
Diophantine approximation

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CONTENTS

1	Continued fractions	3
2	Indices and Tables	27
	Python Module Index	29
	Index	31

The diophantine approximation deals with the approximation of real numbers (or real vectors) with rational numbers (or rational vectors). See the article [Wikipedia article Diophantine_approximation](#) for more information.

CONTINUED FRACTIONS

A continued fraction is a representation of a real number in terms of a sequence of integers denoted $[a_0; a_1, a_2, \dots]$. The well known decimal expansion is another way of representing a real number by a sequence of integers. The value of a continued fraction is defined recursively as:

$$[a_0; a_1, a_2, \dots] = a_0 + \frac{1}{[a_1; a_2, \dots]} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}}$$

In this expansion, all coefficients a_n are integers and only the value a_0 may be non positive. Note that a_0 is nothing else but the floor (this remark provides a way to build the continued fraction expansion from a given real number). As examples

$$\frac{45}{38} = 1 + \frac{1}{5 + \frac{1}{2 + \frac{1}{3}}}$$
$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{\dots}}}}}$$

It is quite remarkable that

- any real number admits a unique continued fraction expansion
- finite expansions correspond to rationals
- ultimately periodic expansions correspond to quadratic numbers (ie numbers of the form $a + b\sqrt{D}$ with a and b rationals and D square free positive integer)
- two real numbers x and y have the same tail (up to a shift) in their continued fraction expansion if and only if there are integers a, b, c, d with $|ad - bc| = 1$ and such that $y = (ax + b)/(cx + d)$.

Moreover, the rational numbers obtained by truncation of the expansion of a real number gives its so-called best approximations. For more informations on continued fractions, you may have a look at [Wikipedia article Continued_fraction](#).

EXAMPLES:

If you want to create the continued fraction of some real number you may either use its method `continued_fraction` (if it exists) or call `continued_fraction()`:

```
sage: (13/27).continued_fraction()
[0; 2, 13]
sage: 0 + 1/(2 + 1/13)
13/27

sage: continued_fraction(22/45)
[0; 2, 22]
sage: 0 + 1/(2 + 1/22)
22/45

sage: continued_fraction(pi)
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: continued_fraction_list(pi, nterms=5)
[3, 7, 15, 1, 292]

sage: K.<cbt5> = NumberField(x^3 - 5, embedding=1.709)
sage: continued_fraction(cbt5)
[1; 1, 2, 2, 4, 3, 3, 1, 5, 1, 1, 4, 10, 17, 1, 14, 1, 1, 3052, 1, ...]
```

It is also possible to create a continued fraction from a list of partial quotients:

```
sage: continued_fraction([-3,1,2,3,4,1,2])
[-3; 1, 2, 3, 4, 1, 2]
```

Even infinite:

```
sage: w = words.ThueMorseWord([1,2])
sage: w
word: 1221211221121221211212211221211221121221...
sage: continued_fraction(w)
[1; 2, 2, 1, 2, 1, 1, 2, 2, 1...]
```

To go back and forth between the value (as a real number) and the partial quotients (seen as a finite or infinite list) you can use the methods `quotients` and `value`:

```
sage: cf = (13/27).continued_fraction()
sage: cf.quotients()
[0, 2, 13]
sage: cf.value()
13/27

sage: cf = continued_fraction(pi)
sage: cf.quotients()
lazy list [3, 7, 15, ...]
sage: cf.value()
pi

sage: w = words.FibonacciWord([1,2])
sage: cf = continued_fraction(w)
sage: cf.quotients()
word: 1211212112112121121121121121121121121121...
sage: v = cf.value()
sage: v
```

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```
1.387954587967143?
sage: v.n(digits=100)
1.
→387954587967142336919313859873185477878152452498532271894917289826418577622648932169885237034242967
sage: v.continued_fraction()
[1; 2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2...]
```

Recall that quadratic numbers correspond to ultimately periodic continued fractions. For them special methods give access to preperiod and period:

```
sage: K.<sqrt2> = QuadraticField(2)
sage: cf = continued_fraction(sqrt2); cf
[1; (2)*]
sage: cf.value()
sqrt2
sage: cf.preperiod()
(1,)
sage: cf.period()
(2,)

sage: cf = (3*sqrt2 + 1/2).continued_fraction(); cf
[4; (1, 2, 1, 7)*]

sage: cf = continued_fraction([(1,2,3),(1,4)]); cf
[1; 2, 3, (1, 4)*]
sage: cf.value()
-2/23*sqrt2 + 36/23
```

On the following we can remark how the tail may change even in the same quadratic field:

```
sage: for i in range(20): print(continued_fraction(i*sqrt2))
[0]
[1; (2)*]
[2; (1, 4)*]
[4; (4, 8)*]
[5; (1, 1, 1, 10)*]
[7; (14)*]
...
[24; (24, 48)*]
[25; (2, 5, 6, 5, 2, 50)*]
[26; (1, 6, 1, 2, 3, 2, 26, 2, 3, 2, 1, 6, 1, 52)*]
```

Nevertheless, the tail is preserved under invertible integer homographies:

