:     for P in P4)
True
```

**order_ideal_toggles** (*I*, *vs*)

Return the result of toggling the elements of the list (or iterable) `vs` (one by one, from left to right) in the order ideal `I`.

See *order_ideal_toggle()* for a definition of toggling.

EXAMPLES:

```
sage: P = Poset({1: [2,3], 2: [4], 3: []})
sage: I = Set({1, 2})
sage: P.order_ideal_toggles(I, [1,2,3,4])
{1, 3}
sage: P.order_ideal_toggles(I, (1,2,3,4))
{1, 3}
```

**principal_lower_set** (*x*)

Return the order ideal generated by an element `x`.

This is also called the lower set generated by this element.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.principal_order_ideal(6)
[0, 2, 4, 6]
```

**principal_order_filter** (*x*)

Return the order filter generated by an element `x`.

This is also called the upper set generated by this element.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.principal_order_filter(2)
[2, 3, 6, 7, 10, 11, 14, 15]
```

**principal_order_ideal** (*x*)

Return the order ideal generated by an element `x`.

This is also called the lower set generated by this element.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.principal_order_ideal(6)
[0, 2, 4, 6]
```

**principal_upper_set** (*x*)

Return the order filter generated by an element `x`.

This is also called the upper set generated by this element.

EXAMPLES:

```
sage: B = posets.BooleanLattice(4)
sage: B.principal_order_filter(2)
[2, 3, 6, 7, 10, 11, 14, 15]
```

**upper_covers** (*x*)

Return the upper covers of $x$, that is, the elements $y$ such that $x < y$ and there exists no $z$ such that $x < z < y$.

EXAMPLES:

```
sage: D = Poset((divisors(30), attrcall("divides")))
sage: D.upper_covers(3)
[6, 15]
```

**example** (*choice=None*)

Return examples of objects of `Posets()`, as per *Category.example()*.

EXAMPLES:

```
sage: Posets().example()
An example of a poset: sets ordered by inclusion

sage: Posets().example("facade")
An example of a facade poset: the positive integers ordered by divisibility
```

**super_categories** ()

Return a list of the (immediate) super categories of `self`, as per *Category.super_categories()*.

EXAMPLES:

```
sage: Posets().super_categories()
[Category of sets]
```

## 3.117 Principal ideal domains

**class** sage.categories.principal_ideal_domains.**PrincipalIdealDomains** (*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of (constructive) principal ideal domains

By constructive, we mean that a single generator can be constructively found for any ideal given by a finite set of generators. Note that this constructive definition only implies that finitely generated ideals are principal. It is not clear what we would mean by an infinitely generated ideal.

EXAMPLES:

```
sage: PrincipalIdealDomains()
Category of principal ideal domains
sage: PrincipalIdealDomains().super_categories()
[Category of unique factorization domains]
```

See also Wikipedia article Principal_ideal_domain

**class ElementMethods**

**class ParentMethods**

**additional_structure** ()

Return `None`.

Indeed, the category of principal ideal domains defines no additional structure: a ring morphism between two principal ideal domains is a principal ideal domain morphism.

EXAMPLES:

```
sage: PrincipalIdealDomains().additional_structure()
```

**super_categories**()
EXAMPLES:

```
sage: PrincipalIdealDomains().super_categories()
[Category of unique factorization domains]
```

## 3.118 Quotient fields

**class** sage.categories.quotient_fields.**QuotientFields**(*s=None*)
Bases: *sage.categories.category_singleton.Category_singleton*

The category of quotient fields over an integral domain

EXAMPLES:

```
sage: QuotientFields()
Category of quotient fields
sage: QuotientFields().super_categories()
[Category of fields]
```

**class ElementMethods**

**denominator**()
Constructor for abstract methods

EXAMPLES:

```
sage: def f(x):
....:     "doc of f"
....:     return 1
...
sage: x = abstract_method(f); x
<abstract method f at ...>
sage: x.__doc__
'doc of f'
sage: x.__name__
'f'
sage: x.__module__
'__main__'
```

**derivative**(**args*)
The derivative of this rational function, with respect to variables supplied in args.

Multiple variables and iteration counts may be supplied; see documentation for the global derivative() function for more details.

**See also:**

`_derivative()`

EXAMPLES:

```
sage: F.<x> = Frac(QQ['x'])
sage: (1/x).derivative()
-1/x^2
```

```
sage: (x+1/x).derivative(x, 2)
2/x^3
```

```
sage: F.<x,y> = Frac(QQ['x,y'])
sage: (1/(x+y)).derivative(x,y)
2/(x^3 + 3*x^2*y + 3*x*y^2 + y^3)
```

**factor**(**args*, ***kwds*)

Return the factorization of `self` over the base ring.

INPUT:

- `*args` - Arbitrary arguments suitable over the base ring
- `**kwds` - Arbitrary keyword arguments suitable over the base ring

OUTPUT:

- Factorization of `self` over the base ring

EXAMPLES:

```
sage: K.<x> = QQ[]
sage: f = (x^3+x)/(x-3)
sage: f.factor()
(x - 3)^-1 * x * (x^2 + 1)
```

Here is an example to show that trac ticket #7868 has been resolved:

```
sage: R.<x,y> = GF(2)[]
sage: f = x*y/(x+y)
sage: f.factor()
(x + y)^-1 * y * x
```

**gcd**(*other*)

Greatest common divisor

**Note:** In a field, the greatest common divisor is not very informative, as it is only determined up to a unit. But in the fraction field of an integral domain that provides both gcd and lcm, it is possible to be a bit more specific and define the gcd uniquely up to a unit of the base ring (rather than in the fraction field).

AUTHOR:

- Simon King (2011-02): See trac ticket #10771

EXAMPLES:

```
sage: R.<x> = QQ['x']
sage: p = (1+x)^3*(1+2*x^2)/(1-x^5)
sage: q = (1+x)^2*(1+3*x^2)/(1-x^4)
sage: factor(p)
(-2) * (x - 1)^-1 * (x + 1)^3 * (x^2 + 1/2) * (x^4 + x^3 + x^2 + x + 1)^-1
sage: factor(q)
(-3) * (x - 1)^-1 * (x + 1) * (x^2 + 1)^-1 * (x^2 + 1/3)
sage: gcd(p,q)
(x + 1)/(x^7 + x^5 - x^2 - 1)
sage: factor(gcd(p,q))
(x - 1)^-1 * (x + 1) * (x^2 + 1)^-1 * (x^4 + x^3 + x^2 + x + 1)^-1
sage: factor(gcd(p,1+x))
(x - 1)^-1 * (x + 1) * (x^4 + x^3 + x^2 + x + 1)^-1
```

```
sage: factor(gcd(1+x,q))
(x - 1)^-1 * (x + 1) * (x^2 + 1)^-1
```

**lcm**(*other*)

Least common multiple

In a field, the least common multiple is not very informative, as it is only determined up to a unit. But in the fraction field of an integral domain that provides both gcd and lcm, it is reasonable to be a bit more specific and to define the least common multiple so that it restricts to the usual least common multiple in the base ring and is unique up to a unit of the base ring (rather than up to a unit of the fraction field).

The least common multiple is easily described in terms of the prime decomposition. A rational number can be written as a product of primes with integer (positive or negative) powers in a unique way. The least common multiple of two rational numbers $x$ and $y$ can then be defined by specifying that the exponent of every prime $p$ in $lcm(x, y)$ is the supremum of the exponents of $p$ in $x$, and the exponent of $p$ in $y$ (where the primes that does not appear in the decomposition of $x$ or $y$ are considered to have exponent zero).

AUTHOR:

- Simon King (2011-02): See trac ticket #10771

EXAMPLES:

```
sage: lcm(2/3, 1/5)
2
```

Indeed $2/3 = 2^1 3^{-1} 5^0$ and $1/5 = 2^0 3^0 5^{-1}$, so $lcm(2/3, 1/5) = 2^1 3^0 5^0 = 2$.

sage: lcm(1/3, 1/5) 1 sage: lcm(1/3, 1/6) 1/3

Some more involved examples:

```
sage: R.<x> = QQ[]
sage: p = (1+x)^3*(1+2*x^2)/(1-x^5)
sage: q = (1+x)^2*(1+3*x^2)/(1-x^4)
sage: factor(p)
(-2) * (x - 1)^-1 * (x + 1)^3 * (x^2 + 1/2) * (x^4 + x^3 + x^2 + x + 1)^-1
sage: factor(q)
(-3) * (x - 1)^-1 * (x + 1) * (x^2 + 1)^-1 * (x^2 + 1/3)
sage: factor(lcm(p,q))
(x - 1)^-1 * (x + 1)^3 * (x^2 + 1/3) * (x^2 + 1/2)
sage: factor(lcm(p,1+x))
(x + 1)^3 * (x^2 + 1/2)
sage: factor(lcm(1+x,q))
(x + 1) * (x^2 + 1/3)
```

**numerator**()

Constructor for abstract methods

EXAMPLES:

```
sage: def f(x):
....:     "doc of f"
....:     return 1
...
sage: x = abstract_method(f); x
<abstract method f at ...>
sage: x.__doc__
'doc of f'
sage: x.__name__
```

```
'f'
sage: x.__module__
'__main__'
```

**partial_fraction_decomposition**(*decompose_powers=True*)

Decomposes fraction field element into a whole part and a list of fraction field elements over prime power denominators.

The sum will be equal to the original fraction.

INPUT:

- **decompose_powers – whether to decompose prime power** denominators as opposed to having a single term for each irreducible factor of the denominator (default: True)

OUTPUT:

- Partial fraction decomposition of self over the base ring.

AUTHORS:

- Robert Bradshaw (2007-05-31)

EXAMPLES:

```
sage: S.<t> = QQ[]
sage: q = 1/(t+1) + 2/(t+2) + 3/(t-3); q
(6*t^2 + 4*t - 6)/(t^3 - 7*t - 6)
sage: whole, parts = q.partial_fraction_decomposition(); parts
[3/(t - 3), 1/(t + 1), 2/(t + 2)]
sage: sum(parts) == q
True
sage: q = 1/(t^3+1) + 2/(t^2+2) + 3/(t-3)^5
sage: whole, parts = q.partial_fraction_decomposition(); parts
[1/3/(t + 1), 3/(t^5 - 15*t^4 + 90*t^3 - 270*t^2 + 405*t - 243), (-1/3*t␣
↪+ 2/3)/(t^2 - t + 1), 2/(t^2 + 2)]
sage: sum(parts) == q
True
sage: q = 2*t / (t + 3)^2
sage: q.partial_fraction_decomposition()
(0, [2/(t + 3), -6/(t^2 + 6*t + 9)])
sage: for p in q.partial_fraction_decomposition()[1]: print(p.factor())
(2) * (t + 3)^-1
(-6) * (t + 3)^-2
sage: q.partial_fraction_decomposition(decompose_powers=False)
(0, [2*t/(t^2 + 6*t + 9)])
```

We can decompose over a given algebraic extension:

```
sage: R.<x> = QQ[sqrt(2)][]
sage: r =  1/(x^4+1)
sage: r.partial_fraction_decomposition()
(0,
 [(-1/4*sqrt2*x + 1/2)/(x^2 - sqrt2*x + 1),
  (1/4*sqrt2*x + 1/2)/(x^2 + sqrt2*x + 1)])

sage: R.<x> = QQ[I][]  # of QQ[sqrt(-1)]
sage: r =  1/(x^4+1)
sage: r.partial_fraction_decomposition()
(0, [(-1/2*I)/(x^2 - I), 1/2*I/(x^2 + I)])
```

We can also ask Sage to find the least extension where the denominator factors in linear terms:

```
sage: R.<x> = QQ[]
sage: r = 1/(x^4+2)
sage: N = r.denominator().splitting_field('a')
sage: N
Number Field in a with defining polynomial x^8 - 8*x^6 + 28*x^4 + 16*x^2␣
↪+ 36
sage: R1.<x1>=N[]
sage: r1 = 1/(x1^4+2)
sage: r1.partial_fraction_decomposition()
(0,
 [(-1/224*a^6 + 13/448*a^4 - 5/56*a^2 - 25/224)/(x1 - 1/28*a^6 + 13/56*a^
↪4 - 5/7*a^2 - 25/28),
  (1/224*a^6 - 13/448*a^4 + 5/56*a^2 + 25/224)/(x1 + 1/28*a^6 - 13/56*a^4␣
↪+ 5/7*a^2 + 25/28),
  (-5/1344*a^7 + 43/1344*a^5 - 85/672*a^3 - 31/672*a)/(x1 - 5/168*a^7 +␣
↪43/168*a^5 - 85/84*a^3 - 31/84*a),
  (5/1344*a^7 - 43/1344*a^5 + 85/672*a^3 + 31/672*a)/(x1 + 5/168*a^7 - 43/
↪168*a^5 + 85/84*a^3 + 31/84*a)])
```

Or we may work directly over an algebraically closed field:

```
sage: R.<x> = QQbar[]
sage: r =  1/(x^4+1)
sage: r.partial_fraction_decomposition()
(0,
 [(-0.1767766952966369? - 0.1767766952966369?*I)/(x - 0.7071067811865475?␣
↪- 0.7071067811865475?*I),
  (-0.1767766952966369? + 0.1767766952966369?*I)/(x - 0.7071067811865475?␣
↪+ 0.7071067811865475?*I),
  (0.1767766952966369? - 0.1767766952966369?*I)/(x + 0.7071067811865475? -
↪ 0.7071067811865475?*I),
  (0.1767766952966369? + 0.1767766952966369?*I)/(x + 0.7071067811865475?␣
↪+ 0.7071067811865475?*I)])
```

We do the best we can over inexact fields:

```
sage: R.<x> = RealField(20)[]
sage: q = 1/(x^2 + x + 2)^2 + 1/(x-1); q
(x^4 + 2.0000*x^3 + 5.0000*x^2 + 5.0000*x + 3.0000)/(x^5 + x^4 + 3.0000*x^
↪3 - x^2 - 4.0000)
sage: whole, parts = q.partial_fraction_decomposition(); parts
[1.0000/(x - 1.0000), 1.0000/(x^4 + 2.0000*x^3 + 5.0000*x^2 + 4.0000*x +␣
↪4.0000)]
sage: sum(parts)
(x^4 + 2.0000*x^3 + 5.0000*x^2 + 5.0000*x + 3.0000)/(x^5 + x^4 + 3.0000*x^
↪3 - x^2 - 4.0000)
```

**xgcd**(*other*)

Return a triple `(g,s,t)` of elements of that field such that `g` is the greatest common divisor of `self` and `other` and `g = s*self + t*other`.

**Note:** In a field, the greatest common divisor is not very informative, as it is only determined up to a unit. But in the fraction field of an integral domain that provides both xgcd and lcm, it is possible to be a bit more specific and define the gcd uniquely up to a unit of the base ring (rather than in the fraction field).

EXAMPLES:

```
sage: QQ(3).xgcd(QQ(2))
(1, 1, -1)
sage: QQ(3).xgcd(QQ(1/2))
(1/2, 0, 1)
sage: QQ(1/3).xgcd(QQ(2))
(1/3, 1, 0)
sage: QQ(3/2).xgcd(QQ(5/2))
(1/2, 2, -1)

sage: R.<x> = QQ['x']
sage: p = (1+x)^3*(1+2*x^2)/(1-x^5)
sage: q = (1+x)^2*(1+3*x^2)/(1-x^4)
sage: factor(p)
(-2) * (x - 1)^-1 * (x + 1)^3 * (x^2 + 1/2) * (x^4 + x^3 + x^2 + x + 1)^-1
sage: factor(q)
(-3) * (x - 1)^-1 * (x + 1) * (x^2 + 1)^-1 * (x^2 + 1/3)
sage: g,s,t = xgcd(p,q)
sage: g
(x + 1)/(x^7 + x^5 - x^2 - 1)
sage: g == s*p + t*q
True
```

An example without a well defined gcd or xgcd on its base ring:

```
sage: K = QuadraticField(5)
sage: O = K.maximal_order()
sage: R = PolynomialRing(O, 'x')
sage: F = R.fraction_field()
sage: x = F.gen(0)
sage: x.gcd(x+1)
1
sage: x.xgcd(x+1)
(1, 1/x, 0)
sage: zero = F.zero()
sage: zero.gcd(x)
1
sage: zero.xgcd(x)
(1, 0, 1/x)
sage: zero.xgcd(zero)
(0, 0, 0)
```

**class ParentMethods**

**super_categories**()

EXAMPLES:

```
sage: QuotientFields().super_categories()
[Category of fields]
```

## 3.119 Regular Crystals

**class** sage.categories.regular_crystals.**RegularCrystals**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of regular crystals.

A crystal is called *regular* if every vertex $b$ satisfies

$$\varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq 0\} \quad \text{and} \quad \varphi_i(b) = \max\{k \mid f_i^k(b) \neq 0\}.$$

**Note:** Regular crystals are sometimes referred to as *normal*. When only one of the conditions (on either $\varphi_i$ or $\varepsilon_i$) holds, these crystals are sometimes called *seminormal* or *semiregular*.

EXAMPLES:

```
sage: C = RegularCrystals()
sage: C
Category of regular crystals
sage: C.super_categories()
[Category of crystals]
sage: C.example()
Highest weight crystal of type A_3 of highest weight omega_1
```

**class ElementMethods**

**demazure_operator_simple**(*i*, *ring=None*)

Return the Demazure operator $D_i$ applied to `self`.

INPUT:

- `i` – an element of the index set of the underlying crystal
- `ring` – (default: `QQ`) a ring

OUTPUT:

An element of the `ring`-free module indexed by the underlying crystal.

Let $r = \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$, then $D_i(b)$ is defined as follows:

- If $r \geq 0$, this returns the sum of the elements obtained from `self` by application of $f_i^k$ for $0 \leq k \leq r$.
- If $r < 0$, this returns the opposite of the sum of the elements obtained by application of $e_i^k$ for $0 < k < -r$.

REFERENCES:

- [Li1995]
- [Ka1993]

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = T(rows=[[1,2],[2]])
sage: t.demazure_operator_simple(2)
B[[[1, 2], [2]]] + B[[[1, 3], [2]]] + B[[[1, 3], [3]]]
sage: t.demazure_operator_simple(2).parent()
Algebra of The crystal of tableaux of type ['A', 2] and shape(s) [[2, 1]]
        over Integer Ring

sage: t.demazure_operator_simple(1)
0

sage: K = crystals.KirillovReshetikhin(['A',2,1],2,1)
sage: t = K(rows=[[3],[2]])
sage: t.demazure_operator_simple(0)
B[[[1, 2]]] + B[[[2, 3]]]
```

**dual_equivalence_class** (*index_set=None*)

Return the dual equivalence class indexed by `index_set` of `self`.

The dual equivalence class of an element $b \in B$ is the set of all elements of $B$ reachable from $b$ via sequences of $i$-elementary dual equivalence relations (i.e., $i$-elementary dual equivalence transformations and their inverses) for $i$ in the index set of $B$.

For this to be well-defined, the element $b$ has to be of weight $0$ with respect to $I$; that is, we need to have $\varepsilon_j(b) = \varphi_j(b)$ for all $j \in I$.

See [As2008]. See also `dual_equivalence_graph()` for a definition of $i$-elementary dual equivalence transformations.

INPUT:

- `index_set` – (optional) the index set $I$ (default: the whole index set of the crystal); this has to be a subset of the index set of the crystal (as a list or tuple)

OUTPUT:

The dual equivalence class of `self` indexed by the subset `index_set`. This class is returned as an undirected edge-colored multigraph. The color of an edge is the index $i$ of the dual equivalence relation it encodes.

**See also:**

- `dual_equivalence_graph()`
- `sage.combinat.partition.Partition.dual_equivalence_graph()`

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',3], shape=[2,2])
sage: G = T(2,1,4,3).dual_equivalence_class()
sage: sorted(G.edges())
[([[1, 3], [2, 4]], [[1, 2], [3, 4]], 2),
 ([[1, 3], [2, 4]], [[1, 2], [3, 4]], 3)]
sage: T = crystals.Tableaux(['A',4], shape=[3,2])
sage: G = T(2,1,4,3,5).dual_equivalence_class()
sage: sorted(G.edges())
[([[1, 3, 5], [2, 4]], [[1, 3, 4], [2, 5]], 4),
 ([[1, 3, 5], [2, 4]], [[1, 2, 5], [3, 4]], 2),
 ([[1, 3, 5], [2, 4]], [[1, 2, 5], [3, 4]], 3),
 ([[1, 3, 4], [2, 5]], [[1, 2, 4], [3, 5]], 2),
 ([[1, 2, 4], [3, 5]], [[1, 2, 3], [4, 5]], 3),
 ([[1, 2, 4], [3, 5]], [[1, 2, 3], [4, 5]], 4)]
```

**epsilon** (*i*)

Return $\varepsilon_i$ of `self`.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).epsilon(1)
0
sage: C(2).epsilon(1)
1
```

**phi** (*i*)

Return $\varphi_i$ of `self`.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).phi(1)
1
sage: C(2).phi(1)
0
```

**stembridgeDel_depth** $(i, j)$

Return the difference in the $j$-depth of `self` and $f_i$ of `self`, where $i$ and $j$ are in the index set of the underlying crystal. This function is useful for checking the Stembridge local axioms for crystal bases.

The $i$-depth of a crystal node $x$ is $\varepsilon_i(x)$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t=T(rows=[[1,1],[2]])
sage: t.stembridgeDel_depth(1,2)
0
sage: s=T(rows=[[1,3],[3]])
sage: s.stembridgeDel_depth(1,2)
-1
```

**stembridgeDel_rise** $(i, j)$

Return the difference in the $j$-rise of `self` and $f_i$ of `self`, where $i$ and $j$ are in the index set of the underlying crystal. This function is useful for checking the Stembridge local axioms for crystal bases.

The $i$-rise of a crystal node $x$ is $\varphi_i(x)$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t=T(rows=[[1,1],[2]])
sage: t.stembridgeDel_rise(1,2)
-1
sage: s=T(rows=[[1,3],[3]])
sage: s.stembridgeDel_rise(1,2)
0
```

**stembridgeDelta_depth** $(i, j)$

Return the difference in the $j$-depth of `self` and $e_i$ of `self`, where $i$ and $j$ are in the index set of the underlying crystal. This function is useful for checking the Stembridge local axioms for crystal bases.

The $i$-depth of a crystal node $x$ is $-\varepsilon_i(x)$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t=T(rows=[[1,2],[2]])
sage: t.stembridgeDelta_depth(1,2)
0
sage: s=T(rows=[[2,3],[3]])
sage: s.stembridgeDelta_depth(1,2)
-1
```

**stembridgeDelta_rise** $(i, j)$

Return the difference in the $j$-rise of `self` and $e_i$ of `self`, where $i$ and $j$ are in the index set of the underlying crystal. This function is useful for checking the Stembridge local axioms for crystal bases.

The $i$-rise of a crystal node $x$ is $\varphi_i(x)$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t=T(rows=[[1,2],[2]])
sage: t.stembridgeDelta_rise(1,2)
-1
sage: s=T(rows=[[2,3],[3]])
sage: s.stembridgeDelta_rise(1,2)
0
```

**stembridgeTriple** $(i, j)$

Let $A$ be the Cartan matrix of the crystal, $x$ a crystal element, and let $i$ and $j$ be in the index set of the crystal. Further, set `b=stembridgeDelta_depth(x,i,j)`, and `c=stembridgeDelta_rise(x,i,j))`. If `x.e(i)` is non-empty, this function returns the triple $(A_{ij}, b, c)$; otherwise it returns `None`. By the Stembridge local characterization of crystal bases, one should have $A_{ij} = b + c$.

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t=T(rows=[[1,1],[2]])
sage: t.stembridgeTriple(1,2)
sage: s=T(rows=[[1,2],[2]])
sage: s.stembridgeTriple(1,2)
(-1, 0, -1)

sage: T = crystals.Tableaux(['B',2], shape=[2,1])
sage: t=T(rows=[[1,2],[2]])
sage: t.stembridgeTriple(1,2)
(-2, 0, -2)
sage: s=T(rows=[[-1,-1],[0]])
sage: s.stembridgeTriple(1,2)
(-2, -2, 0)
sage: u=T(rows=[[0,2],[1]])
sage: u.stembridgeTriple(1,2)
(-2, -1, -1)
```

**weight** ()

Return the weight of this crystal element.

EXAMPLES:

```
sage: C = crystals.Letters(['A',5])
sage: C(1).weight()
(1, 0, 0, 0, 0, 0)
```

**class MorphismMethods**

**is_isomorphism** ()

Check if `self` is a crystal isomorphism, which is true if and only if this is a strict embedding with the same number of connected components.

EXAMPLES:

```
sage: La = RootSystem(['A',2,1]).weight_space(extended=True).fundamental_
↪weights()
sage: B = crystals.LSPaths(La[0])
sage: La = RootSystem(['A',2,1]).weight_lattice(extended=True).
↪fundamental_weights()
```

```
sage: C = crystals.GeneralizedYoungWalls(2, La[0])
sage: H = Hom(B, C)
sage: from sage.categories.highest_weight_crystals import␣
↪HighestWeightCrystalMorphism
sage: class Psi(HighestWeightCrystalMorphism):
....:     def is_strict(self):
....:         return True
sage: psi = Psi(H, C.module_generators)
sage: psi
['A', 2, 1] Crystal morphism:
  From: The crystal of LS paths of type ['A', 2, 1] and weight Lambda[0]
  To:   Highest weight crystal of generalized Young walls of Cartan type [
↪'A', 2, 1]
         and highest weight Lambda[0]
  Defn: (Lambda[0],) |--> []
sage: psi.is_isomorphism()
True
```

**class ParentMethods**

**demazure_operator** (*element*, *reduced_word*)

Returns the application of Demazure operators $D_i$ for $i$ from `reduced_word` on `element`.

INPUT:

- `element` – an element of a free module indexed by the underlying crystal
- `reduced_word` – a reduced word of the Weyl group of the same type as the underlying crystal

OUTPUT:

- an element of the free module indexed by the underlying crystal

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: C = CombinatorialFreeModule(QQ,T)
sage: t = T.highest_weight_vector()
sage: b = 2*C(t)
sage: T.demazure_operator(b,[1,2,1])
2*B[[[1, 1], [2]]] + 2*B[[[1, 2], [2]]] + 2*B[[[1, 3], [2]]] + 2*B[[[1,␣
↪1], [3]]]
+ 2*B[[[1, 2], [3]]] + 2*B[[[1, 3], [3]]] + 2*B[[[2, 2], [3]]] + 2*B[[[2,␣
↪3], [3]]]
```

The Demazure operator is idempotent:

```
sage: T = crystals.Tableaux("A1",shape=[4])
sage: C = CombinatorialFreeModule(QQ,T)
sage: b = C(T.module_generators[0]); b
B[[[1, 1, 1, 1]]]
sage: e = T.demazure_operator(b,[1]); e
B[[[1, 1, 1, 1]]] + B[[[1, 1, 1, 2]]] + B[[[1, 1, 2, 2]]] + B[[[1, 2, 2,␣
↪2]]] + B[[[2, 2, 2, 2]]]
sage: e == T.demazure_operator(e,[1])
True

sage: all(T.demazure_operator(T.demazure_operator(C(t),[1]),[1]) == T.
↪demazure_operator(C(t),[1]) for t in T)
True
```

**demazure_subcrystal** (*element*, *reduced_word*, *only_support=True*)

Return the subcrystal corresponding to the application of Demazure operators $D_i$ for $i$ from `reduced_word` on `element`.

INPUT:

- `element` – an element of a free module indexed by the underlying crystal
- `reduced_word` – a reduced word of the Weyl group of the same type as the underlying crystal
- `only_support` – (default: `True`) only include arrows corresponding to the support of `reduced_word`

OUTPUT:

- the Demazure subcrystal

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',2], shape=[2,1])
sage: t = T.highest_weight_vector()
sage: S = T.demazure_subcrystal(t, [1,2])
sage: list(S)
[[[1, 1], [2]], [[1, 1], [3]], [[1, 2], [2]],
 [[1, 2], [3]], [[2, 2], [3]]]
sage: S = T.demazure_subcrystal(t, [2,1])
sage: list(S)
[[[1, 1], [2]], [[1, 1], [3]], [[1, 2], [2]],
 [[1, 3], [2]], [[1, 3], [3]]]
```

We construct an example where we don't only want the arrows indicated by the support of the reduced word:

```
sage: K = crystals.KirillovReshetikhin(['A',1,1], 1, 2)
sage: mg = K.module_generator()
sage: S = K.demazure_subcrystal(mg, [1])
sage: S.digraph().edges()
[([[1, 1]], [[1, 2]], 1), ([[1, 2]], [[2, 2]], 1)]
sage: S = K.demazure_subcrystal(mg, [1], only_support=False)
sage: S.digraph().edges()
[([[1, 1]], [[1, 2]], 1),
 ([[1, 2]], [[1, 1]], 0),
 ([[1, 2]], [[2, 2]], 1),
 ([[2, 2]], [[1, 2]], 0)]
```

**dual_equivalence_graph**(*X=None*, *index_set=None*, *directed=True*)

Return the dual equivalence graph indexed by `index_set` on the subset `X` of `self`.

Let $b \in B$ be an element of weight $0$, so $\varepsilon_j(b) = \varphi_j(b)$ for all $j \in I$, where $I$ is the indexing set. We say $b'$ is an $i$-elementary dual equivalence transformation of $b$ (where $i \in I$) if

- $\varepsilon_i(b) = 1$ and $\varepsilon_{i-1}(b) = 0$, and
- $b' = f_{i-1} f_i e_{i-1} e_i b$.

We can do the inverse procedure by interchanging $i$ and $i-1$ above.

**Note:** If the index set is not an ordered interval, we let $i-1$ mean the index appearing before $i$ in $I$.

This definition comes from [As2008] Section 4 (where our $\varphi_j(b)$ and $\varepsilon_j(b)$ are denoted by $\epsilon(b,j)$ and $-\delta(b,j)$, respectively).

The dual equivalence graph of $B$ is defined to be the colored graph whose vertices are the elements of $B$ of weight $0$, and whose edges of color $i$ (for $i \in I$) connect pairs $\{b, b'\}$ such that $b'$ is an $i$-elementary dual equivalence transformation of $b$.

**Note:** This dual equivalence graph is a generalization of $\mathcal{G}(\mathcal{X})$ in [As2008] Section 4 except we do not require $\varepsilon_i(b) = 0, 1$ for all $i$.

This definition can be generalized by choosing a subset $X$ of the set of all vertices of $B$ of weight 0, and restricting the dual equivalence graph to the vertex set $X$.

INPUT:

- `X` – (optional) the vertex set $X$ (default: the whole set of vertices of `self` of weight 0)
- `index_set` – (optional) the index set $I$ (default: the whole index set of `self`); this has to be a subset of the index set of `self` (as a list or tuple)
- `directed` – (default: `True`) whether to have the dual equivalence graph be directed, where the head of an edge $b - b'$ is $b$ and the tail is $b' = f_{i-1} f_i e_{i-1} e_i b)$

**See also:**

`sage.combinat.partition.Partition.dual_equivalence_graph()`

EXAMPLES:

```
sage: T = crystals.Tableaux(['A',3], shape=[2,2])
sage: G = T.dual_equivalence_graph()
sage: sorted(G.edges())
[([[1, 3], [2, 4]], [[1, 2], [3, 4]], 2),
 ([[1, 2], [3, 4]], [[1, 3], [2, 4]], 3)]
sage: T = crystals.Tableaux(['A',4], shape=[3,2])
sage: G = T.dual_equivalence_graph()
sage: sorted(G.edges())
[([[1, 3, 5], [2, 4]], [[1, 3, 4], [2, 5]], 4),
 ([[1, 3, 5], [2, 4]], [[1, 2, 5], [3, 4]], 2),
 ([[1, 3, 4], [2, 5]], [[1, 2, 4], [3, 5]], 2),
 ([[1, 2, 5], [3, 4]], [[1, 3, 5], [2, 4]], 3),
 ([[1, 2, 4], [3, 5]], [[1, 2, 3], [4, 5]], 3),
 ([[1, 2, 3], [4, 5]], [[1, 2, 4], [3, 5]], 4)]

sage: T = crystals.Tableaux(['A',4], shape=[3,1])
sage: G = T.dual_equivalence_graph(index_set=[1,2,3])
sage: G.vertices()
[[[1, 3, 4], [2]], [[1, 2, 4], [3]], [[1, 2, 3], [4]]]
sage: G.edges()
[([[1, 3, 4], [2]], [[1, 2, 4], [3]], 2),
 ([[1, 2, 4], [3]], [[1, 2, 3], [4]], 3)]
```

**class TensorProducts**(*category*, **args*)

Bases: *sage.categories.tensor.TensorProductsCategory*

The category of regular crystals constructed by tensor product of regular crystals.

**extra_super_categories**()

EXAMPLES:

```
sage: RegularCrystals().TensorProducts().extra_super_categories()
[Category of regular crystals]
```

**additional_structure**()

Return `None`.

Indeed, the category of regular crystals defines no new structure: it only relates $\varepsilon_a$ and $\varphi_a$ to $e_a$ and $f_a$ respectively.

**See also:**

*Category.additional_structure()*

**Todo:** Should this category be a CategoryWithAxiom?

EXAMPLES:

```
sage: RegularCrystals().additional_structure()
```

**example** (*n=3*)

Returns an example of highest weight crystals, as per *Category.example()*.

EXAMPLES:

```
sage: B = RegularCrystals().example(); B
Highest weight crystal of type A_3 of highest weight omega_1
```

**super_categories** ()

EXAMPLES:

```
sage: RegularCrystals().super_categories()
[Category of crystals]
```

## 3.120 Regular Supercrystals

**class** sage.categories.regular_supercrystals.**RegularSuperCrystals** (*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of crystals for super Lie algebras.

EXAMPLES:

```
sage: from sage.categories.regular_supercrystals import RegularSuperCrystals
sage: C = RegularSuperCrystals()
sage: C
Category of regular super crystals
sage: C.super_categories()
[Category of finite crystals]
```

Parents in this category should implement the following methods:

- either an attribute `_cartan_type` or a method `cartan_type`
- `module_generators`: a list (or container) of distinct elements that generate the crystal using $f_i$ and $e_i$

Furthermore, their elements `x` should implement the following methods:

- `x.e(i)` (returning $e_i(x)$)
- `x.f(i)` (returning $f_i(x)$)
- `x.weight()` (returning $\mathrm{wt}(x)$)

EXAMPLES:

```
sage: from sage.misc.abstract_method import abstract_methods_of_class
sage: from sage.categories.regular_supercrystals import RegularSuperCrystals
sage: abstract_methods_of_class(RegularSuperCrystals().element_class)
{'optional': [], 'required': ['e', 'f', 'weight']}
```

**class ElementMethods**

**epsilon** (*i*)

Return $\varepsilon_i$ of `self`.

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',[1,2]], shape = [2,1])
sage: c = C.an_element(); c
[[-2, -2], [-1]]
sage: c.epsilon(2)
0
sage: c.epsilon(0)
0
sage: c.epsilon(-1)
0
```

**is_genuine_highest_weight** (*index_set=None*)

Return whether `self` is a genuine highest weight element.

INPUT:

- `index_set` – (optional) the index set of the (sub)crystal on which to check

EXAMPLES:

```
sage: B = crystals.Tableaux(['A', [1,1]], shape=[3,2,1])
sage: for b in B.highest_weight_vectors():
....:     print("{} {}".format(b, b.is_genuine_highest_weight()))
[[-2, -2, -2], [-1, 2], [1]] False
[[-2, -2, -2], [-1, -1], [1]] True
[[-2, -2, 2], [-1, -1], [1]] False
sage: [b for b in B if b.is_genuine_highest_weight([-1,0])]
[[[-2, -2, -2], [-1, -1], [1]],
 [[-2, -2, -2], [-1, -1], [2]],
 [[-2, -2, -2], [-1, 2], [2]],
 [[-2, -2, 2], [-1, -1], [2]],
 [[-2, -2, 2], [-1, 2], [2]],
 [[-2, -2, -2], [-1, 2], [1]],
 [[-2, -2, 2], [-1, -1], [1]],
 [[-2, -2, 2], [-1, 2], [1]]]
```

**is_genuine_lowest_weight** (*index_set=None*)

Return whether `self` is a genuine lowest weight element.

INPUT:

- `index_set` – (optional) the index set of the (sub)crystal on which to check

EXAMPLES:

```
sage: B = crystals.Tableaux(['A', [1,1]], shape=[3,2,1])
sage: for b in B.lowest_weight_vectors():
....:     print("{} {}".format(b, b.is_genuine_lowest_weight()))
[[-2, 1, 2], [-1, 2], [1]] False
[[-1, 1, 2], [1, 2], [2]] True
[[-2, 1, 2], [-1, 2], [2]] False
```

```
sage: [b for b in B if b.is_genuine_lowest_weight([-1,0])]
[[[-2, -1, 1], [-1, 1], [1]],
 [[-2, -1, 1], [-1, 1], [2]],
 [[-2, 1, 2], [-1, 1], [2]],
 [[-2, 1, 2], [-1, 1], [1]],
 [[-1, -1, 1], [1, 2], [2]],
 [[-1, -1, 1], [1, 2], [1]],
 [[-1, 1, 2], [1, 2], [2]],
 [[-1, 1, 2], [1, 2], [1]]]
```

**phi**(*i*)

Return $\varphi_i$ of `self`.

EXAMPLES:

```
sage: C = crystals.Tableaux(['A',[1,2]], shape = [2,1])
sage: c = C.an_element(); c
[[-2, -2], [-1]]
sage: c.phi(1)
0
sage: c.phi(2)
0
sage: c.phi(0)
1
```

**class ParentMethods**

**character**()

Return the character of `self`.

---

**Todo:** Once the *WeylCharacterRing* is implemented, make this consistent with the implementation in *sage.categories.classical_crystals.ClassicalCrystals.ParentMethods.character()*.

---

EXAMPLES:

```
sage: B = crystals.Letters(['A',[1,2]])
sage: B.character()
B[(1, 0, 0, 0, 0)] + B[(0, 1, 0, 0, 0)] + B[(0, 0, 1, 0, 0)]
 + B[(0, 0, 0, 1, 0)] + B[(0, 0, 0, 0, 1)]
```

**connected_components**()

Return the connected components of `self` as subcrystals.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.connected_components()
[Subcrystal of The crystal of letters for type ['A', [1, 2]]]

sage: T = B.tensor(B)
sage: T.connected_components()
[Subcrystal of Full tensor product of the crystals
  [The crystal of letters for type ['A', [1, 2]],
   The crystal of letters for type ['A', [1, 2]]],
 Subcrystal of Full tensor product of the crystals
```

```
[The crystal of letters for type ['A', [1, 2]],
 The crystal of letters for type ['A', [1, 2]]]]
```

**connected_components_generators**()

Return the tuple of genuine highest weight elements of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.genuine_highest_weight_vectors()
(-2,)

sage: T = B.tensor(B)
sage: T.genuine_highest_weight_vectors()
([-2, -1], [-2, -2])
sage: s1, s2 = T.connected_components()
sage: s = s1 + s2
sage: s.genuine_highest_weight_vectors()
([-2, -1], [-2, -2])
```

**digraph**()

Return the `DiGraph` associated to `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,3]])
sage: G = B.digraph(); G
Digraph on 6 vertices
```

The edges of the crystal graph are by default colored using blue for edge 1, red for edge 2, green for edge 3, and dashed with the corresponding color for barred edges. Edge 0 is dotted black:

```
sage: view(G)  # optional - dot2tex graphviz, not tested (opens external␣
↪window)
```

**genuine_highest_weight_vectors**()

Return the tuple of genuine highest weight elements of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.genuine_highest_weight_vectors()
(-2,)

sage: T = B.tensor(B)
sage: T.genuine_highest_weight_vectors()
([-2, -1], [-2, -2])
sage: s1, s2 = T.connected_components()
sage: s = s1 + s2
sage: s.genuine_highest_weight_vectors()
([-2, -1], [-2, -2])
```

**genuine_lowest_weight_vectors**()

Return the tuple of genuine lowest weight elements of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.genuine_lowest_weight_vectors()
```

```
(3,)

sage: T = B.tensor(B)
sage: T.genuine_lowest_weight_vectors()
([3, 3], [3, 2])
sage: s1, s2 = T.connected_components()
sage: s = s1 + s2
sage: s.genuine_lowest_weight_vectors()
([3, 3], [3, 2])
```

**highest_weight_vectors**()

Return the highest weight vectors of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.highest_weight_vectors()
(-2,)

sage: T = B.tensor(B)
sage: T.highest_weight_vectors()
([-2, -2], [-2, -1])
```

We give an example from [BKK2000] that has fake highest weight vectors:

```
sage: B = crystals.Tableaux(['A', [1,1]], shape=[3,2,1])
sage: B.highest_weight_vectors()
([[-2, -2, -2], [-1, 2], [1]],
 [[-2, -2, -2], [-1, -1], [1]],
 [[-2, -2, 2], [-1, -1], [1]])

sage: B.genuine_highest_weight_vectors()
([[-2, -2, -2], [-1, -1], [1]],)
```

**lowest_weight_vectors**()

Return the lowest weight vectors of `self`.

