Sage Reference Manual: Manifolds

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The Sage Development Team

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CONTENTS

This is the Sage implementation of manifolds resulting from the SageManifolds project. This section describes only the "manifold" part of SageManifolds; the pure algebraic part is described in the section Tensors on free modules of finite rank.

More documentation (in particular example worksheets) can be found here.

CHAPTER

ONE

TOPOLOGICAL MANIFOLDS

1.1 Topological Manifolds

Given a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$) and a non-negative integer n, a *topological manifold of dimension* n *over K* is a topological space M such that

- M is a Hausdorff space,
- M is second countable,
- every point in M has a neighborhood homeomorphic to K^n.

Topological manifolds are implemented via the class *TopologicalManifold*. Open subsets of topological manifolds are also implemented via *TopologicalManifold*, since they are topological manifolds by themselves.

In the current setting, topological manifolds are mostly described by means of charts (see *Chart*).

TopologicalManifold serves as a base class for more specific manifold classes.

The user interface is provided by the generic function *Manifold()*, with with the argument `structure` set to `'topological'`.

Example 1: the 2-sphere as a topological manifold of dimension 2 over $\mathbf{R}$

One starts by declaring S^2 as a 2-dimensional topological manifold:

```
sage: M = Manifold(2, 'S^2', structure='topological')
sage: M
2-dimensional topological manifold S^2
```

Since the base topological field has not been specified in the argument list of `Manifold`, $\mathbf{R}$ is assumed:

```
sage: M.base_field()
Real Field with 53 bits of precision
sage: dim(M)
2
```

Let us consider the complement of a point, the "North pole" say; this is an open subset of S^2, which we call U:

```
sage: U = M.open_subset('U'); U
Open subset U of the 2-dimensional topological manifold S^2
```

A standard chart on U is provided by the stereographic projection from the North pole to the equatorial plane:

```
sage: stereoN.<x,y> = U.chart(); stereoN
Chart (U, (x, y))
```

Thanks to the operator `<x,y>` on the left-hand side, the coordinates declared in a chart (here x and y), are accessible by their names; they are Sage's symbolic variables:

```
sage: y
y
sage: type(y)
<type 'sage.symbolic.expression.Expression'>
```

The South pole is the point of coordinates $(x, y) = (0, 0)$ in the above chart:

```
sage: S = U.point((0,0), chart=stereoN, name='S'); S
Point S on the 2-dimensional topological manifold S^2
```

Let us call V the open subset that is the complement of the South pole and let us introduce on it the chart induced by the stereographic projection from the South pole to the equatorial plane:

```
sage: V = M.open_subset('V'); V
Open subset V of the 2-dimensional topological manifold S^2
sage: stereoS.<u,v> = V.chart(); stereoS
Chart (V, (u, v))
```

The North pole is the point of coordinates $(u, v) = (0, 0)$ in this chart:

```
sage: N = V.point((0,0), chart=stereoS, name='N'); N
Point N on the 2-dimensional topological manifold S^2
```

To fully construct the manifold, we declare that it is the union of U and V:

```
sage: M.declare_union(U,V)
```

and we provide the transition map between the charts `stereoN` = $(U, (x, y))$ and `stereoS` = $(V, (u, v))$, denoting by W the intersection of U and V (W is the subset of U defined by $x^2 + y^2 \neq 0$, as well as the subset of V defined by $u^2 + v^2 \neq 0$):

```
sage: stereoN_to_S = stereoN.transition_map(stereoS, [x/(x^2+y^2), y/(x^2+y^2)],
....:                          intersection_name='W', restrictions1= x^2+y^2!=0,
....:                                                 restrictions2= u^2+v^2!=0)
sage: stereoN_to_S
Change of coordinates from Chart (W, (x, y)) to Chart (W, (u, v))
sage: stereoN_to_S.display()
u = x/(x^2 + y^2)
v = y/(x^2 + y^2)
```

We give the name `W` to the Python variable representing $W = U \cap V$:

```
sage: W = U.intersection(V)
```

The inverse of the transition map is computed by the method *sage.manifolds.chart.CoordChange.inverse()*:

```
sage: stereoN_to_S.inverse()
Change of coordinates from Chart (W, (u, v)) to Chart (W, (x, y))
sage: stereoN_to_S.inverse().display()
```

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```
x = u/(u^2 + v^2)
y = v/(u^2 + v^2)
```

At this stage, we have four open subsets on S^2:

```
sage: M.list_of_subsets()
[2-dimensional topological manifold S^2,
 Open subset U of the 2-dimensional topological manifold S^2,
 Open subset V of the 2-dimensional topological manifold S^2,
 Open subset W of the 2-dimensional topological manifold S^2]
```

W is the open subset that is the complement of the two poles:

```
sage: N in W or S in W
False
```

The North pole lies in V and the South pole in U:

```
sage: N in V, N in U
(True, False)
sage: S in U, S in V
(True, False)
```

The manifold's (user) atlas contains four charts, two of them being restrictions of charts to a smaller domain:

```
sage: M.atlas()
[Chart (U, (x, y)), Chart (V, (u, v)),
 Chart (W, (x, y)), Chart (W, (u, v))]
```

Let us consider the point of coordinates $(1, 2)$ in the chart `stereoN`:

```
sage: p = M.point((1,2), chart=stereoN, name='p'); p
Point p on the 2-dimensional topological manifold S^2
sage: p.parent()
2-dimensional topological manifold S^2
sage: p in W
True
```

The coordinates of p in the chart `stereoS` are computed by letting the chart act on the point:

```
sage: stereoS(p)
(1/5, 2/5)
```

Given the definition of p, we have of course:

```
sage: stereoN(p)
(1, 2)
```

Similarly:

```
sage: stereoS(N)
(0, 0)
sage: stereoN(S)
(0, 0)
```

A continuous map $S^2 \to \mathbf{R}$ (scalar field):

```
sage: f = M.scalar_field({stereoN: atan(x^2+y^2), stereoS: pi/2-atan(u^2+v^2)},
....:                    name='f')
sage: f
Scalar field f on the 2-dimensional topological manifold S^2
sage: f.display()
f: S^2 --> R
on U: (x, y) |--> arctan(x^2 + y^2)
on V: (u, v) |--> 1/2*pi - arctan(u^2 + v^2)
sage: f(p)
arctan(5)
sage: f(N)
1/2*pi
sage: f(S)
0
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold S^2
sage: f.parent().category()
Category of commutative algebras over Symbolic Ring
```

Example 2: the Riemann sphere as a topological manifold of dimension 1 over $\mathbf{C}$

We declare the Riemann sphere $\mathbf{C}^*$ as a 1-dimensional topological manifold over $\mathbf{C}$:

```
sage: M = Manifold(1, 'C*', structure='topological', field='complex'); M
Complex 1-dimensional topological manifold C*
```

We introduce a first open subset, which is actually $\mathbf{C} = \mathbf{C}^* \setminus \{\infty\}$ if we interpret $\mathbf{C}^*$ as the Alexandroff one-point compactification of $\mathbf{C}$:

```
sage: U = M.open_subset('U')
```

A natural chart on U is then nothing but the identity map of $\mathbf{C}$, hence we denote the associated coordinate by z:

```
sage: Z.<z> = U.chart()
```

The origin of the complex plane is the point of coordinate $z = 0$:

```
sage: O = U.point((0,), chart=Z, name='O'); O
Point O on the Complex 1-dimensional topological manifold C*
```

Another open subset of $\mathbf{C}^*$ is $V = \mathbf{C}^* \setminus \{O\}$:

```
sage: V = M.open_subset('V')
```

We define a chart on V such that the point at infinity is the point of coordinate 0 in this chart:

```
sage: W.<w> = V.chart(); W
Chart (V, (w,))
sage: inf = M.point((0,), chart=W, name='inf', latex_name=r'\infty')
sage: inf
Point inf on the Complex 1-dimensional topological manifold C*
```

To fully construct the Riemann sphere, we declare that it is the union of U and V:

```
sage: M.declare_union(U,V)
```

and we provide the transition map between the two charts as $w = 1/z$ on $A = U \cap V$:

```
sage: Z_to_W = Z.transition_map(W, 1/z, intersection_name='A',
....:                           restrictions1= z!=0, restrictions2= w!=0)
sage: Z_to_W
Change of coordinates from Chart (A, (z,)) to Chart (A, (w,))
sage: Z_to_W.display()
w = 1/z
sage: Z_to_W.inverse()
Change of coordinates from Chart (A, (w,)) to Chart (A, (z,))
sage: Z_to_W.inverse().display()
z = 1/w
```

Let consider the complex number i as a point of the Riemann sphere:

```
sage: i = M((I,), chart=Z, name='i'); i
Point i on the Complex 1-dimensional topological manifold C*
```

Its coordinates w.r.t. the charts `Z` and `W` are:

```
sage: Z(i)
(I,)
sage: W(i)
(-I,)
```

and we have:

```
sage: i in U
True
sage: i in V
True
```

The following subsets and charts have been defined:

```
sage: M.list_of_subsets()
[Open subset A of the Complex 1-dimensional topological manifold C*,
 Complex 1-dimensional topological manifold C*,
 Open subset U of the Complex 1-dimensional topological manifold C*,
 Open subset V of the Complex 1-dimensional topological manifold C*]
sage: M.atlas()
[Chart (U, (z,)), Chart (V, (w,)), Chart (A, (z,)), Chart (A, (w,))]
```

A constant map $\mathbf{C}^* \to \mathbf{C}$:

```
sage: f = M.constant_scalar_field(3+2*I, name='f'); f
Scalar field f on the Complex 1-dimensional topological manifold C*
sage: f.display()
f: C* --> C
on U: z |--> 2*I + 3
on V: w |--> 2*I + 3
sage: f(O)
2*I + 3
sage: f(i)
2*I + 3
sage: f(inf)
2*I + 3
sage: f.parent()
Algebra of scalar fields on the Complex 1-dimensional topological
```

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```
 manifold C*
sage: f.parent().category()
Category of commutative algebras over Symbolic Ring
```

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2015): structure described via `TopologicalStructure` or `RealTopologicalStructure`

REFERENCES:

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`sage.manifolds.manifold.`**`Manifold`**(*dim*, *name*, *latex_name=None*, *field='real'*, *structure='smooth'*, *start_index=0*, ***extra_kwds*)

Construct a manifold of a given type over a topological field.

Given a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$) and a non-negative integer n, a *topological manifold of dimension* n *over K* is a topological space M such that

- M is a Hausdorff space,
- M is second countable, and
- every point in M has a neighborhood homeomorphic to K^n.

A *real manifold* is a manifold over $\mathbf{R}$. A *differentiable* (resp. *smooth*, resp. *analytic*) *manifold* is a manifold such that all transition maps are *differentiable* (resp. *smooth*, resp. *analytic*). A *pseudo-Riemannian manifold* is a real differentiable manifold equipped with a metric tensor g (i.e. a field of non-degenerate symmetric bilinear forms), with the two subcases of *Riemannian manifold* (g positive-definite) and *Lorentzian manifold* (g has signature $n - 2$ or $2 - n$).

INPUT:

- `dim` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none are provided, it is set to `name`
- `field` – (default: `'real'`) field K on which the manifold is defined; allowed values are
  - `'real'` or an object of type `RealField` (e.g. `RR`) for a manifold over $\mathbf{R}$
  - `'complex'` or an object of type `ComplexField` (e.g. `CC`) for a manifold over $\mathbf{C}$
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of manifolds
- `structure` – (default: `'smooth'`) to specify the structure or type of manifold; allowed values are
  - `'topological'` or `'top'` for a topological manifold
  - `'differentiable'` or `'diff'` for a differentiable manifold
  - `'smooth'` for a smooth manifold

  - 'analytic' for an analytic manifold
  - 'pseudo-Riemannian' for a real differentiable manifold equipped with a pseudo-Riemannian metric; the signature is specified via the keyword argument signature (see below)
  - 'Riemannian' for a real differentiable manifold equipped with a Riemannian (i.e. positive definite) metric
  - 'Lorentzian' for a real differentiable manifold equipped with a Lorentzian metric; the signature convention is specified by the keyword argument signature='positive' (default) or 'negative'
- start_index – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g. coordinates in a chart
- extra_kwds – keywords meaningful only for some specific types of manifolds:
  - diff_degree – (only for differentiable manifolds; default: infinity): the degree of differentiability
  - ambient – (only to construct a submanifold): the ambient manifold
  - metric_name – (only for pseudo-Riemannian manifolds; default: 'g') string; name (symbol) given to the metric
  - metric_latex_name – (only for pseudo-Riemannian manifolds; default: None) string; LaTeX symbol to denote the metric; if none is provided, the symbol is set to metric_name
  - signature – (only for pseudo-Riemannian manifolds; default: None) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. negative terms) in any diagonal writing of the metric components; if signature is not provided, S is set to the manifold's dimension (Riemannian signature); for Lorentzian manifolds the values signature='positive' (default) or signature='negative' are allowed to indicate the chosen signature convention.

OUTPUT:

- a manifold of the specified type, as an instance of *TopologicalManifold* or one of its subclasses *DifferentiableManifold* or *PseudoRiemannianManifold*

EXAMPLES:

A 3-dimensional real topological manifold:

```
sage: M = Manifold(3, 'M', structure='topological'); M
3-dimensional topological manifold M
```

Given the default value of the parameter field, the above is equivalent to:

```
sage: M = Manifold(3, 'M', structure='topological', field='real'); M
3-dimensional topological manifold M
```

A complex topological manifold:

```
sage: M = Manifold(3, 'M', structure='topological', field='complex'); M
Complex 3-dimensional topological manifold M
```

A topological manifold over **Q**:

```
sage: M = Manifold(3, 'M', structure='topological', field=QQ); M
3-dimensional topological manifold M over the Rational Field
```

A 3-dimensional real differentiable manifold of class C^4:

```
sage: M = Manifold(3, 'M', field='real', structure='differentiable',
....:              diff_degree=4); M
3-dimensional differentiable manifold M
```

Since the default value of the parameter `field` is `'real'`, the above is equivalent to:

```
sage: M = Manifold(3, 'M', structure='differentiable', diff_degree=4)
sage: M
3-dimensional differentiable manifold M
sage: M.base_field_type()
'real'
```

A 3-dimensional real smooth manifold:

```
sage: M = Manifold(3, 'M', structure='differentiable', diff_degree=+oo)
sage: M
3-dimensional differentiable manifold M
```

Instead of `structure='differentiable', diff_degree=+oo`, it suffices to use `structure='smooth'` to get the same result:

```
sage: M = Manifold(3, 'M', structure='smooth'); M
3-dimensional differentiable manifold M
sage: M.diff_degree()
+Infinity
```

Actually, since `'smooth'` is the default value of the parameter `structure`, the creation of a real smooth manifold can be shortened to:

```
sage: M = Manifold(3, 'M'); M
3-dimensional differentiable manifold M
sage: M.diff_degree()
+Infinity
```

For a complex smooth manifold, we have to set the parameter `field`:

```
sage: M = Manifold(3, 'M', field='complex'); M
3-dimensional complex manifold M
sage: M.diff_degree()
+Infinity
```

See the documentation of classes *TopologicalManifold*, *DifferentiableManifold* and *PseudoRiemannianManifold* for more detailed examples.

Uniqueness of manifold objects

Suppose we construct a manifold named M:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
```

At some point, we change our mind and would like to restart with a new manifold, using the same name M and keeping the previous manifold for reference:

```
sage: M_old = M  # for reference
sage: M = Manifold(2, 'M', structure='topological')
```

This results in a brand new object:

```
sage: M.atlas()
[]
```

The object `M_old` is intact:

```
sage: M_old.atlas()
[Chart (M, (x, y))]
```

Both objects have the same display:

```
sage: M
2-dimensional topological manifold M
sage: M_old
2-dimensional topological manifold M
```

but they are different:

```
sage: M != M_old
True
```

Let us introduce a chart on `M`, using the same coordinate symbols as for `M_old`:

```
sage: X.<x,y> = M.chart()
```

The charts are displayed in the same way:

```
sage: M.atlas()
[Chart (M, (x, y))]
sage: M_old.atlas()
[Chart (M, (x, y))]
```

but they are actually different:

```
sage: M.atlas()[0] != M_old.atlas()[0]
True
```

Moreover, the two manifolds `M` and `M_old` are still considered distinct:

```
sage: M != M_old
True
```

This reflects the fact that the equality of manifold objects holds only for identical objects, i.e. one has `M1 == M2` if, and only if, `M1 is M2`. Actually, the manifold classes inherit from `WithEqualityById`:

```
sage: isinstance(M, sage.misc.fast_methods.WithEqualityById)
True
```

class `sage.manifolds.manifold.`**`TopologicalManifold`**(*n*, *name*, *field*, *structure*, *base_manifold=None*, *latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: *`sage.manifolds.subset.ManifoldSubset`*

Topological manifold over a topological field K.

Given a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$) and a non-negative integer n, a *topological manifold of dimension* n *over* K is a topological space M such that

- M is a Hausdorff space,
- M is second countable, and
- every point in M has a neighborhood homeomorphic to K^n.

This is a Sage *parent* class, the corresponding *element* class being `ManifoldPoint`.

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `field` – field K on which the manifold is defined; allowed values are
  - `'real'` or an object of type `RealField` (e.g., `RR`) for a manifold over $\mathbf{R}$
  - `'complex'` or an object of type `ComplexField` (e.g., `CC`) for a manifold over $\mathbf{C}$
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of manifolds
- `structure` – manifold structure (see `TopologicalStructure` or `RealTopologicalStructure`)
- `base_manifold` – (default: `None`) if not `None`, must be a topological manifold; the created object is then an open subset of `base_manifold`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none are provided, it is set to `name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g., coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(field)` is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from `ManifoldSubset` would return the previously constructed object corresponding to these arguments)

EXAMPLES:

A 4-dimensional topological manifold (over $\mathbf{R}$):

```
sage: M = Manifold(4, 'M', latex_name=r'\mathcal{M}', structure='topological')
sage: M
4-dimensional topological manifold M
sage: latex(M)
\mathcal{M}
sage: type(M)
<class 'sage.manifolds.manifold.TopologicalManifold_with_category'>
sage: M.base_field()
Real Field with 53 bits of precision
sage: dim(M)
4
```

The input parameter `start_index` defines the range of indices on the manifold:

```
sage: M = Manifold(4, 'M', structure='topological')
sage: list(M.irange())
[0, 1, 2, 3]
sage: M = Manifold(4, 'M', structure='topological', start_index=1)
sage: list(M.irange())
[1, 2, 3, 4]
sage: list(Manifold(4, 'M', structure='topological', start_index=-2).irange())
[-2, -1, 0, 1]
```

A complex manifold:

```
sage: N = Manifold(3, 'N', structure='topological', field='complex'); N
Complex 3-dimensional topological manifold N
```

A manifold over $\mathbf{Q}$:

```
sage: N = Manifold(6, 'N', structure='topological', field=QQ); N
6-dimensional topological manifold N over the Rational Field
```

A manifold over $\mathbf{Q}_5$, the field of 5-adic numbers:

```
sage: N = Manifold(2, 'N', structure='topological', field=Qp(5)); N
2-dimensional topological manifold N over the 5-adic Field with capped
 relative precision 20
```

A manifold is a Sage *parent* object, in the category of topological manifolds over a given topological field (see `Manifolds`):

```
sage: isinstance(M, Parent)
True
sage: M.category()
Category of manifolds over Real Field with 53 bits of precision
sage: from sage.categories.manifolds import Manifolds
sage: M.category() is Manifolds(RR)
True
sage: M.category() is Manifolds(M.base_field())
True
sage: M in Manifolds(RR)
True
sage: N in Manifolds(Qp(5))
True
```

The corresponding Sage *elements* are points:

```
sage: X.<t, x, y, z> = M.chart()
sage: p = M.an_element(); p
Point on the 4-dimensional topological manifold M
sage: p.parent()
4-dimensional topological manifold M
sage: M.is_parent_of(p)
True
sage: p in M
True
```

The manifold's points are instances of class *ManifoldPoint*:

```
sage: isinstance(p, sage.manifolds.point.ManifoldPoint)
True
```

Since an open subset of a topological manifold M is itself a topological manifold, open subsets of M are instances of the class *TopologicalManifold*:

```
sage: U = M.open_subset('U'); U
Open subset U of the 4-dimensional topological manifold M
sage: isinstance(U, sage.manifolds.manifold.TopologicalManifold)
True
sage: U.base_field() == M.base_field()
True
sage: dim(U) == dim(M)
True
sage: U.category()
Join of Category of subobjects of sets and Category of manifolds over
 Real Field with 53 bits of precision
```

The manifold passes all the tests of the test suite relative to its category:

```
sage: TestSuite(M).run()
```

See also:

sage.manifolds.manifold

atlas()

Return the list of charts that have been defined on the manifold.

EXAMPLES:

Let us consider $\mathbf{R}^2$ as a 2-dimensional manifold:

```
sage: M = Manifold(2, 'R^2', structure='topological')
```

Immediately after the manifold creation, the atlas is empty, since no chart has been defined yet:

```
sage: M.atlas()
[]
```

Let us introduce the chart of Cartesian coordinates:

```
sage: c_cart.<x,y> = M.chart()
sage: M.atlas()
[Chart (R^2, (x, y))]
```

The complement of the half line $\{y = 0, x \geq 0\}$:

```
sage: U = M.open_subset('U', coord_def={c_cart: (y!=0,x<0)})
sage: U.atlas()
[Chart (U, (x, y))]
sage: M.atlas()
[Chart (R^2, (x, y)), Chart (U, (x, y))]
```

Spherical (polar) coordinates on U:

```
sage: c_spher.<r, ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: U.atlas()
[Chart (U, (x, y)), Chart (U, (r, ph))]
sage: M.atlas()
[Chart (R^2, (x, y)), Chart (U, (x, y)), Chart (U, (r, ph))]
```

See also:

top_charts()

base_field()

Return the field on which the manifold is defined.

OUTPUT:

- a topological field

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: M.base_field()
Real Field with 53 bits of precision
sage: M = Manifold(3, 'M', structure='topological', field='complex')
sage: M.base_field()
Complex Field with 53 bits of precision
sage: M = Manifold(3, 'M', structure='topological', field=QQ)
sage: M.base_field()
Rational Field
```

base_field_type()

Return the type of topological field on which the manifold is defined.

OUTPUT:

- a string describing the field, with three possible values:
  - `'real'` for the real field $\mathbf{R}$
  - `'complex'` for the complex field $\mathbf{C}$
  - `'neither_real_nor_complex'` for a field different from $\mathbf{R}$ and $\mathbf{C}$

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: M.base_field_type()
'real'
sage: M = Manifold(3, 'M', structure='topological', field='complex')
sage: M.base_field_type()
'complex'
sage: M = Manifold(3, 'M', structure='topological', field=QQ)
sage: M.base_field_type()
'neither_real_nor_complex'
```

chart(*coordinates=''*, *names=None*, *calc_method=None*)

Define a chart, the domain of which is the manifold.

A *chart* is a pair (U, φ), where U is the current manifold and $\varphi : U \to V \subset K^n$ is a homeomorphism from U to an open subset V of K^n, K being the field on which the manifold is defined.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in K^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

See *Chart* for a complete documentation.

INPUT:

- `coordinates` – (default: `''` (empty string)) string defining the coordinate symbols, ranges and possible periodicities, see below

- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)
- `calc_method` – (default: `None`) string defining the calculus method to be used on this chart; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the current calculus method defined on the manifold is used (cf. `set_calculus_method()`)

The coordinates declared in the string `coordinates` are separated by `' '` (whitespace) and each coordinate has at most four fields, separated by a colon (`':'`):

1. The coordinate symbol (a letter or a few letters).
2. (optional, only for manifolds over $\mathbf{R}$) The interval I defining the coordinate range: if not provided, the coordinate is assumed to span all $\mathbf{R}$; otherwise I must be provided in the form `(a,b)` (or equivalently `]a,b[`) The bounds `a` and `b` can be `+/-Infinity`, `Inf`, `infinity`, `inf` or `oo`. For *singular* coordinates, non-open intervals such as `[a,b]` and `(a,b]` (or equivalently `]a,b]`) are allowed. Note that the interval declaration must not contain any space character.
3. (optional) Indicator of the periodic character of the coordinate, either as `period=T`, where `T` is the period, or, for manifolds over $\mathbf{R}$ only, as the keyword `periodic` (the value of the period is then deduced from the interval I declared in field 2; see the example below)
4. (optional) The LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used.

The order of fields 2 to 4 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of the backslash character (see examples below). If no interval range, no period and no LaTeX spelling is to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).

OUTPUT:

- the created chart, as an instance of `Chart` or one of its subclasses, like `RealDiffChart` for differentiable manifolds over $\mathbf{R}$.

EXAMPLES:

Chart on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X = M.chart('x y'); X
Chart (M, (x, y))
sage: X[0]
x
sage: X[1]
y
sage: X[:]
(x, y)
```

The declared coordinates are not known at the global level:

```
sage: y
Traceback (most recent call last):
...
NameError: name 'y' is not defined
```

They can be recovered by the operator `[:]` applied to the chart:

```
sage: (x, y) = X[:]
sage: y
y
sage: type(y)
<type 'sage.symbolic.expression.Expression'>
```

But a shorter way to proceed is to use the operator <,> in the left-hand side of the chart declaration (there is then no need to pass the string 'x y' to chart()):

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(); X
Chart (M, (x, y))
```

Indeed, the declared coordinates are then known at the global level:

```
sage: y
y
sage: (x,y) == X[:]
True
```

Actually the instruction `X.<x,y> = M.chart()` is equivalent to the combination of the two instructions `X = M.chart('x y')` and `(x,y) = X[:]`.

As an example of coordinate ranges and LaTeX symbols passed via the string `coordinates` to `chart()`, let us introduce polar coordinates:

```
sage: U = M.open_subset('U', coord_def={X: x^2+y^2 != 0})
sage: P.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):periodic:\phi'); P
Chart (U, (r, ph))
sage: P.coord_range()
r: (0, +oo); ph: [0, 2*pi] (periodic)
sage: latex(P)
\left(U,(r, {\phi})\right)
```

See the documentation of classes *Chart* and *RealChart* for more examples, especially regarding the coordinates ranges and restrictions.

constant_scalar_field(*value*, *name=None*, *latex_name=None*)

Define a constant scalar field on the manifold.

INPUT:

- `value` – constant value of the scalar field, either a numerical value or a symbolic expression not involving any chart coordinates
- `name` – (default: `None`) name given to the scalar field
- `latex_name` – (default: `None`) LaTeX symbol to denote the scalar field; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- instance of *ScalarField* representing the scalar field whose constant value is `value`

EXAMPLES:

A constant scalar field on the 2-sphere:

```
sage: M = Manifold(2, 'M', structure='topological') # the 2-dimensional␣
↪sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1= x^2+y^2!=0,
....:                                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: f = M.constant_scalar_field(-1) ; f
Scalar field on the 2-dimensional topological manifold M
sage: f.display()
M --> R
on U: (x, y) |--> -1
on V: (u, v) |--> -1
```

We have:

```
sage: f.restrict(U) == U.constant_scalar_field(-1)
True
sage: M.constant_scalar_field(0) is M.zero_scalar_field()
True
```

See also:

zero_scalar_field(), *one_scalar_field()*

continuous_map(*codomain*, *coord_functions=None*, *chart1=None*, *chart2=None*, *name=None*, *latex_name=None*)

Define a continuous map from `self` to `codomain`.

INPUT:

- `codomain` – *TopologicalManifold*; the map's codomain
- `coord_functions` – (default: `None`) if not `None`, must be either
  - (i) a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts `(chart1, chart2)` as keys (`chart1` being a chart on `self` and `chart2` a chart on `codomain`);
  - (ii) a single coordinate expression in a given pair of charts, the latter being provided by the arguments `chart1` and `chart2`;

  in both cases, if the dimension of the codomain is 1, a single coordinate expression can be passed instead of a tuple with a single element
- `chart1` – (default: `None`; used only in case (ii) above) chart on `self` defining the start coordinates involved in `coord_functions` for case (ii); if `None`, the coordinates are assumed to refer to the default chart of `self`
- `chart2` – (default: `None`; used only in case (ii) above) chart on `codomain` defining the target coordinates involved in `coord_functions` for case (ii); if `None`, the coordinates are assumed to refer to the default chart of `codomain`
- `name` – (default: `None`) name given to the continuous map
- `latex_name` – (default: `None`) LaTeX symbol to denote the continuous map; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- the continuous map as an instance of `ContinuousMap`

EXAMPLES:

A continuous map between an open subset of S^2 covered by regular spherical coordinates and $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2', structure='topological')
sage: U = M.open_subset('U')
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: N = Manifold(3, 'R^3', latex_name=r'\RR^3', structure='topological')
sage: c_cart.<x,y,z> = N.chart()  # Cartesian coord. on R^3
sage: Phi = U.continuous_map(N, (sin(th)*cos(ph), sin(th)*sin(ph), cos(th)),
....:                         name='Phi', latex_name=r'\Phi')
sage: Phi
Continuous map Phi from the Open subset U of the 2-dimensional topological␣
↪manifold S^2 to the 3-dimensional topological manifold R^3
```

The same definition, but with a dictionary with pairs of charts as keys (case (i) above):

```
sage: Phi1 = U.continuous_map(N,
....:        {(c_spher, c_cart): (sin(th)*cos(ph), sin(th)*sin(ph), cos(th))},
....:        name='Phi', latex_name=r'\Phi')
sage: Phi1 == Phi
True
```

The continuous map acting on a point:

```
sage: p = U.point((pi/2, pi)) ; p
Point on the 2-dimensional topological manifold S^2
sage: Phi(p)
Point on the 3-dimensional topological manifold R^3
sage: Phi(p).coord(c_cart)
(-1, 0, 0)
sage: Phi1(p) == Phi(p)
True
```

See also:

See `ContinuousMap` for the complete documentation and more examples.

Todo: Allow the construction of continuous maps from `self` to the base field (considered as a trivial 1-dimensional manifold).

coord_change (*chart1, chart2*)

Return the change of coordinates (transition map) between two charts defined on the manifold.

The change of coordinates must have been defined previously, for instance by the method `transition_map()`.

INPUT:

- `chart1` – chart 1
- `chart2` – chart 2

OUTPUT:

- instance of `CoordChange` representing the transition map from chart 1 to chart 2

EXAMPLES:

Change of coordinates on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: c_xy.transition_map(c_uv, (x+y, x-y)) # defines the coord. change
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: M.coord_change(c_xy, c_uv) # returns the coord. change defined above
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
```

coord_changes()

Return the changes of coordinates (transition maps) defined on subsets of the manifold.

OUTPUT:

- dictionary of changes of coordinates, with pairs of charts as keys

EXAMPLES:

Various changes of coordinates on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y])
sage: M.coord_changes()
{(Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
sage: uv_to_xy = xy_to_uv.inverse()
sage: M.coord_changes()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
sage: c_rs.<r,s> = M.chart()
sage: uv_to_rs = c_uv.transition_map(c_rs, [-u+2*v, 3*u-v])
sage: M.coord_changes()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (r, s)),
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
sage: xy_to_rs = uv_to_rs * xy_to_uv
sage: M.coord_changes()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (r, s))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (r, s)),
 (Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
```

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```
 Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v)),
 (Chart (M, (x, y)),
 Chart (M, (r, s))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (r, s))}
```

default_chart()

Return the default chart defined on the manifold.

Unless changed via *set_default_chart()*, the *default chart* is the first one defined on a subset of the manifold (possibly itself).

OUTPUT:

- instance of *Chart* representing the default chart

EXAMPLES:

Default chart on a 2-dimensional manifold and on some subsets:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.chart('x y')
Chart (M, (x, y))
sage: M.chart('u v')
Chart (M, (u, v))
sage: M.default_chart()
Chart (M, (x, y))
sage: A = M.open_subset('A')
sage: A.chart('t z')
Chart (A, (t, z))
sage: A.default_chart()
Chart (A, (t, z))
```

dim()

Return the dimension of the manifold over its base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.dimension()
2
```

A shortcut is `dim()`:

```
sage: M.dim()
2
```

The Sage global function `dim` can also be used:

```
sage: dim(M)
2
```

dimension()

Return the dimension of the manifold over its base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.dimension()
2
```

A shortcut is `dim()`:

```
sage: M.dim()
2
```

The Sage global function `dim` can also be used:

```
sage: dim(M)
2
```

get_chart(*coordinates*, *domain=None*)

Get a chart from its coordinates.

The chart must have been previously created by the method `chart()`.

INPUT:

- `coordinates` – single string composed of the coordinate symbols separated by a space
- `domain` – (default: `None`) string containing the name of the chart's domain, which must be a subset of the current manifold; if `None`, the current manifold is assumed

OUTPUT:

- instance of `Chart` (or of the subclass `RealChart`) representing the chart corresponding to the above specifications

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: M.get_chart('x y')
Chart (M, (x, y))
sage: M.get_chart('x y') is X
True
sage: U = M.open_subset('U', coord_def={X: (y!=0,x<0)})
sage: Y.<r, ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: M.atlas()
[Chart (M, (x, y)), Chart (U, (x, y)), Chart (U, (r, ph))]
sage: M.get_chart('x y', domain='U')
Chart (U, (x, y))
sage: M.get_chart('x y', domain='U') is X.restrict(U)
True
sage: U.get_chart('r ph')
Chart (U, (r, ph))
sage: M.get_chart('r ph', domain='U')
Chart (U, (r, ph))
sage: M.get_chart('r ph', domain='U') is Y
True
```

homeomorphism(*codomain*, *coord_functions=None*, *chart1=None*, *chart2=None*, *name=None*, *latex_name=None*)

Define a homeomorphism between the current manifold and another one.

See `ContinuousMap` for a complete documentation.

INPUT:

- codomain – *TopologicalManifold*; codomain of the homeomorphism
- coord_functions – (default: None) if not None, must be either
  - (i) a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts (chart1, chart2) as keys (chart1 being a chart on self and chart2 a chart on codomain);
  - (ii) a single coordinate expression in a given pair of charts, the latter being provided by the arguments chart1 and chart2;

  in both cases, if the dimension of the codomain is 1, a single coordinate expression can be passed instead of a tuple with a single element
- chart1 – (default: None; used only in case (ii) above) chart on self defining the start coordinates involved in coord_functions for case (ii); if None, the coordinates are assumed to refer to the default chart of self
- chart2 – (default: None; used only in case (ii) above) chart on codomain defining the target coordinates involved in coord_functions for case (ii); if None, the coordinates are assumed to refer to the default chart of codomain
- name – (default: None) name given to the homeomorphism
- latex_name – (default: None) LaTeX symbol to denote the homeomorphism; if None, the LaTeX symbol is set to name

OUTPUT:

- the homeomorphism, as an instance of *ContinuousMap*

EXAMPLES:

Homeomorphism between the open unit disk in $\mathbf{R}^2$ and $\mathbf{R}^2$:

```
sage: forget()  # for doctests only
sage: M = Manifold(2, 'M', structure='topological')  # the open unit disk
sage: c_xy.<x,y> = M.chart('x:(-1,1) y:(-1,1)')  # Cartesian coord on M
sage: c_xy.add_restrictions(x^2+y^2<1)
sage: N = Manifold(2, 'N', structure='topological')  # R^2
sage: c_XY.<X,Y> = N.chart()  # canonical coordinates on R^2
sage: Phi = M.homeomorphism(N, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                       name='Phi', latex_name=r'\Phi')
sage: Phi
Homeomorphism Phi from the 2-dimensional topological manifold M to
 the 2-dimensional topological manifold N
sage: Phi.display()
Phi: M --> N
   (x, y) |--> (X, Y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The inverse homeomorphism:

```
sage: Phi^(-1)
Homeomorphism Phi^(-1) from the 2-dimensional topological
 manifold N to the 2-dimensional topological manifold M
sage: (Phi^(-1)).display()
Phi^(-1): N --> M
   (X, Y) |--> (x, y) = (X/sqrt(X^2 + Y^2 + 1), Y/sqrt(X^2 + Y^2 + 1))
```

See the documentation of *ContinuousMap* for more examples.

identity_map()

Identity map of `self`.

The identity map of a topological manifold M is the trivial homeomorphism:

$$\begin{array}{llll} \mathrm{Id}_M : & M & \longrightarrow & M \\ & p & \longmapsto & p \end{array}$$

OUTPUT:

- the identity map as an instance of *ContinuousMap*

EXAMPLES:

Identity map of a complex manifold:

```
sage: M = Manifold(2, 'M', structure='topological', field='complex')
sage: X.<x,y> = M.chart()
sage: id = M.identity_map(); id
Identity map Id_M of the Complex 2-dimensional topological manifold M
sage: id.parent()
Set of Morphisms from Complex 2-dimensional topological manifold M
 to Complex 2-dimensional topological manifold M in Category of
 manifolds over Complex Field with 53 bits of precision
sage: id.display()
Id_M: M --> M
   (x, y) |--> (x, y)
```

The identity map acting on a point:

```
sage: p = M((1+I, 3-I), name='p'); p
Point p on the Complex 2-dimensional topological manifold M
sage: id(p)
Point p on the Complex 2-dimensional topological manifold M
sage: id(p) == p
True
```

See also:

See *ContinuousMap* for the complete documentation.

index_generator(*nb_indices*)

Generator of index series.

INPUT:

- `nb_indices` – number of indices in a series

OUTPUT:

- an iterable index series for a generic component with the specified number of indices

EXAMPLES:

Indices on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological', start_index=1)
sage: list(M.index_generator(2))
[(1, 1), (1, 2), (2, 1), (2, 2)]
```

Loops can be nested:

```
sage: for ind1 in M.index_generator(2):
....:     print("{} : {}".format(ind1, list(M.index_generator(2))))
(1, 1) : [(1, 1), (1, 2), (2, 1), (2, 2)]
(1, 2) : [(1, 1), (1, 2), (2, 1), (2, 2)]
(2, 1) : [(1, 1), (1, 2), (2, 1), (2, 2)]
(2, 2) : [(1, 1), (1, 2), (2, 1), (2, 2)]
```

irange(*start=None*)

Single index generator.

INPUT:

- `start` – (default: `None`) initial value i_0 of the index; if none are provided, the value returned by *start_index()* is assumed

OUTPUT:

- an iterable index, starting from i_0 and ending at $i_0 + n - 1$, where n is the manifold's dimension

EXAMPLES:

Index range on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M', structure='topological')
sage: list(M.irange())
[0, 1, 2, 3]
sage: list(M.irange(2))
[2, 3]
```

Index range on a 4-dimensional manifold with starting index=1:

```
sage: M = Manifold(4, 'M', structure='topological', start_index=1)
sage: list(M.irange())
[1, 2, 3, 4]
sage: list(M.irange(2))
[2, 3, 4]
```

In general, one has always:

```
sage: next(M.irange()) == M.start_index()
True
```

is_manifestly_coordinate_domain()

Return `True` if the manifold is known to be the domain of some coordinate chart and `False` otherwise.

If `False` is returned, either the manifold cannot be the domain of some coordinate chart or no such chart has been declared yet.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: X.<x,y> = U.chart()
sage: U.is_manifestly_coordinate_domain()
True
sage: M.is_manifestly_coordinate_domain()
False
sage: Y.<u,v> = M.chart()
sage: M.is_manifestly_coordinate_domain()
True
```

is_open()

Return if `self` is an open set.

In the present case (manifold or open subset of it), always return `True`.

one_scalar_field()

Return the constant scalar field with value the unit element of the base field of `self`.

OUTPUT:

- a *ScalarField* representing the constant scalar field with value the unit element of the base field of `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.one_scalar_field(); f
Scalar field 1 on the 2-dimensional topological manifold M
sage: f.display()
1: M --> R
   (x, y) |--> 1
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: f is M.scalar_field_algebra().one()
True
```

open_subset(*name*, *latex_name=None*, *coord_def={}*)

Create an open subset of the manifold.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a topological manifold by itself. Hence the returned object is an instance of *TopologicalManifold*.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none are provided, it is set to `name`
- `coord_def` – (default: {}) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts on the manifold and values the symbolic expressions formed by the coordinates to define the subset

OUTPUT:

- the open subset, as an instance of *TopologicalManifold*

EXAMPLES:

Creating an open subset of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.open_subset('A'); A
Open subset A of the 2-dimensional topological manifold M
```

As an open subset of a topological manifold, `A` is itself a topological manifold, on the same topological field and of the same dimension as `M`:

```
sage: isinstance(A, sage.manifolds.manifold.TopologicalManifold)
True
```

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```
sage: A.base_field() == M.base_field()
True
sage: dim(A) == dim(M)
True
sage: A.category() is M.category().Subobjects()
True
```

Creating an open subset of A:

```
sage: B = A.open_subset('B'); B
Open subset B of the 2-dimensional topological manifold M
```

We have then:

```
sage: A.subsets()  # random (set output)
{Open subset B of the 2-dimensional topological manifold M,
 Open subset A of the 2-dimensional topological manifold M}
sage: B.is_subset(A)
True
sage: B.is_subset(M)
True
```

Defining an open subset by some coordinate restrictions: the open unit disk in $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1}); U
Open subset U of the 2-dimensional topological manifold R^2
```

Since the argument `coord_def` has been set, `U` is automatically provided with a chart, which is the restriction of the Cartesian one to `U`:

```
sage: U.atlas()
[Chart (U, (x, y))]
```

Therefore, one can immediately check whether a point belongs to `U`:

```
sage: M.point((0,0)) in U
True
sage: M.point((1/2,1/3)) in U
True
sage: M.point((1,2)) in U
False
```

options(**get_value*, ***set_value*)

Sets and displays the options for manifolds. If no parameters are set, then the function returns a copy of the options dictionary.

The `options` to manifolds can be accessed as the method `Manifold.options`.

OPTIONS:

- `omit_function_arguments` – (default: `False`) Determine whether the arguments of symbolic functions are printed
- `textbook_output` – (default: `True`) textbook-like output instead of the Pynac output for derivatives

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: g = function('g')(x, y)
```

For coordinate functions, the display is more "textbook" like:

```
sage: f = X.function(diff(g, x) + diff(g, y))
sage: f
d(g)/dx + d(g)/dy
sage: latex(f)
\frac{\partial\,g}{\partial x} + \frac{\partial\,g}{\partial y}
```

One can switch to Pynac notation by changing `textbook_output` to `False`:

```
sage: Manifold.options.textbook_output=False
sage: f
diff(g(x, y), x) + diff(g(x, y), y)
sage: latex(f)
\frac{\partial}{\partial x}g\left(x, y\right)
 + \frac{\partial}{\partial y}g\left(x, y\right)
sage: Manifold.options._reset()
```

If there is a clear understanding that u and v are functions of (x, y), the explicit mention of the latter can be cumbersome in lengthy tensor expressions:

```
sage: f = X.function(function('u')(x, y) * function('v')(x, y))
sage: f
u(x, y)*v(x, y)
```

We can switch it off by:

```
sage: M.options.omit_function_arguments=True
sage: f
u*v
sage: M.options._reset()
```

See `GlobalOptions` for more features of these options.

scalar_field(*coord_expression=None*, *chart=None*, *name=None*, *latex_name=None*)

Define a scalar field on the manifold.

See *ScalarField* (or *DiffScalarField* if the manifold is differentiable) for a complete documentation.

INPUT:

- `coord_expression` – (default: `None`) coordinate expression(s) of the scalar field; this can be either
  - a single coordinate expression; if the argument `chart` is `'all'`, this expression is set to all the charts defined on the open set; otherwise, the expression is set in the specific chart provided by the argument `chart`
  - a dictionary of coordinate expressions, with the charts as keys
- `chart` – (default: `None`) chart defining the coordinates used in `coord_expression` when the latter is a single coordinate expression; if `None`, the default chart of the open set is assumed; if `chart=='all'`, `coord_expression` is assumed to be independent of the chart (constant scalar field)

- name – (default: None) name given to the scalar field
- latex_name – (default: None) LaTeX symbol to denote the scalar field; if None, the LaTeX symbol is set to name

If coord_expression is None or does not fully specified the scalar field, other coordinate expressions can be added subsequently by means of the methods *add_expr()*, *add_expr_by_continuation()*, or *set_expr()*

OUTPUT:

- instance of *ScalarField* (or of the subclass *DiffScalarField* if the manifold is differentiable) representing the defined scalar field

EXAMPLES:

A scalar field defined by its coordinate expression in the open set's default chart:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: c_xyz.<x,y,z> = U.chart()
sage: f = U.scalar_field(sin(x)*cos(y) + z, name='F'); f
Scalar field F on the Open subset U of the 3-dimensional topological manifold␣
↪M
sage: f.display()
F: U --> R
   (x, y, z) |--> cos(y)*sin(x) + z
sage: f.parent()
Algebra of scalar fields on the Open subset U of the 3-dimensional␣
↪topological manifold M
sage: f in U.scalar_field_algebra()
True
```

Equivalent definition with the chart specified:

```
sage: f = U.scalar_field(sin(x)*cos(y) + z, chart=c_xyz, name='F')
sage: f.display()
F: U --> R
   (x, y, z) |--> cos(y)*sin(x) + z
```

Equivalent definition with a dictionary of coordinate expression(s):

```
sage: f = U.scalar_field({c_xyz: sin(x)*cos(y) + z}, name='F')
sage: f.display()
F: U --> R
   (x, y, z) |--> cos(y)*sin(x) + z
```

See the documentation of class *ScalarField* for more examples.

See also:

constant_scalar_field(), *zero_scalar_field()*, *one_scalar_field()*

scalar_field_algebra()

Return the algebra of scalar fields defined the manifold.

See *ScalarFieldAlgebra* for a complete documentation.

OUTPUT:

- instance of *ScalarFieldAlgebra* representing the algebra $C^0(U)$ of all scalar fields defined on U = self

EXAMPLES:

Scalar algebra of a 3-dimensional open subset:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: CU = U.scalar_field_algebra() ; CU
Algebra of scalar fields on the Open subset U of the 3-dimensional
↪topological manifold M
sage: CU.category()
Category of commutative algebras over Symbolic Ring
sage: CU.zero()
Scalar field zero on the Open subset U of the 3-dimensional topological
↪manifold M
```

The output is cached:

```
sage: U.scalar_field_algebra() is CU
True
```

set_calculus_method(*method*)

Set the calculus method to be used for coordinate computations on this manifold.