```
sage: apply_homography = lambda m,z: (m[0,0]*z + m[0,1]) / (m[1,0]*z + m[1,1])
sage: m1 = SL2Z([60,13,83,18])
sage: m2 = SL2Z([27,80,28,83])
sage: a = sqrt2/3
sage: a.continued_fraction()
[0; 2, (8, 4)*]
sage: b = apply_homography(m1, a)
sage: b.continued_fraction()
[0; 1, 2, 1, 1, 1, 1, 6, (8, 4)*]
```

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```
sage: c = apply_homography(m2, a)
sage: c.continued_fraction()
[0; 1, 26, 1, 2, 2, (8, 4)*]
sage: d = apply_homography(m1**2*m2**3, a)
sage: d.continued_fraction()
[0; 1, 2, 1, 1, 1, 1, 5, 2, 1, 1, 1, 1, 5, 26, 1, 2, 1, 26, 1, 2, 1, 26, 1, 2, 2, (8, ↵
↵4)*]
```

Todo:

- Improve numerical approximation (the method `_mpfr_()` is quite slow compared to the same method for an element of a number field)
- Make a class for generalized continued fractions of the form $a_0 + b_0/(a_1 + b_1/(...))$ (the standard continued fractions are when all $b_n = 1$ while the Hirzebruch-Jung continued fractions are the one for which $b_n = -1$ for all n). See [Wikipedia article Generalized continued fraction](#).
- look at the function `ContinuedFractionApproximationOfRoot` in GAP

AUTHORS:

- Vincent Delecroix (2014): cleaning, refactorisation, documentation from the old implementation in `contfrac` ([trac ticket #14567](#)).

class `sage.rings.continued_fraction.ContinuedFraction_base`

Bases: `sage.structure.sage_object.SageObject`

Base class for (standard) continued fractions.

If you want to implement your own continued fraction, simply derived from this class and implement the following methods:

- `def quotient(self, n):` return the n -th quotient of `self` as a Sage integer
- `def length(self):` the number of partial quotients of `self` as a Sage integer or `Infinity`.

and optionally:

- `def value(self):` return the value of `self` (an exact real number)

This base class will provide:

- computation of convergents in `convergent()`, `numerator()` and `denominator()`
- comparison with other continued fractions (see `__richcmp__()`)
- elementary arithmetic function `floor()`, `ceil()`, `sign()`
- accurate numerical approximations `_mpfr_()`

All other methods, in particular the ones involving binary operations like sum or product, rely on the optional method `value()` (and not on convergents) and may fail at execution if it is not implemented.

`additive_order()`

Return the additive order of this continued fraction, which we defined to be the additive order of its value.

EXAMPLES:

```
sage: continued_fraction(-1).additive_order()
+Infinity
sage: continued_fraction(0).additive_order()
1
```

apply_homography($a, b, c, d, \text{forward_value}=\text{False}$)

Return the continued fraction of $(ax + b)/(cx + d)$.

This is computed using Gosper's algorithm, see `continued_fraction_gosper`.

INPUT:

- a, b, c, d – integers
- `forward_value` – boolean (default: `False`) whether the returned continued fraction is given the symbolic value of $(ax + b)/(cx + d)$ and not only the list of partial quotients obtained from Gosper's algorithm.

EXAMPLES:

```
sage: (5 * 13/6 - 2) / (3 * 13/6 - 4)
53/15
sage: continued_fraction(13/6).apply_homography(5, -2, 3, -4).value()
53/15
```

We demonstrate now the effect of the optional argument `forward_value`:

```
sage: cf = continued_fraction(pi)
sage: h1 = cf.apply_homography(35, -27, 12, -5)
sage: h1
[2; 1, 1, 6, 3, 1, 2, 1, 5, 3, 1, 1, 1, 1, 9, 12, 1, 1, 1, 3...]
sage: h1.value()
2.536941776086946?

sage: h2 = cf.apply_homography(35, -27, 12, -5, forward_value=True)
sage: h2
[2; 1, 1, 6, 3, 1, 2, 1, 5, 3, 1, 1, 1, 1, 9, 12, 1, 1, 1, 3...]
sage: h2.value()
(35*pi - 27)/(12*pi - 5)
```

REFERENCES:

- [Gos1972]
- [Knu1998] Exercise 4.5.3.15
- [LS1998]

ceil()

Return the ceil of `self`.

EXAMPLES:

```
sage: cf = continued_fraction([2,1,3,4])
sage: cf.ceil()
3
```

convergent(n)

Return the n -th partial convergent to `self`.

EXAMPLES:

```
sage: a = continued_fraction(pi); a
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: a.convergent(3)
355/113
sage: a.convergent(15)
411557987/131002976
```

convergents()

Return the list of partial convergents of `self`.

If `self` is an infinite continued fraction, then the object returned is a `lazy_list_generic` which behave like an infinite list.