EXAMPLES:

```
sage: B = crystals.Letters(['A', [1,2]])
sage: B.lowest_weight_vectors()
(3,)

sage: T = B.tensor(B)
sage: T.lowest_weight_vectors()
([3, 3], [3, 2])
```

We give an example from [BKK2000] that has fake lowest weight vectors:

```
sage: B = crystals.Tableaux(['A', [1,1]], shape=[3,2,1])
sage: B.lowest_weight_vectors()
([[-2, 1, 2], [-1, 2], [1]],
 [[-1, 1, 2], [1, 2], [2]],
 [[-2, 1, 2], [-1, 2], [2]])

sage: B.genuine_lowest_weight_vectors()
([[-1, 1, 2], [1, 2], [2]],)
```

**tensor**(**crystals*, ***options*)

Return the tensor product of `self` with the crystals `B`.

EXAMPLES:

```
sage: B = crystals.Letters(['A',[1,2]])
sage: C = crystals.Tableaux(['A',[1,2]], shape = [2,1])
sage: T = C.tensor(B); T
Full tensor product of the crystals [Crystal of BKK tableaux of shape [2,␣
↪1] of gl(2|3),
The crystal of letters for type ['A', [1, 2]]]
sage: S = B.tensor(C); S
Full tensor product of the crystals [The crystal of letters for type ['A',
↪ [1, 2]],
Crystal of BKK tableaux of shape [2, 1] of gl(2|3)]
sage: G = T.digraph()
sage: H = S.digraph()
sage: G.is_isomorphic(H, edge_labels= True)
True
```

**class TensorProducts**(*category*, **args*)

Bases: *sage.categories.tensor.TensorProductsCategory*

The category of regular crystals constructed by tensor product of regular crystals.

**extra_super_categories**()

EXAMPLES:

```
sage: RegularCrystals().TensorProducts().extra_super_categories()
[Category of regular crystals]
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.regular_supercrystals import RegularSuperCrystals
sage: C = RegularSuperCrystals()
sage: C.super_categories()
[Category of finite crystals]
```

## 3.121 Right modules

**class** sage.categories.right_modules.**RightModules**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of right modules right modules over an rng (ring not necessarily with unit), i.e. an abelian group with right multiplication by elements of the rng

EXAMPLES:

```
sage: RightModules(QQ)
Category of right modules over Rational Field
sage: RightModules(QQ).super_categories()
[Category of commutative additive groups]
```

**class ElementMethods**

**class ParentMethods**

**super_categories**()

EXAMPLES:

```
sage: RightModules(QQ).super_categories()
[Category of commutative additive groups]
```

## 3.122 Ring ideals

**class** sage.categories.ring_ideals.**RingIdeals**(*R*)

Bases: *sage.categories.category_types.Category_ideal*

The category of two-sided ideals in a fixed ring.

EXAMPLES:

```
sage: Ideals(Integers(200))
Category of ring ideals in Ring of integers modulo 200
sage: C = Ideals(IntegerRing()); C
Category of ring ideals in Integer Ring
sage: I = C([8,12,18])
sage: I
Principal ideal (2) of Integer Ring
```

See also: *CommutativeRingIdeals*.

**Todo:**

- If useful, implement `RingLeftIdeals` and `RingRightIdeals` of which `RingIdeals` would be a subcategory.
- Make `RingIdeals(R)`, return `CommutativeRingIdeals(R)` when `R` is commutative.

**super_categories**()

EXAMPLES:

```
sage: RingIdeals(ZZ).super_categories()
[Category of modules over Integer Ring]
sage: RingIdeals(QQ).super_categories()
[Category of vector spaces over Rational Field]
```

## 3.123 Rings

**class** sage.categories.rings.**Rings**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of rings

Associative rings with unit, not necessarily commutative

EXAMPLES:

```
sage: Rings()
Category of rings
sage: sorted(Rings().super_categories(), key=str)
```

```
[Category of rngs, Category of semirings]

sage: sorted(Rings().axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive', 'Unital']

sage: Rings() is (CommutativeAdditiveGroups() & Monoids()).Distributive()
True
sage: Rings() is Rngs().Unital()
True
sage: Rings() is Semirings().AdditiveInverse()
True
```

**Todo:** (see: http://trac.sagemath.org/sage_trac/wiki/CategoriesRoadMap)

- Make Rings() into a subcategory or alias of Algebras(ZZ);
- A parent P in the category `Rings()` should automatically be in the category `Algebras(P)`.

**Commutative**
alias of *CommutativeRings*

**Division**
alias of *DivisionRings*

**class ElementMethods**

**inverse_of_unit**()
Return the inverse of this element if it is a unit.

OUTPUT:

An element in the same ring as this element.

EXAMPLES:

```
sage: R.<x> = ZZ[]
sage: S = R.quo(x^2 + x + 1)
sage: S(1).inverse_of_unit()
1
```

This method fails when the element is not a unit:

```
sage: 2.inverse_of_unit()
Traceback (most recent call last):
...
ArithmeticError: inverse does not exist
```

The inverse returned is in the same ring as this element:

```
sage: a = -1
sage: a.parent()
Integer Ring
sage: a.inverse_of_unit().parent()
Integer Ring
```

Note that this is often not the case when computing inverses in other ways:

```
sage: (~a).parent()
Rational Field
sage: (1/a).parent()
Rational Field
```

**is_unit()**

Return whether this element is a unit in the ring.

**Note:** This is a generic implementation for (non-commutative) rings which only works for the one element, its additive inverse, and the zero element. Most rings should provide a more specialized implementation.

EXAMPLES:

```
sage: MS = MatrixSpace(ZZ, 2)
sage: MS.one().is_unit()
True
sage: MS.zero().is_unit()
False
sage: MS([1,2,3,4]).is_unit()
False
```

**class MorphismMethods**

**is_injective()**

Return whether or not this morphism is injective.

EXAMPLES:

This often raises a `NotImplementedError` as many homomorphisms do not implement this method:

```
sage: R.<x> = QQ[]
sage: f = R.hom([x + 1]); f
Ring endomorphism of Univariate Polynomial Ring in x over Rational Field
  Defn: x |--> x + 1
sage: f.is_injective()
Traceback (most recent call last):
...
NotImplementedError
```

If the domain is a field, the homomorphism is injective:

```
sage: K.<x> = FunctionField(QQ)
sage: L.<y> = FunctionField(QQ)
sage: f = K.hom([y]); f
Function Field morphism:
  From: Rational function field in x over Rational Field
  To:   Rational function field in y over Rational Field
  Defn: x |--> y
sage: f.is_injective()
True
```

Unless the codomain is the zero ring:

```
sage: codomain = Integers(1)
sage: f = QQ.hom([Zmod(1)(0)], check=False)
sage: f.is_injective()
False
```

Homomorphism from rings of characteristic zero to rings of positive characteristic can not be injective:

```
sage: R.<x> = ZZ[]
sage: f = R.hom([GF(3)(1)]); f
Ring morphism:
  From: Univariate Polynomial Ring in x over Integer Ring
  To:   Finite Field of size 3
  Defn: x |--> 1
sage: f.is_injective()
False
```

A morphism whose domain is an order in a number field is injective if the codomain has characteristic zero:

```
sage: K.<x> = FunctionField(QQ)
sage: f = ZZ.hom(K); f
Composite map:
  From: Integer Ring
  To:   Rational function field in x over Rational Field
  Defn:   Conversion via FractionFieldElement_1poly_field map:
          From: Integer Ring
          To:   Fraction Field of Univariate Polynomial Ring in x over␣
↪Rational Field
        then
          Isomorphism morphism:
          From: Fraction Field of Univariate Polynomial Ring in x over␣
↪Rational Field
          To:   Rational function field in x over Rational Field
sage: f.is_injective()
True
```

A coercion to the fraction field is injective:

```
sage: R = ZpFM(3)
sage: R.fraction_field().coerce_map_from(R).is_injective()
True
```

**NoZeroDivisors**
alias of *Domains*

**class ParentMethods**

**bracket** $(x, y)$
Returns the Lie bracket $[x, y] = xy - yx$ of $x$ and $y$.

INPUT:
- x, y – elements of self

EXAMPLES:

```
sage: F = AlgebrasWithBasis(QQ).example()
sage: F
```

```
An example of an algebra with basis: the free algebra on the generators (
↪'a', 'b', 'c') over Rational Field
sage: a,b,c = F.algebra_generators()
sage: F.bracket(a,b)
B[word: ab] - B[word: ba]
```

This measures the default of commutation between $x$ and $y$. $F$ endowed with the bracket operation is a Lie algebra; in particular, it satisfies Jacobi's identity:

```
sage: F.bracket( F.bracket(a,b), c) + F.bracket(F.bracket(b,c),a) + F.
↪bracket(F.bracket(c,a),b)
0
```

**characteristic**()

Return the characteristic of this ring.

EXAMPLES:

```
sage: QQ.characteristic()
0
sage: GF(19).characteristic()
19
sage: Integers(8).characteristic()
8
sage: Zp(5).characteristic()
0
```

**ideal**(**args*, ***kwds*)

Create an ideal of this ring.

NOTE:

The code is copied from the base class `Ring`. This is because there are rings that do not inherit from that class, such as matrix algebras. See trac ticket #7797.

INPUT:

- An element or a list/tuple/sequence of elements.
- `coerce` (optional bool, default `True`): First coerce the elements into this ring.
- `side`, optional string, one of `"twosided"` (default), `"left"`, `"right"`: determines whether the resulting ideal is twosided, a left ideal or a right ideal.

EXAMPLES:

```
sage: MS = MatrixSpace(QQ,2,2)
sage: isinstance(MS,Ring)
False
sage: MS in Rings()
True
sage: MS.ideal(2)
Twosided Ideal
(
  [2 0]
  [0 2]
)
 of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
sage: MS.ideal([MS.0,MS.1],side='right')
Right Ideal
(
  [1 0]
```

```
  [0 0],

  [0 1]
  [0 0]
)
 of Full MatrixSpace of 2 by 2 dense matrices over Rational Field
```

**ideal_monoid**()

The monoid of the ideals of this ring.

NOTE:

The code is copied from the base class of rings. This is since there are rings that do not inherit from that class, such as matrix algebras. See trac ticket #7797.

EXAMPLES:

```
sage: MS = MatrixSpace(QQ,2,2)
sage: isinstance(MS,Ring)
False
sage: MS in Rings()
True
sage: MS.ideal_monoid()
Monoid of ideals of Full MatrixSpace of 2 by 2 dense matrices
over Rational Field
```

Note that the monoid is cached:

```
sage: MS.ideal_monoid() is MS.ideal_monoid()
True
```

**is_ring**()

Return True, since this in an object of the category of rings.

EXAMPLES:

```
sage: Parent(QQ,category=Rings()).is_ring()
True
```

**is_zero**()

Return `True` if this is the zero ring.

EXAMPLES:

```
sage: Integers(1).is_zero()
True
sage: Integers(2).is_zero()
False
sage: QQ.is_zero()
False
sage: R.<x> = ZZ[]
sage: R.quo(1).is_zero()
True
sage: R.<x> = GF(101)[]
sage: R.quo(77).is_zero()
True
sage: R.quo(x^2+1).is_zero()
False
```

**quo** (*I*, *names=None*)

Quotient of a ring by a two-sided ideal.

NOTE:

This is a synonym for `quotient()`.

EXAMPLES:

```
sage: MS = MatrixSpace(QQ,2)
sage: I = MS*MS.gens()*MS
```

`MS` is not an instance of `Ring`.

However it is an instance of the parent class of the category of rings. The quotient method is inherited from there:

```
sage: isinstance(MS,sage.rings.ring.Ring)
False
sage: isinstance(MS,Rings().parent_class)
True
sage: MS.quo(I,names = ['a','b','c','d'])
Quotient of Full MatrixSpace of 2 by 2 dense matrices over Rational Field␣
↪by the ideal
(
  [1 0]
  [0 0],

  [0 1]
  [0 0],

  [0 0]
  [1 0],

  [0 0]
  [0 1]
)
```

**quotient** (*I*, *names=None*)

Quotient of a ring by a two-sided ideal.

INPUT:

- `I`: A twosided ideal of this ring.
- `names`: a list of strings to be used as names for the variables in the quotient ring.

EXAMPLES:

Usually, a ring inherits a method `sage.rings.ring.Ring.quotient()`. So, we need a bit of effort to make the following example work with the category framework:

```
sage: F.<x,y,z> = FreeAlgebra(QQ)
sage: from sage.rings.noncommutative_ideals import Ideal_nc
sage: from itertools import product
sage: class PowerIdeal(Ideal_nc):
....:  def __init__(self, R, n):
....:      self._power = n
....:      Ideal_nc.__init__(self, R, [R.prod(m) for m in product(R.
↪gens(), repeat=n)])
....:  def reduce(self, x):
....:      R = self.ring()
....:      return add([c*R(m) for m,c in x if len(m) < self._power], R(0))
```

```
....:
sage: I = PowerIdeal(F,3)
sage: Q = Rings().parent_class.quotient(F, I); Q
Quotient of Free Algebra on 3 generators (x, y, z) over Rational Field by␣
↪the ideal (x^3, x^2*y, x^2*z, x*y*x, x*y^2, x*y*z, x*z*x, x*z*y, x*z^2,␣
↪y*x^2, y*x*y, y*x*z, y^2*x, y^3, y^2*z, y*z*x, y*z*y, y*z^2, z*x^2,␣
↪z*x*y, z*x*z, z*y*x, z*y^2, z*y*z, z^2*x, z^2*y, z^3)
sage: Q.0
xbar
sage: Q.1
ybar
sage: Q.2
zbar
sage: Q.0*Q.1
xbar*ybar
sage: Q.0*Q.1*Q.0
0
```

**quotient_ring**(*I*, *names=None*)

Quotient of a ring by a two-sided ideal.

NOTE:

This is a synonyme for `quotient()`.

EXAMPLES:

```
sage: MS = MatrixSpace(QQ,2)
sage: I = MS*MS.gens()*MS
```

`MS` is not an instance of `Ring`, but it is an instance of the parent class of the category of rings. The quotient method is inherited from there:

```
sage: isinstance(MS,sage.rings.ring.Ring)
False
sage: isinstance(MS,Rings().parent_class)
True
sage: MS.quotient_ring(I,names = ['a','b','c','d'])
Quotient of Full MatrixSpace of 2 by 2 dense matrices over Rational Field␣
↪by the ideal
(
  [1 0]
  [0 0],

  [0 1]
  [0 0],

  [0 0]
  [1 0],

  [0 0]
  [0 1]
)
```

**class SubcategoryMethods**

**Division**()

Return the full subcategory of the division objects of `self`.

A ring satisfies the *division axiom* if all non-zero elements have multiplicative inverses.

**Note:** This could be generalized to `MagmasAndAdditiveMagmas.Distributive.AdditiveUnital`.

EXAMPLES:

```
sage: Rings().Division()
Category of division rings
sage: Rings().Commutative().Division()
Category of fields
```

**NoZeroDivisors**()

Return the full subcategory of the objects of `self` having no nonzero zero divisors.

A *zero divisor* in a ring $R$ is an element $x \in R$ such that there exists a nonzero element $y \in R$ such that $x \cdot y = 0$ or $y \cdot x = 0$ (see Wikipedia article Zero_divisor).

EXAMPLES:

```
sage: Rings().NoZeroDivisors()
Category of domains
```

**Note:** This could be generalized to `MagmasAndAdditiveMagmas.Distributive.AdditiveUnital`.

## 3.124 Rngs

**class** sage.categories.rngs.**Rngs**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of rngs.

An *rng* $(S, +, *)$ is similar to a ring but not necessarily unital. In other words, it is a combination of a commutative additive group $(S, +)$ and a multiplicative semigroup $(S, *)$, where $*$ distributes over $+$.

EXAMPLES:

```
sage: C = Rngs(); C
Category of rngs
sage: sorted(C.super_categories(), key=str)
[Category of associative additive commutative additive associative additive␣
↪unital distributive magmas and additive magmas,
 Category of commutative additive groups]

sage: sorted(C.axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveInverse',
 'AdditiveUnital', 'Associative', 'Distributive']

sage: C is (CommutativeAdditiveGroups() & Semigroups()).Distributive()
True
sage: C.Unital()
Category of rings
```

**Unital**
alias of *Rings*

## 3.125 R-trivial semigroups

**class** sage.categories.r_trivial_semigroups.**RTrivialSemigroups**(*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**Commutative_extra_super_categories**()
Implement the fact that a commutative $R$-trivial semigroup is $J$-trivial.

EXAMPLES:

```
sage: Semigroups().RTrivial().Commutative_extra_super_categories()
[Category of j trivial semigroups]
```

**extra_super_categories**()
Implement the fact that a $R$-trivial semigroup is $H$-trivial.

EXAMPLES:

```
sage: Semigroups().RTrivial().extra_super_categories()
[Category of h trivial semigroups]
```

## 3.126 Schemes

**class** sage.categories.schemes.**Schemes**(*s=None*)
Bases: *sage.categories.category.Category*

The category of all schemes.

EXAMPLES:

```
sage: Schemes()
Category of schemes
```

`Schemes` can also be used to construct the category of schemes over a given base:

```
sage: Schemes(Spec(ZZ))
Category of schemes over Integer Ring

sage: Schemes(ZZ)
Category of schemes over Integer Ring
```

**Todo:** Make `Schemes()` a singleton category (and remove *Schemes* from the workaround in `category_types.Category_over_base._test_category_over_bases()`).

This is currently incompatible with the dispatching below.

**super_categories**()
EXAMPLES:

```
sage: Schemes().super_categories()
[Category of sets]
```

**class** sage.categories.schemes.**Schemes_over_base**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base*

The category of schemes over a given base scheme.

EXAMPLES:

```
sage: Schemes(Spec(ZZ))
Category of schemes over Integer Ring
```

**base_scheme**()

EXAMPLES:

```
sage: Schemes(Spec(ZZ)).base_scheme()
Spectrum of Integer Ring
```

**super_categories**()

EXAMPLES:

```
sage: Schemes(Spec(ZZ)).super_categories()
[Category of schemes]
```

## 3.127 Semigroups

**class** sage.categories.semigroups.**Semigroups**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of (multiplicative) semigroups.

A *semigroup* is an associative *magma*, that is a set endowed with a multiplicative binary operation $*$ which is associative (see Wikipedia article Semigroup).

The operation $*$ is not required to have a neutral element. A semigroup for which such an element exists is a *monoid*.

EXAMPLES:

```
sage: C = Semigroups(); C
Category of semigroups
sage: C.super_categories()
[Category of magmas]
sage: C.all_super_categories()
[Category of semigroups, Category of magmas,
 Category of sets, Category of sets with partial maps, Category of objects]
sage: C.axioms()
frozenset({'Associative'})
sage: C.example()
An example of a semigroup: the left zero semigroup
```

**class Algebras**(*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**algebra_generators**()

The generators of this algebra, as per *MagmaticAlgebras.ParentMethods.algebra_generators()*.

They correspond to the generators of the semigroup.

EXAMPLES:

```
sage: M = FiniteSemigroups().example(); M
An example of a finite semigroup:
the left regular band generated by ('a', 'b', 'c', 'd')
sage: M.semigroup_generators()
Family ('a', 'b', 'c', 'd')
sage: M.algebra(ZZ).algebra_generators()
Finite family {0: B['a'], 1: B['b'], 2: B['c'], 3: B['d']}
```

**gen**(*i=0*)

Return the i-th generator of self.

EXAMPLES:

```
sage: A = GL(3, GF(7)).algebra(ZZ)
sage: A.gen(0)
[3 0 0]
[0 1 0]
[0 0 1]
```

**gens**()

Return the generators of self.

EXAMPLES:

```
sage: a, b = SL2Z.algebra(ZZ).gens(); a, b
([ 0 -1]
 [ 1  0],
 [1 1]
 [0 1])
sage: 2*a + b
2*[ 0 -1]
  [ 1  0]
+
[1 1]
[0 1]
```

**ngens**()

Return the number of generators of self.

EXAMPLES:

```
sage: SL2Z.algebra(ZZ).ngens()
2
sage: DihedralGroup(4).algebra(RR).ngens()
2
```

**product_on_basis**(*g1, g2*)

Product, on basis elements, as per *MagmaticAlgebras.WithBasis.ParentMethods.product_on_basis()*.

The product of two basis elements is induced by the product of the corresponding elements of the group.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(); S
An example of a finite semigroup: the left regular band generated by (
↪'a', 'b', 'c', 'd')
sage: A = S.algebra(QQ)
sage: a,b,c,d = A.algebra_generators()
sage: a * b + b * d * c * d
B['ab'] + B['bdc']
```

**regular_representation**(*side='left'*)

Return the regular representation of `self`.

INPUT:

- `side` – (default: `"left"`) whether this is the `"left"` or `"right"` regular representation

EXAMPLES:

```
sage: G = groups.permutation.Dihedral(4)
sage: A = G.algebra(QQ)
sage: V = A.regular_representation()
sage: V == G.regular_representation(QQ)
True
```

**trivial_representation**(*side='twosided'*)

Return the trivial representation of `self`.

INPUT:

- `side` – ignored

EXAMPLES:

```
sage: G = groups.permutation.Dihedral(4)
sage: A = G.algebra(QQ)
sage: V = A.trivial_representation()
sage: V == G.trivial_representation(QQ)
True
```

**extra_super_categories**()

Implement the fact that the algebra of a semigroup is indeed a (not necessarily unital) algebra.

EXAMPLES:

```
sage: Semigroups().Algebras(QQ).extra_super_categories()
[Category of semigroups]
sage: Semigroups().Algebras(QQ).super_categories()
[Category of associative algebras over Rational Field,
 Category of magma algebras over Rational Field]
```

**Aperiodic**

alias of `AperiodicSemigroups`

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**extra_super_categories**()

Implement the fact that a Cartesian product of semigroups is a semigroup.

EXAMPLES:

```
sage: Semigroups().CartesianProducts().extra_super_categories()
[Category of semigroups]
```

```
sage: Semigroups().CartesianProducts().super_categories()
[Category of semigroups, Category of Cartesian products of magmas]
```

**class ElementMethods**

**Finite**
alias of *FiniteSemigroups*

**FinitelyGeneratedAsMagma**
alias of `FinitelyGeneratedSemigroups`

**HTrivial**
alias of `HTrivialSemigroups`

**JTrivial**
alias of `JTrivialSemigroups`

**LTrivial**
alias of `LTrivialSemigroups`

**class ParentMethods**

**cayley_graph**(*side='right'*, *simple=False*, *elements=None*, *generators=None*, *connecting_set=None*)
Return the Cayley graph for this finite semigroup.

INPUT:
- `side` – "left", "right", or "twosided": the side on which the generators act (default:"right")
- `simple` – boolean (default:False): if True, returns a simple graph (no loops, no labels, no multiple edges)
- `generators` – a list, tuple, or family of elements of `self` (default: `self.semigroup_generators()`)
- `connecting_set` – alias for `generators`; deprecated
- `elements` – a list (or iterable) of elements of `self`

OUTPUT:
- `DiGraph`

EXAMPLES:

We start with the (right) Cayley graphs of some classical groups:

```
sage: D4 = DihedralGroup(4); D4
Dihedral group of order 8 as a permutation group
sage: G = D4.cayley_graph()
sage: show(G, color_by_label=True, edge_labels=True)
sage: A5 = AlternatingGroup(5); A5
Alternating group of order 5!/2 as a permutation group
sage: G = A5.cayley_graph()
sage: G.show3d(color_by_label=True, edge_size=0.01, edge_size2=0.02,␣
↪vertex_size=0.03)
sage: G.show3d(vertex_size=0.03, edge_size=0.01, edge_size2=0.02, vertex_
↪colors={(1,1,1):G.vertices()}, bgcolor=(0,0,0), color_by_label=True,␣
↪xres=700, yres=700, iterations=200) # long time (less than a minute)
sage: G.num_edges()
120

sage: w = WeylGroup(['A',3])
sage: d = w.cayley_graph(); d
Digraph on 24 vertices
sage: d.show3d(color_by_label=True, edge_size=0.01, vertex_size=0.03)
```

Alternative generators may be specified:

```
sage: G = A5.cayley_graph(generators=[A5.gens()[0]])
sage: G.num_edges()
60
sage: g=PermutationGroup([(i+1,j+1) for i in range(5) for j in range(5)␣
↪if j!=i])
sage: g.cayley_graph(generators=[(1,2),(2,3)])
Digraph on 120 vertices
```

If elements is specified, then only the subgraph induced and those elements is returned. Here we use it to display the Cayley graph of the free monoid truncated on the elements of length at most 3:

```
sage: M = Monoids().example(); M
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
sage: elements = [ M.prod(w) for w in sum((list(Words(M.semigroup_
↪generators(),k)) for k in range(4)),[]) ]
sage: G = M.cayley_graph(elements = elements)
sage: G.num_verts(), G.num_edges()
(85, 84)
sage: G.show3d(color_by_label=True, edge_size=0.001, vertex_size=0.01)
```

We now illustrate the side and simple options on a semigroup:

```
sage: S = FiniteSemigroups().example(alphabet=('a','b'))
sage: g = S.cayley_graph(simple=True)
sage: g.vertices()
['a', 'ab', 'b', 'ba']
sage: g.edges()
[('a', 'ab', None), ('b', 'ba', None)]
```

```
sage: g = S.cayley_graph(side="left", simple=True)
sage: g.vertices()
['a', 'ab', 'b', 'ba']
sage: g.edges()
[('a', 'ba', None), ('ab', 'ba', None), ('b', 'ab', None),
('ba', 'ab', None)]
```

```
sage: g = S.cayley_graph(side="twosided", simple=True)
sage: g.vertices()
['a', 'ab', 'b', 'ba']
sage: g.edges()
[('a', 'ab', None), ('a', 'ba', None), ('ab', 'ba', None),
('b', 'ab', None), ('b', 'ba', None), ('ba', 'ab', None)]
```

```
sage: g = S.cayley_graph(side="twosided")
sage: g.vertices()
['a', 'ab', 'b', 'ba']
sage: g.edges()
[('a', 'a', (0, 'left')), ('a', 'a', (0, 'right')), ('a', 'ab', (1, 'right
↪')), ('a', 'ba', (1, 'left')), ('ab', 'ab', (0, 'left')), ('ab', 'ab',␣
↪(0, 'right')), ('ab', 'ab', (1, 'right')), ('ab', 'ba', (1, 'left')), (
↪'b', 'ab', (0, 'left')), ('b', 'b', (1, 'left')), ('b', 'b', (1, 'right
↪')), ('b', 'ba', (0, 'right')), ('ba', 'ab', (0, 'left')), ('ba', 'ba',␣
↪(0, 'right')), ('ba', 'ba', (1, 'left')), ('ba', 'ba', (1, 'right'))]
```

```
sage: s1 = SymmetricGroup(1); s = s1.cayley_graph(); s.vertices()
[()]
```

**Todo:**

- Add more options for constructing subgraphs of the Cayley graph, handling the standard use cases when exploring large/infinite semigroups (a predicate, generators of an ideal, a maximal length in term of the generators)
- Specify good default layout/plot/latex options in the graph
- Generalize to combinatorial modules with module generators / operators

AUTHORS:

- Bobby Moretti (2007-08-10)
- Robert Miller (2008-05-01): editing
- Nicolas M. Thiery (2008-12): extension to semigroups, `side`, `simple`, and `elements` options, ...

**magma_generators**()

An alias for *semigroup_generators()*.

EXAMPLES:

```
sage: S = Semigroups().example("free"); S
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c',
↪ 'd')
sage: S.magma_generators()
Family ('a', 'b', 'c', 'd')
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

**prod**(*args*)

Return the product of the list of elements `args` inside `self`.

EXAMPLES:

```
sage: S = Semigroups().example("free")
sage: S.prod([S('a'), S('b'), S('c')])
'abc'
sage: S.prod([])
Traceback (most recent call last):
...
AssertionError: Cannot compute an empty product in a semigroup
```

**regular_representation**(*base_ring=None*, *side='left'*)

Return the regular representation of `self` over `base_ring`.

- `side` – (default: `"left"`) whether this is the `"left"` or `"right"` regular representation

EXAMPLES:

```
sage: G = groups.permutation.Dihedral(4)
sage: G.regular_representation()
Left Regular Representation of Dihedral group of order 8
 as a permutation group over Integer Ring
```

**semigroup_generators**()

Return distinguished semigroup generators for `self`.

OUTPUT: a family

This method is optional.

EXAMPLES:

```
sage: S = Semigroups().example("free"); S
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c',
↪ 'd')
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

**subsemigroup**(*generators*, *one=None*, *category=None*)

Return the multiplicative subsemigroup generated by `generators`.

INPUT:

- `generators` – a finite family of elements of `self`, or a list, iterable, ... that can be converted into one (see `Family`).
- `one` – a unit for the subsemigroup, or `None`.
- `category` – a category

This implementation lazily constructs all the elements of the semigroup, and the right Cayley graph relations between them, and uses the latter as an automaton.

See `AutomaticSemigroup` for details.

EXAMPLES:

```
sage: R = IntegerModRing(15)
sage: M = R.subsemigroup([R(3),R(5)]); M
A subsemigroup of (Ring of integers modulo 15) with 2 generators
sage: M.list()
[3, 5, 9, 0, 10, 12, 6]
```

By default, $M$ is just in the category of subsemigroups:

```
sage: M in Semigroups().Subobjects()
True
```

In the following example, we specify that $M$ is a submonoid of the finite monoid $R$ (it shares the same unit), and a group by itself:

```
sage: M = R.subsemigroup([R(-1)],
....:     category=Monoids().Finite().Subobjects() & Groups()); M
A submonoid of (Ring of integers modulo 15) with 1 generators
sage: M.list()
[1, 14]
sage: M.one()
1
```

In the following example $M$ is a group; however its unit does not coincide with that of $R$, so $M$ is only a subsemigroup, and we need to specify its unit explicitly:

```
sage: M = R.subsemigroup([R(5)],
....:     category=Semigroups().Finite().Subobjects() & Groups()); M
Traceback (most recent call last):
...
ValueError: For a monoid which is just a subsemigroup, the unit should be␣
↪specified

sage: M = R.subsemigroup([R(5)], one=R(10),
....:     category=Semigroups().Finite().Subobjects() & Groups()); M
```

```
A subsemigroup of (Ring of integers modulo 15) with 1 generators
sage: M in Groups()
True
sage: M.list()
[10, 5]
sage: M.one()
10
```

**Todo:**

- Fix the failure in TESTS by providing a default implementation of `__invert__` for finite groups (or even finite monoids).
- Provide a default implementation of `one` for a finite monoid, so that we would not need to specify it explicitly?

**trivial_representation**(*base_ring=None*, *side='twosided'*)

Return the trivial representation of `self` over `base_ring`.

INPUT:

- `base_ring` – (optional) the base ring; the default is $\mathbf{Z}$
- `side` – ignored

EXAMPLES:

```
sage: G = groups.permutation.Dihedral(4)
sage: G.trivial_representation()
Trivial representation of Dihedral group of order 8
 as a permutation group over Integer Ring
```

**class Quotients**(*category*, **args*)

Bases: *sage.categories.quotients.QuotientsCategory*

**class ParentMethods**

**semigroup_generators**()

Return semigroup generators for `self` by retracting the semigroup generators of the ambient semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().Quotients().example().semigroup_
↪generators() # todo: not implemented
```

**example**()

Return an example of quotient of a semigroup, as per *Category.example()*.

EXAMPLES:

```
sage: Semigroups().Quotients().example()
An example of a (sub)quotient semigroup: a quotient of the left zero␣
↪semigroup
```

**RTrivial**

alias of `RTrivialSemigroups`

**class SubcategoryMethods**

**Aperiodic**()

Return the full subcategory of the aperiodic objects of `self`.

A (multiplicative) *semigroup* $S$ is *aperiodic* if for any element $s \in S$, the sequence $s, s^2, s^3, \ldots$ eventually stabilizes.

In terms of variety, this can be described by the equation $s^\omega s = s$.

EXAMPLES:

```
sage: Semigroups().Aperiodic()
Category of aperiodic semigroups
```

An aperiodic semigroup is $H$-trivial:

```
sage: Semigroups().Aperiodic().axioms()
frozenset({'Aperiodic', 'Associative', 'HTrivial'})
```

In the finite case, the two notions coincide:

```
sage: Semigroups().Aperiodic().Finite() is Semigroups().HTrivial().
↪Finite()
True
```

**See also:**

- Wikipedia article Aperiodic_semigroup
- *Semigroups.SubcategoryMethods.RTrivial*
- *Semigroups.SubcategoryMethods.LTrivial*
- *Semigroups.SubcategoryMethods.JTrivial*
- *Semigroups.SubcategoryMethods.Aperiodic*

**HTrivial**()

Return the full subcategory of the $H$-trivial objects of `self`.

Let $S$ be (multiplicative) *semigroup*. Two elements of $S$ are in the same $H$-class if they are in the same $L$-class and in the same $R$-class.

The semigroup $S$ is *$H$-trivial* if all its $H$-classes are trivial (that is of cardinality $1$).

EXAMPLES:

```
sage: C = Semigroups().HTrivial(); C
Category of h trivial semigroups
sage: Semigroups().HTrivial().Finite().example()
NotImplemented
```

**See also:**

- Wikipedia article Green's_relations
- *Semigroups.SubcategoryMethods.RTrivial*
- *Semigroups.SubcategoryMethods.LTrivial*
- *Semigroups.SubcategoryMethods.JTrivial*
- *Semigroups.SubcategoryMethods.Aperiodic*

**JTrivial**()

Return the full subcategory of the $J$-trivial objects of `self`.

Let $S$ be (multiplicative) *semigroup*. The *$J$-preorder* $\leq_J$ on $S$ is defined by:

$$x \leq_J y \qquad \Longleftrightarrow \qquad x \in SyS$$

The $J$-*classes* are the equivalence classes for the associated equivalence relation. The semigroup $S$ is $J$-*trivial* if all its $J$-classes are trivial (that is of cardinality 1), or equivalently if the $J$-preorder is in fact a partial order.

EXAMPLES:

```
sage: C = Semigroups().JTrivial(); C
Category of j trivial semigroups
```

A semigroup is $J$-trivial if and only if it is $L$-trivial and $R$-trivial:

```
sage: sorted(C.axioms())
['Associative', 'HTrivial', 'JTrivial', 'LTrivial', 'RTrivial']
sage: Semigroups().LTrivial().RTrivial()
Category of j trivial semigroups
```

For a commutative semigroup, all three axioms are equivalent:

```
sage: Semigroups().Commutative().LTrivial()
Category of commutative j trivial semigroups
sage: Semigroups().Commutative().RTrivial()
Category of commutative j trivial semigroups
```

**See also:**

- Wikipedia article Green's_relations
- *Semigroups.SubcategoryMethods.LTrivial*
- *Semigroups.SubcategoryMethods.RTrivial*
- *Semigroups.SubcategoryMethods.HTrivial*

**LTrivial**()

Return the full subcategory of the $L$-trivial objects of `self`.

Let $S$ be (multiplicative) *semigroup*. The $L$-*preorder* $\leq_L$ on $S$ is defined by:

$$x \leq_L y \qquad \iff \qquad x \in Sy$$

The $L$-*classes* are the equivalence classes for the associated equivalence relation. The semigroup $S$ is $L$-*trivial* if all its $L$-classes are trivial (that is of cardinality 1), or equivalently if the $L$-preorder is in fact a partial order.

EXAMPLES:

```
sage: C = Semigroups().LTrivial(); C
Category of l trivial semigroups
```

A $L$-trivial semigroup is $H$-trivial:

```
sage: sorted(C.axioms())
['Associative', 'HTrivial', 'LTrivial']
```

**See also:**

- Wikipedia article Green's_relations
- *Semigroups.SubcategoryMethods.RTrivial*
- *Semigroups.SubcategoryMethods.JTrivial*
- *Semigroups.SubcategoryMethods.HTrivial*

**RTrivial**()

Return the full subcategory of the $R$-trivial objects of `self`.

Let $S$ be (multiplicative) *semigroup*. The $R$*-preorder* $\leq_R$ on $S$ is defined by:

$$x \leq_R y \iff x \in yS$$

The $R$*-classes* are the equivalence classes for the associated equivalence relation. The semigroup $S$ is $R$*-trivial* if all its $R$-classes are trivial (that is of cardinality 1), or equivalently if the $R$-preorder is in fact a partial order.

EXAMPLES:

```
sage: C = Semigroups().RTrivial(); C
Category of r trivial semigroups
```

An $R$-trivial semigroup is $H$-trivial:

```
sage: sorted(C.axioms())
['Associative', 'HTrivial', 'RTrivial']
```

**See also:**

- Wikipedia article Green's_relations
- *Semigroups.SubcategoryMethods.LTrivial*
- *Semigroups.SubcategoryMethods.JTrivial*
- *Semigroups.SubcategoryMethods.HTrivial*

**class Subquotients**(*category*, **args*)

Bases: *sage.categories.subquotients.SubquotientsCategory*

The category of subquotient semi-groups.

EXAMPLES:

```
sage: Semigroups().Subquotients().all_super_categories()
[Category of subquotients of semigroups,
 Category of semigroups,
 Category of subquotients of magmas,
 Category of magmas,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]

[Category of subquotients of semigroups,
 Category of semigroups,
 Category of subquotients of magmas,
 Category of magmas,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

**example**()

Returns an example of subquotient of a semigroup, as per *Category.example()*.

EXAMPLES:

```
sage: Semigroups().Subquotients().example()
An example of a (sub)quotient semigroup: a quotient of the left zero␣
↪semigroup
```

**Unital**
alias of *Monoids*

**example** (*choice='leftzero'*, ***kwds*)
Returns an example of a semigroup, as per *Category.example()*.

INPUT:

- choice – str (default: 'leftzero'). Can be either 'leftzero' for the left zero semigroup, or 'free' for the free semigroup.
- **kwds – keyword arguments passed onto the constructor for the chosen semigroup.

EXAMPLES:

```
sage: Semigroups().example(choice='leftzero')
An example of a semigroup: the left zero semigroup
sage: Semigroups().example(choice='free')
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c', 'd
↪')
sage: Semigroups().example(choice='free', alphabet=('a','b'))
An example of a semigroup: the free semigroup generated by ('a', 'b')
```

## 3.128 Semirngs

**class** sage.categories.semirings.**Semirings** (*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of semirings.

A semiring $(S, +, *)$ is similar to a ring, but without the requirement that each element must have an additive inverse. In other words, it is a combination of a commutative additive monoid $(S, +)$ and a multiplicative monoid $(S, *)$, where $*$ distributes over $+$.

**See also:**

Wikipedia article Semiring

EXAMPLES:

```
sage: Semirings()
Category of semirings
sage: Semirings().super_categories()
[Category of associative additive commutative additive associative additive␣
↪unital distributive magmas and additive magmas,
 Category of monoids]

sage: sorted(Semirings().axioms())
['AdditiveAssociative', 'AdditiveCommutative', 'AdditiveUnital', 'Associative',
↪'Distributive', 'Unital']

sage: Semirings() is (CommutativeAdditiveMonoids() & Monoids()).Distributive()
True

sage: Semirings().AdditiveInverse()
Category of rings
```

## 3.129 Semisimple Algebras

**class** sage.categories.semisimple_algebras.**SemisimpleAlgebras**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base_ring*

The category of semisimple algebras over a given base ring.

EXAMPLES:

```
sage: from sage.categories.semisimple_algebras import SemisimpleAlgebras
sage: C = SemisimpleAlgebras(QQ); C
Category of semisimple algebras over Rational Field
```

This category is best constructed as:

```
sage: D = Algebras(QQ).Semisimple(); D
Category of semisimple algebras over Rational Field
sage: D is C
True

sage: C.super_categories()
[Category of algebras over Rational Field]
```

Typically, finite group algebras are semisimple:

```
sage: DihedralGroup(5).algebra(QQ) in SemisimpleAlgebras
True
```

Unless the characteristic of the field divides the order of the group:

```
sage: DihedralGroup(5).algebra(IntegerModRing(5)) in SemisimpleAlgebras
False

sage: DihedralGroup(5).algebra(IntegerModRing(7)) in SemisimpleAlgebras
True
```

**See also:**

Wikipedia article Semisimple_algebra

**class FiniteDimensional**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**WithBasis**

alias of FiniteDimensionalSemisimpleAlgebrasWithBasis

**class ParentMethods**

**radical_basis**(***keywords*)

Return a basis of the Jacobson radical of this algebra.

- keywords – for compatibility; ignored.

OUTPUT: the empty list since this algebra is semisimple.

EXAMPLES:

```
sage: A = SymmetricGroup(4).algebra(QQ)
sage: A.radical_basis()
()
```

**super_categories**()

EXAMPLES:

```
sage: Algebras(QQ).Semisimple().super_categories()
[Category of algebras over Rational Field]
```

## 3.130 Sets

**exception** sage.categories.sets_cat.**EmptySetError**

Bases: exceptions.ValueError

Exception raised when some operation can't be performed on the empty set.

EXAMPLES:

```
sage: def first_element(st):
....:  if not st: raise EmptySetError("no elements")
....:  else: return st[0]
sage: first_element(Set((1,2,3)))
1
sage: first_element(Set([]))
Traceback (most recent call last):
...
EmptySetError: no elements
```

**class** sage.categories.sets_cat.**Sets**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of sets.

The base category for collections of elements with = (equality).

This is also the category whose objects are all parents.

EXAMPLES:

```
sage: Sets()
Category of sets
sage: Sets().super_categories()
[Category of sets with partial maps]
sage: Sets().all_super_categories()
[Category of sets, Category of sets with partial maps, Category of objects]
```

Let us consider an example of set:

```
sage: P = Sets().example("inherits")
sage: P
Set of prime numbers
```

See `P??` for the code.

P is in the category of sets:

```
sage: P.category()
Category of sets
```

and therefore gets its methods from the following classes:

```
sage: for cl in P.__class__.mro(): print(cl)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits_with_category'>
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits'>
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Abstract'>
<class 'sage.structure.unique_representation.UniqueRepresentation'>
<class 'sage.structure.unique_representation.CachedRepresentation'>
<type 'sage.misc.fast_methods.WithEqualityById'>
<type 'sage.structure.parent.Parent'>
<type 'sage.structure.category_object.CategoryObject'>
<type 'sage.structure.sage_object.SageObject'>
<class 'sage.categories.sets_cat.Sets.parent_class'>
<class 'sage.categories.sets_with_partial_maps.SetsWithPartialMaps.parent_class'>
<class 'sage.categories.objects.Objects.parent_class'>
<... 'object'>
```

We run some generic checks on P:

```
sage: TestSuite(P).run(verbose=True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

Now, we manipulate some elements of P:

```
sage: P.an_element()
47
sage: x = P(3)
sage: x.parent()
Set of prime numbers
sage: x in P, 4 in P
(True, False)
sage: x.is_prime()
True
```

They get their methods from the following classes:

```
sage: for cl in x.__class__.mro(): print(cl)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits_with_category.
↪element_class'>
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits.Element'>
<type 'sage.rings.integer.IntegerWrapper'>
```

```
<type 'sage.rings.integer.Integer'>
<type 'sage.structure.element.EuclideanDomainElement'>
<type 'sage.structure.element.PrincipalIdealDomainElement'>
<type 'sage.structure.element.DedekindDomainElement'>
<type 'sage.structure.element.IntegralDomainElement'>
<type 'sage.structure.element.CommutativeRingElement'>
<type 'sage.structure.element.RingElement'>
<type 'sage.structure.element.ModuleElement'>
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Abstract.Element'>
<type 'sage.structure.element.Element'>
<type 'sage.structure.sage_object.SageObject'>
<class 'sage.categories.sets_cat.Sets.element_class'>
<class 'sage.categories.sets_with_partial_maps.SetsWithPartialMaps.element_class'>
<class 'sage.categories.objects.Objects.element_class'>
<... 'object'>
```

FIXME: Objects.element_class is not very meaningful . . .