The provided method is transmitted to all coordinate charts defined on the manifold.

INPUT:

- `method` – string specifying the method to be used for coordinate computations on this manifold; one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy

EXAMPLES:

Let us consider a scalar field `f` on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field(x^2 + cos(y)*sin(x), name='F')
```

By default, the coordinate expression of `f` returned by *expr()* is a Sage's symbolic expression:

```
sage: f.expr()
x^2 + cos(y)*sin(x)
sage: type(f.expr())
<type 'sage.symbolic.expression.Expression'>
sage: parent(f.expr())
Symbolic Ring
sage: f.display()
F: M --> R
   (x, y) |--> x^2 + cos(y)*sin(x)
```

If we change the calculus method to SymPy, it becomes a SymPy object instead:

```
sage: M.set_calculus_method('sympy')
sage: f.expr()
x**2 + sin(x)*cos(y)
sage: type(f.expr())
<class 'sympy.core.add.Add'>
```

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```
sage: parent(f.expr())
<class 'sympy.core.add.Add'>
sage: f.display()
F: M --> R
   (x, y) |--> x**2 + sin(x)*cos(y)
```

Back to the Symbolic Ring:

```
sage: M.set_calculus_method('SR')
sage: f.display()
F: M --> R
   (x, y) |--> x^2 + cos(y)*sin(x)
```

See also:

`calculus_method()` for a control of the calculus method chart by chart

`set_default_chart`(*chart*)

Changing the default chart on `self`.

INPUT:

- `chart` – a chart (must be defined on some subset `self`)

EXAMPLES:

Charts on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: M.default_chart()
Chart (M, (x, y))
sage: M.set_default_chart(c_uv)
sage: M.default_chart()
Chart (M, (u, v))
```

`set_simplify_function`(*simplifying_func*, *method=None*)

Set the simplifying function associated to a given coordinate calculus method in all the charts defined on `self`.

INPUT:

- `simplifying_func` – either the string `'default'` for restoring the default simplifying function or a function `f` of a single argument `expr` such that `f(expr)` returns an object of the same type as `expr` (hopefully the simplified version of `expr`), this type being
  - `Expression` if `method` = `'SR'`
  - a SymPy type if `method` = `'sympy'`
- `method` – (default: `None`) string defining the calculus method for which `simplifying_func` is provided; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the currently active calculus method on each chart is assumed

See also:

calculus_method() and *sage.manifolds.calculus_method.CalculusMethod.simplify()* for a control of the calculus method chart by chart

EXAMPLES:

Les us add two scalar fields on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field((x+y)^2 + cos(x)^2)
sage: g = M.scalar_field(-x^2-2*x*y-y^2 + sin(x)^2)
sage: f.expr()
(x + y)^2 + cos(x)^2
sage: g.expr()
-x^2 - 2*x*y - y^2 + sin(x)^2
sage: s = f + g
```

The outcome is automatically simplified:

```
sage: s.expr()
1
```

The simplification is performed thanks to the default simplifying function on chart `X`, which is *simplify_chain_real()* in the present case (real manifold and `SR` calculus):

```
sage: X.calculus_method().simplify_function() is \
....: sage.manifolds.utilities.simplify_chain_real
True
```

Let us change it to the generic Sage function `simplify()`:

```
sage: M.set_simplify_function(simplify)
sage: X.calculus_method().simplify_function() is simplify
True
```

`simplify()` is faster, but it does not do much:

```
sage: s = f + g
sage: s.expr()
(x + y)^2 - x^2 - 2*x*y - y^2 + cos(x)^2 + sin(x)^2
```

We can replaced it by any user defined function, for instance:

```
sage: def simpl_trig(a):
....:     return a.simplify_trig()
....:
sage: M.set_simplify_function(simpl_trig)
sage: s = f + g
sage: s.expr()
1
```

The default simplifying function is restored via:

```
sage: M.set_simplify_function('default')
```

Then we are back to:

```
sage: X.calculus_method().simplify_function() is \
....: sage.manifolds.utilities.simplify_chain_real
True
```

Thanks to the argument `method`, one can specify a simplifying function for a calculus method distinct from the current one. For instance, let us define a simplifying function for SymPy (note that `trigsimp()` is a SymPy method only):

```
sage: def simpl_trig_sympy(a):
....:     return a.trigsimp()
....:
sage: M.set_simplify_function(simpl_trig_sympy, method='sympy')
```

Then, it becomes active as soon as we change the calculus engine to SymPy:

```
sage: M.set_calculus_method('sympy')
sage: X.calculus_method().simplify_function() is simpl_trig_sympy
True
```

We have then:

```
sage: s = f + g
sage: s.expr()
1
sage: type(s.expr())
<class 'sympy.core.numbers.One'>
```

start_index()

Return the first value of the index range used on the manifold.

This is the parameter `start_index` passed at the construction of the manifold.

OUTPUT:

- the integer i_0 such that all indices of indexed objects on the manifold range from i_0 to $i_0 + n - 1$, where n is the manifold's dimension

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: M.start_index()
0
sage: M = Manifold(3, 'M', structure='topological', start_index=1)
sage: M.start_index()
1
```

top_charts()

Return the list of charts defined on subsets of the current manifold that are not subcharts of charts on larger subsets.

OUTPUT:

- list of charts defined on open subsets of the manifold but not on larger subsets

EXAMPLES:

Charts on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: Y.<u,v> = U.chart()
sage: M.top_charts()
[Chart (M, (x, y)), Chart (U, (u, v))]
```

Note that the (user) atlas contains one more chart: `(U, (x,y))`, which is not a "top" chart:

```
sage: M.atlas()
[Chart (M, (x, y)), Chart (U, (x, y)), Chart (U, (u, v))]
```

See also:

`atlas()` for the complete list of charts defined on the manifold.

zero_scalar_field()

Return the zero scalar field defined on `self`.

OUTPUT:

- a `ScalarField` representing the constant scalar field with value 0

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.zero_scalar_field() ; f
Scalar field zero on the 2-dimensional topological manifold M
sage: f.display()
zero: M --> R
   (x, y) |--> 0
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: f is M.scalar_field_algebra().zero()
True
```

1.2 Subsets of Topological Manifolds

The class `ManifoldSubset` implements generic subsets of a topological manifold. Open subsets are implemented by the class `TopologicalManifold` (since an open subset of a manifold is a manifold by itself), which inherits from `ManifoldSubset`.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Travis Scrimshaw (2015): review tweaks; removal of facade parents

REFERENCES:

- [?]

EXAMPLES:

Two subsets on a manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A'); a
Subset A of the 2-dimensional topological manifold M
sage: b = M.subset('B'); b
Subset B of the 2-dimensional topological manifold M
sage: M.list_of_subsets()
[Subset A of the 2-dimensional topological manifold M,
 Subset B of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M]
```

The intersection of the two subsets:

```
sage: c = a.intersection(b); c
Subset A_inter_B of the 2-dimensional topological manifold M
```

Their union:

```
sage: d = a.union(b); d
Subset A_union_B of the 2-dimensional topological manifold M
```

Lists of subsets after the above operations:

```
sage: M.list_of_subsets()
[Subset A of the 2-dimensional topological manifold M,
 Subset A_inter_B of the 2-dimensional topological manifold M,
 Subset A_union_B of the 2-dimensional topological manifold M,
 Subset B of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M]
sage: a.list_of_subsets()
[Subset A of the 2-dimensional topological manifold M,
 Subset A_inter_B of the 2-dimensional topological manifold M]
sage: c.list_of_subsets()
[Subset A_inter_B of the 2-dimensional topological manifold M]
sage: d.list_of_subsets()
[Subset A of the 2-dimensional topological manifold M,
 Subset A_inter_B of the 2-dimensional topological manifold M,
 Subset A_union_B of the 2-dimensional topological manifold M,
 Subset B of the 2-dimensional topological manifold M]
```

class `sage.manifolds.subset.`**`ManifoldSubset`**(*manifold*, *name*, *latex_name=None*, *category=None*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

Subset of a topological manifold.

The class *`ManifoldSubset`* inherits from the generic class `Parent`. The corresponding element class is *`ManifoldPoint`*.

Note that open subsets are not implemented directly by this class, but by the derived class *`TopologicalManifold`* (an open subset of a topological manifold being itself a topological manifold).

INPUT:

- `manifold` – topological manifold on which the subset is defined
- `name` – string; name (symbol) given to the subset
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the subset; if none are provided, it is set to `name`

- `category` – (default: `None`) to specify the category; if `None`, the category for generic subsets is used

EXAMPLES:

A subset of a manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: from sage.manifolds.subset import ManifoldSubset
sage: A = ManifoldSubset(M, 'A', latex_name=r'\mathcal{A}')
sage: A
Subset A of the 2-dimensional topological manifold M
sage: latex(A)
\mathcal{A}
sage: A.is_subset(M)
True
```

Instead of importing *ManifoldSubset* in the global namespace, it is recommended to use the method *subset()* to create a new subset:

```
sage: B = M.subset('B', latex_name=r'\mathcal{B}'); B
Subset B of the 2-dimensional topological manifold M
sage: M.list_of_subsets()
[Subset A of the 2-dimensional topological manifold M,
 Subset B of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M]
```

The manifold is itself a subset:

```
sage: isinstance(M, ManifoldSubset)
True
sage: M in M.subsets()
True
```

Instances of *ManifoldSubset* are parents:

```
sage: isinstance(A, Parent)
True
sage: A.category()
Category of subobjects of sets
sage: p = A.an_element(); p
Point on the 2-dimensional topological manifold M
sage: p.parent()
Subset A of the 2-dimensional topological manifold M
sage: p in A
True
sage: p in M
True
```

Element
alias of *sage.manifolds.point.ManifoldPoint*

ambient()
Return the ambient manifold of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: A.manifold()
```

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```
2-dimensional topological manifold M
sage: A.manifold() is M
True
sage: B = A.subset('B')
sage: B.manifold() is M
True
```

An alias is `ambient`:

```
sage: A.ambient() is A.manifold()
True
```

declare_union (*dom1, dom2*)

Declare that the current subset is the union of two subsets.

Suppose U is the current subset, then this method declares that U

$$U = U_1 \cup U_2,$$

where $U_1 \subset U$ and $U_2 \subset U$.

INPUT:

- `dom1` – the subset U_1
- `dom2` – the subset U_2

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: B = M.subset('B')
sage: M.declare_union(A, B)
sage: A.union(B)
2-dimensional topological manifold M
```

get_subset (*name*)

Get a subset by its name.

The subset must have been previously created by the method *subset()* (or *open_subset()*)

INPUT:

- `name` – (string) name of the subset

OUTPUT:

- instance of *ManifoldSubset* (or of the derived class *TopologicalManifold* for an open subset) representing the subset whose name is `name`

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='topological')
sage: A = M.subset('A')
sage: B = A.subset('B')
sage: U = M.open_subset('U')
sage: M.list_of_subsets()
[Subset A of the 4-dimensional topological manifold M,
 Subset B of the 4-dimensional topological manifold M,
 4-dimensional topological manifold M,
```

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```
 Open subset U of the 4-dimensional topological manifold M]
sage: M.get_subset('A')
Subset A of the 4-dimensional topological manifold M
sage: M.get_subset('A') is A
True
sage: M.get_subset('B') is B
True
sage: A.get_subset('B') is B
True
sage: M.get_subset('U')
Open subset U of the 4-dimensional topological manifold M
sage: M.get_subset('U') is U
True
```

intersection(*other*, *name=None*, *latex_name=None*)

Return the intersection of the current subset with another subset.

INPUT:

- `other` – another subset of the same manifold
- `name` – (default: `None`) name given to the intersection in the case the latter has to be created; the default is `self._name` inter `other._name`
- `latex_name` – (default: `None`) LaTeX symbol to denote the intersection in the case the latter has to be created; the default is built upon the symbol $\cap$

OUTPUT:

- instance of *ManifoldSubset* representing the subset that is the intersection of the current subset with `other`

EXAMPLES:

Intersection of two subsets:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A')
sage: b = M.subset('B')
sage: c = a.intersection(b); c
Subset A_inter_B of the 2-dimensional topological manifold M
sage: a.list_of_subsets()
[Subset A of the 2-dimensional topological manifold M,
 Subset A_inter_B of the 2-dimensional topological manifold M]
sage: b.list_of_subsets()
[Subset A_inter_B of the 2-dimensional topological manifold M,
 Subset B of the 2-dimensional topological manifold M]
sage: c._supersets  # random (set output)
{Subset B of the 2-dimensional topological manifold M,
 Subset A_inter_B of the 2-dimensional topological manifold M,
 Subset A of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M}
```

Some checks:

```
sage: (a.intersection(b)).is_subset(a)
True
sage: (a.intersection(b)).is_subset(a)
True
```

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```
sage: a.intersection(b) is b.intersection(a)
True
sage: a.intersection(a.intersection(b)) is a.intersection(b)
True
sage: (a.intersection(b)).intersection(a) is a.intersection(b)
True
sage: M.intersection(a) is a
True
sage: a.intersection(M) is a
True
```

is_open()

Return if `self` is an open set.

This method always returns `False`, since open subsets must be constructed as instances of the subclass *TopologicalManifold* (which redefines `is_open`)

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: A.is_open()
False
```

is_subset(*other*)

Return `True` if and only if `self` is included in `other`.

EXAMPLES:

Subsets on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A')
sage: b = a.subset('B')
sage: c = M.subset('C')
sage: a.is_subset(M)
True
sage: b.is_subset(a)
True
sage: b.is_subset(M)
True
sage: a.is_subset(b)
False
sage: c.is_subset(a)
False
```

lift(*p*)

Return the lift of `p` to the ambient manifold of `self`.

INPUT:

- `p` – point of the subset

OUTPUT:

- the same point, considered as a point of the ambient manifold

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: A = M.open_subset('A', coord_def={X: x>0})
sage: p = A((1, -2)); p
Point on the 2-dimensional topological manifold M
sage: p.parent()
Open subset A of the 2-dimensional topological manifold M
sage: q = A.lift(p); q
Point on the 2-dimensional topological manifold M
sage: q.parent()
2-dimensional topological manifold M
sage: q.coord()
(1, -2)
sage: (p == q) and (q == p)
True
```

list_of_subsets()

Return the list of subsets that have been defined on the current subset.

The list is sorted by the alphabetical names of the subsets.

OUTPUT:

- a list containing all the subsets that have been defined on the current subset

Note: To get the subsets as a Python set, used the method *subsets()* instead.

EXAMPLES:

Subsets of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = M.subset('V')
sage: M.list_of_subsets()
[2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M,
 Subset V of the 2-dimensional topological manifold M]
```

The method *subsets()* returns a set instead of a list:

```
sage: M.subsets()  # random (set output)
{Subset V of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M}
```

manifold()

Return the ambient manifold of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: A = M.subset('A')
sage: A.manifold()
2-dimensional topological manifold M
sage: A.manifold() is M
True
```

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```
sage: B = A.subset('B')
sage: B.manifold() is M
True
```

An alias is `ambient`:

```
sage: A.ambient() is A.manifold()
True
```

open_covers()

Return the list of open covers of the current subset.

If the current subset, A say, is a subset of the manifold M, an *open cover* of A is list (indexed set) $(U_i)_{i\in I}$ of open subsets of M such that

$$A \subset \bigcup_{i\in I} U_i.$$

If A is open, we ask that the above inclusion is actually an identity:

$$A = \bigcup_{i\in I} U_i.$$

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: M.open_covers()
[[2-dimensional topological manifold M]]
sage: U = M.open_subset('U')
sage: U.open_covers()
[[Open subset U of the 2-dimensional topological manifold M]]
sage: A = U.open_subset('A')
sage: B = U.open_subset('B')
sage: U.declare_union(A,B)
sage: U.open_covers()
[[Open subset U of the 2-dimensional topological manifold M],
 [Open subset A of the 2-dimensional topological manifold M,
  Open subset B of the 2-dimensional topological manifold M]]
sage: V = M.open_subset('V')
sage: M.declare_union(U,V)
sage: M.open_covers()
[[2-dimensional topological manifold M],
 [Open subset U of the 2-dimensional topological manifold M,
  Open subset V of the 2-dimensional topological manifold M],
 [Open subset A of the 2-dimensional topological manifold M,
  Open subset B of the 2-dimensional topological manifold M,
  Open subset V of the 2-dimensional topological manifold M]]
```

point(*coords=None*, *chart=None*, *name=None*, *latex_name=None*)

Define a point in `self`.

See *ManifoldPoint* for a complete documentation.

INPUT:

- `coords` – the point coordinates (as a tuple or a list) in the chart specified by `chart`
- `chart` – (default: `None`) chart in which the point coordinates are given; if `None`, the coordinates are assumed to refer to the default chart of the current subset

- `name` – (default: `None`) name given to the point
- `latex_name` – (default: `None`) LaTeX symbol to denote the point; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- the declared point, as an instance of *ManifoldPoint*

EXAMPLES:

Points on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: p = M.point((1,2), name='p'); p
Point p on the 2-dimensional topological manifold M
sage: p in M
True
sage: a = M.open_subset('A')
sage: c_uv.<u,v> = a.chart()
sage: q = a.point((-1,0), name='q'); q
Point q on the 2-dimensional topological manifold M
sage: q in a
True
sage: p._coordinates
{Chart (M, (x, y)): (1, 2)}
sage: q._coordinates
{Chart (A, (u, v)): (-1, 0)}
```

retract (*p*)

Return the retract of `p` to `self`.

INPUT:

- `p` – point of the ambient manifold

OUTPUT:

- the same point, considered as a point of the subset

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: A = M.open_subset('A', coord_def={X: x>0})
sage: p = M((1, -2)); p
Point on the 2-dimensional topological manifold M
sage: p.parent()
2-dimensional topological manifold M
sage: q = A.retract(p); q
Point on the 2-dimensional topological manifold M
sage: q.parent()
Open subset A of the 2-dimensional topological manifold M
sage: q.coord()
(1, -2)
sage: (q == p) and (p == q)
True
```

Of course, if the point does not belong to `A`, the `retract` method fails:

```
sage: p = M((-1, 3))  #  x < 0, so that p is not in A
sage: q = A.retract(p)
Traceback (most recent call last):
...
ValueError: the Point on the 2-dimensional topological manifold M
 is not in Open subset A of the 2-dimensional topological manifold M
```

subset (*name*, *latex_name=None*, *is_open=False*)

Create a subset of the current subset.

INPUT:

- `name` – name given to the subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none are provided, it is set to `name`
- `is_open` – (default: `False`) if `True`, the created subset is assumed to be open with respect to the manifold's topology

OUTPUT:

- the subset, as an instance of *ManifoldSubset*, or of the derived class *TopologicalManifold* if `is_open` is `True`

EXAMPLES:

Creating a subset of a manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A'); a
Subset A of the 2-dimensional topological manifold M
```

Creating a subset of `A`:

```
sage: b = a.subset('B', latex_name=r'\mathcal{B}'); b
Subset B of the 2-dimensional topological manifold M
sage: latex(b)
\mathcal{B}
```

We have then:

```
sage: b.is_subset(a)
True
sage: b in a.subsets()
True
```

subsets ()

Return the set of subsets that have been defined on the current subset.

OUTPUT:

- a Python set containing all the subsets that have been defined on the current subset

Note: To get the subsets as a list, used the method *list_of_subsets()* instead.

EXAMPLES:

Subsets of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: V = M.subset('V')
sage: M.subsets()   # random (set output)
{Subset V of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M}
sage: type(M.subsets())
<... 'frozenset'>
sage: U in M.subsets()
True
```

The method `list_of_subsets()` returns a list (sorted alphabetically by the subset names) instead of a set:

```
sage: M.list_of_subsets()
[2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M,
 Subset V of the 2-dimensional topological manifold M]
```

superset (*name*, *latex_name=None*, *is_open=False*)

Create a superset of the current subset.

A *superset* is a manifold subset in which the current subset is included.

INPUT:

- `name` – name given to the superset
- `latex_name` – (default: `None`) LaTeX symbol to denote the superset; if none are provided, it is set to `name`
- `is_open` – (default: `False`) if `True`, the created subset is assumed to be open with respect to the manifold's topology

OUTPUT:

- the superset, as an instance of `ManifoldSubset` or of the derived class `TopologicalManifold` if `is_open` is `True`

EXAMPLES:

Creating some superset of a given subset:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A')
sage: b = a.superset('B'); b
Subset B of the 2-dimensional topological manifold M
sage: b.list_of_subsets()
[Subset A of the 2-dimensional topological manifold M,
 Subset B of the 2-dimensional topological manifold M]
sage: a._supersets # random (set output)
{Subset B of the 2-dimensional topological manifold M,
 Subset A of the 2-dimensional topological manifold M,
 2-dimensional topological manifold M}
```

The superset of the whole manifold is itself:

```
sage: M.superset('SM') is M
True
```

Two supersets of a given subset are a priori different:

```
sage: c = a.superset('C')
sage: c == b
False
```

union (*other*, *name=None*, *latex_name=None*)

Return the union of the current subset with another subset.

INPUT:

- `other` – another subset of the same manifold
- `name` – (default: `None`) name given to the union in the case the latter has to be created; the default is `self._name` union `other._name`
- `latex_name` – (default: `None`) LaTeX symbol to denote the union in the case the latter has to be created; the default is built upon the symbol $\cup$

OUTPUT:

- instance of *ManifoldSubset* representing the subset that is the union of the current subset with `other`

EXAMPLES:

Union of two subsets:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: a = M.subset('A')
sage: b = M.subset('B')
sage: c = a.union(b); c
Subset A_union_B of the 2-dimensional topological manifold M
sage: a._supersets  # random (set output)
set([subset 'A_union_B' of the 2-dimensional manifold 'M',
     2-dimensional manifold 'M',
     subset 'A' of the 2-dimensional manifold 'M'])
sage: b._supersets  # random (set output)
set([subset 'B' of the 2-dimensional manifold 'M',
     2-dimensional manifold 'M',
     subset 'A_union_B' of the 2-dimensional manifold 'M'])
sage: c._subsets  # random (set output)
set([subset 'A_union_B' of the 2-dimensional manifold 'M',
    subset 'A' of the 2-dimensional manifold 'M',
    subset 'B' of the 2-dimensional manifold 'M'])
```

Some checks:

```
sage: a.is_subset(a.union(b))
True
sage: b.is_subset(a.union(b))
True
sage: a.union(b) is b.union(a)
True
sage: a.union(a.union(b)) is a.union(b)
True
sage: (a.union(b)).union(a) is a.union(b)
True
sage: a.union(M) is M
True
```

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```
sage: M.union(a) is M
True
```

1.3 Manifold Structures

These classes encode the structure of a manifold.

AUTHORS:

- Travis Scrimshaw (2015-11-25): Initial version
- Eric Gourgoulhon (2015): add `DifferentialStructure` and `RealDifferentialStructure`
- Eric Gourgoulhon (2018): add `PseudoRiemannianStructure`, `RiemannianStructure` and `LorentzianStructure`

class sage.manifolds.structure.**DifferentialStructure**

Bases: `sage.misc.fast_methods.Singleton`

The structure of a differentiable manifold over a general topological field.

chart

alias of `sage.manifolds.differentiable.chart.DiffChart`

homset

alias of `sage.manifolds.differentiable.manifold_homset.DifferentiableManifoldHomset`

scalar_field_algebra

alias of `sage.manifolds.differentiable.scalarfield_algebra.DiffScalarFieldAlgebra`

subcategory(*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import DifferentialStructure
sage: from sage.categories.manifolds import Manifolds
sage: DifferentialStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class sage.manifolds.structure.**LorentzianStructure**

Bases: `sage.misc.fast_methods.Singleton`

The structure of a Lorentzian manifold.

chart

alias of `sage.manifolds.differentiable.chart.RealDiffChart`

homset

alias of `sage.manifolds.differentiable.manifold_homset.DifferentiableManifoldHomset`

scalar_field_algebra

alias of `sage.manifolds.differentiable.scalarfield_algebra.DiffScalarFieldAlgebra`

subcategory (*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import LorentzianStructure
sage: from sage.categories.manifolds import Manifolds
sage: LorentzianStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class `sage.manifolds.structure.`**PseudoRiemannianStructure**

Bases: `sage.misc.fast_methods.Singleton`

The structure of a pseudo-Riemannian manifold.

chart

alias of *sage.manifolds.differentiable.chart.RealDiffChart*

homset

alias of *sage.manifolds.differentiable.manifold_homset.DifferentiableManifoldHomset*

scalar_field_algebra

alias of *sage.manifolds.differentiable.scalarfield_algebra.DiffScalarFieldAlgebra*

subcategory (*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import PseudoRiemannianStructure
sage: from sage.categories.manifolds import Manifolds
sage: PseudoRiemannianStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class `sage.manifolds.structure.`**RealDifferentialStructure**

Bases: `sage.misc.fast_methods.Singleton`

The structure of a differentiable manifold over $\mathbf{R}$.

chart

alias of *sage.manifolds.differentiable.chart.RealDiffChart*

homset

alias of *sage.manifolds.differentiable.manifold_homset.DifferentiableManifoldHomset*

scalar_field_algebra

alias of *sage.manifolds.differentiable.scalarfield_algebra.DiffScalarFieldAlgebra*

subcategory (*cat*)

Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import RealDifferentialStructure
sage: from sage.categories.manifolds import Manifolds
sage: RealDifferentialStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class sage.manifolds.structure.**RealTopologicalStructure**
Bases: sage.misc.fast_methods.Singleton

The structure of a topological manifold over $\mathbf{R}$.

chart
alias of *sage.manifolds.chart.RealChart*

homset
alias of *sage.manifolds.manifold_homset.TopologicalManifoldHomset*

scalar_field_algebra
alias of *sage.manifolds.scalarfield_algebra.ScalarFieldAlgebra*

subcategory (*cat*)
Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import RealTopologicalStructure
sage: from sage.categories.manifolds import Manifolds
sage: RealTopologicalStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class sage.manifolds.structure.**RiemannianStructure**
Bases: sage.misc.fast_methods.Singleton

The structure of a Riemannian manifold.

chart
alias of *sage.manifolds.differentiable.chart.RealDiffChart*

homset
alias of *sage.manifolds.differentiable.manifold_homset.DifferentiableManifoldHomset*

scalar_field_algebra
alias of *sage.manifolds.differentiable.scalarfield_algebra.DiffScalarFieldAlgebra*

subcategory (*cat*)
Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import RiemannianStructure
sage: from sage.categories.manifolds import Manifolds
sage: RiemannianStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

class sage.manifolds.structure.**TopologicalStructure**
Bases: sage.misc.fast_methods.Singleton

The structure of a topological manifold over a general topological field.

chart
alias of *sage.manifolds.chart.Chart*

homset
alias of *sage.manifolds.manifold_homset.TopologicalManifoldHomset*

scalar_field_algebra
alias of *sage.manifolds.scalarfield_algebra.ScalarFieldAlgebra*

subcategory(*cat*)
Return the subcategory of `cat` corresponding to the structure of `self`.

EXAMPLES:

```
sage: from sage.manifolds.structure import TopologicalStructure
sage: from sage.categories.manifolds import Manifolds
sage: TopologicalStructure().subcategory(Manifolds(RR))
Category of manifolds over Real Field with 53 bits of precision
```

1.4 Points of Topological Manifolds

The class *ManifoldPoint* implements points of a topological manifold.

A *ManifoldPoint* object can have coordinates in various charts defined on the manifold. Two points are declared equal if they have the same coordinates in the same chart.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version

REFERENCES:

- [?]
- [?]

EXAMPLES:

Defining a point in $\mathbf{R}^3$ by its spherical coordinates:

```
sage: M = Manifold(3, 'R^3', structure='topological')
sage: U = M.open_subset('U')  # the domain of spherical coordinates
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):periodic:\phi')
```

We construct the point in the coordinates in the default chart of `U` (`c_spher`):

```
sage: p = U((1, pi/2, pi), name='P')
sage: p
Point P on the 3-dimensional topological manifold R^3
sage: latex(p)
P
sage: p in U
True
sage: p.parent()
Open subset U of the 3-dimensional topological manifold R^3
sage: c_spher(p)
(1, 1/2*pi, pi)
sage: p.coordinates(c_spher) # equivalent to above
(1, 1/2*pi, pi)
```

Computing the coordinates of `p` in a new chart:

```
sage: c_cart.<x,y,z> = U.chart() # Cartesian coordinates on U
sage: spher_to_cart = c_spher.transition_map(c_cart,
....:                     [r*sin(th)*cos(ph), r*sin(th)*sin(ph), r*cos(th)])
sage: c_cart(p)  # evaluate P's Cartesian coordinates
(-1, 0, 0)
```

Points can be compared:

```
sage: p1 = U((1, pi/2, pi))
sage: p1 == p
True
sage: q = U((2, pi/2, pi))
sage: q == p
False
```

even if they were initially not defined within the same coordinate chart:

```
sage: p2 = U((-1,0,0), chart=c_cart)
sage: p2 == p
True
```

The 2π-periodicity of the ϕ coordinate is also taken into account for the comparison:

```
sage: p3 = U((1, pi/2, 5*pi))
sage: p3 == p
True
sage: p4 = U((1, pi/2, -pi))
sage: p4 == p
True
```

class sage.manifolds.point.**ManifoldPoint**(*parent*, *coords=None*, *chart=None*, *name=None*, *latex_name=None*, *check_coords=True*)

Bases: `sage.structure.element.Element`

Point of a topological manifold.

This is a Sage *element* class, the corresponding *parent* class being *TopologicalManifold* or *ManifoldSubset*.

INPUT:

- `parent` – the manifold subset to which the point belongs
- `coords` – (default: `None`) the point coordinates (as a tuple or a list) in the chart `chart`
- `chart` – (default: `None`) chart in which the coordinates are given; if `None`, the coordinates are assumed to refer to the default chart of `parent`
- `name` – (default: `None`) name given to the point
- `latex_name` – (default: `None`) LaTeX symbol to denote the point; if `None`, the LaTeX symbol is set to `name`
- `check_coords` – (default: `True`) determines whether `coords` are valid coordinates for the chart `chart`; for symbolic coordinates, it is recommended to set `check_coords` to `False`

EXAMPLES:

A point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: (a, b) = var('a b') # generic coordinates for the point
sage: p = M.point((a, b), name='P'); p
Point P on the 2-dimensional topological manifold M
sage: p.coordinates()   # coordinates of P in the subset's default chart
(a, b)
```

Since points are Sage *elements*, the *parent* of which being the subset on which they are defined, it is equivalent to write:

```
sage: p = M((a, b), name='P'); p
Point P on the 2-dimensional topological manifold M
```

A point is an element of the manifold subset in which it has been defined:

```
sage: p in M
True
sage: p.parent()
2-dimensional topological manifold M
sage: U = M.open_subset('U', coord_def={c_xy: x>0})
sage: q = U.point((2,1), name='q')
sage: q.parent()
Open subset U of the 2-dimensional topological manifold M
sage: q in U
True
sage: q in M
True
```

By default, the LaTeX symbol of the point is deduced from its name:

```
sage: latex(p)
P
```

But it can be set to any value:

```
sage: p = M.point((a, b), name='P', latex_name=r'\mathcal{P}')
sage: latex(p)
\mathcal{P}
```

Points can be drawn in 2D or 3D graphics thanks to the method `plot()`.

add_coord(*coords*, *chart=None*)

Adds some coordinates in the specified chart.

The previous coordinates with respect to other charts are kept. To clear them, use `set_coord()` instead.

INPUT:

- `coords` – the point coordinates (as a tuple or a list)
- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart

Warning: If the point has already coordinates in other charts, it is the user's responsibility to make sure that the coordinates to be added are consistent with them.

EXAMPLES:

Setting coordinates to a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point()
```

We give the point some coordinates in the manifold's default chart:

```
sage: p.add_coordinates((2,-3))
sage: p.coordinates()
(2, -3)
sage: X(p)
(2, -3)
```

A shortcut for `add_coordinates` is `add_coord`:

```
sage: p.add_coord((2,-3))
sage: p.coord()
(2, -3)
```

Let us introduce a second chart on the manifold:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
```

If we add coordinates for `p` in chart `Y`, those in chart `X` are kept:

```
sage: p.add_coordinates((-1,5), chart=Y)
sage: p._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (-1, 5), Chart (M, (x, y)): (2, -3)}
```

On the contrary, with the method *`set_coordinates()`*, the coordinates in charts different from `Y` would be lost:

```
sage: p.set_coordinates((-1,5), chart=Y)
sage: p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

add_coordinates(*coords*, *chart=None*)

Adds some coordinates in the specified chart.

The previous coordinates with respect to other charts are kept. To clear them, use *`set_coord()`* instead.

INPUT:

- `coords` – the point coordinates (as a tuple or a list)
- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart

> **Warning:** If the point has already coordinates in other charts, it is the user's responsibility to make sure that the coordinates to be added are consistent with them.

EXAMPLES:

Setting coordinates to a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point()
```

We give the point some coordinates in the manifold's default chart:

```
sage: p.add_coordinates((2,-3))
sage: p.coordinates()
(2, -3)
sage: X(p)
(2, -3)
```

A shortcut for `add_coordinates` is `add_coord`:

```
sage: p.add_coord((2,-3))
sage: p.coord()
(2, -3)
```

Let us introduce a second chart on the manifold:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
```

If we add coordinates for `p` in chart `Y`, those in chart `X` are kept:

```
sage: p.add_coordinates((-1,5), chart=Y)
sage: p._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (-1, 5), Chart (M, (x, y)): (2, -3)}
```

On the contrary, with the method *`set_coordinates()`*, the coordinates in charts different from `Y` would be lost:

```
sage: p.set_coordinates((-1,5), chart=Y)
sage: p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

`coord`(*chart=None*, *old_chart=None*)

Return the point coordinates in the specified chart.

If these coordinates are not already known, they are computed from known ones by means of change-of-chart formulas.

An equivalent way to get the coordinates of a point is to let the chart acting on the point, i.e. if `X` is a chart and `p` a point, one has `p.coordinates(chart=X) == X(p)`.

INPUT:

- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart
- `old_chart` – (default: `None`) chart from which the coordinates in `chart` are to be computed; if `None`, a chart in which the point's coordinates are already known will be picked, privileging the subset's default chart

EXAMPLES:

Spherical coordinates of a point on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):\phi') # spherical coordinates
sage: p = M.point((1, pi/2, pi))
sage: p.coordinates()  # coordinates in the manifold's default chart
(1, 1/2*pi, pi)
```

Since the default chart of `M` is `c_spher`, it is equivalent to write:

```
sage: p.coordinates(c_spher)
(1, 1/2*pi, pi)
```

An alternative way to get the coordinates is to let the chart act on the point (from the very definition of a chart):

```
sage: c_spher(p)
(1, 1/2*pi, pi)
```

A shortcut for `coordinates` is `coord`:

```
sage: p.coord()
(1, 1/2*pi, pi)
```

Computing the Cartesian coordinates from the spherical ones:

```
sage: c_cart.<x,y,z> = M.chart()  # Cartesian coordinates
sage: c_spher.transition_map(c_cart, [r*sin(th)*cos(ph),
....:                                 r*sin(th)*sin(ph), r*cos(th)])
Change of coordinates from Chart (M, (r, th, ph)) to Chart (M, (x, y, z))
```

The computation is performed by means of the above change of coordinates:

```
sage: p.coord(c_cart)
(-1, 0, 0)
sage: p.coord(c_cart) == c_cart(p)
True
```

Coordinates of a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: (a, b) = var('a b') # generic coordinates for the point
sage: P = M.point((a, b), name='P')
```

Coordinates of `P` in the manifold's default chart:

```
sage: P.coord()
(a, b)
```

Coordinates of `P` in a new chart:

```
sage: c_uv.<u,v> = M.chart()
sage: ch_xy_uv = c_xy.transition_map(c_uv, [x-y, x+y])
sage: P.coord(c_uv)
(a - b, a + b)
```

Coordinates of `P` in a third chart:

```
sage: c_wz.<w,z> = M.chart()
sage: ch_uv_wz = c_uv.transition_map(c_wz, [u^3, v^3])
sage: P.coord(c_wz, old_chart=c_uv)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
```

Actually, in the present case, it is not necessary to specify `old_chart='uv'`. Note that the first command erases all the coordinates except those in the chart `c_uv`:

```
sage: P.set_coord((a-b, a+b), c_uv)
sage: P._coordinates
{Chart (M, (u, v)): (a - b, a + b)}
sage: P.coord(c_wz)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
sage: P._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (a - b, a + b),
 Chart (M, (w, z)): (a^3 - 3*a^2*b + 3*a*b^2 - b^3,
                     a^3 + 3*a^2*b + 3*a*b^2 + b^3)}
```

coordinates (*chart=None*, *old_chart=None*)

Return the point coordinates in the specified chart.

If these coordinates are not already known, they are computed from known ones by means of change-of-chart formulas.

An equivalent way to get the coordinates of a point is to let the chart acting on the point, i.e. if `X` is a chart and `p` a point, one has `p.coordinates(chart=X) == X(p)`.

INPUT:

- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart
- `old_chart` – (default: `None`) chart from which the coordinates in `chart` are to be computed; if `None`, a chart in which the point's coordinates are already known will be picked, privileging the subset's default chart

EXAMPLES:

Spherical coordinates of a point on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):\phi') # spherical coordinates
sage: p = M.point((1, pi/2, pi))
sage: p.coordinates()   # coordinates in the manifold's default chart
(1, 1/2*pi, pi)
```

Since the default chart of `M` is `c_spher`, it is equivalent to write:

```
sage: p.coordinates(c_spher)
(1, 1/2*pi, pi)
```

An alternative way to get the coordinates is to let the chart act on the point (from the very definition of a chart):

```
sage: c_spher(p)
(1, 1/2*pi, pi)
```

A shortcut for `coordinates` is `coord`:

```
sage: p.coord()
(1, 1/2*pi, pi)
```

Computing the Cartesian coordinates from the spherical ones:

```
sage: c_cart.<x,y,z> = M.chart()   # Cartesian coordinates
sage: c_spher.transition_map(c_cart, [r*sin(th)*cos(ph),
```

(continues on next page)

(continued from previous page)

```
....:                                  r*sin(th)*sin(ph), r*cos(th)])
Change of coordinates from Chart (M, (r, th, ph)) to Chart (M, (x, y, z))
```

The computation is performed by means of the above change of coordinates:

```
sage: p.coord(c_cart)
(-1, 0, 0)
sage: p.coord(c_cart) == c_cart(p)
True
```

Coordinates of a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: (a, b) = var('a b') # generic coordinates for the point
sage: P = M.point((a, b), name='P')
```

Coordinates of `P` in the manifold's default chart:

```
sage: P.coord()
(a, b)
```

Coordinates of `P` in a new chart:

```
sage: c_uv.<u,v> = M.chart()
sage: ch_xy_uv = c_xy.transition_map(c_uv, [x-y, x+y])
sage: P.coord(c_uv)
(a - b, a + b)
```

Coordinates of `P` in a third chart:

```
sage: c_wz.<w,z> = M.chart()
sage: ch_uv_wz = c_uv.transition_map(c_wz, [u^3, v^3])
sage: P.coord(c_wz, old_chart=c_uv)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
```

Actually, in the present case, it is not necessary to specify `old_chart='uv'`. Note that the first command erases all the coordinates except those in the chart `c_uv`:

```
sage: P.set_coord((a-b, a+b), c_uv)
sage: P._coordinates
{Chart (M, (u, v)): (a - b, a + b)}
sage: P.coord(c_wz)
(a^3 - 3*a^2*b + 3*a*b^2 - b^3, a^3 + 3*a^2*b + 3*a*b^2 + b^3)
sage: P._coordinates  # random (dictionary output)
{Chart (M, (u, v)): (a - b, a + b),
 Chart (M, (w, z)): (a^3 - 3*a^2*b + 3*a*b^2 - b^3,
                     a^3 + 3*a^2*b + 3*a*b^2 + b^3)}
```

plot(*chart=None*, *ambient_coords=None*, *mapping=None*, *label=None*, *parameters=None*, *color='black'*, *label_offset=0.1*, *fontsize=10*, *label_color=None*, *size=10*, ***kwds*)

For real manifolds, plot `self` in a Cartesian graph based on the coordinates of some ambient chart.

The point is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of a given chart, called hereafter the *ambient chart*. The domain of the ambient chart must contain the point, or its image by a continuous manifold map Φ.

INPUT:

- chart – (default: None) the ambient chart (see above); if None, the ambient chart is set the default chart of self.parent()
- ambient_coords – (default: None) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if None, all the coordinates of the ambient chart are considered
- mapping – (default: None) *ContinuousMap*; continuous manifold map Φ providing the link between the current point p and the ambient chart chart: the domain of chart must contain $\Phi(p)$; if None, the identity map is assumed
- label – (default: None) label printed next to the point; if None, the point's name is used
- parameters – (default: None) dictionary giving the numerical values of the parameters that may appear in the point coordinates
- size – (default: 10) size of the point once drawn as a small disk or sphere
- color – (default: 'black') color of the point
- label_color – (default: None) color to print the label; if None, the value of color is used
- fontsize – (default: 10) size of the font used to print the label
- label_offset – (default: 0.1) determines the separation between the point and its label

OUTPUT:

- a graphic object, either an instance of Graphics for a 2D plot (i.e. based on 2 coordinates of the ambient chart) or an instance of Graphics3d for a 3D plot (i.e. based on 3 coordinates of the ambient chart)

EXAMPLES:

Drawing a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point((1,3), name='p')
sage: g = p.plot(X)
sage: print(g)
Graphics object consisting of 2 graphics primitives
sage: gX = X.plot(max_range=4) # plot of the coordinate grid
sage: g + gX # display of the point atop the coordinate grid
Graphics object consisting of 20 graphics primitives
```

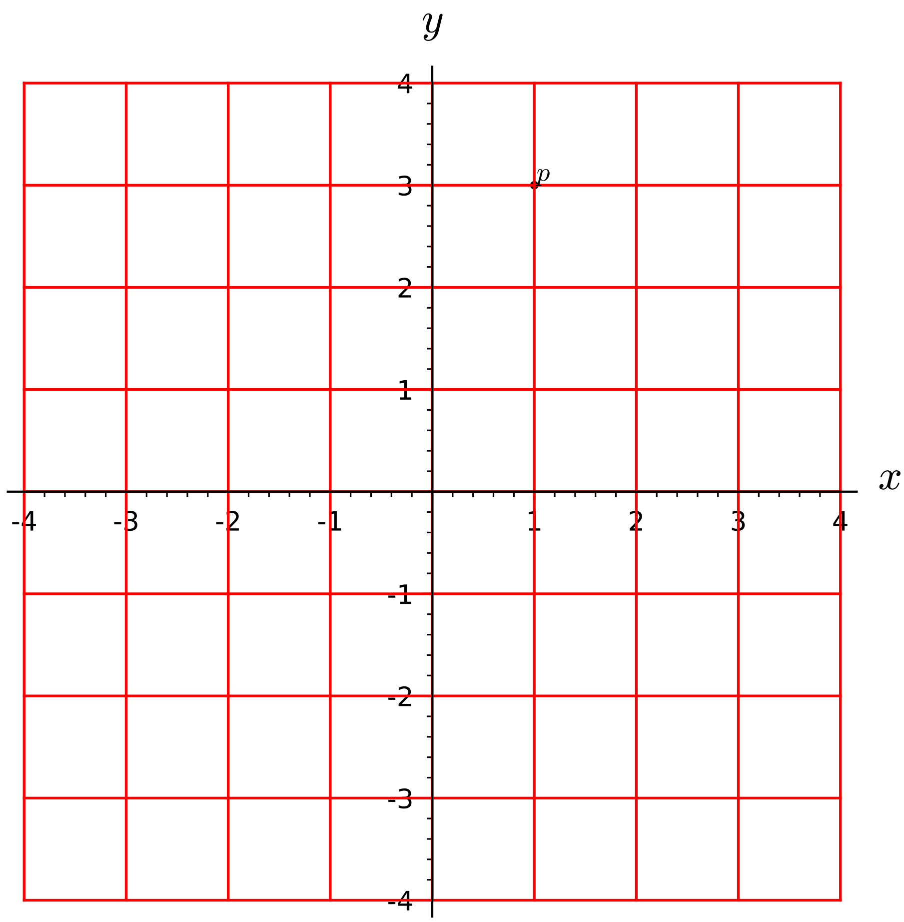
Actually, since X is the default chart of the open set in which p has been defined, it can be skipped in the arguments of plot:

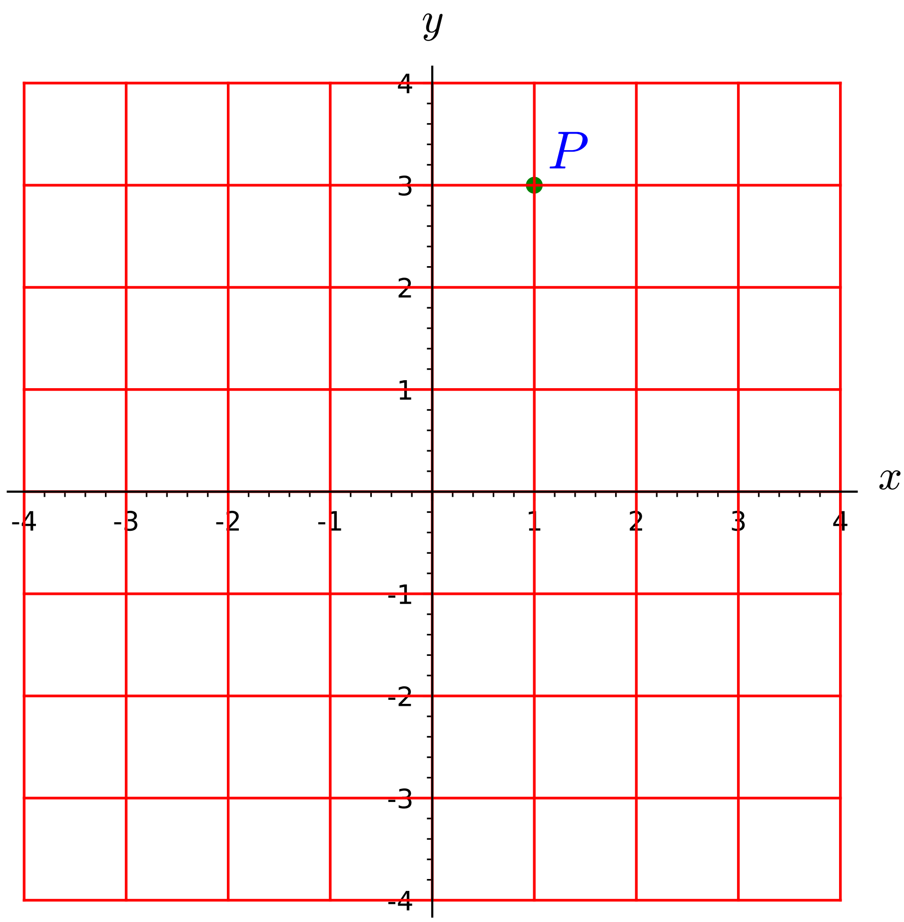
```
sage: g = p.plot()
sage: g + gX
Graphics object consisting of 20 graphics primitives
```

Call with some options:

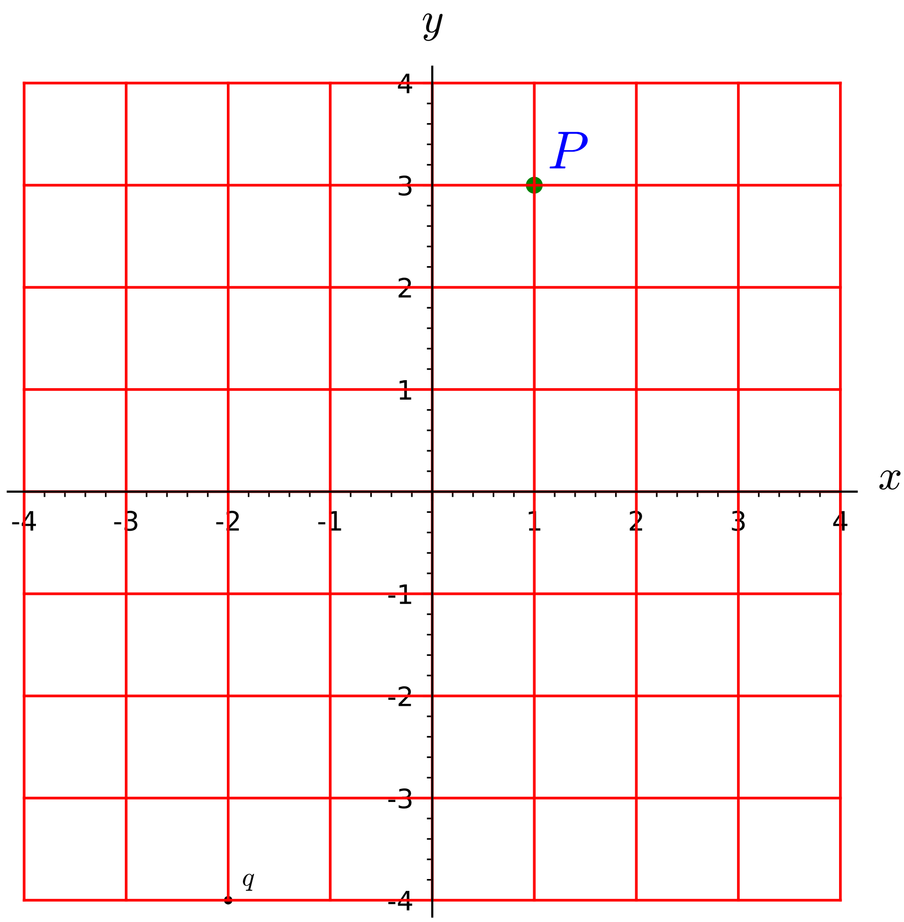
```
sage: g = p.plot(chart=X, size=40, color='green', label='$P$',
....:            label_color='blue', fontsize=20, label_offset=0.3)
sage: g + gX
Graphics object consisting of 20 graphics primitives
```

Use of the parameters option to set a numerical value of some symbolic variable:


y
x
p
4
3
2
1
-1
-2
-3
-4
-4
-3
-2
-1
1
2
3
4


y
x
P
4
3
2
1
-1
-2
-3
-4
-4
-3
-2
-1
1
2
3
4

```
sage: a = var('a')
sage: q = M.point((a,2*a), name='q')
sage: gq = q.plot(parameters={a:-2}, label_offset=0.2)
sage: g + gX + gq
Graphics object consisting of 22 graphics primitives
```



The numerical value is used only for the plot:

```
sage: q.coord()
(a, 2*a)
```

Drawing a point on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: X.<x,y,z> = M.chart()
sage: p = M.point((2,1,3), name='p')
sage: g = p.plot()
sage: print(g)
Graphics3d Object
sage: gX = X.plot(number_values=5) # coordinate mesh cube
sage: g + gX # display of the point atop the coordinate mesh
Graphics3d Object
```

Call with some options:

```
sage: g = p.plot(chart=X, size=40, color='green', label='P_1',
....:             label_color='blue', fontsize=20, label_offset=0.3)
sage: g + gX
Graphics3d Object
```

An example of plot via a mapping: plot of a point on a 2-sphere viewed in the 3-dimensional space M:

```
sage: S2 = Manifold(2, 'S^2', structure='topological')
sage: U = S2.open_subset('U') # the open set covered by spherical coord.
sage: XS.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: p = U.point((pi/4, pi/8), name='p')
sage: F = S2.continuous_map(M, {(XS, X): [sin(th)*cos(ph),
....:                            sin(th)*sin(ph), cos(th)]}, name='F')
sage: F.display()
F: S^2 --> M
on U: (th, ph) |--> (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
sage: g = p.plot(chart=X, mapping=F)
sage: gS2 = XS.plot(chart=X, mapping=F, number_values=9)
sage: g + gS2
Graphics3d Object
```

Use of the option ambient_coords for plots on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M', structure='topological')
sage: X.<t,x,y,z> = M.chart()
sage: p = M.point((1,2,3,4), name='p')
sage: g = p.plot(X, ambient_coords=(t,x,y), label_offset=0.4)  # the␣
↪coordinate z is skipped
sage: gX = X.plot(X, ambient_coords=(t,x,y), number_values=5)  # long time
sage: g + gX # 3D plot  # long time
Graphics3d Object
sage: g = p.plot(X, ambient_coords=(t,y,z), label_offset=0.4)  # the␣
↪coordinate x is skipped
sage: gX = X.plot(X, ambient_coords=(t,y,z), number_values=5)  # long time
sage: g + gX # 3D plot  # long time
Graphics3d Object
sage: g = p.plot(X, ambient_coords=(y,z), label_offset=0.4)  # the␣
↪coordinates t and x are skipped
sage: gX = X.plot(X, ambient_coords=(y,z))
sage: g + gX # 2D plot
Graphics object consisting of 20 graphics primitives
```

set_coord(*coords*, *chart=None*)

Sets the point coordinates in the specified chart.

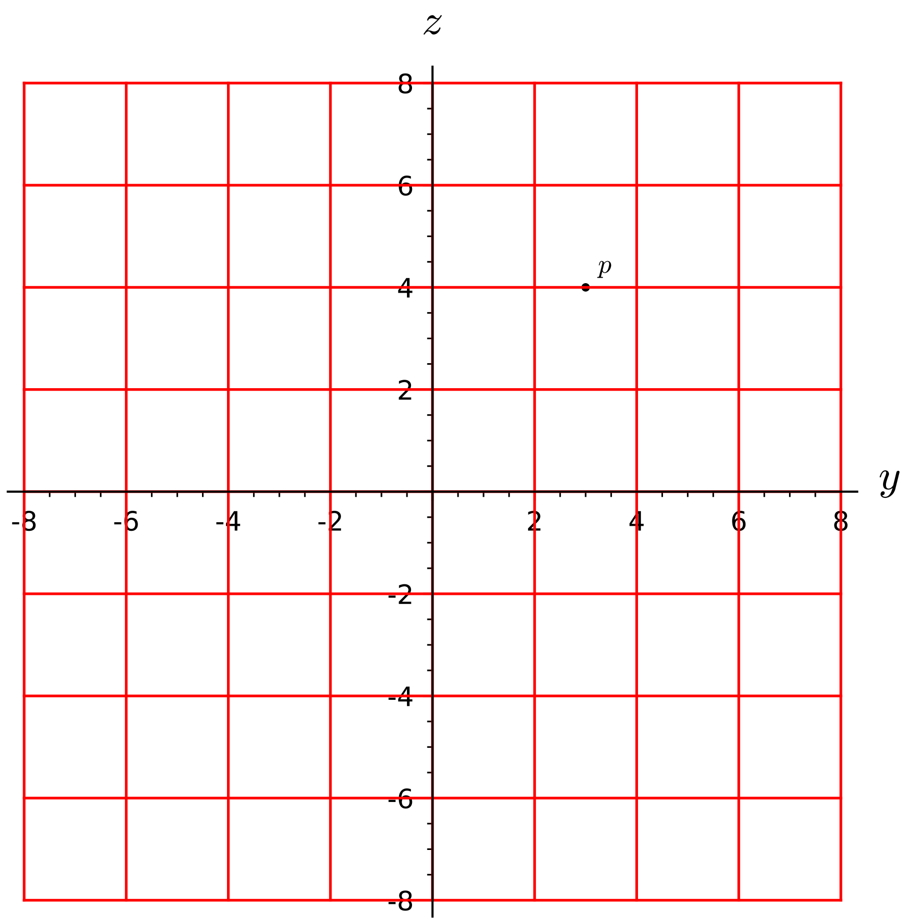
Coordinates with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use the method *add_coord()* instead.

INPUT:

- coords – the point coordinates (as a tuple or a list)
- chart – (default: None) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart

EXAMPLES:

Setting coordinates to a point on a 2-dimensional manifold:



z
8
6
4
2
-2
-4
-6
-8
y
-8
-6
-4
-2
2
4
6
8
p

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point()
```

We set the coordinates in the manifold's default chart:

```
sage: p.set_coordinates((2,-3))
sage: p.coordinates()
(2, -3)
sage: X(p)
(2, -3)
```

A shortcut for `set_coordinates` is `set_coord`:

```
sage: p.set_coord((2,-3))
sage: p.coord()
(2, -3)
```

Let us introduce a second chart on the manifold:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
```

If we set the coordinates of `p` in chart `Y`, those in chart `X` are lost:

```
sage: Y(p)
(-1, 5)
sage: p.set_coord(Y(p), chart=Y)
sage: p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

set_coordinates(*coords*, *chart=None*)

Sets the point coordinates in the specified chart.

Coordinates with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use the method *add_coord()* instead.

INPUT:

- `coords` – the point coordinates (as a tuple or a list)
- `chart` – (default: `None`) chart in which the coordinates are given; if none are provided, the coordinates are assumed to refer to the subset's default chart

EXAMPLES:

Setting coordinates to a point on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: p = M.point()
```

We set the coordinates in the manifold's default chart:

```
sage: p.set_coordinates((2,-3))
sage: p.coordinates()
(2, -3)
sage: X(p)
(2, -3)
```

A shortcut for `set_coordinates` is `set_coord`:

```
sage: p.set_coord((2,-3))
sage: p.coord()
(2, -3)
```

Let us introduce a second chart on the manifold:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
```

If we set the coordinates of `p` in chart `Y`, those in chart `X` are lost:

```
sage: Y(p)
(-1, 5)
sage: p.set_coord(Y(p), chart=Y)
sage: p._coordinates
{Chart (M, (u, v)): (-1, 5)}
```

1.5 Coordinate Charts

1.5.1 Coordinate Charts

The class *Chart* implements coordinate charts on a topological manifold over a topological field K. The subclass *RealChart* is devoted to the case $K = \mathbf{R}$, for which the concept of coordinate range is meaningful. Moreover, *RealChart* is endowed with some plotting capabilities (cf. method *plot()*).

Transition maps between charts are implemented via the class *CoordChange*.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Travis Scrimshaw (2015): review tweaks
- Eric Gourgoulhon (2019): periodic coordinates, add *calculus_method()*

REFERENCES:

- Chap. 2 of [**?**]
- Chap. 1 of [**?**]

class sage.manifolds.chart.**Chart**(*domain*, *coordinates=''*, *names=None*, *calc_method=None*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.sage_object.SageObject`

Chart on a topological manifold.

Given a topological manifold M of dimension n over a topological field K, a *chart* on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to V \subset K^n$ is a homeomorphism from U to an open subset V of K^n.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in K^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

INPUT:

- `domain` – open subset U on which the chart is defined (must be an instance of *TopologicalManifold*)

- `coordinates` – (default: '' (empty string)) single string defining the coordinate symbols, with `' '` (whitespace) as a separator; each item has at most three fields, separated by a colon (`:`):

  1. the coordinate symbol (a letter or a few letters)
  2. (optional) the period of the coordinate if the coordinate is periodic; the period field must be written as `period=T`, where `T` is the period (see examples below)
  3. (optional) the LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used

  The order of fields 2 and 3 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of LaTeX's backslash character (see examples below). If no period and no LaTeX spelling are to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).

- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

- `calc_method` – (default: `None`) string defining the calculus method for computations involving coordinates of the chart; must be one of

  – `'SR'`: Sage's default symbolic engine (Symbolic Ring)

  – `'sympy'`: SymPy

  – `None`: the default of *`CalculusMethod`* will be used

EXAMPLES:

A chart on a complex 2-dimensional topological manifold:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X = M.chart('x y'); X
Chart (M, (x, y))
sage: latex(X)
\left(M,(x, y)\right)
sage: type(X)
<class 'sage.manifolds.chart.Chart'>
```

To manipulate the coordinates (x, y) as global variables, one has to set:

```
sage: x,y = X[:]
```

However, a shortcut is to use the declarator `<x,y>` in the left-hand side of the chart declaration (there is then no need to pass the string `'x y'` to `chart()`):

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart(); X
Chart (M, (x, y))
```

The coordinates are then immediately accessible:

```
sage: y
y
sage: x is X[0] and y is X[1]
True
```

Note that `x` and `y` declared in `<x,y>` are mere Python variable names and do not have to coincide with the coordinate symbols; for instance, one may write:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x1,y1> = M.chart('x y'); X
Chart (M, (x, y))
```

Then y is not known as a global Python variable and the coordinate y is accessible only through the global variable y1:

```
sage: y1
y
sage: latex(y1)
y
sage: y1 is X[1]
True
```

However, having the name of the Python variable coincide with the coordinate symbol is quite convenient; so it is recommended to declare:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart()
```

In the above example, the chart X covers entirely the manifold M:

```
sage: X.domain()
Complex 2-dimensional topological manifold M
```

Of course, one may declare a chart only on an open subset of M:

```
sage: U = M.open_subset('U')
sage: Y.<z1, z2> = U.chart(r'z1:\zeta_1 z2:\zeta_2'); Y
Chart (U, (z1, z2))
sage: Y.domain()
Open subset U of the Complex 2-dimensional topological manifold M
```

In the above declaration, we have also specified some LaTeX writing of the coordinates different from the text one:

```
sage: latex(z1)
{\zeta_1}
```

Note the prefix r in front of the string r'z1:\zeta_1 z2:\zeta_2'; it makes sure that the backslash character is treated as an ordinary character, to be passed to the LaTeX interpreter.

Periodic coordinates are declared through the keyword period= in the coordinate field:

```
sage: N = Manifold(2, 'N', field='complex', structure='topological')
sage: XN.<Z1,Z2> = N.chart('Z1:period=1+2*I Z2')
sage: XN.periods()
{0: 2*I + 1}
```

Coordinates are Sage symbolic variables (see sage.symbolic.expression):

```
sage: type(z1)
<type 'sage.symbolic.expression.Expression'>
```

In addition to the Python variable name provided in the operator <., .>, the coordinates are accessible by their indices:

```
sage: Y[0], Y[1]
(z1, z2)
```

The index range is that declared during the creation of the manifold. By default, it starts at 0, but this can be changed via the parameter `start_index`:

```
sage: M1 = Manifold(2, 'M_1', field='complex', structure='topological',
....:                start_index=1)
sage: Z.<u,v> = M1.chart()
sage: Z[1], Z[2]
(u, v)
```

The full set of coordinates is obtained by means of the slice operator `[:]`:

```
sage: Y[:]
(z1, z2)
```

Some partial sets of coordinates:

```
sage: Y[:1]
(z1,)
sage: Y[1:]
(z2,)
```

Each constructed chart is automatically added to the manifold's user atlas:

```
sage: M.atlas()
[Chart (M, (x, y)), Chart (U, (z1, z2))]
```

and to the atlas of the chart's domain:

```
sage: U.atlas()
[Chart (U, (z1, z2))]
```

Manifold subsets have a *default chart*, which, unless changed via the method `set_default_chart()`, is the first defined chart on the subset (or on a open subset of it):

```
sage: M.default_chart()
Chart (M, (x, y))
sage: U.default_chart()
Chart (U, (z1, z2))
```

The default charts are not privileged charts on the manifold, but rather charts whose name can be skipped in the argument list of functions having an optional `chart=` argument.

The chart map φ acting on a point is obtained by passing it as an input to the map:

```
sage: p = M.point((1+i, 2), chart=X); p
Point on the Complex 2-dimensional topological manifold M
sage: X(p)
(I + 1, 2)
sage: X(p) == p.coord(X)
True
```

See also:

`sage.manifolds.chart.RealChart` for charts on topological manifolds over $\mathbf{R}$.

add_restrictions (*restrictions*)

Add some restrictions on the coordinates.

INPUT:

- `restrictions` – list of restrictions on the coordinates, in addition to the ranges declared by the intervals specified in the chart constructor

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.add_restrictions(abs(x) > 1)
sage: X.valid_coordinates(2+i, 1)
True
sage: X.valid_coordinates(i, 1)
False
```

calculus_method ()

Return the interface governing the calculus engine for expressions involving coordinates of this chart.

The calculus engine can be one of the following:

- Sage's symbolic engine (Pynac + Maxima), implemented via the Symbolic Ring `SR`
- SymPy

See also:

`CalculusMethod` for a complete documentation.

OUTPUT:

- an instance of *`CalculusMethod`*

EXAMPLES:

The default calculus method relies on Sage's Symbolic Ring:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.calculus_method()
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

Accordingly the method *`expr()`* of a function `f` defined on the chart `X` returns a Sage symbolic expression:

```
sage: f = X.function(x^2 + cos(y)*sin(x))
sage: f.expr()
x^2 + cos(y)*sin(x)
sage: type(f.expr())
<type 'sage.symbolic.expression.Expression'>
sage: parent(f.expr())
Symbolic Ring
sage: f.display()
(x, y) |--> x^2 + cos(y)*sin(x)
```

Changing to SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.expr()
x**2 + sin(x)*cos(y)
sage: type(f.expr())
<class 'sympy.core.add.Add'>
sage: parent(f.expr())
<class 'sympy.core.add.Add'>
sage: f.display()
(x, y) |--> x**2 + sin(x)*cos(y)
```

Back to the Symbolic Ring:

```
sage: X.calculus_method().set('SR')
sage: f.display()
(x, y) |--> x^2 + cos(y)*sin(x)
```

domain()

Return the open subset on which the chart is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.domain()
2-dimensional topological manifold M
sage: U = M.open_subset('U')
sage: Y.<u,v> = U.chart()
sage: Y.domain()
Open subset U of the 2-dimensional topological manifold M
```

function(*expression*, *calc_method=None*, *expansion_symbol=None*, *order=None*)

Define a coordinate function to the base field.

If the current chart belongs to the atlas of a n-dimensional manifold over a topological field K, a *coordinate function* is a map

$$\begin{array}{llcl} f: & V \subset K^n & \longrightarrow & K \\ & (x^1, \ldots, x^n) & \longmapsto & f(x^1, \ldots, x^n), \end{array}$$

where V is the chart codomain and $(x^1, \ldots, x^n)$ are the chart coordinates.

INPUT:

- `expression` – a symbolic expression involving the chart coordinates, to represent $f(x^1, \ldots, x^n)$
- `calc_method` – string (default: `None`): the calculus method with respect to which the internal expression of the function must be initialized from `expression`; one of

    - 'SR': Sage's default symbolic engine (Symbolic Ring)
    - 'sympy': SymPy
    - None: the chart current calculus method is assumed

- expansion_symbol – (default: None) symbolic variable (the "small parameter") with respect to which the coordinate expression is expanded in power series (around the zero value of this variable)
- order – integer (default: None); the order of the expansion if expansion_symbol is not None; the *order* is defined as the degree of the polynomial representing the truncated power series in expansion_symbol.

Warning: The value of order is $n-1$, where n is the order of the big O in the power series expansion

OUTPUT:

- instance of *ChartFunction* representing the coordinate function f

EXAMPLES:

A symbolic coordinate function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(sin(x*y))
sage: f
sin(x*y)
sage: type(f)
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
→class'>
sage: f.display()
(x, y) |--> sin(x*y)
sage: f(2,3)
sin(6)
```

Using SymPy for the internal representation of the function (dictionary _express):

```
sage: g = X.function(x^2 + x*cos(y), calc_method='sympy')
sage: g._express
{'sympy': x**2 + x*cos(y)}
```

On the contrary, for f, only the SR part has been initialized:

```
sage: f._express
{'SR': sin(x*y)}
```

See *ChartFunction* for more examples.

function_ring()

Return the ring of coordinate functions on self.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.function_ring()
Ring of chart functions on Chart (M, (x, y))
```

manifold()

Return the manifold on which the chart is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: X.<x,y> = U.chart()
sage: X.manifold()
2-dimensional topological manifold M
sage: X.domain()
Open subset U of the 2-dimensional topological manifold M
```

multifunction(**expressions*)

Define a coordinate function to some Cartesian power of the base field.

If n and m are two positive integers and (U, φ) is a chart on a topological manifold M of dimension n over a topological field K, a *multi-coordinate function* associated to (U, φ) is a map

$$\begin{array}{llll} f: & V \subset K^n & \longrightarrow & K^m \\ & (x^1, \ldots, x^n) & \longmapsto & (f_1(x^1, \ldots, x^n), \ldots, f_m(x^1, \ldots, x^n)), \end{array}$$

where V is the codomain of φ. In other words, f is a K^m-valued function of the coordinates associated to the chart (U, φ).

See `MultiCoordFunction` for a complete documentation.

INPUT:

- `expressions` – list (or tuple) of m elements to construct the coordinate functions f_i ($1 \leq i \leq m$); for symbolic coordinate functions, this must be symbolic expressions involving the chart coordinates, while for numerical coordinate functions, this must be data file names

OUTPUT:

- a `MultiCoordFunction` representing f

EXAMPLES:

Function of two coordinates with values in $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x+y, sin(x*y), x^2 + 3*y); f
Coordinate functions (x + y, sin(x*y), x^2 + 3*y) on the Chart (M, (x, y))
sage: f(2,3)
(5, sin(6), 13)
```

one_function()

Return the constant function of the coordinates equal to one.

If the current chart belongs to the atlas of a n-dimensional manifold over a topological field K, the "one" coordinate function is the map

$$\begin{array}{llll} f: & V \subset K^n & \longrightarrow & K \\ & (x^1, \ldots, x^n) & \longmapsto & 1, \end{array}$$

where V is the chart codomain.

See class `ChartFunction` for a complete documentation.

OUTPUT:

- a `ChartFunction` representing the one coordinate function f

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.one_function()
1
sage: X.one_function().display()
(x, y) |--> 1
sage: type(X.one_function())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

The result is cached:

```
sage: X.one_function() is X.one_function()
True
```

One function on a p-adic manifold:

```
sage: M = Manifold(2, 'M', structure='topological', field=Qp(5)); M
2-dimensional topological manifold M over the 5-adic Field with
 capped relative precision 20
sage: X.<x,y> = M.chart()
sage: X.one_function()
1 + O(5^20)
sage: X.one_function().display()
(x, y) |--> 1 + O(5^20)
```

periods()

Return the coordinate periods as a dictionary, possibly empty if no coordinate is periodic.

OUTPUT:

- a dictionary with keys the indices of the periodic coordinates and with values the periods.

EXAMPLES:

A chart without any periodic coordinate:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.periods()
{}
```

Charts with a periodic coordinate:

```
sage: Y.<u,v> = M.chart("u v:(0,2*pi):periodic")
sage: Y.periods()
{1: 2*pi}
sage: Z.<a,b> = M.chart(r"a:period=sqrt(2):\alpha b:\beta")
sage: Z.periods()
{0: sqrt(2)}
```

The key in the output dictionary takes into account the index range declared on the manifold with `start_index`:

```
sage: M = Manifold(2, 'M', structure='topological', start_index=1)
sage: Y.<u,v> = M.chart("u v:(0,2*pi):periodic")
sage: Y[2]
v
sage: Y.periods()
{2: 2*pi}
sage: Z.<a,b> = M.chart(r"a:period=sqrt(2):\alpha b:\beta")
sage: Z[1]
a
sage: Z.periods()
{1: sqrt(2)}
```

Complex manifod with a periodic coordinate:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological',
....:              start_index=1)
sage: X.<x,y> = M.chart("x y:period=1+I")
sage: X.periods()
{2: I + 1}
```

restrict (*subset*, *restrictions=None*)

Return the restriction of `self` to some open subset of its domain.

If the current chart is (U, φ), a *restriction* (or *subchart*) is a chart (V, ψ) such that $V \subset U$ and $\psi = \varphi|_V$.

If such subchart has not been defined yet, it is constructed here.

The coordinates of the subchart bare the same names as the coordinates of the current chart.

INPUT:

- `subset` – open subset V of the chart domain U (must be an instance of `TopologicalManifold`)
- `restrictions` – (default: `None`) list of coordinate restrictions defining the subset V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- chart (V, ψ) as a `Chart`

EXAMPLES:

Coordinates on the unit open ball of $\mathbf{C}^2$ as a subchart of the global coordinates of $\mathbf{C}^2$:

```
sage: M = Manifold(2, 'C^2', field='complex', structure='topological')
sage: X.<z1, z2> = M.chart()
sage: B = M.open_subset('B')
sage: X_B = X.restrict(B, abs(z1)^2 + abs(z2)^2 < 1); X_B
Chart (B, (z1, z2))
```

transition_map(*other*, *transformations*, *intersection_name=None*, *restrictions1=None*, *restrictions2=None*)

Construct the transition map between the current chart, (U, φ) say, and another one, (V, ψ) say.

If n is the manifold's dimension, the *transition map* is the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset K^n \to \psi(U \cap V) \subset K^n,$$

where K is the manifold's base field. In other words, the transition map expresses the coordinates $(y^1, \ldots, y^n)$ of (V, ψ) in terms of the coordinates $(x^1, \ldots, x^n)$ of (U, φ) on the open subset where the two charts intersect, i.e. on $U \cap V$.

INPUT:

- `other` – the chart (V, ψ)
- `transformations` – tuple (or list) $(Y_1, \ldots, Y_n)$, where Y_i is the symbolic expression of the coordinate y^i in terms of the coordinates $(x^1, \ldots, x^n)$
- `intersection_name` – (default: `None`) name to be given to the subset $U \cap V$ if the latter differs from U or V
- `restrictions1` – (default: `None`) list of conditions on the coordinates of the current chart that define $U \cap V$ if the latter differs from U
- `restrictions2` – (default: `None`) list of conditions on the coordinates of the chart (V, ψ) that define $U \cap V$ if the latter differs from V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- the transition map $\psi \circ \varphi^{-1}$ defined on $U \cap V$ as a `CoordChange`

EXAMPLES:

Transition map between two stereographic charts on the circle S^1:

```
sage: M = Manifold(1, 'S^1', structure='topological')
sage: U = M.open_subset('U') # Complement of the North pole
sage: cU.<x> = U.chart() # Stereographic chart from the North pole
sage: V = M.open_subset('V') # Complement of the South pole
sage: cV.<y> = V.chart() # Stereographic chart from the South pole
sage: M.declare_union(U,V)    # S^1 is the union of U and V
sage: trans = cU.transition_map(cV, 1/x, intersection_name='W',
....:                           restrictions1= x!=0, restrictions2 = y!=0)
sage: trans
Change of coordinates from Chart (W, (x,)) to Chart (W, (y,))
sage: trans.display()
y = 1/x
```

The subset W, intersection of U and V, has been created by `transition_map()`:

```
sage: M.list_of_subsets()
[1-dimensional topological manifold S^1,
 Open subset U of the 1-dimensional topological manifold S^1,
 Open subset V of the 1-dimensional topological manifold S^1,
 Open subset W of the 1-dimensional topological manifold S^1]
sage: W = M.list_of_subsets()[3]
sage: W is U.intersection(V)
True
sage: M.atlas()
[Chart (U, (x,)), Chart (V, (y,)), Chart (W, (x,)), Chart (W, (y,))]
```

Transition map between the spherical chart and the Cartesian one on $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart()
sage: U = M.open_subset('U') # the complement of the half line {y=0, x >= 0}
sage: c_spher.<r,phi> = U.chart(r'r:(0,+oo) phi:(0,2*pi):\phi')
sage: trans = c_spher.transition_map(c_cart, (r*cos(phi), r*sin(phi)),
....:                                restrictions2=(y!=0, x<0))
sage: trans
Change of coordinates from Chart (U, (r, phi)) to Chart (U, (x, y))
sage: trans.display()
x = r*cos(phi)
y = r*sin(phi)
```

In this case, no new subset has been created since $U \cap M = U$:

```
sage: M.list_of_subsets()
[2-dimensional topological manifold R^2,
 Open subset U of the 2-dimensional topological manifold R^2]
```

but a new chart has been created: $(U, (x, y))$:

```
sage: M.atlas()
[Chart (R^2, (x, y)), Chart (U, (r, phi)), Chart (U, (x, y))]
```

valid_coordinates(**coordinates*, ***kwds*)

Check whether a tuple of coordinates can be the coordinates of a point in the chart domain.

INPUT:

- `*coordinates` – coordinate values
- `**kwds` – options:
  - `parameters=None`, dictionary to set numerical values to some parameters (see example below)

OUTPUT:

- `True` if the coordinate values are admissible in the chart image, `False` otherwise

EXAMPLES:

```
sage: M = Manifold(2, 'M', field='complex', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.add_restrictions([abs(x)<1, y!=0])
sage: X.valid_coordinates(0, i)
True
```

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```
sage: X.valid_coordinates(i, 1)
False
sage: X.valid_coordinates(i/2, 1)
True
sage: X.valid_coordinates(i/2, 0)
False
sage: X.valid_coordinates(2, 0)
False
```

Example of use with the keyword `parameters` to set a specific value to a parameter appearing in the coordinate restrictions:

```
sage: var('a')  # the parameter is a symbolic variable
a
sage: Y.<u,v> = M.chart()
sage: Y.add_restrictions(abs(v)<a)
sage: Y.valid_coordinates(1, i, parameters={a: 2})  # setting a=2
True
sage: Y.valid_coordinates(1, 2*i, parameters={a: 2})
False
```

zero_function()

Return the zero function of the coordinates.

If the current chart belongs to the atlas of a n-dimensional manifold over a topological field K, the zero coordinate function is the map

$$f: \begin{array}{ccc} V\subset K^n & \longrightarrow & K \\ (x^1,\ldots,x^n) & \longmapsto & 0, \end{array}$$

where V is the chart codomain.

See class *ChartFunction* for a complete documentation.

OUTPUT:

- a *ChartFunction* representing the zero coordinate function f

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.zero_function()
0
sage: X.zero_function().display()
(x, y) |--> 0
sage: type(X.zero_function())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

The result is cached:

```
sage: X.zero_function() is X.zero_function()
True
```

Zero function on a p-adic manifold:

```
sage: M = Manifold(2, 'M', structure='topological', field=Qp(5)); M
2-dimensional topological manifold M over the 5-adic Field with
 capped relative precision 20
sage: X.<x,y> = M.chart()
sage: X.zero_function()
0
sage: X.zero_function().display()
(x, y) |--> 0
```

class sage.manifolds.chart.**CoordChange**(*chart1*, *chart2*, **transformations*)

Bases: sage.structure.sage_object.SageObject

Transition map between two charts of a topological manifold.

Giving two coordinate charts (U, φ) and (V, ψ) on a topological manifold M of dimension n over a topological field K, the *transition map from* (U, φ) *to* (V, ψ) is the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset K^n \to \psi(U \cap V) \subset K^n.$$

In other words, the transition map $\psi \circ \varphi^{-1}$ expresses the coordinates $(y^1, \ldots, y^n)$ of (V, ψ) in terms of the coordinates $(x^1, \ldots, x^n)$ of (U, φ) on the open subset where the two charts intersect, i.e. on $U \cap V$.

INPUT:

- `chart1` – chart (U, φ)
- `chart2` – chart (V, ψ)
- `transformations` – tuple (or list) $(Y_1, \ldots, Y_2)$, where Y_i is the symbolic expression of the coordinate y^i in terms of the coordinates $(x^1, \ldots, x^n)$

EXAMPLES:

Transition map on a 2-dimensional topological manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
sage: X_to_Y
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: type(X_to_Y)
<class 'sage.manifolds.chart.CoordChange'>
sage: X_to_Y.display()
u = x + y
v = x - y
```

disp()

Display of the coordinate transformation.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

From spherical coordinates to Cartesian ones in the plane:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: U = M.open_subset('U') # the complement of the half line {y=0, x>= 0}
sage: c_cart.<x,y> = U.chart()
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
```

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```
sage: spher_to_cart.display()
x = r*cos(ph)
y = r*sin(ph)
sage: latex(spher_to_cart.display())
\left\{\begin{array}{lcl} x & = & r \cos\left({\phi}\right) \\
 y & = & r \sin\left({\phi}\right) \end{array}\right.
```

A shortcut is `disp()`:

```
sage: spher_to_cart.disp()
x = r*cos(ph)
y = r*sin(ph)
```

display()

Display of the coordinate transformation.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

From spherical coordinates to Cartesian ones in the plane:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: U = M.open_subset('U') # the complement of the half line {y=0, x>= 0}
sage: c_cart.<x,y> = U.chart()
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
sage: spher_to_cart.display()
x = r*cos(ph)
y = r*sin(ph)
sage: latex(spher_to_cart.display())
\left\{\begin{array}{lcl} x & = & r \cos\left({\phi}\right) \\
 y & = & r \sin\left({\phi}\right) \end{array}\right.
```

A shortcut is `disp()`:

```
sage: spher_to_cart.disp()
x = r*cos(ph)
y = r*sin(ph)
```

inverse()

Compute the inverse coordinate transformation.

OUTPUT:

- an instance of *CoordChange* representing the inverse of the current coordinate transformation

EXAMPLES:

Inverse of a coordinate transformation corresponding to a rotation in the Cartesian plane:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: phi = var('phi', domain='real')
sage: xy_to_uv = c_xy.transition_map(c_uv,
....:                                [cos(phi)*x + sin(phi)*y,
....:                                 -sin(phi)*x + cos(phi)*y])
sage: M.coord_changes()
```

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```
{(Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
sage: uv_to_xy = xy_to_uv.inverse(); uv_to_xy
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: uv_to_xy.display()
x = u*cos(phi) - v*sin(phi)
y = v*cos(phi) + u*sin(phi)
sage: M.coord_changes()  # random (dictionary output)
{(Chart (M, (u, v)),
  Chart (M, (x, y))): Change of coordinates from Chart (M, (u, v)) to Chart␣
↪(M, (x, y)),
 (Chart (M, (x, y)),
  Chart (M, (u, v))): Change of coordinates from Chart (M, (x, y)) to Chart␣
↪(M, (u, v))}
```

restrict (*dom1*, *dom2=None*)

Restriction to subsets.

INPUT:

- `dom1` – open subset of the domain of `chart1`
- `dom2` – (default: `None`) open subset of the domain of `chart2`; if `None`, `dom1` is assumed

OUTPUT:

- the transition map between the charts restricted to the specified subsets

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
sage: U = M.open_subset('U', coord_def={X: x>0, Y: u+v>0})
sage: X_to_Y_U = X_to_Y.restrict(U); X_to_Y_U
Change of coordinates from Chart (U, (x, y)) to Chart (U, (u, v))
sage: X_to_Y_U.display()
u = x + y
v = x - y
```

The result is cached:

```
sage: X_to_Y.restrict(U) is X_to_Y_U
True
```

set_inverse (**transformations*, ***kwds*)

Sets the inverse of the coordinate transformation.

This is useful when the automatic computation via `inverse()` fails.

INPUT:

- `transformations` – the inverse transformations expressed as a list of the expressions of the "old" coordinates in terms of the "new" ones
- `kwds` – keyword arguments: only `verbose=True` or `verbose=False` (default) are meaningful; it determines whether the provided transformations are checked to be indeed the inverse coordinate transformations

EXAMPLES:

From spherical coordinates to Cartesian ones in the plane:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: U = M.open_subset('U') # the complement of the half line {y=0, x>= 0}
sage: c_cart.<x,y> = U.chart()
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
sage: spher_to_cart.set_inverse(sqrt(x^2+y^2), atan2(y,x))
sage: spher_to_cart.inverse()
Change of coordinates from Chart (U, (x, y)) to Chart (U, (r, ph))
sage: spher_to_cart.inverse().display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
sage: M.coord_changes()  # random (dictionary output)
{(Chart (U, (r, ph)),
  Chart (U, (x, y))): Change of coordinates from Chart (U, (r, ph)) to Chart␣
↪(U, (x, y)),
 (Chart (U, (x, y)),
  Chart (U, (r, ph))): Change of coordinates from Chart (U, (x, y)) to Chart␣
↪(U, (r, ph))}
```

Introducing a wrong inverse transformation (note the `x^3` typo) is revealed by setting `verbose` to `True`:

```
sage: spher_to_cart.set_inverse(sqrt(x^3+y^2), atan2(y,x), verbose=True)
Check of the inverse coordinate transformation:
   r == sqrt(r*cos(ph)^3 + sin(ph)^2)*r
   ph == arctan2(r*sin(ph), r*cos(ph))
   x == sqrt(x^3 + y^2)*x/sqrt(x^2 + y^2)
   y == sqrt(x^3 + y^2)*y/sqrt(x^2 + y^2)
```

class `sage.manifolds.chart.`**`RealChart`**(*domain*, *coordinates=''*, *names=None*, *calc_method=None*)

Bases: *`sage.manifolds.chart.Chart`*

Chart on a topological manifold over $\mathbf{R}$.

Given a topological manifold M of dimension n over $\mathbf{R}$, a *chart* on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to V \subset \mathbf{R}^n$ is a homeomorphism from U to an open subset V of $\mathbf{R}^n$.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in \mathbf{R}^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

INPUT:

- `domain` – open subset U on which the chart is defined
- `coordinates` – (default: '' (empty string)) single string defining the coordinate symbols, with `' '` (whitespace) as a separator; each item has at most four fields, separated by a colon (`:`):
  1. the coordinate symbol (a letter or a few letters)
  2. (optional) the interval I defining the coordinate range: if not provided, the coordinate is assumed to span all $\mathbf{R}$; otherwise I must be provided in the form `(a,b)` (or equivalently `]a,b[`); the bounds `a` and `b` can be `+/-Infinity`, `Inf`, `infinity`, `inf` or `oo`; for *singular* coordinates, non-open intervals such as `[a,b]` and `(a,b]` (or equivalently `]a,b]`) are allowed; note that the interval declaration must not contain any whitespace
  3. (optional) indicator of the periodic character of the coordinate, either as `period=T`, where `T` is the period, or as the keyword `periodic` (the value of the period is then deduced from the interval I declared in field 2; see examples below)

4. (optional) the LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used

The order of fields 2 to 4 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of LaTeX's backslash character (see examples below). If interval range, no period and no LaTeX spelling are to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).

- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)
- `calc_method` – (default: `None`) string defining the calculus method for computations involving coordinates of the chart; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the default of `CalculusMethod` will be used

EXAMPLES:

Cartesian coordinates on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', structure='topological',
....:              start_index=1)
sage: c_cart = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
sage: type(c_cart)
<class 'sage.manifolds.chart.RealChart'>
```

To have the coordinates accessible as global variables, one has to set:

```
sage: (x,y,z) = c_cart[:]
```

However, a shortcut is to use the declarator `<x,y,z>` in the left-hand side of the chart declaration (there is then no need to pass the string `'x y z'` to `chart()`):

```
sage: M = Manifold(3, 'R^3', r'\RR^3', structure='topological',
....:              start_index=1)
sage: c_cart.<x,y,z> = M.chart(); c_cart
Chart (R^3, (x, y, z))
```

The coordinates are then immediately accessible:

```
sage: y
y
sage: y is c_cart[2]
True
```

Note that `x, y, z` declared in `<x,y,z>` are mere Python variable names and do not have to coincide with the coordinate symbols; for instance, one may write:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', structure='topological',
....:              start_index=1)
sage: c_cart.<x1,y1,z1> = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
```

Then `y` is not known as a global variable and the coordinate y is accessible only through the global variable `y1`:

```
sage: y1
y
sage: y1 is c_cart[2]
True
```

However, having the name of the Python variable coincide with the coordinate symbol is quite convenient; so it is recommended to declare:

```
sage: forget()   # for doctests only
sage: M = Manifold(3, 'R^3', r'\RR^3', structure='topological', start_index=1)
sage: c_cart.<x,y,z> = M.chart()
```

Spherical coordinates on the subset U of $\mathbf{R}^3$ that is the complement of the half-plane $\{y = 0, x \geq 0\}$:

```
sage: U = M.open_subset('U')
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: c_spher
Chart (U, (r, th, ph))
```

Note the prefix ‘r’ for the string defining the coordinates in the arguments of `chart`.

Coordinates are Sage symbolic variables (see `sage.symbolic.expression`):

```
sage: type(th)
<type 'sage.symbolic.expression.Expression'>
sage: latex(th)
{\theta}
sage: assumptions(th)
[th is real, th > 0, th < pi]
```

Coordinate are also accessible by their indices:

```
sage: x1 = c_spher[1]; x2 = c_spher[2]; x3 = c_spher[3]
sage: [x1, x2, x3]
[r, th, ph]
sage: (x1, x2, x3) == (r, th, ph)
True
```

The full set of coordinates is obtained by means of the slice `[:]`:

```
sage: c_cart[:]
(x, y, z)
sage: c_spher[:]
(r, th, ph)
```

Let us check that the declared coordinate ranges have been taken into account:

```
sage: c_cart.coord_range()
x: (-oo, +oo); y: (-oo, +oo); z: (-oo, +oo)
sage: c_spher.coord_range()
r: (0, +oo); th: (0, pi); ph: (0, 2*pi)
sage: bool(th>0 and th<pi)
True
sage: assumptions()   # list all current symbolic assumptions
[x is real, y is real, z is real, r is real, r > 0, th is real,
 th > 0, th < pi, ph is real, ph > 0, ph < 2*pi]
```

The coordinate ranges are used for simplifications:

```
sage: simplify(abs(r)) # r has been declared to lie in the interval (0,+oo)
r
sage: simplify(abs(x)) # no positive range has been declared for x
abs(x)
```

A coordinate can be declared periodic by adding the keyword `periodic` to its range:

```
sage: V = M.open_subset('V')
sage: c_spher1.<r,th,ph1> = \
....: V.chart(r'r:(0,+oo) th:(0,pi):\theta ph1:(0,2*pi):periodic:\phi_1')
sage: c_spher1.periods()
{3: 2*pi}
sage: c_spher1.coord_range()
r: (0, +oo); th: (0, pi); ph1: [0, 2*pi] (periodic)
```

It is equivalent to give the period as `period=2*pi`, skipping the coordinate range:

```
sage: c_spher2.<r,th,ph2> = \
....: V.chart(r'r:(0,+oo) th:(0,pi):\theta ph2:period=2*pi:\phi_2')
sage: c_spher2.periods()
{3: 2*pi}
sage: c_spher2.coord_range()
r: (0, +oo); th: (0, pi); ph2: [0, 2*pi] (periodic)
```

Each constructed chart is automatically added to the manifold's user atlas:

```
sage: M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2))]
```

and to the atlas of its domain:

```
sage: U.atlas()
[Chart (U, (r, th, ph))]
```

Manifold subsets have a *default chart*, which, unless changed via the method `set_default_chart()`, is the first defined chart on the subset (or on a open subset of it):

```
sage: M.default_chart()
Chart (R^3, (x, y, z))
sage: U.default_chart()
Chart (U, (r, th, ph))
```

The default charts are not privileged charts on the manifold, but rather charts whose name can be skipped in the argument list of functions having an optional `chart=` argument.