EXAMPLES:

```
sage: a = continued_fraction(23/157); a
[0; 6, 1, 4, 1, 3]
sage: a.convergents()
[0, 1/6, 1/7, 5/34, 6/41, 23/157]
```

Todo: Add an example with infinite list.

denominator(*n*)

Return the denominator of the *n*-th partial convergent of `self`.

EXAMPLES:

```
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c.denominator(0)
1
sage: c.denominator(12)
25510582
sage: c.denominator(152)
1255341492699841451528811722575401081588363886480089431843026103930863337221076748
```

floor()

Return the floor of `self`.

EXAMPLES:

```
sage: cf = continued_fraction([2,1,2,3])
sage: cf.floor()
2
```

is_minus_one()

Test whether `self` is minus one.

EXAMPLES:

```
sage: continued_fraction(-1).is_minus_one()
True
sage: continued_fraction(1).is_minus_one()
```

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```
False
sage: continued_fraction(0).is_minus_one()
False
sage: continued_fraction(-2).is_minus_one()
False
sage: continued_fraction([-1,1]).is_minus_one()
False
```

is_one()

Test whether `self` is one.

EXAMPLES:

```
sage: continued_fraction(1).is_one()
True
sage: continued_fraction(5/4).is_one()
False
sage: continued_fraction(0).is_one()
False
sage: continued_fraction(pi).is_one()
False
```

is_zero()

Test whether `self` is zero.

EXAMPLES:

```
sage: continued_fraction(0).is_zero()
True
sage: continued_fraction((0,1)).is_zero()
False
sage: continued_fraction(-1/2).is_zero()
False
sage: continued_fraction(pi).is_zero()
False
```

multiplicative_order()

Return the multiplicative order of this continued fraction, which we defined to be the multiplicative order of its value.

EXAMPLES:

```
sage: continued_fraction(-1).multiplicative_order()
2
sage: continued_fraction(1).multiplicative_order()
1
sage: continued_fraction(pi).multiplicative_order()
+Infinity
```

n(*prec=None, digits=None, algorithm=None*)

Return a numerical approximation of this continued fraction with `prec` bits (or decimal `digits`) of precision.

INPUT:

- `prec` – precision in bits

- `digits` – precision in decimal digits (only used if `prec` is not given)
- `algorithm` – ignored for continued fractions

If neither `prec` nor `digits` is given, the default precision is 53 bits (roughly 16 digits).

EXAMPLES:

```
sage: w = words.FibonacciWord([1,3])
sage: cf = continued_fraction(w)
sage: cf
[1; 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3...]
sage: cf.numerical_approx(prec=53)
1.28102513329557
```

The method `n` is a shortcut to this one:

```
sage: cf.n(digits=25)
1.281025133295569815552930
sage: cf.n(digits=33)
1.28102513329556981555293038097590
```

numerator(*n*)

Return the numerator of the n -th partial convergent of `self`.

EXAMPLES:

```
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c.numerator(0)
3
sage: c.numerator(12)
80143857
sage: c.numerator(152)
3943771611212266962743738812600748213157266596588744951727393497446921245353005283
```

numerical_approx(*prec=None, digits=None, algorithm=None*)

Return a numerical approximation of this continued fraction with `prec` bits (or decimal `digits`) of precision.

INPUT:

- `prec` – precision in bits
- `digits` – precision in decimal digits (only used if `prec` is not given)
- `algorithm` – ignored for continued fractions

If neither `prec` nor `digits` is given, the default precision is 53 bits (roughly 16 digits).

EXAMPLES:

```
sage: w = words.FibonacciWord([1,3])
sage: cf = continued_fraction(w)
sage: cf
[1; 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3...]
sage: cf.numerical_approx(prec=53)
1.28102513329557
```

The method `n` is a shortcut to this one:

```
sage: cf.n(digits=25)
1.281025133295569815552930
sage: cf.n(digits=33)
1.28102513329556981555293038097590
```

p(n)

Return the numerator of the n -th partial convergent of `self`.

EXAMPLES:

```
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c.numerator(0)
3
sage: c.numerator(12)
80143857
sage: c.numerator(152)
3943771611212266962743738812600748213157266596588744951727393497446921245353005283
```

q(n)

Return the denominator of the n -th partial convergent of `self`.

EXAMPLES:

```
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c.denominator(0)
1
sage: c.denominator(12)
25510582
sage: c.denominator(152)
1255341492699841451528811722575401081588363886480089431843026103930863337221076748
```

quotients()

Return the list of partial quotients of `self`.

If `self` is an infinite continued fraction, then the object returned is a `lazy_list_generic` which behaves like an infinite list.

EXAMPLES:

```
sage: a = continued_fraction(23/157); a
[0; 6, 1, 4, 1, 3]
sage: a.quotients()
[0, 6, 1, 4, 1, 3]
```

Todo: Add an example with infinite list.

sign()

Return the sign of `self` as an Integer.

The sign is defined to be 0 if `self` is 0, 1 if `self` is positive and -1 if `self` is negative.

EXAMPLES:

```

sage: continued_fraction(tan(pi/7)).sign()
1
sage: continued_fraction(-34/2115).sign()
-1
sage: continued_fraction([0]).sign()
0
    
```

str(*nterms=10, unicode=False, join=True*)

Return a string representing this continued fraction.

INPUT:

- *nterms* – the maximum number of terms to use
- *unicode* – (default False) whether to use unicode character
- *join* – (default True) if False instead of returning a string return a list of string, each of them representing a line

EXAMPLES:

```

sage: print(continued_fraction(pi).str())
3 + -----
      1
      7 + -----
            1
            15 + -----
                  1
                  1 + -----
                        1
                        292 + -----
                              1
                              1 + -----
                                    1
                                    1 + -----
                                          1
                                          1 + -----
                                                1
                                                2 + -----
                                                      1 + ...

sage: print(continued_fraction(pi).str(nterms=1))
3 + ...

sage: print(continued_fraction(pi).str(nterms=2))
      1
3 + -----
      7 + ...

sage: print(continued_fraction(243/354).str())
      1
-----
      1
1 + -----
      1
      2 + -----
    
```

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```

          1
      5 + ----
          1
        3 + ---
          2
sage: continued_fraction(243/354).str(join=False)
['          1          ',
 '-----',
 '          1          ',
 '1 + ----',
 '          1          ',
 '2 + ----',
 '          1          ',
 '5 + ----',
 '          1          ',
 '3 + ---',
 '          2          ']

sage: print(continued_fraction(243/354).str(unicode=True))
          1
-----
1 + ----
          1
      2 + ----
          1
      5 + ---
          1
          3 + -
              2
    
```

class sage.rings.continued_fraction.ContinuedFraction_infinite(*w*, *value=None*, *check=True*)

Bases: `sage.rings.continued_fraction.ContinuedFraction_base`

A continued fraction defined by an infinite sequence of partial quotients.

EXAMPLES:

```

sage: t = continued_fraction(words.ThueMorseWord([1,2])); t
[1; 2, 2, 1, 2, 1, 1, 2, 2, 1...]
sage: t.n(digits=100)
1.
↪ 4223887368827854883415471160245658253068791089917118293118924529164567472725658833124554129620720
    
```

We check that comparisons work well:

```

sage: t > continued_fraction(1) and t < continued_fraction(3/2)
True
sage: t < continued_fraction(1) or t > continued_fraction(2)
False
    
```

Can also be called with a value option:

```
sage: def f(n):
.....:     if n % 3 == 2: return 2*(n+1)//3
.....:     return 1
sage: w = Word(f, alphabet=NN)
sage: w
word: 1,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,1,16,1,1,18,1,1,20,1,1,22,1,1,
↪24,1,1,26,1,...
sage: cf = continued_fraction(w, value=e-1)
sage: cf
[1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1...]
```

In that case a small check is done on the input:

```
sage: cf = continued_fraction(w, value=pi)
Traceback (most recent call last):
...
ValueError: value evaluates to 3.141592653589794? while the continued
fraction evaluates to 1.718281828459046? in Real Interval Field
with 53 bits of precision.
```

length()

Return infinity.

EXAMPLES:

```
sage: w = words.FibonacciWord([3,13])
sage: cf = continued_fraction(w)
sage: cf.length()
+Infinity
```

quotient(n)

Return the n-th partial quotient of self.

INPUT:

- n – an integer

EXAMPLES:

```
sage: w = words.FibonacciWord([1,3])
sage: cf = continued_fraction(w)
sage: cf.quotient(0)
1
sage: cf.quotient(1)
3
sage: cf.quotient(2)
1
```

quotients()

Return the infinite list from which this continued fraction was built.

EXAMPLES:

```
sage: w = words.FibonacciWord([1,5])
sage: cf = continued_fraction(w)
```

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```
sage: cf.quotients()
word: 1511515115115151151151151151151151151151151151...
```

value()

Return the value of `self`.

If this value was provided on initialization, just return this value otherwise return an element of the real lazy field.

EXAMPLES:

```
sage: def f(n):
....:     if n % 3 == 2: return 2*(n+1)//3
....:     return 1
sage: w = Word(f, alphabet=NN)
sage: w
word: 1,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,1,16,1,1,18,1,1,20,1,1,22,
↪1,1,24,1,1,26,1,...
sage: cf = continued_fraction(w, value=e-1)
sage: cf
[1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1...]
sage: cf.value()
e - 1

sage: w = words.FibonacciWord([2,5])
sage: cf = continued_fraction(w)
sage: cf
[2; 5, 2, 2, 5, 2, 5, 2, 2, 5, 2, 2, 5, 2, 5, 2, 2, 5, 2, 5...]
sage: cf.value()
2.184951302409338?
```

class `sage.rings.continued_fraction.ContinuedFraction_periodic`(*x1*, *x2=None*, *check=True*)

Bases: `sage.rings.continued_fraction.ContinuedFraction_base`

Continued fraction associated with rational or quadratic number.

A rational number has a finite continued fraction expansion (or ultimately 0). The one of a quadratic number, ie a number of the form $a + b\sqrt{D}$ with a and b rational, is ultimately periodic.

Note: This class stores a tuple `_x1` for the preperiod and a tuple `_x2` for the period. In the purely periodic case `_x1` is empty while in the rational case `_x2` is the tuple `(0,)`.

length()

Return the number of partial quotients of `self`.

EXAMPLES:

```
sage: continued_fraction(2/5).length()
3
sage: cf = continued_fraction([(0,1),(2,)]); cf
[0; 1, (2)*]
sage: cf.length()
+Infinity
```

period()

Return the periodic part of self.

EXAMPLES:

```
sage: K.<sqrt3> = QuadraticField(3)
sage: cf = continued_fraction(sqrt3); cf
[1; (1, 2)*]
sage: cf.period()
(1, 2)

sage: for k in xrange(2,40):
.....:     if not k.is_square():
.....:         s = QuadraticField(k).gen()
.....:         cf = continued_fraction(s)
.....:         print('%2d %d %s' % (k, len(cf.period()), cf))
 2 1 [1; (2)*]
 3 2 [1; (1, 2)*]
 5 1 [2; (4)*]
 6 2 [2; (2, 4)*]
 7 4 [2; (1, 1, 1, 4)*]
 8 2 [2; (1, 4)*]
10 1 [3; (6)*]
11 2 [3; (3, 6)*]
12 2 [3; (2, 6)*]
13 5 [3; (1, 1, 1, 1, 6)*]
14 4 [3; (1, 2, 1, 6)*]
...
35 2 [5; (1, 10)*]
37 1 [6; (12)*]
38 2 [6; (6, 12)*]
39 2 [6; (4, 12)*]
```

period_length()

Return the number of partial quotients of the preperiodic part of self.

EXAMPLES:

```
sage: continued_fraction(2/5).period_length()
1
sage: cf = continued_fraction([(0,1),(2,)]); cf
[0; 1, (2)*]
sage: cf.period_length()
1
```

preperiod()

Return the preperiodic part of self.

EXAMPLES:

```
sage: K.<sqrt3> = QuadraticField(3)
sage: cf = continued_fraction(sqrt3); cf
[1; (1, 2)*]
sage: cf.preperiod()
(1,)
```

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```
sage: cf = continued_fraction(sqrt3/7); cf
[0; 4, (24, 8)*]
sage: cf.preperiod()
(0, 4)
```

preperiod_length()

Return the number of partial quotients of the preperiodic part of `self`.

EXAMPLES:

```
sage: continued_fraction(2/5).preperiod_length()
3
sage: cf = continued_fraction([(0,1),(2,)]); cf
[0; 1, (2)*]
sage: cf.preperiod_length()
2
```

quotient(*n*)

Return the *n*-th partial quotient of `self`.

EXAMPLES:

```
sage: cf = continued_fraction([(12,5),(1,3)])
sage: [cf.quotient(i) for i in range(10)]
[12, 5, 1, 3, 1, 3, 1, 3, 1, 3]
```

value()

Return the value of `self` as a quadratic number (with square free discriminant).

EXAMPLES:

Some purely periodic examples:

```
sage: cf = continued_fraction([( ),(2,)]); cf
[(2)*]
sage: v = cf.value(); v
sqrt2 + 1
sage: v.continued_fraction()
[(2)*]

sage: cf = continued_fraction([( ),(1,2)]); cf
[(1, 2)*]
sage: v = cf.value(); v
1/2*sqrt3 + 1/2
sage: v.continued_fraction()
[(1, 2)*]
```

The number `sqrt3` that appear above is actually internal to the continued fraction. In order to be access it from the console:

```
sage: cf.value().parent().inject_variables()
Defining sqrt3
sage: sqrt3
sqrt3
```

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```
sage: ((sqrt3+1)/2).continued_fraction()
[1, 2)*]
```

Some ultimately periodic but non periodic examples:

```
sage: cf = continued_fraction([(1,),(2,)]); cf
[1; (2)*]
sage: v = cf.value(); v
sqrt2
sage: v.continued_fraction()
[1; (2)*]

sage: cf = continued_fraction([(1,3),(1,2)]); cf
[1; 3, (1, 2)*]
sage: v = cf.value(); v
-sqrt3 + 3
sage: v.continued_fraction()
[1; 3, (1, 2)*]

sage: cf = continued_fraction([(-5,18), (1,3,1,5)])
sage: cf.value().continued_fraction() == cf
True
sage: cf = continued_fraction([(-1,),(1,)])
sage: cf.value().continued_fraction() == cf
True
```

class sage.rings.continued_fraction.ContinuedFraction_real(*x*)

Bases: [sage.rings.continued_fraction.ContinuedFraction_base](#)

Continued fraction of a real (exact) number.

This class simply wraps a real number into an attribute (that can be accessed through the method [value\(\)](#)). The number is assumed to be irrational.

EXAMPLES:

```
sage: cf = continued_fraction(pi)
sage: cf
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: cf.value()
pi

sage: cf = continued_fraction(e)
sage: cf
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, ...]
sage: cf.value()
e
```

length()

Return infinity

EXAMPLES:

```
sage: continued_fraction(pi).length()
+Infinity
```

quotient(*n*)

Return the *n*-th quotient of self.

EXAMPLES:

```
sage: cf = continued_fraction(pi)
sage: cf.quotient(27)
13
sage: cf.quotient(2552)
152
sage: cf.quotient(10000)  # long time
5
```

The algorithm is not efficient with element of the symbolic ring and, if possible, one can always prefer number fields elements. The reason is that, given a symbolic element *x*, there is no automatic way to evaluate in RIF an expression of the form $(a*x+b)/(c*x+d)$ where both the numerator and the denominator are extremely small:

```
sage: a1 = pi
sage: c1 = continued_fraction(a1)
sage: p0 = c1.numerator(12); q0 = c1.denominator(12)
sage: p1 = c1.numerator(13); q1 = c1.denominator(13)
sage: num = (q0*a1 - p0); num.n()
1.49011611938477e-8
sage: den = (q1*a1 - p1); den.n()
-2.98023223876953e-8
sage: a1 = -num/den
sage: RIF(a1)
[-infinity .. +infinity]
```

The same computation with an element of a number field instead of *pi* gives a very satisfactory answer:

```
sage: K.<a2> = NumberField(x^3 - 2, embedding=1.25)
sage: c2 = continued_fraction(a2)
sage: p0 = c2.numerator(111); q0 = c2.denominator(111)
sage: p1 = c2.numerator(112); q1 = c2.denominator(112)
sage: num = (q0*a2 - p0); num.n()
-4.56719261665907e46
sage: den = (q1*a2 - p1); den.n()
-3.65375409332726e47
sage: a2 = -num/den
sage: b2 = RIF(a2); b2
1.002685823312715?
sage: b2.absolute_diameter()
8.88178419700125e-16
```

The consequence is that the precision needed with *c1* grows when we compute larger and larger partial quotients:

```
sage: c1.quotient(100)
2
sage: c1._xa.parent()
Real Interval Field with 353 bits of precision
sage: c1.quotient(200)
3
```

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```
sage: c1._xa.parent()
Real Interval Field with 753 bits of precision
sage: c1.quotient(300)
5
sage: c1._xa.parent()
Real Interval Field with 1053 bits of precision

sage: c2.quotient(200)
6
sage: c2._xa.parent()
Real Interval Field with 53 bits of precision
sage: c2.quotient(500)
1
sage: c2._xa.parent()
Real Interval Field with 53 bits of precision
sage: c2.quotient(1000)
1
sage: c2._xa.parent()
Real Interval Field with 53 bits of precision
```

value()

Return the value of `self` (the number from which it was built).

EXAMPLES:

```
sage: cf = continued_fraction(e)
sage: cf.value()
e
```

`sage.rings.continued_fraction.check_and_reduce_pair(x1, x2=None)`

There are often two ways to represent a given continued fraction. This function makes it canonical.

In the very special case of the number 0 we return the pair $((0,), (0,))$.

`sage.rings.continued_fraction.continued_fraction(x, value=None)`

Return the continued fraction of `x`.

INPUT:

- x – a number or a list of partial quotients (for finite development) or two list of partial quotients (preperiod and period for ultimately periodic development)

EXAMPLES:

A finite continued fraction may be initialized by a number or by its list of partial quotients:

```
sage: continued_fraction(12/571)
[0; 47, 1, 1, 2, 2]
sage: continued_fraction([3,2,1,4])
[3; 2, 1, 4]
```

It can be called with elements defined from symbolic values, in which case the partial quotients are evaluated in a lazy way:

```
sage: c = continued_fraction(golden_ratio); c
[1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...]
```

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```

sage: c.convergent(12)
377/233
sage: fibonacci(14)/fibonacci(13)
377/233

sage: continued_fraction(pi)
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: c = continued_fraction(pi); c
[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...]
sage: a = c.convergent(3); a
355/113
sage: a.n()
3.14159292035398
sage: pi.n()
3.14159265358979

sage: continued_fraction(sqrt(2))
[1; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, ...]
sage: continued_fraction(tan(1))
[1; 1, 1, 3, 1, 5, 1, 7, 1, 9, 1, 11, 1, 13, 1, 15, 1, 17, 1, 19, ...]
sage: continued_fraction(tanh(1))
[0; 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, ...]
sage: continued_fraction(e)
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, ...]

```

If you want to play with quadratic numbers (such as `golden_ratio` and `sqrt(2)` above), it is much more convenient to use number fields as follows since preperiods and periods are computed:

```

sage: K.<sqrt5> = NumberField(x^2-5, embedding=2.23)
sage: my_golden_ratio = (1 + sqrt5)/2
sage: cf = continued_fraction((1+sqrt5)/2); cf
[(1)*]
sage: cf.convergent(12)
377/233
sage: cf.period()
(1,)
sage: cf = continued_fraction(2/3+sqrt5/5); cf
[1; 8, (1, 3, 1, 1, 3, 9)*]
sage: cf.preperiod()
(1, 8)
sage: cf.period()
(1, 3, 1, 1, 3, 9)

sage: L.<sqrt2> = NumberField(x^2-2, embedding=1.41)
sage: cf = continued_fraction(sqrt2); cf
[1; (2)*]
sage: cf.period()
(2,)
sage: cf = continued_fraction(sqrt2/3); cf
[0; 2, (8, 4)*]
sage: cf.period()
(8, 4)

```

It is also possible to go the other way around, build a ultimately periodic continued fraction from its preperiod and its period and get its value back:

```
sage: cf = continued_fraction([(1,1),(2,8)]); cf
[1; 1, (2, 8)*]
sage: cf.value()
2/11*sqrt(5) + 14/11
```

It is possible to deal with higher degree number fields but in that case the continued fraction expansion is known to be aperiodic:

```
sage: K.<a> = NumberField(x^3-2, embedding=1.25)
sage: cf = continued_fraction(a); cf
[1; 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, ...]
```

Note that initial rounding can result in incorrect trailing partial quotients:

```
sage: continued_fraction(RealField(39)(e))
[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 2]
```

Note the value returned for floating point number is the continued fraction associated to the rational number you obtain with a conversion:

```
sage: for _ in range(10):
.....:     x = RR.random_element()
.....:     cff = continued_fraction(x)
.....:     cfe = QQ(x).continued_fraction()
.....:     assert cff == cfe, "%s %s %s"%(x,cff,cfe)
```

`sage.rings.continued_fraction.continued_fraction_list(x, type='std', partial_convergents=False, bits=None, nterms=None)`

Return the (finite) continued fraction of x as a list.

The continued fraction expansion of x are the coefficients a_i in

$$x = a_0 + 1/(a_1 + 1/(...))$$

with a_0 integer and a_1, \dots positive integers. The Hirzebruch-Jung continued fraction is the one for which the $+$ signs are replaced with $-$ signs

$$x = a_0 - 1/(a_1 - 1/(...))$$

See also:

[`continued_fraction\(\)`](#)

INPUT:

- x – exact rational or floating-point number. The number to compute the continued fraction of.
- `type` – either “std” (default) for standard continued fractions or “hj” for Hirzebruch-Jung ones.
- `partial_convergents` – boolean. Whether to return the partial convergents.
- `bits` – an optional integer that specify a precision for the real interval field that is used internally.
- `nterms` – integer. The upper bound on the number of terms in the continued fraction expansion to return.

OUTPUT:

A list of integers, the coefficients in the continued fraction expansion of x . If `partial_convergents` is set to `True`, then return a pair containing the coefficient list and the partial convergents list is returned.

EXAMPLES:

```
sage: continued_fraction_list(45/19)
[2, 2, 1, 2, 2]
sage: 2 + 1/(2 + 1/(1 + 1/(2 + 1/2)))
45/19

sage: continued_fraction_list(45/19,type="hj")
[3, 2, 3, 2, 3]
sage: 3 - 1/(2 - 1/(3 - 1/(2 - 1/3)))
45/19
```

Specifying `bits` or `nterms` modify the length of the output:

```
sage: continued_fraction_list(e, bits=20)
[2, 1, 2, 1, 1, 4, 2]
sage: continued_fraction_list(sqrt(2)+sqrt(3), bits=30)
[3, 6, 1, 5, 7, 2]
sage: continued_fraction_list(pi, bits=53)
[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14]

sage: continued_fraction_list(log(3/2), nterms=15)
[0, 2, 2, 6, 1, 11, 2, 1, 2, 2, 1, 4, 3, 1, 1]
sage: continued_fraction_list(tan(sqrt(pi)), nterms=20)
[-5, 9, 4, 1, 1, 1, 1, 1, 1, 5, 1, 1, 1, 1, 1, 2, 4, 3, 1, 63]
```

When the continued fraction is infinite (ie x is an irrational number) and the parameters `bits` and `nterms` are not specified then a warning is raised:

```
sage: continued_fraction_list(sqrt(2))
doctest:...: UserWarning: the continued fraction of sqrt(2) seems infinite, return
↳ only the first 20 terms
[1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2]
sage: continued_fraction_list(sqrt(4/19))
doctest:...: UserWarning: the continued fraction of 2*sqrt(1/19) seems infinite,
↳ return only the first 20 terms
[0, 2, 5, 1, 1, 2, 1, 16, 1, 2, 1, 1, 5, 4, 5, 1, 1, 2, 1, 16]
```

An examples with the list of partial convergents:

```
sage: continued_fraction_list(RR(pi), partial_convergents=True)
([3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 3],
 [(3, 1),
  (22, 7),
  (333, 106),
  (355, 113),
  (103993, 33102),
  (104348, 33215),
  (208341, 66317),
  (312689, 99532),
```

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```
(833719, 265381),
(1146408, 364913),
(4272943, 1360120),
(5419351, 1725033),
(80143857, 25510582),
(245850922, 78256779)])
```

`sage.rings.continued_fraction.convergents(x)`

Return the (partial) convergents of the number x.