**class Algebras** (*category*, **args*)

Bases: *sage.categories.algebra_functor.AlgebrasCategory*

**class ParentMethods**

**construction** ()

Return the functorial construction of `self`.

EXAMPLES:

```
sage: A = GroupAlgebra(KleinFourGroup(), QQ)
sage: A.construction()
(GroupAlgebraFunctor, Rational Field)
```

**extra_super_categories** ()

EXAMPLES:

```
sage: Sets().Algebras(ZZ).super_categories()
[Category of modules with basis over Integer Ring]

sage: Sets().Algebras(QQ).extra_super_categories()
[Category of vector spaces with basis over Rational Field]

sage: Sets().example().algebra(ZZ).categories()
[Category of set algebras over Integer Ring,
 Category of modules with basis over Integer Ring,
 ...
 Category of objects]
```

**class CartesianProducts** (*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example()
sage: C
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the non negative integers,
 An example of a finite enumerated set: {1,2,3})
sage: C.category()
```

```
Category of Cartesian products of sets
sage: C.categories()
[Category of Cartesian products of sets, Category of sets,
 Category of sets with partial maps,
 Category of objects]
sage: TestSuite(C).run()
```

**class ElementMethods**

**cartesian_factors**()

Return the Cartesian factors of `self`.

EXAMPLES:

```
sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.__custom_name = "F"
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.__custom_name = "G"
sage: H = CombinatorialFreeModule(ZZ, [4,7]); H.__custom_name = "H"
sage: S = cartesian_product([F, G, H])
sage: x = S.monomial((0,4)) + 2 * S.monomial((0,5)) + 3 * S.
↪monomial((1,6)) + 4 * S.monomial((2,4)) + 5 * S.monomial((2,7))
sage: x.cartesian_factors()
(B[4] + 2*B[5], 3*B[6], 4*B[4] + 5*B[7])
sage: [s.parent() for s in x.cartesian_factors()]
[F, G, H]
sage: S.zero().cartesian_factors()
(0, 0, 0)
sage: [s.parent() for s in S.zero().cartesian_factors()]
[F, G, H]
```

**cartesian_projection**(*i*)

Return the projection of `self` onto the $i$-th factor of the Cartesian product.

INPUT:

- `i` – the index of a factor of the Cartesian product

EXAMPLES:

```
sage: F = CombinatorialFreeModule(ZZ, [4,5]); F.__custom_name = "F"
sage: G = CombinatorialFreeModule(ZZ, [4,6]); G.__custom_name = "G"
sage: S = cartesian_product([F, G])
sage: x = S.monomial((0,4)) + 2 * S.monomial((0,5)) + 3 * S.
↪monomial((1,6))
sage: x.cartesian_projection(0)
B[4] + 2*B[5]
sage: x.cartesian_projection(1)
3*B[6]
```

**summand_projection**(**args, **kwds*)

Deprecated: Use *cartesian_projection()* instead. See trac ticket #10963 for details.

**summand_split**(**args, **kwds*)

Deprecated: Use *cartesian_factors()* instead. See trac ticket #10963 for details.

**class ParentMethods**

**an_element**()

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the non negative integers,
 An example of a finite enumerated set: {1,2,3})
sage: C.an_element()
(47, 42, 1)
```

**cardinality**()

Return the cardinality of self.

EXAMPLES:

```
sage: E = FiniteEnumeratedSet([1,2,3])
sage: C = cartesian_product([E,SymmetricGroup(4)])
sage: C.cardinality()
72

sage: E = FiniteEnumeratedSet([])
sage: C = cartesian_product([E, ZZ, QQ])
sage: C.cardinality()
0

sage: C = cartesian_product([ZZ, QQ])
sage: C.cardinality()
+Infinity

sage: cartesian_product([GF(5), Permutations(10)]).cardinality()
18144000
sage: cartesian_product([GF(71)]*20).cardinality() == 71**20
True
```

**cartesian_factors**()

Return the Cartesian factors of `self`.

EXAMPLES:

```
sage: cartesian_product([QQ, ZZ, ZZ]).cartesian_factors()
(Rational Field, Integer Ring, Integer Ring)
```

**cartesian_projection**($i$)

Return the natural projection onto the $i$-th Cartesian factor of `self`.

INPUT:

- `i` – the index of a Cartesian factor of `self`

EXAMPLES:

```
sage: C = Sets().CartesianProducts().example(); C
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the non negative integers,
 An example of a finite enumerated set: {1,2,3})
sage: x = C.an_element(); x
(47, 42, 1)
sage: pi = C.cartesian_projection(1)
sage: pi(x)
42
```

**is_empty**()

Return whether this set is empty.

EXAMPLES:

```
sage: S1 = FiniteEnumeratedSet([1,2,3])
sage: S2 = Set([])
sage: cartesian_product([S1,ZZ]).is_empty()
False
sage: cartesian_product([S1,S2,S1]).is_empty()
True
```

**is_finite**()

Return whether this set is finite.

EXAMPLES:

```
sage: E = FiniteEnumeratedSet([1,2,3])
sage: C = cartesian_product([E, SymmetricGroup(4)])
sage: C.is_finite()
True

sage: cartesian_product([ZZ,ZZ]).is_finite()
False
sage: cartesian_product([ZZ, Set(), ZZ]).is_finite()
True
```

**random_element**(**args*)

Return a random element of this Cartesian product.

The extra arguments are passed down to each of the factors of the Cartesian product.

EXAMPLES:

```
sage: C = cartesian_product([Permutations(10)]*5)
sage: C.random_element()           # random
([2, 9, 4, 7, 1, 8, 6, 10, 5, 3],
 [8, 6, 5, 7, 1, 4, 9, 3, 10, 2],
 [5, 10, 3, 8, 2, 9, 1, 4, 7, 6],
 [9, 6, 10, 3, 2, 1, 5, 8, 7, 4],
 [8, 5, 2, 9, 10, 3, 7, 1, 4, 6])

sage: C = cartesian_product([ZZ]*10)
sage: c1 = C.random_element()
sage: c1                        # random
(3, 1, 4, 1, 1, -3, 0, -4, -17, 2)
sage: c2 = C.random_element(4,7)
sage: c2                        # random
(6, 5, 6, 4, 5, 6, 6, 4, 5, 5)
sage: all(4 <= i < 7 for i in c2)
True
```

**example**()

EXAMPLES:

```
sage: Sets().CartesianProducts().example()
The Cartesian product of (Set of prime numbers (basic implementation),
 An example of an infinite enumerated set: the non negative integers,
 An example of a finite enumerated set: {1,2,3})
```

**extra_super_categories**()

A Cartesian product of sets is a set.

EXAMPLES:

```
sage: Sets().CartesianProducts().extra_super_categories()
[Category of sets]
sage: Sets().CartesianProducts().super_categories()
[Category of sets]
```

**class ElementMethods**

**cartesian_product**(**elements*)

Return the Cartesian product of its arguments, as an element of the Cartesian product of the parents of those elements.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ)
sage: A = C.example()
sage: (a,b,c) = A.algebra_generators()
sage: a.cartesian_product(b, c)
B[(0, word: a)] + B[(1, word: b)] + B[(2, word: c)]
```

FIXME: is this a policy that we want to enforce on all parents?

**Enumerated**

alias of *EnumeratedSets*

**Facade**

alias of `FacadeSets`

**Finite**

alias of *FiniteSets*

**class Infinite**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class ParentMethods**

**cardinality**()

Count the elements of the enumerated set.

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN.cardinality()
+Infinity
```

**is_empty**()

Return whether this set is empty.

Since this set is infinite this always returns `False`.

EXAMPLES:

```
sage: C = InfiniteEnumeratedSets().example()
sage: C.is_empty()
False
```

**is_finite**()

Return whether this set is finite.

Since this set is infinite this always returns False.

EXAMPLES:

```
sage: C = InfiniteEnumeratedSets().example()
sage: C.is_finite()
False
```

**class IsomorphicObjects**(*category*, **args*)

Bases: *sage.categories.isomorphic_objects.IsomorphicObjectsCategory*

A category for isomorphic objects of sets.

EXAMPLES:

```
sage: Sets().IsomorphicObjects()
Category of isomorphic objects of sets
sage: Sets().IsomorphicObjects().all_super_categories()
[Category of isomorphic objects of sets,
 Category of subobjects of sets, Category of quotients of sets,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

**class ParentMethods**

**Metric**

alias of MetricSpaces

**class MorphismMethods**

**is_injective**()

Return whether this map is injective.

EXAMPLES:

```
sage: f = ZZ.hom(GF(3)); f
Natural morphism:
  From: Integer Ring
  To:   Finite Field of size 3
sage: f.is_injective()
False
```

Note that many maps do not implement this method:

```
sage: R.<x> = ZZ[]
sage: f = R.hom([x])
sage: f.is_injective()
Traceback (most recent call last):
...
NotImplementedError
```

**class ParentMethods**

**CartesianProduct**

alias of *CartesianProduct*

**algebra** (*base_ring*, *category=None*, ***kwds*)

Return the algebra of `self` over `base_ring`.

INPUT:

- `self` – a parent $S$
- `base_ring` – a ring $K$
- `category` – a super category of the category of $S$, or `None`

This returns the space of formal linear combinations of elements of $G$ with coefficients in $R$, endowed with whatever structure can be induced from that of $S$. See the documentation of `sage.categories.algebra_functor` for details.

EXAMPLES:

If $S$ is a `group`, the result is its group algebra $KS$:

```
sage: S = DihedralGroup(4); S
    Dihedral group of order 8 as a permutation group
sage: A = S.algebra(QQ); A
Algebra of Dihedral group of order 8 as a permutation group
        over Rational Field
sage: A.category()
Category of finite group algebras over Rational Field
sage: a = A.an_element(); a
() + 4*(1,2,3,4) + 2*(1,4)(2,3)
```

This space is endowed with an algebra structure, obtained by extending by bilinearity the multiplication of $G$ to a multiplication on $RG$:

```
sage: a * a
5*() + 8*(2,4) + 8*(1,2,3,4) + 8*(1,3) + 16*(1,3)(2,4) + 4*(1,4)(2,3)
```

If $S$ is a `monoid`, the result is its monoid algebra $KS$:

```
sage: S = Monoids().example(); S
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ); A
Algebra of An example of a monoid: the free monoid generated by ('a', 'b',
↪ 'c', 'd')
        over Rational Field
sage: A.category()
Category of monoid algebras over Rational Field
```

Similarly, we can construct algebras for additive magmas, monoids, and groups.

One may specify for which category one takes the algebra; here we build the algebra of the additive group $GF_3$:

```
sage: from sage.categories.additive_groups import AdditiveGroups
sage: S = GF(7)
sage: A = S.algebra(QQ, category=AdditiveGroups()); A
Algebra of Finite Field of size 7 over Rational Field
sage: A.category()
Category of finite dimensional additive group algebras
         over Rational Field

sage: a = A(S(1))
sage: a
1
```

```
sage: 1 + a * a * a
0 + 3
```

Note that the category keyword needs to be fed with the structure on $S$ to be used, not the induced structure on the result.

**an_element**()

Return a (preferably typical) element of this parent.

This is used both for illustration and testing purposes. If the set self is empty, *an_element()* should raise the exception *EmptySetError*.

This default implementation calls _an_element_() and caches the result. Any parent should implement either *an_element()* or _an_element_().

EXAMPLES:

```
sage: CDF.an_element()
1.0*I
sage: ZZ[['t']].an_element()
t
```

**cartesian_product**(**parents*, ***kwargs*)

Return the Cartesian product of the parents.

INPUT:

- parents – a list (or other iterable) of parents.
- category – (default: None) the category the Cartesian product belongs to. If None is passed, then category_from_parents() is used to determine the category.
- extra_category – (default: None) a category that is added to the Cartesian product in addition to the categories obtained from the parents.
- other keyword arguments will passed on to the class used for this Cartesian product (see also CartesianProduct).

OUTPUT:

The Cartesian product.

EXAMPLES:

```
sage: C = AlgebrasWithBasis(QQ)
sage: A = C.example(); A.rename("A")
sage: A.cartesian_product(A,A)
A (+) A (+) A
sage: ZZ.cartesian_product(GF(2), FiniteEnumeratedSet([1,2,3]))
The Cartesian product of (Integer Ring, Finite Field of size 2, {1, 2, 3})

sage: C = ZZ.cartesian_product(A); C
The Cartesian product of (Integer Ring, A)
```

```
sage: cartesian_product([ZZ, ZZ], category=Sets()).category()
Category of sets
sage: cartesian_product([ZZ, ZZ]).category()
Join of
Category of Cartesian products of commutative rings and
Category of Cartesian products of enumerated sets
sage: cartesian_product([ZZ, ZZ], extra_category=Posets()).category()
Join of
Category of Cartesian products of commutative rings and
```

```
Category of posets and
Category of Cartesian products of enumerated sets
```

**construction**()

Return a pair (functor, parent) such that functor(parent) returns self. If self does not have a functorial construction, return None.

EXAMPLES:

```
sage: QQ.construction()
(FractionField, Integer Ring)
sage: f, R = QQ['x'].construction()
sage: f
Poly[x]
sage: R
Rational Field
sage: f(R)
Univariate Polynomial Ring in x over Rational Field
```

**is_parent_of**(*element*)

Return whether self is the parent of element.

INPUT:

- element – any object

EXAMPLES:

```
sage: S = ZZ
sage: S.is_parent_of(1)
True
sage: S.is_parent_of(2/1)
False
```

This method differs from __contains__() because it does not attempt any coercion:

```
sage: 2/1 in S, S.is_parent_of(2/1)
(True, False)
sage: int(1) in S, S.is_parent_of(int(1))
(True, False)
```

**some_elements**()

Return a list (or iterable) of elements of self.

This is typically used for running generic tests (see TestSuite).

This default implementation calls *an_element()*.

EXAMPLES:

```
sage: S = Sets().example(); S
Set of prime numbers (basic implementation)
sage: S.an_element()
47
sage: S.some_elements()
[47]
sage: S = Set([])
sage: S.some_elements()
[]
```

This method should return an iterable, *not* an iterator.

**class Quotients**(*category*, **args*)
Bases: *sage.categories.quotients.QuotientsCategory*

A category for quotients of sets.

**See also:**

`Sets().Quotients()`

EXAMPLES:

```
sage: Sets().Quotients()
Category of quotients of sets
sage: Sets().Quotients().all_super_categories()
[Category of quotients of sets,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

**class ParentMethods**

**class Realizations**(*category*, **args*)
Bases: *sage.categories.realizations.RealizationsCategory*

**class ParentMethods**

**realization_of**()
Return the parent this is a realization of.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: In = A.In(); In
The subset algebra of {1, 2, 3} over Rational Field in the In basis
sage: In.realization_of()
The subset algebra of {1, 2, 3} over Rational Field
```

**class SubcategoryMethods**

**Algebras**(*base_ring*)
Return the category of objects constructed as algebras of objects of `self` over `base_ring`.

INPUT:

- `base_ring` – a ring

See *Sets.ParentMethods.algebra()* for the precise meaning in Sage of the *algebra of an object*.

EXAMPLES:

```
sage: Monoids().Algebras(QQ)
Category of monoid algebras over Rational Field

sage: Groups().Algebras(QQ)
Category of group algebras over Rational Field

sage: AdditiveMagmas().AdditiveAssociative().Algebras(QQ)
Category of additive semigroup algebras over Rational Field
```

```
sage: Monoids().Algebras(Rings())
Category of monoid algebras over Category of rings
```

**See also:**

- *algebra_functor.AlgebrasCategory*
- *CovariantFunctorialConstruction*

**CartesianProducts**()

Return the full subcategory of the objects of `self` constructed as Cartesian products.

**See also:**

- *cartesian_product.CartesianProductFunctor*
- `RegressiveCovariantFunctorialConstruction`

EXAMPLES:

```
sage: Sets().CartesianProducts()
Category of Cartesian products of sets
sage: Semigroups().CartesianProducts()
Category of Cartesian products of semigroups
sage: EuclideanDomains().CartesianProducts()
Category of Cartesian products of commutative rings
```

**Enumerated**()

Return the full subcategory of the enumerated objects of `self`.

An enumerated object can be iterated to get its elements.

EXAMPLES:

```
sage: Sets().Enumerated()
Category of enumerated sets
sage: Rings().Finite().Enumerated()
Category of finite enumerated rings
sage: Rings().Infinite().Enumerated()
Category of infinite enumerated rings
```

**Facade**()

Return the full subcategory of the facade objects of `self`.

### What is a facade set?

Recall that, in Sage, *sets are modelled by *parents**, and their elements know which distinguished set they belong to. For example, the ring of integers $\mathbf{Z}$ is modelled by the parent `ZZ`, and integers know that they belong to this set:

```
sage: ZZ
Integer Ring
sage: 42.parent()
Integer Ring
```

Sometimes, it is convenient to represent the elements of a parent `P` by elements of some other parent. For example, the elements of the set of prime numbers are represented by plain integers:

```
sage: Primes()
Set of all prime numbers: 2, 3, 5, 7, ...
```

```
sage: p = Primes().an_element(); p
43
sage: p.parent()
Integer Ring
```

In this case, P is called a *facade set*.

This feature is advertised through the category of $P$:

```
sage: Primes().category()
Category of facade infinite enumerated sets
sage: Sets().Facade()
Category of facade sets
```

Typical use cases include modeling a subset of an existing parent:

```
sage: Set([4,6,9])                    # random
{4, 6, 9}
sage: Sets().Facade().example()
An example of facade set: the monoid of positive integers
```

or the union of several parents:

```
sage: Sets().Facade().example("union")
An example of a facade set: the integers completed by +-infinity
```

or endowing an existing parent with more (or less!) structure:

```
sage: Posets().example("facade")
An example of a facade poset: the positive integers ordered by␣
↪divisibility
```

Let us investigate in detail a close variant of this last example: let $P$ be set of divisors of 12 partially ordered by divisibility. There are two options for representing its elements:

1. as plain integers:

```
sage: P = Poset((divisors(12), attrcall("divides")), facade=True)
```

2. as integers, modified to be aware that their parent is $P$:

```
sage: Q = Poset((divisors(12), attrcall("divides")), facade=False)
```

The advantage of option 1. is that one needs not do conversions back and forth between $P$ and **Z**. The disadvantage is that this introduces an ambiguity when writing $2 < 3$: does this compare 2 and 3 w.r.t. the natural order on integers or w.r.t. divisibility?:

```
sage: 2 < 3
True
```

To raise this ambiguity, one needs to explicitly specify the underlying poset as in $2 <_P 3$:

```
sage: P = Posets().example("facade")
sage: P.lt(2,3)
False
```

On the other hand, with option 2. and once constructed, the elements know unambiguously how to compare themselves:

```
sage: Q(2) < Q(3)
False
sage: Q(2) < Q(6)
True
```

Beware that `P(2)` is still the integer 2. Therefore `P(2) < P(3)` still compares 2 and 3 as integers!:

```
sage: P(2) < P(3)
True
```

In short $P$ being a facade parent is one of the programmatic counterparts (with e.g. coercions) of the usual mathematical idiom: "for ease of notation, we identify an element of $P$ with the corresponding integer". Too many identifications lead to confusion; the lack thereof leads to heavy, if not obfuscated, notations. Finding the right balance is an art, and even though there are common guidelines, it is ultimately up to the writer to choose which identifications to do. This is no different in code.

**See also:**

The following examples illustrate various ways to implement subsets like the set of prime numbers; look at their code for details:

```
sage: Sets().example("facade")
Set of prime numbers (facade implementation)
sage: Sets().example("inherits")
Set of prime numbers
sage: Sets().example("wrapper")
Set of prime numbers (wrapper implementation)
```

### Specifications

A parent which is a facade must either:

- call `Parent.__init__()` using the `facade` parameter to specify a parent, or tuple thereof.
- overload the method `facade_for()`.

**Note:** The concept of facade parents was originally introduced in the computer algebra system MuPAD.

**Facades**(**args*, ***kwds*)
Deprecated: Use `Facade()` instead. See trac ticket #17073 for details.

**Finite**()
Return the full subcategory of the finite objects of `self`.

EXAMPLES:

```
sage: Sets().Finite()
Category of finite sets
sage: Rings().Finite()
Category of finite rings
```

**Infinite**()
Return the full subcategory of the infinite objects of `self`.

EXAMPLES:

```
sage: Sets().Infinite()
Category of infinite sets
sage: Rings().Infinite()
Category of infinite rings
```

**IsomorphicObjects**()

Return the full subcategory of the objects of `self` constructed by isomorphism.

Given a concrete category `As()` (i.e. a subcategory of `Sets()`), `As().IsomorphicObjects()` returns the category of objects of `As()` endowed with a distinguished description as the image of some other object of `As()` by an isomorphism in this category.

See *Subquotients()* for background.

EXAMPLES:

In the following example, $A$ is defined as the image by $x \mapsto x^2$ of the finite set $B = \{1, 2, 3\}$:

```
sage: A = FiniteEnumeratedSets().IsomorphicObjects().example(); A
The image by some isomorphism of An example of a finite enumerated set:
↪{1,2,3}
```

Since $B$ is a finite enumerated set, so is $A$:

```
sage: A in FiniteEnumeratedSets()
True
sage: A.cardinality()
3
sage: A.list()
[1, 4, 9]
```

The isomorphism from $B$ to $A$ is available as:

```
sage: A.retract(3)
9
```

and its inverse as:

```
sage: A.lift(9)
3
```

It often is natural to declare those morphisms as coercions so that one can do `A(b)` and `B(a)` to go back and forth between $A$ and $B$ (TODO: refer to a category example where the maps are declared as a coercion). This is not done by default. Indeed, in many cases one only wants to transport part of the structure of $B$ to $A$. Assume for example, that one wants to construct the set of integers $B = ZZ$, endowed with `max` as addition, and + as multiplication instead of the usual + and $*$. One can construct $A$ as isomorphic to $B$ as an infinite enumerated set. However $A$ is *not* isomorphic to $B$ as a ring; for example, for $a \in A$ and $a \in B$, the expressions $a + A(b)$ and $B(a) + b$ give completely different results; hence we would not want the expression $a + b$ to be implicitly resolved to any one of above two, as the coercion mechanism would do.

Coercions also cannot be used with facade parents (see *Sets.Facade*) like in the example above.

We now look at a category of isomorphic objects:

```
sage: C = Sets().IsomorphicObjects(); C
Category of isomorphic objects of sets

sage: C.super_categories()
```

```
[Category of subobjects of sets, Category of quotients of sets]

sage: C.all_super_categories()
[Category of isomorphic objects of sets,
 Category of subobjects of sets,
 Category of quotients of sets,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

Unless something specific about isomorphic objects is implemented for this category, one actually get an optimized super category:

```
sage: C = Semigroups().IsomorphicObjects(); C
Join of Category of quotients of semigroups
    and Category of isomorphic objects of sets
```

**See also:**

- *Subquotients()* for background
- *isomorphic_objects.IsomorphicObjectsCategory*
- RegressiveCovariantFunctorialConstruction

**Metric**()

Return the subcategory of the metric objects of `self`.

**Quotients**()

Return the full subcategory of the objects of `self` constructed as quotients.

Given a concrete category `As()` (i.e. a subcategory of `Sets()`), `As().Quotients()` returns the category of objects of `As()` endowed with a distinguished description as quotient (in fact homomorphic image) of some other object of `As()`.

Implementing an object of `As().Quotients()` is done in the same way as for `As().Subquotients()`; namely by providing an ambient space and a lift and a retract map. See *Subquotients()* for detailed instructions.

**See also:**

- *Subquotients()* for background
- *quotients.QuotientsCategory*
- RegressiveCovariantFunctorialConstruction

EXAMPLES:

```
sage: C = Semigroups().Quotients(); C
Category of quotients of semigroups
sage: C.super_categories()
[Category of subquotients of semigroups, Category of quotients of sets]
sage: C.all_super_categories()
[Category of quotients of semigroups,
 Category of subquotients of semigroups,
 Category of semigroups,
 Category of subquotients of magmas,
 Category of magmas,
 Category of quotients of sets,
 Category of subquotients of sets,
 Category of sets,
```

```
Category of sets with partial maps,
Category of objects]
```

The caller is responsible for checking that the given category admits a well defined category of quotients:

```
sage: EuclideanDomains().Quotients()
Join of Category of euclidean domains
    and Category of subquotients of monoids
    and Category of quotients of semigroups
```

**Subobjects**()

Return the full subcategory of the objects of `self` constructed as subobjects.

Given a concrete category `As()` (i.e. a subcategory of `Sets()`), `As().Subobjects()` returns the category of objects of `As()` endowed with a distinguished embedding into some other object of `As()`.

Implementing an object of `As().Subobjects()` is done in the same way as for `As().Subquotients()`; namely by providing an ambient space and a lift and a retract map. In the case of a trivial embedding, the two maps will typically be identity maps that just change the parent of their argument. See *Subquotients()* for detailed instructions.

**See also:**

- *Subquotients()* for background
- *subobjects.SubobjectsCategory*
- `RegressiveCovariantFunctorialConstruction`

EXAMPLES:

```
sage: C = Sets().Subobjects(); C
Category of subobjects of sets

sage: C.super_categories()
[Category of subquotients of sets]

sage: C.all_super_categories()
[Category of subobjects of sets,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

Unless something specific about subobjects is implemented for this category, one actually gets an optimized super category:

```
sage: C = Semigroups().Subobjects(); C
Join of Category of subquotients of semigroups
    and Category of subobjects of sets
```

The caller is responsible for checking that the given category admits a well defined category of subobjects.

**Subquotients**()

Return the full subcategory of the objects of `self` constructed as subquotients.

Given a concrete category `self == As()` (i.e. a subcategory of `Sets()`), `As().Subquotients()` returns the category of objects of `As()` endowed with a distinguished description as subquotient of some other object of `As()`.

EXAMPLES:

```
sage: Monoids().Subquotients()
Category of subquotients of monoids
```

A parent $A$ in `As()` is further in `As().Subquotients()` if there is a distinguished parent $B$ in `As()`, called the *ambient set*, a subobject $B'$ of $B$, and a pair of maps:

$$l : A \to B' \text{ and } r : B' \to A$$

called respectively the *lifting map* and *retract map* such that $r \circ l$ is the identity of $A$ and $r$ is a morphism in `As()`.

**Todo:** Draw the typical commutative diagram.

It follows that, for each operation *op* of the category, we have some property like:

$$op_A(e) = r(op_B(l(e))), \text{ for all } e \in A$$

This allows for implementing the operations on $A$ from those on $B$.

The two most common use cases are:

- *homomorphic images* (or *quotients*), when $B' = B$, $r$ is an homomorphism from $B$ to $A$ (typically a canonical quotient map), and $l$ a section of it (not necessarily a homomorphism); see `Quotients()`;
- *subobjects* (up to an isomorphism), when $l$ is an embedding from $A$ into $B$; in this case, $B'$ is typically isomorphic to $A$ through the inverse isomorphisms $r$ and $l$; see `Subobjects()`;

**Note:**

- The usual definition of "subquotient" (Wikipedia article Subquotient) does not involve the lifting map $l$. This map is required in Sage's context to make the definition constructive. It is only used in computations and does not affect their results. This is relatively harmless since the category is a concrete category (i.e., its objects are sets and its morphisms are set maps).
- In mathematics, especially in the context of quotients, the retract map $r$ is often referred to as a *projection map* instead.
- Since $B'$ is not specified explicitly, it is possible to abuse the framework with situations where $B'$ is not quite a subobject and $r$ not quite a morphism, as long as the lifting and retract maps can be used as above to compute all the operations in $A$. Use at your own risk!

Assumptions:

- For any category `As()`, `As().Subquotients()` is a subcategory of `As()`.

  Example: a subquotient of a group is a group (e.g., a left or right quotient of a group by a non-normal subgroup is not in this category).
- This construction is covariant: if `As()` is a subcategory of `Bs()`, then `As().Subquotients()` is a subcategory of `Bs().Subquotients()`.

  Example: if $A$ is a subquotient of $B$ in the category of groups, then it is also a subquotient of $B$ in the category of monoids.
- If the user (or a program) calls `As().Subquotients()`, then it is assumed that subquotients are well defined in this category. This is not checked, and probably never will be. Note that, if a category `As()` does not specify anything about its subquotients, then its subquotient category looks like this:

```
sage: EuclideanDomains().Subquotients()
Join of Category of euclidean domains
    and Category of subquotients of monoids
```

Interface: the ambient set $B$ of $A$ is given by `A.ambient()`. The subset $B'$ needs not be specified, so the retract map is handled as a partial map from $B$ to $A$.

The lifting and retract map are implemented respectively as methods `A.lift(a)` and `A.retract(b)`. As a shorthand for the former, one can use alternatively `a.lift()`:

```
sage: S = Semigroups().Subquotients().example(); S
An example of a (sub)quotient semigroup: a quotient of the left zero␣
↪semigroup
sage: S.ambient()
An example of a semigroup: the left zero semigroup
sage: S(3).lift().parent()
An example of a semigroup: the left zero semigroup
sage: S(3) * S(1) == S.retract( S(3).lift() * S(1).lift() )
True
```

See `S?` for more.

**Todo:** use a more interesting example, like $\mathbf{Z}/n\mathbf{Z}$.

**See also:**

- *`Quotients()`*, *`Subobjects()`*, *`IsomorphicObjects()`*
- *`subquotients.SubquotientsCategory`*
- `RegressiveCovariantFunctorialConstruction`

**`Topological`**()

Return the subcategory of the topological objects of `self`.

**class `Subobjects`**(*category*, **args*)

Bases: *`sage.categories.subobjects.SubobjectsCategory`*

A category for subobjects of sets.

**See also:**

`Sets().Subobjects()`

EXAMPLES:

```
sage: Sets().Subobjects()
Category of subobjects of sets
sage: Sets().Subobjects().all_super_categories()
[Category of subobjects of sets,
 Category of subquotients of sets,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

**class `ParentMethods`**

**class `Subquotients`**(*category*, **args*)

Bases: *`sage.categories.subquotients.SubquotientsCategory`*

A category for subquotients of sets.

**See also:**

`Sets().Subquotients()`

EXAMPLES:

```
sage: Sets().Subquotients()
Category of subquotients of sets
sage: Sets().Subquotients().all_super_categories()
[Category of subquotients of sets, Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

**class ElementMethods**

**lift**()

Lift `self` to the ambient space for its parent.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: s = S.an_element()
sage: s, s.parent()
(42, An example of a (sub)quotient semigroup: a quotient of the left␣
↪zero semigroup)
sage: S.lift(s), S.lift(s).parent()
(42, An example of a semigroup: the left zero semigroup)
sage: s.lift(), s.lift().parent()
(42, An example of a semigroup: the left zero semigroup)
```

**class ParentMethods**

**ambient**()

Return the ambient space for `self`.

EXAMPLES:

```
sage: Semigroups().Subquotients().example().ambient()
An example of a semigroup: the left zero semigroup
```

**See also:**

*Sets.SubcategoryMethods.Subquotients()* for the specifications and *lift()* and *retract()*.

**lift**($x$)

Lift $x$ to the ambient space for `self`.

INPUT:

- `x` – an element of `self`

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: s = S.an_element()
sage: s, s.parent()
(42, An example of a (sub)quotient semigroup: a quotient of the left␣
↪zero semigroup)
sage: S.lift(s), S.lift(s).parent()
(42, An example of a semigroup: the left zero semigroup)
```

```
sage: s.lift(), s.lift().parent()
(42, An example of a semigroup: the left zero semigroup)
```

**See also:**

*Sets.SubcategoryMethods.Subquotients* for the specifications, *ambient()*, *retract()*, and also *Sets.Subquotients.ElementMethods.lift()*.

**retract** (*x*)

Retract x to self.

INPUT:

- x – an element of the ambient space for self

**See also:**

*Sets.SubcategoryMethods.Subquotients* for the specifications, *ambient()*, *retract()*, and also Sets.Subquotients.ElementMethods.retract().

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: s = S.ambient().an_element()
sage: s, s.parent()
(42, An example of a semigroup: the left zero semigroup)
sage: S.retract(s), S.retract(s).parent()
(42, An example of a (sub)quotient semigroup: a quotient of the left
↪zero semigroup)
```

**Topological**

alias of TopologicalSpaces

**class WithRealizations** (*category*, **args*)

Bases: *sage.categories.with_realizations.WithRealizationsCategory*

**class ParentMethods**

**class Realizations** (*parent_with_realization*)

Bases: *sage.categories.realizations.Category_realization_of_parent*

**super_categories** ()

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.Realizations().super_categories()
[Category of realizations of sets]
```

**a_realization** ()

Return a realization of self.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.a_realization()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental
↪basis
```

**facade_for** ()

Return the parents self is a facade for, that is the realizations of self

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.facade_for()
[The subset algebra of {1, 2, 3} over Rational Field in the
↪Fundamental basis, The subset algebra of {1, 2, 3} over Rational
↪Field in the In basis, The subset algebra of {1, 2, 3} over Rational
↪Field in the Out basis]

sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: f = A.F().an_element(); f
F[{}] + 2*F[{1}] + 3*F[{2}] + F[{1, 2}]
sage: i = A.In().an_element(); i
In[{}] + 2*In[{1}] + 3*In[{2}] + In[{1, 2}]
sage: o = A.Out().an_element(); o
Out[{}] + 2*Out[{1}] + 3*Out[{2}] + Out[{1, 2}]
sage: f in A, i in A, o in A
(True, True, True)
```

**inject_shorthands** (*shorthands=None*, *verbose=True*)

Import standard shorthands into the global namespace.

INPUT:

- `shorthands` – a list (or iterable) of strings (default: `self._shorthands`) or `"all"` (for `self._shorthands_all`)
- **verbose – boolean (default True);** whether to print the defined shorthands

EXAMPLES:

When computing with a set with multiple realizations, like `SymmetricFunctions` or *SubsetAlgebra*, it is convenient to define shorthands for the various realizations, but cumbersome to do it by hand:

```
sage: S = SymmetricFunctions(ZZ); S
Symmetric Functions over Integer Ring
sage: s = S.s(); s
Symmetric Functions over Integer Ring in the Schur basis
sage: e = S.e(); e
Symmetric Functions over Integer Ring in the elementary basis
```

This method automatizes the process:

```
sage: S.inject_shorthands()
Defining e as shorthand for Symmetric Functions over Integer Ring in
↪the elementary basis
Defining f as shorthand for Symmetric Functions over Integer Ring in
↪the forgotten basis
Defining h as shorthand for Symmetric Functions over Integer Ring in
↪the homogeneous basis
Defining m as shorthand for Symmetric Functions over Integer Ring in
↪the monomial basis
Defining p as shorthand for Symmetric Functions over Integer Ring in
↪the powersum basis
Defining s as shorthand for Symmetric Functions over Integer Ring in
↪the Schur basis
sage: s[1] + e[2] * p[1,1] + 2*h[3] + m[2,1]
s[1] - 2*s[1, 1, 1] + s[1, 1, 1, 1] + s[2, 1] + 2*s[2, 1, 1] + s[2, 2]
↪+ 2*s[3] + s[3, 1]
```

```
sage: e
Symmetric Functions over Integer Ring in the elementary basis
sage: p
Symmetric Functions over Integer Ring in the powersum basis
sage: s
Symmetric Functions over Integer Ring in the Schur basis
```

Sometimes, like for symmetric functions, one can request for all shorthands to be defined, including less common ones:

```
sage: S.inject_shorthands("all")
Defining e as shorthand for Symmetric Functions over Integer Ring in␣
↪the elementary basis
Defining f as shorthand for Symmetric Functions over Integer Ring in␣
↪the forgotten basis
Defining h as shorthand for Symmetric Functions over Integer Ring in␣
↪the homogeneous basis
Defining ht as shorthand for Symmetric Functions over Integer Ring in␣
↪the induced trivial character basis
Defining m as shorthand for Symmetric Functions over Integer Ring in␣
↪the monomial basis
Defining o as shorthand for Symmetric Functions over Integer Ring in␣
↪the orthogonal basis
Defining p as shorthand for Symmetric Functions over Integer Ring in␣
↪the powersum basis
Defining s as shorthand for Symmetric Functions over Integer Ring in␣
↪the Schur basis
Defining sp as shorthand for Symmetric Functions over Integer Ring in␣
↪the symplectic basis
Defining st as shorthand for Symmetric Functions over Integer Ring in␣
↪the irreducible symmetric group character basis
Defining w as shorthand for Symmetric Functions over Integer Ring in␣
↪the Witt basis
```

The messages can be silenced by setting `verbose=False`:

```
sage: Q = QuasiSymmetricFunctions(ZZ)
sage: Q.inject_shorthands(verbose=False)

sage: F[1,2,1] + 5*M[1,3] + F[2]^2
5*F[1, 1, 1, 1] - 5*F[1, 1, 2] - 3*F[1, 2, 1] + 6*F[1, 3] +
2*F[2, 2] + F[3, 1] + F[4]

sage: F
Quasisymmetric functions over the Integer Ring in the
 Fundamental basis
sage: M
Quasisymmetric functions over the Integer Ring in the
 Monomial basis
```

One can also just import a subset of the shorthands:

```
sage: SQ = SymmetricFunctions(QQ)
sage: SQ.inject_shorthands(['p', 's'], verbose=False)
sage: p
Symmetric Functions over Rational Field in the powersum basis
sage: s
```

```
Symmetric Functions over Rational Field in the Schur basis
```

Note that e is left unchanged:

```
sage: e
Symmetric Functions over Integer Ring in the elementary basis
```

**realizations**()

Return all the realizations of self that self is aware of.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.realizations()
[The subset algebra of {1, 2, 3} over Rational Field in the␣
↪Fundamental basis, The subset algebra of {1, 2, 3} over Rational␣
↪Field in the In basis, The subset algebra of {1, 2, 3} over Rational␣
↪Field in the Out basis]
```

**Note:** Constructing a parent P in the category A.Realizations() automatically adds P to this list by calling A._register_realization(A)

**example**(*base_ring=None*, *set=None*)

Return an example of set with multiple realizations, as per *Category.example()*.

EXAMPLES:

```
sage: Sets().WithRealizations().example()
The subset algebra of {1, 2, 3} over Rational Field

sage: Sets().WithRealizations().example(ZZ, Set([1,2]))
The subset algebra of {1, 2} over Integer Ring
```

**extra_super_categories**()

A set with multiple realizations is a facade parent.

EXAMPLES:

```
sage: Sets().WithRealizations().extra_super_categories()
[Category of facade sets]
sage: Sets().WithRealizations().super_categories()
[Category of facade sets]
```

**example**(*choice=None*)

Returns examples of objects of Sets(), as per *Category.example()*.