The chart map φ acting on a point is obtained by means of the call operator, i.e. the operator `()`:

```
sage: p = M.point((1,0,-2)); p
Point on the 3-dimensional topological manifold R^3
sage: c_cart(p)
(1, 0, -2)
sage: c_cart(p) == p.coord(c_cart)
True
sage: q = M.point((2,pi/2,pi/3), chart=c_spher) # point defined by its spherical
↪coordinates
sage: c_spher(q)
```

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```
(2, 1/2*pi, 1/3*pi)
sage: c_spher(q) == q.coord(c_spher)
True
sage: a = U.point((1,pi/2,pi)) # the default coordinates on U are the spherical␣
↪ones
sage: c_spher(a)
(1, 1/2*pi, pi)
sage: c_spher(a) == a.coord(c_spher)
True
```

Cartesian coordinates on U as an example of chart construction with coordinate restrictions: since U is the complement of the half-plane $\{y = 0, x \geq 0\}$, we must have $y \neq 0$ or $x < 0$ on U. Accordingly, we set:

```
sage: c_cartU.<x,y,z> = U.chart()
sage: c_cartU.add_restrictions((y!=0, x<0))
sage: U.atlas()
[Chart (U, (r, th, ph)), Chart (U, (x, y, z))]
sage: M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2)),
 Chart (U, (x, y, z))]
sage: c_cartU.valid_coordinates(-1,0,2)
True
sage: c_cartU.valid_coordinates(1,0,2)
False
sage: c_cart.valid_coordinates(1,0,2)
True
```

Note that, as an example, the following would have meant $y \neq 0$ *and* $x < 0$:

```
c_cartU.add_restrictions([y!=0, x<0])
```

Chart grids can be drawn in 2D or 3D graphics thanks to the method `plot()`.

add_restrictions (*restrictions*)

Add some restrictions on the coordinates.

INPUT:

- `restrictions` – list of restrictions on the coordinates, in addition to the ranges declared by the intervals specified in the chart constructor

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

EXAMPLES:

Cartesian coordinates on the open unit disc in $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'M', structure='topological') # the open unit disc
sage: X.<x,y> = M.chart()
sage: X.add_restrictions(x^2+y^2<1)
sage: X.valid_coordinates(0,2)
False
sage: X.valid_coordinates(0,1/3)
True
```

The restrictions are transmitted to subcharts:

```
sage: A = M.open_subset('A') # annulus 1/2 < r < 1
sage: X_A = X.restrict(A, x^2+y^2 > 1/4)
sage: X_A._restrictions
[x^2 + y^2 < 1, x^2 + y^2 > (1/4)]
sage: X_A.valid_coordinates(0,1/3)
False
sage: X_A.valid_coordinates(2/3,1/3)
True
```

If appropriate, the restrictions are transformed into bounds on the coordinate ranges:

```
sage: U = M.open_subset('U')
sage: X_U = X.restrict(U)
sage: X_U.coord_range()
x: (-oo, +oo); y: (-oo, +oo)
sage: X_U.add_restrictions([x<0, y>1/2])
sage: X_U.coord_range()
x: (-oo, 0); y: (1/2, +oo)
```

coord_bounds (*i=None*)

Return the lower and upper bounds of the range of a coordinate.

For a nicely formatted output, use `coord_range()` instead.

INPUT:

- `i` – (default: `None`) index of the coordinate; if `None`, the bounds of all the coordinates are returned

OUTPUT:

- the coordinate bounds as the tuple `((xmin, min_included), (xmax, max_included))` where
  - `xmin` is the coordinate lower bound
  - `min_included` is a boolean, indicating whether the coordinate can take the value `xmin`, i.e. `xmin` is a strict lower bound iff `min_included` is `False`
  - `xmin` is the coordinate upper bound
  - `max_included` is a boolean, indicating whether the coordinate can take the value `xmax`, i.e. `xmax` is a strict upper bound iff `max_included` is `False`

EXAMPLES:

Some coordinate bounds on a 2-dimensional manifold:

```
sage: forget()  # for doctests only
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart('x y:[0,1)')
sage: c_xy.coord_bounds(0)  # x in (-oo,+oo) (the default)
```

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```
((-Infinity, False), (+Infinity, False))
sage: c_xy.coord_bounds(1)  # y in [0,1)
((0, True), (1, False))
sage: c_xy.coord_bounds()
(((-Infinity, False), (+Infinity, False)), ((0, True), (1, False)))
sage: c_xy.coord_bounds() == (c_xy.coord_bounds(0), c_xy.coord_bounds(1))
True
```

The coordinate bounds can also be recovered via the method `coord_range()`:

```
sage: c_xy.coord_range()
x: (-oo, +oo); y: [0, 1)
sage: c_xy.coord_range(y)
y: [0, 1)
```

or via Sage's function `sage.symbolic.assumptions.assumptions()`:

```
sage: assumptions(x)
[x is real]
sage: assumptions(y)
[y is real, y >= 0, y < 1]
```

coord_range(*xx=None*)

Display the range of a coordinate (or all coordinates), as an interval.

INPUT:

- `xx` – (default: `None`) symbolic expression corresponding to a coordinate of the current chart; if `None`, the ranges of all coordinates are displayed

EXAMPLES:

Ranges of coordinates on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: X.coord_range()
x: (-oo, +oo); y: (-oo, +oo)
sage: X.coord_range(x)
x: (-oo, +oo)
sage: U = M.open_subset('U', coord_def={X: [x>1, y<pi]})
sage: XU = X.restrict(U)  # restriction of chart X to U
sage: XU.coord_range()
x: (1, +oo); y: (-oo, pi)
sage: XU.coord_range(x)
x: (1, +oo)
sage: XU.coord_range(y)
y: (-oo, pi)
```

The output is LaTeX-formatted for the notebook:

```
sage: latex(XU.coord_range(y))
y :\ \left( -\infty, \pi \right)
```

plot(*chart=None, ambient_coords=None, mapping=None, fixed_coords=None, ranges=None, number_values=None, steps=None, parameters=None, max_range=8, style='-', label_axes=True, color='red', plot_points=75, thickness=1, **kwds*)

Plot `self` as a grid in a Cartesian graph based on the coordinates of some ambient chart.

The grid is formed by curves along which a chart coordinate varies, the other coordinates being kept fixed. It is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of another chart, called hereafter the *ambient chart*.

The ambient chart is related to the current chart either by a transition map if both charts are defined on the same manifold, or by the coordinate expression of some continuous map (typically an immersion). In the latter case, the two charts may be defined on two different manifolds.

INPUT:

- `chart` – (default: `None`) the ambient chart (see above); if `None`, the ambient chart is set to the current chart
- `ambient_coords` – (default: `None`) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if `None`, all the coordinates of the ambient chart are considered
- `mapping` – (default: `None`) *ContinuousMap*; continuous manifold map providing the link between the current chart and the ambient chart (cf. above); if `None`, both charts are supposed to be defined on the same manifold and related by some transition map (see `transition_map()`)
- `fixed_coords` – (default: `None`) dictionary with keys the chart coordinates that are not drawn and with values the fixed value of these coordinates; if `None`, all the coordinates of the current chart are drawn
- `ranges` – (default: `None`) dictionary with keys the coordinates to be drawn and values tuples `(x_min, x_max)` specifying the coordinate range for the plot; if `None`, the entire coordinate range declared during the chart construction is considered (with `-Infinity` replaced by `-max_range` and `+Infinity` by `max_range`)
- `number_values` – (default: `None`) either an integer or a dictionary with keys the coordinates to be drawn and values the number of constant values of the coordinate to be considered; if `number_values` is a single integer, it represents the number of constant values for all coordinates; if `number_values` is `None`, it is set to 9 for a 2D plot and to 5 for a 3D plot
- `steps` – (default: `None`) dictionary with keys the coordinates to be drawn and values the step between each constant value of the coordinate; if `None`, the step is computed from the coordinate range (specified in `ranges`) and `number_values`. On the contrary if the step is provided for some coordinate, the corresponding number of constant values is deduced from it and the coordinate range.
- `parameters` – (default: `None`) dictionary giving the numerical values of the parameters that may appear in the relation between the two coordinate systems
- `max_range` – (default: 8) numerical value substituted to +Infinity if the latter is the upper bound of the range of a coordinate for which the plot is performed over the entire coordinate range (i.e. for which no specific plot range has been set in `ranges`); similarly `-max_range` is the numerical valued substituted for `-Infinity`
- `color` – (default: `'red'`) either a single color or a dictionary of colors, with keys the coordinates to be drawn, representing the colors of the lines along which the coordinate varies, the other being kept constant; if `color` is a single color, it is used for all coordinate lines
- `style` – (default: `'-'`) either a single line style or a dictionary of line styles, with keys the coordinates to be drawn, representing the style of the lines along which the coordinate varies, the other being kept constant; if `style` is a single style, it is used for all coordinate lines; NB: `style` is effective only for 2D plots
- `thickness` – (default: 1) either a single line thickness or a dictionary of line thicknesses, with keys the coordinates to be drawn, representing the thickness of the lines along which the coordinate varies, the other being kept constant; if `thickness` is a single value, it is used for all coordinate lines

- plot_points – (default: 75) either a single number of points or a dictionary of integers, with keys the coordinates to be drawn, representing the number of points to plot the lines along which the coordinate varies, the other being kept constant; if plot_points is a single integer, it is used for all coordinate lines
- label_axes – (default: True) boolean determining whether the labels of the ambient coordinate axes shall be added to the graph; can be set to False if the graph is 3D and must be superposed with another graph

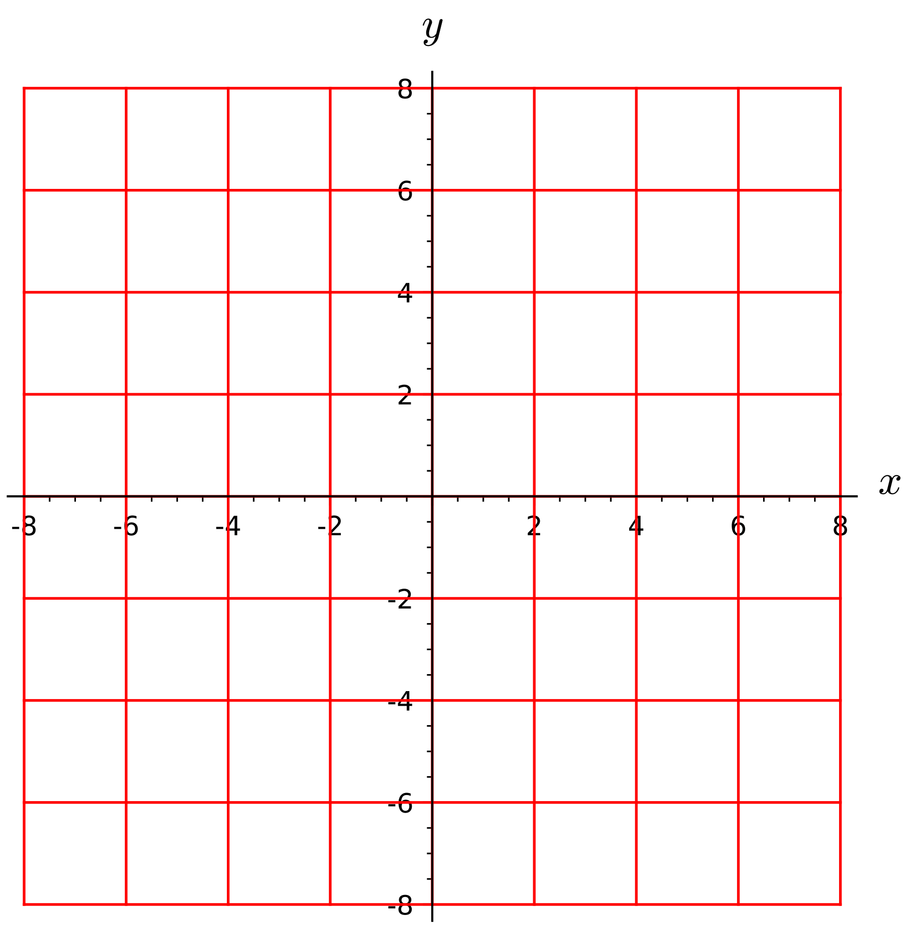
OUTPUT:

- a graphic object, either a Graphics for a 2D plot (i.e. based on 2 coordinates of the ambient chart) or a Graphics3d for a 3D plot (i.e. based on 3 coordinates of the ambient chart)

EXAMPLES:

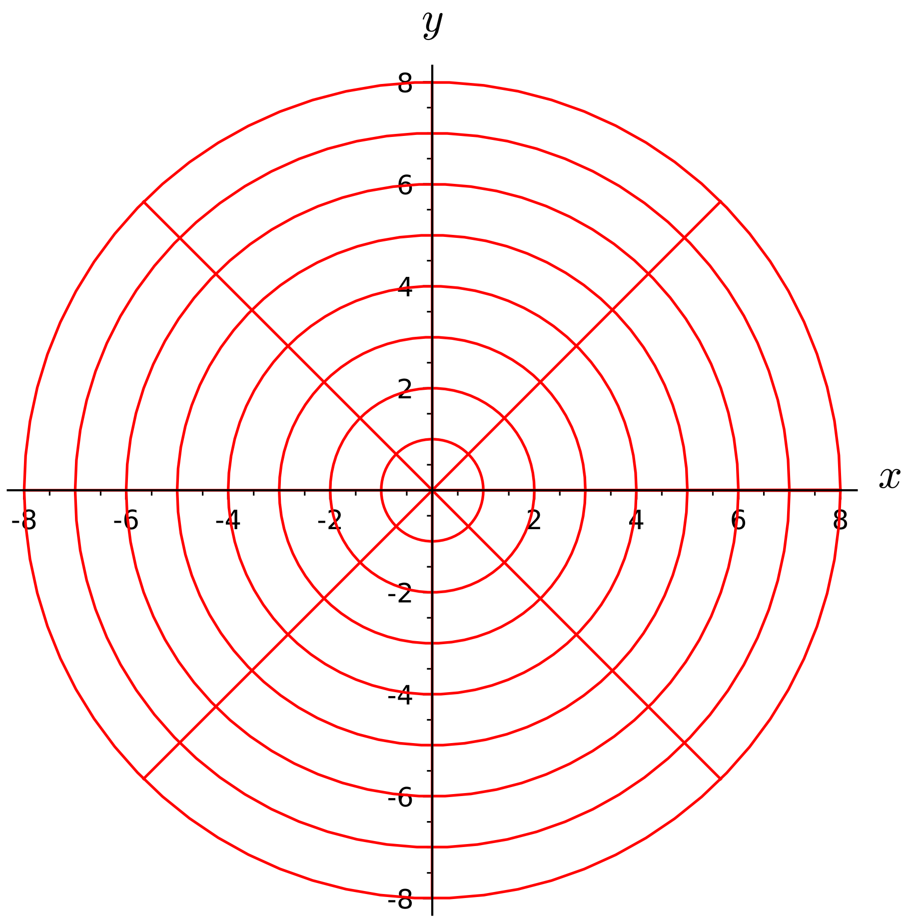
A 2-dimensional chart plotted in terms of itself results in a rectangular grid:

```
sage: R2 = Manifold(2, 'R^2', structure='topological') # the Euclidean plane
sage: c_cart.<x,y> = R2.chart() # Cartesian coordinates
sage: g = c_cart.plot()  # equivalent to c_cart.plot(c_cart)
sage: g
Graphics object consisting of 18 graphics primitives
```



Grid of polar coordinates in terms of Cartesian coordinates in the Euclidean plane:

```
sage: U = R2.open_subset('U', coord_def={c_cart: (y!=0, x<0)}) # the␣
↪complement of the segment y=0 and x>0
sage: c_pol.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # polar␣
↪coordinates on U
sage: pol_to_cart = c_pol.transition_map(c_cart, [r*cos(ph), r*sin(ph)])
sage: g = c_pol.plot(c_cart)
sage: g
Graphics object consisting of 18 graphics primitives
```



Call with non-default values:

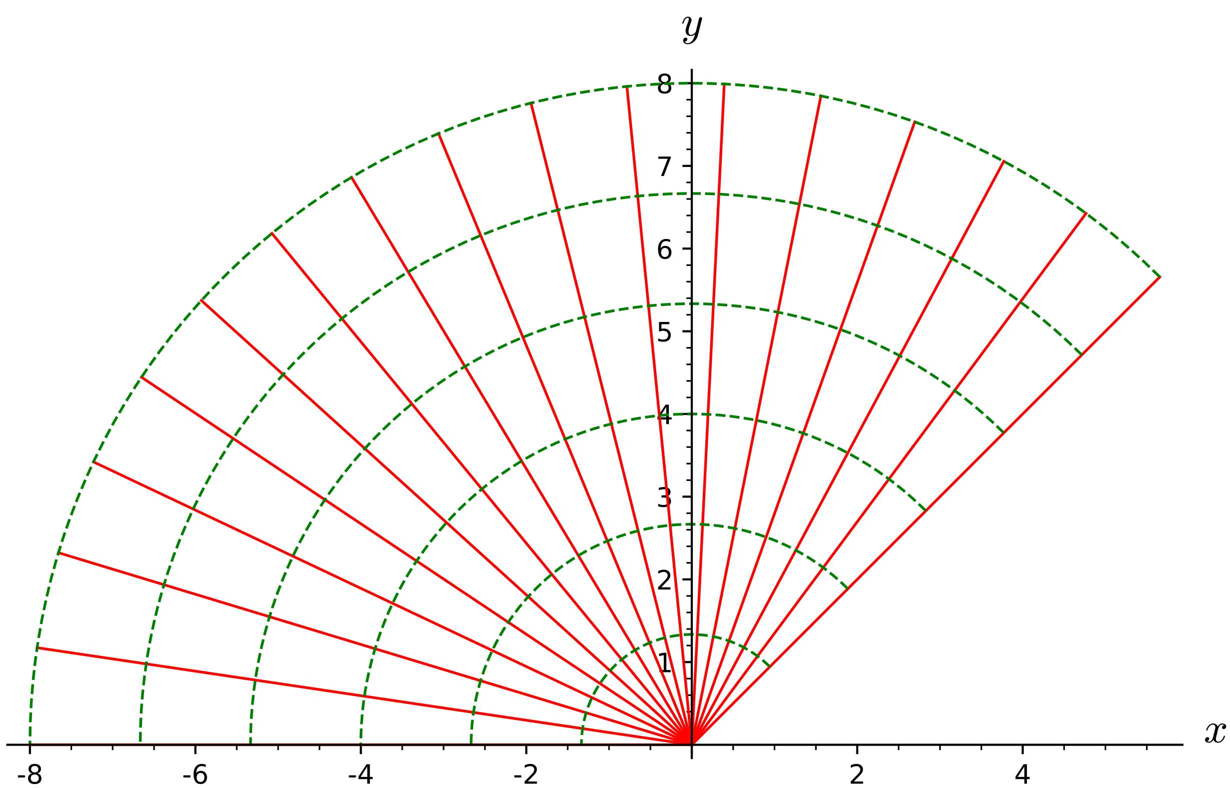
```
sage: g = c_pol.plot(c_cart, ranges={ph:(pi/4,pi)},
....:                number_values={r:7, ph:17},
....:                color={r:'red', ph:'green'},
....:                style={r:'-', ph:'--'})
```

A single coordinate line can be drawn:

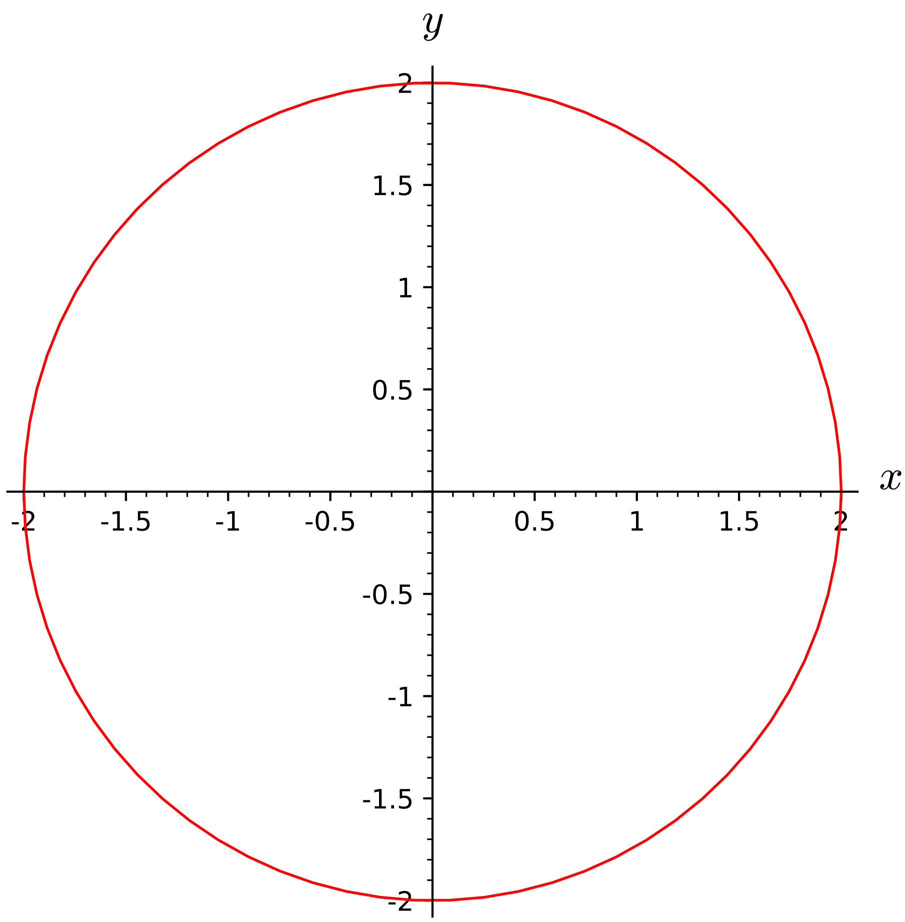
```
sage: g = c_pol.plot(c_cart, fixed_coords={r: 2}) # draw a circle of radius␣
↪r=2
```

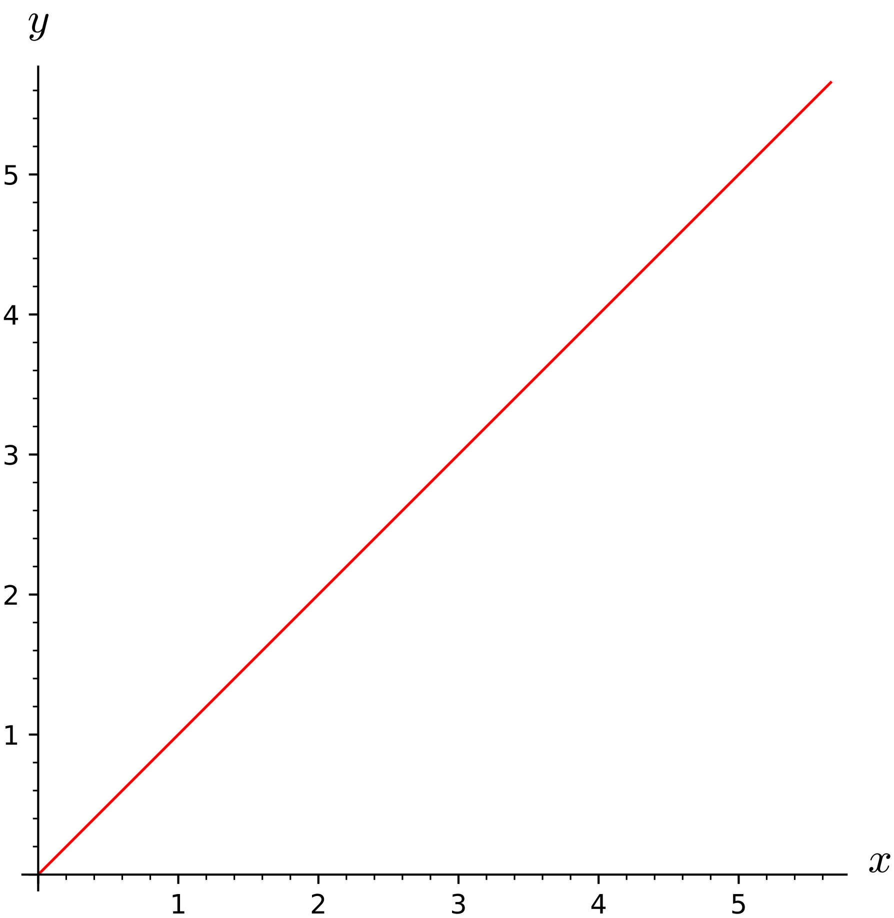
```
sage: g = c_pol.plot(c_cart, fixed_coords={ph: pi/4}) # draw a segment at␣
↪phi=pi/4
```

An example with the ambient chart lying in an another manifold (the plot is then performed via some



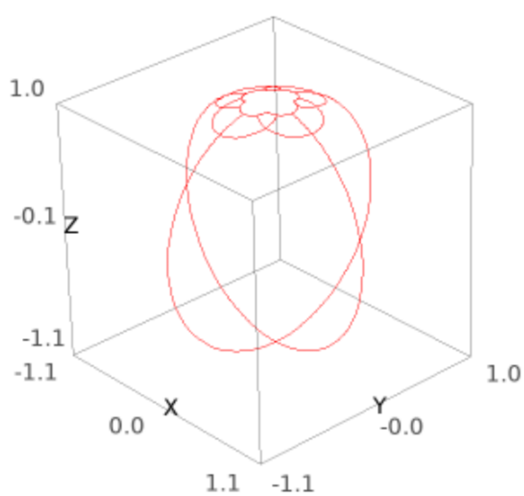
y
x
8
7
6
5
4
3
2
1
-8
-6
-4
-2
2
4


y
x
2
1.5
1
0.5
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0.5
1
1.5
2


y
5
4
3
2
1
x
1
2
3
4
5

manifold map passed as the argument `mapping`): 3D plot of the stereographic charts on the 2-sphere:

```
sage: S2 = Manifold(2, 'S^2', structure='topological') # the 2-sphere
sage: U = S2.open_subset('U') ; V = S2.open_subset('V') # complement of the␣
↪North and South pole, respectively
sage: S2.declare_union(U,V)
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                   intersection_name='W', restrictions1= x^2+y^2!=0,
....:                   restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: R3 = Manifold(3, 'R^3', structure='topological') # the Euclidean space␣
↪R^3
sage: c_cart.<X,Y,Z> = R3.chart()  # Cartesian coordinates on R^3
sage: Phi = S2.continuous_map(R3, {(c_xy, c_cart): [2*x/(1+x^2+y^2),
....:                              2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^2+y^2)],
....:                              (c_uv, c_cart): [2*u/(1+u^2+v^2),
....:                              2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^2+v^2)]},
....:                             name='Phi', latex_name=r'\Phi') # Embedding of␣
↪S^2 in R^3
sage: g = c_xy.plot(c_cart, mapping=Phi)
sage: g
Graphics3d Object
```



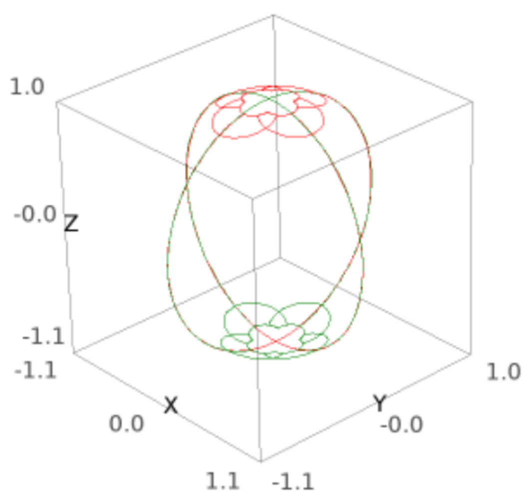
NB: to get a better coverage of the whole sphere, one should increase the coordinate sampling via the argument `number_values` or the argument `steps` (only the default value, `number_values = 5`, is used here, which is pretty low).

The same plot without the `(X,Y,Z)` axes labels:

```
sage: g = c_xy.plot(c_cart, mapping=Phi, label_axes=False)
```

The North and South stereographic charts on the same plot:

```
sage: g2 = c_uv.plot(c_cart, mapping=Phi, color='green')
sage: g + g2
Graphics3d Object
```



South stereographic chart drawned in terms of the North one (we split the plot in four parts to avoid the singularity at $(u, v) = (0, 0)$):

```
sage: W = U.intersection(V) # the subset common to both charts
sage: c_uvW = c_uv.restrict(W) # chart (W,(u,v))
sage: gSN1 = c_uvW.plot(c_xy, ranges={u:[-6.,-0.02], v:[-6.,-0.02]})  # long␣
↪time
sage: gSN2 = c_uvW.plot(c_xy, ranges={u:[-6.,-0.02], v:[0.02,6.]})  # long␣
↪time
sage: gSN3 = c_uvW.plot(c_xy, ranges={u:[0.02,6.], v:[-6.,-0.02]})  # long␣
↪time
sage: gSN4 = c_uvW.plot(c_xy, ranges={u:[0.02,6.], v:[0.02,6.]})  # long time
```

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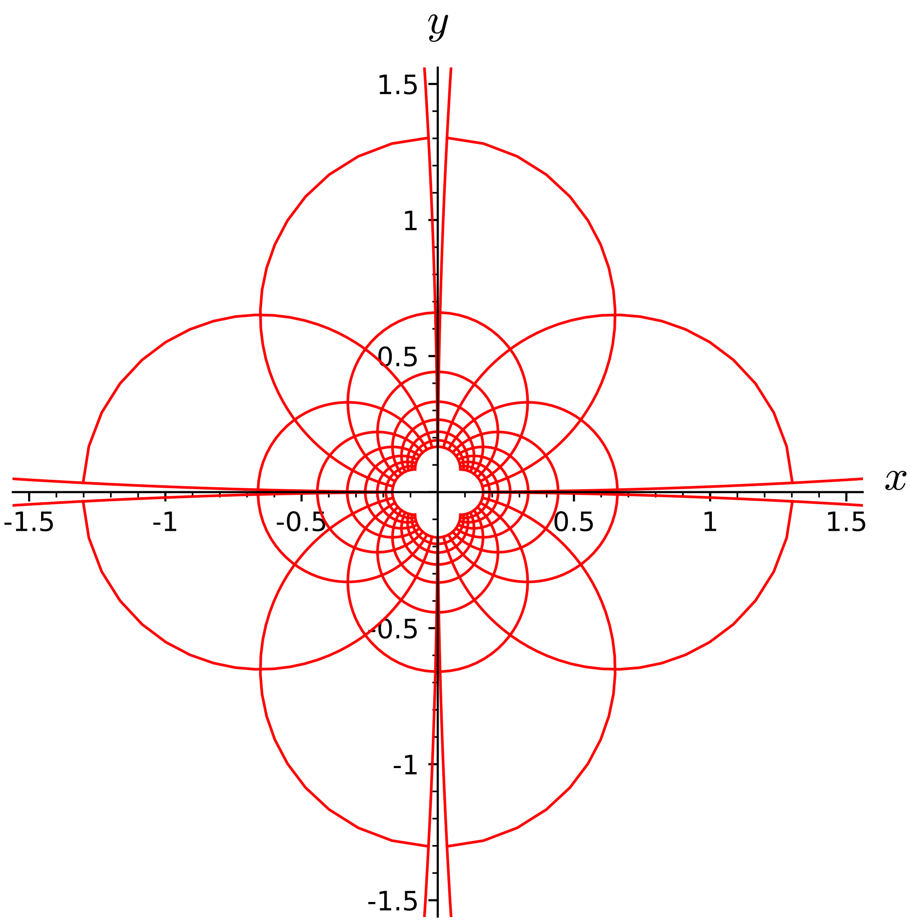
```
sage: show(gSN1+gSN2+gSN3+gSN4, xmin=-1.5, xmax=1.5, ymin=-1.5, ymax=1.5)  #␣
↪long time
```



The coordinate line $u = 1$ (red) and the coordinate line $v = 1$ (green) on the same plot:

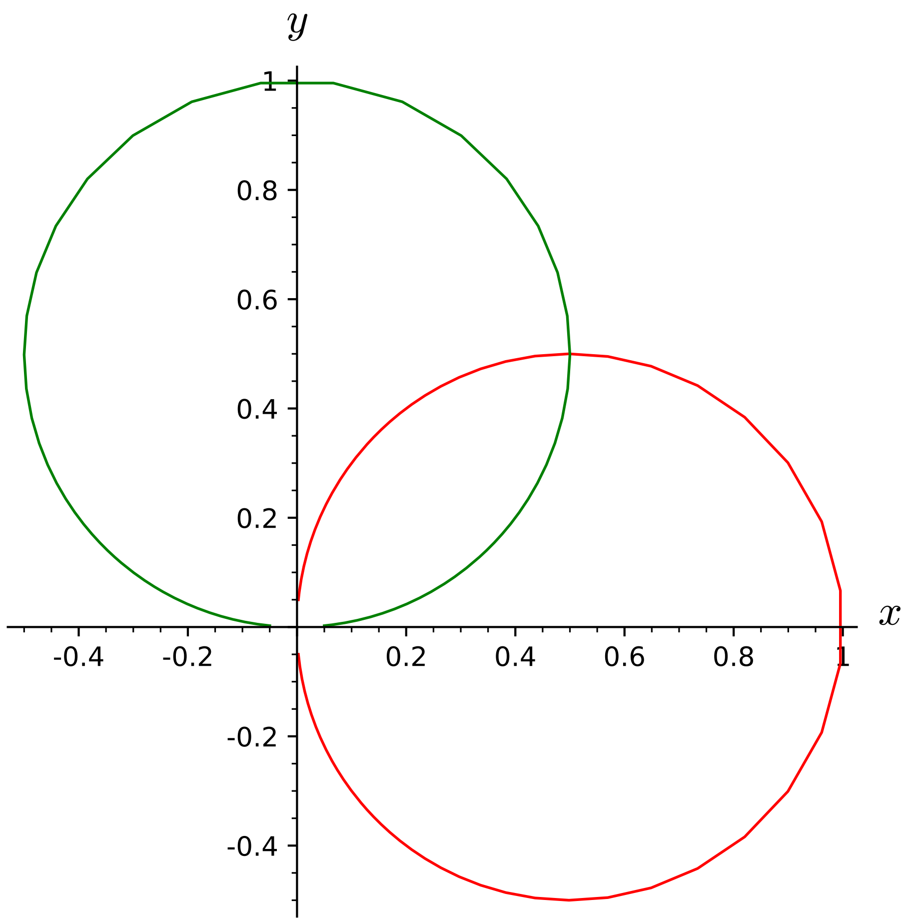
```
sage: gu1 = c_uvW.plot(c_xy, fixed_coords={u: 1}, max_range=20, plot_
↪points=300)  # long time
sage: gv1 = c_uvW.plot(c_xy, fixed_coords={v: 1}, max_range=20, plot_
↪points=300, color='green')  # long time
sage: gu1 + gv1  # long time
Graphics object consisting of 2 graphics primitives
```

Note that we have set `max_range=20` to have a wider range for the coordinates u and v, i.e. to have $[-20, 20]$ instead of the default $[-8, 8]$.

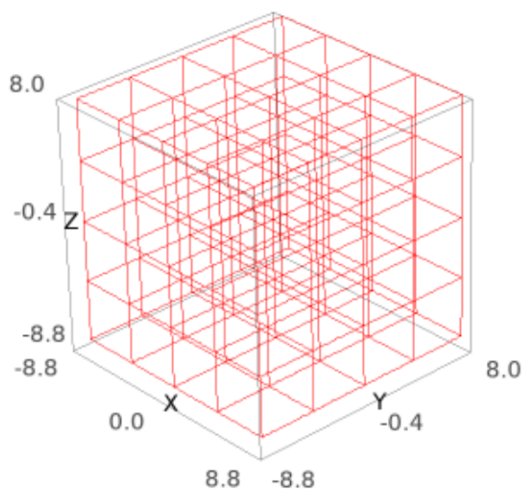
A 3-dimensional chart plotted in terms of itself results in a 3D rectangular grid:

```
sage: g = c_cart.plot() # equivalent to c_cart.plot(c_cart)  # long time
sage: g  # long time
Graphics3d Object
```

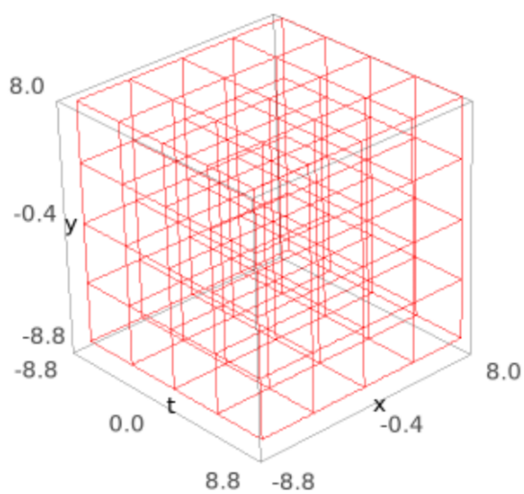
A 4-dimensional chart plotted in terms of itself (the plot is performed for at most 3 coordinates, which must be specified via the argument `ambient_coords`):



y
x
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.4
-0.2
0.2
0.4
0.6
0.8
1


8.0
-0.4
Z
-8.8
-8.8
0.0
X
8.8
-8.8
Y
-0.4
8.0

```
sage: M = Manifold(4, 'M', structure='topological')
sage: X.<t,x,y,z> = M.chart()
sage: g = X.plot(ambient_coords=(t,x,y)) # the coordinate z is not depicted
↪# long time
sage: g  # long time
Graphics3d Object
```




```
sage: g = X.plot(ambient_coords=(t,y)) # the coordinates x and z are not␣
↪depicted
sage: g
Graphics object consisting of 18 graphics primitives
```

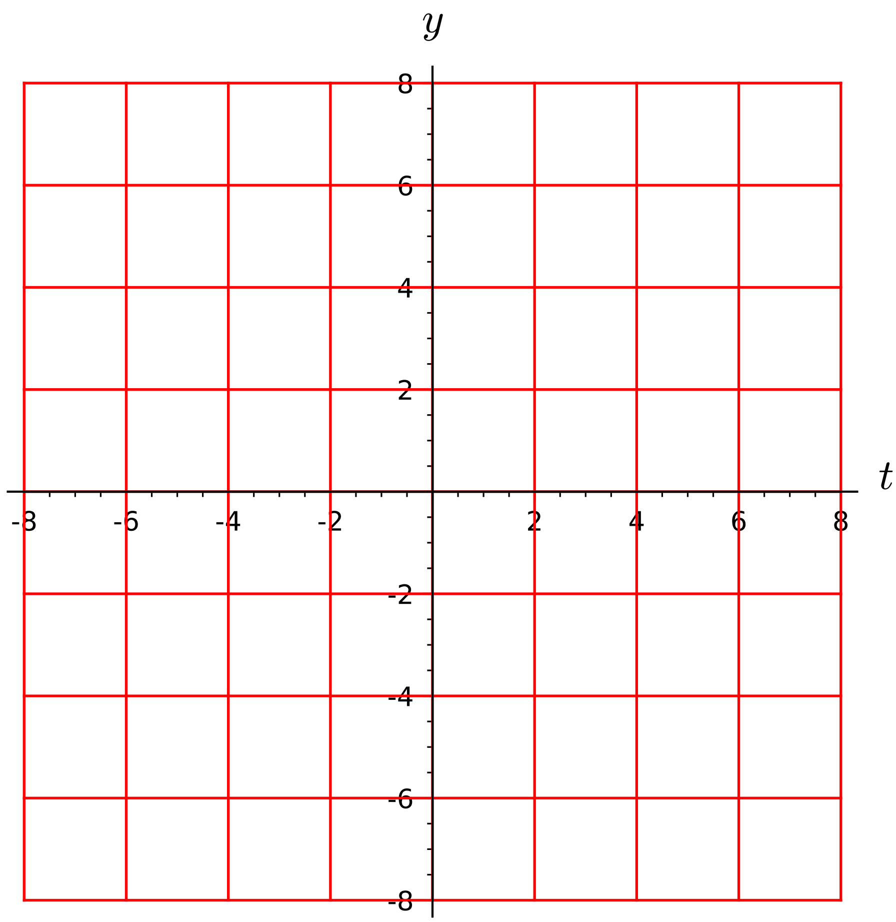
Note that the default values of some arguments of the method `plot` are stored in the dictionary `plot.options`:

```
sage: X.plot.options  # random (dictionary output)
{'color': 'red', 'label_axes': True, 'max_range': 8,
 'plot_points': 75, 'style': '-', 'thickness': 1}
```

so that they can be adjusted by the user:

```
sage: X.plot.options['color'] = 'blue'
```

From now on, all chart plots will use blue as the default color. To restore the original default options, it suffices to type:


y
t
8
6
4
2
-2
-4
-6
-8
-8
-6
-4
-2
2
4
6
8

```
sage: X.plot.reset()
```

restrict (*subset*, *restrictions=None*)

Return the restriction of the chart to some open subset of its domain.

If the current chart is (U, φ), a *restriction* (or *subchart*) is a chart (V, ψ) such that $V \subset U$ and $\psi = \varphi|_V$.

If such subchart has not been defined yet, it is constructed here.

The coordinates of the subchart bare the same names as the coordinates of the current chart.