EXAMPLES:

```
sage: from sage.rings.continued_fraction import convergents
sage: convergents(143/255)
[0, 1, 1/2, 4/7, 5/9, 9/16, 14/25, 23/41, 60/107, 143/255]
```

`sage.rings.continued_fraction.last_two_convergents(x)`

Given the list x that consists of numbers, return the two last convergents $p_{n-1}, q_{n-1}, p_n, q_n$.

This function is principally used to compute the value of a ultimately periodic continued fraction.

OUTPUT: a 4-tuple of Sage integers

EXAMPLES:

```
sage: from sage.rings.continued_fraction import last_two_convergents
sage: last_two_convergents([])
(0, 1, 1, 0)
sage: last_two_convergents([0])
(1, 0, 0, 1)
sage: last_two_convergents([-1,1,3,2])
(-1, 4, -2, 9)
```

`sage.rings.continued_fraction.rat_interval_cf_list(r1, r2)`

Return the common prefix of the rationals r1 and r2 seen as continued fractions.

OUTPUT: a list of Sage integers.

EXAMPLES:

```
sage: from sage.rings.continued_fraction import rat_interval_cf_list
sage: rat_interval_cf_list(257/113, 5224/2297)
[2, 3, 1, 1, 1, 4]
sage: for prec in range(10,54):
.....:     R = RealIntervalField(prec)
.....:     for _ in range(100):
.....:         x = R.random_element() * R.random_element() + R.random_element() / 100
.....:         l = x.lower().exact_rational()
.....:         u = x.upper().exact_rational()
.....:         if l.floor() != u.floor():
.....:             continue
.....:         cf = rat_interval_cf_list(l,u)
.....:         a = continued_fraction(cf).value()
.....:         b = continued_fraction(cf+[1]).value()
.....:         if a > b:
```

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(continued from previous page)

.....:	a,b = b,a
.....:	assert a <= l
.....:	assert b >= u

INDICES AND TABLES

- [Index](#)
- [Module Index](#)
- [Search Page](#)

PYTHON MODULE INDEX

r

`sage.rings.continued_fraction`, 3

INDEX

A

`additive_order()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 6

`apply_homography()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 7

C

`ceil()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 7

`check_and_reduce_pair()` (in module
sage.rings.continued_fraction), 20

`continued_fraction()` (in module
sage.rings.continued_fraction), 20

`continued_fraction_list()` (in module
sage.rings.continued_fraction), 22

`ContinuedFraction_base` (class in
sage.rings.continued_fraction), 6

`ContinuedFraction_infinite` (class in
sage.rings.continued_fraction), 13

`ContinuedFraction_periodic` (class in
sage.rings.continued_fraction), 15

`ContinuedFraction_real` (class in
sage.rings.continued_fraction), 18

`convergent()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 7

`convergents()` (in module
sage.rings.continued_fraction), 24

`convergents()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 8

D

`denominator()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 8

F

`floor()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 8

I

`is_minus_one()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 8

`is_one()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 9

`is_zero()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 9

L

`last_two_convergents()` (in module
sage.rings.continued_fraction), 24

`length()` (*sage.rings.continued_fraction.ContinuedFraction_infinite*
method), 14

`length()` (*sage.rings.continued_fraction.ContinuedFraction_periodic*
method), 15

`length()` (*sage.rings.continued_fraction.ContinuedFraction_real*
method), 18

M

module

`multiplicative_order()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 9

N

`numerical_base_approx()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 10

`numerator()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 10

P

`period()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 11

`period()` (*sage.rings.continued_fraction.ContinuedFraction_periodic*
method), 15

`period_length()` (*sage.rings.continued_fraction.ContinuedFraction_periodic*
method), 16

`preperiod()` (*sage.rings.continued_fraction.ContinuedFraction_periodic*
method), 16

`preperiod_length()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 17

Q

`q()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 11

`quotient()` (*sage.rings.continued_fraction.ContinuedFraction_infinite*
method), 14

`quotient()` (*sage.rings.continued_fraction.ContinuedFraction_periodic*
method), 17

`quotient()` (*sage.rings.continued_fraction.ContinuedFraction_real*
method), 18

`quotients()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 11

`quotients()` (*sage.rings.continued_fraction.ContinuedFraction_infinite*
method), 14

R

`rat_interval_cf_list()` (in *module*
sage.rings.continued_fraction), 24

S

`sage.rings.continued_fraction`
module, 3

`sign()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 11

`str()` (*sage.rings.continued_fraction.ContinuedFraction_base*
method), 12

V

`value()` (*sage.rings.continued_fraction.ContinuedFraction_infinite*
method), 15

`value()` (*sage.rings.continued_fraction.ContinuedFraction_periodic*
method), 17

`value()` (*sage.rings.continued_fraction.ContinuedFraction_real*
method), 20