EXAMPLES:

```
sage: Sets().example()
Set of prime numbers (basic implementation)

sage: Sets().example("inherits")
Set of prime numbers

sage: Sets().example("facade")
Set of prime numbers (facade implementation)
```

```
sage: Sets().example("wrapper")
Set of prime numbers (wrapper implementation)
```

**super_categories**()

We include SetsWithPartialMaps between Sets and Objects so that we can define morphisms between sets that are only partially defined. This is also to have the Homset constructor not complain that SetsWithPartialMaps is not a supercategory of Fields, for example.

EXAMPLES:

```
sage: Sets().super_categories()
[Category of sets with partial maps]
```

sage.categories.sets_cat.**print_compare**(*x*, *y*)

Helper method used in `Sets.ParentMethods._test_elements_eq_symmetric()`, `Sets.ParentMethods._test_elements_eq_tranisitive()`.

INPUT:

- `x` – an element
- `y` – an element

EXAMPLES:

```
sage: from sage.categories.sets_cat import print_compare
sage: print_compare(1,2)
1 != 2
sage: print_compare(1,1)
1 == 1
```

## 3.131 Sets With a Grading

**class** sage.categories.sets_with_grading.**SetsWithGrading**(*s=None*)

Bases: `sage.categories.category.Category`

The category of sets with a grading.

A *set with a grading* is a set $S$ equipped with a grading by some other set $I$ (by default the set $\mathbf{N}$ of the non-negative integers):

$$S = \biguplus_{i \in I} S_i$$

where the *graded components* $S_i$ are (usually finite) sets. The *grading* function maps each element $s$ of $S$ to its *grade* $i$, so that $s \in S_i$.

From implementation point of view, if the graded set is enumerated then each graded component should be enumerated (there is a check in the method `_test_graded_components()`). The contrary needs not be true.

To implement this category, a parent must either implement `graded_component()` or `subset()`. If only `subset()` is implemented, the first argument must be the grading for compatibility with `graded_component()`. Additionally either the parent must implement `grading()` or its elements must implement a method `grade()`. See the example `sage.categories.examples.sets_with_grading.NonNegativeIntegers`.

Finally, if the graded set is enumerated (see *EnumeratedSets*) then each graded component should be enumerated. The contrary needs not be true.

EXAMPLES:

A typical example of a set with a grading is the set of non-negative integers graded by themselves:

```
sage: N = SetsWithGrading().example(); N
Non negative integers
sage: N.category()
Category of facade sets with grading
sage: N.grading_set()
Non negative integers
```

The *grading function* is given by `N.grading`:

```
sage: N.grading(4)
4
```

The graded component $S_i$ is the set of all integer partitions of $i$:

```
sage: N.graded_component(grade = 5)
{5}
sage: N.graded_component(grade = 42)
{42}
```

Here are some information about this category:

```
sage: SetsWithGrading()
Category of sets with grading
sage: SetsWithGrading().super_categories()
[Category of sets]
sage: SetsWithGrading().all_super_categories()
[Category of sets with grading,
 Category of sets,
 Category of sets with partial maps,
 Category of objects]
```

**Todo:**

- This should be moved to `Sets().WithGrading()`.
- Should the grading set be a parameter for this category?
- Does the enumeration need to be compatible with the grading? Be careful that the fact that graded components are allowed to be finite or infinite make the answer complicated.

**class ParentMethods**

**generating_series**()

Default implementation for generating series.

OUTPUT:

A series, indexed by the grading set.

EXAMPLES:

```
sage: N = SetsWithGrading().example(); N
Non negative integers
sage: N.generating_series()
1/(-z + 1)
```

**graded_component** (*grade*)

Return the graded component of `self` with grade `grade`.

The default implementation just calls the method *subset()* with the first argument `grade`.

EXAMPLES:

```
sage: N = SetsWithGrading().example(); N
Non negative integers
sage: N.graded_component(3)
{3}
```

**grading** (*elt*)

Return the grading of the element `elt` of `self`.

This default implementation calls `elt.grade()`.

EXAMPLES:

```
sage: N = SetsWithGrading().example(); N
Non negative integers
sage: N.grading(4)
4
```

**grading_set** ()

Return the set `self` is graded by. By default, this is the set of non-negative integers.

EXAMPLES:

```
sage: SetsWithGrading().example().grading_set()
Non negative integers
```

**subset** (**args, **options*)

Return the subset of `self` described by the given parameters.

**See also:**

-*graded_component()*

EXAMPLES:

```
sage: W = WeightedIntegerVectors([3,2,1]); W
Integer vectors weighted by [3, 2, 1]
sage: W.subset(4)
Integer vectors of 4 weighted by [3, 2, 1]
```

**super_categories** ()

EXAMPLES:

```
sage: SetsWithGrading().super_categories()
[Category of sets]
```

## 3.132 SetsWithPartialMaps

**class** sage.categories.sets_with_partial_maps.**SetsWithPartialMaps**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category whose objects are sets and whose morphisms are maps that are allowed to raise a ValueError on some inputs.

This category is equivalent to the category of pointed sets, via the equivalence sending an object X to X union {error}, a morphism f to the morphism of pointed sets that sends x to f(x) if f does not raise an error on x, or to error if it does.

EXAMPLES:

```
sage: SetsWithPartialMaps()
Category of sets with partial maps

sage: SetsWithPartialMaps().super_categories()
[Category of objects]
```

**super_categories**()

EXAMPLES:

```
sage: SetsWithPartialMaps().super_categories()
[Category of objects]
```

## 3.133 Shephard Groups

**class** sage.categories.shephard_groups.**ShephardGroups**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of Shephard groups.

EXAMPLES:

```
sage: from sage.categories.shephard_groups import ShephardGroups
sage: C = ShephardGroups(); C
Category of shephard groups
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.shephard_groups import ShephardGroups
sage: ShephardGroups().super_categories()
[Category of finite generalized coxeter groups]
```

## 3.134 Simplicial Complexes

**class** sage.categories.simplicial_complexes.**SimplicialComplexes**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of abstract simplicial complexes.

An abstract simplicial complex $A$ is a collection of sets $X$ such that:

- $\emptyset \in A$,
- if $X \subset Y \in A$, then $X \in A$.

---

**Todo:** Implement the category of simplicial complexes considered as *CW complexes* and rename this to the category of `AbstractSimplicialComplexes` with appropriate functors.

---

EXAMPLES:

```
sage: from sage.categories.simplicial_complexes import SimplicialComplexes
sage: C = SimplicialComplexes(); C
Category of simplicial complexes
```

**class Finite**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

Category of finite simplicial complexes.

**class ParentMethods**

**dimension**()

Return the dimension of `self`.

EXAMPLES:

```
sage: S = SimplicialComplex([[1,3,4], [1,2],[2,5],[4,5]])
sage: S.dimension()
2
```

**class ParentMethods**

**faces**()

Return the faces of `self`.

EXAMPLES:

```
sage: S = SimplicialComplex([[1,3,4], [1,2],[2,5],[4,5]])
sage: S.faces()
{-1: {()},
 0: {(1,), (2,), (3,), (4,), (5,)},
 1: {(1, 2), (1, 3), (1, 4), (2, 5), (3, 4), (4, 5)},
 2: {(1, 3, 4)}}
```

**facets**()

Return the facets of `self`.

EXAMPLES:

```
sage: S = SimplicialComplex([[1,3,4], [1,2],[2,5],[4,5]])
sage: S.facets()
{(1, 2), (1, 3, 4), (2, 5), (4, 5)}
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.simplicial_complexes import SimplicialComplexes
sage: SimplicialComplexes().super_categories()
[Category of sets]
```

## 3.135 Simplicial Sets

**class** sage.categories.simplicial_sets.**SimplicialSets**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of simplicial sets.

A simplicial set $X$ is a collection of sets $X_i$, indexed by the non-negative integers, together with maps

$$d_i : X_n \to X_{n-1}, \quad 0 \leq i \leq n \quad \text{(face maps)}$$
$$s_j : X_n \to X_{n+1}, \quad 0 \leq j \leq n \quad \text{(degeneracy maps)}$$

satisfying the *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{if } i < j \\ d_i s_j &= s_{j-1} d_i \quad \text{if } i < j \\ d_j s_j &= 1 = d_{j+1} s_j \\ d_i s_j &= s_j d_{i-1} \quad \text{if } i > j+1 \\ s_i s_j &= s_{j+1} s_i \quad \text{if } i \leq j \end{aligned}$$

Morphisms are sequences of maps $f_i : X_i \to Y_i$ which commute with the face and degeneracy maps.

EXAMPLES:

```
sage: from sage.categories.simplicial_sets import SimplicialSets
sage: C = SimplicialSets(); C
Category of simplicial sets
```

**class Finite**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

Category of finite simplicial sets.

The objects are simplicial sets with finitely many non-degenerate simplices.

**class Homsets**(*category*, **args*)

Bases: *sage.categories.homsets.HomsetsCategory*

**class Endset**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**class ParentMethods**

**one**()

Return the identity morphism in $\mathrm{Hom}(S, S)$.

EXAMPLES:

```
sage: T = simplicial_sets.Torus()
sage: Hom(T, T).identity()
Simplicial set endomorphism of Torus
  Defn: Identity map
```

**class ParentMethods**

**is_finite**()

Return `True` if this simplicial set is finite, i.e., has a finite number of nondegenerate simplices.

EXAMPLES:

```
sage: simplicial_sets.Torus().is_finite()
True
sage: C5 = groups.misc.MultiplicativeAbelian([5])
sage: simplicial_sets.ClassifyingSpace(C5).is_finite()
False
```

**is_pointed**()

Return `True` if this simplicial set is pointed, i.e., has a base point.

EXAMPLES:

```
sage: from sage.homology.simplicial_set import AbstractSimplex,␣
↪SimplicialSet
sage: v = AbstractSimplex(0)
sage: w = AbstractSimplex(0)
sage: e = AbstractSimplex(1)
sage: X = SimplicialSet({e: (v, w)})
sage: Y = SimplicialSet({e: (v, w)}, base_point=w)
sage: X.is_pointed()
False
sage: Y.is_pointed()
True
```

**set_base_point**(*point*)

Return a copy of this simplicial set in which the base point is set to `point`.

INPUT:

- `point` – a 0-simplex in this simplicial set

EXAMPLES:

```
sage: from sage.homology.simplicial_set import AbstractSimplex,␣
↪SimplicialSet
sage: v = AbstractSimplex(0, name='v_0')
sage: w = AbstractSimplex(0, name='w_0')
sage: e = AbstractSimplex(1)
sage: X = SimplicialSet({e: (v, w)})
sage: Y = SimplicialSet({e: (v, w)}, base_point=w)
sage: Y.base_point()
w_0
sage: X_star = X.set_base_point(w)
sage: X_star.base_point()
w_0
sage: Y_star = Y.set_base_point(v)
sage: Y_star.base_point()
v_0
```

**class Pointed**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**class Finite**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

**class ParentMethods**

**fat_wedge** (*n*)

Return the $n$-th fat wedge of this pointed simplicial set.

This is the subcomplex of the $n$-fold product $X^n$ consisting of those points in which at least one factor is the base point. Thus when $n = 2$, this is the wedge of the simplicial set with itself, but when $n$ is larger, the fat wedge is larger than the $n$-fold wedge.

EXAMPLES:

```
sage: S1 = simplicial_sets.Sphere(1)
sage: S1.fat_wedge(0)
Point
sage: S1.fat_wedge(1)
S^1
sage: S1.fat_wedge(2).fundamental_group()
Finitely presented group < e0, e1 |  >
sage: S1.fat_wedge(4).homology()
{0: 0, 1: Z x Z x Z x Z, 2: Z^6, 3: Z x Z x Z x Z}
```

**smash_product** (**others*)

Return the smash product of this simplicial set with `others`.

INPUT:

- `others` – one or several simplicial sets

EXAMPLES:

```
sage: S1 = simplicial_sets.Sphere(1)
sage: RP2 = simplicial_sets.RealProjectiveSpace(2)
sage: X = S1.smash_product(RP2)
sage: X.homology(base_ring=GF(2))
{0: Vector space of dimension 0 over Finite Field of size 2,
 1: Vector space of dimension 0 over Finite Field of size 2,
 2: Vector space of dimension 1 over Finite Field of size 2,
 3: Vector space of dimension 1 over Finite Field of size 2}

sage: T = S1.product(S1)
sage: X = T.smash_product(S1)
sage: X.homology(reduced=False)
{0: Z, 1: 0, 2: Z x Z, 3: Z}
```

**unset_base_point** ()

Return a copy of this simplicial set in which the base point has been forgotten.

EXAMPLES:

```
sage: from sage.homology.simplicial_set import AbstractSimplex,␣
↪SimplicialSet
sage: v = AbstractSimplex(0, name='v_0')
sage: w = AbstractSimplex(0, name='w_0')
sage: e = AbstractSimplex(1)
sage: Y = SimplicialSet({e: (v, w)}, base_point=w)
sage: Y.is_pointed()
True
sage: Y.base_point()
w_0
sage: Z = Y.unset_base_point()
```

```
sage: Z.is_pointed()
False
```

**class ParentMethods**

**base_point**()

Return this simplicial set's base point

EXAMPLES:

```
sage: from sage.homology.simplicial_set import AbstractSimplex,␣
↪SimplicialSet
sage: v = AbstractSimplex(0, name='*')
sage: e = AbstractSimplex(1)
sage: S1 = SimplicialSet({e: (v, v)}, base_point=v)
sage: S1.is_pointed()
True
sage: S1.base_point()
*
```

**base_point_map**(*domain=None*)

Return a map from a one-point space to this one, with image the base point.

This raises an error if this simplicial set does not have a base point.

INPUT:

- `domain` – optional, default `None`. Use this to specify a particular one-point space as the domain. The default behavior is to use the `sage.homology.simplicial_set.Point()` function to use a standard one-point space.

EXAMPLES:

```
sage: T = simplicial_sets.Torus()
sage: f = T.base_point_map(); f
Simplicial set morphism:
  From: Point
  To:   Torus
  Defn: Constant map at (v_0, v_0)
sage: S3 = simplicial_sets.Sphere(3)
sage: g = S3.base_point_map()
sage: f.domain() == g.domain()
True
sage: RP3 = simplicial_sets.RealProjectiveSpace(3)
sage: temp = simplicial_sets.Simplex(0)
sage: pt = temp.set_base_point(temp.n_cells(0)[0])
sage: h = RP3.base_point_map(domain=pt)
sage: f.domain() == h.domain()
False

sage: C5 = groups.misc.MultiplicativeAbelian([5])
sage: BC5 = simplicial_sets.ClassifyingSpace(C5)
sage: BC5.base_point_map()
Simplicial set morphism:
  From: Point
  To:   Classifying space of Multiplicative Abelian group isomorphic␣
↪to C5
  Defn: Constant map at 1
```

**connectivity**(*max_dim=None*)

Return the connectivity of this pointed simplicial set.

INPUT:

- `max_dim` – specify a maximum dimension through which to check. This is required if this simplicial set is simply connected and not finite.

The dimension of the first nonzero homotopy group. If simply connected, this is the same as the dimension of the first nonzero homology group.

> **Warning:** See the warning for the *is_simply_connected()* method.

The connectivity of a contractible space is `+Infinity`.

EXAMPLES:

```
sage: simplicial_sets.Sphere(3).connectivity()
2
sage: simplicial_sets.Sphere(0).connectivity()
-1
sage: K = simplicial_sets.Simplex(4)
sage: K = K.set_base_point(K.n_cells(0)[0])
sage: K.connectivity()
+Infinity
sage: X = simplicial_sets.Torus().suspension(2)
sage: X.connectivity()
2

sage: C2 = groups.misc.MultiplicativeAbelian([2])
sage: BC2 = simplicial_sets.ClassifyingSpace(C2)
sage: BC2.connectivity()
0
```

**fundamental_group**(*simplify=True*)

Return the fundamental group of this pointed simplicial set.

INPUT:

- `simplify` (bool, optional `True`) – if `False`, then return a presentation of the group in terms of generators and relations. If `True`, the default, simplify as much as GAP is able to.

Algorithm: we compute the edge-path group – see Section 19 of [Kan1958] and Wikipedia article Fundamental_group. Choose a spanning tree for the connected component of the 1-skeleton containing the base point, and then the group's generators are given by the non-degenerate edges. There are two types of relations: $e = 1$ if $e$ is in the spanning tree, and for every 2-simplex, if its faces are $e_0$, $e_1$, and $e_2$, then we impose the relation $e_0 e_1^{-1} e_2 = 1$, where we first set $e_i = 1$ if $e_i$ is degenerate.

EXAMPLES:

```
sage: S1 = simplicial_sets.Sphere(1)
sage: eight = S1.wedge(S1)
sage: eight.fundamental_group() # free group on 2 generators
Finitely presented group < e0, e1 |  >
```

The fundamental group of a disjoint union of course depends on the choice of base point:

```
sage: T = simplicial_sets.Torus()
sage: K = simplicial_sets.KleinBottle()
sage: X = T.disjoint_union(K)
```

```
sage: X_0 = X.set_base_point(X.n_cells(0)[0])
sage: X_0.fundamental_group().is_abelian()
True
sage: X_1 = X.set_base_point(X.n_cells(0)[1])
sage: X_1.fundamental_group().is_abelian()
False

sage: RP3 = simplicial_sets.RealProjectiveSpace(3)
sage: RP3.fundamental_group()
Finitely presented group < e | e^2 >
```

Compute the fundamental group of some classifying spaces:

```
sage: C5 = groups.misc.MultiplicativeAbelian([5])
sage: BC5 = C5.nerve()
sage: BC5.fundamental_group()
Finitely presented group < e0 | e0^5 >

sage: Sigma3 = groups.permutation.Symmetric(3)
sage: BSigma3 = Sigma3.nerve()
sage: pi = BSigma3.fundamental_group(); pi
Finitely presented group < e0, e1 | e0^2, e1^3, (e0*e1^-1)^2 >
sage: pi.order()
6
sage: pi.is_abelian()
False
```

**is_simply_connected**()

Return `True` if this pointed simplicial set is simply connected.

> **Warning:** Determining simple connectivity is not always possible, because it requires determining when a group, as given by generators and relations, is trivial. So this conceivably may give a false negative in some cases.

EXAMPLES:

```
sage: T = simplicial_sets.Torus()
sage: T.is_simply_connected()
False
sage: T.suspension().is_simply_connected()
True
sage: simplicial_sets.KleinBottle().is_simply_connected()
False

sage: S2 = simplicial_sets.Sphere(2)
sage: S3 = simplicial_sets.Sphere(3)
sage: (S2.wedge(S3)).is_simply_connected()
True
sage: X = S2.disjoint_union(S3)
sage: X = X.set_base_point(X.n_cells(0)[0])
sage: X.is_simply_connected()
False

sage: C3 = groups.misc.MultiplicativeAbelian([3])
sage: BC3 = simplicial_sets.ClassifyingSpace(C3)
```

```
sage: BC3.is_simply_connected()
False
```

**class SubcategoryMethods**

**Pointed**()

A simplicial set is *pointed* if it has a distinguished base point.

EXAMPLES:

```
sage: from sage.categories.simplicial_sets import SimplicialSets
sage: SimplicialSets().Pointed().Finite()
Category of finite pointed simplicial sets
sage: SimplicialSets().Finite().Pointed()
Category of finite pointed simplicial sets
```

**super_categories**()

EXAMPLES:

```
sage: from sage.categories.simplicial_sets import SimplicialSets
sage: SimplicialSets().super_categories()
[Category of sets]
```

## 3.136 Super Algebras

**class** sage.categories.super_algebras.**SuperAlgebras**(*base_category*)

Bases: *sage.categories.super_modules.SuperModulesCategory*

The category of super algebras.

An $R$*-super algebra* is an $R$-super module $A$ endowed with an $R$-algebra structure satisfying

$$A_0 A_0 \subseteq A_0, \qquad A_0 A_1 \subseteq A_1, \qquad A_1 A_0 \subseteq A_1, \qquad A_1 A_1 \subseteq A_0$$

and $1 \in A_0$.

EXAMPLES:

```
sage: Algebras(ZZ).Super()
Category of super algebras over Integer Ring
```

**class ParentMethods**

**graded_algebra**()

Return the associated graded algebra to `self`.

**Warning:** Because a super module $M$ is naturally $\mathbf{Z}/2\mathbf{Z}$-graded, and graded modules have a natural filtration induced by the grading, if $M$ has a different filtration, then the associated graded module $\operatorname{gr} M \neq M$. This is most apparent with super algebras, such as the `differential Weyl algebra`, and the multiplication may not coincide.

**extra_super_categories**()

EXAMPLES:

```
sage: Algebras(ZZ).Super().super_categories() # indirect doctest
[Category of graded algebras over Integer Ring,
 Category of super modules over Integer Ring]
```

## 3.137 Super algebras with basis

**class** sage.categories.super_algebras_with_basis.**SuperAlgebrasWithBasis**(*base_category*)

Bases: *sage.categories.super_modules.SuperModulesCategory*

The category of super algebras with a distinguished basis

EXAMPLES:

```
sage: C = Algebras(ZZ).WithBasis().Super(); C
Category of super algebras with basis over Integer Ring
```

**class ParentMethods**

**graded_algebra**()

Return the associated graded module to `self`.

See `AssociatedGradedAlgebra` for the definition and the properties of this.

**See also:**

`graded_algebra()`

EXAMPLES:

```
sage: W.<x,y> = algebras.DifferentialWeyl(QQ)
sage: W.graded_algebra()
Graded Algebra of Differential Weyl algebra of
 polynomials in x, y over Rational Field
```

**extra_super_categories**()

EXAMPLES:

```
sage: C = Algebras(ZZ).WithBasis().Super()
sage: sorted(C.super_categories(), key=str) # indirect doctest
[Category of graded algebras with basis over Integer Ring,
 Category of super algebras over Integer Ring,
 Category of super modules with basis over Integer Ring]
```

## 3.138 Super Hopf algebras with basis

**class** sage.categories.super_hopf_algebras_with_basis.**SuperHopfAlgebrasWithBasis**(*base_category*)

Bases: *sage.categories.super_modules.SuperModulesCategory*

The category of super Hopf algebras with a distinguished basis.

EXAMPLES:

```
sage: C = HopfAlgebras(ZZ).WithBasis().Super(); C
Category of super hopf algebras with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of super algebras with basis over Integer Ring,
 Category of super coalgebras with basis over Integer Ring,
 Category of super hopf algebras over Integer Ring]
```

## 3.139 Super modules

**class** sage.categories.super_modules.**SuperModules**(*base_category*)

Bases: *sage.categories.super_modules.SuperModulesCategory*

The category of super modules.

An $R$-*super module* (where $R$ is a ring) is an $R$-module $M$ equipped with a decomposition $M = M_0 \oplus M_1$ into two $R$-submodules $M_0$ and $M_1$ (called the *even part* and the *odd part* of $M$, respectively).

Thus, an $R$-super module automatically becomes a $\mathbf{Z}/2\mathbf{Z}$-graded $R$-module, with $M_0$ being the degree-0 component and $M_1$ being the degree-1 component.

EXAMPLES:

```
sage: Modules(ZZ).Super()
Category of super modules over Integer Ring
sage: Modules(ZZ).Super().super_categories()
[Category of graded modules over Integer Ring]
```

The category of super modules defines the super structure which shall be preserved by morphisms:

```
sage: Modules(ZZ).Super().additional_structure()
Category of super modules over Integer Ring
```

**class ElementMethods**

**is_even**()

Return if `self` is an even element.

EXAMPLES:

```
sage: cat = Algebras(QQ).WithBasis().Super()
sage: C = CombinatorialFreeModule(QQ, Partitions(), category=cat)
sage: C.degree_on_basis = sum
sage: C.basis()[2,2,1].is_even()
False
sage: C.basis()[2,2].is_even()
True
```

**is_even_odd**()

Return `0` if `self` is an even element or `1` if an odd element.

**Note:** The default implementation assumes that the even/odd is determined by the parity of `degree()`.

Overwrite this method if the even/odd behavior is desired to be independent.

EXAMPLES:

```
sage: cat = Algebras(QQ).WithBasis().Super()
sage: C = CombinatorialFreeModule(QQ, Partitions(), category=cat)
sage: C.degree_on_basis = sum
sage: C.basis()[2,2,1].is_even_odd()
1
sage: C.basis()[2,2].is_even_odd()
0
```

**is_odd**()

Return if `self` is an odd element.

EXAMPLES:

```
sage: cat = Algebras(QQ).WithBasis().Super()
sage: C = CombinatorialFreeModule(QQ, Partitions(), category=cat)
sage: C.degree_on_basis = sum
sage: C.basis()[2,2,1].is_odd()
True
sage: C.basis()[2,2].is_odd()
False
```

**class ParentMethods**

**extra_super_categories**()

Adds *VectorSpaces* to the super categories of `self` if the base ring is a field.

EXAMPLES:

```
sage: Modules(QQ).Super().extra_super_categories()
[Category of vector spaces over Rational Field]
sage: Modules(ZZ).Super().extra_super_categories()
[]
```

This makes sure that `Modules(QQ).Super()` returns an instance of *SuperModules* and not a join category of an instance of this class and of `VectorSpaces(QQ)`:

```
sage: type(Modules(QQ).Super())
<class 'sage.categories.super_modules.SuperModules_with_category'>
```

---

**Todo:** Get rid of this workaround once there is a more systematic approach for the alias `Modules(QQ)` -> `VectorSpaces(QQ)`. Probably the latter should be a category with axiom, and covariant constructions should play well with axioms.

---

**super_categories**()

EXAMPLES:

```
sage: Modules(ZZ).Super().super_categories()
[Category of graded modules over Integer Ring]
```

Nota bene:

```
sage: Modules(QQ).Super()
Category of super modules over Rational Field
sage: Modules(QQ).Super().super_categories()
[Category of graded modules over Rational Field]
```

**class** sage.categories.super_modules.**SuperModulesCategory**(*base_category*)

Bases: *sage.categories.covariant_functorial_construction.CovariantConstructionCategory*, *sage.categories.category_types.Category_over_base_ring*

EXAMPLES:

```
sage: C = Algebras(QQ).Super()
sage: C
Category of super algebras over Rational Field
sage: C.base_category()
Category of algebras over Rational Field
sage: sorted(C.super_categories(), key=str)
[Category of graded algebras over Rational Field,
 Category of super modules over Rational Field]

sage: AlgebrasWithBasis(QQ).Super().base_ring()
Rational Field
sage: HopfAlgebrasWithBasis(QQ).Super().base_ring()
Rational Field
```

**classmethod default_super_categories**(*category*, **args*)

Return the default super categories of $F_{Cat}(A, B, ...)$ for $A, B, ...$ parents in $Cat$.

INPUT:

- `cls` – the category class for the functor $F$
- `category` – a category $Cat$
- `*args` – further arguments for the functor

OUTPUT:

A join category.

This implements the property that subcategories constructed by the set of whitelisted axioms is a subcategory.

EXAMPLES:

```
sage: HopfAlgebras(ZZ).WithBasis().FiniteDimensional().Super() # indirect␣
↪doctest
Category of finite dimensional super hopf algebras with basis over Integer␣
↪Ring
```

## 3.140 Super modules with basis

**class** sage.categories.super_modules_with_basis.**SuperModulesWithBasis**(*base_category*)

Bases: *sage.categories.super_modules.SuperModulesCategory*

The category of super modules with a distinguished basis.

An $R$-*super module with a distinguished basis* is an $R$-super module equipped with an $R$-module basis whose elements are homogeneous.

EXAMPLES:

```
sage: C = GradedModulesWithBasis(ZZ); C
Category of graded modules with basis over Integer Ring
sage: sorted(C.super_categories(), key=str)
[Category of filtered modules with basis over Integer Ring,
 Category of graded modules over Integer Ring]
sage: C is ModulesWithBasis(ZZ).Graded()
True
```

**class ElementMethods**

**even_component**()

Return the even component of `self`.

EXAMPLES:

```
sage: Q = QuadraticForm(QQ, 2, [1,2,3])
sage: C.<x,y> = CliffordAlgebra(Q)
sage: a = x*y + x - 3*y + 4
sage: a.even_component()
x*y + 4
```

**is_even_odd**()

Return `0` if `self` is an even element and `1` if `self` is an odd element.

EXAMPLES:

```
sage: Q = QuadraticForm(QQ, 2, [1,2,3])
sage: C.<x,y> = CliffordAlgebra(Q)
sage: a = x + y
sage: a.is_even_odd()
1
sage: a = x*y + 4
sage: a.is_even_odd()
0
sage: a = x + 4
sage: a.is_even_odd()
Traceback (most recent call last):
...
ValueError: element is not homogeneous

sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: (x*y).is_even_odd()
0
```

**is_super_homogeneous**()

Return whether this element is homogeneous, in the sense of a super module (i.e., is even or odd).

EXAMPLES:

```
sage: Q = QuadraticForm(QQ, 2, [1,2,3])
sage: C.<x,y> = CliffordAlgebra(Q)
sage: a = x + y
sage: a.is_super_homogeneous()
True
sage: a = x*y + 4
sage: a.is_super_homogeneous()
True
sage: a = x*y + x - 3*y + 4
```

```
sage: a.is_super_homogeneous()
False
```

The exterior algebra has a $\mathbf{Z}$ grading, which induces the $\mathbf{Z}/2\mathbf{Z}$ grading. However the definition of homogeneous elements differs because of the different gradings:

```
sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: a = x*y + 4
sage: a.is_super_homogeneous()
True
sage: a.is_homogeneous()
False
```

**odd_component**()

Return the odd component of `self`.

EXAMPLES:

```
sage: Q = QuadraticForm(QQ, 2, [1,2,3])
sage: C.<x,y> = CliffordAlgebra(Q)
sage: a = x*y + x - 3*y + 4
sage: a.odd_component()
x - 3*y
```

**class ParentMethods**

## 3.141 Topological Spaces

**class** sage.categories.topological_spaces.**TopologicalSpaces**(*category*, **args*)

Bases: *sage.categories.topological_spaces.TopologicalSpacesCategory*

The category of topological spaces.

EXAMPLES:

```
sage: Sets().Topological()
Category of topological spaces
sage: Sets().Topological().super_categories()
[Category of sets]
```

The category of topological spaces defines the topological structure, which shall be preserved by morphisms:

```
sage: Sets().Topological().additional_structure()
Category of topological spaces
```

**class Compact**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

The category of compact topological spaces.

**class Connected**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom*

The category of connected topological spaces.

**class SubcategoryMethods**

**Compact**()
Return the subcategory of the compact objects of self.

EXAMPLES:

```
sage: Sets().Topological().Compact()
Category of compact topological spaces
```

**Connected**()
Return the full subcategory of the connected objects of self.

EXAMPLES:

```
sage: Sets().Topological().Connected()
Category of connected topological spaces
```

**class** sage.categories.topological_spaces.**TopologicalSpacesCategory**(*category*, **args*)
Bases: *sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory*

## 3.142 Unique factorization domains

**class** sage.categories.unique_factorization_domains.**UniqueFactorizationDomains**(*s=None*)
Bases: *sage.categories.category_singleton.Category_singleton*

The category of unique factorization domains constructive unique factorization domains, i.e. where one can constructively factor members into a product of a finite number of irreducible elements

EXAMPLES:

```
sage: UniqueFactorizationDomains()
Category of unique factorization domains
sage: UniqueFactorizationDomains().super_categories()
[Category of gcd domains]
```

**class ElementMethods**

**class ParentMethods**

**is_unique_factorization_domain**(*proof=True*)
Return True, since this in an object of the category of unique factorization domains.

EXAMPLES:

```
sage: Parent(QQ,category=UniqueFactorizationDomains()).is_unique_
↪factorization_domain()
True
```

**additional_structure**()
Return whether self is a structure category.

**See also:**

*Category.additional_structure()*

The category of unique factorization domains does not define additional structure: a ring morphism between unique factorization domains is a unique factorization domain morphism.

EXAMPLES:

```
sage: UniqueFactorizationDomains().additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: UniqueFactorizationDomains().super_categories()
[Category of gcd domains]
```

## 3.143 Unital algebras

**class** sage.categories.unital_algebras.**UnitalAlgebras**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

The category of non-associative algebras over a given base ring.

A non-associative algebra over a ring $R$ is a module over $R$ which s also a unital magma.

> **Warning:** Until trac ticket #15043 is implemented, *Algebras* is the category of associative unital algebras; thus, unlike the name suggests, *UnitalAlgebras* is not a subcategory of *Algebras* but of *MagmaticAlgebras*.

EXAMPLES:

```
sage: from sage.categories.unital_algebras import UnitalAlgebras
sage: C = UnitalAlgebras(ZZ); C
Category of unital algebras over Integer Ring
```

**class ParentMethods**

**from_base_ring**(*r*)

Return the canonical embedding of `r` into `self`.

INPUT:

- `r` – an element of `self.base_ring()`

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example(); A
An example of an algebra with basis: the free algebra on the generators (
↪'a', 'b', 'c') over Rational Field
sage: A.from_base_ring(1)
B[word: ]
```

**class WithBasis**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class ParentMethods**

**from_base_ring**()

**from_base_ring_from_one_basis**(*r*)

Implement the canonical embedding from the ground ring.

INPUT:

- r – an element of the coefficient ring

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()
sage: A.from_base_ring_from_one_basis(3)
3*B[word: ]
sage: A.from_base_ring(3)
3*B[word: ]
sage: A(3)
3*B[word: ]
```

**one**()

Return the multiplicative unit element.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()
sage: A.one_basis()
word:
sage: A.one()
B[word: ]
```

**one_basis**()

When the one of an algebra with basis is an element of this basis, this optional method can return the index of this element. This is used to provide a default implementation of *one()*, and an optimized default implementation of *from_base_ring()*.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()
sage: A.one_basis()
word:
sage: A.one()
B[word: ]
sage: A.from_base_ring(4)
4*B[word: ]
```

**one_from_one_basis**()

Return the one of the algebra, as per `Monoids.ParentMethods.one()`

By default, this is implemented from *one_basis()*, if available.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()
sage: A.one_basis()
word:
sage: A.one_from_one_basis()
B[word: ]
sage: A.one()
B[word: ]
```

Even if called in the wrong order, they should returns their respective one:

```
sage: Bone().parent() is B
True
sage: Aone().parent() is A
True
```

## 3.144 Vector Spaces

**class** sage.categories.vector_spaces.**VectorSpaces**(*K*)

Bases: *sage.categories.category_types.Category_module*

The category of (abstract) vector spaces over a given field

??? with an embedding in an ambient vector space ???

EXAMPLES:

```
sage: VectorSpaces(QQ)
Category of vector spaces over Rational Field
sage: VectorSpaces(QQ).super_categories()
[Category of modules over Rational Field]
```

**class CartesianProducts**(*category*, **args*)

Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**extra_super_categories**()

The category of vector spaces is closed under Cartesian products:

```
sage: C = VectorSpaces(QQ)
sage: C.CartesianProducts()
Category of Cartesian products of vector spaces over Rational Field
sage: C in C.CartesianProducts().super_categories()
True
```

**class DualObjects**(*category*, **args*)

Bases: *sage.categories.dual.DualObjectsCategory*

**extra_super_categories**()

Returns the dual category

EXAMPLES:

The category of algebras over the Rational Field is dual to the category of coalgebras over the same field:

```
sage: C = VectorSpaces(QQ)
sage: C.dual()
Category of duals of vector spaces over Rational Field
sage: C.dual().super_categories() # indirect doctest
[Category of vector spaces over Rational Field]
```

**class ElementMethods**

**class ParentMethods**

**class TensorProducts**(*category*, **args*)

Bases: *sage.categories.tensor.TensorProductsCategory*

**extra_super_categories**()

The category of vector spaces is closed under tensor products:

```
sage: C = VectorSpaces(QQ)
sage: C.TensorProducts()
Category of tensor products of vector spaces over Rational Field
sage: C in C.TensorProducts().super_categories()
True
```

**class WithBasis** (*base_category*)
Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_over_base_ring*

**class CartesianProducts** (*category*, **args*)
Bases: *sage.categories.cartesian_product.CartesianProductsCategory*

**extra_super_categories** ()
The category of vector spaces with basis is closed under Cartesian products:

```
sage: C = VectorSpaces(QQ).WithBasis()
sage: C.CartesianProducts()
Category of Cartesian products of vector spaces with basis over␣
↪Rational Field
sage: C in C.CartesianProducts().super_categories()
True
```

**class TensorProducts** (*category*, **args*)
Bases: *sage.categories.tensor.TensorProductsCategory*

**extra_super_categories** ()
The category of vector spaces with basis is closed under tensor products:

```
sage: C = VectorSpaces(QQ).WithBasis()
sage: C.TensorProducts()
Category of tensor products of vector spaces with basis over Rational␣
↪Field
sage: C in C.TensorProducts().super_categories()
True
```

**is_abelian** ()
Return whether this category is abelian.

This is always `True` since the base ring is a field.

EXAMPLES:

```
sage: VectorSpaces(QQ).WithBasis().is_abelian()
True
```

**additional_structure** ()
Return `None`.

Indeed, the category of vector spaces defines no additional structure: a bimodule morphism between two vector spaces is a vector space morphism.

**See also:**

*Category.additional_structure()*

---

**Todo:** Should this category be a `CategoryWithAxiom`?

---

EXAMPLES:

```
sage: VectorSpaces(QQ).additional_structure()
```

**base_field** ()
Returns the base field over which the vector spaces of this category are all defined.

EXAMPLES:

```
sage: VectorSpaces(QQ).base_field()
Rational Field
```

**super_categories**()

EXAMPLES:

```
sage: VectorSpaces(QQ).super_categories()
[Category of modules over Rational Field]
```

## 3.145 Weyl Groups

**class** sage.categories.weyl_groups.**WeylGroups**(*s=None*)

Bases: *sage.categories.category_singleton.Category_singleton*

The category of Weyl groups

See the Wikipedia page of Weyl Groups.

EXAMPLES:

```
sage: WeylGroups()
Category of weyl groups
sage: WeylGroups().super_categories()
[Category of coxeter groups]
```

Here are some examples:

```
sage: WeylGroups().example()            # todo: not implemented
sage: FiniteWeylGroups().example()
The symmetric group on {0, ..., 3}
sage: AffineWeylGroups().example()      # todo: not implemented
sage: WeylGroup(["B", 3])
Weyl Group of type ['B', 3] (as a matrix group acting on the ambient space)
```

This one will eventually be also in this category:

```
sage: SymmetricGroup(4)
Symmetric group of order 4! as a permutation group
```

**class ElementMethods**

**bruhat_lower_covers_coroots**()

Return all 2-tuples (v, $\alpha$) where v is covered by self and $\alpha$ is the positive coroot such that self = v $s_\alpha$ where $s_\alpha$ is the reflection orthogonal to $\alpha$.

ALGORITHM:

See *bruhat_lower_covers()* and *bruhat_lower_covers_reflections()* for Coxeter groups.

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.bruhat_lower_covers_coroots()
[(s1*s2*s1, alphacheck[1] + alphacheck[2] + alphacheck[3]),
 (s3*s2*s1, alphacheck[2]), (s3*s1*s2, alphacheck[1])]
```

**bruhat_upper_covers_coroots**()

Returns all 2-tuples (v, $\alpha$) where v is covers self and $\alpha$ is the positive coroot such that self = v $s_\alpha$ where $s_\alpha$ is the reflection orthogonal to $\alpha$.

ALGORITHM:

See *bruhat_upper_covers()* and *bruhat_upper_covers_reflections()* for Coxeter groups.

EXAMPLES:

```
sage: W = WeylGroup(['A',4], prefix="s")
sage: w = W.from_reduced_word([3,1,2,1])
sage: w.bruhat_upper_covers_coroots()
[(s1*s2*s3*s2*s1, alphacheck[3]),
 (s2*s3*s1*s2*s1, alphacheck[2] + alphacheck[3]),
 (s3*s4*s1*s2*s1, alphacheck[4]),
 (s4*s3*s1*s2*s1, alphacheck[1] + alphacheck[2] + alphacheck[3] +
↪alphacheck[4])]
```

**inversion_arrangement**(*side='right'*)

Return the inversion hyperplane arrangement of self.

INPUT:

- side – 'right' (default) or 'left'

OUTPUT:

A (central) hyperplane arrangement whose hyperplanes correspond to the inversions of self given as roots.

The side parameter determines on which side to compute the inversions.

EXAMPLES:

```
sage: W = WeylGroup(['A',3])
sage: w = W.from_reduced_word([1, 2, 3, 1, 2])
sage: A = w.inversion_arrangement(); A
Arrangement of 5 hyperplanes of dimension 3 and rank 3
sage: A.hyperplanes()
(Hyperplane 0*a1 + 0*a2 + a3 + 0,
 Hyperplane 0*a1 + a2 + 0*a3 + 0,
 Hyperplane 0*a1 + a2 + a3 + 0,
 Hyperplane a1 + a2 + 0*a3 + 0,
 Hyperplane a1 + a2 + a3 + 0)
```

The identity element gives the empty arrangement:

```
sage: W = WeylGroup(['A',3])
sage: W.one().inversion_arrangement()
Empty hyperplane arrangement of dimension 3
```

**inversions**(*side='right'*, *inversion_type='reflections'*)

Returns the set of inversions of self.