INPUT:

- `subset` – open subset V of the chart domain U (must be an instance of `TopologicalManifold`)
- `restrictions` – (default: `None`) list of coordinate restrictions defining the subset V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- the chart (V, ψ) as a `RealChart`

EXAMPLES:

Cartesian coordinates on the unit open disc in $\mathbf{R}^2$ as a subchart of the global Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', structure='topological')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: D = M.open_subset('D') # the unit open disc
sage: c_cart_D = c_cart.restrict(D, x^2+y^2<1)
sage: p = M.point((1/2, 0))
sage: p in D
True
sage: q = M.point((1, 2))
sage: q in D
False
```

Cartesian coordinates on the annulus $1 < \sqrt{x^2 + y^2} < 2$:

```
sage: A = M.open_subset('A')
sage: c_cart_A = c_cart.restrict(A, [x^2+y^2>1, x^2+y^2<4])
sage: p in A, q in A
(False, False)
sage: a = M.point((3/2,0))
sage: a in A
True
```

valid_coordinates (**coordinates*, ***kwds*)

Check whether a tuple of coordinates can be the coordinates of a point in the chart domain.

INPUT:

- `*coordinates` – coordinate values
- `**kwds` – options:
  - `tolerance=0`, to set the absolute tolerance in the test of coordinate ranges
  - `parameters=None`, to set some numerical values to parameters

OUTPUT:

- `True` if the coordinate values are admissible in the chart range and `False` otherwise

EXAMPLES:

Cartesian coordinates on a square interior:

```
sage: forget()  # for doctest only
sage: M = Manifold(2, 'M', structure='topological')  # the square interior
sage: X.<x,y> = M.chart('x:(-2,2) y:(-2,2)')
sage: X.valid_coordinates(0,1)
True
sage: X.valid_coordinates(-3/2,5/4)
True
sage: X.valid_coordinates(0,3)
False
```

The unit open disk inside the square:

```
sage: D = M.open_subset('D', coord_def={X: x^2+y^2<1})
sage: XD = X.restrict(D)
sage: XD.valid_coordinates(0,1)
False
sage: XD.valid_coordinates(-3/2,5/4)
False
sage: XD.valid_coordinates(-1/2,1/2)
True
sage: XD.valid_coordinates(0,0)
True
```

Another open subset of the square, defined by $x^2 + y^2 < 1$ or ($x > 0$ and $|y| < 1$):

```
sage: B = M.open_subset('B',
....:                   coord_def={X: (x^2+y^2<1,
....:                                  [x>0, abs(y)<1])})
sage: XB = X.restrict(B)
sage: XB.valid_coordinates(-1/2, 0)
True
sage: XB.valid_coordinates(-1/2, 3/2)
False
sage: XB.valid_coordinates(3/2, 1/2)
True
```

valid_coordinates_numerical(**coordinates*)

Check whether a tuple of float coordinates can be the coordinates of a point in the chart domain.

This version is optimized for float numbers, and cannot accept parameters nor tolerance. The chart restriction must also be specified in CNF (i.e. a list of tuples).

INPUT:

- `*coordinates` – coordinate values

OUTPUT:

- `True` if the coordinate values are admissible in the chart range and `False` otherwise

EXAMPLES:

Cartesian coordinates on a square interior:

```
sage: forget()  # for doctest only
sage: M = Manifold(2, 'M', structure='topological')  # the square interior
sage: X.<x,y> = M.chart('x:(-2,2) y:(-2,2)')
sage: X.valid_coordinates_numerical(0,1)
True
sage: X.valid_coordinates_numerical(-3/2,5/4)
True
sage: X.valid_coordinates_numerical(0,3)
False
```

The unit open disk inside the square:

```
sage: D = M.open_subset('D', coord_def={X: x^2+y^2<1})
sage: XD = X.restrict(D)
sage: XD.valid_coordinates_numerical(0,1)
False
sage: XD.valid_coordinates_numerical(-3/2,5/4)
False
sage: XD.valid_coordinates_numerical(-1/2,1/2)
True
sage: XD.valid_coordinates_numerical(0,0)
True
```

Another open subset of the square, defined by $x^2 + y^2 < 1$ or ($x > 0$ and $|y| < 1$):

```
sage: B = M.open_subset('B',coord_def={X: [(x^2+y^2<1, x>0),
....:                      (x^2+y^2<1,  abs(y)<1)]})
sage: XB = X.restrict(B)
sage: XB.valid_coordinates_numerical(-1/2, 0)
True
sage: XB.valid_coordinates_numerical(-1/2, 3/2)
False
sage: XB.valid_coordinates_numerical(3/2, 1/2)
True
```

1.5.2 Chart Functions

In the context of a topological manifold M over a topological field K, a *chart function* is a function from a chart codomain to K. In other words, a chart function is a K-valued function of the coordinates associated to some chart. The internal coordinate expressions of chart functions and calculus on them are taken in charge by different calculus methods, at the choice of the user:

- Sage's default symbolic engine (Pynac + Maxima), implemented via the Symbolic Ring (`SR`)
- SymPy engine, denoted `sympy` hereafter

See *CalculusMethod* for details.

AUTHORS:

- Marco Mancini (2017) : initial version

- Eric Gourgoulhon (2015) : for a previous class implementing only SR calculus (CoordFunctionSymb)
- Florentin Jaffredo (2018) : series expansion with respect to a given parameter

class `sage.manifolds.chart_func.`**`ChartFunction`**(*parent*, *expression=None*, *calc_method=None*, *expansion_symbol=None*, *order=None*)

Bases: `sage.structure.element.AlgebraElement`

Function of coordinates of a given chart.

If (U, φ) is a chart on a topological manifold M of dimension n over a topological field K, a *chart function* associated to (U, φ) is a map

$$\begin{array}{llll} f: & V \subset K^n & \longrightarrow & K \\ & (x^1, \ldots, x^n) & \longmapsto & f(x^1, \ldots, x^n), \end{array}$$

where V is the codomain of φ. In other words, f is a K-valued function of the coordinates associated to the chart (U, φ).

The chart function f can be represented by expressions pertaining to different calculus methods; the currently implemented ones are

- `SR` (Sage's Symbolic Ring)
- `SymPy`

See `expr()` for details.

INPUT:

- `parent` – the algebra of chart functions on the chart (U, φ)
- `expression` – (default: `None`) a symbolic expression representing $f(x^1, \ldots, x^n)$, where $(x^1, \ldots, x^n)$ are the coordinates of the chart (U, φ)
- `calc_method` – string (default: `None`): the calculus method with respect to which the internal expression of `self` must be initialized from `expression`; one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the chart current calculus method is assumed
- `expansion_symbol` – (default: `None`) symbolic variable (the "small parameter") with respect to which the coordinate expression is expanded in power series (around the zero value of this variable)
- `order` – integer (default: `None`); the order of the expansion if `expansion_symbol` is not `None`; the *order* is defined as the degree of the polynomial representing the truncated power series in `expansion_symbol`

> **Warning:** The value of `order` is $n-1$, where n is the order of the big O in the power series expansion

EXAMPLES:

A symbolic chart function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2+3*y+1)
sage: type(f)
```

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```
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: f.display()
(x, y) |--> x^2 + 3*y + 1
sage: f(x,y)
x^2 + 3*y + 1
```

The symbolic expression is returned when asking for the direct display of the function:

```
sage: f
x^2 + 3*y + 1
sage: latex(f)
x^{2} + 3 \, y + 1
```

A similar output is obtained by means of the method `expr()`:

```
sage: f.expr()
x^2 + 3*y + 1
```

The expression returned by `expr()` is by default a Sage symbolic expression:

```
sage: type(f.expr())
<type 'sage.symbolic.expression.Expression'>
```

A SymPy expression can also be asked for:

```
sage: f.expr('sympy')
x**2 + 3*y + 1
sage: type(f.expr('sympy'))
<class 'sympy.core.add.Add'>
```

The value of the function at specified coordinates is obtained by means of the standard parentheses notation:

```
sage: f(2,-1)
2
sage: var('a b')
(a, b)
sage: f(a,b)
a^2 + 3*b + 1
```

An unspecified chart function:

```
sage: g = X.function(function('G')(x, y))
sage: g
G(x, y)
sage: g.display()
(x, y) |--> G(x, y)
sage: g.expr()
G(x, y)
sage: g(2,3)
G(2, 3)
```

Coordinate functions can be compared to other values:

```
sage: f = X.function(x^2+3*y+1)
sage: f == 2
False
```

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```
sage: f == x^2 + 3*y + 1
True
sage: g = X.function(x*y)
sage: f == g
False
sage: h = X.function(x^2+3*y+1)
sage: f == h
True
```

Expansion to a given order with respect to a small parameter:

```
sage: t = var('t')  # the small parameter
sage: f = X.function(cos(t)*x*y, expansion_symbol=t, order=2)
```

The expansion is triggered by the call to `simplify()`:

```
sage: f
x*y*cos(t)
sage: f.simplify()
-1/2*t^2*x*y + x*y
```

Differences between `ChartFunction` and callable symbolic expressions

Callable symbolic expressions are defined directly from symbolic expressions of the coordinates:

```
sage: f0(x,y) = x^2 + 3*y + 1
sage: type(f0)
<type 'sage.symbolic.expression.Expression'>
sage: f0
(x, y) |--> x^2 + 3*y + 1
sage: f0(x,y)
x^2 + 3*y + 1
```

To get an output similar to that of `f0` for a chart function, we must use the method `display()`:

```
sage: f = X.function(x^2+3*y+1)
sage: f
x^2 + 3*y + 1
sage: f.display()
(x, y) |--> x^2 + 3*y + 1
sage: f(x,y)
x^2 + 3*y + 1
```

More importantly, instances of `ChartFunction` differ from callable symbolic expression by the automatic simplifications in all operations. For instance, adding the two callable symbolic expressions:

```
sage: f0(x,y,z) = cos(x)^2 ; g0(x,y,z) = sin(x)^2
```

results in:

```
sage: f0 + g0
(x, y, z) |--> cos(x)^2 + sin(x)^2
```

To get 1, one has to call `simplify_trig()`:

```
sage: (f0 + g0).simplify_trig()
(x, y, z) |--> 1
```

On the contrary, the sum of the corresponding *ChartFunction* instances is automatically simplified (see *simplify_chain_real()* and *simplify_chain_generic()* for details):

```
sage: f = X.function(cos(x)^2) ; g = X.function(sin(x)^2)
sage: f + g
1
```

Another difference regards the display of partial derivatives: for callable symbolic functions, it involves `diff`:

```
sage: g = function('g')(x, y)
sage: f0(x,y) = diff(g, x) + diff(g, y)
sage: f0
(x, y) |--> diff(g(x, y), x) + diff(g(x, y), y)
```

while for chart functions, the display is more "textbook" like:

```
sage: f = X.function(diff(g, x) + diff(g, y))
sage: f
d(g)/dx + d(g)/dy
```

The difference is even more dramatic on LaTeX outputs:

```
sage: latex(f0)
\left( x, y \right) \ {\mapsto} \ \frac{\partial}{\partial x}g\left(x, y\right) +
↪\frac{\partial}{\partial y}g\left(x, y\right)
sage: latex(f)
\frac{\partial\,g}{\partial x} + \frac{\partial\,g}{\partial y}
```

Note that this regards only the display of coordinate functions: internally, the `diff` notation is still used, as we can check by asking for the symbolic expression stored in `f`:

```
sage: f.expr()
diff(g(x, y), x) + diff(g(x, y), y)
```

One can switch to Pynac notation by changing the options:

```
sage: Manifold.options.textbook_output=False
sage: latex(f)
\frac{\partial}{\partial x}g\left(x, y\right) + \frac{\partial}{\partial y}
↪g\left(x, y\right)
sage: Manifold.options._reset()
sage: latex(f)
\frac{\partial\,g}{\partial x} + \frac{\partial\,g}{\partial y}
```

Another difference between *ChartFunction* and callable symbolic expression is the possibility to switch off the display of the arguments of unspecified functions. Consider for instance:

```
sage: f = X.function(function('u')(x, y) * function('v')(x, y))
sage: f
u(x, y)*v(x, y)
sage: f0(x,y) = function('u')(x, y) * function('v')(x, y)
sage: f0
(x, y) |--> u(x, y)*v(x, y)
```

If there is a clear understanding that u and v are functions of (x, y), the explicit mention of the latter can be cumbersome in lengthy tensor expressions. We can switch it off by:

```
sage: Manifold.options.omit_function_arguments=True
sage: f
u*v
```

Note that neither the callable symbolic expression `f0` nor the internal expression of `f` is affected by the above command:

```
sage: f0
(x, y) |--> u(x, y)*v(x, y)
sage: f.expr()
u(x, y)*v(x, y)
```

We revert to the default behavior by:

```
sage: Manifold.options._reset()
sage: f
u(x, y)*v(x, y)
```

__call__(**coords*, ***options*)

Compute the value of the function at specified coordinates.

INPUT:

- `*coords` – list of coordinates $(x^1, \ldots, x^n)$, where the function f is to be evaluated
- `**options` – allows to pass `simplify=False` to disable the call of the simplification chain on the result

OUTPUT:

- the value $f(x^1, \ldots, x^n)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(sin(x*y))
sage: f.__call__(-2, 3)
-sin(6)
sage: f(-2, 3)
-sin(6)
sage: var('a b')
(a, b)
sage: f.__call__(a, b)
sin(a*b)
sage: f(a,b)
sin(a*b)
sage: f.__call__(pi, 1)
0
sage: f.__call__(pi, 1/2)
1
```

With SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f(-2,3)
-sin(6)
```

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```
sage: type(f(-2,3))
<class 'sympy.core.mul.Mul'>
sage: f(a,b)
sin(a*b)
sage: type(f(a,b))
sin
sage: type(f(pi,1))
<class 'sympy.core.numbers.Zero'>
sage: f(pi, 1/2)
1
sage: type(f(pi, 1/2))
<class 'sympy.core.numbers.One'>
```

arccos()

Arc cosine of `self`.

OUTPUT:

- chart function $\arccos(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arccos()
arccos(x*y)
sage: arccos(f)  # equivalent to f.arccos()
arccos(x*y)
sage: acos(f)  # equivalent to f.arccos()
arccos(x*y)
sage: arccos(f).display()
(x, y) |--> arccos(x*y)
sage: arccos(X.zero_function()).display()
(x, y) |--> 1/2*pi
```

The same test with SymPy:

```
sage: M.set_calculus_method('sympy')
sage: f = X.function(x*y)
sage: f.arccos()
acos(x*y)
sage: arccos(f)  # equivalent to f.arccos()
acos(x*y)
sage: acos(f)  # equivalent to f.arccos()
acos(x*y)
sage: arccos(f).display()
(x, y) |--> acos(x*y)
```

arccosh()

Inverse hyperbolic cosine of `self`.

OUTPUT:

- chart function $\operatorname{arccosh}(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arccosh()
arccosh(x*y)
sage: arccosh(f)  # equivalent to f.arccosh()
arccosh(x*y)
sage: acosh(f)  # equivalent to f.arccosh()
arccosh(x*y)
sage: arccosh(f).display()
(x, y) |--> arccosh(x*y)
sage: arccosh(X.function(1)) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arccosh()
acosh(x*y)
sage: arccosh(f)  # equivalent to f.arccosh()
acosh(x*y)
sage: acosh(f)  # equivalent to f.arccosh()
acosh(x*y)
```

arcsin()

Arc sine of `self`.

OUTPUT:

- chart function $\arcsin(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arcsin()
arcsin(x*y)
sage: arcsin(f)  # equivalent to f.arcsin()
arcsin(x*y)
sage: asin(f)  # equivalent to f.arcsin()
arcsin(x*y)
sage: arcsin(f).display()
(x, y) |--> arcsin(x*y)
sage: arcsin(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arcsin()
asin(x*y)
sage: arcsin(f)  # equivalent to f.arcsin()
asin(x*y)
sage: asin(f)  # equivalent to f.arcsin()
asin(x*y)
```

arcsinh()

Inverse hyperbolic sine of `self`.

OUTPUT:

- chart function $\operatorname{arcsinh}(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arcsinh()
arcsinh(x*y)
sage: arcsinh(f)  # equivalent to f.arcsinh()
arcsinh(x*y)
sage: asinh(f)  # equivalent to f.arcsinh()
arcsinh(x*y)
sage: arcsinh(f).display()
(x, y) |--> arcsinh(x*y)
sage: arcsinh(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arcsinh()
asinh(x*y)
sage: arcsinh(f)  # equivalent to f.arcsinh()
asinh(x*y)
sage: asinh(f)  # equivalent to f.arcsinh()
asinh(x*y)
```

arctan()

Arc tangent of `self`.

OUTPUT:

- chart function $\arctan(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arctan()
arctan(x*y)
sage: arctan(f)  # equivalent to f.arctan()
arctan(x*y)
sage: atan(f)  # equivalent to f.arctan()
arctan(x*y)
sage: arctan(f).display()
(x, y) |--> arctan(x*y)
sage: arctan(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arctan()
atan(x*y)
sage: arctan(f)  # equivalent to f.arctan()
atan(x*y)
```

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```
sage: atan(f)   # equivalent to f.arctan()
atan(x*y)
```

arctanh()

Inverse hyperbolic tangent of `self`.

OUTPUT:

- chart function $\operatorname{arctanh}(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.arctanh()
arctanh(x*y)
sage: arctanh(f)   # equivalent to f.arctanh()
arctanh(x*y)
sage: atanh(f)   # equivalent to f.arctanh()
arctanh(x*y)
sage: arctanh(f).display()
(x, y) |--> arctanh(x*y)
sage: arctanh(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.arctanh()
atanh(x*y)
sage: arctanh(f)   # equivalent to f.arctanh()
atanh(x*y)
sage: atanh(f)   # equivalent to f.arctanh()
atanh(x*y)
```

chart()

Return the chart with respect to which `self` is defined.

OUTPUT:

- a `Chart`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(1+x+y^2)
sage: f.chart()
Chart (M, (x, y))
sage: f.chart() is X
True
```

collect(s)

Collect the coefficients of s in the expression of `self` into a group.

INPUT:

- `s` – the symbol whose coefficients will be collected

OUTPUT:

- `self` with the coefficients of `s` grouped in its expression

EXAMPLES:

Action on a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2*y + x*y + (x*y)^2)
sage: f.display()
(x, y) |--> x^2*y^2 + x^2*y + x*y
sage: f.collect(y)
x^2*y^2 + (x^2 + x)*y
```

The method `collect()` has changed the expression of `f`:

```
sage: f.display()
(x, y) |--> x^2*y^2 + (x^2 + x)*y
```

The same test with SymPy

```
sage: X.calculus_method().set('sympy')
sage: f = X.function(x^2*y + x*y + (x*y)^2)
sage: f.display()
(x, y) |--> x**2*y**2 + x**2*y + x*y
sage: f.collect(y)
x**2*y**2 + y*(x**2 + x)
```

collect_common_factors()

Collect common factors in the expression of `self`.

This method does not perform a full factorization but only looks for factors which are already explicitly present.

OUTPUT:

- `self` with the common factors collected in its expression

EXAMPLES:

Action on a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x/(x^2*y + x*y))
sage: f.display()
(x, y) |--> x/(x^2*y + x*y)
sage: f.collect_common_factors()
1/((x + 1)*y)
```

The method `collect_common_factors()` has changed the expression of `f`:

```
sage: f.display()
(x, y) |--> 1/((x + 1)*y)
```

The same test with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: g = X.function(x/(x^2*y + x*y))
sage: g.display()
(x, y) |--> x/(x**2*y + x*y)
sage: g.collect_common_factors()
1/(y*(x + 1))
```

copy()

Return an exact copy of the object.

OUTPUT:

- a `ChartFunctionSymb`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x+y^2)
sage: g = f.copy(); g
y^2 + x
```

By construction, `g` is identical to `f`:

```
sage: type(g) == type(f)
True
sage: g == f
True
```

but it is not the same object:

```
sage: g is f
False
```

cos()

Cosine of `self`.

OUTPUT:

- chart function $\cos(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.cos()
cos(x*y)
sage: cos(f)  # equivalent to f.cos()
cos(x*y)
sage: cos(f).display()
(x, y) |--> cos(x*y)
sage: cos(X.zero_function()).display()
(x, y) |--> 1
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.cos()
```

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```
cos(x*y)
sage: cos(f)   # equivalent to f.cos()
cos(x*y)
```

cosh()

Hyperbolic cosine of `self`.

OUTPUT:

- chart function $\cosh(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.cosh()
cosh(x*y)
sage: cosh(f)   # equivalent to f.cosh()
cosh(x*y)
sage: cosh(f).display()
(x, y) |--> cosh(x*y)
sage: cosh(X.zero_function()).display()
(x, y) |--> 1
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.cosh()
cosh(x*y)
sage: cosh(f)   # equivalent to f.cosh()
cosh(x*y)
```

diff(*coord*)

Partial derivative with respect to a coordinate.

INPUT:

- `coord` – either the coordinate x^i with respect to which the derivative of the chart function f is to be taken, or the index i labelling this coordinate (with the index convention defined on the chart domain via the parameter `start_index`)

OUTPUT:

- a *ChartFunction* representing the partial derivative $\frac{\partial f}{\partial x^i}$

EXAMPLES:

Partial derivatives of a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(calc_method='SR')
sage: f = X.function(x^2+3*y+1); f
x^2 + 3*y + 1
sage: f.diff(x)
2*x
sage: f.diff(y)
3
```

Each partial derivatives is itself a chart function:

```
sage: type(f.diff(x))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

An index can be used instead of the coordinate symbol:

```
sage: f.diff(0)
2*x
sage: f.diff(1)
3
```

The index range depends on the convention used on the chart's domain:

```
sage: M = Manifold(2, 'M', structure='topological', start_index=1)
sage: X.<x,y> = M.chart(calc_method='sympy')
sage: f = X.function(x**2+3*y+1)
sage: f.diff(0)
Traceback (most recent call last):
...
ValueError: coordinate index out of range
sage: f.diff(1)
2*x
sage: f.diff(2)
3
```

The same test with SymPy:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(calc_method='sympy')
sage: f = X.function(x^2+3*y+1); f
x**2 + 3*y + 1
sage: f.diff(x)
2*x
sage: f.diff(y)
3
```

disp()

Display `self` in arrow notation. For display the standard `SR` representation is used.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Coordinate function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(cos(x*y/2))
sage: f.display()
(x, y) |--> cos(1/2*x*y)
sage: latex(f.display())
\left(x, y\right) \mapsto \cos\left(\frac{1}{2} \, x y\right)
```

A shortcut is `disp()`:

```
sage: f.disp()
(x, y) |--> cos(1/2*x*y)
```

Display of the zero function:

```
sage: X.zero_function().display()
(x, y) |--> 0
```

display()

Display self in arrow notation. For display the standard SR representation is used.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Coordinate function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(cos(x*y/2))
sage: f.display()
(x, y) |--> cos(1/2*x*y)
sage: latex(f.display())
\left(x, y\right) \mapsto \cos\left(\frac{1}{2} \, x y\right)
```

A shortcut is disp():

```
sage: f.disp()
(x, y) |--> cos(1/2*x*y)
```

Display of the zero function:

```
sage: X.zero_function().display()
(x, y) |--> 0
```

exp()

Exponential of self.

OUTPUT:

- chart function $\exp(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x+y)
sage: f.exp()
e^(x + y)
sage: exp(f) # equivalent to f.exp()
e^(x + y)
sage: exp(f).display()
(x, y) |--> e^(x + y)
sage: exp(X.zero_function())
1
```

The same test with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f = X.function(x+y)
sage: f.exp()
exp(x + y)
sage: exp(f) # equivalent to f.exp()
exp(x + y)
```

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```
sage: exp(f).display()
(x, y) |--> exp(x + y)
sage: exp(X.zero_function())
1
```

expand()

Expand the coordinate expression of `self`.

OUTPUT:

- `self` with its expression expanded

EXAMPLES:

Expanding a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function((x - y)^2)
sage: f.display()
(x, y) |--> (x - y)^2
sage: f.expand()
x^2 - 2*x*y + y^2
```

The method `expand()` has changed the expression of `f`:

```
sage: f.display()
(x, y) |--> x^2 - 2*x*y + y^2
```

The same test with SymPy

```
sage: X.calculus_method().set('sympy')
sage: g = X.function((x - y)^2)
sage: g.expand()
x**2 - 2*x*y + y**2
```

expr(*method=None*)

Return the symbolic expression of `self` in terms of the chart coordinates, as an object of a specified calculus method.

INPUT:

- `method` – string (default: `None`): the calculus method which the returned expression belongs to; one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the chart current calculus method is assumed

OUTPUT:

- a `Sage symbolic expression` if `method` is `'SR'`
- a SymPy object if `method` is `'sympy'`

EXAMPLES:

Chart function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2+y)
sage: f.expr()
x^2 + y
sage: type(f.expr())
<type 'sage.symbolic.expression.Expression'>
```

Asking for the SymPy expression:

```
sage: f.expr('sympy')
x**2 + y
sage: type(f.expr('sympy'))
<class 'sympy.core.add.Add'>
```

The default corresponds to the current calculus method, here the one based on the Symbolic Ring `SR`:

```
sage: f.expr() is f.expr('SR')
True
```

If we change the current calculus method on chart `X`, we change the default:

```
sage: X.calculus_method().set('sympy')
sage: f.expr()
x**2 + y
sage: f.expr() is f.expr('sympy')
True
sage: X.calculus_method().set('SR')  # revert back to SR
```

Internally, the expressions corresponding to various calculus methods are stored in the dictionary `_express`:

```
sage: for method in sorted(f._express):
....:     print("'{}': {}".format(method, f._express[method]))
....:
'SR': x^2 + y
'sympy': x**2 + y
```

The method *expr()* is useful for accessing to all the symbolic expression functionalities in Sage; for instance:

```
sage: var('a')
a
sage: f = X.function(a*x*y); f.display()
(x, y) |--> a*x*y
sage: f.expr()
a*x*y
sage: f.expr().subs(a=2)
2*x*y
```

Note that for substituting the value of a coordinate, the function call can be used as well:

```
sage: f(x,3)
3*a*x
sage: bool( f(x,3) == f.expr().subs(y=3) )
True
```

factor()

Factorize the coordinate expression of `self`.

OUTPUT:

- `self` with its expression factorized

EXAMPLES:

Factorization of a 2-dimensional chart function:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x^2 + 2*x*y + y^2)
sage: f.display()
(x, y) |--> x^2 + 2*x*y + y^2
sage: f.factor()
(x + y)^2
```

The method `factor()` has changed the expression of `f`:

```
sage: f.display()
(x, y) |--> (x + y)^2
```

The same test with SymPy

```
sage: X.calculus_method().set('sympy')
sage: g = X.function(x^2 + 2*x*y + y^2)
sage: g.display()
(x, y) |--> x**2 + 2*x*y + y**2
sage: g.factor()
(x + y)**2
```

is_trivial_zero()

Check if `self` is trivially equal to zero without any simplification.

This method is supposed to be fast as compared with `self.is_zero()` or `self == 0` and is intended to be used in library code where trying to obtain a mathematically correct result by applying potentially expensive rewrite rules is not desirable.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(0)
sage: f.is_trivial_zero()
True
sage: f = X.function(float(0.0))
sage: f.is_trivial_zero()
True
sage: f = X.function(x-x)
sage: f.is_trivial_zero()
True
sage: X.zero_function().is_trivial_zero()
True
```

No simplification is attempted, so that `False` is returned for non-trivial cases:

```
sage: f = X.function(cos(x)^2 + sin(x)^2 - 1)
sage: f.is_trivial_zero()
False
```

On the contrary, the method `is_zero()` and the direct comparison to zero involve some simplification algorithms and return `True`:

```
sage: f.is_zero()
True
sage: f == 0
True
```

log(*base=None*)

Logarithm of `self`.

INPUT:

- `base` – (default: `None`) base of the logarithm; if `None`, the natural logarithm (i.e. logarithm to base e) is returned

OUTPUT:

- chart function $\log_a(f)$, where f is the current chart function and a is the base

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x+y)
sage: f.log()
log(x + y)
sage: log(f) # equivalent to f.log()
log(x + y)
sage: log(f).display()
(x, y) |--> log(x + y)
sage: f.log(2)
log(x + y)/log(2)
sage: log(f, 2)
log(x + y)/log(2)
```

The same test with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f = X.function(x+y)
sage: f.log()
log(x + y)
sage: log(f) # equivalent to f.log()
log(x + y)
sage: log(f).display()
(x, y) |--> log(x + y)
sage: f.log(2)
log(x + y)/log(2)
sage: log(f, 2)
log(x + y)/log(2)
```

scalar_field(*name=None*, *latex_name=None*)

Construct the scalar field that has `self` as coordinate expression.

The domain of the scalar field is the open subset covered by the chart on which `self` is defined.

INPUT:

- name – (default: None) name given to the scalar field
- latex_name – (default: None) LaTeX symbol to denote the scalar field; if None, the LaTeX symbol is set to name

OUTPUT:

- a *ScalarField*

EXAMPLES:

Construction of a scalar field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: fc = c_xy.function(x+2*y^3)
sage: f = fc.scalar_field() ; f
Scalar field on the 2-dimensional topological manifold M
sage: f.display()
M --> R
(x, y) |--> 2*y^3 + x
sage: f.coord_function(c_xy) is fc
True
```

set_expr(*calc_method*, *expression*)

Add an expression in a particular calculus method self. Some control is done to verify the consistency between the different representations of the same expression.

INPUT:

- calc_method – calculus method
- expression – symbolic expression

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(1+x^2)
sage: f._repr_()
'x^2 + 1'
sage: f.set_expr('sympy','x**2+1')
sage: f  # indirect doctest
x^2 + 1

sage: g = X.function(1+x^3)
sage: g._repr_()
'x^3 + 1'
sage: g.set_expr('sympy','x**2+y')
Traceback (most recent call last):
...
ValueError: Expressions are not equal
```

simplify()

Simplify the coordinate expression of self.

For details about the employed chain of simplifications for the SR calculus method, see *simplify_chain_real()* for chart functions on real manifolds and *simplify_chain_generic()* for the generic case.

If `self` has been defined with the small parameter `expansion_symbol` and some truncation order, the coordinate expression of `self` will be expanded in power series of that parameter and truncated to the the given order.

OUTPUT:

- `self` with its coordinate expression simplified

EXAMPLES:

Simplification of a chart function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(cos(x)^2 + sin(x)^2 + sqrt(x^2))
sage: f.display()
(x, y) |--> cos(x)^2 + sin(x)^2 + abs(x)
sage: f.simplify()
abs(x) + 1
```

The method `simplify()` has changed the expression of `f`:

```
sage: f.display()
(x, y) |--> abs(x) + 1
```

Another example:

```
sage: f = X.function((x^2-1)/(x+1)); f
(x^2 - 1)/(x + 1)
sage: f.simplify()
x - 1
```

Examples taking into account the declared range of a coordinate:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(1,+oo) y')
sage: f = X.function(sqrt(x^2-2*x+1)); f
sqrt(x^2 - 2*x + 1)
sage: f.simplify()
x - 1
```

```
sage: forget()  # to clear the previous assumption on x
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(-oo,0) y')
sage: f = X.function(sqrt(x^2-2*x+1)); f
sqrt(x^2 - 2*x + 1)
sage: f.simplify()
-x + 1
```

The same tests with SymPy:

```
sage: forget()  # to clear the previous assumption on x
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart(calc_method='sympy')
sage: f = X.function(cos(x)^2 + sin(x)^2 + sqrt(x^2)); f
sin(x)**2 + cos(x)**2 + Abs(x)
sage: f.simplify()
Abs(x) + 1
```

```
sage: f = X.function((x^2-1)/(x+1)); f
(x**2 - 1)/(x + 1)
sage: f.simplify()
x - 1
```

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(1,+oo) y', calc_method='sympy')
sage: f = X.function(sqrt(x^2-2*x+1)); f
sqrt(x**2 - 2*x + 1)
sage: f.simplify()
x - 1
```

```
sage: forget()  # to clear the previous assumption on x
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(-oo,0) y', calc_method='sympy')
sage: f = X.function(sqrt(x^2-2*x+1)); f
sqrt(x**2 - 2*x + 1)
sage: f.simplify()
1 - x
```

Power series expansion with respect to a small parameter t (at the moment, this is implemented only for the `SR` calculus backend, hence the first line below):

```
sage: X.calculus_method().set('SR')
sage: t = var('t')
sage: f = X.function(exp(t*x), expansion_symbol=t, order=3)
```

At this stage, f is not expanded in power series:

```
sage: f
e^(t*x)
```

Invoking `simplify()` triggers the expansion to the given order:

```
sage: f.simplify()
1/6*t^3*x^3 + 1/2*t^2*x^2 + t*x + 1
sage: f.display()
(x, y) |--> 1/6*t^3*x^3 + 1/2*t^2*x^2 + t*x + 1
```

sin()

Sine of `self`.

OUTPUT:

- chart function $\sin(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.sin()
sin(x*y)
sage: sin(f)  # equivalent to f.sin()
sin(x*y)
sage: sin(f).display()
(x, y) |--> sin(x*y)
```

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```
sage: sin(X.zero_function()) == X.zero_function()
True
sage: f = X.function(2-cos(x)^2+y)
sage: g = X.function(-sin(x)^2+y)
sage: (f+g).simplify()
2*y + 1
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f = X.function(x*y)
sage: f.sin()
sin(x*y)
sage: sin(f)  # equivalent to f.sin()
sin(x*y)
```

sinh()

Hyperbolic sine of `self`.

OUTPUT:

- chart function $\sinh(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.sinh()
sinh(x*y)
sage: sinh(f)  # equivalent to f.sinh()
sinh(x*y)
sage: sinh(f).display()
(x, y) |--> sinh(x*y)
sage: sinh(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.sinh()
sinh(x*y)
sage: sinh(f)  # equivalent to f.sinh()
sinh(x*y)
```

sqrt()

Square root of `self`.

OUTPUT:

- chart function $\sqrt{f}$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x+y)
sage: f.sqrt()
```

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```
sqrt(x + y)
sage: sqrt(f)  # equivalent to f.sqrt()
sqrt(x + y)
sage: sqrt(f).display()
(x, y) |--> sqrt(x + y)
sage: sqrt(X.zero_function()).display()
(x, y) |--> 0
```

tan()

Tangent of `self`.

OUTPUT:

- chart function $\tan(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.tan()
sin(x*y)/cos(x*y)
sage: tan(f)  # equivalent to f.tan()
sin(x*y)/cos(x*y)
sage: tan(f).display()
(x, y) |--> sin(x*y)/cos(x*y)
sage: tan(X.zero_function()) == X.zero_function()
True
```

The same test with SymPy:

```
sage: M.set_calculus_method('sympy')
sage: g = X.function(x*y)
sage: g.tan()
tan(x*y)
sage: tan(g)  # equivalent to g.tan()
tan(x*y)
sage: tan(g).display()
(x, y) |--> tan(x*y)
```

tanh()

Hyperbolic tangent of `self`.

OUTPUT:

- chart function $\tanh(f)$, where f is the current chart function

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.function(x*y)
sage: f.tanh()
sinh(x*y)/cosh(x*y)
sage: tanh(f)  # equivalent to f.tanh()
sinh(x*y)/cosh(x*y)
sage: tanh(f).display()
(x, y) |--> sinh(x*y)/cosh(x*y)
```

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```
sage: tanh(X.zero_function()) == X.zero_function()
True
```

The same tests with SymPy:

```
sage: X.calculus_method().set('sympy')
sage: f.tanh()
tanh(x*y)
sage: tanh(f)   # equivalent to f.tanh()
tanh(x*y)
```

class sage.manifolds.chart_func.**ChartFunctionRing**(*chart*)

Bases: sage.structure.parent.Parent, sage.structure.unique_representation.UniqueRepresentation

Ring of all chart functions on a chart.

INPUT:

- chart – a coordinate chart, as an instance of class *Chart*

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring(); FR
Ring of chart functions on Chart (M, (x, y))
sage: type(FR)
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category'>
sage: FR.category()
Category of commutative algebras over Symbolic Ring
```

Element

alias of *ChartFunction*

is_field()

Return False as self is not an integral domain.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring()
sage: FR.is_integral_domain()
False
sage: FR.is_field()
False
```

is_integral_domain()

Return False as self is not an integral domain.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring()
sage: FR.is_integral_domain()
False
```

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```
sage: FR.is_field()
False
```

one()

Return the constant function 1 in `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring()
sage: FR.one()
1

sage: M = Manifold(2, 'M', structure='topological', field=Qp(5))
sage: X.<x,y> = M.chart()
sage: X.function_ring().one()
1 + O(5^20)
```

zero()

Return the constant function 0 in `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: FR = X.function_ring()
sage: FR.zero()
0

sage: M = Manifold(2, 'M', structure='topological', field=Qp(5))
sage: X.<x,y> = M.chart()
sage: X.function_ring().zero()
0
```

class sage.manifolds.chart_func.**MultiCoordFunction**(*chart*, *expressions*)

Bases: `sage.structure.sage_object.SageObject`

Coordinate function to some Cartesian power of the base field.

If n and m are two positive integers and (U, φ) is a chart on a topological manifold M of dimension n over a topological field K, a *multi-coordinate function* associated to (U, φ) is a map

$$\begin{array}{llcl} f: & V \subset K^n & \longrightarrow & K^m \\ & (x^1, \ldots, x^n) & \longmapsto & (f_1(x^1, \ldots, x^n), \ldots, f_m(x^1, \ldots, x^n)), \end{array}$$

where V is the codomain of φ. In other words, f is a K^m-valued function of the coordinates associated to the chart (U, φ). Each component f_i $(1 \leq i \leq m)$ is a coordinate function and is therefore stored as a `ChartFunction`.

INPUT:

- `chart` – the chart (U, φ)
- `expressions` – list (or tuple) of length m of elements to construct the coordinate functions f_i $(1 \leq i \leq m)$; for symbolic coordinate functions, this must be symbolic expressions involving the chart coordinates, while for numerical coordinate functions, this must be data file names

EXAMPLES:

A function $f : V \subset \mathbf{R}^2 \longrightarrow \mathbf{R}^3$:

```
sage: forget()  # to clear the previous assumption on x
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, cos(x)*exp(y)); f
Coordinate functions (x - y, x*y, cos(x)*e^y) on the Chart (M, (x, y))
sage: type(f)
<class 'sage.manifolds.chart_func.MultiCoordFunction'>
sage: f(x,y)
(x - y, x*y, cos(x)*e^y)
sage: latex(f)
\left(x - y, x y, \cos\left(x\right) e^{y}\right)
```

Each real-valued function f_i ($1 \leq i \leq m$) composing f can be accessed via the square-bracket operator, by providing $i-1$ as an argument:

```
sage: f[0]
x - y
sage: f[1]
x*y
sage: f[2]
cos(x)*e^y
```

We can give a more verbose explanation of each function:

```
sage: f[0].display()
(x, y) |--> x - y
```

Each `f[i-1]` is an instance of *ChartFunction*:

```
sage: isinstance(f[0], sage.manifolds.chart_func.ChartFunction)
True
```

A class *MultiCoordFunction* can represent a real-valued function (case $m = 1$), although one should rather employ the class *ChartFunction* for this purpose:

```
sage: g = X.multifunction(x*y^2)
sage: g(x,y)
(x*y^2,)
```

Evaluating the functions at specified coordinates:

```
sage: f(1,2)
(-1, 2, cos(1)*e^2)
sage: var('a b')
(a, b)
sage: f(a,b)
(a - b, a*b, cos(a)*e^b)
sage: g(1,2)
(4,)
```

chart()

Return the chart with respect to which `self` is defined.

OUTPUT:

- a *Chart*

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, cos(x)*exp(y))
sage: f.chart()
Chart (M, (x, y))
sage: f.chart() is X
True
```

expr(*method=None*)

Return a tuple of data, the item no. i being sufficient to reconstruct the coordinate function no. i.

In other words, if `f` is a multi-coordinate function, then `f.chart().multifunction(*(f.expr()))` results in a multi-coordinate function identical to `f`.

INPUT:

- `method` – string (default: `None`): the calculus method which the returned expressions belong to; one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the chart current calculus method is assumed

OUTPUT:

- a tuple of the symbolic expressions of the chart functions composing `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, cos(x)*exp(y))
sage: f.expr()
(x - y, x*y, cos(x)*e^y)
sage: type(f.expr()[0])
<type 'sage.symbolic.expression.Expression'>
```

A SymPy output:

```
sage: f.expr('sympy')
(x - y, x*y, exp(y)*cos(x))
sage: type(f.expr('sympy')[0])
<class 'sympy.core.add.Add'>
```

One shall always have:

```
sage: f.chart().multifunction(*(f.expr())) == f
True
```

jacobian()

Return the Jacobian matrix of the system of coordinate functions.

`jacobian()` is a 2-dimensional array of size $m \times n$, where m is the number of functions and n the number of coordinates, the generic element being $J_{ij} = \frac{\partial f_i}{\partial x^j}$ with $1 \leq i \leq m$ (row index) and $1 \leq j \leq n$ (column index).

OUTPUT:

- Jacobian matrix as a 2-dimensional array `J` of coordinate functions with `J[i-1][j-1]` being $J_{ij} = \frac{\partial f_i}{\partial x^j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$

EXAMPLES:

Jacobian of a set of 3 functions of 2 coordinates:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, y^3*cos(x))
sage: f.jacobian()
[           1           -1]
[           y            x]
[ -y^3*sin(x) 3*y^2*cos(x)]
```

Each element of the result is a *chart function*:

```
sage: type(f.jacobian()[2,0])
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
sage: f.jacobian()[2,0].display()
(x, y) |--> -y^3*sin(x)
```

Test of the computation:

```
sage: [[f.jacobian()[i,j] == f[i].diff(j) for j in range(2)] for i in␣
↪range(3)]
[[True, True], [True, True], [True, True]]
```

Test with `start_index = 1`:

```
sage: M = Manifold(2, 'M', structure='topological', start_index=1)
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y, y^3*cos(x))
sage: f.jacobian()
[           1           -1]
[           y            x]
[ -y^3*sin(x) 3*y^2*cos(x)]
sage: [[f.jacobian()[i,j] == f[i].diff(j+1) for j in range(2)]  # note the j+1
....:                                        for i in range(3)]
[[True, True], [True, True], [True, True]]
```

jacobian_det()

Return the Jacobian determinant of the system of functions.

The number m of coordinate functions must equal the number n of coordinates.

OUTPUT:

- a *ChartFunction* representing the determinant

EXAMPLES:

Jacobian determinant of a set of 2 functions of 2 coordinates:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = X.multifunction(x-y, x*y)
sage: f.jacobian_det()
x + y
```

The output of *jacobian_det()* is an instance of *ChartFunction* and can therefore be called on specific values of the coordinates, e.g. $(x, y) = (1, 2)$:

```
sage: type(f.jacobian_det())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
sage: f.jacobian_det().display()
(x, y) |--> x + y
sage: f.jacobian_det()(1,2)
3
```

The result is cached:

```
sage: f.jacobian_det() is f.jacobian_det()
True
```

We verify the determinant of the Jacobian:

```
sage: f.jacobian_det() == det(matrix([[f[i].diff(j).expr() for j in range(2)]
....:                             for i in range(2)]))
True
```

An example using SymPy:

```
sage: M.set_calculus_method('sympy')
sage: g = X.multifunction(x*y^3, e^x)
sage: g.jacobian_det()
-3*x*y**2*exp(x)
sage: type(g.jacobian_det().expr())
<class 'sympy.core.mul.Mul'>
```

Jacobian determinant of a set of 3 functions of 3 coordinates:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: X.<x,y,z> = M.chart()
sage: f = X.multifunction(x*y+z^2, z^2*x+y^2*z, (x*y*z)^3)
sage: f.jacobian_det().display()
(x, y, z) |--> 6*x^3*y^5*z^3 - 3*x^4*y^3*z^4 - 12*x^2*y^4*z^5 + 6*x^3*y^2*z^6
```

We verify the determinant of the Jacobian:

```
sage: f.jacobian_det() == det(matrix([[f[i].diff(j).expr() for j in range(3)]
....:                             for i in range(3)]))
True
```

1.5.3 Coordinate calculus methods

The class `CalculusMethod` governs the calculus methods (symbolic and numerical) used for coordinate computations on manifolds.

AUTHORS:

- Marco Mancini (2017): initial version
- Eric Gourgoulhon (2019): add `set_simplify_function()` and various accessors

class `sage.manifolds.calculus_method.`**`CalculusMethod`**(*current=None*, *base_field_type='real'*)

Bases: `sage.structure.sage_object.SageObject`

Control of calculus backends used on coordinate charts of manifolds.

This class stores the possible calculus methods and permits to switch between them, as well as to change the simplifying functions associated with them. For the moment, only two calculus backends are implemented:

- Sage's symbolic engine (Pynac + Maxima), implemented via the Symbolic Ring `SR`
- SymPy engine, denoted `sympy` hereafter

INPUT:

- `current` – (default: `None`) string defining the calculus method that will be considered as the active one, until it is changed by `set()`; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the default calculus method (`'SR'`)
- `base_field_type` – (default: `'real'`) base field type of the manifold (cf. `base_field_type()`)

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod()
```

In the display, the currently active method is pointed out with a star:

```
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

It can be changed with `set()`:

```
sage: cm.set('sympy')
sage: cm
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
```

while `reset()` brings back to the default:

```
sage: cm.reset()
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

See `simplify_function()` for the default simplification algorithms associated with each calculus method and `set_simplify_function()` for introducing a new simplification algorithm.

current()

Return the active calculus method as a string.

OUTPUT:

- string defining the calculus method, one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(); cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.current()
'SR'
sage: cm.set('sympy')
sage: cm.current()
'sympy'
```

is_trivial_zero(*expression*, *method=None*)

Check if an expression is trivially equal to zero without any simplification.

INPUT:

- `expression` – expression
- `method` – (default: `None`) string defining the calculus method to use; if `None` the current calculus method of `self` is used.

OUTPUT:

- `True` is expression is trivially zero, `False` elsewhere.

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(base_field_type='real')
sage: f = sin(x) - sin(x)
sage: cm.is_trivial_zero(f)
True
sage: cm.is_trivial_zero(f._sympy_(), method='sympy')
True
```

```
sage: f = sin(x)^2 + cos(x)^2 - 1
sage: cm.is_trivial_zero(f)
False
sage: cm.is_trivial_zero(f._sympy_(), method='sympy')
False
```

reset()

Set the current calculus method to the default one.

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(base_field_type='complex')
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.set('sympy')
sage: cm
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
sage: cm.reset()
sage: cm
```

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```
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
```

set (*method*)

Set the currently active calculus method.

- `method` – string defining the calculus method

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(base_field_type='complex')
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.set('sympy')
sage: cm
Available calculus methods (* = current):
 - SR (default)
 - sympy (*)
sage: cm.set('lala')
Traceback (most recent call last):
...
NotImplementedError: method lala not implemented
```

set_simplify_function (*simplifying_func*, *method=None*)

Set the simplifying function associated to a given calculus method.

INPUT:

- `simplifying_func` – either the string `'default'` for restoring the default simplifying function or a function `f` of a single argument `expr` such that `f(expr)` returns an object of the same type as `expr` (hopefully the simplified version of `expr`), this type being
  - `Expression` if `method` = `'SR'`
  - a SymPy type if `method` = `'sympy'`
- `method` – (default: `None`) string defining the calculus method for which `simplifying_func` is provided; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the currently active calculus method of `self` is assumed

EXAMPLES:

On a real manifold, the default simplifying function is *simplify_chain_real()* when the calculus method is `SR`:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(base_field_type='real'); cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.simplify_function() is \
```

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```
....: sage.manifolds.utilities.simplify_chain_real
True
```

Let us change it to `simplify()`:

```
sage: cm.set_simplify_function(simplify)
sage: cm.simplify_function() is simplify
True
```

Since `SR` is the current calculus method, the above is equivalent to:

```
sage: cm.set_simplify_function(simplify, method='SR')
sage: cm.simplify_function(method='SR') is simplify
True
```

We revert to the default simplifying function by:

```
sage: cm.set_simplify_function('default')
```

Then we are back to:

```
sage: cm.simplify_function() is \
....: sage.manifolds.utilities.simplify_chain_real
True
```

simplify(*expression*, *method=None*)

Apply the simplifying function associated with a given calculus method to a symbolic expression.

INPUT:

- `expression` – symbolic expression to simplify
- `method` – (default: `None`) string defining the calculus method to use; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the current calculus method of `self` is used.