INPUT:

- side – 'right' (default) or 'left'
- inversion_type – 'reflections' (default), 'roots', or 'coroots'.

OUTPUT:

For reflections, the set of reflections r in the Weyl group such that `self r < self`. For (co)roots, the set of positive (co)roots that are sent by `self` to negative (co)roots; their associated reflections are described above.

If `side` is 'left', the inverse Weyl group element is used.

EXAMPLES:

```
sage: W=WeylGroup(['C',2], prefix="s")
sage: w=W.from_reduced_word([1,2])
sage: w.inversions()
[s2, s2*s1*s2]
sage: w.inversions(inversion_type = 'reflections')
[s2, s2*s1*s2]
sage: w.inversions(inversion_type = 'roots')
[alpha[2], alpha[1] + alpha[2]]
sage: w.inversions(inversion_type = 'coroots')
[alphacheck[2], alphacheck[1] + 2*alphacheck[2]]
sage: w.inversions(side = 'left')
[s1, s1*s2*s1]
sage: w.inversions(side = 'left', inversion_type = 'roots')
[alpha[1], 2*alpha[1] + alpha[2]]
sage: w.inversions(side = 'left', inversion_type = 'coroots')
[alphacheck[1], alphacheck[1] + alphacheck[2]]
```

**is_pieri_factor**()

Returns whether `self` is a Pieri factor, as used for computing Stanley symmetric functions.

**See also:**

- *stanley_symmetric_function()*
- *WeylGroups.ParentMethods.pieri_factors()*

EXAMPLES:

```
sage: W = WeylGroup(['A',5,1])
sage: W.from_reduced_word([3,2,5]).is_pieri_factor()
True
sage: W.from_reduced_word([3,2,4,5]).is_pieri_factor()
False

sage: W = WeylGroup(['C',4,1])
sage: W.from_reduced_word([0,2,1]).is_pieri_factor()
True
sage: W.from_reduced_word([0,2,1,0]).is_pieri_factor()
False

sage: W = WeylGroup(['B',3])
sage: W.from_reduced_word([3,2,3]).is_pieri_factor()
False
sage: W.from_reduced_word([2,1,2]).is_pieri_factor()
True
```

**left_pieri_factorizations**(*max_length=+Infinity*)

Returns all factorizations of `self` as $uv$, where $u$ is a Pieri factor and $v$ is an element of the Weyl group.

**See also:**

- *WeylGroups.ParentMethods.pieri_factors()*
- `sage.combinat.root_system.pieri_factors`

EXAMPLES:

If we take $w = w_0$ the maximal element of a strict parabolic subgroup of type $A_{n_1} \times \cdots \times A_{n_k}$, then the Pieri factorizations are in correspondence with all Pieri factors, and there are $\prod 2^{n_i}$ of them:

```
sage: W = WeylGroup(['A', 4, 1])
sage: W.from_reduced_word([]).left_pieri_factorizations().cardinality()
1
sage: W.from_reduced_word([1]).left_pieri_factorizations().cardinality()
2
sage: W.from_reduced_word([1,2,1]).left_pieri_factorizations().
↪cardinality()
4
sage: W.from_reduced_word([1,2,3,1,2,1]).left_pieri_factorizations().
↪cardinality()
8

sage: W.from_reduced_word([1,3]).left_pieri_factorizations().cardinality()
4
sage: W.from_reduced_word([1,3,4,3]).left_pieri_factorizations().
↪cardinality()
8

sage: W.from_reduced_word([2,1]).left_pieri_factorizations().cardinality()
3
sage: W.from_reduced_word([1,2]).left_pieri_factorizations().cardinality()
2
sage: [W.from_reduced_word([1,2]).left_pieri_factorizations(max_length=i).
↪cardinality() for i in [-1, 0, 1, 2]]
[0, 1, 2, 2]

sage: W = WeylGroup(['C',4,1])
sage: w = W.from_reduced_word([0,3,2,1,0])
sage: w.left_pieri_factorizations().cardinality()
7
sage: [(u.reduced_word(),v.reduced_word()) for (u,v) in w.left_pieri_
↪factorizations()]
[([], [3, 2, 0, 1, 0]),
([0], [3, 2, 1, 0]),
([3], [2, 0, 1, 0]),
([3, 0], [2, 1, 0]),
([3, 2], [0, 1, 0]),
([3, 2, 0], [1, 0]),
([3, 2, 0, 1], [0])]

sage: W = WeylGroup(['B',4,1])
sage: W.from_reduced_word([0,2,1,0]).left_pieri_factorizations().
↪cardinality()
6
```

**quantum_bruhat_successors**(*index_set=None*, *roots=False*, *quantum_only=False*)

Return the successors of `self` in the quantum Bruhat graph on the parabolic quotient of the Weyl group determined by the subset of Dynkin nodes `index_set`.

INPUT:

- `self` – a Weyl group element, which is assumed to be of minimum length in its coset with respect to the parabolic subgroup
- `index_set` – (default: `None`) indicates the set of simple reflections used to generate the parabolic subgroup; the default value indicates that the subgroup is the identity

- `roots` – (default: `False`) if `True`, returns the list of 2-tuples ($w$, $\alpha$) where `w` is a successor and $\alpha$ is the positive root associated with the successor relation
- `quantum_only` – (default: `False`) if `True`, returns only the quantum successors

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: w = W.from_reduced_word([3,1,2])
sage: w.quantum_bruhat_successors([1], roots = True)
[(s3, alpha[2]), (s1*s2*s3*s2, alpha[3]),
 (s2*s3*s1*s2, alpha[1] + alpha[2] + alpha[3])]
sage: w.quantum_bruhat_successors([1,3])
[1, s2*s3*s1*s2]
sage: w.quantum_bruhat_successors(roots = True)
[(s3*s1*s2*s1, alpha[1]),
 (s3*s1, alpha[2]),
 (s1*s2*s3*s2, alpha[3]),
 (s2*s3*s1*s2, alpha[1] + alpha[2] + alpha[3])]
sage: w.quantum_bruhat_successors()
[s3*s1*s2*s1, s3*s1, s1*s2*s3*s2, s2*s3*s1*s2]
sage: w.quantum_bruhat_successors(quantum_only = True)
[s3*s1]
sage: w = W.from_reduced_word([2,3])
sage: w.quantum_bruhat_successors([1,3])
Traceback (most recent call last):
...
ValueError: s2*s3 is not of minimum length in its coset of the parabolic␣
↪subgroup generated by the reflections (1, 3)
```

**reflection_to_coroot**()

Returns the coroot associated with the reflection `self`.

EXAMPLES:

```
sage: W=WeylGroup(['C',2],prefix="s")
sage: W.from_reduced_word([1,2,1]).reflection_to_coroot()
alphacheck[1] + alphacheck[2]
sage: W.from_reduced_word([1,2]).reflection_to_coroot()
Traceback (most recent call last):
...
ValueError: s1*s2 is not a reflection
sage: W.long_element().reflection_to_coroot()
Traceback (most recent call last):
...
ValueError: s2*s1*s2*s1 is not a reflection
```

**reflection_to_root**()

Returns the root associated with the reflection `self`.

EXAMPLES:

```
sage: W=WeylGroup(['C',2],prefix="s")
sage: W.from_reduced_word([1,2,1]).reflection_to_root()
2*alpha[1] + alpha[2]
sage: W.from_reduced_word([1,2]).reflection_to_root()
Traceback (most recent call last):
...
ValueError: s1*s2 is not a reflection
sage: W.long_element().reflection_to_root()
Traceback (most recent call last):
```

```
...
ValueError: s2*s1*s2*s1 is not a reflection
```

**stanley_symmetric_function**()

Return the affine Stanley symmetric function indexed by `self`.

INPUT:

- `self` – an element $w$ of a Weyl group

Returns the affine Stanley symmetric function indexed by $w$. Stanley symmetric functions are defined as generating series of the factorizations of $w$ into Pieri factors and weighted by a statistic on Pieri factors.

**See also:**

- *stanley_symmetric_function_as_polynomial()*
- *WeylGroups.ParentMethods.pieri_factors()*
- `sage.combinat.root_system.pieri_factors`

EXAMPLES:

```
sage: W = WeylGroup(['A', 3, 1])
sage: W.from_reduced_word([3,1,2,0,3,1,0]).stanley_symmetric_function()
8*m[1, 1, 1, 1, 1, 1, 1] + 4*m[2, 1, 1, 1, 1, 1] + 2*m[2, 2, 1, 1, 1] +
↪m[2, 2, 2, 1]
sage: A = AffinePermutationGroup(['A',3,1])
sage: A.from_reduced_word([3,1,2,0,3,1,0]).stanley_symmetric_function()
8*m[1, 1, 1, 1, 1, 1, 1] + 4*m[2, 1, 1, 1, 1, 1] + 2*m[2, 2, 1, 1, 1] +
↪m[2, 2, 2, 1]

sage: W = WeylGroup(['C',3,1])
sage: W.from_reduced_word([0,2,1,0]).stanley_symmetric_function()
32*m[1, 1, 1, 1] + 16*m[2, 1, 1] + 8*m[2, 2] + 4*m[3, 1]

sage: W = WeylGroup(['B',3,1])
sage: W.from_reduced_word([3,2,1]).stanley_symmetric_function()
2*m[1, 1, 1] + m[2, 1] + 1/2*m[3]

sage: W = WeylGroup(['B',4])
sage: w = W.from_reduced_word([3,2,3,1])
sage: w.stanley_symmetric_function()  # long time (6s on sage.math, 2011)
48*m[1, 1, 1, 1] + 24*m[2, 1, 1] + 12*m[2, 2] + 8*m[3, 1] + 2*m[4]

sage: A = AffinePermutationGroup(['A',4,1])
sage: a = A([-2,0,1,4,12])
sage: a.stanley_symmetric_function()
6*m[1, 1, 1, 1, 1, 1, 1, 1] + 5*m[2, 1, 1, 1, 1, 1, 1] + 4*m[2, 2, 1, 1,
↪1, 1]
+ 3*m[2, 2, 2, 1, 1] + 2*m[2, 2, 2, 2] + 4*m[3, 1, 1, 1, 1, 1] + 3*m[3, 2,
↪ 1, 1, 1]
+ 2*m[3, 2, 2, 1] + 2*m[3, 3, 1, 1] + m[3, 3, 2] + 3*m[4, 1, 1, 1, 1] +
↪2*m[4, 2, 1, 1]
+ m[4, 2, 2] + m[4, 3, 1]
```

One more example (trac ticket #14095):

```
sage: G = SymmetricGroup(4)
sage: w = G.from_reduced_word([3,2,3,1])
sage: w.stanley_symmetric_function()
3*m[1, 1, 1, 1] + 2*m[2, 1, 1] + m[2, 2] + m[3, 1]
```

REFERENCES:

- [BH1994]
- [Lam2008]
- [LSS2009]
- [Pon2010]

**stanley_symmetric_function_as_polynomial**(*max_length=+Infinity*)

Returns a multivariate generating function for the number of factorizations of a Weyl group element into Pieri factors of decreasing length, weighted by a statistic on Pieri factors.

**See also:**

- *stanley_symmetric_function()*
- *WeylGroups.ParentMethods.pieri_factors()*
- sage.combinat.root_system.pieri_factors

INPUT:

- self – an element $w$ of a Weyl group $W$
- max_length – a non negative integer or infinity (default: infinity)

Returns the generating series for the Pieri factorizations $w = u_1 \cdots u_k$, where $u_i$ is a Pieri factor for all $i$, $l(w) = \sum_{i=1}^{k} l(u_i)$ and max_length $\geq l(u_1) \geq \cdots \geq l(u_k)$.

A factorization $u_1 \cdots u_k$ contributes a monomial of the form $\prod_i x_{l(u_i)}$, with coefficient given by $\prod_i 2^{c(u_i)}$, where $c$ is a type-dependent statistic on Pieri factors, as returned by the method u[i].stanley_symm_poly_weight().

EXAMPLES:

```
sage: W = WeylGroup(['A', 3, 1])
sage: W.from_reduced_word([]).stanley_symmetric_function_as_polynomial()
1
sage: W.from_reduced_word([1]).stanley_symmetric_function_as_polynomial()
x1
sage: W.from_reduced_word([1,2]).stanley_symmetric_function_as_
↪polynomial()
x1^2
sage: W.from_reduced_word([2,1]).stanley_symmetric_function_as_
↪polynomial()
x1^2 + x2
sage: W.from_reduced_word([1,2,1]).stanley_symmetric_function_as_
↪polynomial()
2*x1^3 + x1*x2
sage: W.from_reduced_word([1,2,1,0]).stanley_symmetric_function_as_
↪polynomial()
3*x1^4 + 2*x1^2*x2 + x2^2 + x1*x3
sage: W.from_reduced_word([1,2,3,1,2,1,0]).stanley_symmetric_function_as_
↪polynomial() # long time
22*x1^7 + 11*x1^5*x2 + 5*x1^3*x2^2 + 3*x1^4*x3 + 2*x1*x2^3 + x1^2*x2*x3
sage: W.from_reduced_word([3,1,2,0,3,1,0]).stanley_symmetric_function_as_
↪polynomial() # long time
8*x1^7 + 4*x1^5*x2 + 2*x1^3*x2^2 + x1*x2^3

sage: W = WeylGroup(['C',3,1])
sage: W.from_reduced_word([0,2,1,0]).stanley_symmetric_function_as_
↪polynomial()
32*x1^4 + 16*x1^2*x2 + 8*x2^2 + 4*x1*x3

sage: W = WeylGroup(['B',3,1])
```

```
sage: W.from_reduced_word([3,2,1]).stanley_symmetric_function_as_
↪polynomial()
2*x1^3 + x1*x2 + 1/2*x3
```

Algorithm: Induction on the left Pieri factors. Note that this induction preserves subsets of $W$ which are stable by taking right factors, and in particular Grassmanian elements.

**Finite**
alias of *FiniteWeylGroups*

**class ParentMethods**

**coxeter_matrix**()
Return the Coxeter matrix associated to `self`.

EXAMPLES:

```
sage: G = WeylGroup(['A',3])
sage: G.coxeter_matrix()
[1 3 2]
[3 1 3]
[2 3 1]
```

**pieri_factors**(**args*, ***keywords*)
Returns the set of Pieri factors in this Weyl group.

For any type, the set of Pieri factors forms a lower ideal in Bruhat order, generated by all the conjugates of some special element of the Weyl group. In type $A_n$, this special element is $s_n \cdots s_1$, and the conjugates are obtained by rotating around this reduced word.

These are used to compute Stanley symmetric functions.

**See also:**

- *WeylGroups.ElementMethods.stanley_symmetric_function()*
- `sage.combinat.root_system.pieri_factors`

EXAMPLES:

```
sage: W = WeylGroup(['A',5,1])
sage: PF = W.pieri_factors()
sage: PF.cardinality()
63

sage: W = WeylGroup(['B',3])
sage: PF = W.pieri_factors()
sage: [w.reduced_word() for w in PF]
[[1, 2, 3, 2, 1],
 [1, 2, 3, 2],
 [2, 3, 2],
 [3, 1, 2],
 [1, 2, 3, 1],
 [1, 2, 1],
 [3, 1],
 [2, 1],
 [2, 3, 2, 1],
 [1, 2, 3],
 [3, 1, 2, 1],
 [2, 3],
```

```
 [3, 2],
 [1, 2],
 [3],
 [],
 [2],
 [3, 2, 1],
 [2, 3, 1],
 [1]]
sage: W = WeylGroup(['C',4,1])
sage: PF = W.pieri_factors()
sage: W.from_reduced_word([3,2,0]) in PF
True
```

**quantum_bruhat_graph**(*index_set=()*)

Return the quantum Bruhat graph of the quotient of the Weyl group by a parabolic subgroup $W_J$.

INPUT:

- `index_set` – (default: ()) a tuple $J$ of nodes of the Dynkin diagram

By default, the value for `index_set` indicates that the subgroup is trivial and the quotient is the full Weyl group.

EXAMPLES:

```
sage: W = WeylGroup(['A',3], prefix="s")
sage: g = W.quantum_bruhat_graph((1,3))
sage: g
Parabolic Quantum Bruhat Graph of Weyl Group of type ['A', 3] (as a␣
↪matrix group acting on the ambient space) for nodes (1, 3): Digraph on␣
↪6 vertices
sage: g.vertices()
[s2*s3*s1*s2, s3*s1*s2, s1*s2, s3*s2, s2, 1]
sage: g.edges()
[(s2*s3*s1*s2, s2, alpha[2]),
 (s3*s1*s2, s2*s3*s1*s2, alpha[1] + alpha[2] + alpha[3]),
 (s3*s1*s2, 1, alpha[2]),
 (s1*s2, s3*s1*s2, alpha[2] + alpha[3]),
 (s3*s2, s3*s1*s2, alpha[1] + alpha[2]),
 (s2, s1*s2, alpha[1] + alpha[2]),
 (s2, s3*s2, alpha[2] + alpha[3]),
 (1, s2, alpha[2])]
sage: W = WeylGroup(['A',3,1], prefix="s")
sage: g = W.quantum_bruhat_graph()
Traceback (most recent call last):
...
ValueError: the Cartan type ['A', 3, 1] is not finite
```

**additional_structure**()

Return `None`.

Indeed, the category of Weyl groups defines no additional structure: Weyl groups are a special class of Coxeter groups.

**See also:**

*Category.additional_structure()*

**Todo:** Should this category be a `CategoryWithAxiom`?

EXAMPLES:

```
sage: WeylGroups().additional_structure()
```

**super_categories**()

EXAMPLES:

```
sage: WeylGroups().super_categories()
[Category of coxeter groups]
```

## 3.146 Technical Categories

### 3.146.1 Facade Sets

For background, see *What is a facade set?*.

**class** sage.categories.facade_sets.**FacadeSets**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

**class ParentMethods**

**facade_for**()

Returns the parents this set is a facade for

This default implementation assumes that `self` has an attribute `_facade_for`, typically initialized by `Parent.__init__()`. If the attribute is not present, the method raises a NotImplementedError.

EXAMPLES:

```
sage: S = Sets().Facade().example(); S
An example of facade set: the monoid of positive integers
sage: S.facade_for()
(Integer Ring,)
```

Check that trac ticket #13801 is corrected:

```
sage: class A(Parent):
....:     def __init__(self):
....:         Parent.__init__(self, category=Sets(), facade=True)
sage: a = A()
sage: a.facade_for()
Traceback (most recent call last):
...
NotImplementedError: this parent did not specify which parents it is a␣
↪facade for
```

**is_parent_of**(*element*)

Returns whether `self` is the parent of `element`

INPUT:

- `element` – any object

Since `self` is a facade domain, this actually tests whether the parent of `element` is any of the parent `self` is a facade for.

EXAMPLES:

```
sage: S = Sets().Facade().example(); S
An example of facade set: the monoid of positive integers
sage: S.is_parent_of(1)
True
sage: S.is_parent_of(1/2)
False
```

This method differs from `__contains__()` in two ways. First, this does not take into account the fact that `self` may be a strict subset of the parent(s) it is a facade for:

```
sage: -1 in S, S.is_parent_of(-1)
(False, True)
```

Furthermore, there is no coercion attempted:

```
sage: int(1) in S, S.is_parent_of(int(1))
(True, False)
```

> **Warning:** this implementation does not handle facade parents of facade parents. Is this a feature we want generically?

**example**(*choice='subset'*)

Returns an example of facade set, as per `Category.example()`.

INPUT:

- `choice` – 'union' or 'subset' (default: 'subset').

EXAMPLES:

```
sage: Sets().Facade().example()
An example of facade set: the monoid of positive integers
sage: Sets().Facade().example(choice='union')
An example of a facade set: the integers completed by +-infinity
sage: Sets().Facade().example(choice='subset')
An example of facade set: the monoid of positive integers
```

CHAPTER

FOUR

# FUNCTORIAL CONSTRUCTIONS

## 4.1 Covariant Functorial Constructions

A *functorial construction* is a collection of functors $(F_{Cat})_{Cat}$ (indexed by a collection of categories) which associate to a sequence of parents $(A, B, ...)$ in a category $Cat$ a parent $F_{Cat}(A, B, ...)$. Typical examples of functorial constructions are `cartesian_product` and `tensor_product`.

The category of $F_{Cat}(A, B, ...)$, which only depends on $Cat$, is called the (functorial) construction category.

A functorial construction is *(category)-covariant* if for every categories $Cat$ and $SuperCat$, the category of $F_{Cat}(A, B, ...)$ is a subcategory of the category of $F_{SuperCat}(A, B, ...)$ whenever $Cat$ is a subcategory of $SuperCat$. A functorial construction is *(category)-regressive* if the category of $F_{Cat}(A, B, ...)$ is a subcategory of $Cat$.

The goal of this module is to provide generic support for covariant functorial constructions. In particular, given some parents $A$, $B$, $\ldots$, in respective categories $Cat_A$, $Cat_B$, $\ldots$, it provides tools for calculating the best known category for the parent $F(A, B, ...)$. For examples, knowing that Cartesian products of semigroups (resp. monoids, groups) have a semigroup (resp. monoid, group) structure, and given a group $B$ and two monoids $A$ and $C$ it can calculate that $A \times B \times C$ is naturally endowed with a monoid structure.

See *CovariantFunctorialConstruction*, *CovariantConstructionCategory* and *RegressiveCovariantConstructionCategory* for more details.

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

**class** sage.categories.covariant_functorial_construction.**CovariantConstructionCategory**(*catego* **args*)

Bases: *sage.categories.covariant_functorial_construction.FunctorialConstructionCategory*

Abstract class for categories $F_{Cat}$ obtained through a covariant functorial construction

**additional_structure**()

Return the additional structure defined by `self`.

By default, a functorial construction category `A.F()` defines additional structure if and only if $A$ is the category defining $F$. The rationale is that, for a subcategory $B$ of $A$, the fact that $B.F()$ morphisms shall preserve the $F$-specific structure is already imposed by $A.F()$.

**See also:**

- *Category.additional_structure()*.
- *is_construction_defined_by_base()*.

EXAMPLES:

sage: Modules(ZZ).Graded().additional_structure() Category of graded modules over Integer Ring sage: Algebras(ZZ).Graded().additional_structure()

**classmethod default_super_categories** (*category*, **args*)

Return the default super categories of $F_{Cat}(A, B, ...)$ for $A, B, ...$ parents in $Cat$.

INPUT:

- `cls` – the category class for the functor $F$
- `category` – a category $Cat$
- `*args` – further arguments for the functor

OUTPUT: a (join) category

The default implementation is to return the join of the categories of $F(A, B, ...)$ for $A, B, ...$ in turn in each of the super categories of `category`.

This is implemented as a class method, in order to be able to reconstruct the functorial category associated to each of the super categories of `category`.

EXAMPLES:

Bialgebras are both algebras and coalgebras:

```
sage: Bialgebras(QQ).super_categories()
[Category of algebras over Rational Field, Category of coalgebras over
↪Rational Field]
```

Hence tensor products of bialgebras are tensor products of algebras and tensor products of coalgebras:

```
sage: Bialgebras(QQ).TensorProducts().super_categories()
[Category of tensor products of algebras over Rational Field, Category of
↪tensor products of coalgebras over Rational Field]
```

Here is how `default_super_categories()` was called internally:

```
sage: sage.categories.tensor.TensorProductsCategory.default_super_
↪categories(Bialgebras(QQ))
Join of Category of tensor products of algebras over Rational Field and
↪Category of tensor products of coalgebras over Rational Field
```

We now show a similar example, with the `Algebra` functor which takes a parameter $\mathbf{Q}$:

```
sage: FiniteMonoids().super_categories()
[Category of monoids, Category of finite semigroups]
sage: sorted(FiniteMonoids().Algebras(QQ).super_categories(), key=str)
[Category of finite dimensional algebras with basis over Rational Field,
 Category of finite set algebras over Rational Field,
 Category of monoid algebras over Rational Field]
```

Note that neither the category of *finite* semigroup algebras nor that of monoid algebras appear in the result; this is because there is currently nothing specific implemented about them.

Here is how `default_super_categories()` was called internally:

```
sage: sage.categories.algebra_functor.AlgebrasCategory.default_super_
↪categories(FiniteMonoids(), QQ)
Join of Category of finite dimensional algebras with basis over Rational Field
    and Category of monoid algebras over Rational Field
    and Category of finite set algebras over Rational Field
```

**is_construction_defined_by_base**()

Return whether the construction is defined by the base of `self`.

EXAMPLES:

The graded functorial construction is defined by the modules category. Hence this method returns `True` for graded modules and `False` for other graded xxx categories:

```
sage: Modules(ZZ).Graded().is_construction_defined_by_base()
True
sage: Algebras(QQ).Graded().is_construction_defined_by_base()
False
sage: Modules(ZZ).WithBasis().Graded().is_construction_defined_by_base()
False
```

This is implemented as follows: given the base category $A$ and the construction $F$ of `self`, that is `self=A.F()`, check whether no super category of $A$ has $F$ defined.

**Note:** Recall that, when $A$ does not implement the construction `F`, a join category is returned. Therefore, in such cases, this method is not available:

```
sage: Coalgebras(QQ).Graded().is_construction_defined_by_base()
Traceback (most recent call last):
...
AttributeError: 'JoinCategory_with_category' object has no attribute 'is_
↪construction_defined_by_base'
```

**class** sage.categories.covariant_functorial_construction.**CovariantFunctorialConstruction**

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.sage_object.SageObject`

An abstract class for construction functors $F$ (eg $F$ = Cartesian product, tensor product, $\mathbf{Q}$-algebra, . . . ) such that:

- Each category $Cat$ (eg $Cat =$ `Groups()`) can provide a category $F_{Cat}$ for parents constructed via this functor (e.g. $F_{Cat} =$ `CartesianProductsOf(Groups())`).
- For every category $Cat$, $F_{Cat}$ is a subcategory of $F_{SuperCat}$ for every super category $SuperCat$ of $Cat$ (the functorial construction is (category)-covariant).
- For parents $A$, $B$, . . . , respectively in the categories $Cat_A$, $Cat_B$, . . . , the category of $F(A, B, ...)$ is $F_{Cat}$ where $Cat$ is the meet of the categories $Cat_A$, $Cat_B$, . . . ,.

This covers two slightly different use cases:

- In the first use case, one uses directly the construction functor to create new parents:

```
sage: tensor()   # todo: not implemented (add an example)
```

  or even new elements, which indirectly constructs the corresponding parent:

```
sage: tensor(...) # todo: not implemented
```

- In the second use case, one implements a parent, and then put it in the category $F_{Cat}$ to specify supplementary mathematical information about that parent.

The main purpose of this class is to handle automatically the trivial part of the category hierarchy. For example, `CartesianProductsOf(Groups())` is set automatically as a subcategory of `CartesianProductsOf(Monoids())`.

In practice, each subclass of this class should provide the following attributes:

- `_functor_category` - a string which should match the name of the nested category class to be used in each category to specify information and generic operations for elements of this category.
- `_functor_name` - an string which specifies the name of the functor, and also (when relevant) of the method on parents and elements used for calling the construction.

TODO: What syntax do we want for $F_{Cat}$? For example, for the tensor product construction, which one of the followings do we want (see chat on IRC, on 07/12/2009):

- `tensor(Cat)`
- `tensor((Cat, Cat))`
- `tensor.of((Cat, Cat))`
- `tensor.category_from_categories((Cat, Cat, Cat))`
- `Cat.TensorProducts()`

The syntax `Cat.TensorProducts()` does not supports well multivariate constructions like `tensor.of([Algebras(), HopfAlgebras(), ...])`. Also it forces every category to be (somehow) aware of all the tensorial construction that could apply to it, even those which are only induced from super categories.

Note: for each functorial construction, there probably is one (or several) largest categories on which it applies. For example, the `CartesianProducts()` construction makes only sense for concrete categories, that is subcategories of `Sets()`. Maybe we want to model this one way or the other.

**category_from_categories**(*categories*)

Return the category of $F(A, B, ...)$ for $A, B, ...$ parents in the given categories.

INPUT:

- `self`: a functor $F$
- `categories`: a non empty tuple of categories

EXAMPLES:

```
sage: Cat1 = Rings()
sage: Cat2 = Groups()
sage: cartesian_product.category_from_categories((Cat1, Cat1, Cat1))
Join of Category of rings and ...
    and Category of Cartesian products of monoids
    and Category of Cartesian products of commutative additive groups

sage: cartesian_product.category_from_categories((Cat1, Cat2))
Category of Cartesian products of monoids
```

**category_from_category**(*category*)

Return the category of $F(A, B, ...)$ for $A, B, ...$ parents in `category`.

INPUT:

- `self`: a functor $F$
- `category`: a category

EXAMPLES:

```
sage: tensor.category_from_category(ModulesWithBasis(QQ))
Category of tensor products of vector spaces with basis over Rational Field
```

# TODO: add support for parametrized functors

**category_from_parents** (*parents*)

Return the category of $F(A, B, ...)$ for $A, B, ...$ parents.

INPUT:

- self: a functor F
- parents: a list (or iterable) of parents.

EXAMPLES:

```
sage: E = CombinatorialFreeModule(QQ, ["a", "b", "c"])
sage: tensor.category_from_parents((E, E, E))
Category of tensor products of vector spaces with basis over Rational Field
```

**class** sage.categories.covariant_functorial_construction.**FunctorialConstructionCategory** (*categ*
**args*

Bases: *sage.categories.category.Category*

Abstract class for categories $F_{Cat}$ obtained through a functorial construction

**base_category** ()

Return the base category of the category self.

For any category B = $F_{Cat}$ obtained through a functorial construction $F$, the call B.base_category() returns the category $Cat$.

EXAMPLES:

```
sage: Semigroups().Quotients().base_category()
Category of semigroups
```

**classmethod category_of** (*category*, **args*)

Return the image category of the functor $F_{Cat}$.

This is the main entry point for constructing the category $F_{Cat}$ of parents $F(A, B, ...)$ constructed from parents $A, B, ...$ in $Cat$.

INPUT:

- cls – the category class for the functorial construction $F$
- category – a category $Cat$
- *args – further arguments for the functor

EXAMPLES:

```
sage: sage.categories.tensor.TensorProductsCategory.category_
↪of(ModulesWithBasis(QQ))
Category of tensor products of vector spaces with basis over Rational Field

sage: sage.categories.algebra_functor.AlgebrasCategory.category_
↪of(FiniteMonoids(), QQ)
Join of Category of finite dimensional algebras with basis over Rational Field
    and Category of monoid algebras over Rational Field
    and Category of finite set algebras over Rational Field
```

**extra_super_categories** ()

Return the extra super categories of a construction category.

Default implementation which returns [].

EXAMPLES:

```
sage: Sets().Subquotients().extra_super_categories()
[]
sage: Semigroups().Quotients().extra_super_categories()
[]
```

**super_categories**()

Return the super categories of a construction category.

EXAMPLES:

```
sage: Sets().Subquotients().super_categories()
[Category of sets]
sage: Semigroups().Quotients().super_categories()
[Category of subquotients of semigroups, Category of quotients of sets]
```

**class** sage.categories.covariant_functorial_construction.**RegressiveCovariantConstructionCate**

Bases: *sage.categories.covariant_functorial_construction.CovariantConstructionCategory*

Abstract class for categories $F_{Cat}$ obtained through a regressive covariant functorial construction

**classmethod default_super_categories**(*category*, **args*)

Return the default super categories of $F_{Cat}(A, B, ...)$ for $A, B, ...$ parents in $Cat$.

INPUT:

- cls – the category class for the functor $F$
- category – a category $Cat$
- *args – further arguments for the functor

OUTPUT:

A join category.

This implements the property that an induced subcategory is a subcategory.

EXAMPLES:

A subquotient of a monoid is a monoid, and a subquotient of semigroup:

```
sage: Monoids().Subquotients().super_categories()
[Category of monoids, Category of subquotients of semigroups]
```

## 4.2 Cartesian Product Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2008-2010): initial revision and refactorization

**class** sage.categories.cartesian_product.**CartesianProductFunctor**

Bases: *sage.categories.covariant_functorial_construction.CovariantFunctorialConstruction*, *sage.categories.pushout.MultivariateConstructionFunctor*

A singleton class for the Cartesian product functor.

EXAMPLES:

```
sage: cartesian_product
The cartesian_product functorial construction
```

`cartesian_product` takes a finite collection of sets, and constructs the Cartesian product of those sets:

```
sage: A = FiniteEnumeratedSet(['a','b','c'])
sage: B = FiniteEnumeratedSet([1,2])
sage: C = cartesian_product([A, B]); C
The Cartesian product of ({'a', 'b', 'c'}, {1, 2})
sage: C.an_element()
('a', 1)
sage: C.list()          # todo: not implemented
[['a', 1], ['a', 2], ['b', 1], ['b', 2], ['c', 1], ['c', 2]]
```

If those sets are endowed with more structure, say they are monoids (hence in the category $Monoids()$), then the result is automatically endowed with its natural monoid structure:

```
sage: M = Monoids().example()
sage: M
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
sage: M.rename('M')
sage: C = cartesian_product([M, ZZ, QQ])
sage: C
The Cartesian product of (M, Integer Ring, Rational Field)
sage: C.an_element()
('abcd', 1, 1/2)
sage: C.an_element()^2
('abcdabcd', 1, 1/4)
sage: C.category()
Category of Cartesian products of monoids

sage: Monoids().CartesianProducts()
Category of Cartesian products of monoids
```

The Cartesian product functor is covariant: if `A` is a subcategory of `B`, then `A.CartesianProducts()` is a subcategory of `B.CartesianProducts()` (see also *CovariantFunctorialConstruction*):

```
sage: C.categories()
[Category of Cartesian products of monoids,
 Category of monoids,
 Category of Cartesian products of semigroups,
 Category of semigroups,
 Category of Cartesian products of unital magmas,
 Category of Cartesian products of magmas,
 Category of unital magmas,
 Category of magmas,
 Category of Cartesian products of sets,
 Category of sets, ...]

[Category of Cartesian products of monoids,
 Category of monoids,
 Category of Cartesian products of semigroups,
 Category of semigroups,
 Category of Cartesian products of magmas,
 Category of unital magmas,
 Category of magmas,
 Category of Cartesian products of sets,
 Category of sets,
```

```
Category of sets with partial maps,
Category of objects]
```

Hence, the role of `Monoids().CartesianProducts()` is solely to provide mathematical information and algorithms which are relevant to Cartesian product of monoids. For example, it specifies that the result is again a monoid, and that its multiplicative unit is the Cartesian product of the units of the underlying sets:

```
sage: C.one()
('', 1, 1)
```

Those are implemented in the nested class *Monoids.CartesianProducts* of `Monoids(QQ)`. This nested class is itself a subclass of *CartesianProductsCategory*.

**class** `sage.categories.cartesian_product.`**CartesianProductsCategory**(*category*, **args*)

Bases: *sage.categories.covariant_functorial_construction.CovariantConstructionCategory*

An abstract base class for all `CartesianProducts` categories.

**CartesianProducts**()

Return the category of (finite) Cartesian products of objects of `self`.

By associativity of Cartesian products, this is `self` (a Cartesian product of Cartesian products of $A$'s is a Cartesian product of $A$'s).

EXAMPLES:

```
sage: ModulesWithBasis(QQ).CartesianProducts().CartesianProducts()
Category of Cartesian products of vector spaces with basis over Rational Field
```

**base_ring**()

The base ring of a Cartesian product is the base ring of the underlying category.

EXAMPLES:

```
sage: Algebras(ZZ).CartesianProducts().base_ring()
Integer Ring
```

## 4.3 Tensor Product Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2008-2010): initial revision and refactorization

**class** `sage.categories.tensor.`**TensorProductFunctor**

Bases: *sage.categories.covariant_functorial_construction.CovariantFunctorialConstruction*

A singleton class for the tensor functor.

This functor takes a collection of vector spaces (or modules with basis), and constructs the tensor product of those vector spaces. If this vector space is in a subcategory, say that of `Algebras(QQ)`, it is automatically endowed with its natural algebra structure, thanks to the category `Algebras(QQ).TensorProducts()` of tensor products of algebras.

The tensor functor is covariant: if `A` is a subcategory of `B`, then `A.TensorProducts()` is a subcategory of `B.TensorProducts()` (see also *CovariantFunctorialConstruction*). Hence, the role of `Algebras(QQ).TensorProducts()` is solely to provide mathematical information and algorithms which are relevant to tensor product of algebras.

Those are implemented in the nested class *TensorProducts* of `Algebras(QQ)`. This nested class is itself a subclass of *TensorProductsCategory*.

**class** sage.categories.tensor.**TensorProductsCategory**(*category*, **args*)
Bases: *sage.categories.covariant_functorial_construction.CovariantConstructionCategory*

An abstract base class for all TensorProducts's categories

**TensorProducts**()
Returns the category of tensor products of objects of `self`

By associativity of tensor products, this is `self` (a tensor product of tensor products of $Cat$'s is a tensor product of $Cat$'s)

EXAMPLES:

```
sage: ModulesWithBasis(QQ).TensorProducts().TensorProducts()
Category of tensor products of vector spaces with basis over Rational Field
```

**base**()
The base of a tensor product is the base (usually a ring) of the underlying category.

EXAMPLES:

```
sage: ModulesWithBasis(ZZ).TensorProducts().base()
Integer Ring
```

sage.categories.tensor.**tensor = The tensor functorial construction**
The tensor product functorial construction

See *TensorProductFunctor* for more information

EXAMPLES:

```
sage: tensor
The tensor functorial construction
```

## 4.4 Dual functorial construction

AUTHORS:

- Nicolas M. Thiery (2009-2010): initial revision

**class** sage.categories.dual.**DualFunctor**
Bases: *sage.categories.covariant_functorial_construction.CovariantFunctorialConstruction*

A singleton class for the dual functor

**class** sage.categories.dual.**DualObjectsCategory**(*category*, **args*)
Bases: *sage.categories.covariant_functorial_construction.CovariantConstructionCategory*

## 4.5 Group algebras and beyond: the Algebra functorial construction

### 4.5.1 Introduction: group algebras

Let $G$ be a group and $R$ be a ring. For example:

```
sage: G = DihedralGroup(3)
sage: R = QQ
```

The *group algebra* $A = RG$ of $G$ over $R$ is the space of formal linear combinations of elements of $group$ with coefficients in $R$:

```
sage: A = G.algebra(R); A
Algebra of Dihedral group of order 6 as a permutation group
        over Rational Field
sage: a = A.an_element(); a
() + 4*(1,2,3) + 2*(1,3)
```

This space is endowed with an algebra structure, obtained by extending by bilinearity the multiplication of $G$ to a multiplication on $RG$:

```
sage: A in Algebras
True
sage: a * a
5*() + 8*(2,3) + 8*(1,2) + 8*(1,2,3) + 16*(1,3,2) + 4*(1,3)
```

In particular, the product of two basis elements is induced by the product of the corresponding elements of the group, and the unit of the group algebra is indexed by the unit of the group:

```
sage: (s, t) = A.algebra_generators()
sage: s*t
(1,2)
sage: A.one_basis()
()
sage: A.one()
()
```

For the user convenience and backward compatibility, the group algebra can also be constructed with:

```
sage: GroupAlgebra(G, R)
Algebra of Dihedral group of order 6 as a permutation group
        over Rational Field
```

Since trac ticket #18700, both constructions are strictly equivalent:

```
sage: GroupAlgebra(G, R) is G.algebra(R)
True
```

Group algebras are further endowed with a Hopf algebra structure; see below.

### 4.5.2 Generalizations

The above construction extends to weaker multiplicative structures than groups: magmas, semigroups, monoids. For a monoid $S$, we obtain the monoid algebra $RS$, which is defined exactly as above:

```
sage: S = Monoids().example(); S
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
sage: A = S.algebra(QQ); A
Algebra of An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
        over Rational Field
sage: A.category()
Category of monoid algebras over Rational Field
```

This construction also extends to additive structures: magmas, semigroups, monoids, or groups:

```
sage: S = CommutativeAdditiveMonoids().example(); S
An example of a commutative monoid:
 the free commutative monoid generated by ('a', 'b', 'c', 'd')
sage: U = S.algebra(QQ); U
Algebra of An example of a commutative monoid:
           the free commutative monoid generated by ('a', 'b', 'c', 'd')
        over Rational Field
```

Despite saying "free module", this is really an algebra, whose multiplication is induced by the addition of elements of $S$:

```
sage: U in Algebras(QQ)
True
sage: (a,b,c,d) = S.additive_semigroup_generators()
sage: U(a) * U(b)
B[a + b]
```

To catter uniformly for the use cases above and some others, for $S$ a set and $K$ a ring, we define in Sage the *algebra of '$S$'* as the $K$-free module with basis indexed by $S$, endowed with whatever algebraic structure can be induced from that of $S$.

**Warning:** In most use cases, the result is actually an algebra, hence the name of this construction. In other cases this name is misleading:

```
sage: A = Sets().example().algebra(QQ); A
Algebra of Set of prime numbers (basic implementation)
        over Rational Field
sage: A.category()
Category of set algebras over Rational Field
sage: A in Algebras(QQ)
False
```

Suggestions for a uniform, meaningful, and non misleading name are welcome!

To achieve this flexibility, the features are implemented as a *Covariant Functorial Constructions* that is essentially a hierarchy of categories each providing the relevant additional features:

```
sage: A = DihedralGroup(3).algebra(QQ)
sage: A.categories()
[Category of finite group algebras over Rational Field,
 ...
 Category of group algebras over Rational Field,
 ...
 Category of monoid algebras over Rational Field,
 ...
 Category of semigroup algebras over Rational Field,
```

```
...
Category of unital magma algebras over Rational Field,
...
Category of magma algebras over Rational Field,
...
Category of set algebras over Rational Field,
...]
```

### 4.5.3 Specifying the algebraic structure

Constructing the algebra of a set endowed with both an additive and a multiplicative structure is ambiguous:

```
sage: Z3 = IntegerModRing(3)
sage: A = Z3.algebra(QQ)
Traceback (most recent call last):
...
TypeError:  `S = Ring of integers modulo 3` is both
 an additive and a multiplicative semigroup.
 Constructing its algebra is ambiguous.
 Please use, e.g., S.algebra(QQ, category=Semigroups())
```

This ambiguity can be resolved using the `category` argument of the construction:

```
sage: A = Z3.algebra(QQ, category=Monoids()); A
Algebra of Ring of integers modulo 3 over Rational Field
sage: A.category()
Category of finite dimensional monoid algebras over Rational Field

sage: A = Z3.algebra(QQ, category=CommutativeAdditiveGroups()); A
Algebra of Ring of integers modulo 3 over Rational Field
sage: A.category()
Category of finite dimensional commutative additive group algebras
 over Rational Field
```

In general, the `category` argument can be used to specify which structure of $S$ shall be extended to $KS$.

### 4.5.4 Group algebras, continued

Let us come back to the case of a group algebra $A = RG$. It is endowed with more structure and in particular that of a *Hopf algebra*:

```
sage: G = DihedralGroup(3)
sage: A = G.algebra(R); A
Algebra of Dihedral group of order 6 as a permutation group
        over Rational Field
sage: A in HopfAlgebras(R).FiniteDimensional().WithBasis()
True
```

The basis elements are *group-like* for the coproduct: $\Delta(g) = g \otimes g$:

```
sage: s
(1,2,3)
sage: s.coproduct()
(1,2,3) # (1,2,3)
```

The counit is the constant function 1 on the basis elements:

```
sage: A = GroupAlgebra(DihedralGroup(6), QQ)
sage: [A.counit(g) for g in A.basis()]
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```

The antipode is given on basis elements by $\chi(g) = g^{-1}$:

```
sage: A = GroupAlgebra(DihedralGroup(3), QQ)
sage: s
(1,2,3)
sage: s.antipode()
(1,3,2)
```

By Maschke's theorem, for a finite group whose cardinality does not divide the characteristic of the base field, the algebra is semisimple:

```
sage: SymmetricGroup(5).algebra(QQ) in Algebras(QQ).Semisimple()
True
sage: CyclicPermutationGroup(10).algebra(FiniteField(7)) in Algebras.Semisimple
True
sage: CyclicPermutationGroup(10).algebra(FiniteField(5)) in Algebras.Semisimple
False
```

### 4.5.5 Coercions

Let $RS$ be the algebra of some structure $S$. Then $RS$ admits the natural coercion from any other algebra $R'S'$ of some structure $S'$, as long as $R'$ coerces into $R$ and $S'$ coerces into $S$.

For example, since there is a natural inclusion from the dihedral group $D_2$ of order 4 into the symmetric group $S_4$ of order 4!, and since there is a natural map from the integers to the rationals, there is a natural map from $\mathbf{Z}[D_2]$ to $\mathbf{Q}[S_4]$:

```
sage: A = DihedralGroup(2).algebra(ZZ)
sage: B = SymmetricGroup(4).algebra(QQ)
sage: a = A.an_element(); a
() + 3*(3,4) + 3*(1,2)
sage: b = B.an_element(); b
() + 2*(1,2) + 4*(1,2,3,4)
sage: B(a)
() + 3*(3,4) + 3*(1,2)
sage: a * b  # a is automatically converted to an element of B
7*() + 3*(3,4) + 5*(1,2) + 6*(1,2)(3,4) + 12*(1,2,3) + 4*(1,2,3,4) + 12*(1,3,4)
sage: parent(a * b)
Symmetric group algebra of order 4 over Rational Field
```

There is no obvious map in the other direction, though:

```
sage: A(b)
Traceback (most recent call last):
...
TypeError: do not know how to make x (= () + 2*(1,2) + 4*(1,2,3,4))
 an element of self
 (=Algebra of Dihedral group of order 4 as a permutation group
          over Integer Ring)
```

If $S$ is a unital (additive) magma, then $RS$ is a unital algebra, and thus admits a coercion from its base ring $R$ and any ring that coerces into $R$.

```
sage: G = DihedralGroup(2)
sage: A = G.algebra(ZZ)
sage: A(2)
2*()
```

If $S$ is a multiplicative group, then $RS$ admits a coercion from $S$ and from any group which coerce into $S$:

```
sage: g = DihedralGroup(2).gen(0); g
(3,4)
sage: A(g)
(3,4)
sage: A(2) * g
2*(3,4)
```

Note that there is an ambiguity if $S'$ is a group which coerces into both $R$ and $S$. For example) if $S$ is the additive group $(\mathbf{Z}, +)$, and $A = RS$ is its group algebra, then the integer 2 can be coerced into $A$ in two ways – via $S$, or via the base ring $R$ – and *the answers are different*. It that case the coercion to $R$ takes precedence. In particular, if $\mathbf{Z}$ is the ring (or group) of integers, then $\mathbf{Z}$ will coerce to any $RS$, by sending $\mathbf{Z}$ to $R$. In generic code, it is therefore recommented to always explicitly use `A.monomial(g)` to convert an element of the group into $A$.

AUTHORS:

- David Loeffler (2008-08-24): initial version
- Martin Raum (2009-08): update to use new coercion model – see trac ticket #6670.
- John Palmieri (2011-07): more updates to coercion, categories, etc., group algebras constructed using CombinatorialFreeModule – see trac ticket #6670.
- Nicolas M. Thiéry (2010-2017), Travis Scrimshaw (2017): generalization to a covariant functorial construction for monoid algebras, and beyond – see e.g. trac ticket #18700.

**class** sage.categories.algebra_functor.**AlgebraFunctor**(*base_ring*)

Bases: *sage.categories.covariant_functorial_construction.CovariantFunctorialConstruction*

For a fixed ring, a functor sending a group/... to the corresponding group/... algebra.

EXAMPLES:

```
sage: from sage.categories.algebra_functor import AlgebraFunctor
sage: F = AlgebraFunctor(QQ); F
The algebra functorial construction
sage: F(DihedralGroup(3))
Algebra of Dihedral group of order 6 as a permutation group
        over Rational Field
```

**base_ring**()

Return the base ring for this functor.

EXAMPLES:

```
sage: from sage.categories.algebra_functor import AlgebraFunctor
sage: AlgebraFunctor(QQ).base_ring()
Rational Field
```

**class** sage.categories.algebra_functor.**AlgebrasCategory**(*category*, **args*)

Bases: *sage.categories.covariant_functorial_construction.CovariantConstructionCategory*, *sage.categories.category_types.Category_over_base_ring*

An abstract base class for categories of monoid algebras, groups algebras, and the like.

**See also:**

- *Sets.ParentMethods.algebra()*
- *Sets.SubcategoryMethods.Algebras()*
- *CovariantFunctorialConstruction*

INPUT:

- base_ring – a ring

EXAMPLES:

```
sage: C = Groups().Algebras(QQ); C
Category of group algebras over Rational Field
sage: C = Monoids().Algebras(QQ); C
Category of monoid algebras over Rational Field

sage: C._short_name()
'Algebras'
sage: latex(C) # todo: improve that
\mathbf{Algebras}(\mathbf{Monoids})
```

**class** sage.categories.algebra_functor.**GroupAlgebraFunctor**(*group*)

Bases: *sage.categories.pushout.ConstructionFunctor*

For a fixed group, a functor sending a commutative ring to the corresponding group algebra.

INPUT:

- group – the group associated to each group algebra under consideration

EXAMPLES:

```
sage: from sage.categories.algebra_functor import GroupAlgebraFunctor
sage: F = GroupAlgebraFunctor(KleinFourGroup()); F
GroupAlgebraFunctor
sage: A = F(QQ); A
Algebra of The Klein 4 group of order 4, as a permutation group over Rational
↪Field
```

**group**()

Return the group which is associated to this functor.

EXAMPLES:

```
sage: from sage.categories.algebra_functor import GroupAlgebraFunctor
sage: GroupAlgebraFunctor(CyclicPermutationGroup(17)).group() ==
↪CyclicPermutationGroup(17)
True
```

## 4.6 Subquotient Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

**class** sage.categories.subquotients.**SubquotientsCategory**(*category*, **args*)
Bases: *sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory*

## 4.7 Quotients Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

**class** sage.categories.quotients.**QuotientsCategory**(*category*, **args*)
Bases: *sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory*

**classmethod default_super_categories**(*category*)
Returns the default super categories of `category.Quotients()`

Mathematical meaning: if $A$ is a quotient of $B$ in the category $C$, then $A$ is also a subquotient of $B$ in the category $C$.

INPUT:

- `cls` – the class `QuotientsCategory`
- `category` – a category $Cat$

OUTPUT: a (join) category

In practice, this returns `category.Subquotients()`, joined together with the result of the method *RegressiveCovariantConstructionCategory.default_super_categories()* (that is the join of `category` and `cat.Quotients()` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Groups()`, which has `cat=Monoids()` as super category. Then, a subgroup of a group $G$ is simultaneously a subquotient of $G$, a group by itself, and a quotient monoid of `G`:

```
sage: Groups().Quotients().super_categories()
[Category of groups, Category of subquotients of monoids, Category of␣
↪quotients of semigroups]
```

Mind the last item above: there is indeed currently nothing implemented about quotient monoids.

This resulted from the following call:

```
sage: sage.categories.quotients.QuotientsCategory.default_super_
↪categories(Groups())
Join of Category of groups and Category of subquotients of monoids and␣
↪Category of quotients of semigroups
```

## 4.8 Subobjects Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

**class** sage.categories.subobjects.**SubobjectsCategory**(*category*, **args*)

Bases: *sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory*

**classmethod default_super_categories**(*category*)

Returns the default super categories of `category.Subobjects()`

Mathematical meaning: if $A$ is a subobject of $B$ in the category $C$, then $A$ is also a subquotient of $B$ in the category $C$.

INPUT:

- `cls` – the class `SubobjectsCategory`
- `category` – a category $Cat$

OUTPUT: a (join) category

In practice, this returns `category.Subquotients()`, joined together with the result of the method *RegressiveCovariantConstructionCategory.default_super_categories()* (that is the join of `category` and `cat.Subobjects()` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Groups()`, which has `cat=Monoids()` as super category. Then, a subgroup of a group $G$ is simultaneously a subquotient of $G$, a group by itself, and a submonoid of $G$:

```
sage: Groups().Subobjects().super_categories()
[Category of groups, Category of subquotients of monoids, Category of␣
↪subobjects of sets]
```

Mind the last item above: there is indeed currently nothing implemented about submonoids.

This resulted from the following call:

```
sage: sage.categories.subobjects.SubobjectsCategory.default_super_
↪categories(Groups())
Join of Category of groups and Category of subquotients of monoids and␣
↪Category of subobjects of sets
```

## 4.9 Isomorphic Objects Functorial Construction

AUTHORS:

- Nicolas M. Thiery (2010): initial revision

**class** sage.categories.isomorphic_objects.**IsomorphicObjectsCategory**(*category*, **args*)

Bases: *sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory*

**classmethod default_super_categories**(*category*)

Returns the default super categories of `category.IsomorphicObjects()`

Mathematical meaning: if $A$ is the image of $B$ by an isomorphism in the category $C$, then $A$ is both a subobject of $B$ and a quotient of $B$ in the category $C$.

INPUT:

- `cls` – the class `IsomorphicObjectsCategory`

- `category` – a category $Cat$

OUTPUT: a (join) category

In practice, this returns `category.Subobjects()` and `category.Quotients()`, joined together with the result of the method `RegressiveCovariantConstructionCategory.default_super_categories()` (that is the join of `category` and `cat.IsomorphicObjects()` for each `cat` in the super categories of `category`).

EXAMPLES:

Consider `category=Groups()`, which has `cat=Monoids()` as super category. Then, the image of a group $G'$ by a group isomorphism is simultaneously a subgroup of $G$, a subquotient of $G$, a group by itself, and the image of $G$ by a monoid isomorphism:

```
sage: Groups().IsomorphicObjects().super_categories()
[Category of groups,
 Category of subquotients of monoids,
 Category of quotients of semigroups,
 Category of isomorphic objects of sets]
```

Mind the last item above: there is indeed currently nothing implemented about isomorphic objects of monoids.

This resulted from the following call:

```
sage: sage.categories.isomorphic_objects.IsomorphicObjectsCategory.default_
↪super_categories(Groups())
Join of Category of groups and
Category of subquotients of monoids and
Category of quotients of semigroups and
Category of isomorphic objects of sets
```

## 4.10 Homset categories

**class** `sage.categories.homsets.`**`Homsets`**(*s=None*)

Bases: `sage.categories.category_singleton.Category_singleton`

The category of all homsets.

EXAMPLES:

```
sage: from sage.categories.homsets import Homsets
sage: Homsets()
Category of homsets
```

This is a subcategory of `Sets()`:

```
sage: Homsets().super_categories()
[Category of sets]
```

By this, we assume that all homsets implemented in Sage are sets, or equivalently that we only implement locally small categories. See Wikipedia article Category_(mathematics).

trac ticket #17364: every homset category shall be a subcategory of the category of all homsets:

```
sage: Schemes().Homsets().is_subcategory(Homsets())
True
```

```
sage: AdditiveMagmas().Homsets().is_subcategory(Homsets())
True
sage: AdditiveMagmas().AdditiveUnital().Homsets().is_subcategory(Homsets())
True
```

This is tested in `HomsetsCategory._test_homsets_category()`.

**class Endset**(*base_category*)

Bases: *sage.categories.category_with_axiom.CategoryWithAxiom_singleton*

The category of all endomorphism sets.

This category serves too purposes: making sure that the `Endset` axiom is implemented in the category where it's defined, namely `Homsets`, and specifying that `Endsets` are monoids.

EXAMPLES:

```
sage: from sage.categories.homsets import Homsets
sage: Homsets().Endset()
Category of endsets
```

**class ParentMethods**

**is_endomorphism_set**()

Return `True` as `self` is in the category of `Endsets`.

EXAMPLES:

```
sage: P.<t> = ZZ[]
sage: E = End(P)
sage: E.is_endomorphism_set()
True
```

**extra_super_categories**()

Implement the fact that endsets are monoids.

**See also:**

`CategoryWithAxiom.extra_super_categories()`

EXAMPLES:

```
sage: from sage.categories.homsets import Homsets
sage: Homsets().Endset().extra_super_categories()
[Category of monoids]
```

**class ParentMethods**

**is_endomorphism_set**()

Return `True` if the domain and codomain of `self` are the same object.

EXAMPLES:

```
sage: P.<t> = ZZ[]
sage: f = P.hom([1/2*t])
sage: f.parent().is_endomorphism_set()
False
sage: g = P.hom([2*t])
```

```
sage: g.parent().is_endomorphism_set()
True
```

**class SubcategoryMethods**

**Endset()**
Return the subcategory of the homsets of `self` that are endomorphism sets.

EXAMPLES:

```
sage: Sets().Homsets().Endset()
Category of endsets of sets

sage: Posets().Homsets().Endset()
Category of endsets of posets
```

**super_categories()**
Return the super categories of `self`.

EXAMPLES:

```
sage: from sage.categories.homsets import Homsets
sage: Homsets()
Category of homsets
```

**class** sage.categories.homsets.**HomsetsCategory**(*category*, **args*)
Bases: *sage.categories.covariant_functorial_construction.FunctorialConstructionCategory*

**base()**
If this homsets category is subcategory of a category with a base, return that base.

**Todo:** Is this really useful?

EXAMPLES:

```
sage: ModulesWithBasis(ZZ).Homsets().base()
Integer Ring
```

**classmethod default_super_categories**(*category*)
Return the default super categories of `category.Homsets()`.

INPUT:

- `cls` – the category class for the functor $F$
- `category` – a category $Cat$

OUTPUT: a category

As for the other functorial constructions, if `category` implements a nested `Homsets` class, this method is used in combination with `category.Homsets().extra_super_categories()` to compute the super categories of `category.Homsets()`.

EXAMPLES:

If `category` has one or more full super categories, then the join of their respective homsets category is returned. In this example, this join consists of a single category:

```
sage: from sage.categories.homsets import HomsetsCategory
sage: from sage.categories.additive_groups import AdditiveGroups

sage: C = AdditiveGroups()
sage: C.full_super_categories()
[Category of additive inverse additive unital additive magmas,
 Category of additive monoids]
sage: H = HomsetsCategory.default_super_categories(C); H
Category of homsets of additive monoids
sage: type(H)
<class 'sage.categories.additive_monoids.AdditiveMonoids.Homsets_with_category
↪'>
```

and, given that nothing specific is currently implemented for homsets of additive groups, `H` is directly the category thereof:

```
sage: C.Homsets()
Category of homsets of additive monoids
```

Similarly for rings: a ring homset is just a homset of unital magmas and additive magmas:

```
sage: Rings().Homsets()
Category of homsets of unital magmas and additive unital additive magmas
```

Otherwise, if `category` implements a nested class `Homsets`, this method returns the category of all homsets:

```
sage: AdditiveMagmas.Homsets
<class 'sage.categories.additive_magmas.AdditiveMagmas.Homsets'>
sage: HomsetsCategory.default_super_categories(AdditiveMagmas())
Category of homsets
```

which gives one of the super categories of `category.Homsets()`:

```
sage: AdditiveMagmas().Homsets().super_categories()
[Category of additive magmas, Category of homsets]
sage: AdditiveMagmas().AdditiveUnital().Homsets().super_categories()
[Category of additive unital additive magmas, Category of homsets]
```

the other coming from `category.Homsets().extra_super_categories()`:

```
sage: AdditiveMagmas().Homsets().extra_super_categories()
[Category of additive magmas]
```

Finally, as a last resort, this method returns a stub category modelling the homsets of this category:

```
sage: hasattr(Posets, "Homsets")
False
sage: H = HomsetsCategory.default_super_categories(Posets()); H
Category of homsets of posets
sage: type(H)
<class 'sage.categories.homsets.HomsetsOf_with_category'>
sage: Posets().Homsets()
Category of homsets of posets
```

**class** sage.categories.homsets.**HomsetsOf**(*category*, **args*)
Bases: *sage.categories.homsets.HomsetsCategory*

Default class for homsets of a category.

This is used when a category $C$ defines some additional structure but not a homset category of its own. Indeed, unlike for covariant functorial constructions, we cannot represent the homset category of $C$ by just the join of the homset categories of its super categories.

EXAMPLES:

```
sage: C = (Magmas() & Posets()).Homsets(); C
Category of homsets of magmas and posets
sage: type(C)
<class 'sage.categories.homsets.HomsetsOf_with_category'>
```

**super_categories**()

Return the super categories of `self`.

A stub homset category admits a single super category, namely the category of all homsets.

EXAMPLES:

```
sage: C = (Magmas() & Posets()).Homsets(); C
Category of homsets of magmas and posets
sage: type(C)
<class 'sage.categories.homsets.HomsetsOf_with_category'>
sage: C.super_categories()
[Category of homsets]
```

## 4.11 Realizations Covariant Functorial Construction

**See also:**

- *Sets().WithRealizations* for an introduction to *realizations* and *with realizations*.
- *sage.categories.covariant_functorial_construction* for an introduction to covariant functorial constructions.
- *sage.categories.examples.with_realizations* for an example.

**class** sage.categories.realizations.**Category_realization_of_parent**(*parent_with_realization*)

Bases: *sage.categories.category_types.Category_over_base*, sage.misc.bindable_class.BindableClass

An abstract base class for categories of all realizations of a given parent

INPUT:

- `parent_with_realization` – a parent

**See also:**

*Sets().WithRealizations*

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
```

The role of this base class is to implement some technical goodies, like the binding `A.Realizations()` when a subclass `Realizations` is implemented as a nested class in `A` (see the *code of the example*):

```
sage: C = A.Realizations(); C
Category of realizations of The subset algebra of {1, 2, 3} over Rational Field
```

as well as the name for that category.

sage.categories.realizations.**Realizations**(*self*)

Return the category of realizations of the parent `self` or of objects of the category `self`

INPUT:

- `self` – a parent or a concrete category

**Note:** this *function* is actually inserted as a *method* in the class `Category` (see `Realizations()`). It is defined here for code locality reasons.

EXAMPLES:

The category of realizations of some algebra:

```
sage: Algebras(QQ).Realizations()
Join of Category of algebras over Rational Field and Category of realizations of␣
↪unital magmas
```

The category of realizations of a given algebra:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.Realizations()
Category of realizations of The subset algebra of {1, 2, 3} over Rational Field

sage: C = GradedHopfAlgebrasWithBasis(QQ).Realizations(); C
Join of Category of graded hopf algebras with basis over Rational Field and␣
↪Category of realizations of hopf algebras over Rational Field
sage: C.super_categories()
[Category of graded hopf algebras with basis over Rational Field, Category of␣
↪realizations of hopf algebras over Rational Field]

sage: TestSuite(C).run()
```

**See also:**

- `Sets().WithRealizations`
- `ClasscallMetaclass`

**Todo:** Add an optional argument to allow for:

```
sage: Realizations(A, category = Blahs()) # todo: not implemented
```

**class** sage.categories.realizations.**RealizationsCategory**(*category*, **args*)

Bases: `sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory`

An abstract base class for all categories of realizations category

Relization are implemented as *RegressiveCovariantConstructionCategory*. See there for the documentation of how the various bindings such as `Sets().Realizations()` and `P.Realizations()`, where `P` is a parent, work.

**See also:**

*Sets().WithRealizations*

## 4.12 With Realizations Covariant Functorial Construction

**See also:**

- *Sets().WithRealizations* for an introduction to *realizations* and *with realizations*.
- *sage.categories.covariant_functorial_construction* for an introduction to covariant functorial constructions.

`sage.categories.with_realizations.`**`WithRealizations`**`(`*self*`)`

Return the category of parents in `self` endowed with multiple realizations.

INPUT:

- `self` – a category

**See also:**

- The documentation and code (*sage.categories.examples.with_realizations*) of `Sets().WithRealizations().example()` for more on how to use and implement a parent with several realizations.
- Various use cases:
  - `SymmetricFunctions`
  - `QuasiSymmetricFunctions`
  - `NonCommutativeSymmetricFunctions`
  - `SymmetricFunctionsNonCommutingVariables`
  - `DescentAlgebra`
  - `algebras.Moebius`
  - `IwahoriHeckeAlgebra`
  - `ExtendedAffineWeylGroup`
- The Implementing Algebraic Structures thematic tutorial.
- *sage.categories.realizations*

**Note:** this *function* is actually inserted as a *method* in the class *Category* (see *WithRealizations()*). It is defined here for code locality reasons.

EXAMPLES:

```
sage: Sets().WithRealizations()
Category of sets with realizations
```

### Parent with realizations

Let us now explain the concept of realizations. A *parent with realizations* is a facade parent (see `Sets.Facade`) admitting multiple concrete realizations where its elements are represented. Consider for example an algebra $A$ which admits several natural bases:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
```

For each such basis $B$ one implements a parent $P_B$ which realizes $A$ with its elements represented by expanding them on the basis $B$:

```
sage: A.F()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
sage: A.Out()
The subset algebra of {1, 2, 3} over Rational Field in the Out basis
sage: A.In()
The subset algebra of {1, 2, 3} over Rational Field in the In basis

sage: A.an_element()
F[{}] + 2*F[{1}] + 3*F[{2}] + F[{1, 2}]
```

If $B$ and $B'$ are two bases, then the change of basis from $B$ to $B'$ is implemented by a canonical coercion between $P_B$ and $P_{B'}$:

```
sage: F = A.F(); In = A.In(); Out = A.Out()
sage: i = In.an_element(); i
In[{}] + 2*In[{1}] + 3*In[{2}] + In[{1, 2}]
sage: F(i)
7*F[{}] + 3*F[{1}] + 4*F[{2}] + F[{1, 2}]
sage: F.coerce_map_from(Out)
Generic morphism:
  From: The subset algebra of {1, 2, 3} over Rational Field in the Out basis
  To:   The subset algebra of {1, 2, 3} over Rational Field in the Fundamental␣
↪basis
```

allowing for mixed arithmetic:

```
sage: (1 + Out.from_set(1)) * In.from_set(2,3)
Out[{}] + 2*Out[{1}] + 2*Out[{2}] + 2*Out[{3}] + 2*Out[{1, 2}] + 2*Out[{1, 3}] +␣
↪4*Out[{2, 3}] + 4*Out[{1, 2, 3}]
```

In our example, there are three realizations:

```
sage: A.realizations()
[The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis,
 The subset algebra of {1, 2, 3} over Rational Field in the In basis,
 The subset algebra of {1, 2, 3} over Rational Field in the Out basis]
```

Instead of manually defining the shorthands `F`, `In`, and `Out`, as above one can just do:

```
sage: A.inject_shorthands()
Defining F as shorthand for The subset algebra of {1, 2, 3} over Rational Field␣
↪in the Fundamental basis
Defining In as shorthand for The subset algebra of {1, 2, 3} over Rational Field␣
↪in the In basis
Defining Out as shorthand for The subset algebra of {1, 2, 3} over Rational Field␣
↪in the Out basis
```

## Rationale

Besides some goodies described below, the role of $A$ is threefold:

- To provide, as illustrated above, a single entry point for the algebra as a whole: documentation, access to its properties and different realizations, etc.
- To provide a natural location for the initialization of the bases and the coercions between, and other methods that are common to all bases.
- To let other objects refer to $A$ while allowing elements to be represented in any of the realizations.

We now illustrate this second point by defining the polynomial ring with coefficients in $A$:

```
sage: P = A['x']; P
Univariate Polynomial Ring in x over The subset algebra of {1, 2, 3} over
↪Rational Field
sage: x = P.gen()
```

In the following examples, the coefficients turn out to be all represented in the $F$ basis:

```
sage: P.one()
F[{}]
sage: (P.an_element() + 1)^2
F[{}]*x^2 + 2*F[{}]*x + F[{}]
```

However we can create a polynomial with mixed coefficients, and compute with it:

```
sage: p = P([1, In[{1}], Out[{2}] ]); p
Out[{2}]*x^2 + In[{1}]*x + F[{}]
sage: p^2
Out[{2}]*x^4
+ (-8*In[{}] + 4*In[{1}] + 8*In[{2}] + 4*In[{3}] - 4*In[{1, 2}] - 2*In[{1, 3}] -
↪4*In[{2, 3}] + 2*In[{1, 2, 3}])*x^3
+ (F[{}] + 3*F[{1}] + 2*F[{2}] - 2*F[{1, 2}] - 2*F[{2, 3}] + 2*F[{1, 2, 3}])*x^2
+ (2*F[{}] + 2*F[{1}])*x
+ F[{}]
```

Note how each coefficient involves a single basis which need not be that of the other coefficients. Which basis is used depends on how coercion happened during mixed arithmetic and needs not be deterministic.

One can easily coerce all coefficient to a given basis with:

```
sage: p.map_coefficients(In)
(-4*In[{}] + 2*In[{1}] + 4*In[{2}] + 2*In[{3}] - 2*In[{1, 2}] - In[{1, 3}] - 2*In[
↪{2, 3}] + In[{1, 2, 3}])*x^2 + In[{1}]*x + In[{}]
```

Alas, the natural notation for constructing such polynomials does not yet work:

```
sage: In[{1}] * x
Traceback (most recent call last):
...
TypeError: unsupported operand parent(s) for *: 'The subset algebra of {1, 2, 3}
↪over Rational Field in the In basis' and 'Univariate Polynomial Ring in x over
↪The subset algebra of {1, 2, 3} over Rational Field'
```

### The category of realizations of $A$

The set of all realizations of $A$, together with the coercion morphisms is a category (whose class inherits from *Category_realization_of_parent*):

```
sage: A.Realizations()
Category of realizations of The subset algebra of {1, 2, 3} over Rational Field
```

The various parent realizing $A$ belong to this category:

```
sage: A.F() in A.Realizations()
True
```

$A$ itself is in the category of algebras with realizations:

```
sage: A in Algebras(QQ).WithRealizations()
True
```

The (mostly technical) `WithRealizations` categories are the analogs of the `*WithSeveralBases` categories in MuPAD-Combinat. They provide support tools for handling the different realizations and the morphisms between them.

Typically, `VectorSpaces(QQ).FiniteDimensional().WithRealizations()` will eventually be in charge, whenever a coercion $\phi : A \mapsto B$ is registered, to register $\phi^{-1}$ as coercion $B \mapsto A$ if there is none defined yet. To achieve this, `FiniteDimensionalVectorSpaces` would provide a nested class `WithRealizations` implementing the appropriate logic.

`WithRealizations` is a *regressive covariant functorial construction*. On our example, this simply means that $A$ is automatically in the category of rings with realizations (covariance):

```
sage: A in Rings().WithRealizations()
True
```

and in the category of algebras (regressiveness):

```
sage: A in Algebras(QQ)
True
```

---

**Note:** For `C` a category, `C.WithRealizations()` in fact calls `sage.categories.with_realizations.WithRealizations(C)`. The later is responsible for building the hierarchy of the categories with realizations in parallel to that of their base categories, optimizing away those categories that do not provide a `WithRealizations` nested class. See *sage.categories.covariant_functorial_construction* for the technical details.

---

**Note:** Design question: currently `WithRealizations` is a regressive construction. That is `self.WithRealizations()` is a subcategory of `self` by default:

```
sage: Algebras(QQ).WithRealizations().super_categories()
[Category of algebras over Rational Field,
 Category of monoids with realizations,
 Category of additive unital additive magmas with realizations]
```

Is this always desirable? For example, `AlgebrasWithBasis(QQ).WithRealizations()` should certainly be a subcategory of `Algebras(QQ)`, but not of `AlgebrasWithBasis(QQ)`. This is because

`AlgebrasWithBasis(QQ)` is specifying something about the concrete realization.

**class** `sage.categories.with_realizations.`**`WithRealizationsCategory`**(*category*, **args*)

Bases: *`sage.categories.covariant_functorial_construction.RegressiveCovariantConstructionCategory`*

An abstract base class for all categories of parents with multiple realizations.

**See also:**

*`Sets().WithRealizations`*

The role of this base class is to implement some technical goodies, such as the name for that category.

CHAPTER

FIVE

# EXAMPLES OF PARENTS USING CATEGORIES

## 5.1 Examples of algebras with basis

`sage.categories.examples.algebras_with_basis.Example`

alias of *FreeAlgebra*

**class** `sage.categories.examples.algebras_with_basis.FreeAlgebra`(*R*, *alphabet=('a', 'b', 'c')*)

Bases: `sage.combinat.free_module.CombinatorialFreeModule`

An example of an algebra with basis: the free algebra

This class illustrates a minimal implementation of an algebra with basis.

**algebra_generators**()

Return the generators of this algebra, as per *algebra_generators()*.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example(); A
An example of an algebra with basis: the free algebra on the generators ('a',
↪'b', 'c') over Rational Field
sage: A.algebra_generators()
Family (B[word: a], B[word: b], B[word: c])
```

**one_basis**()

Returns the empty word, which index the one of this algebra, as per `AlgebrasWithBasis.ParentMethods.one_basis()`.

EXAMPLES::r

sage: A = AlgebrasWithBasis(QQ).example() sage: A.one_basis() word: sage: A.one() B[word: ]

**product_on_basis**(*w1*, *w2*)

Product of basis elements, as per `AlgebrasWithBasis.ParentMethods.product_on_basis()`.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()
sage: Words = A.basis().keys()
sage: A.product_on_basis(Words("acb"), Words("cba"))
B[word: acbcba]
sage: (a,b,c) = A.algebra_generators()
```

```
sage: a * (1-b)^2 * c
B[word: abbc] - 2*B[word: abc] + B[word: ac]
```

## 5.2 Examples of commutative additive monoids

sage.categories.examples.commutative_additive_monoids.**Example**

alias of *FreeCommutativeAdditiveMonoid*

**class** sage.categories.examples.commutative_additive_monoids.**FreeCommutativeAdditiveMonoid**(

Bases: *sage.categories.examples.commutative_additive_semigroups.FreeCommutativeAdditiveSemigroup*

An example of a commutative additive monoid: the free commutative monoid

This class illustrates a minimal implementation of a commutative monoid.

EXAMPLES:

```
sage: S = CommutativeAdditiveMonoids().example(); S
An example of a commutative monoid: the free commutative monoid generated by ('a',
↪ 'b', 'c', 'd')

sage: S.category()
Category of commutative additive monoids
```

This is the free semigroup generated by:

```
sage: S.additive_semigroup_generators()
Family (a, b, c, d)
```

with product rule given by $a \times b = a$ for all $a, b$:

```
sage: (a,b,c,d) = S.additive_semigroup_generators()
```

We conclude by running systematic tests on this commutative monoid:

```
sage: TestSuite(S).run(verbose = True)
running ._test_additive_associativity() . . . pass
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
```

```
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
running ._test_zero() . . . pass
```

**class Element** (*parent*, *iterable*)

Bases: *sage.categories.examples.commutative_additive_semigroups.FreeCommutativeAdditiveSemigroup.Element*

**zero** ()

Returns the zero of this additive monoid, as per `CommutativeAdditiveMonoids.ParentMethods.zero()`.

EXAMPLES:

```
sage: M = CommutativeAdditiveMonoids().example(); M
An example of a commutative monoid: the free commutative monoid generated by (
↪'a', 'b', 'c', 'd')
sage: M.zero()
0
```

## 5.3 Examples of commutative additive semigroups

`sage.categories.examples.commutative_additive_semigroups.`**`Example`**

alias of *FreeCommutativeAdditiveSemigroup*

**class** `sage.categories.examples.commutative_additive_semigroups.`**`FreeCommutativeAdditiveSemi`**

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

An example of a commutative additive monoid: the free commutative monoid

This class illustrates a minimal implementation of a commutative additive monoid.

EXAMPLES:

```
sage: S = CommutativeAdditiveSemigroups().example(); S
An example of a commutative monoid: the free commutative monoid generated by ('a',
↪ 'b', 'c', 'd')

sage: S.category()
Category of commutative additive semigroups
```

This is the free semigroup generated by:

```
sage: S.additive_semigroup_generators()
Family (a, b, c, d)
```

with product rule given by $a \times b = a$ for all $a, b$:

```
sage: (a,b,c,d) = S.additive_semigroup_generators()
```

We conclude by running systematic tests on this commutative monoid:

```
sage: TestSuite(S).run(verbose = True)
running ._test_additive_associativity() . . . pass
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

**class Element** (*parent*, *iterable*)

Bases: `sage.structure.element_wrapper.ElementWrapper`

EXAMPLES:

```
sage: F = CommutativeAdditiveSemigroups().example()
sage: x = F.element_class(F, (('a',4), ('b', 0), ('a', 2), ('c', 1), ('d',␣
↪5)))
sage: x
2*a + c + 5*d
sage: x.value
{'a': 2, 'b': 0, 'c': 1, 'd': 5}
sage: x.parent()
An example of a commutative monoid: the free commutative monoid generated by (
↪'a', 'b', 'c', 'd')
```

Internally, elements are represented as dense dictionaries which associate to each generator of the monoid its multiplicity. In order to get an element, we wrap the dictionary into an element via `ElementWrapper`:

```
sage: x.value
{'a': 2, 'b': 0, 'c': 1, 'd': 5}
```

**additive_semigroup_generators** ()

Returns the generators of the semigroup.

EXAMPLES:

```
sage: F = CommutativeAdditiveSemigroups().example()
sage: F.additive_semigroup_generators()
Family (a, b, c, d)
```

**an_element**()

Returns an element of the semigroup.

EXAMPLES:

```
sage: F = CommutativeAdditiveSemigroups().example()
sage: F.an_element()
a + 3*c + 2*b + 4*d
```

**summation**(*x*, *y*)

Returns the product of `x` and `y` in the semigroup, as per `CommutativeAdditiveSemigroups.ParentMethods.summation()`.

EXAMPLES:

```
sage: F = CommutativeAdditiveSemigroups().example()
sage: (a,b,c,d) = F.additive_semigroup_generators()
sage: F.summation(a,b)
a + b
sage: (a+b) + (a+c)
2*a + c + b
```

## 5.4 Examples of Coxeter groups

## 5.5 Example of a crystal

**class** sage.categories.examples.crystals.**HighestWeightCrystalOfTypeA**(*n=3*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

An example of a crystal: the highest weight crystal of type $A_n$ of highest weight $\omega_1$.

The purpose of this class is to provide a minimal template for implementing crystals. See `CrystalOfLetters` for a full featured and optimized implementation.

EXAMPLES:

```
sage: C = Crystals().example()
sage: C
Highest weight crystal of type A_3 of highest weight omega_1
sage: C.category()
Category of classical crystals
```

The elements of this crystal are in the set $\{1, \ldots, n+1\}$:

```
sage: C.list()
[1, 2, 3, 4]
sage: C.module_generators[0]
1
```

The crystal operators themselves correspond to the elementary transpositions:

```
sage: b = C.module_generators[0]
sage: b.f(1)
2
```

```
sage: b.f(1).e(1) == b
True
```

Only the following basic operations are implemented:

- `cartan_type()` or an attribute _cartan_type
- an attribute module_generators
- `Element.e()`
- `Element.f()`

All the other usual crystal operations are inherited from the categories; for example:

```
sage: C.cardinality()
4
```

**class Element**

Bases: `sage.structure.element_wrapper.ElementWrapper`

**e**(*i*)

Returns the action of $e_i$ on `self`.

EXAMPLES:

```
sage: C = Crystals().example(4)
sage: [[c,i,c.e(i)] for i in C.index_set() for c in C if c.e(i) is not␣
↪None]
[[2, 1, 1], [3, 2, 2], [4, 3, 3], [5, 4, 4]]
```

**f**(*i*)

Returns the action of $f_i$ on `self`.

EXAMPLES:

```
sage: C = Crystals().example(4)
sage: [[c,i,c.f(i)] for i in C.index_set() for c in C if c.f(i) is not␣
↪None]
[[1, 1, 2], [2, 2, 3], [3, 3, 4], [4, 4, 5]]
```

**class** `sage.categories.examples.crystals.`**NaiveCrystal**

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

This is an example of a "crystal" which does not come from any kind of representation, designed primarily to test the Stembridge local rules with. The crystal has vertices labeled 0 through 5, with 0 the highest weight.

The code here could also possibly be generalized to create a class that automatically builds a crystal from an edge-colored digraph, if someone feels adventurous.

Currently, only the methods `highest_weight_vector()`, `e()`, and `f()` are guaranteed to work.

EXAMPLES:

```
sage: C = Crystals().example(choice='naive')
sage: C.highest_weight_vector()
0
```

**class Element**

Bases: `sage.structure.element_wrapper.ElementWrapper`

**e** (*i*)
Returns the action of $e_i$ on `self`.

EXAMPLES:

```
sage: C = Crystals().example(choice='naive')
sage: [[c,i,c.e(i)] for i in C.index_set() for c in [C(j) for j in [0..
↪5]] if c.e(i) is not None]
[[1, 1, 0], [2, 1, 1], [3, 1, 2], [5, 1, 3], [4, 2, 0], [5, 2, 4]]
```

**f** (*i*)
Returns the action of $f_i$ on `self`.

EXAMPLES:

```
sage: C = Crystals().example(choice='naive')
sage: [[c,i,c.f(i)] for i in C.index_set() for c in [C(j) for j in [0..
↪5]] if c.f(i) is not None]
[[0, 1, 1], [1, 1, 2], [2, 1, 3], [3, 1, 5], [0, 2, 4], [4, 2, 5]]
```

## 5.6 Examples of CW complexes

`sage.categories.examples.cw_complexes.`**`Example`**
alias of `Surface`

**class** `sage.categories.examples.cw_complexes.`**`Surface`** (*bdy=(1, 2, 1, 2)*)
Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

An example of a CW complex: a (2-dimensional) surface.