OUTPUT:

- the simplified version of `expression`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x, y> = M.chart()
sage: f = x^2 + sin(x)^2 + cos(x)^2
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod(base_field_type='real')
sage: cm.simplify(f)
x^2 + 1
```

Using a weaker simplifying function, like `simplify()`, does not succeed in this case:

```
sage: cm.set_simplify_function(simplify)
sage: cm.simplify(f)
x^2 + cos(x)^2 + sin(x)^2
```

Back to the default simplifying function (`simplify_chain_real()` in the present case):

```
sage: cm.set_simplify_function('default')
sage: cm.simplify(f)
x^2 + 1
```

A `SR` expression, such as `f`, cannot be simplified when the current calculus method is `sympy`:

```
sage: cm.set('sympy')
sage: cm.simplify(f)
Traceback (most recent call last):
...
AttributeError: 'sage.symbolic.expression.Expression' object has no attribute
↪'combsimp'
```

In the present case, one should either transform `f` to a SymPy object:

```
sage: cm.simplify(f._sympy_())
x**2 + 1
```

or force the calculus method to be `'SR'`:

```
sage: cm.simplify(f, method='SR')
x^2 + 1
```

simplify_function (*method=None*)

Return the simplifying function associated to a given calculus method.

The simplifying function is that used in all computations involved with the calculus method.

INPUT:

- `method` – (default: `None`) string defining the calculus method for which `simplifying_func` is provided; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the currently active calculus method of `self` is assumed

OUTPUT:

- the simplifying function

EXAMPLES:

```
sage: from sage.manifolds.calculus_method import CalculusMethod
sage: cm = CalculusMethod()
sage: cm
Available calculus methods (* = current):
 - SR (default) (*)
 - sympy
sage: cm.simplify_function()  # random (memory address)
<function simplify_chain_real at 0x7f958d5b6758>
```

The output stands for the function `simplify_chain_real()`:

```
sage: cm.simplify_function() is \
....: sage.manifolds.utilities.simplify_chain_real
True
```

Since SR is the default calculus method, we have:

```
sage: cm.simplify_function() is cm.simplify_function(method='SR')
True
```

The simplifying function associated with sympy is *simplify_chain_real_sympy()*:

```
sage: cm.simplify_function(method='sympy')  # random (memory address)
<function simplify_chain_real_sympy at 0x7f0b35a578c0>
sage: cm.simplify_function(method='sympy') is \
....: sage.manifolds.utilities.simplify_chain_real_sympy
True
```

On complex manifolds, the simplifying functions are *simplify_chain_generic()* and *simplify_chain_generic_sympy()* for respectively SR and sympy:

```
sage: cmc = CalculusMethod(base_field_type='complex')
sage: cmc.simplify_function(method='SR') is \
....: sage.manifolds.utilities.simplify_chain_generic
True
sage: cmc.simplify_function(method='sympy') is \
....: sage.manifolds.utilities.simplify_chain_generic_sympy
True
```

Note that the simplifying functions can be customized via *set_simplify_function()*.

1.6 Scalar Fields

1.6.1 Algebra of Scalar Fields

The class *ScalarFieldAlgebra* implements the commutative algebra $C^0(M)$ of scalar fields on a topological manifold M over a topological field K. By *scalar field*, it is meant a continuous function $M \to K$. The set $C^0(M)$ is an algebra over K, whose ring product is the pointwise multiplication of K-valued functions, which is clearly commutative.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [?]
- [?]

class sage.manifolds.scalarfield_algebra.**ScalarFieldAlgebra**(*domain*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

Commutative algebra of scalar fields on a topological manifold.

If M is a topological manifold over a topological field K, the commutative algebra of scalar fields on M is the set $C^0(M)$ of all continuous maps $M \to K$. The set $C^0(M)$ is an algebra over K, whose ring product is the pointwise multiplication of K-valued functions, which is clearly commutative.

If $K = \mathbf{R}$ or $K = \mathbf{C}$, the field K over which the algebra $C^0(M)$ is constructed is represented by the Symbolic Ring SR, since there is no exact representation of $\mathbf{R}$ nor $\mathbf{C}$.

INPUT:

- `domain` – the topological manifold M on which the scalar fields are defined

EXAMPLES:

Algebras of scalar fields on the sphere S^2 and on some open subsets of it:

```
sage: M = Manifold(2, 'M', structure='topological') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U')  # complement of the North pole
sage: c_xy.<x,y> = U.chart()  # stereographic coordinates from the North pole
sage: V = M.open_subset('V')  # complement of the South pole
sage: c_uv.<u,v> = V.chart()  # stereographic coordinates from the South pole
sage: M.declare_union(U,V)    # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1= x^2+y^2!=0,
....:                                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: CM = M.scalar_field_algebra(); CM
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: W = U.intersection(V)  # S^2 minus the two poles
sage: CW = W.scalar_field_algebra(); CW
Algebra of scalar fields on the Open subset W of the
 2-dimensional topological manifold M
```

$C^0(M)$ and $C^0(W)$ belong to the category of commutative algebras over $\mathbf{R}$ (represented here by `SymbolicRing`):

```
sage: CM.category()
Category of commutative algebras over Symbolic Ring
sage: CM.base_ring()
Symbolic Ring
sage: CW.category()
Category of commutative algebras over Symbolic Ring
sage: CW.base_ring()
Symbolic Ring
```

The elements of $C^0(M)$ are scalar fields on M:

```
sage: CM.an_element()
Scalar field on the 2-dimensional topological manifold M
sage: CM.an_element().display()   # this sample element is a constant field
M --> R
on U: (x, y) |--> 2
on V: (u, v) |--> 2
```

Those of $C^0(W)$ are scalar fields on W:

```
sage: CW.an_element()
Scalar field on the Open subset W of the 2-dimensional topological
 manifold M
sage: CW.an_element().display()   # this sample element is a constant field
W --> R
(x, y) |--> 2
(u, v) |--> 2
```

The zero element:

```
sage: CM.zero()
Scalar field zero on the 2-dimensional topological manifold M
sage: CM.zero().display()
zero: M --> R
on U: (x, y) |--> 0
on V: (u, v) |--> 0
```

```
sage: CW.zero()
Scalar field zero on the Open subset W of the 2-dimensional
 topological manifold M
sage: CW.zero().display()
zero: W --> R
   (x, y) |--> 0
   (u, v) |--> 0
```

The unit element:

```
sage: CM.one()
Scalar field 1 on the 2-dimensional topological manifold M
sage: CM.one().display()
1: M --> R
on U: (x, y) |--> 1
on V: (u, v) |--> 1
```

```
sage: CW.one()
Scalar field 1 on the Open subset W of the 2-dimensional topological
 manifold M
sage: CW.one().display()
1: W --> R
  (x, y) |--> 1
  (u, v) |--> 1
```

A generic element can be constructed by using a dictionary of the coordinate expressions defining the scalar field:

```
sage: f = CM({c_xy: atan(x^2+y^2), c_uv: pi/2 - atan(u^2+v^2)}); f
Scalar field on the 2-dimensional topological manifold M
sage: f.display()
M --> R
on U: (x, y) |--> arctan(x^2 + y^2)
on V: (u, v) |--> 1/2*pi - arctan(u^2 + v^2)
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
```

Specific elements can also be constructed in this way:

```
sage: CM(0) == CM.zero()
True
sage: CM(1) == CM.one()
True
```

Note that the zero scalar field is cached:

```
sage: CM(0) is CM.zero()
True
```

Elements can also be constructed by means of the method *scalar_field()* acting on the domain (this allows one to set the name of the scalar field at the construction):

```
sage: f1 = M.scalar_field({c_xy: atan(x^2+y^2), c_uv: pi/2 - atan(u^2+v^2)},
....:                     name='f')
sage: f1.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: f1 == f
True
sage: M.scalar_field(0, chart='all') == CM.zero()
True
```

The algebra $C^0(M)$ coerces to $C^0(W)$ since W is an open subset of M:

```
sage: CW.has_coerce_map_from(CM)
True
```

The reverse is of course false:

```
sage: CM.has_coerce_map_from(CW)
False
```

The coercion map is nothing but the restriction to W of scalar fields on M:

```
sage: fW = CW(f) ; fW
Scalar field on the Open subset W of the
 2-dimensional topological manifold M
sage: fW.display()
W --> R
   (x, y) |--> arctan(x^2 + y^2)
   (u, v) |--> 1/2*pi - arctan(u^2 + v^2)
```

```
sage: CW(CM.one()) == CW.one()
True
```

The coercion map allows for the addition of elements of $C^0(W)$ with elements of $C^0(M)$, the result being an element of $C^0(W)$:

```
sage: s = fW + f
sage: s.parent()
Algebra of scalar fields on the Open subset W of the
 2-dimensional topological manifold M
sage: s.display()
W --> R
   (x, y) |--> 2*arctan(x^2 + y^2)
   (u, v) |--> pi - 2*arctan(u^2 + v^2)
```

Another coercion is that from the Symbolic Ring. Since the Symbolic Ring is the base ring for the algebra `CM`, the coercion of a symbolic expression `s` is performed by the operation `s*CM.one()`, which invokes the (reflected) multiplication operator. If the symbolic expression does not involve any chart coordinate, the outcome is a constant scalar field:

```
sage: h = CM(pi*sqrt(2)) ; h
Scalar field on the 2-dimensional topological manifold M
sage: h.display()
M --> R
on U: (x, y) |--> sqrt(2)*pi
```

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```
on V: (u, v) |--> sqrt(2)*pi
sage: a = var('a')
sage: h = CM(a); h.display()
M --> R
on U: (x, y) |--> a
on V: (u, v) |--> a
```

If the symbolic expression involves some coordinate of one of the manifold's charts, the outcome is initialized only on the chart domain:

```
sage: h = CM(a+x); h.display()
M --> R
on U: (x, y) |--> a + x
sage: h = CM(a+u); h.display()
M --> R
on V: (u, v) |--> a + u
```

If the symbolic expression involves coordinates of different charts, the scalar field is created as a Python object, but is not initialized, in order to avoid any ambiguity:

```
sage: h = CM(x+u); h.display()
M --> R
```

```
sage: s = f - h ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> -sqrt(2)*pi + arctan(x^2 + y^2)
on V: (u, v) |--> -1/2*pi*(2*sqrt(2) - 1) - arctan(u^2 + v^2)
```

```
sage: s = f*h ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> sqrt(2)*pi*arctan(x^2 + y^2)
on V: (u, v) |--> 1/2*sqrt(2)*(pi^2 - 2*pi*arctan(u^2 + v^2))
```

```
sage: s = f/h ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> 1/2*sqrt(2)*arctan(x^2 + y^2)/pi
on V: (u, v) |--> 1/4*sqrt(2)*(pi - 2*arctan(u^2 + v^2))/pi
```

```
sage: f*(h+f) == f*h + f*f
True
```

Ring laws with coercion:

```
sage: f - fW == CW.zero()
True
sage: f/fW == CW.one()
True
sage: s = f*fW ; s
Scalar field on the Open subset W of the 2-dimensional topological
```

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```
 manifold M
sage: s.display()
W --> R
(x, y) |--> arctan(x^2 + y^2)^2
(u, v) |--> 1/4*pi^2 - pi*arctan(u^2 + v^2) + arctan(u^2 + v^2)^2
sage: s/f == fW
True
```

Multiplication by a real number:

```
sage: s = 2*f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> 2*arctan(x^2 + y^2)
on V: (u, v) |--> pi - 2*arctan(u^2 + v^2)
```

```
sage: 0*f == CM.zero()
True
sage: 1*f == f
True
sage: 2*(f/2) == f
True
sage: (f+2*f)/3 == f
True
sage: 1/3*(f+2*f) == f
True
```

The Sage test suite for algebras is passed:

```
sage: TestSuite(CM).run()
```

It is passed also for $C^0(W)$:

```
sage: TestSuite(CW).run()
```

Element
alias of *sage.manifolds.scalarfield.ScalarField*

one()
Return the unit element of the algebra.

This is nothing but the constant scalar field 1 on the manifold, where 1 is the unit element of the base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: CM = M.scalar_field_algebra()
sage: h = CM.one(); h
Scalar field 1 on the 2-dimensional topological manifold M
sage: h.display()
1: M --> R
   (x, y) |--> 1
```

The result is cached:

```
sage: CM.one() is h
True
```

zero()

Return the zero element of the algebra.

This is nothing but the constant scalar field 0 on the manifold, where 0 is the zero element of the base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: CM = M.scalar_field_algebra()
sage: z = CM.zero(); z
Scalar field zero on the 2-dimensional topological manifold M
sage: z.display()
zero: M --> R
   (x, y) |--> 0
```

The result is cached:

```
sage: CM.zero() is z
True
```

1.6.2 Scalar Fields

Given a topological manifold M over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a *scalar field* on M is a continuous map

$$f : M \longrightarrow K$$

Scalar fields are implemented by the class `ScalarField`.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Travis Scrimshaw (2016): review tweaks
- Marco Mancini (2017): SymPy as an optional symbolic engine, alternative to SR
- Florentin Jaffredo (2018) : series expansion with respect to a given parameter

REFERENCES:

- [?]
- [?]

class sage.manifolds.scalarfield.**ScalarField**(*parent*, *coord_expression=None*, *chart=None*, *name=None*, *latex_name=None*)

Bases: `sage.structure.element.CommutativeAlgebraElement`

Scalar field on a topological manifold.

Given a topological manifold M over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a *scalar field on* M is a continuous map

$$f : M \longrightarrow K.$$

A scalar field on M is an element of the commutative algebra $C^0(M)$ (see *ScalarFieldAlgebra*).

INPUT:

- `parent` – the algebra of scalar fields containing the scalar field (must be an instance of class *ScalarFieldAlgebra*)
- `coord_expression` – (default: `None`) coordinate expression(s) of the scalar field; this can be either
  - a dictionary of coordinate expressions in various charts on the domain, with the charts as keys;
  - a single coordinate expression; if the argument `chart` is `'all'`, this expression is set to all the charts defined on the open set; otherwise, the expression is set in the specific chart provided by the argument `chart`
- `chart` – (default: `None`) chart defining the coordinates used in `coord_expression` when the latter is a single coordinate expression; if none is provided (default), the default chart of the open set is assumed. If `chart=='all'`, `coord_expression` is assumed to be independent of the chart (constant scalar field).
- `name` – (default: `None`) string; name (symbol) given to the scalar field
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the scalar field; if none is provided, the LaTeX symbol is set to `name`

If `coord_expression` is `None` or incomplete, coordinate expressions can be added after the creation of the object, by means of the methods *add_expr()*, *add_expr_by_continuation()* and *set_expr()*.

EXAMPLES:

A scalar field on the 2-sphere:

```
sage: M = Manifold(2, 'M', structure='topological') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1= x^2+y^2!=0,
....:                                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                    name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

For scalar fields defined by a single coordinate expression, the latter can be passed instead of the dictionary over the charts:

```
sage: g = U.scalar_field(x*y, chart=c_xy, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional topological
 manifold M
```

The above is indeed equivalent to:

```
sage: g = U.scalar_field({c_xy: x*y}, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional topological
 manifold M
```

Since `c_xy` is the default chart of `U`, the argument `chart` can be skipped:

```
sage: g = U.scalar_field(x*y, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional topological
 manifold M
```

The scalar field g is defined on U and has an expression in terms of the coordinates (u, v) on $W = U \cap V$:

```
sage: g.display()
g: U --> R
   (x, y) |--> x*y
on W: (u, v) |--> u*v/(u^4 + 2*u^2*v^2 + v^4)
```

Scalar fields on M can also be declared with a single chart:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
```

Their definition must then be completed by providing the expressions on other charts, via the method *add_expr()*, to get a global cover of the manifold:

```
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

We can even first declare the scalar field without any coordinate expression and provide them subsequently:

```
sage: f = M.scalar_field(name='f')
sage: f.add_expr(1/(1+x^2+y^2), chart=c_xy)
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

We may also use the method *add_expr_by_continuation()* to complete the coordinate definition using the analytic continuation from domains in which charts overlap:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
sage: f.add_expr_by_continuation(c_uv, U.intersection(V))
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

A scalar field can also be defined by some unspecified function of the coordinates:

```
sage: h = U.scalar_field(function('H')(x, y), name='h') ; h
Scalar field h on the Open subset U of the 2-dimensional topological
 manifold M
```

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```
sage: h.display()
h: U --> R
   (x, y) |--> H(x, y)
on W: (u, v) |--> H(u/(u^2 + v^2), v/(u^2 + v^2))
```

We may use the argument `latex_name` to specify the LaTeX symbol denoting the scalar field if the latter is different from `name`:

```
sage: latex(f)
f
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                    name='f', latex_name=r'\mathcal{F}')
sage: latex(f)
\mathcal{F}
```

The coordinate expression in a given chart is obtained via the method *expr()*, which returns a symbolic expression:

```
sage: f.expr(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
sage: type(f.expr(c_uv))
<type 'sage.symbolic.expression.Expression'>
```

The method *coord_function()* returns instead a function of the chart coordinates, i.e. an instance of *ChartFunction*:

```
sage: f.coord_function(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
sage: type(f.coord_function(c_uv))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: f.coord_function(c_uv).display()
(u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

The value returned by the method *expr()* is actually the coordinate expression of the chart function:

```
sage: f.expr(c_uv) is f.coord_function(c_uv).expr()
True
```

A constant scalar field is declared by setting the argument `chart` to `'all'`:

```
sage: c = M.scalar_field(2, chart='all', name='c') ; c
Scalar field c on the 2-dimensional topological manifold M
sage: c.display()
c: M --> R
on U: (x, y) |--> 2
on V: (u, v) |--> 2
```

A shortcut is to use the method *constant_scalar_field()*:

```
sage: c == M.constant_scalar_field(2)
True
```

The constant value can be some unspecified parameter:

```
sage: var('a')
a
```

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```
sage: c = M.constant_scalar_field(a, name='c') ; c
Scalar field c on the 2-dimensional topological manifold M
sage: c.display()
c: M --> R
on U: (x, y) |--> a
on V: (u, v) |--> a
```

A special case of constant field is the zero scalar field:

```
sage: zer = M.constant_scalar_field(0) ; zer
Scalar field zero on the 2-dimensional topological manifold M
sage: zer.display()
zero: M --> R
on U: (x, y) |--> 0
on V: (u, v) |--> 0
```

It can be obtained directly by means of the function `zero_scalar_field()`:

```
sage: zer is M.zero_scalar_field()
True
```

A third way is to get it as the zero element of the algebra $C^0(M)$ of scalar fields on M (see below):

```
sage: zer is M.scalar_field_algebra().zero()
True
```

By definition, a scalar field acts on the manifold's points, sending them to elements of the manifold's base field (real numbers in the present case):

```
sage: N = M.point((0,0), chart=c_uv) # the North pole
sage: S = M.point((0,0), chart=c_xy) # the South pole
sage: E = M.point((1,0), chart=c_xy) # a point at the equator
sage: f(N)
0
sage: f(S)
1
sage: f(E)
1/2
sage: h(E)
H(1, 0)
sage: c(E)
a
sage: zer(E)
0
```

A scalar field can be compared to another scalar field:

```
sage: f == g
False
```

. . . to a symbolic expression:

```
sage: f == x*y
False
sage: g == x*y
True
```

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```
sage: c == a
True
```

...to a number:

```
sage: f == 2
False
sage: zer == 0
True
```

...to anything else:

```
sage: f == M
False
```

Standard mathematical functions are implemented:

```
sage: sqrt(f)
Scalar field sqrt(f) on the 2-dimensional topological manifold M
sage: sqrt(f).display()
sqrt(f): M --> R
on U: (x, y) |--> 1/sqrt(x^2 + y^2 + 1)
on V: (u, v) |--> sqrt(u^2 + v^2)/sqrt(u^2 + v^2 + 1)
```

```
sage: tan(f)
Scalar field tan(f) on the 2-dimensional topological manifold M
sage: tan(f).display()
tan(f): M --> R
on U: (x, y) |--> sin(1/(x^2 + y^2 + 1))/cos(1/(x^2 + y^2 + 1))
on V: (u, v) |--> sin((u^2 + v^2)/(u^2 + v^2 + 1))/cos((u^2 + v^2)/(u^2 + v^2 +
→1))
```

Arithmetics of scalar fields

Scalar fields on M (resp. U) belong to the algebra $C^0(M)$ (resp. $C^0(U)$):

```
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: f.parent() is M.scalar_field_algebra()
True
sage: g.parent()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold M
sage: g.parent() is U.scalar_field_algebra()
True
```

Consequently, scalar fields can be added:

```
sage: s = f + c ; s
Scalar field f+c on the 2-dimensional topological manifold M
sage: s.display()
f+c: M --> R
on U: (x, y) |--> (a*x^2 + a*y^2 + a + 1)/(x^2 + y^2 + 1)
on V: (u, v) |--> ((a + 1)*u^2 + (a + 1)*v^2 + a)/(u^2 + v^2 + 1)
```

and subtracted:

```
sage: s = f - c ; s
Scalar field f-c on the 2-dimensional topological manifold M
sage: s.display()
f-c: M --> R
on U: (x, y) |--> -(a*x^2 + a*y^2 + a - 1)/(x^2 + y^2 + 1)
on V: (u, v) |--> -((a - 1)*u^2 + (a - 1)*v^2 + a)/(u^2 + v^2 + 1)
```

Some tests:

```
sage: f + zer == f
True
sage: f - f == zer
True
sage: f + (-f) == zer
True
sage: (f+c)-f == c
True
sage: (f-c)+c == f
True
```

We may add a number (interpreted as a constant scalar field) to a scalar field:

```
sage: s = f + 1 ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
on V: (u, v) |--> (2*u^2 + 2*v^2 + 1)/(u^2 + v^2 + 1)
sage: (f+1)-1 == f
True
```

The number can represented by a symbolic variable:

```
sage: s = a + f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s == c + f
True
```

However if the symbolic variable is a chart coordinate, the addition is performed only on the chart domain:

```
sage: s = f + x; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> (x^3 + x*y^2 + x + 1)/(x^2 + y^2 + 1)
sage: s = f + u; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on V: (u, v) |--> (u^3 + (u + 1)*v^2 + u^2 + u)/(u^2 + v^2 + 1)
```

The addition of two scalar fields with different domains is possible if the domain of one of them is a subset of the domain of the other; the domain of the result is then this subset:

```
sage: f.domain()
2-dimensional topological manifold M
```

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```
sage: g.domain()
Open subset U of the 2-dimensional topological manifold M
sage: s = f + g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.domain()
Open subset U of the 2-dimensional topological manifold M
sage: s.display()
U --> R
(x, y) |--> (x*y^3 + (x^3 + x)*y + 1)/(x^2 + y^2 + 1)
on W: (u, v) |--> (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6 + u*v^3
 + (u^3 + u)*v)/(u^6 + v^6 + (3*u^2 + 1)*v^4 + u^4 + (3*u^4 + 2*u^2)*v^2)
```

The operation actually performed is $f|_U + g$:

```
sage: s == f.restrict(U) + g
True
```

In Sage framework, the addition of f and g is permitted because there is a *coercion* of the parent of f, namely $C^0(M)$, to the parent of g, namely $C^0(U)$ (see `ScalarFieldAlgebra`):

```
sage: CM = M.scalar_field_algebra()
sage: CU = U.scalar_field_algebra()
sage: CU.has_coerce_map_from(CM)
True
```

The coercion map is nothing but the restriction to domain U:

```
sage: CU.coerce(f) == f.restrict(U)
True
```

Since the algebra $C^0(M)$ is a vector space over $\mathbf{R}$, scalar fields can be multiplied by a number, either an explicit one:

```
sage: s = 2*f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> 2/(x^2 + y^2 + 1)
on V: (u, v) |--> 2*(u^2 + v^2)/(u^2 + v^2 + 1)
```

or a symbolic one:

```
sage: s = a*f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> a/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)*a/(u^2 + v^2 + 1)
```

However, if the symbolic variable is a chart coordinate, the multiplication is performed only in the corresponding chart:

```
sage: s = x*f; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
```

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```
M --> R
on U: (x, y) |--> x/(x^2 + y^2 + 1)
sage: s = u*f; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on V: (u, v) |--> (u^2 + v^2)*u/(u^2 + v^2 + 1)
```

Some tests:

```
sage: 0*f == 0
True
sage: 0*f == zer
True
sage: 1*f == f
True
sage: (-2)*f == - f - f
True
```

The ring multiplication of the algebras $C^0(M)$ and $C^0(U)$ is the pointwise multiplication of functions:

```
sage: s = f*f ; s
Scalar field f*f on the 2-dimensional topological manifold M
sage: s.display()
f*f: M --> R
on U: (x, y) |--> 1/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) |--> (u^4 + 2*u^2*v^2 + v^4)/(u^4 + v^4 + 2*(u^2 + 1)*v^2
 + 2*u^2 + 1)
sage: s = g*h ; s
Scalar field g*h on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
g*h: U --> R
   (x, y) |--> x*y*H(x, y)
on W: (u, v) |--> u*v*H(u/(u^2 + v^2), v/(u^2 + v^2))/(u^4 + 2*u^2*v^2 + v^4)
```

Thanks to the coercion $C^0(M) \to C^0(U)$ mentioned above, it is possible to multiply a scalar field defined on M by a scalar field defined on U, the result being a scalar field defined on U:

```
sage: f.domain(), g.domain()
(2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M)
sage: s = f*g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> x*y/(x^2 + y^2 + 1)
on W: (u, v) |--> u*v/(u^4 + v^4 + (2*u^2 + 1)*v^2 + u^2)
sage: s == f.restrict(U)*g
True
```

Scalar fields can be divided (pointwise division):

```
sage: s = f/c ; s
Scalar field f/c on the 2-dimensional topological manifold M
```

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```
sage: s.display()
f/c: M --> R
on U: (x, y) |--> 1/(a*x^2 + a*y^2 + a)
on V: (u, v) |--> (u^2 + v^2)/(a*u^2 + a*v^2 + a)
sage: s = g/h ; s
Scalar field g/h on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
g/h: U --> R
   (x, y) |--> x*y/H(x, y)
on W: (u, v) |--> u*v/((u^4 + 2*u^2*v^2 + v^4)*H(u/(u^2 + v^2), v/(u^2 + v^2)))
sage: s = f/g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> 1/(x*y^3 + (x^3 + x)*y)
on W: (u, v) |--> (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)/(u*v^3 + (u^3 + u)*v)
sage: s == f.restrict(U)/g
True
```

For scalar fields defined on a single chart domain, we may perform some arithmetics with symbolic expressions involving the chart coordinates:

```
sage: s = g + x^2 - y ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> x^2 + (x - 1)*y
on W: (u, v) |--> -(v^3 - u^2 + (u^2 - u)*v)/(u^4 + 2*u^2*v^2 + v^4)
```

```
sage: s = g*x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> x^2*y
on W: (u, v) |--> u^2*v/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)
```

```
sage: s = g/x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> y
on W: (u, v) |--> v/(u^2 + v^2)
sage: s = x/g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> 1/y
on W: (u, v) |--> (u^2 + v^2)/v
```

Examples with SymPy as the symbolic engine

From now on, we ask that all symbolic calculus on manifold M are performed by SymPy:

```
sage: M.set_calculus_method('sympy')
```

We define f as above:

```
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                     name='f') ; f
Scalar field f on the 2-dimensional topological manifold M
sage: f.display()  # notice the SymPy display of exponents
f: M --> R
on U: (x, y) |--> 1/(x**2 + y**2 + 1)
on V: (u, v) |--> (u**2 + v**2)/(u**2 + v**2 + 1)
sage: type(f.coord_function(c_xy).expr())
<class 'sympy.core.power.Pow'>
```

The scalar field g defined on U:

```
sage: g = U.scalar_field({c_xy: x*y}, name='g')
sage: g.display() # again notice the SymPy display of exponents
g: U --> R
    (x, y) |--> x*y
on W: (u, v) |--> u*v/(u**4 + 2*u**2*v**2 + v**4)
```

Definition on a single chart and subsequent completion:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f')
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x**2 + y**2 + 1)
on V: (u, v) |--> (u**2 + v**2)/(u**2 + v**2 + 1)
```

Defintion without any coordinate expression and subsequent completion:

```
sage: f = M.scalar_field(name='f')
sage: f.add_expr(1/(1+x^2+y^2), chart=c_xy)
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x**2 + y**2 + 1)
on V: (u, v) |--> (u**2 + v**2)/(u**2 + v**2 + 1)
```

Use of *add_expr_by_continuation()*:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f')
sage: f.add_expr_by_continuation(c_uv, U.intersection(V))
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x**2 + y**2 + 1)
on V: (u, v) |--> (u**2 + v**2)/(u**2 + v**2 + 1)
```

A scalar field defined by some unspecified function of the coordinates:

```
sage: h = U.scalar_field(function('H')(x, y), name='h') ; h
Scalar field h on the Open subset U of the 2-dimensional topological
```

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```
 manifold M
sage: h.display()
h: U --> R
   (x, y) |--> H(x, y)
on W: (u, v) |--> H(u/(u**2 + v**2), v/(u**2 + v**2))
```

The coordinate expression in a given chart is obtained via the method `expr()`, which in the present context, returns a SymPy object:

```
sage: f.expr(c_uv)
(u**2 + v**2)/(u**2 + v**2 + 1)
sage: type(f.expr(c_uv))
<class 'sympy.core.mul.Mul'>
```

The method `coord_function()` returns instead a function of the chart coordinates, i.e. an instance of `ChartFunction`:

```
sage: f.coord_function(c_uv)
(u**2 + v**2)/(u**2 + v**2 + 1)
sage: type(f.coord_function(c_uv))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: f.coord_function(c_uv).display()
(u, v) |--> (u**2 + v**2)/(u**2 + v**2 + 1)
```

The value returned by the method `expr()` is actually the coordinate expression of the chart function:

```
sage: f.expr(c_uv) is f.coord_function(c_uv).expr()
True
```

We may ask for the `SR` representation of the coordinate function:

```
sage: f.coord_function(c_uv).expr('SR')
(u^2 + v^2)/(u^2 + v^2 + 1)
```

A constant scalar field with SymPy representation:

```
sage: c = M.constant_scalar_field(2, name='c')
sage: c.display()
c: M --> R
on U: (x, y) |--> 2
on V: (u, v) |--> 2
sage: type(c.expr(c_xy))
<class 'sympy.core.numbers.Integer'>
```

The constant value can be some unspecified parameter:

```
sage: var('a')
a
sage: c = M.constant_scalar_field(a, name='c')
sage: c.display()
c: M --> R
on U: (x, y) |--> a
on V: (u, v) |--> a
sage: type(c.expr(c_xy))
<class 'sympy.core.symbol.Symbol'>
```

The zero scalar field:

```
sage: zer = M.constant_scalar_field(0) ; zer
Scalar field zero on the 2-dimensional topological manifold M
sage: zer.display()
zero: M --> R
on U: (x, y) |--> 0
on V: (u, v) |--> 0
sage: type(zer.expr(c_xy))
<class 'sympy.core.numbers.Zero'>
sage: zer is M.zero_scalar_field()
True
```

Action of scalar fields on manifold's points:

```
sage: N = M.point((0,0), chart=c_uv) # the North pole
sage: S = M.point((0,0), chart=c_xy) # the South pole
sage: E = M.point((1,0), chart=c_xy) # a point at the equator
sage: f(N)
0
sage: f(S)
1
sage: f(E)
1/2
sage: h(E)
H(1, 0)
sage: c(E)
a
sage: zer(E)
0
```

A scalar field can be compared to another scalar field:

```
sage: f == g
False
```

. . . to a symbolic expression:

```
sage: f == x*y
False
sage: g == x*y
True
sage: c == a
True
```

. . . to a number:

```
sage: f == 2
False
sage: zer == 0
True
```

. . . to anything else:

```
sage: f == M
False
```

Standard mathematical functions are implemented:

```
sage: sqrt(f)
Scalar field sqrt(f) on the 2-dimensional topological manifold M
sage: sqrt(f).display()
sqrt(f): M --> R
on U: (x, y) |--> 1/sqrt(x**2 + y**2 + 1)
on V: (u, v) |--> sqrt(u**2 + v**2)/sqrt(u**2 + v**2 + 1)
```

```
sage: tan(f)
Scalar field tan(f) on the 2-dimensional topological manifold M
sage: tan(f).display()
tan(f): M --> R
on U: (x, y) |--> tan(1/(x**2 + y**2 + 1))
on V: (u, v) |--> tan((u**2 + v**2)/(u**2 + v**2 + 1))
```

Arithmetics of scalar fields with SymPy

Scalar fields on M (resp. U) belong to the algebra $C^0(M)$ (resp. $C^0(U)$):

```
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological manifold M
sage: f.parent() is M.scalar_field_algebra()
True
sage: g.parent()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold M
sage: g.parent() is U.scalar_field_algebra()
True
```

Consequently, scalar fields can be added:

```
sage: s = f + c ; s
Scalar field f+c on the 2-dimensional topological manifold M
sage: s.display()
f+c: M --> R
on U: (x, y) |--> (a*x**2 + a*y**2 + a + 1)/(x**2 + y**2 + 1)
on V: (u, v) |--> (a*u**2 + a*v**2 + a + u**2 + v**2)/(u**2 + v**2 + 1)
```

and subtracted:

```
sage: s = f - c ; s
Scalar field f-c on the 2-dimensional topological manifold M
sage: s.display()
f-c: M --> R
on U: (x, y) |--> (-a*x**2 - a*y**2 - a + 1)/(x**2 + y**2 + 1)
on V: (u, v) |--> (-a*u**2 - a*v**2 - a + u**2 + v**2)/(u**2 + v**2 + 1)
```

Some tests:

```
sage: f + zer == f
True
sage: f - f == zer
True
sage: f + (-f) == zer
True
sage: (f+c)-f == c
```

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```
True
sage: (f-c)+c == f
True
```

We may add a number (interpreted as a constant scalar field) to a scalar field:

```
sage: s = f + 1 ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> (x**2 + y**2 + 2)/(x**2 + y**2 + 1)
on V: (u, v) |--> (2*u**2 + 2*v**2 + 1)/(u**2 + v**2 + 1)
sage: (f+1)-1 == f
True
```

The number can represented by a symbolic variable:

```
sage: s = a + f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s == c + f
True
```

However if the symbolic variable is a chart coordinate, the addition is performed only on the chart domain:

```
sage: s = f + x; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> (x**3 + x*y**2 + x + 1)/(x**2 + y**2 + 1)
sage: s = f + u; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on V: (u, v) |--> (u**3 + u**2 + u*v**2 + u + v**2)/(u**2 + v**2 + 1)
```

The addition of two scalar fields with different domains is possible if the domain of one of them is a subset of the domain of the other; the domain of the result is then this subset:

```
sage: f.domain()
2-dimensional topological manifold M
sage: g.domain()
Open subset U of the 2-dimensional topological manifold M
sage: s = f + g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.domain()
Open subset U of the 2-dimensional topological manifold M
sage: s.display()
U --> R
(x, y) |--> (x**3*y + x*y**3 + x*y + 1)/(x**2 + y**2 + 1)
on W: (u, v) |--> (u**6 + 3*u**4*v**2 + u**3*v + 3*u**2*v**4 + u*v**3 + u*v +
↪v**6)/(u**6 + 3*u**4*v**2 + u**4 + 3*u**2*v**4 + 2*u**2*v**2 + v**6 + v**4)
```

The operation actually performed is $f|_U + g$:

```
sage: s == f.restrict(U) + g
True
```

Since the algebra $C^0(M)$ is a vector space over $\mathbf{R}$, scalar fields can be multiplied by a number, either an explicit one:

```
sage: s = 2*f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> 2/(x**2 + y**2 + 1)
on V: (u, v) |--> 2*(u**2 + v**2)/(u**2 + v**2 + 1)
```

or a symbolic one:

```
sage: s = a*f ; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> a/(x**2 + y**2 + 1)
on V: (u, v) |--> a*(u**2 + v**2)/(u**2 + v**2 + 1)
```

However, if the symbolic variable is a chart coordinate, the multiplication is performed only in the corresponding chart:

```
sage: s = x*f; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on U: (x, y) |--> x/(x**2 + y**2 + 1)
sage: s = u*f; s
Scalar field on the 2-dimensional topological manifold M
sage: s.display()
M --> R
on V: (u, v) |--> u*(u**2 + v**2)/(u**2 + v**2 + 1)
```

Some tests:

```
sage: 0*f == 0
True
sage: 0*f == zer
True
sage: 1*f == f
True
sage: (-2)*f == - f - f
True
```

The ring multiplication of the algebras $C^0(M)$ and $C^0(U)$ is the pointwise multiplication of functions:

```
sage: s = f*f ; s
Scalar field f*f on the 2-dimensional topological manifold M
sage: s.display()
f*f: M --> R
on U: (x, y) |--> 1/(x**4 + 2*x**2*y**2 + 2*x**2 + y**4 + 2*y**2 + 1)
on V: (u, v) |--> (u**4 + 2*u**2*v**2 + v**4)/(u**4 + 2*u**2*v**2 + 2*u**2 + v**4
↪+ 2*v**2 + 1)

sage: s = g*h ; s
Scalar field g*h on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
```

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```
g*h: U --> R
   (x, y) |--> x*y*H(x, y)
on W: (u, v) |--> u*v*H(u/(u**2 + v**2), v/(u**2 + v**2))/(u**4 + 2*u**2*v**2 +␣
↪v**4)
```

Thanks to the coercion $C^0(M) \to C^0(U)$ mentioned above, it is possible to multiply a scalar field defined on M by a scalar field defined on U, the result being a scalar field defined on U:

```
sage: f.domain(), g.domain()
(2-dimensional topological manifold M,
 Open subset U of the 2-dimensional topological manifold M)
sage: s = f*g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> x*y/(x**2 + y**2 + 1)
on W: (u, v) |--> u*v/(u**4 + 2*u**2*v**2 + u**2 + v**4 + v**2)

sage: s == f.restrict(U)*g
True
```

Scalar fields can be divided (pointwise division):

```
sage: s = f/c ; s
Scalar field f/c on the 2-dimensional topological manifold M
sage: s.display()
f/c: M --> R
on U: (x, y) |--> 1/(a*(x**2 + y**2 + 1))
on V: (u, v) |--> (u**2 + v**2)/(a*(u**2 + v**2 + 1))
sage: s = g/h ; s
Scalar field g/h on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
g/h: U --> R
   (x, y) |--> x*y/H(x, y)
on W: (u, v) |--> u*v/((u**4 + 2*u**2*v**2 + v**4)*H(u/(u**2 + v**2), v/(u**2 +␣
↪v**2)))

sage: s = f/g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> 1/(x*y*(x**2 + y**2 + 1))
on W: (u, v) |--> (u**6 + 3*u**4*v**2 + 3*u**2*v**4 + v**6)/(u*v*(u**2 + v**2 +␣
↪1))
sage: s == f.restrict(U)/g
True
```

For scalar fields defined on a single chart domain, we may perform some arithmetics with symbolic expressions involving the chart coordinates:

```
sage: s = g + x^2 - y ; s
Scalar field on the Open subset U of the 2-dimensional topological manifold M
sage: s.display()
```

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```
U --> R
(x, y) |--> x**2 + x*y - y
on W: (u, v) |--> (-u**2*v + u**2 + u*v - v**3)/(u**4 + 2*u**2*v**2 + v**4)
```

```
sage: s = g*x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> x**2*y
on W: (u, v) |--> u**2*v/(u**6 + 3*u**4*v**2 + 3*u**2*v**4 + v**6)
```

```
sage: s = g/x ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> y
on W: (u, v) |--> v/(u**2 + v**2)
sage: s = x/g ; s
Scalar field on the Open subset U of the 2-dimensional topological
 manifold M
sage: s.display()
U --> R
(x, y) |--> 1/y
on W: (u, v) |--> u**2/v + v
```

The test suite is passed:

```
sage: TestSuite(f).run()
sage: TestSuite(zer).run()
```

add_expr (*coord_expression*, *chart=None*)

Add some coordinate expression to the scalar field.

The previous expressions with respect to other charts are kept. To clear them, use `set_expr()` instead.

INPUT:

- `coord_expression` – coordinate expression of the scalar field
- `chart` – (default: `None`) chart in which `coord_expression` is defined; if `None`, the default chart of the scalar field's domain is assumed

> **Warning:** If the scalar field has already expressions in other charts, it is the user's responsibility to make sure that the expression to be added is consistent with them.

EXAMPLES:

Adding scalar field expressions on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x^2 + 2*x*y +1)
sage: f._express
```

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```
{Chart (M, (x, y)): x^2 + 2*x*y + 1}
sage: f.add_expr(3*y)
sage: f._express  # the (x,y) expression has been changed:
{Chart (M, (x, y)): 3*y}
sage: c_uv.<u,v> = M.chart()
sage: f.add_expr(cos(u)-sin(v), c_uv)
sage: f._express # random (dict. output); f has now 2 expressions:
{Chart (M, (x, y)): 3*y, Chart (M, (u, v)): cos(u) - sin(v)}
```

add_expr_by_continuation(*chart*, *subdomain*)

Set coordinate expression in a chart by continuation of the coordinate expression in a subchart.

The continuation is performed by demanding that the coordinate expression is identical to that in the restriction of the chart to a given subdomain.

INPUT:

- `chart` – coordinate chart $(U, (x^i))$ in which the expression of the scalar field is to set
- `subdomain` – open subset $V \subset U$ in which the expression in terms of the restriction of the coordinate chart $(U, (x^i))$ to V is already known or can be evaluated by a change of coordinates.

EXAMPLES:

Scalar field on the sphere S^2:

```
sage: M = Manifold(2, 'S^2', structure='topological')
sage: U = M.open_subset('U') ; V = M.open_subset('V') # the complement of␣
↪resp. N pole and S pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic␣
↪coordinates
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W =  U.intersection(V)  # S^2 minus the two poles
sage: f = M.scalar_field(atan(x^2+y^2), chart=c_xy, name='f')
```

The scalar field has been defined only on the domain covered by the chart `c_xy`, i.e. U:

```
sage: f.display()
f: S^2 --> R
on U: (x, y) |--> arctan(x^2 + y^2)
```

We note that on $W = U \cap V$, the expression of f in terms of coordinates (u, v) can be deduced from that in the coordinates (x, y) thanks to the transition map between the two charts:

```
sage: f.display(c_uv.restrict(W))
f: S^2 --> R
on W: (u, v) |--> arctan(1/(u^2 + v^2))
```

We use this fact to extend the definition of f to the open subset V, covered by the chart `c_uv`:

```
sage: f.add_expr_by_continuation(c_uv, W)
```

Then, f is known on the whole sphere:

```
sage: f.display()
f: S^2 --> R
on U: (x, y) |--> arctan(x^2 + y^2)
on V: (u, v) |--> arctan(1/(u^2 + v^2))
```

arccos()

Arc cosine of the scalar field.

OUTPUT:

- the scalar field $\arccos f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arccos(f) ; g
Scalar field arccos(f) on the 2-dimensional topological manifold M
sage: latex(g)
\arccos\left(\Phi\right)
sage: g.display()
arccos(f): M --> R
   (x, y) |--> arccos(x*y)
```

The notation `acos` can be used as well:

```
sage: acos(f)
Scalar field arccos(f) on the 2-dimensional topological manifold M
sage: acos(f) == g
True
```

Some tests:

```
sage: cos(g) == f
True
sage: arccos(M.constant_scalar_field(1)) == M.zero_scalar_field()
True
sage: arccos(M.zero_scalar_field()) == M.constant_scalar_field(pi/2)
True
```

arccosh()

Inverse hyperbolic cosine of the scalar field.

OUTPUT:

- the scalar field $\operatorname{arccosh} f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arccosh(f) ; g
Scalar field arccosh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\,\mathrm{arccosh}\left(\Phi\right)
sage: g.display()
arccosh(f): M --> R
   (x, y) |--> arccosh(x*y)
```

The notation acosh can be used as well:

```
sage: acosh(f)
Scalar field arccosh(f) on the 2-dimensional topological manifold M
sage: acosh(f) == g
True
```

Some tests:

```
sage: cosh(g) == f
True
sage: arccosh(M.constant_scalar_field(1)) == M.zero_scalar_field()
True
```

arcsin()

Arc sine of the scalar field.

OUTPUT:

- the scalar field $\arcsin f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arcsin(f) ; g
Scalar field arcsin(f) on the 2-dimensional topological manifold M
sage: latex(g)
\arcsin\left(\Phi\right)
sage: g.display()
arcsin(f): M --> R
   (x, y) |--> arcsin(x*y)
```

The notation asin can be used as well:

```
sage: asin(f)
Scalar field arcsin(f) on the 2-dimensional topological manifold M
sage: asin(f) == g
True
```

Some tests:

```
sage: sin(g) == f
True
sage: arcsin(M.zero_scalar_field()) == M.zero_scalar_field()
True
sage: arcsin(M.constant_scalar_field(1)) == M.constant_scalar_field(pi/2)
True
```

arcsinh()

Inverse hyperbolic sine of the scalar field.

OUTPUT:

- the scalar field $\operatorname{arcsinh} f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arcsinh(f) ; g
Scalar field arcsinh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\,\mathrm{arcsinh}\left(\Phi\right)
sage: g.display()
arcsinh(f): M --> R
   (x, y) |--> arcsinh(x*y)
```

The notation `asinh` can be used as well:

```
sage: asinh(f)
Scalar field arcsinh(f) on the 2-dimensional topological manifold M
sage: asinh(f) == g
True
```

Some tests:

```
sage: sinh(g) == f
True
sage: arcsinh(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

arctan()

Arc tangent of the scalar field.

OUTPUT:

- the scalar field $\arctan f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arctan(f) ; g
Scalar field arctan(f) on the 2-dimensional topological manifold M
sage: latex(g)
\arctan\left(\Phi\right)
sage: g.display()
arctan(f): M --> R
   (x, y) |--> arctan(x*y)
```

The notation `atan` can be used as well:

```
sage: atan(f)
Scalar field arctan(f) on the 2-dimensional topological manifold M
sage: atan(f) == g
True
```

Some tests:

```
sage: tan(g) == f
True
sage: arctan(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

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```
sage: arctan(M.constant_scalar_field(1)) == M.constant_scalar_field(pi/4)
True
```

arctanh()

Inverse hyperbolic tangent of the scalar field.