This class illustrates a minimal implementation of a CW complex.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example(); X
An example of a CW complex: the surface given by the boundary map (1, 2, 1, 2)

sage: X.category()
Category of finite finite dimensional CW complexes
```

We conclude by running systematic tests on this manifold:

```
sage: TestSuite(X).run()
```

**class Element** (*parent, dim, name*)
Bases: `sage.structure.element.Element`

A cell in a CW complex.

**dimension** ()
Return the dimension of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: f = X.an_element()
```

```
sage: f.dimension()
2
```

**an_element**()

Return an element of the CW complex, as per *Sets.ParentMethods.an_element()*.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: X.an_element()
2-cell f
```

**cells**()

Return the cells of `self`.

EXAMPLES:

```
sage: from sage.categories.cw_complexes import CWComplexes
sage: X = CWComplexes().example()
sage: C = X.cells()
sage: sorted((d, C[d]) for d in C.keys())
[(0, (0-cell v,)),
 (1, (0-cell e1, 0-cell e2)),
 (2, (2-cell f,))]
```

## 5.7 Example of facade set

**class** sage.categories.examples.facade_sets.**IntegersCompletion**

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of a facade parent: the set of integers completed with $+-\infty$

This class illustrates a minimal implementation of a facade parent that models the union of several other parents.

EXAMPLES:

```
sage: S = Sets().Facade().example("union"); S
An example of a facade set: the integers completed by +-infinity
```

**class** sage.categories.examples.facade_sets.**PositiveIntegerMonoid**

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of a facade parent: the positive integers viewed as a multiplicative monoid

This class illustrates a minimal implementation of a facade parent which models a subset of a set.

EXAMPLES:

```
sage: S = Sets().Facade().example(); S
An example of facade set: the monoid of positive integers
```

## 5.8 Examples of finite Coxeter groups

**class** sage.categories.examples.finite_coxeter_groups.**DihedralGroup**(*n=5*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of finite Coxeter group: the $n$-th dihedral group of order $2n$.

The purpose of this class is to provide a minimal template for implementing finite Coxeter groups. See DihedralGroup for a full featured and optimized implementation.

EXAMPLES:

```
sage: G = FiniteCoxeterGroups().example()
```

This group is generated by two simple reflections $s_1$ and $s_2$ subject to the relation $(s_1 s_2)^n = 1$:

```
sage: G.simple_reflections()
Finite family {1: (1,), 2: (2,)}

sage: s1, s2 = G.simple_reflections()
sage: (s1*s2)^5 == G.one()
True
```

An element is represented by its reduced word (a tuple of elements of $self.index_set()$):

```
sage: G.an_element()
(1, 2)

sage: list(G)
[(),
 (1,),
 (2,),
 (1, 2),
 (2, 1),
 (1, 2, 1),
 (2, 1, 2),
 (1, 2, 1, 2),
 (2, 1, 2, 1),
 (1, 2, 1, 2, 1)]
```

This reduced word is unique, except for the longest element where the choosen reduced word is $(1, 2, 1, 2 \dots)$:

```
sage: G.long_element()
(1, 2, 1, 2, 1)
```

**class Element**

Bases: sage.structure.element_wrapper.ElementWrapper

**apply_simple_reflection_right**(*i*)

Implements CoxeterGroups.ElementMethods.apply_simple_reflection().

EXAMPLES:

```
sage: D5 = FiniteCoxeterGroups().example(5)
sage: [i^2 for i in D5]   # indirect doctest
[(), (), (), (1, 2, 1, 2), (2, 1, 2, 1), (), (), (2, 1), (1, 2), ()]
sage: [i^5 for i in D5]   # indirect doctest
[(), (1,), (2,), (), (), (1, 2, 1), (2, 1, 2), (), (), (1, 2, 1, 2, 1)]
```

**has_right_descent** (*i*, *positive=False*, *side='right'*)

Implements `SemiGroups.ElementMethods.has_right_descent()`.

EXAMPLES:

```
sage: D6 = FiniteCoxeterGroups().example(6)
sage: s = D6.simple_reflections()
sage: s[1].has_descent(1)
True
sage: s[1].has_descent(1)
True
sage: s[1].has_descent(2)
False
sage: D6.one().has_descent(1)
False
sage: D6.one().has_descent(2)
False
sage: D6.long_element().has_descent(1)
True
sage: D6.long_element().has_descent(2)
True
```

**coxeter_matrix** ()

Return the Coxeter matrix of `self`.

EXAMPLES:

```
sage: FiniteCoxeterGroups().example(6).coxeter_matrix()
[1 6]
[6 1]
```

**degrees** ()

Return the degrees of `self`.

EXAMPLES:

```
sage: FiniteCoxeterGroups().example(6).degrees()
(2, 6)
```

**index_set** ()

Implements *CoxeterGroups.ParentMethods.index_set()*.

EXAMPLES:

```
sage: D4 = FiniteCoxeterGroups().example(4)
sage: D4.index_set()
(1, 2)
```

**one** ()

Implements `Monoids.ParentMethods.one()`.

EXAMPLES:

```
sage: D6 = FiniteCoxeterGroups().example(6)
sage: D6.one()
()
```

`sage.categories.examples.finite_coxeter_groups.`**Example**

alias of *DihedralGroup*

## 5.9 Example of a finite dimensional algebra with basis

`sage.categories.examples.finite_dimensional_algebras_with_basis.Example`
alias of *KroneckerQuiverPathAlgebra*

**class** `sage.categories.examples.finite_dimensional_algebras_with_basis.KroneckerQuiverPathAl`
Bases: `sage.combinat.free_module.CombinatorialFreeModule`

An example of a finite dimensional algebra with basis: the path algebra of the Kronecker quiver.

This class illustrates a minimal implementation of a finite dimensional algebra with basis. See `sage.quivers.algebra.PathAlgebra` for a full-featured implementation of path algebras.

**algebra_generators**()
Return algebra generators for this algebra.

**See also:**

`Algebras.ParentMethods.algebra_generators()`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example(); A
An example of a finite dimensional algebra with basis:
the path algebra of the Kronecker quiver
(containing the arrows a:x->y and b:x->y) over Rational Field
sage: A.algebra_generators()
Finite family {'y': y, 'x': x, 'b': b, 'a': a}
```

**one**()
Return the unit of this algebra.

**See also:**

`AlgebrasWithBasis.ParentMethods.one_basis()`

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example()
sage: A.one()
x + y
```

**product_on_basis**(*w1*, *w2*)
Return the product of the two basis elements indexed by `w1` and `w2`.

**See also:**

`AlgebrasWithBasis.ParentMethods.product_on_basis()`.

EXAMPLES:

```
sage: A = FiniteDimensionalAlgebrasWithBasis(QQ).example()
```

Here is the multiplication table for the algebra:

```
sage: matrix([[p*q for q in A.basis()] for p in A.basis()])
[x 0 a b]
[0 y 0 0]
[0 a 0 0]
[0 b 0 0]
```

Here we take some products of linear combinations of basis elements:

```
sage: x, y, a, b = A.basis()
sage: a * (1-b)^2 * x
0
sage: x*a + b*y
a + b
sage: x*x
x
sage: x*y
0
sage: x*a*y
a
```

## 5.10 Examples of finite enumerated sets

**class** sage.categories.examples.finite_enumerated_sets.**Example**

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of a finite enumerated set: $\{1, 2, 3\}$

This class provides a minimal implementation of a finite enumerated set.

See FiniteEnumeratedSet for a full featured implementation.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().example()
sage: C.cardinality()
3
sage: C.list()
[1, 2, 3]
sage: C.an_element()
1
```

This checks that the different methods of the enumerated set $C$ return consistent results:

```
sage: TestSuite(C).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
running ._test_eq() . . . pass
```

```
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

**class** sage.categories.examples.finite_enumerated_sets.**IsomorphicObjectOfFiniteEnumeratedSet**

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

**ambient**()

Returns the ambient space for self, as per *Sets.Subquotients.ParentMethods.ambient()*.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().IsomorphicObjects().example(); C
The image by some isomorphism of An example of a finite enumerated set: {1,2,
↪3}
sage: C.ambient()
An example of a finite enumerated set: {1,2,3}
```

**lift**(*x*)

**INPUT:**

- x – an element of self

Lifts x to the ambient space for self, as per *Sets.Subquotients.ParentMethods.lift()*.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().IsomorphicObjects().example(); C
The image by some isomorphism of An example of a finite enumerated set: {1,2,
↪3}
sage: C.lift(9)
3
```

**retract**(*x*)

**INPUT:**

- x – an element of the ambient space for self

Retracts x from the ambient space to self, as per *Sets.Subquotients.ParentMethods.retract()*.

EXAMPLES:

```
sage: C = FiniteEnumeratedSets().IsomorphicObjects().example(); C
The image by some isomorphism of An example of a finite enumerated set: {1,2,
↪3}
sage: C.retract(3)
9
```

## 5.11 Examples of a finite dimensional Lie algebra with basis

**class** sage.categories.examples.finite_dimensional_lie_algebras_with_basis.**AbelianLieAlgebra**

Bases: sage.structure.parent.Parent, sage.structure.unique_representation.UniqueRepresentation

An example of a finite dimensional Lie algebra with basis: the abelian Lie algebra.

Let $R$ be a commutative ring, and $M$ an $R$-module. The *abelian Lie algebra* on $M$ is the $R$-Lie algebra obtained by endowing $M$ with the trivial Lie bracket ($[a, b] = 0$ for all $a, b \in M$).

This class illustrates a minimal implementation of a finite dimensional Lie algebra with basis.

INPUT:

- R – base ring
- n – (optional) a nonnegative integer (default: None)
- M – an $R$-module (default: the free $R$-module of rank n) to serve as the ground space for the Lie algebra
- ambient – (optional) a Lie algebra; if this is set, then the resulting Lie algebra is declared a Lie subalgebra of ambient

OUTPUT:

The abelian Lie algebra on $M$.

**class Element**

Bases: *sage.categories.examples.lie_algebras.LieAlgebraFromAssociative.Element*

**lift**()

Return the lift of self to the universal enveloping algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: elt = 2*a + 2*b + 3*c
sage: elt.lift()
2*b0 + 2*b1 + 3*b2
```

**monomial_coefficients**(*copy=True*)

Return the monomial coefficients of self.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: elt = 2*a + 2*b + 3*c
sage: elt.monomial_coefficients()
{0: 2, 1: 2, 2: 3}
```

**to_vector**()

Return `self` as a vector in `self.parent().module()`.

See the docstring of the latter method for the meaning of this.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: elt = 2*a + 2*b + 3*c
sage: elt.to_vector()
(2, 2, 3)
```

**ambient**()

Return the ambient Lie algebra of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: S = L.subalgebra([2*a+b, b + c])
sage: S.ambient() == L
True
```

**basis**()

Return the basis of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.basis()
Finite family {0: (1, 0, 0), 1: (0, 1, 0), 2: (0, 0, 1)}
```

**basis_matrix**()

Return the basis matrix of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.basis_matrix()
[1 0 0]
[0 1 0]
[0 0 1]
```

**from_vector**(*v*)

Return the element of `self` corresponding to the vector `v` in `self.module()`.

Implement this if you implement *module()*; see the documentation of `sage.categories.lie_algebras.LieAlgebras.module()` for how this is to be done.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: u = L.from_vector(vector(QQ, (1, 0, 0))); u
```

```
(1, 0, 0)
sage: parent(u) is L
True
```

**gens**()

Return the generators of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.gens()
((1, 0, 0), (0, 1, 0), (0, 0, 1))
```

**ideal**(*gens*)

Return the Lie subalgebra of `self` generated by the elements of the iterable `gens`.

This currently requires the ground ring $R$ to be a field.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.subalgebra([2*a+b, b + c])
An example of a finite dimensional Lie algebra with basis:
 the 2-dimensional abelian Lie algebra over Rational Field with
 basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

**is_ideal**(*A*)

Return if `self` is an ideal of the ambient space `A`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.is_ideal(L)
True
sage: S1 = L.subalgebra([2*a+b, b + c])
sage: S1.is_ideal(L)
True
sage: S2 = L.subalgebra([2*a+b])
sage: S2.is_ideal(S1)
True
sage: S1.is_ideal(S2)
False
```

**lie_algebra_generators**()

Return the basis of `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.basis()
Finite family {0: (1, 0, 0), 1: (0, 1, 0), 2: (0, 0, 1)}
```

**module**()

Return an $R$-module which is isomorphic to the underlying $R$-module of `self`.

See `sage.categories.lie_algebras.LieAlgebras.module()` for an explanation.

In this particular example, this returns the module $M$ that was used to construct `self`.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.module()
Vector space of dimension 3 over Rational Field

sage: a, b, c = L.lie_algebra_generators()
sage: S = L.subalgebra([2*a+b, b + c])
sage: S.module()
Vector space of degree 3 and dimension 2 over Rational Field
Basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

**subalgebra**(*gens*)

Return the Lie subalgebra of `self` generated by the elements of the iterable `gens`.

This currently requires the ground ring $R$ to be a field.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: a, b, c = L.lie_algebra_generators()
sage: L.subalgebra([2*a+b, b + c])
An example of a finite dimensional Lie algebra with basis:
 the 2-dimensional abelian Lie algebra over Rational Field with
 basis matrix:
[   1    0 -1/2]
[   0    1    1]
```

**zero**()

Return the zero element.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).FiniteDimensional().WithBasis().example()
sage: L.zero()
(0, 0, 0)
```

`sage.categories.examples.finite_dimensional_lie_algebras_with_basis.`**Example**

alias of *AbelianLieAlgebra*

## 5.12 Examples of finite monoids

`sage.categories.examples.finite_monoids.`**Example**

alias of *IntegerModMonoid*

**class** `sage.categories.examples.finite_monoids.`**IntegerModMonoid**(*n=12*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

An example of a finite monoid: the integers mod $n$

This class illustrates a minimal implementation of a finite monoid.

EXAMPLES:

```
sage: S = FiniteMonoids().example(); S
An example of a finite multiplicative monoid: the integers modulo 12

sage: S.category()
Category of finitely generated finite enumerated monoids
```

We conclude by running systematic tests on this monoid:

```
sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
```

**class Element**

Bases: `sage.structure.element_wrapper.ElementWrapper`

**wrapped_class**

alias of `Integer`

**an_element**()

Returns an element of the monoid, as per *Sets.ParentMethods.an_element()*.

EXAMPLES:

```
sage: M = FiniteMonoids().example()
sage: M.an_element()
6
```

**one**()

Return the one of the monoid, as per `Monoids.ParentMethods.one()`.

EXAMPLES:

```
sage: M = FiniteMonoids().example()
sage: M.one()
1
```

**product** (*x*, *y*)

Return the product of two elements $x$ and $y$ of the monoid, as per `Semigroups.ParentMethods.product()`.

EXAMPLES:

```
sage: M = FiniteMonoids().example()
sage: M.product(M(3), M(5))
3
```

**semigroup_generators** ()

Returns a set of generators for `self`, as per *Semigroups.ParentMethods.semigroup_generators()*. Currently this returns all integers mod $n$, which is of course far from optimal!

EXAMPLES:

```
sage: M = FiniteMonoids().example()
sage: M.semigroup_generators()
Family (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
```

## 5.13 Examples of finite semigroups

`sage.categories.examples.finite_semigroups.`**Example**

alias of *LeftRegularBand*

**class** `sage.categories.examples.finite_semigroups.`**LeftRegularBand** (*alphabet=('a', 'b', 'c', 'd')*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

An example of a finite semigroup

This class provides a minimal implementation of a finite semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(); S
An example of a finite semigroup: the left regular band generated by ('a', 'b', 'c
↪', 'd')
```

This is the semigroup generated by:

```
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

such that $x^2 = x$ and $xyx = xy$ for any $x$ and $y$ in $S$:

```
sage: S('dab')
'dab'
sage: S('dab') * S('acb')
'dabc'
```

It follows that the elements of $S$ are strings without repetitions over the alphabet $a$, $b$, $c$, $d$:

```
sage: sorted(S.list())
['a', 'ab', 'abc', 'abcd', 'abd', 'abdc', 'ac', 'acb', 'acbd', 'acd',
 'acdb', 'ad', 'adb', 'adbc', 'adc', 'adcb', 'b', 'ba', 'bac',
```

```
'bacd', 'bad', 'badc', 'bc', 'bca', 'bcad', 'bcd', 'bcda', 'bd',
'bda', 'bdac', 'bdc', 'bdca', 'c', 'ca', 'cab', 'cabd', 'cad',
'cadb', 'cb', 'cba', 'cbad', 'cbd', 'cbda', 'cd', 'cda', 'cdab',
'cdb', 'cdba', 'd', 'da', 'dab', 'dabc', 'dac', 'dacb', 'db',
'dba', 'dbac', 'dbc', 'dbca', 'dc', 'dca', 'dcab', 'dcb', 'dcba']
```

It also follows that there are finitely many of them:

```
sage: S.cardinality()
64
```

Indeed:

```
sage: 4 * ( 1 + 3 * (1 + 2 * (1 + 1)))
64
```

As expected, all the elements of $S$ are idempotents:

```
sage: all( x.is_idempotent() for x in S )
True
```

Now, let us look at the structure of the semigroup:

```
sage: S = FiniteSemigroups().example(alphabet = ('a','b','c'))
sage: S.cayley_graph(side="left", simple=True).plot()
Graphics object consisting of 60 graphics primitives
sage: S.j_transversal_of_idempotents() # random (arbitrary choice)
['acb', 'ac', 'ab', 'bc', 'a', 'c', 'b']
```

We conclude by running systematic tests on this semigroup:

```
sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

**class Element**
Bases: sage.structure.element_wrapper.ElementWrapper

**an_element** ()
Returns an element of the semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S.an_element()
'cdab'

sage: S = FiniteSemigroups().example(("b"))
sage: S.an_element()
'b'
```

**product** (*x*, *y*)
Returns the product of two elements of the semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().example()
sage: S('a') * S('b')
'ab'
sage: S('a') * S('b') * S('a')
'ab'
sage: S('a') * S('a')
'a'
```

**semigroup_generators** ()
Returns the generators of the semigroup.

EXAMPLES:

```
sage: S = FiniteSemigroups().example(alphabet=('x','y'))
sage: S.semigroup_generators()
Family ('x', 'y')
```

## 5.14 Examples of finite Weyl groups

sage.categories.examples.finite_weyl_groups.**Example**
alias of *SymmetricGroup*

**class** sage.categories.examples.finite_weyl_groups.**SymmetricGroup** (*n=4*)
Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of finite Weyl group: the symmetric group, with elements in list notation.

The purpose of this class is to provide a minimal template for implementing finite Weyl groups. See SymmetricGroup for a full featured and optimized implementation.

EXAMPLES:

```
sage: S = FiniteWeylGroups().example()
sage: S
The symmetric group on {0, ..., 3}
```

```
sage: S.category()
Category of finite weyl groups
```

The elements of this group are permutations of the set $\{0, \ldots, 3\}$:

```
sage: S.one()
(0, 1, 2, 3)
sage: S.an_element()
(1, 2, 3, 0)
```

The group itself is generated by the elementary transpositions:

```
sage: S.simple_reflections()
Finite family {0: (1, 0, 2, 3), 1: (0, 2, 1, 3), 2: (0, 1, 3, 2)}
```

Only the following basic operations are implemented:

- *one()*
- *product()*
- *simple_reflection()*
- *Element.has_right_descent()*.

All the other usual Weyl group operations are inherited from the categories:

```
sage: S.cardinality()
24
sage: S.long_element()
(3, 2, 1, 0)
sage: S.cayley_graph(side = "left").plot()
Graphics object consisting of 120 graphics primitives
```

Alternatively, one could have implemented `sage.categories.coxeter_groups.CoxeterGroups.ElementMethods.apply_simple_reflection()` instead of *simple_reflection()* and *product()*. See `CoxeterGroups().example()`.

**class Element**

Bases: `sage.structure.element_wrapper.ElementWrapper`

**has_right_descent** (*i*)

Implements *CoxeterGroups.ElementMethods.has_right_descent()*.

EXAMPLES:

```
sage: S = FiniteWeylGroups().example()
sage: s = S.simple_reflections()
sage: (s[1] * s[2]).has_descent(2)
True
sage: S._test_has_descent()
```

**degrees** ()

Return the degrees of `self`.

EXAMPLES:

```
sage: W = FiniteWeylGroups().example()
sage: W.degrees()
(2, 3, 4)
```

**index_set**()
Implements *CoxeterGroups.ParentMethods.index_set()*.

EXAMPLES:

```
sage: FiniteWeylGroups().example().index_set()
[0, 1, 2]
```

**one**()
Implements Monoids.ParentMethods.one().

EXAMPLES:

```
sage: FiniteWeylGroups().example().one()
(0, 1, 2, 3)
```

**product**(*x*, *y*)
Implements Semigroups.ParentMethods.product().

EXAMPLES:

```
sage: s = FiniteWeylGroups().example().simple_reflections()
sage: s[1] * s[2]
(0, 2, 3, 1)
```

**simple_reflection**(*i*)
Implement CoxeterGroups.ParentMethods.simple_reflection() by returning the transposition $(i, i+1)$.

EXAMPLES:

```
sage: FiniteWeylGroups().example().simple_reflection(2)
(0, 1, 3, 2)
```

## 5.15 Examples of graded connected Hopf algebras with basis

sage.categories.examples.graded_connected_hopf_algebras_with_basis.**Example**
alias of *GradedConnectedCombinatorialHopfAlgebraWithPrimitiveGenerator*

**class** sage.categories.examples.graded_connected_hopf_algebras_with_basis.**GradedConnectedCor**
Bases: sage.combinat.free_module.CombinatorialFreeModule

This class illustrates an implementation of a graded Hopf algebra with basis that has one primitive generator of degree 1 and basis elements indexed by non-negative integers.

This Hopf algebra example differs from what topologists refer to as a graded Hopf algebra because the twist operation in the tensor rule satisfies

$$(\mu \otimes \mu) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) = \Delta \circ \mu$$

where $\tau(x \otimes y) = y \otimes x$.

**coproduct_on_basis**(*i*)
The coproduct of a basis element.

$$\Delta(P_i) = \sum_{j=0}^{i} P_{i-j} \otimes P_j$$

INPUT:

- i – a non-negative integer

OUTPUT:

- an element of the tensor square of self

**degree_on_basis**(*i*)

The degree of a non-negative integer is itself

INPUT:

- i – a non-negative integer

OUTPUT:

- a non-negative integer

**one_basis**()

Returns 0, which index the unit of the Hopf algebra.

OUTPUT:

- the non-negative integer 0

EXAMPLES:

```
sage: H = GradedHopfAlgebrasWithBasis(QQ).Connected().example()
sage: H.one_basis()
0
sage: H.one()
P0
```

**product_on_basis**(*i*, *j*)

The product of two basis elements.

The product of elements of degree i and j is an element of degree i+j.

INPUT:

- i, j – non-negative integers

OUTPUT:

- a basis element indexed by i+j

## 5.16 Examples of graded modules with basis

sage.categories.examples.graded_modules_with_basis.**Example**

alias of *GradedPartitionModule*

**class** sage.categories.examples.graded_modules_with_basis.**GradedPartitionModule**(*base_ring*)

Bases: sage.combinat.free_module.CombinatorialFreeModule

This class illustrates an implementation of a graded module with basis: the free module over partitions.

INPUT:

- R – base ring

The implementation involves the following:

- A choice of how to represent elements. In this case, the basis elements are partitions. The algebra is constructed as a CombinatorialFreeModule on the set of partitions, so it inherits all of the methods for such objects, and has operations like addition already defined.

```
sage: A = GradedModulesWithBasis(QQ).example()
```

- A basis function - this module is graded by the non-negative integers, so there is a function defined in this module, creatively called *basis()*, which takes an integer $d$ as input and returns a family of partitions representing a basis for the algebra in degree $d$.

```
sage: A.basis(2)
Lazy family (Term map from Partitions to An example of a graded module with␣
↪basis: the free module on partitions over Rational Field(i))_{i in␣
↪Partitions of the integer 2}
sage: A.basis(6)[Partition([3,2,1])]
P[3, 2, 1]
```

- If the algebra is called `A`, then its basis function is stored as `A.basis`. Thus the function can be used to find a basis for the degree $d$ piece: essentially, just call `A.basis(d)`. More precisely, call `x` for each `x` in `A.basis(d)`.

```
sage: [m for m in A.basis(4)]
[P[4], P[3, 1], P[2, 2], P[2, 1, 1], P[1, 1, 1, 1]]
```

- For dealing with basis elements: *degree_on_basis()*, and `_repr_term()`. The first of these defines the degree of any monomial, and then the `degree` method for elements – see the next item – uses it to compute the degree for a linear combination of monomials. The last of these determines the print representation for monomials, which automatically produces the print representation for general elements.

```
sage: A.degree_on_basis(Partition([4,3]))
7
sage: A._repr_term(Partition([4,3]))
'P[4, 3]'
```

- There is a class for elements, which inherits from `IndexedFreeModuleElement`. An element is determined by a dictionary whose keys are partitions and whose corresponding values are the coefficients. The class implements two things: an `is_homogeneous` method and a `degree` method.

```
sage: p = A.monomial(Partition([3,2,1])); p
P[3, 2, 1]
sage: p.is_homogeneous()
True
sage: p.degree()
6
```

**basis**(*d=None*)

Return the basis for (the `d`-th homogeneous component of) `self`.

INPUT:

- `d` – (optional, default `None`) nonnegative integer or `None`

OUTPUT:

If `d` is `None`, returns the basis of the module. Otherwise, returns the basis of the homogeneous component of degree `d` (i.e., the subfamily of the basis of the whole module which consists only of the basis vectors lying in $F_d \setminus \bigcup_{i<d} F_i$).

The basis is always returned as a family.

EXAMPLES:

```
sage: A = ModulesWithBasis(ZZ).Filtered().example()
sage: A.basis(4)
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions of the integer 4}
```

Without arguments, the full basis is returned:

```
sage: A.basis()
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions}
sage: A.basis()
Lazy family (Term map from Partitions to An example of a
 filtered module with basis: the free module on partitions
 over Integer Ring(i))_{i in Partitions}
```

Checking this method on a filtered algebra. Note that this will typically raise a `NotImplementedError` when this feature is not implemented.

```
sage: A = AlgebrasWithBasis(ZZ).Filtered().example()
sage: A.basis(4)
Traceback (most recent call last):
...
NotImplementedError: infinite set
```

Without arguments, the full basis is returned:

```
sage: A.basis()
Lazy family (Term map from Free abelian monoid indexed by
 {'x', 'y', 'z'} to An example of a filtered algebra with
 basis: the universal enveloping algebra of Lie algebra
 of RR^3 with cross product over Integer Ring(i))_{i in
 Free abelian monoid indexed by {'x', 'y', 'z'}}
```

An example with a graded algebra:

```
sage: E.<x,y> = ExteriorAlgebra(QQ)
sage: E.basis()
Lazy family (Term map from Subsets of {0, 1} to
 The exterior algebra of rank 2 over Rational Field(i))_{i in
 Subsets of {0, 1}}
```

**degree_on_basis**(*t*)

The degree of the element determined by the partition `t` in this graded module.

INPUT:

- `t` – the index of an element of the basis of this module, i.e. a partition

OUTPUT: an integer, the degree of the corresponding basis element

EXAMPLES:

```
sage: A = GradedModulesWithBasis(QQ).example()
sage: A.degree_on_basis(Partition((2,1)))
3
sage: A.degree_on_basis(Partition((4,2,1,1,1,1)))
10
```

```
sage: type(A.degree_on_basis(Partition((1,1))))
<type 'sage.rings.integer.Integer'>
```

## 5.17 Examples of graphs

**class** sage.categories.examples.graphs.**Cycle**(*n=5*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of a graph: the cycle of length $n$.

This class illustrates a minimal implementation of a graph.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example(); C
An example of a graph: the 5-cycle

sage: C.category()
Category of graphs
```

We conclude by running systematic tests on this graph:

```
sage: TestSuite(C).run()
```

**class Element**

Bases: sage.structure.element_wrapper.ElementWrapper

**dimension**()

Return the dimension of self.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: e = C.edges()[0]
sage: e.dimension()
2
sage: v = C.vertices()[0]
sage: v.dimension()
1
```

**an_element**()

Return an element of the graph, as per *Sets.ParentMethods.an_element()*.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.an_element()
0
```

**edges**()

Return the edges of self.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.edges()
[(0, 1), (1, 2), (2, 3), (3, 4), (4, 0)]
```

**vertices**()

Return the vertices of `self`.

EXAMPLES:

```
sage: from sage.categories.graphs import Graphs
sage: C = Graphs().example()
sage: C.vertices()
[0, 1, 2, 3, 4]
```

sage.categories.examples.graphs.**Example**

alias of *Cycle*

## 5.18 Examples of algebras with basis

**class** sage.categories.examples.hopf_algebras_with_basis.**MyGroupAlgebra**(*R*, *G*)

Bases: sage.combinat.free_module.CombinatorialFreeModule

An of a Hopf algebra with basis: the group algebra of a group

This class illustrates a minimal implementation of a Hopf algebra with basis.

**algebra_generators**()

Return the generators of this algebra, as per *algebra_generators()*.

They correspond to the generators of the group.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example(); A
An example of Hopf algebra with basis: the group algebra of the Dihedral␣
↪group of order 6 as a permutation group over Rational Field
sage: A.algebra_generators()
Finite family {(1,3): B[(1,3)], (1,2,3): B[(1,2,3)]}
```

**antipode_on_basis**(*g*)

Antipode, on basis elements, as per *HopfAlgebrasWithBasis.ParentMethods.antipode_on_basis()*.

It is given, on basis elements, by $\nu(g) = g^{-1}$

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: (a, b) = A._group.gens()
sage: A.antipode_on_basis(a)
B[(1,3,2)]
```

**coproduct_on_basis**(*g*)

Coproduct, on basis elements, as per `HopfAlgebrasWithBasis.ParentMethods.coproduct_on_basis()`.

The basis elements are group like: $\Delta(g) = g \otimes g$.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: (a, b) = A._group.gens()
sage: A.coproduct_on_basis(a)
B[(1,2,3)] # B[(1,2,3)]
```

**counit_on_basis**(*g*)
Counit, on basis elements, as per `HopfAlgebrasWithBasis.ParentMethods.counit_on_basis()`.

The counit on the basis elements is 1.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: (a, b) = A._group.gens()
sage: A.counit_on_basis(a)
1
```

**one_basis**()
Returns the one of the group, which index the one of this algebra, as per `AlgebrasWithBasis.ParentMethods.one_basis()`.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: A.one_basis()
()
sage: A.one()
B[()]
```

**product_on_basis**(*g1*, *g2*)
Product, on basis elements, as per `AlgebrasWithBasis.ParentMethods.product_on_basis()`.

The product of two basis elements is induced by the product of the corresponding elements of the group.

EXAMPLES:

```
sage: A = HopfAlgebrasWithBasis(QQ).example()
sage: (a, b) = A._group.gens()
sage: a*b
(1,2)
sage: A.product_on_basis(a, b)
B[(1,2)]
```

## 5.19 Examples of infinite enumerated sets

`sage.categories.examples.infinite_enumerated_sets.`**`Example`**
alias of `NonNegativeIntegers`

**class** `sage.categories.examples.infinite_enumerated_sets.`**`NonNegativeIntegers`**
Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

An example of infinite enumerated set: the non negative integers

This class provides a minimal implementation of an infinite enumerated set.

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN
An example of an infinite enumerated set: the non negative integers
sage: NN.cardinality()
+Infinity
sage: NN.list()
Traceback (most recent call last):
...
NotImplementedError: cannot list an infinite set
sage: NN.element_class
<type 'sage.rings.integer.Integer'>
sage: it = iter(NN)
sage: [next(it), next(it), next(it), next(it), next(it)]
[0, 1, 2, 3, 4]
sage: x = next(it); type(x)
<type 'sage.rings.integer.Integer'>
sage: x.parent()
Integer Ring
sage: x+3
8
sage: NN(15)
15
sage: NN.first()
0
```

This checks that the different methods of $NN$ return consistent results:

```
sage: TestSuite(NN).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_enumerated_set_contains() . . . pass
running ._test_enumerated_set_iter_cardinality() . . . pass
running ._test_enumerated_set_iter_list() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

**Element**

alias of `Integer`

**an_element**()

EXAMPLES:

```
sage: InfiniteEnumeratedSets().example().an_element()
42
```

**next**(*o*)

EXAMPLES:

```
sage: NN = InfiniteEnumeratedSets().example()
sage: NN.next(3)
4
```

## 5.20 Examples of manifolds

sage.categories.examples.manifolds.**Example**

alias of *Plane*

**class** sage.categories.examples.manifolds.**Plane**(*n=3*, *base_ring=None*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of a manifold: the $n$-dimensional plane.

This class illustrates a minimal implementation of a manifold.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: M = Manifolds(QQ).example(); M
An example of a Rational Field manifold: the 3-dimensional plane

sage: M.category()
Category of manifolds over Rational Field
```

We conclude by running systematic tests on this manifold:

```
sage: TestSuite(M).run()
```

**Element**

alias of ElementWrapper

**an_element**()

Return an element of the manifold, as per *Sets.ParentMethods.an_element()*.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: M = Manifolds(QQ).example()
sage: M.an_element()
(0, 0, 0)
```

**dimension**()

Return the dimension of self.

EXAMPLES:

```
sage: from sage.categories.manifolds import Manifolds
sage: M = Manifolds(QQ).example()
sage: M.dimension()
3
```

## 5.21 Examples of a Lie algebra

sage.categories.examples.lie_algebras.**Example**
alias of *LieAlgebraFromAssociative*

**class** sage.categories.examples.lie_algebras.**LieAlgebraFromAssociative**(*gens*)
Bases: sage.structure.parent.Parent, sage.structure.unique_representation.UniqueRepresentation

An example of a Lie algebra: a Lie algebra generated by a set of elements of an associative algebra.

This class illustrates a minimal implementation of a Lie algebra.

Let $R$ be a commutative ring, and $A$ an associative $R$-algebra. The Lie algebra $A$ (sometimes denoted $A^-$) is defined to be the $R$-module $A$ with Lie bracket given by the commutator in $A$: that is, $[a, b] := ab - ba$ for all $a, b \in A$.

What this class implements is not precisely $A^-$, however; it is the Lie subalgebra of $A^-$ generated by the elements of the iterable gens. This specific implementation does not provide a reasonable containment test (i.e., it does not allow you to check if a given element $a$ of $A^-$ belongs to this Lie subalgebra); it, however, allows computing inside it.

INPUT:

- gens – a nonempty iterable consisting of elements of an associative algebra $A$

OUTPUT:

The Lie subalgebra of $A^-$ generated by the elements of gens

EXAMPLES:

We create a model of $\mathfrak{sl}_2$ using matrices:

```
sage: gens = [matrix([[0,1],[0,0]]), matrix([[0,0],[1,0]]), matrix([[1,0],[0,-
↪1]])]
sage: for g in gens:
....:     g.set_immutable()
sage: L = LieAlgebras(QQ).example(gens)
sage: e,f,h = L.lie_algebra_generators()
sage: e.bracket(f) == h
True
sage: h.bracket(e) == 2*e
True
sage: h.bracket(f) == -2*f
True
```

**class Element**
Bases: sage.structure.element_wrapper.ElementWrapper

Wrap an element as a Lie algebra element.

**lie_algebra_generators**()
Return the generators of self as a Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()
sage: L.lie_algebra_generators()
Family ([2, 1, 3], [2, 3, 1])
```

**zero**()

Return the element 0.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).example()
sage: L.zero()
0
```

## 5.22 Examples of a Lie algebra with basis

**class** sage.categories.examples.lie_algebras_with_basis.**AbelianLieAlgebra**(*R*, *gens*)

Bases: sage.combinat.free_module.CombinatorialFreeModule

An example of a Lie algebra: the abelian Lie algebra.

This class illustrates a minimal implementation of a Lie algebra with a distinguished basis.

**class Element**

Bases: sage.modules.with_basis.indexed_element.IndexedFreeModuleElement

**lift**()

Return the lift of self to the universal enveloping algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: elt = L.an_element()
sage: elt.lift()
3*P[F[2]] + 2*P[F[1]] + 2*P[F[]]
```

**bracket_on_basis**(*x*, *y*)

Return the Lie bracket on basis elements indexed by x and y.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: L.bracket_on_basis(Partition([4,1]), Partition([2,2,1]))
0
```

**lie_algebra_generators**()

Return the generators of self as a Lie algebra.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: L.lie_algebra_generators()
Lazy family (Term map from Partitions to
 An example of a Lie algebra: the abelian Lie algebra on the
 generators indexed by Partitions over Rational
 Field(i))_{i in Partitions}
```

sage.categories.examples.lie_algebras_with_basis.**Example**
alias of *AbelianLieAlgebra*

**class** sage.categories.examples.lie_algebras_with_basis.**IndexedPolynomialRing**(*R*, *indices*, ***kwds*)
Bases: sage.combinat.free_module.CombinatorialFreeModule

Polynomial ring whose generators are indexed by an arbitrary set.

**Todo:** Currently this is just used as the universal enveloping algebra for the example of the abelian Lie algebra. This should be factored out into a more complete class.

**algebra_generators**()
Return the algebra generators of self.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: UEA = L.universal_enveloping_algebra()
sage: UEA.algebra_generators()
Lazy family (algebra generator map(i))_{i in Partitions}
```

**one_basis**()
Return the index of element 1.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: UEA = L.universal_enveloping_algebra()
sage: UEA.one_basis()
1
sage: UEA.one_basis().parent()
Free abelian monoid indexed by Partitions
```

**product_on_basis**(*x*, *y*)
Return the product of the monomials indexed by x and y.

EXAMPLES:

```
sage: L = LieAlgebras(QQ).WithBasis().example()
sage: UEA = L.universal_enveloping_algebra()
sage: I = UEA._indices
sage: UEA.product_on_basis(I.an_element(), I.an_element())
P[F[]^4*F[1]^4*F[2]^6]
```

## 5.23 Examples of monoids

sage.categories.examples.monoids.**Example**
alias of *FreeMonoid*

**class** sage.categories.examples.monoids.**FreeMonoid**(*alphabet=('a', 'b', 'c', 'd')*)
Bases: *sage.categories.examples.semigroups.FreeSemigroup*

An example of a monoid: the free monoid

This class illustrates a minimal implementation of a monoid. For a full featured implementation of free monoids, see *FreeMonoid()*.

EXAMPLES:

```
sage: S = Monoids().example(); S
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')

sage: S.category()
Category of monoids
```

This is the free semigroup generated by:

```
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

with product rule given by concatenation of words:

```
sage: S('dab') * S('acb')
'dabacb'
```

and unit given by the empty word:

```
sage: S.one()
''
```

We conclude by running systematic tests on this monoid:

```
sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_one() . . . pass
running ._test_pickling() . . . pass
running ._test_prod() . . . pass
running ._test_some_elements() . . . pass
```

**class Element**

Bases: `sage.structure.element_wrapper.ElementWrapper`

**monoid_generators**()

Return the generators of this monoid.

EXAMPLES:

```
sage: M = Monoids().example(); M
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
sage: M.monoid_generators()
Finite family {'a': 'a', 'c': 'c', 'b': 'b', 'd': 'd'}
sage: a,b,c,d = M.monoid_generators()
sage: a*d*c*b
'adcb'
```

**one**()

Returns the one of the monoid, as per `Monoids.ParentMethods.one()`.

EXAMPLES:

```
sage: M = Monoids().example(); M
An example of a monoid: the free monoid generated by ('a', 'b', 'c', 'd')
sage: M.one()
''
```

## 5.24 Examples of posets

**class** sage.categories.examples.posets.**FiniteSetsOrderedByInclusion**

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

An example of a poset: finite sets ordered by inclusion

This class provides a minimal implementation of a poset

EXAMPLES:

```
sage: P = Posets().example(); P
An example of a poset: sets ordered by inclusion
```

We conclude by running systematic tests on this poset:

```
sage: TestSuite(P).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

**class Element**
Bases: `sage.structure.element_wrapper.ElementWrapper`

**wrapped_class**
alias of `Set_object_enumerated`

**an_element()**
Returns an element of this poset

EXAMPLES:

```
sage: B = Posets().example()
sage: B.an_element()
{1, 4, 6}
```

**le(x, y)**
Returns whether $x$ is a subset of $y$

EXAMPLES:

```
sage: P = Posets().example()
sage: P.le( P(Set([1,3])), P(Set([1,2,3])) )
True
sage: P.le( P(Set([1,3])), P(Set([1,3])) )
True
sage: P.le( P(Set([1,2])), P(Set([1,3])) )
False
```

**class** sage.categories.examples.posets.**PositiveIntegersOrderedByDivisibilityFacade**
Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

An example of a facade poset: the positive integers ordered by divisibility

This class provides a minimal implementation of a facade poset

EXAMPLES:

```
sage: P = Posets().example("facade"); P
An example of a facade poset: the positive integers ordered by divisibility

sage: P(5)
5
sage: P(0)
Traceback (most recent call last):
...
ValueError: Can't coerce `0` in any parent `An example of a facade poset: the␣
↪positive integers ordered by divisibility` is a facade for

sage: 3 in P
True
sage: 0 in P
False
```

**class element_class(X)**
Bases: `sage.sets.set.Set_object_enumerated`, `sage.categories.finite_sets.FiniteSets.parent_class`

A finite enumerated set.

**le** (*x*, *y*)
Returns whether $x$ is divisible by $y$

EXAMPLES:

```
sage: P = Posets().example("facade")
sage: P.le(3, 6)
True
sage: P.le(3, 3)
True
sage: P.le(3, 7)
False
```

## 5.25 Examples of semigroups in cython

**class** sage.categories.examples.semigroups_cython.**IdempotentSemigroups** (*s=None*)
Bases: *sage.categories.category.Category*

**class ElementMethods**

**is_idempotent** ()
EXAMPLES:

```
sage: from sage.categories.examples.semigroups_cython import␣
↪LeftZeroSemigroup
sage: S = LeftZeroSemigroup()
sage: S(2).is_idempotent()
True
```

**super_categories** ()
EXAMPLES:

```
sage: from sage.categories.examples.semigroups_cython import␣
↪IdempotentSemigroups
sage: IdempotentSemigroups().super_categories()
[Category of semigroups]
```

**class** sage.categories.examples.semigroups_cython.**LeftZeroSemigroup**
Bases: *sage.categories.examples.semigroups.LeftZeroSemigroup*

An example of semigroup

This class illustrates a minimal implementation of a semi-group where the element class is an extension type, and still gets code from the category. The category itself must be a Python class though.

This is purely a proof of concept. The code obviously needs refactorisation!

Comments:

- one cannot play ugly class surgery tricks (as with _mul_parent). available operations should really be declared to the coercion model!

EXAMPLES:

```
sage: from sage.categories.examples.semigroups_cython import LeftZeroSemigroup
sage: S = LeftZeroSemigroup(); S
An example of a semigroup: the left zero semigroup
```

This is the semigroup which contains all sort of objects:

```
sage: S.some_elements()
[3, 42, 'a', 3.4, 'raton laveur']
```

with product rule is given by $a \times b = a$ for all $a, b$.

```
sage: S('hello') * S('world')
'hello'

sage: S(3)*S(1)*S(2)
3

sage: S(3)^12312321312321
3

sage: TestSuite(S).run(verbose = True)
running ._test_an_element() . . . pass
running ._test_associativity() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

That's really the only method which is obtained from the category …

```
sage: S(42).is_idempotent
<bound method IdempotentSemigroups.element_class.is_idempotent of 42>
sage: S(42).is_idempotent()
True

sage: S(42)._pow_int
<bound method IdempotentSemigroups.element_class._pow_int of 42>
sage: S(42)^10
42

sage: S(42).is_idempotent
<bound method IdempotentSemigroups.element_class.is_idempotent of 42>
sage: S(42).is_idempotent()
True
```

**Element**

alias of *LeftZeroSemigroupElement*

**class** sage.categories.examples.semigroups_cython.**LeftZeroSemigroupElement**

Bases: sage.structure.element.Element

EXAMPLES:

```
sage: from sage.categories.examples.semigroups_cython import LeftZeroSemigroup
sage: S = LeftZeroSemigroup()
sage: x = S(3)
sage: TestSuite(x).run()
```

## 5.26 Examples of semigroups

**class** sage.categories.examples.semigroups.**FreeSemigroup**(*alphabet=('a', 'b', 'c', 'd')*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of semigroup.

The purpose of this class is to provide a minimal template for implementing of a semigroup.

EXAMPLES:

```
sage: S = Semigroups().example("free"); S
An example of a semigroup: the free semigroup generated by ('a', 'b', 'c', 'd')
```

This is the free semigroup generated by:

```
sage: S.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

and with product given by concatenation:

```
sage: S('dab') * S('acb')
'dabacb'
```

**class Element**

Bases: sage.structure.element_wrapper.ElementWrapper

The class for elements of the free semigroup.

**an_element**()

Returns an element of the semigroup.

EXAMPLES:

```
sage: F = Semigroups().example('free')
sage: F.an_element()
'abcd'
```

**product**(*x, y*)

Returns the product of x and y in the semigroup, as per Semigroups.ParentMethods.product().

EXAMPLES:

```
sage: F = Semigroups().example('free')
sage: F.an_element() * F('a')^5
'abcdaaaaa'
```

**semigroup_generators**()

Returns the generators of the semigroup.

EXAMPLES:

```
sage: F = Semigroups().example('free')
sage: F.semigroup_generators()
Family ('a', 'b', 'c', 'd')
```

**class** sage.categories.examples.semigroups.**IncompleteSubquotientSemigroup**(*category=None*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An incompletely implemented subquotient semigroup, for testing purposes

EXAMPLES:

```
sage: S = sage.categories.examples.semigroups.IncompleteSubquotientSemigroup()
sage: S
A subquotient of An example of a semigroup: the left zero semigroup
```

**class Element**

Bases: sage.structure.element_wrapper.ElementWrapper

**ambient**()

Returns the ambient semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.ambient()
An example of a semigroup: the left zero semigroup
```

**class** sage.categories.examples.semigroups.**LeftZeroSemigroup**

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of a semigroup.

This class illustrates a minimal implementation of a semigroup.

EXAMPLES:

```
sage: S = Semigroups().example(); S
An example of a semigroup: the left zero semigroup
```

This is the semigroup that contains all sorts of objects:

```
sage: S.some_elements()
[3, 42, 'a', 3.4, 'raton laveur']
```

with product rule given by $a \times b = a$ for all $a, b$:

```
sage: S('hello') * S('world')
'hello'
sage: S(3)*S(1)*S(2)
```

```
3
sage: S(3)^12312321312321
3
```

**class Element**

Bases: sage.structure.element_wrapper.ElementWrapper

**is_idempotent()**

Trivial implementation of Semigroups.Element.is_idempotent since all elements of this semigroup are idempotent!

EXAMPLES:

```
sage: S = Semigroups().example()
sage: S.an_element().is_idempotent()
True
sage: S(17).is_idempotent()
True
```

**an_element()**

Returns an element of the semigroup.

EXAMPLES:

```
sage: Semigroups().example().an_element()
42
```

**product** (*x*, *y*)

Returns the product of x and y in the semigroup, as per Semigroups.ParentMethods.product().

EXAMPLES:

```
sage: S = Semigroups().example()
sage: S('hello') * S('world')
'hello'
sage: S(3)*S(1)*S(2)
3
```

**some_elements()**

Returns a list of some elements of the semigroup.

EXAMPLES:

```
sage: Semigroups().example().some_elements()
[3, 42, 'a', 3.4, 'raton laveur']
```

**class** sage.categories.examples.semigroups.**QuotientOfLeftZeroSemigroup**(*category=None*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

Example of a quotient semigroup

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example(); S
An example of a (sub)quotient semigroup: a quotient of the left zero semigroup
```

This is the quotient of:

```
sage: S.ambient()
An example of a semigroup: the left zero semigroup
```

obtained by setting $x = 42$ for any $x \geq 42$:

```
sage: S(100)
42
sage: S(100) == S(42)
True
```

The product is inherited from the ambient semigroup:

```
sage: S(1)*S(2) == S(1)
True
```

**class Element**

Bases: `sage.structure.element_wrapper.ElementWrapper`

**ambient()**

Returns the ambient semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.ambient()
An example of a semigroup: the left zero semigroup
```

**an_element()**

Returns an element of the semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.an_element()
42
```

**lift(*x*)**

Lift the element `x` into the ambient semigroup.

INPUT:

- `x` – an element of `self`.

OUTPUT:

- an element of `self.ambient()`.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: x = S.an_element(); x
42
sage: S.lift(x)
42
sage: S.lift(x) in S.ambient()
True
sage: y = S.ambient()(100); y
100
sage: S.lift(S(y))
42
```

**retract** ($x$)
Returns the retract x onto an element of this semigroup.

INPUT:

- x – an element of the ambient semigroup (self.ambient()).

OUTPUT:

- an element of self.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: L = S.ambient()
sage: S.retract(L(17))
17
sage: S.retract(L(42))
42
sage: S.retract(L(171))
42
```

**some_elements** ()
Returns a list of some elements of the semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.some_elements()
[1, 2, 3, 8, 42, 42]
```

**the_answer** ()
Returns the Answer to Life, the Universe, and Everything as an element of this semigroup.

EXAMPLES:

```
sage: S = Semigroups().Subquotients().example()
sage: S.the_answer()
42
```

## 5.27 Examples of sets

**class** sage.categories.examples.sets_cat.**PrimeNumbers**
Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

An example of parent in the category of sets: the set of prime numbers.

The elements are represented as plain integers in $\mathbf{Z}$ (facade implementation).

This is a minimal implementations. For more advanced examples of implementations, see also:

```
sage: P = Sets().example("facade")
sage: P = Sets().example("inherits")
sage: P = Sets().example("wrapper")
```

EXAMPLES:

```
sage: P = Sets().example()
sage: P(12)
Traceback (most recent call last):
...
AssertionError: 12 is not a prime number
sage: a = P.an_element()
sage: a.parent()
Integer Ring
sage: x = P(13); x
13
sage: type(x)
<type 'sage.rings.integer.Integer'>
sage: x.parent()
Integer Ring
sage: 13 in P
True
sage: 12 in P
False
sage: y = x+1; y
14
sage: type(y)
<type 'sage.rings.integer.Integer'>

sage: TestSuite(P).run(verbose=True)
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_nonzero_equal() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

**an_element**()

Implements *Sets.ParentMethods.an_element()*.

**element_class**

alias of `Integer`

**class** sage.categories.examples.sets_cat.**PrimeNumbers_Abstract**

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

This class shows how to write a parent while keeping the choice of the datastructure for the children open. Different class with fixed datastructure will then be constructed by inheriting from *PrimeNumbers_Abstract*.

This is used by:

sage: P = Sets().example("facade") sage: P = Sets().example("inherits") sage: P = Sets().example("wrapper")

**class Element**

Bases: `sage.structure.element.Element`

**is_prime()**

Return whether `self` is a prime number.

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: x = P.an_element()
sage: P.an_element().is_prime()
True
```

**next()**

Return the next prime number.

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: p = P.an_element(); p
47
sage: p.next()
53
```

**Note:** This method is not meant to implement the protocol iterator, and thus not subject to Python 2 vs Python 3 incompatibilities.

**an_element()**

Implements *`Sets.ParentMethods.an_element()`*.

**next(*i*)**

Return the next prime number.

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: x = P.next(P.an_element()); x
53
sage: x.parent()
Set of prime numbers
```

**some_elements()**

Return some prime numbers.

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: P.some_elements()
[47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97]
```

**class** sage.categories.examples.sets_cat.**PrimeNumbers_Facade**

Bases: *`sage.categories.examples.sets_cat.PrimeNumbers_Abstract`*

An example of parent in the category of sets: the set of prime numbers.

In this alternative implementation, the elements are represented as plain integers in **Z** (facade implementation).

EXAMPLES:

```
sage: P = Sets().example("facade")
sage: P(12)
Traceback (most recent call last):
...
ValueError: 12 is not a prime number
sage: a = P.an_element()
sage: a.parent()
Integer Ring
sage: x = P(13); x
13
sage: type(x)
<type 'sage.rings.integer.Integer'>
sage: x.parent()
Integer Ring
sage: 13 in P
True
sage: 12 in P
False
sage: y = x+1; y
14
sage: type(y)
<type 'sage.rings.integer.Integer'>

sage: z = P.next(x); z
17
sage: type(z)
<type 'sage.rings.integer.Integer'>
sage: z.parent()
Integer Ring
```

The disadvantage of this implementation is that the elements do not know that they are prime, so that prime testing is slow:

```
sage: pf = Sets().example("facade").an_element()
sage: timeit("pf.is_prime()") #    random
625 loops, best of 3: 4.1 us per loop
```

compared to the other implementations where prime testing is only done if needed during the construction of the element, and later on the elements "know" that they are prime:

```
sage: pw = Sets().example("wrapper").an_element()
sage: timeit("pw.is_prime()")    # random
625 loops, best of 3: 859 ns per loop

sage: pi = Sets().example("inherits").an_element()
sage: timeit("pw.is_prime()")    # random
625 loops, best of 3: 854 ns per loop
```

Note also that the `next` method for the elements does not exist:

```
sage: pf.next()
Traceback (most recent call last):
...
AttributeError: 'sage.rings.integer.Integer' object has no attribute 'next'
```

unlike in the other implementations:

```
sage: pw.next()
53
sage: pi.next()
53
```

**element_class**
alias of Integer

**class** sage.categories.examples.sets_cat.**PrimeNumbers_Inherits**
Bases: *sage.categories.examples.sets_cat.PrimeNumbers_Abstract*

An example of parent in the category of sets: the set of prime numbers. In this implementation, the element are stored as object of a new class which inherits from the class Integer (technically `IntegerWrapper`).

EXAMPLES:

```
sage: P = Sets().example("inherits")
sage: P
Set of prime numbers
sage: P(12)
Traceback (most recent call last):
...
ValueError: 12 is not a prime number
sage: a = P.an_element()
sage: a.parent()
Set of prime numbers
sage: x = P(13); x
13
sage: x.is_prime()
True
sage: type(x)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits_with_category.
↪element_class'>
sage: x.parent()
Set of prime numbers
sage: P(13) in P
True
sage: y = x+1; y
14
sage: type(y)
<type 'sage.rings.integer.Integer'>
sage: y.parent()
Integer Ring
sage: type(P(13)+P(17))
<type 'sage.rings.integer.Integer'>
sage: type(P(2)+P(3))
<type 'sage.rings.integer.Integer'>

sage: z = P.next(x); z
17
sage: type(z)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Inherits_with_category.
↪element_class'>
sage: z.parent()
Set of prime numbers

sage: TestSuite(P).run(verbose=True)
```

```
running ._test_an_element() . . . pass
running ._test_cardinality() . . . pass
running ._test_category() . . . pass
running ._test_elements() . . .
  Running the test suite of self.an_element()
  running ._test_category() . . . pass
  running ._test_eq() . . . pass
  running ._test_new() . . . pass
  running ._test_not_implemented_methods() . . . pass
  running ._test_pickling() . . . pass
  pass
running ._test_elements_eq_reflexive() . . . pass
running ._test_elements_eq_symmetric() . . . pass
running ._test_elements_eq_transitive() . . . pass
running ._test_elements_neq() . . . pass
running ._test_eq() . . . pass
running ._test_new() . . . pass
running ._test_not_implemented_methods() . . . pass
running ._test_pickling() . . . pass
running ._test_some_elements() . . . pass
```

See also:

```
sage: P = Sets().example("facade")
sage: P = Sets().example("inherits")
sage: P = Sets().example("wrapper")
```

**class Element**(*parent*, *p*)

Bases: `sage.rings.integer.IntegerWrapper`, *sage.categories.examples.sets_cat.PrimeNumbers_Abstract.Element*

**class** sage.categories.examples.sets_cat.**PrimeNumbers_Wrapper**

Bases: *sage.categories.examples.sets_cat.PrimeNumbers_Abstract*

An example of parent in the category of sets: the set of prime numbers.

In this second alternative implementation, the prime integer are stored as a attribute of a sage object by inheriting from *ElementWrapper*. In this case we need to ensure conversion and coercion from this parent and its element to `ZZ` and `Integer`.

EXAMPLES:

```
sage: P = Sets().example("wrapper")
sage: P(12)
Traceback (most recent call last):
...
ValueError: 12 is not a prime number
sage: a = P.an_element()
sage: a.parent()
Set of prime numbers (wrapper implementation)
sage: x = P(13); x
13
sage: type(x)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Wrapper_with_category.
↪element_class'>
sage: x.parent()
Set of prime numbers (wrapper implementation)
sage: 13 in P
True
```

```
sage: 12 in P
False
sage: y = x+1; y
14
sage: type(y)
<type 'sage.rings.integer.Integer'>

sage: z = P.next(x); z
17
sage: type(z)
<class 'sage.categories.examples.sets_cat.PrimeNumbers_Wrapper_with_category.
↪element_class'>
sage: z.parent()
Set of prime numbers (wrapper implementation)
```

**class Element**

Bases: `sage.structure.element_wrapper.ElementWrapper`, *sage.categories.examples.sets_cat.PrimeNumbers_Abstract.Element*

**ElementWrapper**

alias of *ElementWrapper*

## 5.28 Example of a set with grading

sage.categories.examples.sets_with_grading.**Example**

alias of *NonNegativeIntegers*

**class** sage.categories.examples.sets_with_grading.**NonNegativeIntegers**

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

Non negative integers graded by themselves.

EXAMPLES:

```
sage: E = SetsWithGrading().example()
sage: E
Non negative integers
sage: E.graded_component(0)
{0}
sage: E.graded_component(100)
{100}
```

**an_element**()

Returns 0.

EXAMPLES:

```
sage: SetsWithGrading().example().an_element()
0
```

**generating_series**(*var='z'*)

Returns $1/(1-z)$.

EXAMPLES:

```
sage: N = SetsWithGrading().example(); N
Non negative integers
sage: f = N.generating_series(); f
1/(-z + 1)
sage: LaurentSeriesRing(ZZ,'z')(f)
1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9 + z^10 + z^11 + z^12 +
↪z^13 + z^14 + z^15 + z^16 + z^17 + z^18 + z^19 + O(z^20)
```

**graded_component** (*grade*)
Returns the component with grade `grade`.

EXAMPLES:

```
sage: N = SetsWithGrading().example()
sage: N.graded_component(65)
{65}
```

**grading** (*elt*)
Returns the grade of `elt`.

EXAMPLES:

```
sage: N = SetsWithGrading().example()
sage: N.grading(10)
10
```

## 5.29 Examples of parents endowed with multiple realizations

**class** sage.categories.examples.with_realizations.**SubsetAlgebra** (*R, S*)
Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

An example of parent endowed with several realizations

We consider an algebra $A(S)$ whose bases are indexed by the subsets $s$ of a given set $S$. We consider three natural basis of this algebra: `F`, `In`, and `Out`. In the first basis, the product is given by the union of the indexing sets. That is, for any $s, t \subset S$

$$F_s F_t = F_{s \cup t}$$

The `In` basis and `Out` basis are defined respectively by:

$$In_s = \sum_{t \subset s} F_t \qquad \text{and} \qquad F_s = \sum_{t \supset s} Out_t$$

Each such basis gives a realization of $A$, where the elements are represented by their expansion in this basis.

This parent, and its code, demonstrate how to implement this algebra and its three realizations, with coercions and mixed arithmetic between them.

**See also:**

- *Sets().WithRealizations*
- the Implementing Algebraic Structures thematic tutorial.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.base_ring()
Rational Field
```

The three bases of A:

```
sage: F   = A.F()   ; F
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
sage: In  = A.In() ; In
The subset algebra of {1, 2, 3} over Rational Field in the In basis
sage: Out = A.Out(); Out
The subset algebra of {1, 2, 3} over Rational Field in the Out basis
```

One can quickly define all the bases using the following shortcut:

```
sage: A.inject_shorthands()
Defining F as shorthand for The subset algebra of {1, 2, 3} over Rational Field␣
↪in the Fundamental basis
Defining In as shorthand for The subset algebra of {1, 2, 3} over Rational Field␣
↪in the In basis
Defining Out as shorthand for The subset algebra of {1, 2, 3} over Rational Field␣
↪in the Out basis
```

Accessing the basis elements is done with basis() method:

```
sage: F.basis().list()
[F[{}], F[{1}], F[{2}], F[{3}], F[{1, 2}], F[{1, 3}], F[{2, 3}], F[{1, 2, 3}]]
```

To access a particular basis element, you can use the from_set() method:

```
sage: F.from_set(2,3)
F[{2, 3}]
sage: In.from_set(1,3)
In[{1, 3}]
```

or as a convenient shorthand, one can use the following notation:

```
sage: F[2,3]
F[{2, 3}]
sage: In[1,3]
In[{1, 3}]
```

Some conversions:

```
sage: F(In[2,3])
F[{}] + F[{2}] + F[{3}] + F[{2, 3}]
sage: In(F[2,3])
In[{}] - In[{2}] - In[{3}] + In[{2, 3}]

sage: Out(F[3])
Out[{3}] + Out[{1, 3}] + Out[{2, 3}] + Out[{1, 2, 3}]
sage: F(Out[3])
F[{3}] - F[{1, 3}] - F[{2, 3}] + F[{1, 2, 3}]

sage: Out(In[2,3])
Out[{}] + Out[{1}] + 2*Out[{2}] + 2*Out[{3}] + 2*Out[{1, 2}] + 2*Out[{1, 3}] +␣
↪4*Out[{2, 3}] + 4*Out[{1, 2, 3}]
```

We can now mix expressions:

```
sage: (1 + Out[1]) * In[2,3]
Out[{}] + 2*Out[{1}] + 2*Out[{2}] + 2*Out[{3}] + 2*Out[{1, 2}] + 2*Out[{1, 3}] +
↪4*Out[{2, 3}] + 4*Out[{1, 2, 3}]
```

**class Bases**(*parent_with_realization*)

Bases: *sage.categories.realizations.Category_realization_of_parent*

The category of the realizations of the subset algebra

**class ParentMethods**

**from_set**(**args*)

Construct the monomial indexed by the set containing the elements passed as arguments.

EXAMPLES:

```
sage: In = Sets().WithRealizations().example().In(); In
The subset algebra of {1, 2, 3} over Rational Field in the In basis
sage: In.from_set(2,3)
In[{2, 3}]
```

As a shorthand, one can construct elements using the following notation:

```
sage: In[2,3]
In[{2, 3}]
```

**one**()

Returns the unit of this algebra.

This default implementation takes the unit in the fundamental basis, and coerces it in `self`.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: In = A.In(); Out = A.Out()
sage: In.one()
In[{}]
sage: Out.one()
Out[{}] + Out[{1}] + Out[{2}] + Out[{3}] + Out[{1, 2}] + Out[{1, 3}] +
↪Out[{2, 3}] + Out[{1, 2, 3}]
```

**super_categories**()

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: C = A.Bases(); C
Category of bases of The subset algebra of {1, 2, 3} over Rational Field
sage: C.super_categories()
[Category of realizations of The subset algebra of {1, 2, 3} over
↪Rational Field,
 Join of Category of algebras with basis over Rational Field and
         Category of commutative algebras over Rational Field and
         Category of realizations of unital magmas]
```

**class Fundamental**(*A*)

Bases: `sage.combinat.free_module.CombinatorialFreeModule`, `sage.misc.bindable_class.BindableClass`

The Subset algebra, in the fundamental basis

INPUT:

- `A` – a parent with realization in *SubsetAlgebra*

EXAMPLES:

```
sage: A = Sets().WithRealizations().example()
sage: A.F()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
sage: A.Fundamental()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
```

**one**()

Return the multiplicative unit element.

EXAMPLES:

```
sage: A = AlgebrasWithBasis(QQ).example()
sage: A.one_basis()
word:
sage: A.one()
B[word: ]
```

**one_basis**()

Returns the index of the basis element which is equal to '1'.

EXAMPLES:

```
sage: F = Sets().WithRealizations().example().F(); F
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental␣
↪basis
sage: F.one_basis()
{}
sage: F.one()
F[{}]
```

**product_on_basis**(*left*, *right*)

Product of basis elements, as per `AlgebrasWithBasis.ParentMethods.product_on_basis()`.

INPUT:

- `left, right` – sets indexing basis elements

EXAMPLES:

```
sage: F = Sets().WithRealizations().example().F(); F
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental␣
↪basis
sage: S = F.basis().keys(); S
Subsets of {1, 2, 3}
sage: F.product_on_basis(S([]), S([]))
F[{}]
sage: F.product_on_basis(S({1}), S({3}))
F[{1, 3}]
```

```
sage: F.product_on_basis(S({1,2}), S({2,3}))
F[{1, 2, 3}]
```

**class In**(*A*)

Bases: sage.combinat.free_module.CombinatorialFreeModule, sage.misc.bindable_class.BindableClass

The Subset Algebra, in the In basis

INPUT:

- A – a parent with realization in *SubsetAlgebra*

EXAMPLES:

```
sage: A = Sets().WithRealizations().example()
sage: A.In()
The subset algebra of {1, 2, 3} over Rational Field in the In basis
```

**class Out**(*A*)

Bases: sage.combinat.free_module.CombinatorialFreeModule, sage.misc.bindable_class.BindableClass

The Subset Algebra, in the *Out* basis

INPUT:

- A – a parent with realization in *SubsetAlgebra*

EXAMPLES:

```
sage: A = Sets().WithRealizations().example()
sage: A.Out()
The subset algebra of {1, 2, 3} over Rational Field in the Out basis
```

**a_realization**()

Returns the default realization of self

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.a_realization()
The subset algebra of {1, 2, 3} over Rational Field in the Fundamental basis
```

**base_set**()

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.base_set()
{1, 2, 3}
```

**indices**()

The objects that index the basis elements of this algebra.

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
```

```
sage: A.indices()
Subsets of {1, 2, 3}
```

**indices_key** (*x*)

A key function on a set which gives a linear extension of the inclusion order.

INPUT:

- x – set

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: sorted(A.indices(), key=A.indices_key)
[{}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}]
```

**supsets** (*set*)

Returns all the subsets of $S$ containing set

INPUT:

- set – a subset of the base set $S$ of self

EXAMPLES:

```
sage: A = Sets().WithRealizations().example(); A
The subset algebra of {1, 2, 3} over Rational Field
sage: A.supsets(Set((2,)))
[{1, 2, 3}, {2, 3}, {1, 2}, {2}]
```

CHAPTER

SIX

# INTERNALS

## 6.1 Specific category classes

This is placed in a separate file from categories.py to avoid circular imports (as morphisms must be very low in the hierarchy with the new coercion model).

**class** sage.categories.category_types.**AbelianCategory**(*s=None*)

Bases: *sage.categories.category.Category*

**is_abelian**()

Return True as self is an abelian category.

EXAMPLES:

```
sage: CommutativeAdditiveGroups().is_abelian()
True
```

**class** sage.categories.category_types.**Category_ideal**(*ambient*, *name=None*)

Bases: *sage.categories.category_types.Category_in_ambient*

**classmethod an_instance**()

Return an instance of this class.

EXAMPLES:

```
sage: AlgebraIdeals.an_instance()
Category of algebra ideals in Univariate Polynomial Ring in x over Rational␣
↪Field
```

**ring**()

Return the ambient ring used to describe objects self.

EXAMPLES:

```
sage: C = Ideals(IntegerRing())
sage: C.ring()
Integer Ring
```

**class** sage.categories.category_types.**Category_in_ambient**(*ambient*, *name=None*)

Bases: *sage.categories.category.Category*

Initialize self.

EXAMPLES:

```
sage: C = Ideals(IntegerRing())
sage: TestSuite(C).run()
```

**ambient**()

Return the ambient object in which objects of this category are embedded.

EXAMPLES:

```
sage: C = Ideals(IntegerRing())
sage: C.ambient()
Integer Ring
```

**class** sage.categories.category_types.**Category_module**(*base*, *name=None*)

Bases: *sage.categories.category_types.AbelianCategory*, *sage.categories.category_types.Category_over_base_ring*

**class** sage.categories.category_types.**Category_over_base**(*base*, *name=None*)

Bases: *sage.categories.category.CategoryWithParameters*

A base class for categories over some base object

INPUT:

- base – a category $C$ or an object of such a category

Assumption: the classes for the parents, elements, morphisms, of self should only depend on $C$. See trac ticket #11935 for details.

EXAMPLES:

```
sage: Algebras(GF(2)).element_class is Algebras(GF(3)).element_class
True

sage: C = GF(2).category()
sage: Algebras(GF(2)).parent_class is Algebras(C).parent_class
True

sage: C = ZZ.category()
sage: Algebras(ZZ).element_class is Algebras(C).element_class
True
```

**classmethod an_instance**()

Returns an instance of this class

EXAMPLES:

```
sage: Algebras.an_instance()
Category of algebras over Rational Field
```

**base**()

Return the base over which elements of this category are defined.

EXAMPLES:

```
sage: C = Algebras(QQ)
sage: C.base()
Rational Field
```

**class** sage.categories.category_types.**Category_over_base_ring**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_over_base*

Initialize `self`.

EXAMPLES:

```
sage: C = Algebras(GF(2)); C
Category of algebras over Finite Field of size 2
sage: TestSuite(C).run()
```

**base_ring**()

Return the base ring over which elements of this category are defined.

EXAMPLES:

```
sage: C = Algebras(GF(2))
sage: C.base_ring()
Finite Field of size 2
```

**class** sage.categories.category_types.**ChainComplexes**(*base*, *name=None*)

Bases: *sage.categories.category_types.Category_module*

The category of all chain complexes over a base ring.

EXAMPLES:

```
sage: ChainComplexes(RationalField())
Category of chain complexes over Rational Field

sage: ChainComplexes(Integers(9))
Category of chain complexes over Ring of integers modulo 9
```

**super_categories**()

EXAMPLES:

```
sage: ChainComplexes(Integers(9)).super_categories()
[Category of modules over Ring of integers modulo 9]
```

**class** sage.categories.category_types.**Elements**(*object*)

Bases: *sage.categories.category.Category*

The category of all elements of a given parent.

EXAMPLES:

```
sage: a = IntegerRing()(5)
sage: C = a.category(); C
Category of elements of Integer Ring
sage: a in C
True
sage: 2/3 in C
False
sage: loads(C.dumps()) == C
True
```

**classmethod an_instance**()

Returns an instance of this class

EXAMPLES:

```
sage: Elements.an_instance()
Category of elements of Rational Field
```

**object**()

EXAMPLES:

```
sage: Elements(ZZ).object()
Integer Ring
```

**super_categories**()

EXAMPLES:

```
sage: Elements(ZZ).super_categories()
[Category of objects]
```

**Todo:** Check that this is what we want.

## 6.2 Singleton categories

**class** sage.categories.category_singleton.**Category_contains_method_by_parent_class**

Bases: object

Returns whether x is an object in this category.

More specifically, returns True if and only if x has a category which is a subcategory of this one.

EXAMPLES:

```
sage: ZZ in Sets()
True
```

**class** sage.categories.category_singleton.**Category_singleton**(*s=None*)

Bases: *sage.categories.category.Category*

A base class for implementing singleton category

A *singleton* category is a category whose class takes no parameters like Fields() or Rings(). See also the Singleton design pattern.

This is a subclass of *Category*, with a couple optimizations for singleton categories.

The main purpose is to make the idioms:

```
sage: QQ in Fields()
True
sage: ZZ in Fields()
False
```

as fast as possible, and in particular competitive to calling a constant Python method, in order to foster its systematic use throughout the Sage library. Such tests are time critical, in particular when creating a lot of polynomial rings over small fields like in the elliptic curve code.

EXAMPLES:

```
sage: from sage.categories.category_singleton import Category_singleton
sage: class MyRings(Category):
....:     def super_categories(self): return Rings().super_categories()
sage: class MyRingsSingleton(Category_singleton):
....:     def super_categories(self): return Rings().super_categories()
```

We create three rings. One of them is contained in the usual category of rings, one in the category of "my rings" and the third in the category of "my rings singleton":

```
sage: R = QQ['x,y']
sage: R1 = Parent(category = MyRings())
sage: R2 = Parent(category = MyRingsSingleton())
sage: R in MyRings()
False
sage: R1 in MyRings()
True
sage: R1 in MyRingsSingleton()
False
sage: R2 in MyRings()
False
sage: R2 in MyRingsSingleton()
True
```

One sees that containment tests for the singleton class is a lot faster than for a usual class:

```
sage: timeit("R in MyRings()", number=10000)                  # not tested
10000 loops, best of 3: 7.12 µs per loop
sage: timeit("R1 in MyRings()", number=10000)                 # not tested
10000 loops, best of 3: 6.98 µs per loop
sage: timeit("R in MyRingsSingleton()", number=10000)         # not tested
10000 loops, best of 3: 3.08 µs per loop
sage: timeit("R2 in MyRingsSingleton()", number=10000)        # not tested
10000 loops, best of 3: 2.99 µs per loop
```

So this is an improvement, but not yet competitive with a pure Cython method:

```
sage: timeit("R.is_ring()", number=10000)                     # not tested
10000 loops, best of 3: 383 ns per loop
```

However, it is competitive with a Python method. Actually it is faster, if one stores the category in a variable:

```
sage: _Rings = Rings()
sage: R3 = Parent(category = _Rings)
sage: R3.is_ring.__module__
'sage.categories.rings'
sage: timeit("R3.is_ring()", number=10000)                    # not tested
10000 loops, best of 3: 2.64 µs per loop
sage: timeit("R3 in Rings()", number=10000)                   # not tested
10000 loops, best of 3: 3.01 µs per loop
sage: timeit("R3 in _Rings", number=10000)                    # not tested
10000 loops, best of 3: 652 ns per loop
```

This might not be easy to further optimize, since the time is consumed in many different spots:

```
sage: timeit("MyRingsSingleton.__classcall__()", number=10000)# not tested
10000 loops, best of 3: 306 ns per loop

sage: X = MyRingsSingleton()
sage: timeit("R in X  ", number=10000)                        # not tested
10000 loops, best of 3: 699 ns per loop

sage: c = MyRingsSingleton().__contains__
sage: timeit("c(R)", number = 10000)                          # not tested
10000 loops, best of 3: 661 ns per loop
```

**Warning:** A singleton concrete class $A$ should not have a subclass $B$ (necessarily concrete). Otherwise, creating an instance $a$ of $A$ and an instance $b$ of $B$ would break the singleton principle: $A$ would have two instances $a$ and $b$.

With the current implementation only direct subclasses of *Category_singleton* are supported:

```
sage: class MyRingsSingleton(Category_singleton):
....:     def super_categories(self): return Rings().super_categories()
sage: class Disaster(MyRingsSingleton): pass
sage: Disaster()
Traceback (most recent call last):
...
AssertionError: <class '__main__.Disaster'> is not a direct subclass of <class
↪'sage.categories.category_singleton.Category_singleton'>
```

However, it is acceptable for a direct subclass $R$ of *Category_singleton* to create its unique instance as an instance of a subclass of itself (in which case, its the subclass of $R$ which is concrete, not $R$ itself). This is used for example to plug in extra category code via a dynamic subclass:

```
sage: from sage.categories.category_singleton import Category_singleton
sage: class R(Category_singleton):
....:     def super_categories(self): return [Sets()]
sage: R()
Category of r
sage: R().__class__
<class '__main__.R_with_category'>
sage: R().__class__.mro()
[<class '__main__.R_with_category'>,
 <class '__main__.R'>,
 <class 'sage.categories.category_singleton.Category_singleton'>,
 <class 'sage.categories.category.Category'>,
 <class 'sage.structure.unique_representation.UniqueRepresentation'>,
 <class 'sage.structure.unique_representation.CachedRepresentation'>,
 <type 'sage.misc.fast_methods.WithEqualityById'>,
 <type 'sage.structure.sage_object.SageObject'>,
 <class '__main__.R.subcategory_class'>,
 <class 'sage.categories.sets_cat.Sets.subcategory_class'>,
 <class 'sage.categories.sets_with_partial_maps.SetsWithPartialMaps.
↪subcategory_class'>,
 <class 'sage.categories.objects.Objects.subcategory_class'>,
 <... 'object'>]
sage: R() is R()
True
sage: R() is R().__class__()
True
```

In that case, `R` is an abstract class and has a single concrete subclass, so this does not break the Singleton design pattern.

**See also:**

*Category.__classcall__()*, *Category.__init__()*

**Note:** The `_test_category` test is failing because `MyRingsSingleton()` is not a subcategory of the join of its super categories:

```
sage: C = MyRingsSingleton()
sage: C.super_categories()
```

```
[Category of rngs, Category of semirings]
sage: Rngs() & Semirings()
Category of rings
sage: C.is_subcategory(Rings())
False
```

Oh well; it's not really relevant for those tests.

## 6.3 Fast functions for the category framework

AUTHOR:

- Simon King (initial version)

**class** sage.categories.category_cy_helper.**AxiomContainer**

Bases: dict

A fast container for axioms.

This is derived from dict. A key is the name of an axiom. The corresponding value is the "rank" of this axiom, that is used to order the axioms in *canonicalize_axioms()*.

EXAMPLES:

```
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: isinstance(all_axioms, sage.categories.category_with_axiom.AxiomContainer)
True
```

**add**(*axiom*)

Add a new axiom name, of the next rank.

EXAMPLES:

```
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: m = max(all_axioms.values())
sage: all_axioms.add('Awesome')
sage: all_axioms['Awesome'] == m + 1
True
```

To avoid side effects, we remove the added axiom:

```
sage: del all_axioms['Awesome']
```

sage.categories.category_cy_helper.**canonicalize_axioms**(*all_axioms*, *axioms*)

Canonicalize a set of axioms.

INPUT:

- all_axioms – all available axioms
- axioms – a set (or iterable) of axioms

**Note:** *AxiomContainer* provides a fast container for axioms, and the collection of axioms is stored in *sage.categories.category_with_axiom*. In order to avoid circular imports, we expect that the collection of all axioms is provided as an argument to this auxiliary function.

OUTPUT:

A set of axioms as a tuple sorted according to the order of the tuple `all_axioms` in *sage.categories.category_with_axiom*.

EXAMPLES:

```
sage: from sage.categories.category_with_axiom import canonicalize_axioms, all_
↪axioms
sage: canonicalize_axioms(all_axioms, ["Commutative", "Connected", "WithBasis",
↪"Finite"])
('Finite', 'Connected', 'WithBasis', 'Commutative')
sage: canonicalize_axioms(all_axioms, ["Commutative", "Connected", "Commutative",
↪"WithBasis", "Finite"])
('Finite', 'Connected', 'WithBasis', 'Commutative')
```

`sage.categories.category_cy_helper.`**`category_sort_key`**(*category*)

Return `category._cmp_key`.

This helper function is used for sorting lists of categories.

It is semantically equivalent to `operator.attrgetter()` `("_cmp_key")`, but currently faster.

EXAMPLES:

```
sage: from sage.categories.category_cy_helper import category_sort_key
sage: category_sort_key(Rings()) is Rings()._cmp_key
True
```

`sage.categories.category_cy_helper.`**`get_axiom_index`**(*all_axioms*, *axiom*)

Helper function: Return the rank of an axiom.

INPUT:

- `all_axioms` – the axiom collection
- `axiom` – string, name of an axiom

EXAMPLES:

```
sage: all_axioms = sage.categories.category_with_axiom.all_axioms
sage: from sage.categories.category_cy_helper import get_axiom_index
sage: get_axiom_index(all_axioms, 'AdditiveCommutative') == all_axioms[
↪'AdditiveCommutative']
True
```

`sage.categories.category_cy_helper.`**`join_as_tuple`**(*categories*, *axioms*, *ignore_axioms*)

Helper for *join()*.

INPUT:

- `categories` – tuple of categories to be joined,
- `axioms` – tuple of strings; the names of some supplementary axioms.
- `ignore_axioms` – tuple of pairs `(cat, axiom)`, such that `axiom` will not be applied to `cat`, should `cat` occur in the algorithm.

EXAMPLES:

```
sage: from sage.categories.category_cy_helper import join_as_tuple
sage: T = (Coalgebras(QQ), Sets().Finite(), Algebras(ZZ), SimplicialComplexes())
sage: join_as_tuple(T,(),())
```

```
(Category of algebras over Integer Ring,
 Category of finite monoids,
 Category of finite additive groups,
 Category of coalgebras over Rational Field,
 Category of finite simplicial complexes)
sage: join_as_tuple(T,('WithBasis',),())
(Category of algebras with basis over Integer Ring,
 Category of finite monoids,
 Category of coalgebras with basis over Rational Field,
 Category of finite additive groups,
 Category of finite simplicial complexes)
sage: join_as_tuple(T,(),((Monoids(),'Finite'),))
(Category of algebras over Integer Ring,
 Category of finite additive groups,
 Category of coalgebras over Rational Field,
 Category of finite simplicial complexes)
```

## 6.4 Coercion methods for categories

The purpose of this Cython module is to hold special coercion methods, which are inserted by their respective categories.

## 6.5 Poor Man's map

**class** sage.categories.poor_man_map.**PoorManMap**(*function*, *domain=None*, *codomain=None*, *name=None*)

Bases: `sage.structure.sage_object.SageObject`

A class for maps between sets which are not (yet) modeled by parents

Could possibly disappear when all combinatorial classes / enumerated sets will be parents

INPUT:

- `function` – a callable or an iterable of callables. This represents the underlying function used to implement this map. If it is an iterable, then the callables will be composed to implement this map.
- `domain` – the domain of this map or `None` if the domain is not known or should remain unspecified
- `codomain` – the codomain of this map or `None` if the codomain is not known or should remain unspecified
- `name` – a name for this map or `None` if this map has no particular name

EXAMPLES:

```
sage: from sage.categories.poor_man_map import PoorManMap
sage: f = PoorManMap(factorial, domain = (1, 2, 3), codomain = (1, 2, 6))
sage: f
A map from (1, 2, 3) to (1, 2, 6)
sage: f(3)
6
```

The composition of several functions can be created by passing in a tuple of functions:

```
sage: i = PoorManMap((factorial, sqrt), domain= (1, 4, 9), codomain = (1, 2, 6))
```

However, the same effect can also be achieved by just composing maps:

```
sage: g = PoorManMap(factorial, domain = (1, 2, 3), codomain = (1, 2, 6))
sage: h = PoorManMap(sqrt, domain = (1, 4, 9), codomain = (1, 2, 3))
sage: i == g*h
True
```

**codomain**()

Returns the codomain of `self`

EXAMPLES:

```
sage: from sage.categories.poor_man_map import PoorManMap
sage: PoorManMap(lambda x: x+1, domain = (1,2,3), codomain = (2,3,4)).
↪codomain()
(2, 3, 4)
```

**domain**()

Returns the domain of `self`

EXAMPLES:

```
sage: from sage.categories.poor_man_map import PoorManMap
sage: PoorManMap(lambda x: x+1, domain = (1,2,3), codomain = (2,3,4)).domain()
(1, 2, 3)
```

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