OUTPUT:

- the scalar field $\operatorname{arctanh} f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = arctanh(f) ; g
Scalar field arctanh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\,\mathrm{arctanh}\left(\Phi\right)
sage: g.display()
arctanh(f): M --> R
   (x, y) |--> arctanh(x*y)
```

The notation `atanh` can be used as well:

```
sage: atanh(f)
Scalar field arctanh(f) on the 2-dimensional topological manifold M
sage: atanh(f) == g
True
```

Some tests:

```
sage: tanh(g) == f
True
sage: arctanh(M.zero_scalar_field()) == M.zero_scalar_field()
True
sage: arctanh(M.constant_scalar_field(1/2)) == M.constant_scalar_field(log(3)/
↪2)
True
```

common_charts(*other*)

Find common charts for the expressions of the scalar field and `other`.

INPUT:

- `other` – a scalar field

OUTPUT:

- list of common charts; if no common chart is found, `None` is returned (instead of an empty list)

EXAMPLES:

Search for common charts on a 2-dimensional manifold with 2 overlapping domains:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: U = M.open_subset('U')
sage: c_xy.<x,y> = U.chart()
sage: V = M.open_subset('V')
```

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```
sage: c_uv.<u,v> = V.chart()
sage: M.declare_union(U,V)   # M is the union of U and V
sage: f = U.scalar_field(x^2)
sage: g = M.scalar_field(x+y)
sage: f.common_charts(g)
[Chart (U, (x, y))]
sage: g.add_expr(u, c_uv)
sage: f._express
{Chart (U, (x, y)): x^2}
sage: g._express  # random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u}
sage: f.common_charts(g)
[Chart (U, (x, y))]
```

Common charts found as subcharts: the subcharts are introduced via a transition map between charts c_xy and c_uv on the intersecting subdomain $W = U \cap V$:

```
sage: trans = c_xy.transition_map(c_uv, (x+y, x-y), 'W', x<0, u+v<0)
sage: M.atlas()
[Chart (U, (x, y)), Chart (V, (u, v)), Chart (W, (x, y)),
 Chart (W, (u, v))]
sage: c_xy_W = M.atlas()[2]
sage: c_uv_W = M.atlas()[3]
sage: trans.inverse()
Change of coordinates from Chart (W, (u, v)) to Chart (W, (x, y))
sage: f.common_charts(g)
[Chart (U, (x, y))]
sage: f.expr(c_xy_W)
x^2
sage: f._express  # random (dictionary output)
{Chart (U, (x, y)): x^2, Chart (W, (x, y)): x^2}
sage: g._express  # random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u}
sage: g.common_charts(f)  # c_xy_W is not returned because it is subchart of
↪'xy'
[Chart (U, (x, y))]
sage: f.expr(c_uv_W)
1/4*u^2 + 1/2*u*v + 1/4*v^2
sage: f._express  # random (dictionary output)
{Chart (U, (x, y)): x^2, Chart (W, (x, y)): x^2,
 Chart (W, (u, v)): 1/4*u^2 + 1/2*u*v + 1/4*v^2}
sage: g._express  # random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u}
sage: f.common_charts(g)
[Chart (U, (x, y)), Chart (W, (u, v))]
sage: # the expressions have been updated on the subcharts
sage: g._express #  random (dictionary output)
{Chart (U, (x, y)): x + y, Chart (V, (u, v)): u,
 Chart (W, (u, v)): u}
```

Common charts found by computing some coordinate changes:

```
sage: W = U.intersection(V)
sage: f = W.scalar_field(x^2, c_xy_W)
sage: g = W.scalar_field(u+1, c_uv_W)
sage: f._express
```

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```
{Chart (W, (x, y)): x^2}
sage: g._express
{Chart (W, (u, v)): u + 1}
sage: f.common_charts(g)
[Chart (W, (u, v)), Chart (W, (x, y))]
sage: f._express # random (dictionary output)
{Chart (W, (u, v)): 1/4*u^2 + 1/2*u*v + 1/4*v^2,
 Chart (W, (x, y)): x^2}
sage: g._express # random (dictionary output)
{Chart (W, (u, v)): u + 1, Chart (W, (x, y)): x + y + 1}
```

coord_function(*chart=None*, *from_chart=None*)

Return the function of the coordinates representing the scalar field in a given chart.

INPUT:

- `chart` – (default: `None`) chart with respect to which the coordinate expression is to be returned; if `None`, the default chart of the scalar field's domain will be used
- `from_chart` – (default: `None`) chart from which the required expression is computed if it is not known already in the chart `chart`; if `None`, a chart is picked in the known expressions

OUTPUT:

- instance of *ChartFunction* representing the coordinate function of the scalar field in the given chart

EXAMPLES:

Coordinate function on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x*y^2)
sage: f.coord_function()
x*y^2
sage: f.coord_function(c_xy)  # equivalent form (since c_xy is the default␣
↪chart)
x*y^2
sage: type(f.coord_function())
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>
```

Expression via a change of coordinates:

```
sage: c_uv.<u,v> = M.chart()
sage: c_uv.transition_map(c_xy, [u+v, u-v])
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: f._express # at this stage, f is expressed only in terms of (x,y)␣
↪coordinates
{Chart (M, (x, y)): x*y^2}
sage: f.coord_function(c_uv) # forces the computation of the expression of f␣
↪in terms of (u,v) coordinates
u^3 - u^2*v - u*v^2 + v^3
sage: f.coord_function(c_uv) == (u+v)*(u-v)^2  # check
True
sage: f._express  # random (dict. output); f has now 2 coordinate expressions:
{Chart (M, (x, y)): x*y^2, Chart (M, (u, v)): u^3 - u^2*v - u*v^2 + v^3}
```

Usage in a physical context (simple Lorentz transformation - boost in x direction, with relative velocity v between o1 and o2 frames):

```
sage: M = Manifold(2, 'M', structure='topological')
sage: o1.<t,x> = M.chart()
sage: o2.<T,X> = M.chart()
sage: f = M.scalar_field(x^2 - t^2)
sage: f.coord_function(o1)
-t^2 + x^2
sage: v = var('v'); gam = 1/sqrt(1-v^2)
sage: o2.transition_map(o1, [gam*(T - v*X), gam*(X - v*T)])
Change of coordinates from Chart (M, (T, X)) to Chart (M, (t, x))
sage: f.coord_function(o2)
-T^2 + X^2
```

copy()

Return an exact copy of the scalar field.

EXAMPLES:

Copy on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x*y^2)
sage: g = f.copy()
sage: type(g)
<class 'sage.manifolds.scalarfield_algebra.ScalarFieldAlgebra_with_category.
↪element_class'>
sage: g.expr()
x*y^2
sage: g == f
True
sage: g is f
False
```

cos()

Cosine of the scalar field.

OUTPUT:

- the scalar field $\cos f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = cos(f) ; g
Scalar field cos(f) on the 2-dimensional topological manifold M
sage: latex(g)
\cos\left(\Phi\right)
sage: g.display()
cos(f): M --> R
   (x, y) |--> cos(x*y)
```

Some tests:

```
sage: cos(M.zero_scalar_field()) == M.constant_scalar_field(1)
True
sage: cos(M.constant_scalar_field(pi/2)) == M.zero_scalar_field()
True
```

cosh()

Hyperbolic cosine of the scalar field.

OUTPUT:

- the scalar field $\cosh f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = cosh(f) ; g
Scalar field cosh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\cosh\left(\Phi\right)
sage: g.display()
cosh(f): M --> R
   (x, y) |--> cosh(x*y)
```

Some test:

```
sage: cosh(M.zero_scalar_field()) == M.constant_scalar_field(1)
True
```

disp(*chart=None*)

Display the expression of the scalar field in a given chart.

Without any argument, this function displays the expressions of the scalar field in all the charts defined on the scalar field's domain that are not restrictions of another chart to some subdomain (the "top charts").

INPUT:

- `chart` – (default: `None`) chart with respect to which the coordinate expression is to be displayed; if `None`, the display is performed in all the top charts in which the coordinate expression is known

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Various displays:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(sqrt(x+1), name='f')
sage: f.display()
f: M --> R
   (x, y) |--> sqrt(x + 1)
sage: latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ & \left(x,
↪y\right) & \longmapsto & \sqrt{x + 1} \end{array}
sage: g = M.scalar_field(function('G')(x, y), name='g')
sage: g.display()
g: M --> R
   (x, y) |--> G(x, y)
```

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```
sage: latex(g.display())
\begin{array}{llcl} g:& M & \longrightarrow & \mathbb{R} \\ & \left(x,␣
↪y\right) & \longmapsto & G\left(x, y\right) \end{array}
```

A shortcut of `display()` is `disp()`:

```
sage: f.disp()
f: M --> R
   (x, y) |--> sqrt(x + 1)
```

display(*chart=None*)

Display the expression of the scalar field in a given chart.

Without any argument, this function displays the expressions of the scalar field in all the charts defined on the scalar field's domain that are not restrictions of another chart to some subdomain (the "top charts").

INPUT:

- `chart` – (default: `None`) chart with respect to which the coordinate expression is to be displayed; if `None`, the display is performed in all the top charts in which the coordinate expression is known

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Various displays:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(sqrt(x+1), name='f')
sage: f.display()
f: M --> R
   (x, y) |--> sqrt(x + 1)
sage: latex(f.display())
\begin{array}{llcl} f:& M & \longrightarrow & \mathbb{R} \\ & \left(x,␣
↪y\right) & \longmapsto & \sqrt{x + 1} \end{array}
sage: g = M.scalar_field(function('G')(x, y), name='g')
sage: g.display()
g: M --> R
   (x, y) |--> G(x, y)
sage: latex(g.display())
\begin{array}{llcl} g:& M & \longrightarrow & \mathbb{R} \\ & \left(x,␣
↪y\right) & \longmapsto & G\left(x, y\right) \end{array}
```

A shortcut of `display()` is `disp()`:

```
sage: f.disp()
f: M --> R
   (x, y) |--> sqrt(x + 1)
```

domain()

Return the open subset on which the scalar field is defined.

OUTPUT:

- instance of class *TopologicalManifold* representing the manifold's open subset on which the scalar field is defined

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x+2*y)
sage: f.domain()
2-dimensional topological manifold M
sage: U = M.open_subset('U', coord_def={c_xy: x<0})
sage: g = f.restrict(U)
sage: g.domain()
Open subset U of the 2-dimensional topological manifold M
```

exp()

Exponential of the scalar field.

OUTPUT:

- the scalar field $\exp f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y}, name='f', latex_name=r"\Phi")
sage: g = exp(f) ; g
Scalar field exp(f) on the 2-dimensional topological manifold M
sage: g.display()
exp(f): M --> R
   (x, y) |--> e^(x + y)
sage: latex(g)
\exp\left(\Phi\right)
```

Automatic simplifications occur:

```
sage: f = M.scalar_field({X: 2*ln(1+x^2)}, name='f')
sage: exp(f).display()
exp(f): M --> R
   (x, y) |--> x^4 + 2*x^2 + 1
```

The inverse function is `log()`:

```
sage: log(exp(f)) == f
True
```

Some tests:

```
sage: exp(M.zero_scalar_field()) == M.constant_scalar_field(1)
True
sage: exp(M.constant_scalar_field(1)) == M.constant_scalar_field(e)
True
```

expr(*chart=None*, *from_chart=None*)

Return the coordinate expression of the scalar field in a given chart.

INPUT:

- `chart` – (default: `None`) chart with respect to which the coordinate expression is required; if `None`, the default chart of the scalar field's domain will be used
- `from_chart` – (default: `None`) chart from which the required expression is computed if it is not known already in the chart `chart`; if `None`, a chart is picked in `self._express`

OUTPUT:

- the coordinate expression of the scalar field in the given chart, either as a Sage's symbolic expression or as a SymPy object, depending on the symbolic calculus method used on the chart

EXAMPLES:

Expression of a scalar field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x*y^2)
sage: f.expr()
x*y^2
sage: f.expr(c_xy)  # equivalent form (since c_xy is the default chart)
x*y^2
```

Expression via a change of coordinates:

```
sage: c_uv.<u,v> = M.chart()
sage: c_uv.transition_map(c_xy, [u+v, u-v])
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: f._express # at this stage, f is expressed only in terms of (x,y)␣
↪coordinates
{Chart (M, (x, y)): x*y^2}
sage: f.expr(c_uv) # forces the computation of the expression of f in terms␣
↪of (u,v) coordinates
u^3 - u^2*v - u*v^2 + v^3
sage: bool( f.expr(c_uv) == (u+v)*(u-v)^2 ) # check
True
sage: f._express  # random (dict. output); f has now 2 coordinate expressions:
{Chart (M, (x, y)): x*y^2, Chart (M, (u, v)): u^3 - u^2*v - u*v^2 + v^3}
```

Note that the object returned by `expr()` depends on the symbolic backend used for coordinate computations:

```
sage: type(f.expr())
<type 'sage.symbolic.expression.Expression'>
sage: M.set_calculus_method('sympy')
sage: type(f.expr())
<class 'sympy.core.mul.Mul'>
sage: f.expr()  # note the SymPy exponent notation
x*y**2
```

is_trivial_zero()

Check if `self` is trivially equal to zero without any simplification.

This method is supposed to be fast as compared with `self.is_zero()` or `self == 0` and is intended to be used in library code where trying to obtain a mathematically correct result by applying potentially expensive rewrite rules is not desirable.

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: 0})
sage: f.is_trivial_zero()
True
sage: f = M.scalar_field(0)
```

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```
sage: f.is_trivial_zero()
True
sage: M.zero_scalar_field().is_trivial_zero()
True
sage: f = M.scalar_field({X: x+y})
sage: f.is_trivial_zero()
False
```

Scalar field defined by means of two charts:

```
sage: U1 = M.open_subset('U1'); X1.<x1,y1> = U1.chart()
sage: U2 = M.open_subset('U2'); X2.<x2,y2> = U2.chart()
sage: f = M.scalar_field({X1: 0, X2: 0})
sage: f.is_trivial_zero()
True
sage: f = M.scalar_field({X1: 0, X2: 1})
sage: f.is_trivial_zero()
False
```

No simplification is attempted, so that `False` is returned for non-trivial cases:

```
sage: f = M.scalar_field({X: cos(x)^2 + sin(x)^2 - 1})
sage: f.is_trivial_zero()
False
```

On the contrary, the method `is_zero()` and the direct comparison to zero involve some simplification algorithms and return `True`:

```
sage: f.is_zero()
True
sage: f == 0
True
```

log()

Natural logarithm of the scalar field.

OUTPUT:

- the scalar field $\ln f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y}, name='f', latex_name=r"\Phi")
sage: g = log(f) ; g
Scalar field ln(f) on the 2-dimensional topological manifold M
sage: g.display()
ln(f): M --> R
   (x, y) |--> log(x + y)
sage: latex(g)
\ln\left(\Phi\right)
```

The inverse function is *exp()*:

```
sage: exp(log(f)) == f
True
```

restrict (*subdomain*)

Restriction of the scalar field to an open subset of its domain of definition.

INPUT:

- `subdomain` – an open subset of the scalar field's domain

OUTPUT:

- instance of *ScalarField* representing the restriction of the scalar field to `subdomain`

EXAMPLES:

Restriction of a scalar field defined on $\mathbf{R}^2$ to the unit open disc:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()  # Cartesian coordinates
sage: U = M.open_subset('U', coord_def={X: x^2+y^2 < 1}) # U unit open disc
sage: f = M.scalar_field(cos(x*y), name='f')
sage: f_U = f.restrict(U) ; f_U
Scalar field f on the Open subset U of the 2-dimensional
 topological manifold M
sage: f_U.display()
f: U --> R
   (x, y) |--> cos(x*y)
sage: f.parent()
Algebra of scalar fields on the 2-dimensional topological
 manifold M
sage: f_U.parent()
Algebra of scalar fields on the Open subset U of the 2-dimensional
 topological manifold M
```

The restriction to the whole domain is the identity:

```
sage: f.restrict(M) is f
True
sage: f_U.restrict(U) is f_U
True
```

Restriction of the zero scalar field:

```
sage: M.zero_scalar_field().restrict(U)
Scalar field zero on the Open subset U of the 2-dimensional
 topological manifold M
sage: M.zero_scalar_field().restrict(U) is U.zero_scalar_field()
True
```

set_calc_order (*symbol*, *order*, *truncate=False*)

Trigger a power series expansion with respect to a small parameter in computations involving the scalar field.

This property is propagated by usual operations. The internal representation must be `SR` for this to take effect.

If the small parameter is ϵ and f is `self`, the power series expansion to order n is

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots + \epsilon^n f_n + O(\epsilon^{n+1}),$$

where $f_0, f_1, \ldots, f_n$ are $n+1$ scalar fields that do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the coordinate expressions of `self` in various charts are expanded in power series (around the zero value of this variable)
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`

> **Warning:** The order of the big O in the power series expansion is $n + 1$, where n is `order`.

- `truncate` – (default: `False`) determines whether the coordinate expressions of `self` are replaced by their expansions to the given order

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: t = var('t')  # the small parameter
sage: f = M.scalar_field(exp(-t*x))
sage: f.expr()
e^(-t*x)
sage: f.set_calc_order(t, 2, truncate=True)
sage: f.expr()
1/2*t^2*x^2 - t*x + 1
```

set_expr (*coord_expression*, *chart=None*)
Set the coordinate expression of the scalar field.

The expressions with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use *add_expr()* instead.

INPUT:

- `coord_expression` – coordinate expression of the scalar field
- `chart` – (default: `None`) chart in which `coord_expression` is defined; if `None`, the default chart of the scalar field's domain is assumed

EXAMPLES:

Setting scalar field expressions on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x^2 + 2*x*y +1)
sage: f._express
{Chart (M, (x, y)): x^2 + 2*x*y + 1}
sage: f.set_expr(3*y)
sage: f._express  # the (x,y) expression has been changed:
{Chart (M, (x, y)): 3*y}
sage: c_uv.<u,v> = M.chart()
sage: f.set_expr(cos(u)-sin(v), c_uv)
sage: f._express # the (x,y) expression has been lost:
{Chart (M, (u, v)): cos(u) - sin(v)}
sage: f.set_expr(3*y)
sage: f._express # the (u,v) expression has been lost:
{Chart (M, (x, y)): 3*y}
```

set_name (*name=None*, *latex_name=None*)
Set (or change) the text name and LaTeX name of the scalar field.

INPUT:

- `name` – (string; default: `None`) name given to the scalar field
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the scalar field; if `None` while `name` is provided, the LaTeX symbol is set to `name`

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y})
sage: f = M.scalar_field({X: x+y}); f
Scalar field on the 2-dimensional topological manifold M
sage: f.set_name('f'); f
Scalar field f on the 2-dimensional topological manifold M
sage: latex(f)
f
sage: f.set_name('f', latex_name=r'\Phi'); f
Scalar field f on the 2-dimensional topological manifold M
sage: latex(f)
\Phi
```

sin()

Sine of the scalar field.

OUTPUT:

- the scalar field $\sin f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = sin(f) ; g
Scalar field sin(f) on the 2-dimensional topological manifold M
sage: latex(g)
\sin\left(\Phi\right)
sage: g.display()
sin(f): M --> R
   (x, y) |--> sin(x*y)
```

Some tests:

```
sage: sin(M.zero_scalar_field()) == M.zero_scalar_field()
True
sage: sin(M.constant_scalar_field(pi/2)) == M.constant_scalar_field(1)
True
```

sinh()

Hyperbolic sine of the scalar field.

OUTPUT:

- the scalar field $\sinh f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
```

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```
sage: g = sinh(f) ; g
Scalar field sinh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\sinh\left(\Phi\right)
sage: g.display()
sinh(f): M --> R
   (x, y) |--> sinh(x*y)
```

Some test:

```
sage: sinh(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

sqrt()

Square root of the scalar field.

OUTPUT:

- the scalar field $\sqrt{f}$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: 1+x^2+y^2}, name='f',
....:                     latex_name=r"\Phi")
sage: g = sqrt(f) ; g
Scalar field sqrt(f) on the 2-dimensional topological manifold M
sage: latex(g)
\sqrt{\Phi}
sage: g.display()
sqrt(f): M --> R
   (x, y) |--> sqrt(x^2 + y^2 + 1)
```

Some tests:

```
sage: g^2 == f
True
sage: sqrt(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

tan()

Tangent of the scalar field.

OUTPUT:

- the scalar field $\tan f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = tan(f) ; g
Scalar field tan(f) on the 2-dimensional topological manifold M
sage: latex(g)
\tan\left(\Phi\right)
sage: g.display()
```

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```
tan(f): M --> R
   (x, y) |--> sin(x*y)/cos(x*y)
```

Some tests:

```
sage: tan(f) == sin(f) / cos(f)
True
sage: tan(M.zero_scalar_field()) == M.zero_scalar_field()
True
sage: tan(M.constant_scalar_field(pi/4)) == M.constant_scalar_field(1)
True
```

tanh()

Hyperbolic tangent of the scalar field.

OUTPUT:

- the scalar field $\tanh f$, where f is the current scalar field

EXAMPLES:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x*y}, name='f', latex_name=r"\Phi")
sage: g = tanh(f) ; g
Scalar field tanh(f) on the 2-dimensional topological manifold M
sage: latex(g)
\tanh\left(\Phi\right)
sage: g.display()
tanh(f): M --> R
   (x, y) |--> sinh(x*y)/cosh(x*y)
```

Some tests:

```
sage: tanh(f) == sinh(f) / cosh(f)
True
sage: tanh(M.zero_scalar_field()) == M.zero_scalar_field()
True
```

1.7 Continuous Maps

1.7.1 Sets of Morphisms between Topological Manifolds

The class *TopologicalManifoldHomset* implements sets of morphisms between two topological manifolds over the same topological field K, a morphism being a *continuous map* for the category of topological manifolds.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [?]
- [?]

class sage.manifolds.manifold_homset.**TopologicalManifoldHomset**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.categories.homset.Homset`

Set of continuous maps between two topological manifolds.

Given two topological manifolds M and N over a topological field K, the class *TopologicalManifoldHomset* implements the set $\mathrm{Hom}(M, N)$ of morphisms (i.e. continuous maps) $M \to N$.

This is a Sage *parent* class, whose *element* class is *ContinuousMap*.

INPUT:

- `domain` – *TopologicalManifold*; the domain topological manifold M of the morphisms
- `codomain` – *TopologicalManifold*; the codomain topological manifold N of the morphisms
- `name` – (default: `None`) string; the name of `self`; if `None`, `Hom(M,N)` will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote `self`; if `None`, $\mathrm{Hom}(M, N)$ will be used

EXAMPLES:

Set of continuous maps between a 2-dimensional manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: N = Manifold(3, 'N', structure='topological')
sage: Y.<u,v,w> = N.chart()
sage: H = Hom(M, N) ; H
Set of Morphisms from 2-dimensional topological manifold M to
 3-dimensional topological manifold N in Category of manifolds over
 Real Field with 53 bits of precision
sage: type(H)
<class 'sage.manifolds.manifold_homset.TopologicalManifoldHomset_with_category'>
sage: H.category()
Category of homsets of topological spaces
sage: latex(H)
\mathrm{Hom}\left(M,N\right)
sage: H.domain()
2-dimensional topological manifold M
sage: H.codomain()
3-dimensional topological manifold N
```

An element of `H` is a continuous map from `M` to `N`:

```
sage: H.Element
<class 'sage.manifolds.continuous_map.ContinuousMap'>
sage: f = H.an_element() ; f
Continuous map from the 2-dimensional topological manifold M to the
 3-dimensional topological manifold N
sage: f.display()
M --> N
   (x, y) |--> (u, v, w) = (0, 0, 0)
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

When the codomain coincides with the domain, the homset is a set of *endomorphisms* in the category of topological manifolds:

```
sage: E = Hom(M, M) ; E
Set of Morphisms from 2-dimensional topological manifold M to
 2-dimensional topological manifold M in Category of manifolds over
 Real Field with 53 bits of precision
sage: E.category()
Category of endsets of topological spaces
sage: E.is_endomorphism_set()
True
sage: E is End(M)
True
```

In this case, the homset is a monoid for the law of morphism composition:

```
sage: E in Monoids()
True
```

This was of course not the case of `H = Hom(M, N)`:

```
sage: H in Monoids()
False
```

The identity element of the monoid is of course the identity map of `M`:

```
sage: E.one()
Identity map Id_M of the 2-dimensional topological manifold M
sage: E.one() is M.identity_map()
True
sage: E.one().display()
Id_M: M --> M
   (x, y) |--> (x, y)
```

The test suite is passed by `E`:

```
sage: TestSuite(E).run()
```

This test suite includes more tests than in the case of `H`, since `E` has some extra structure (monoid).

Element
alias of *sage.manifolds.continuous_map.ContinuousMap*

one()
Return the identity element of `self` considered as a monoid (case of a set of endomorphisms).

This applies only when the codomain of the homset is equal to its domain, i.e. when the homset is of the type $\mathrm{Hom}(M, M)$. Indeed, $\mathrm{Hom}(M, M)$ equipped with the law of morphisms composition is a monoid, whose identity element is nothing but the identity map of M.

OUTPUT:

- the identity map of M, as an instance of *ContinuousMap*

EXAMPLES:

The identity map of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: H = Hom(M, M) ; H
Set of Morphisms from 2-dimensional topological manifold M to
 2-dimensional topological manifold M in Category of manifolds over
 Real Field with 53 bits of precision
sage: H in Monoids()
True
sage: H.one()
Identity map Id_M of the 2-dimensional topological manifold M
sage: H.one().parent() is H
True
sage: H.one().display()
Id_M: M --> M
   (x, y) |--> (x, y)
```

The identity map is cached:

```
sage: H.one() is H.one()
True
```

If the homset is not a set of endomorphisms, the identity element is meaningless:

```
sage: N = Manifold(3, 'N', structure='topological')
sage: Y.<u,v,w> = N.chart()
sage: Hom(M, N).one()
Traceback (most recent call last):
...
TypeError: Set of Morphisms
 from 2-dimensional topological manifold M
 to 3-dimensional topological manifold N
 in Category of manifolds over Real Field with 53 bits of precision
 is not a monoid
```

1.7.2 Continuous Maps Between Topological Manifolds

`ContinuousMap` implements continuous maps from a topological manifold M to some topological manifold N over the same topological field K as M.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- Chap. 1 of [?]
- [?]

class `sage.manifolds.continuous_map.`**`ContinuousMap`**(*parent, coord_functions=None, name=None, latex_name=None, is_isomorphism=False, is_identity=False*)

Bases: `sage.categories.morphism.Morphism`

Continuous map between two topological manifolds.

This class implements continuous maps of the type

$$\Phi : M \longrightarrow N,$$

where M and N are topological manifolds over the same topological field K.

Continuous maps are the morphisms of the category of topological manifolds. The set of all continuous maps from M to N is therefore the homset between M and N, which is denoted by $\mathrm{Hom}(M, N)$.

The class `ContinuousMap` is a Sage *element* class, whose *parent* class is `TopologicalManifoldHomset`.

INPUT:

- `parent` – homset $\mathrm{Hom}(M, N)$ to which the continuous map belongs
- `coord_functions` – a dictionary of the coordinate expressions (as lists or tuples of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts `(chart1, chart2)` as keys (`chart1` being a chart on M and `chart2` a chart on N)
- `name` – (default: `None`) name given to `self`
- `latex_name` – (default: `None`) LaTeX symbol to denote the continuous map; if `None`, the LaTeX symbol is set to `name`
- `is_isomorphism` – (default: `False`) determines whether the constructed object is a isomorphism (i.e. a homeomorphism); if set to `True`, then the manifolds M and N must have the same dimension
- `is_identity` – (default: `False`) determines whether the constructed object is the identity map; if set to `True`, then N must be M and the entry `coord_functions` is not used

---

Note: If the information passed by means of the argument `coord_functions` is not sufficient to fully specify the continuous map, further coordinate expressions, in other charts, can be subsequently added by means of the method `add_expr()`.

EXAMPLES:

The standard embedding of the sphere S^2 into $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2', structure='topological') # the 2-dimensional sphere␣
↪S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1=x^2+y^2!=0,
....:                                restrictions2=u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: N = Manifold(3, 'R^3', latex_name=r'\RR^3', structure='topological')  # R^3
sage: c_cart.<X,Y,Z> = N.chart()  # Cartesian coordinates on R^3
sage: Phi = M.continuous_map(N,
....:   {(c_xy, c_cart): [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^2+y^
↪2)],
....:    (c_uv, c_cart): [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^2+v^
↪2)]},
....:   name='Phi', latex_name=r'\Phi')
```

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```
sage: Phi
Continuous map Phi from the 2-dimensional topological manifold S^2
 to the 3-dimensional topological manifold R^3
sage: Phi.parent()
Set of Morphisms from 2-dimensional topological manifold S^2
 to 3-dimensional topological manifold R^3
 in Category of manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(M, N)
True
sage: type(Phi)
<class 'sage.manifolds.manifold_homset.TopologicalManifoldHomset_with_category.
↪element_class'>
sage: Phi.display()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^
↪2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

It is possible to create the map using *continuous_map()* with only in a single pair of charts. The argument `coord_functions` is then a mere list of coordinate expressions (and not a dictionary) and the arguments `chart1` and `chart2` have to be provided if the charts differ from the default ones on the domain and/or codomain:

```
sage: Phi1 = M.continuous_map(N, [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/
↪(1+x^2+y^2)],
....:                           chart1=c_xy, chart2=c_cart,
....:                           name='Phi', latex_name=r'\Phi')
```

Since `c_xy` and `c_cart` are the default charts on respectively `M` and `N`, they can be omitted, so that the above declaration is equivalent to:

```
sage: Phi1 = M.continuous_map(N, [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/
↪(1+x^2+y^2)],
....:                           name='Phi', latex_name=r'\Phi')
```

With such a declaration, the continuous map `Phi1` is only partially defined on the manifold S^2 as it is known in only one chart:

```
sage: Phi1.display()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^
↪2 - 1)/(x^2 + y^2 + 1))
```

The definition can be completed by using *add_expr()*:

```
sage: Phi1.add_expr(c_uv, c_cart, [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/
↪(1+u^2+v^2)])
sage: Phi1.display()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 + y^
↪2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^2 +␣
↪v^2 - 1)/(u^2 + v^2 + 1))
```

At this stage, `Phi1` and `Phi` are fully equivalent:

```
sage: Phi1 == Phi
True
```

The map acts on points:

```
sage: np = M.point((0,0), chart=c_uv)  # the North pole
sage: Phi(np)
Point on the 3-dimensional topological manifold R^3
sage: Phi(np).coord()  # Cartesian coordinates
(0, 0, 1)
sage: sp = M.point((0,0), chart=c_xy)  # the South pole
sage: Phi(sp).coord()  # Cartesian coordinates
(0, 0, -1)
```

The test suite is passed:

```
sage: TestSuite(Phi).run()
sage: TestSuite(Phi1).run()
```

Continuous maps can be composed by means of the operator $*$. Let us introduce the map $\mathbf{R}^3 \to \mathbf{R}^2$ corresponding to the projection from the point $(X, Y, Z) = (0, 0, 1)$ onto the equatorial plane $Z = 0$:

```
sage: P = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological') # R^2␣
↪(equatorial plane)
sage: cP.<xP, yP> = P.chart()
sage: Psi = N.continuous_map(P, (X/(1-Z), Y/(1-Z)), name='Psi',
....:                        latex_name=r'\Psi')
sage: Psi
Continuous map Psi from the 3-dimensional topological manifold R^3
 to the 2-dimensional topological manifold R^2
sage: Psi.display()
Psi: R^3 --> R^2
   (X, Y, Z) |--> (xP, yP) = (-X/(Z - 1), -Y/(Z - 1))
```

Then we compose `Psi` with `Phi`, thereby getting a map $S^2 \to \mathbf{R}^2$:

```
sage: ster = Psi * Phi ; ster
Continuous map from the 2-dimensional topological manifold S^2
 to the 2-dimensional topological manifold R^2
```

Let us test on the South pole (`sp`) that `ster` is indeed the composite of `Psi` and `Phi`:

```
sage: ster(sp) == Psi(Phi(sp))
True
```

Actually `ster` is the stereographic projection from the North pole, as its coordinate expression reveals:

```
sage: ster.display()
S^2 --> R^2
on U: (x, y) |--> (xP, yP) = (x, y)
on V: (u, v) |--> (xP, yP) = (u/(u^2 + v^2), v/(u^2 + v^2))
```

If the codomain of a continuous map is 1-dimensional, the map can be defined by a single symbolic expression for each pair of charts and not by a list/tuple with a single element:

```
sage: N = Manifold(1, 'N', structure='topological')
sage: c_N = N.chart('X')
```

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```
sage: Phi = M.continuous_map(N, {(c_xy, c_N): x^2+y^2,
....:                            (c_uv, c_N): 1/(u^2+v^2)})

sage: Psi = M.continuous_map(N, {(c_xy, c_N): [x^2+y^2],
....:                            (c_uv, c_N): [1/(u^2+v^2)]})
sage: Phi == Psi
True
```

Next we construct an example of continuous map $\mathbf{R} \to \mathbf{R}^2$:

```
sage: R = Manifold(1, 'R', structure='topological')  # field R
sage: T.<t> = R.chart()  # canonical chart on R
sage: R2 = Manifold(2, 'R^2', structure='topological')  # R^2
sage: c_xy.<x,y> = R2.chart() # Cartesian coordinates on R^2
sage: Phi = R.continuous_map(R2, [cos(t), sin(t)], name='Phi'); Phi
Continuous map Phi from the 1-dimensional topological manifold R
 to the 2-dimensional topological manifold R^2
sage: Phi.parent()
Set of Morphisms from 1-dimensional topological manifold R
 to 2-dimensional topological manifold R^2
 in Category of manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(R, R2)
True
sage: Phi.display()
Phi: R --> R^2
   t |--> (x, y) = (cos(t), sin(t))
```

An example of homeomorphism between the unit open disk and the Euclidean plane $\mathbf{R}^2$:

```
sage: D = R2.open_subset('D', coord_def={c_xy: x^2+y^2<1}) # the open unit disk
sage: Phi = D.homeomorphism(R2, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                        name='Phi', latex_name=r'\Phi')
sage: Phi
Homeomorphism Phi from the Open subset D of the 2-dimensional
 topological manifold R^2 to the 2-dimensional topological manifold R^2
sage: Phi.parent()
Set of Morphisms from Open subset D of the 2-dimensional topological
 manifold R^2 to 2-dimensional topological manifold R^2 in Join of
 Category of subobjects of sets and Category of manifolds over Real
 Field with 53 bits of precision
sage: Phi.parent() is Hom(D, R2)
True
sage: Phi.display()
Phi: D --> R^2
   (x, y) |--> (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The image of a point:

```
sage: p = D.point((1/2,0))
sage: q = Phi(p) ; q
Point on the 2-dimensional topological manifold R^2
sage: q.coord()
(1/3*sqrt(3), 0)
```

The inverse homeomorphism is computed by `inverse()`:

```
sage: Phi.inverse()
Homeomorphism Phi^(-1) from the 2-dimensional topological manifold R^2
 to the Open subset D of the 2-dimensional topological manifold R^2
sage: Phi.inverse().display()
Phi^(-1): R^2 --> D
   (x, y) |--> (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

Equivalently, one may use the notations `^(-1)` or `~` to get the inverse:

```
sage: Phi^(-1) is Phi.inverse()
True
sage: ~Phi is Phi.inverse()
True
```

Check that `~Phi` is indeed the inverse of `Phi`:

```
sage: (~Phi)(q) == p
True
sage: Phi * ~Phi == R2.identity_map()
True
sage: ~Phi * Phi == D.identity_map()
True
```

The coordinate expression of the inverse homeomorphism:

```
sage: (~Phi).display()
Phi^(-1): R^2 --> D
   (x, y) |--> (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

A special case of homeomorphism: the identity map of the open unit disk:

```
sage: id = D.identity_map() ; id
Identity map Id_D of the Open subset D of the 2-dimensional topological
 manifold R^2
sage: latex(id)
\mathrm{Id}_{D}
sage: id.parent()
Set of Morphisms from Open subset D of the 2-dimensional topological
 manifold R^2 to Open subset D of the 2-dimensional topological
 manifold R^2 in Join of Category of subobjects of sets and Category of
 manifolds over Real Field with 53 bits of precision
sage: id.parent() is Hom(D, D)
True
sage: id is Hom(D,D).one()   # the identity element of the monoid Hom(D,D)
True
```

The identity map acting on a point:

```
sage: id(p)
Point on the 2-dimensional topological manifold R^2
sage: id(p) == p
True
sage: id(p) is p
True
```

The coordinate expression of the identity map:

```
sage: id.display()
Id_D: D --> D
   (x, y) |--> (x, y)
```

The identity map is its own inverse:

```
sage: id^(-1) is id
True
sage: ~id is id
True
```

add_expr (*chart1*, *chart2*, *coord_functions*)
Set a new coordinate representation of `self`.

The previous expressions with respect to other charts are kept. To clear them, use *set_expr()* instead.

INPUT:

- `chart1` – chart for the coordinates on the map's domain
- `chart2` – chart for the coordinates on the map's codomain
- `coord_functions` – the coordinate symbolic expression of the map in the above charts: list (or tuple) of the coordinates of the image expressed in terms of the coordinates of the considered point; if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

Warning: If the map has already expressions in other charts, it is the user's responsibility to make sure that the expression to be added is consistent with them.

EXAMPLES:

Polar representation of a planar rotation initially defined in Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')  #␣
↪the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement␣
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
```

We construct the links between spherical coordinates and Cartesian ones:

```
sage: ch_cart_spher = c_cart.transition_map(c_spher, [sqrt(x*x+y*y), atan2(y,
↪x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph), verbose=True)
Check of the inverse coordinate transformation:
   x == x
   y == y
   r == r
   ph == arctan2(r*sin(ph), r*cos(ph))
sage: rot = U.continuous_map(U, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                        name='R')
sage: rot.display(c_cart, c_cart)
R: U --> U
   (x, y) |--> (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

If we calculate the expression in terms of spherical coordinates, via the method *display()*, we notice some difficulties in `arctan2` simplifications:

```
sage: rot.display(c_spher, c_spher)
R: U --> U
   (r, ph) |--> (r, arctan2(1/2*(sqrt(3)*cos(ph) + sin(ph))*r, -1/
↪2*(sqrt(3)*sin(ph) - cos(ph))*r))
```

Therefore, we use the method *add_expr()* to set the spherical-coordinate expression by hand:

```
sage: rot.add_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot.display(c_spher, c_spher)
R: U --> U
   (r, ph) |--> (r, 1/3*pi + ph)
```

The call to *add_expr()* has not deleted the expression in terms of Cartesian coordinates, as we can check by printing the internal dictionary `_coord_expression`, which stores the various internal representations of the continuous map:

```
sage: rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
  Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

If, on the contrary, we use *set_expr()*, the expression in Cartesian coordinates is lost:

```
sage: rot.set_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot._coord_expression
{(Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

It is recovered (thanks to the known change of coordinates) by a call to *display()*:

```
sage: rot.display(c_cart, c_cart)
R: U --> U
   (x, y) |--> (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

sage: rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

The rotation can be applied to a point by means of either coordinate system:

```
sage: p = M.point((1,2))   #  p defined by its Cartesian coord.
sage: q = rot(p)   # q is computed by means of Cartesian coord.
sage: p1 = M.point((sqrt(5), arctan(2)), chart=c_spher) # p1 is defined only␣
↪in terms of c_spher
sage: q1 = rot(p1) # computation by means of spherical coordinates
sage: q1 == q
True
```

add_expression(*chart1*, *chart2*, *coord_functions*)
Set a new coordinate representation of `self`.

The previous expressions with respect to other charts are kept. To clear them, use *set_expr()* instead.

INPUT:

- `chart1` – chart for the coordinates on the map's domain
- `chart2` – chart for the coordinates on the map's codomain
- `coord_functions` – the coordinate symbolic expression of the map in the above charts: list (or tuple) of the coordinates of the image expressed in terms of the coordinates of the considered point; if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

> **Warning:** If the map has already expressions in other charts, it is the user's responsibility to make sure that the expression to be added is consistent with them.

EXAMPLES:

Polar representation of a planar rotation initially defined in Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')  #␣
↪the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement␣
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
```

We construct the links between spherical coordinates and Cartesian ones:

```
sage: ch_cart_spher = c_cart.transition_map(c_spher, [sqrt(x*x+y*y), atan2(y,
↪x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph), verbose=True)
Check of the inverse coordinate transformation:
   x == x
   y == y
   r == r
   ph == arctan2(r*sin(ph), r*cos(ph))
sage: rot = U.continuous_map(U, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                        name='R')
sage: rot.display(c_cart, c_cart)
R: U --> U
   (x, y) |--> (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

If we calculate the expression in terms of spherical coordinates, via the method *display()*, we notice some difficulties in `arctan2` simplifications:

```
sage: rot.display(c_spher, c_spher)
R: U --> U
   (r, ph) |--> (r, arctan2(1/2*(sqrt(3)*cos(ph) + sin(ph))*r, -1/
↪2*(sqrt(3)*sin(ph) - cos(ph))*r))
```

Therefore, we use the method *add_expr()* to set the spherical-coordinate expression by hand:

```
sage: rot.add_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot.display(c_spher, c_spher)
```

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```
R: U --> U
   (r, ph) |--> (r, 1/3*pi + ph)
```

The call to `add_expr()` has not deleted the expression in terms of Cartesian coordinates, as we can check by printing the internal dictionary `_coord_expression`, which stores the various internal representations of the continuous map:

```
sage: rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
  Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

If, on the contrary, we use `set_expr()`, the expression in Cartesian coordinates is lost:

```
sage: rot.set_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot._coord_expression
{(Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

It is recovered (thanks to the known change of coordinates) by a call to `display()`:

```
sage: rot.display(c_cart, c_cart)
R: U --> U
   (x, y) |--> (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

sage: rot._coord_expression # random (dictionary output)
{(Chart (U, (x, y)), Chart (U, (x, y))):
 Coordinate functions (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
  on the Chart (U, (x, y)),
 (Chart (U, (r, ph)), Chart (U, (r, ph))):
 Coordinate functions (r, 1/3*pi + ph) on the Chart (U, (r, ph))}
```

The rotation can be applied to a point by means of either coordinate system:

```
sage: p = M.point((1,2))   #  p defined by its Cartesian coord.
sage: q = rot(p)   # q is computed by means of Cartesian coord.
sage: p1 = M.point((sqrt(5), arctan(2)), chart=c_spher) # p1 is defined only␣
↪in terms of c_spher
sage: q1 = rot(p1) # computation by means of spherical coordinates
sage: q1 == q
True
```

coord_functions(*chart1=None*, *chart2=None*)

Return the functions of the coordinates representing `self` in a given pair of charts.

If these functions are not already known, they are computed from known ones by means of change-of-chart formulas.

INPUT:

- `chart1` – (default: `None`) chart on the domain of `self`; if `None`, the domain's default chart is assumed
- `chart2` – (default: `None`) chart on the codomain of `self`; if `None`, the codomain's default chart is assumed

OUTPUT:

- a *MultiCoordFunction* representing the continuous map in the above two charts

EXAMPLES:

Continuous map from a 2-dimensional manifold to a 3-dimensional one:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: N = Manifold(3, 'N', structure='topological')
sage: c_uv.<u,v> = M.chart()
sage: c_xyz.<x,y,z> = N.chart()
sage: Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
....:                        latex_name=r'\Phi')
sage: Phi.display()
Phi: M --> N
   (u, v) |--> (x, y, z) = (u*v, u/v, u + v)
sage: Phi.coord_functions(c_uv, c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
sage: Phi.coord_functions() # equivalent to above since 'uv' and 'xyz' are
↪default charts
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
sage: type(Phi.coord_functions())
<class 'sage.manifolds.chart_func.MultiCoordFunction'>
```

Coordinate representation in other charts:

```
sage: c_UV.<U,V> = M.chart()  # new chart on M
sage: ch_uv_UV = c_uv.transition_map(c_UV, [u-v, u+v])
sage: ch_uv_UV.inverse()(U,V)
(1/2*U + 1/2*V, -1/2*U + 1/2*V)
sage: c_XYZ.<X,Y,Z> = N.chart() # new chart on N
sage: ch_xyz_XYZ = c_xyz.transition_map(c_XYZ,
....:                                   [2*x-3*y+z, y+z-x, -x+2*y-z])
sage: ch_xyz_XYZ.inverse()(X,Y,Z)
(3*X + Y + 4*Z, 2*X + Y + 3*Z, X + Y + Z)
sage: Phi.coord_functions(c_UV, c_xyz)
Coordinate functions (-1/4*U^2 + 1/4*V^2, -(U + V)/(U - V), V) on
 the Chart (M, (U, V))
sage: Phi.coord_functions(c_uv, c_XYZ)
Coordinate functions (((2*u + 1)*v^2 + u*v - 3*u)/v,
 -((u - 1)*v^2 - u*v - u)/v, -((u + 1)*v^2 + u*v - 2*u)/v) on the
 Chart (M, (u, v))
sage: Phi.coord_functions(c_UV, c_XYZ)
Coordinate functions
 (-1/2*(U^3 - (U - 2)*V^2 + V^3 - (U^2 + 2*U + 6)*V - 6*U)/(U - V),
  1/4*(U^3 - (U + 4)*V^2 + V^3 - (U^2 - 4*U + 4)*V - 4*U)/(U - V),
  1/4*(U^3 - (U - 4)*V^2 + V^3 - (U^2 + 4*U + 8)*V - 8*U)/(U - V))
 on the Chart (M, (U, V))
```

Coordinate representation with respect to a subchart in the domain:

```
sage: A = M.open_subset('A', coord_def={c_uv: u>0})
sage: Phi.coord_functions(c_uv.restrict(A), c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (A, (u, v))
```

Coordinate representation with respect to a superchart in the codomain:

```
sage: B = N.open_subset('B', coord_def={c_xyz: x<0})
sage: c_xyz_B = c_xyz.restrict(B)
sage: Phi1 = M.continuous_map(B, {(c_uv, c_xyz_B): (u*v, u/v, u+v)})
```

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```
sage: Phi1.coord_functions(c_uv, c_xyz_B) # definition charts
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
sage: Phi1.coord_functions(c_uv, c_xyz) # c_xyz = superchart of c_xyz_B
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
```

Coordinate representation with respect to a pair `(subchart, superchart)`:

```
sage: Phi1.coord_functions(c_uv.restrict(A), c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (A, (u, v))
```

Same example with SymPy as the symbolic calculus engine:

```
sage: M.set_calculus_method('sympy')
sage: N.set_calculus_method('sympy')
sage: Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
....:                        latex_name=r'\Phi')
sage: Phi.coord_functions(c_uv, c_xyz)
Coordinate functions (u*v, u/v, u + v) on the Chart (M, (u, v))
sage: Phi.coord_functions(c_UV, c_xyz)
Coordinate functions (-U**2/4 + V**2/4, -(U + V)/(U - V), V)
 on the Chart (M, (U, V))
sage: Phi.coord_functions(c_UV, c_XYZ)
Coordinate functions ((-U**3 + U**2*V + U*V**2 + 2*U*V + 6*U - V**3
 - 2*V**2 + 6*V)/(2*(U - V)), (U**3/4 - U**2*V/4 - U*V**2/4 + U*V
 - U + V**3/4 - V**2 - V)/(U - V), (U**3 - U**2*V - U*V**2 - 4*U*V
 - 8*U + V**3 + 4*V**2 - 8*V)/(4*(U - V))) on the Chart (M, (U, V))
```

disp(*chart1=None*, *chart2=None*)

Display the expression of `self` in one or more pair of charts.

If the expression is not known already, it is computed from some expression in other charts by means of change-of-coordinate formulas.

INPUT:

- `chart1` – (default: `None`) chart on the domain of `self`; if `None`, the display is performed on all the charts on the domain in which the map is known or computable via some change of coordinates
- `chart2` – (default: `None`) chart on the codomain of `self`; if `None`, the display is performed on all the charts on the codomain in which the map is known or computable via some change of coordinates

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Standard embedding of the sphere S^2 in $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2', structure='topological') # the 2-dimensional␣
↪sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: N = Manifold(3, 'R^3', latex_name=r'\RR^3', structure='topological')  #␣
↪R^3
sage: c_cart.<X,Y,Z> = N.chart()  # Cartesian coordinates on R^3
sage: Phi = M.continuous_map(N,
....:   {(c_xy, c_cart): [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^
↪2+y^2)],
```

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```
....:     (c_uv, c_cart): [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^
↪2+v^2)]},
....:    name='Phi', latex_name=r'\Phi')
sage: Phi.display(c_xy, c_cart)
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
sage: Phi.display(c_uv, c_cart)
Phi: S^2 --> R^3
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^
↪2 + v^2 - 1)/(u^2 + v^2 + 1))
```

The LaTeX output:

```
sage: latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \mbox{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 \, x}{x^{2} + y^{2} + 1},
   \frac{2 \, y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

If the argument `chart2` is not specified, the display is performed on all the charts on the codomain in which the map is known or computable via some change of coordinates (here only one chart: `c_cart`):

```
sage: Phi.display(c_xy)
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
```

Similarly, if the argument `chart1` is omitted, the display is performed on all the charts on the domain of `Phi` in which the map is known or computable via some change of coordinates:

```
sage: Phi.display(chart2=c_cart)
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^
↪2 + v^2 - 1)/(u^2 + v^2 + 1))
```

If neither `chart1` nor `chart2` is specified, the display is performed on all the pair of charts in which `Phi` is known or computable via some change of coordinates:

```
sage: Phi.display()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^
↪2 + v^2 - 1)/(u^2 + v^2 + 1))
```

If a chart covers entirely the map's domain, the mention "on . . ." is omitted:

```
sage: Phi.restrict(U).display()
Phi: U --> R^3
   (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
```

A shortcut of `display()` is `disp()`:

```
sage: Phi.disp()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^
↪2 + v^2 - 1)/(u^2 + v^2 + 1))
```

Display when SymPy is the symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: N.set_calculus_method('sympy')
sage: Phi.display(c_xy, c_cart)
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x**2 + y**2 + 1),
 2*y/(x**2 + y**2 + 1), (x**2 + y**2 - 1)/(x**2 + y**2 + 1))
sage: latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \mbox{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 x}{x^{2} + y^{2} + 1},
   \frac{2 y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

display(*chart1=None*, *chart2=None*)

Display the expression of `self` in one or more pair of charts.

If the expression is not known already, it is computed from some expression in other charts by means of change-of-coordinate formulas.

INPUT:

- `chart1` – (default: `None`) chart on the domain of `self`; if `None`, the display is performed on all the charts on the domain in which the map is known or computable via some change of coordinates
- `chart2` – (default: `None`) chart on the codomain of `self`; if `None`, the display is performed on all the charts on the codomain in which the map is known or computable via some change of coordinates

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

EXAMPLES:

Standard embedding of the sphere S^2 in $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2', structure='topological') # the 2-dimensional␣
↪sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: N = Manifold(3, 'R^3', latex_name=r'\RR^3', structure='topological')  #␣
↪R^3
sage: c_cart.<X,Y,Z> = N.chart()  # Cartesian coordinates on R^3
sage: Phi = M.continuous_map(N,
....:   {(c_xy, c_cart): [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^
↪2+y^2)],
....:    (c_uv, c_cart): [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^
↪2+v^2)]},
```

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```
....:    name='Phi', latex_name=r'\Phi')
sage: Phi.display(c_xy, c_cart)
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
sage: Phi.display(c_uv, c_cart)
Phi: S^2 --> R^3
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^
↪2 + v^2 - 1)/(u^2 + v^2 + 1))
```

The LaTeX output:

```
sage: latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \mbox{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 \, x}{x^{2} + y^{2} + 1},
   \frac{2 \, y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

If the argument `chart2` is not specified, the display is performed on all the charts on the codomain in which the map is known or computable via some change of coordinates (here only one chart: `c_cart`):

```
sage: Phi.display(c_xy)
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
```

Similarly, if the argument `chart1` is omitted, the display is performed on all the charts on the domain of `Phi` in which the map is known or computable via some change of coordinates:

```
sage: Phi.display(chart2=c_cart)
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^
↪2 + v^2 - 1)/(u^2 + v^2 + 1))
```

If neither `chart1` nor `chart2` is specified, the display is performed on all the pair of charts in which `Phi` is known or computable via some change of coordinates:

```
sage: Phi.display()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^
↪2 + v^2 - 1)/(u^2 + v^2 + 1))
```

If a chart covers entirely the map's domain, the mention "on . . . " is omitted:

```
sage: Phi.restrict(U).display()
Phi: U --> R^3
   (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2 +␣
↪y^2 - 1)/(x^2 + y^2 + 1))
```

A shortcut of `display()` is `disp()`:

```
sage: Phi.disp()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1), (x^2␣
↪+ y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1), -(u^
↪2 + v^2 - 1)/(u^2 + v^2 + 1))
```

Display when SymPy is the symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: N.set_calculus_method('sympy')
sage: Phi.display(c_xy, c_cart)
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x**2 + y**2 + 1),
 2*y/(x**2 + y**2 + 1), (x**2 + y**2 - 1)/(x**2 + y**2 + 1))
sage: latex(Phi.display(c_xy, c_cart))
\begin{array}{llcl} \Phi:& S^2 & \longrightarrow & \RR^3
 \\ \mbox{on}\ U : & \left(x, y\right) & \longmapsto
 & \left(X, Y, Z\right) = \left(\frac{2 x}{x^{2} + y^{2} + 1},
   \frac{2 y}{x^{2} + y^{2} + 1},
   \frac{x^{2} + y^{2} - 1}{x^{2} + y^{2} + 1}\right)
 \end{array}
```

expr(*chart1=None*, *chart2=None*)

Return the expression of `self` in terms of specified coordinates.

If the expression is not already known, it is computed from some known expression by means of change-of-chart formulas.

INPUT:

- `chart1` – (default: `None`) chart on the map's domain; if `None`, the domain's default chart is assumed
- `chart2` – (default: `None`) chart on the map's codomain; if `None`, the codomain's default chart is assumed

OUTPUT:

- symbolic expression representing the continuous map in the above two charts

EXAMPLES:

Continuous map from a 2-dimensional manifold to a 3-dimensional one:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: N = Manifold(3, 'N', structure='topological')
sage: c_uv.<u,v> = M.chart()
sage: c_xyz.<x,y,z> = N.chart()
sage: Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
....:                        latex_name=r'\Phi')
sage: Phi.display()
Phi: M --> N
   (u, v) |--> (x, y, z) = (u*v, u/v, u + v)
sage: Phi.expr(c_uv, c_xyz)
(u*v, u/v, u + v)
sage: Phi.expr()  # equivalent to above since 'uv' and 'xyz' are default␣
↪charts
(u*v, u/v, u + v)
sage: type(Phi.expr()[0])
<type 'sage.symbolic.expression.Expression'>
```

Expressions in other charts:

```
sage: c_UV.<U,V> = M.chart()  # new chart on M
sage: ch_uv_UV = c_uv.transition_map(c_UV, [u-v, u+v])
sage: ch_uv_UV.inverse()(U,V)
(1/2*U + 1/2*V, -1/2*U + 1/2*V)
sage: c_XYZ.<X,Y,Z> = N.chart() # new chart on N
sage: ch_xyz_XYZ = c_xyz.transition_map(c_XYZ,
....:                                   [2*x-3*y+z, y+z-x, -x+2*y-z])
sage: ch_xyz_XYZ.inverse()(X,Y,Z)
(3*X + Y + 4*Z, 2*X + Y + 3*Z, X + Y + Z)
sage: Phi.expr(c_UV, c_xyz)
(-1/4*U^2 + 1/4*V^2, -(U + V)/(U - V), V)
sage: Phi.expr(c_uv, c_XYZ)
(((2*u + 1)*v^2 + u*v - 3*u)/v,
 -((u - 1)*v^2 - u*v - u)/v,
 -((u + 1)*v^2 + u*v - 2*u)/v)
sage: Phi.expr(c_UV, c_XYZ)
 (-1/2*(U^3 - (U - 2)*V^2 + V^3 - (U^2 + 2*U + 6)*V - 6*U)/(U - V),
  1/4*(U^3 - (U + 4)*V^2 + V^3 - (U^2 - 4*U + 4)*V - 4*U)/(U - V),
  1/4*(U^3 - (U - 4)*V^2 + V^3 - (U^2 + 4*U + 8)*V - 8*U)/(U - V))
```

A rotation in some Euclidean plane:

```
sage: M = Manifold(2, 'M', structure='topological') # the plane (minus a␣
↪segment to have global regular spherical coordinates)
sage: c_spher.<r,ph> = M.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on the plane
sage: rot = M.continuous_map(M, (r, ph+pi/3), name='R') # pi/3 rotation␣
↪around r=0
sage: rot.expr()
(r, 1/3*pi + ph)
```

Expression of the rotation in terms of Cartesian coordinates:

```
sage: c_cart.<x,y> = M.chart() # Declaration of Cartesian coordinates
sage: ch_spher_cart = c_spher.transition_map(c_cart,
....:                 [r*cos(ph), r*sin(ph)]) # relation to spherical␣
↪coordinates
sage: ch_spher_cart.set_inverse(sqrt(x^2+y^2), atan2(y,x), verbose=True)
Check of the inverse coordinate transformation:
   r == r
   ph == arctan2(r*sin(ph), r*cos(ph))
   x == x
   y == y
sage: rot.expr(c_cart, c_cart)
(-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

expression(*chart1=None*, *chart2=None*)

Return the expression of `self` in terms of specified coordinates.

If the expression is not already known, it is computed from some known expression by means of change-of-chart formulas.

INPUT:

- `chart1` – (default: `None`) chart on the map's domain; if `None`, the domain's default chart is assumed
- `chart2` – (default: `None`) chart on the map's codomain; if `None`, the codomain's default chart is assumed

OUTPUT:

- symbolic expression representing the continuous map in the above two charts

EXAMPLES:

Continuous map from a 2-dimensional manifold to a 3-dimensional one:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: N = Manifold(3, 'N', structure='topological')
sage: c_uv.<u,v> = M.chart()
sage: c_xyz.<x,y,z> = N.chart()
sage: Phi = M.continuous_map(N, (u*v, u/v, u+v), name='Phi',
....:                        latex_name=r'\Phi')
sage: Phi.display()
Phi: M --> N
   (u, v) |--> (x, y, z) = (u*v, u/v, u + v)
sage: Phi.expr(c_uv, c_xyz)
(u*v, u/v, u + v)
sage: Phi.expr()  # equivalent to above since 'uv' and 'xyz' are default␣
↪charts
(u*v, u/v, u + v)
sage: type(Phi.expr()[0])
<type 'sage.symbolic.expression.Expression'>
```

Expressions in other charts:

```
sage: c_UV.<U,V> = M.chart()  # new chart on M
sage: ch_uv_UV = c_uv.transition_map(c_UV, [u-v, u+v])
sage: ch_uv_UV.inverse()(U,V)
(1/2*U + 1/2*V, -1/2*U + 1/2*V)
sage: c_XYZ.<X,Y,Z> = N.chart() # new chart on N
sage: ch_xyz_XYZ = c_xyz.transition_map(c_XYZ,
....:                                   [2*x-3*y+z, y+z-x, -x+2*y-z])
sage: ch_xyz_XYZ.inverse()(X,Y,Z)
(3*X + Y + 4*Z, 2*X + Y + 3*Z, X + Y + Z)
sage: Phi.expr(c_UV, c_xyz)
(-1/4*U^2 + 1/4*V^2, -(U + V)/(U - V), V)
sage: Phi.expr(c_uv, c_XYZ)
(((2*u + 1)*v^2 + u*v - 3*u)/v,
 -((u - 1)*v^2 - u*v - u)/v,
 -((u + 1)*v^2 + u*v - 2*u)/v)
sage: Phi.expr(c_UV, c_XYZ)
 (-1/2*(U^3 - (U - 2)*V^2 + V^3 - (U^2 + 2*U + 6)*V - 6*U)/(U - V),
  1/4*(U^3 - (U + 4)*V^2 + V^3 - (U^2 - 4*U + 4)*V - 4*U)/(U - V),
  1/4*(U^3 - (U - 4)*V^2 + V^3 - (U^2 + 4*U + 8)*V - 8*U)/(U - V))
```

A rotation in some Euclidean plane:

```
sage: M = Manifold(2, 'M', structure='topological') # the plane (minus a␣
↪segment to have global regular spherical coordinates)
sage: c_spher.<r,ph> = M.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on the plane
sage: rot = M.continuous_map(M, (r, ph+pi/3), name='R') # pi/3 rotation␣
↪around r=0
sage: rot.expr()
(r, 1/3*pi + ph)
```

Expression of the rotation in terms of Cartesian coordinates:

```
sage: c_cart.<x,y> = M.chart() # Declaration of Cartesian coordinates
sage: ch_spher_cart = c_spher.transition_map(c_cart,
....:                   [r*cos(ph), r*sin(ph)]) # relation to spherical␣
↪coordinates
sage: ch_spher_cart.set_inverse(sqrt(x^2+y^2), atan2(y,x), verbose=True)
Check of the inverse coordinate transformation:
   r == r
   ph == arctan2(r*sin(ph), r*cos(ph))
   x == x
   y == y
sage: rot.expr(c_cart, c_cart)
(-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

inverse()

Return the inverse of `self` if it is an isomorphism.

OUTPUT:

- the inverse isomorphism

EXAMPLES:

The inverse of a rotation in the Euclidean plane:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')
sage: c_cart.<x,y> = M.chart()
sage: # A pi/3 rotation around the origin:
sage: rot = M.homeomorphism(M, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                       name='R')
sage: rot.inverse()
Homeomorphism R^(-1) of the 2-dimensional topological manifold R^2
sage: rot.inverse().display()
R^(-1): R^2 --> R^2
   (x, y) |--> (1/2*sqrt(3)*y + 1/2*x, -1/2*sqrt(3)*x + 1/2*y)
```

Checking that applying successively the homeomorphism and its inverse results in the identity:

```
sage: (a, b) = var('a b')
sage: p = M.point((a,b)) # a generic point on M
sage: q = rot(p)
sage: p1 = rot.inverse()(q)
sage: p1 == p
True
```

The result is cached:

```
sage: rot.inverse() is rot.inverse()
True
```

The notations `^(-1)` or `~` can also be used for the inverse:

```
sage: rot^(-1) is rot.inverse()
True
sage: ~rot is rot.inverse()
True
```

An example with multiple charts: the equatorial symmetry on the 2-sphere:

```
sage: M = Manifold(2, 'M', structure='topological') # the 2-dimensional␣
↪sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1=x^2+y^2!=0,
....:                                restrictions2=u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: s = M.homeomorphism(M, {(c_xy, c_uv): [x, y], (c_uv, c_xy): [u, v]},
....:                     name='s')
sage: s.display()
s: M --> M
on U: (x, y) |--> (u, v) = (x, y)
on V: (u, v) |--> (x, y) = (u, v)
sage: si = s.inverse(); si
Homeomorphism s^(-1) of the 2-dimensional topological manifold M
sage: si.display()
s^(-1): M --> M
on U: (x, y) |--> (u, v) = (x, y)
on V: (u, v) |--> (x, y) = (u, v)
```

The equatorial symmetry is of course an involution:

```
sage: si == s
True
```

is_identity()

Check whether `self` is an identity map.

EXAMPLES:

Tests on continuous maps of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart()
sage: M.identity_map().is_identity()  # obviously...
True
sage: Hom(M, M).one().is_identity()  # a variant of the obvious
True
sage: a = M.continuous_map(M, coord_functions={(X,X): (x, y)})
sage: a.is_identity()
True
sage: a = M.continuous_map(M, coord_functions={(X,X): (x, y+1)})
sage: a.is_identity()
False
```

Of course, if the codomain of the map does not coincide with its domain, the outcome is `False`:

```
sage: N = Manifold(2, 'N', structure='topological')
sage: Y.<u,v> = N.chart()
sage: a = M.continuous_map(N, {(X,Y): (x, y)})
sage: a.display()
M --> N
   (x, y) |--> (u, v) = (x, y)
```

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```
sage: a.is_identity()
False
```

restrict (*subdomain*, *subcodomain=None*)

Restriction of `self` to some open subset of its domain of definition.

INPUT:

- `subdomain` – *TopologicalManifold*; an open subset of the domain of `self`
- `subcodomain` – (default: `None`) an open subset of the codomain of `self`; if `None`, the codomain of `self` is assumed

OUTPUT:

- a *ContinuousMap* that is the restriction of `self` to `subdomain`

EXAMPLES:

Restriction to an annulus of a homeomorphism between the open unit disk and $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2', structure='topological')  # R^2
sage: c_xy.<x,y> = M.chart()  # Cartesian coord. on R^2
sage: D = M.open_subset('D', coord_def={c_xy: x^2+y^2<1}) # the open unit disk
sage: Phi = D.continuous_map(M, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                        name='Phi', latex_name=r'\Phi')
sage: Phi.display()
Phi: D --> R^2
   (x, y) |--> (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
sage: c_xy_D = c_xy.restrict(D)
sage: U = D.open_subset('U', coord_def={c_xy_D: x^2+y^2>1/2}) # the annulus 1/
↪2 < r < 1
sage: Phi.restrict(U)
Continuous map Phi
 from the Open subset U of the 2-dimensional topological manifold R^2
 to the 2-dimensional topological manifold R^2
sage: Phi.restrict(U).parent()
Set of Morphisms
 from Open subset U of the 2-dimensional topological manifold R^2
 to 2-dimensional topological manifold R^2
 in Join of Category of subobjects of sets
    and Category of manifolds over Real Field with 53 bits of precision
sage: Phi.domain()
Open subset D of the 2-dimensional topological manifold R^2
sage: Phi.restrict(U).domain()
Open subset U of the 2-dimensional topological manifold R^2
sage: Phi.restrict(U).display()
Phi: U --> R^2
   (x, y) |--> (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The result is cached:

```
sage: Phi.restrict(U) is Phi.restrict(U)
True
```

The restriction of the identity map:

```
sage: id = D.identity_map() ; id
Identity map Id_D of the Open subset D of the 2-dimensional
```

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```
 topological manifold R^2
sage: id.restrict(U)
Identity map Id_U of the Open subset U of the 2-dimensional
 topological manifold R^2
sage: id.restrict(U) is U.identity_map()
True
```

The codomain can be restricted (i.e. made tighter):

```
sage: Phi = D.continuous_map(M, [x/sqrt(1+x^2+y^2), y/sqrt(1+x^2+y^2)])
sage: Phi
Continuous map from
 the Open subset D of the 2-dimensional topological manifold R^2
 to the 2-dimensional topological manifold R^2
sage: Phi.restrict(D, subcodomain=D)
Continuous map from the Open subset D of the 2-dimensional
 topological manifold R^2 to itself
```

set_expr(*chart1*, *chart2*, *coord_functions*)

Set a new coordinate representation of `self`.

The expressions with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use *add_expr()* instead.

INPUT:

- `chart1` – chart for the coordinates on the domain of `self`
- `chart2` – chart for the coordinates on the codomain of `self`
- `coord_functions` – the coordinate symbolic expression of the map in the above charts: list (or tuple) of the coordinates of the image expressed in terms of the coordinates of the considered point; if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

EXAMPLES:

Polar representation of a planar rotation initially defined in Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')  #␣
↪the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement␣
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
sage: # Links between spherical coordinates and Cartesian ones:
sage: ch_cart_spher = c_cart.transition_map(c_spher,
....:                                        [sqrt(x*x+y*y), atan2(y,x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph), verbose=True)
Check of the inverse coordinate transformation:
   x == x
   y == y
   r == r
   ph == arctan2(r*sin(ph), r*cos(ph))
sage: rot = U.continuous_map(U, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                        name='R')
sage: rot.display(c_cart, c_cart)
```

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```
R: U --> U
   (x, y) |--> (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

Let us use the method `set_expr()` to set the spherical-coordinate expression by hand:

```
sage: rot.set_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot.display(c_spher, c_spher)
R: U --> U
   (r, ph) |--> (r, 1/3*pi + ph)
```

The expression in Cartesian coordinates has been erased:

```
sage: rot._coord_expression
{(Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

It is recovered (thanks to the known change of coordinates) by a call to `display()`:

```
sage: rot.display(c_cart, c_cart)
R: U --> U
   (x, y) |--> (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

sage: rot._coord_expression  # random (dictionary output)
{(Chart (U, (x, y)),
  Chart (U, (x, y))): Coordinate functions (-1/2*sqrt(3)*y + 1/2*x,
  1/2*sqrt(3)*x + 1/2*y) on the Chart (U, (x, y)),
 (Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

set_expression(*chart1*, *chart2*, *coord_functions*)

Set a new coordinate representation of `self`.

The expressions with respect to other charts are deleted, in order to avoid any inconsistency. To keep them, use `add_expr()` instead.

INPUT:

- `chart1` – chart for the coordinates on the domain of `self`
- `chart2` – chart for the coordinates on the codomain of `self`
- `coord_functions` – the coordinate symbolic expression of the map in the above charts: list (or tuple) of the coordinates of the image expressed in terms of the coordinates of the considered point; if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

EXAMPLES:

Polar representation of a planar rotation initially defined in Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2', latex_name=r'\RR^2', structure='topological')  #
↪the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
```

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```
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
sage: # Links between spherical coordinates and Cartesian ones:
sage: ch_cart_spher = c_cart.transition_map(c_spher,
....:                                        [sqrt(x*x+y*y), atan2(y,x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph), verbose=True)
Check of the inverse coordinate transformation:
   x == x
   y == y
   r == r
   ph == arctan2(r*sin(ph), r*cos(ph))
sage: rot = U.continuous_map(U, ((x - sqrt(3)*y)/2, (sqrt(3)*x + y)/2),
....:                        name='R')
sage: rot.display(c_cart, c_cart)
R: U --> U
   (x, y) |--> (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)
```

Let us use the method `set_expr()` to set the spherical-coordinate expression by hand:

```
sage: rot.set_expr(c_spher, c_spher, (r, ph+pi/3))
sage: rot.display(c_spher, c_spher)
R: U --> U
   (r, ph) |--> (r, 1/3*pi + ph)
```

The expression in Cartesian coordinates has been erased:

```
sage: rot._coord_expression
{(Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

It is recovered (thanks to the known change of coordinates) by a call to `display()`:

```
sage: rot.display(c_cart, c_cart)
R: U --> U
   (x, y) |--> (-1/2*sqrt(3)*y + 1/2*x, 1/2*sqrt(3)*x + 1/2*y)

sage: rot._coord_expression  # random (dictionary output)
{(Chart (U, (x, y)),
  Chart (U, (x, y))): Coordinate functions (-1/2*sqrt(3)*y + 1/2*x,
  1/2*sqrt(3)*x + 1/2*y) on the Chart (U, (x, y)),
 (Chart (U, (r, ph)),
  Chart (U, (r, ph))): Coordinate functions (r, 1/3*pi + ph)
  on the Chart (U, (r, ph))}
```

1.8 Submanifolds of topological manifolds

Given a topological manifold M over a topological field K, a *topological submanifold of* M is defined by a topological manifold N over the same field K of dimension lower than the dimension of M and a topological embedding ϕ from N to M (i.e. ϕ is a homeomorphism onto its image).

In the case where the map ϕ is only an embedding locally, it is called an *topological immersion*, and defines an *immersed submanifold*.

The global embedding property cannot be checked in sage, so the immersed or embedded aspect of the manifold must be declared by the user, by calling either `set_embedding()` or `set_immersion()` while declaring the map ϕ.

The map $\phi : N \to M$ can also depend on one or multiple parameters. As long as ϕ remains injective in these parameters, it represents a *foliation*. The *dimension* of the foliation is defined as the number of parameters.

AUTHORS:

- Florentin Jaffredo (2018): initial version

REFERENCES:

- [**?**]

class sage.manifolds.topological_submanifold.**TopologicalSubmanifold**(*n*, *name*, *field*, *structure*, *ambient=None*, *base_manifold=None*, *latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: `sage.manifolds.manifold.TopologicalManifold`

Submanifold of a topological manifold.

Given a topological manifold M over a topological field K, a *topological submanifold of* M is defined by a topological manifold N over the same field K of dimension lower than the dimension of M and a topological embedding ϕ from N to M (i.e. ϕ is an homeomorphism onto its image).

In the case where ϕ is only an topological immersion (i.e. is only locally an embedding), one says that N is an *immersed submanifold*.

The map ϕ can also depend on one or multiple parameters. As long as ϕ remains injective in these parameters, it represents a *foliation*. The *dimension* of the foliation is defined as the number of parameters.

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `field` – field K on which the manifold is defined; allowed values are
  - `'real'` or an object of type `RealField` (e.g., `RR`) for a manifold over $\mathbf{R}$
  - `'complex'` or an object of type `ComplexField` (e.g., `CC`) for a manifold over $\mathbf{C}$
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of manifolds
- `structure` – manifold structure (see `TopologicalStructure` or `RealTopologicalStructure`)
- `ambient` – (default: `None`) manifold of destination of the immersion. If `None`, set to `self`
- `base_manifold` – (default: `None`) if not `None`, must be a topological manifold; the created object is then an open subset of `base_manifold`

- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none are provided, it is set to `name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g., coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(field)` is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from `ManifoldSubset` would return the previously constructed object corresponding to these arguments)

EXAMPLES:

Let N be a 2-dimensional submanifold of a 3-dimensional manifold M:

```
sage: M = Manifold(3, 'M', structure="topological")
sage: N = Manifold(2, 'N', ambient=M, structure="topological")
sage: N
2-dimensional submanifold N embedded in 3-dimensional manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
```

Let us define a 1-dimensional foliation indexed by t. The inverse map is needed in order to compute the adapted chart in the ambient manifold:

```
sage: t = var('t')
sage: phi = N.continuous_map(M, {(CN,CM):[u, v, t+u**2+v**2]}); phi
Continuous map from the 2-dimensional submanifold N embedded in
 3-dimensional manifold M to the 3-dimensional topological manifold M
sage: phi_inv = M.continuous_map(N, {(CM, CN):[x, y]})
sage: phi_inv_t = M.scalar_field({CM: z-x**2-y**2})
```

ϕ can then be declared as an embedding $N \to M$:

```
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
```

The foliation can also be used to find new charts on the ambient manifold that are adapted to the foliation, i.e. in which the expression of the immersion is trivial. At the same time, the appropriate coordinate changes are computed:

```
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
sage: len(M.coord_changes())
2
```

The foliations parameters are always added as the last coordinates.

See also:

`manifold`

`adapted_chart`(*index=''*, *latex_index=''*)

Create charts and changes of charts in the ambient manifold adapted to the foliation.

A manifold M of dimension m can be foliated by submanifolds N of dimension n. The corresponding embedding needs $m-n$ free parameters to describe the whole manifold.

A set of coordinates adapted to a foliation is a set of coordinates $(x_1, ..., x_n, t_1, ...t_{m-n})$ such that $(x_1, ...x_n)$ are coordinates of N and $(t_1, ...t_{m-n})$ are the $m - n$ free parameters of the foliation.

Provided that an embedding with free variables is already defined, this function constructs such charts and coordinates changes whenever it is possible.

If there are restrictions of the coordinates on the starting chart, these restrictions are also propagated.

INPUT:

- `index` – (default: `""`) string defining the name of the coordinates in the new chart. This string will be added at the end of the names of the old coordinates. By default, it is replaced by `"_"+self._ambient._name`
- `latex_index` – (default: `""`) string defining the latex name of the coordinates in the new chart. This string will be added at the end of the latex names of the old coordinates. By default, it is replaced by `"_"+self._ambient._latex_()`

OUTPUT:

- list of charts created from the charts of `self`

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure="topological")
sage: N = Manifold(2, 'N', ambient=M, structure="topological")
sage: N
2-dimensional submanifold N embedded in 3-dimensional manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M,{(CN,CM):[u,v,t+u**2+v**2]})
sage: phi_inv = M.continuous_map(N,{(CM,CN):[x,y]})
sage: phi_inv_t = M.scalar_field({CM:z-x**2-y**2})
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t:phi_inv_t})
sage: N.declare_embedding()
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
```

ambient()

Return the ambient manifold in which `self` is immersed or embedded.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure="topological")
sage: N = Manifold(2, 'N', ambient=M, structure="topological")
sage: N.ambient()
3-dimensional topological manifold M
```

declare_embedding()

Declare that the immersion provided by *set_immersion()* is in fact an embedding.

A *topological embedding* is a continuous map that is a homeomorphism onto its image. A *differentiable embedding* is a topological embedding that is also a differentiable immersion.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure="topological")
sage: N = Manifold(2, 'N', ambient=M, structure="topological")
sage: N
```

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```
2-dimensional submanifold N embedded in 3-dimensional manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M,{(CN,CM):[u,v,t+u**2+v**2]}); phi
Continuous map from the 2-dimensional submanifold N embedded in
 3-dimensional manifold M to the 3-dimensional topological
 manifold M
sage: phi_inv = M.continuous_map(N,{(CM,CN):[x,y]})
sage: phi_inv_t = M.scalar_field({CM:z-x**2-y**2})
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
sage: N._immersed
True
sage: N._embedded
False
sage: N.declare_embedding()
sage: N._immersed
True
sage: N._embedded
True
```

embedding()

Return the embedding of the submanifold.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure="topological")
sage: N = Manifold(2, 'N', ambient=M, structure="topological")
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M,{(CN,CM):[u,v,t+u**2+v**2]})
sage: phi_inv = M.continuous_map(N,{(CM,CN):[x,y]})
sage: phi_inv_t = M.scalar_field({CM:z-x**2-y**2})
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
sage: N.embedding()
Homeomorphism from the 2-dimensional submanifold N embedded in
 3-dimensional manifold M to the 3-dimensional topological manifold
 M
```

immersion()

Return the immersion of the submanifold.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure="topological")
sage: N = Manifold(2, 'N', ambient=M, structure="topological")
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M,{(CN,CM):[u,v,t+u**2+v**2]})
sage: phi_inv = M.continuous_map(N,{(CM,CN):[x,y]})
sage: phi_inv_t = M.scalar_field({CM:z-x**2-y**2})
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
```

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```
sage: N.immersion()
Continuous map from the 2-dimensional submanifold N embedded in
 3-dimensional manifold M to the 3-dimensional topological
 manifold M
```

plot(*param*, *u*, *v*, *chart1=None*, *chart2=None*, ***kwargs*)

Plot an embedding.

Plot the embedding defined by the foliation and a set of values for the free parameters. This function can only plot 2-dimensional surfaces embedded in 3-dimensional manifolds. It ultimately calls `ParametricSurface`.

INPUT:

- `param` – dictionary of values indexed by the free variables appearing in the foliation.
- `u` – iterable of the values taken by the first coordinate of the surface to plot
- `v` – iterable of the values taken by the second coordinate of the surface to plot
- `chart1` – (default: `None`) chart in which `u` and `v` are considered. By default, the default chart of the submanifold is used
- `chart1` – (default: `None`) destination chart. By default, the default chart of the manifold is used
- `**kwargs` – other arguments as used in `ParametricSurface`

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure="topological")
sage: N = Manifold(2, 'N', ambient = M, structure="topological")
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M,{(CN,CM):[u,v,t+u**2+v**2]})
sage: phi_inv = M.continuous_map(N,{(CM,CN):[x,y]})
sage: phi_inv_t = M.scalar_field({CM:z-x**2-y**2})
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                 t_inverse = {t:phi_inv_t})
sage: N.declare_embedding()
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
sage: P0 = N.plot({t:0}, srange(-1, 1, 0.1), srange(-1, 1, 0.1),
....:             CN, CM, opacity=0.3, mesh=True)
sage: P1 = N.plot({t:1}, srange(-1, 1, 0.1), srange(-1, 1, 0.1),
....:             CN, CM, opacity=0.3, mesh=True)
sage: P2 = N.plot({t:2}, srange(-1, 1, 0.1), srange(-1, 1, 0.1),
....:             CN, CM, opacity=0.3, mesh=True)
sage: P3 = N.plot({t:3}, srange(-1, 1, 0.1), srange(-1, 1, 0.1),
....:             CN, CM, opacity=0.3, mesh=True)
sage: show(P0+P1+P2+P3)
```

See also:

`ParametricSurface`

set_embedding(*phi*, *inverse=None*, *var=None*, *t_inverse=None*)

Register the embedding of an embedded submanifold.

A *topological embedding* is a continuous map that is a homeomorphism onto its image. A *differentiable embedding* is a topological embedding that is also a differentiable immersion.

INPUT:

- phi – continuous map ϕ from self to self._ambient
- inverse – (default: None) inverse of ϕ onto its image, used for computing changes of chart from or to adapted charts. No verification is made
- var – (default: None) list of parameters appearing in ϕ
- t_inverse – (default: None) dictionary of scalar field on self._ambient indexed by elements of var representing the missing information in inverse

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure="topological")
sage: N = Manifold(2, 'N', ambient=M, structure="topological")
sage: N
2-dimensional submanifold N embedded in 3-dimensional manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M,{(CN,CM):[u,v,t+u**2+v**2]}); phi
Continuous map from the 2-dimensional submanifold N embedded in
 3-dimensional manifold M to the 3-dimensional topological
 manifold M
sage: phi_inv = M.continuous_map(N,{(CM,CN):[x,y]})
sage: phi_inv_t = M.scalar_field({CM:z-x**2-y**2})
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
```

set_immersion (*phi, inverse=None, var=None, t_inverse=None*)

Register the immersion of the immersed submanifold.

A *topological immersion* is a continuous map that is locally a topological embedding (i.e. a homeomorphism onto its image). A *differentiable immersion* is a differentiable map whose differential is injective at each point.

If an inverse of the immersion onto its image exists, it can be registered at the same time. If the immersion depends on parameters, they must also be declared here.

INPUT:

- phi – continuous map ϕ from self to self._ambient
- inverse – (default: None) inverse of ϕ onto its image, used for computing changes of chart from or to adapted charts. No verification is made
- var – (default: None) list of parameters appearing in ϕ
- t_inverse – (default: None) dictionary of scalar field on self._ambient indexed by elements of var representing the missing information in inverse

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure="topological")
sage: N = Manifold(2, 'N', ambient=M, structure="topological")
sage: N
2-dimensional submanifold N embedded in 3-dimensional manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
sage: t = var('t')
sage: phi = N.continuous_map(M,{(CN,CM):[u,v,t+u**2+v**2]}); phi
```

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```
Continuous map from the 2-dimensional submanifold N embedded in
 3-dimensional manifold M to the 3-dimensional topological
 manifold M
sage: phi_inv = M.continuous_map(N,{(CM,CN):[x,y]})
sage: phi_inv_t = M.scalar_field({CM:z-x**2-y**2})
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                  t_inverse={t: phi_inv_t})
```

CHAPTER

TWO

DIFFERENTIABLE MANIFOLDS

2.1 Differentiable Manifolds

Given a non-discrete topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$; see however [**?**] for $K = \mathbf{Q}_p$ and [**?**] for other fields), a *differentiable manifold over* K is a topological manifold M over K equipped with an atlas whose transitions maps are of class C^k (i.e. k-times continuously differentiable) for a fixed positive integer k (possibly $k = \infty$). M is then called a C^k*-manifold over* K.

Note that

- if the mention of K is omitted, then $K = \mathbf{R}$ is assumed;
- if $K = \mathbf{C}$, any C^k-manifold with $k \geq 1$ is actually a C^∞-manifold (even an analytic manifold);
- if $K = \mathbf{R}$, any C^k-manifold with $k \geq 1$ admits a compatible C^∞-structure (Whitney's smoothing theorem).

Differentiable manifolds are implemented via the class `DifferentiableManifold`. Open subsets of differentiable manifolds are also implemented via `DifferentiableManifold`, since they are differentiable manifolds by themselves.

The user interface is provided by the generic function `Manifold()`, with the argument `structure` set to `'differentiable'` and the argument `diff_degree` set to k, or the argument `structure` set to `'smooth'` (the default value).

Example 1: the 2-sphere as a differentiable manifold of dimension 2 over $\mathbf{R}$

One starts by declaring S^2 as a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'S^2')
sage: M
2-dimensional differentiable manifold S^2
```

Since the base topological field has not been specified in the argument list of `Manifold`, $\mathbf{R}$ is assumed:

```
sage: M.base_field()
Real Field with 53 bits of precision
sage: dim(M)
2
```

By default, the created object is a smooth manifold:

```
sage: M.diff_degree()
+Infinity
```

Let us consider the complement of a point, the "North pole" say; this is an open subset of S^2, which we call U:

```
sage: U = M.open_subset('U'); U
Open subset U of the 2-dimensional differentiable manifold S^2
```

A standard chart on U is provided by the stereographic projection from the North pole to the equatorial plane:

```
sage: stereoN.<x,y> = U.chart(); stereoN
Chart (U, (x, y))
```

Thanks to the operator `<x,y>` on the left-hand side, the coordinates declared in a chart (here x and y), are accessible by their names; they are Sage's symbolic variables:

```
sage: y
y
sage: type(y)
<type 'sage.symbolic.expression.Expression'>
```

The South pole is the point of coordinates $(x, y) = (0, 0)$ in the above chart:

```
sage: S = U.point((0,0), chart=stereoN, name='S'); S
Point S on the 2-dimensional differentiable manifold S^2
```

Let us call V the open subset that is the complement of the South pole and let us introduce on it the chart induced by the stereographic projection from the South pole to the equatorial plane:

```
sage: V = M.open_subset('V'); V
Open subset V of the 2-dimensional differentiable manifold S^2
sage: stereoS.<u,v> = V.chart(); stereoS
Chart (V, (u, v))
```

The North pole is the point of coordinates $(u, v) = (0, 0)$ in this chart:

```
sage: N = V.point((0,0), chart=stereoS, name='N'); N
Point N on the 2-dimensional differentiable manifold S^2
```

To fully construct the manifold, we declare that it is the union of U and V:

```
sage: M.declare_union(U,V)
```

and we provide the transition map between the charts `stereoN` = $(U, (x, y))$ and `stereoS` = $(V, (u, v))$, denoting by W the intersection of U and V (W is the subset of U defined by $x^2 + y^2 \neq 0$, as well as the subset of V defined by $u^2 + v^2 \neq 0$):

```
sage: stereoN_to_S = stereoN.transition_map(stereoS,
....:                [x/(x^2+y^2), y/(x^2+y^2)], intersection_name='W',
....:                restrictions1= x^2+y^2!=0, restrictions2= u^2+v^2!=0)
sage: stereoN_to_S
Change of coordinates from Chart (W, (x, y)) to Chart (W, (u, v))
sage: stereoN_to_S.display()
u = x/(x^2 + y^2)
v = y/(x^2 + y^2)
```

We give the name `W` to the Python variable representing $W = U \cap V$:

```
sage: W = U.intersection(V)
```

The inverse of the transition map is computed by the method `inverse()`:

```
sage: stereoN_to_S.inverse()
Change of coordinates from Chart (W, (u, v)) to Chart (W, (x, y))
sage: stereoN_to_S.inverse().display()
x = u/(u^2 + v^2)
y = v/(u^2 + v^2)
```

At this stage, we have four open subsets on S^2:

```
sage: M.list_of_subsets()
[2-dimensional differentiable manifold S^2,
 Open subset U of the 2-dimensional differentiable manifold S^2,
 Open subset V of the 2-dimensional differentiable manifold S^2,
 Open subset W of the 2-dimensional differentiable manifold S^2]
```

W is the open subset that is the complement of the two poles:

```
sage: N in W or S in W
False
```

The North pole lies in V and the South pole in U:

```
sage: N in V, N in U
(True, False)
sage: S in U, S in V
(True, False)
```

The manifold's (user) atlas contains four charts, two of them being restrictions of charts to a smaller domain:

```
sage: M.atlas()
[Chart (U, (x, y)), Chart (V, (u, v)), Chart (W, (x, y)), Chart (W, (u, v))]
```

Let us consider the point of coordinates (1,2) in the chart `stereoN`:

```
sage: p = M.point((1,2), chart=stereoN, name='p'); p
Point p on the 2-dimensional differentiable manifold S^2
sage: p.parent()
2-dimensional differentiable manifold S^2
sage: p in W
True
```

The coordinates of p in the chart `stereoS` are computed by letting the chart act on the point:

```
sage: stereoS(p)
(1/5, 2/5)
```

Given the definition of p, we have of course:

```
sage: stereoN(p)
(1, 2)
```

Similarly:

```
sage: stereoS(N)
(0, 0)
sage: stereoN(S)
(0, 0)
```

A differentiable scalar field on the sphere:

```
sage: f = M.scalar_field({stereoN: atan(x^2+y^2), stereoS: pi/2-atan(u^2+v^2)},
....:                    name='f')
sage: f
Scalar field f on the 2-dimensional differentiable manifold S^2
sage: f.display()
f: S^2 --> R
on U: (x, y) |--> arctan(x^2 + y^2)
on V: (u, v) |--> 1/2*pi - arctan(u^2 + v^2)
sage: f(p)
arctan(5)
sage: f(N)
1/2*pi
sage: f(S)
0
sage: f.parent()
Algebra of differentiable scalar fields on the 2-dimensional differentiable
 manifold S^2
sage: f.parent().category()
Category of commutative algebras over Symbolic Ring
```

A differentiable manifold has a default vector frame, which, unless otherwise specified, is the coordinate frame associated with the first defined chart:

```
sage: M.default_frame()
Coordinate frame (U, (d/dx,d/dy))
sage: latex(M.default_frame())
\left(U, \left(\frac{\partial}{\partial x },\frac{\partial}{\partial y }\right)\right)
sage: M.default_frame() is stereoN.frame()
True
```

A vector field on the sphere:

```
sage: w = M.vector_field(name='w')
sage: w[stereoN.frame(), :] = [x, y]
sage: w.add_comp_by_continuation(stereoS.frame(), W, stereoS)
sage: w.display() # display in the default frame (stereoN.frame())
w = x d/dx + y d/dy
sage: w.display(stereoS.frame())
w = -u d/du - v d/dv
sage: w.parent()
Module X(S^2) of vector fields on the 2-dimensional differentiable
 manifold S^2
sage: w.parent().category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold S^2
```

Vector fields act on scalar fields:

```
sage: w(f)
Scalar field w(f) on the 2-dimensional differentiable manifold S^2
sage: w(f).display()
w(f): S^2 --> R
on U: (x, y) |--> 2*(x^2 + y^2)/(x^4 + 2*x^2*y^2 + y^4 + 1)
on V: (u, v) |--> 2*(u^2 + v^2)/(u^4 + 2*u^2*v^2 + v^4 + 1)
sage: w(f) == f.differential()(w)
True
```

The value of the vector field at point p is a vector tangent to the sphere:

```
sage: w.at(p)
Tangent vector w at Point p on the 2-dimensional differentiable manifold S^2
sage: w.at(p).display()
w = d/dx + 2 d/dy
sage: w.at(p).parent()
Tangent space at Point p on the 2-dimensional differentiable manifold S^2
```

A 1-form on the sphere:

```
sage: df = f.differential() ; df
1-form df on the 2-dimensional differentiable manifold S^2
sage: df.display()
df = 2*x/(x^4 + 2*x^2*y^2 + y^4 + 1) dx + 2*y/(x^4 + 2*x^2*y^2 + y^4 + 1) dy
sage: df.display(stereoS.frame())
df = -2*u/(u^4 + 2*u^2*v^2 + v^4 + 1) du - 2*v/(u^4 + 2*u^2*v^2 + v^4 + 1) dv
sage: df.parent()
Module Omega^1(S^2) of 1-forms on the 2-dimensional differentiable
 manifold S^2
sage: df.parent().category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold S^2
```

The value of the 1-form at point p is a linear form on the tangent space at p:

```
sage: df.at(p)
Linear form df on the Tangent space at Point p on the 2-dimensional
 differentiable manifold S^2
sage: df.at(p).display()
df = 1/13 dx + 2/13 dy
sage: df.at(p).parent()
Dual of the Tangent space at Point p on the 2-dimensional differentiable
 manifold S^2
```

Example 2: the Riemann sphere as a differentiable manifold of dimension 1 over $\mathbf{C}$

We declare the Riemann sphere $\mathbf{C}^*$ as a 1-dimensional differentiable manifold over $\mathbf{C}$:

```
sage: M = Manifold(1, 'C*', field='complex'); M
1-dimensional complex manifold C*
```

We introduce a first open subset, which is actually $\mathbf{C} = \mathbf{C}^* \setminus \{\infty\}$ if we interpret $\mathbf{C}^*$ as the Alexandroff one-point compactification of $\mathbf{C}$:

```
sage: U = M.open_subset('U')
```

A natural chart on U is then nothing but the identity map of $\mathbf{C}$, hence we denote the associated coordinate by z:

```
sage: Z.<z> = U.chart()
```

The origin of the complex plane is the point of coordinate $z = 0$:

```
sage: O = U.point((0,), chart=Z, name='O'); O
Point O on the 1-dimensional complex manifold C*
```

Another open subset of $\mathbf{C}^*$ is $V = \mathbf{C}^* \setminus \{O\}$:

```
sage: V = M.open_subset('V')
```

We define a chart on V such that the point at infinity is the point of coordinate 0 in this chart:

```
sage: W.<w> = V.chart(); W
Chart (V, (w,))
sage: inf = M.point((0,), chart=W, name='inf', latex_name=r'\infty')
sage: inf
Point inf on the 1-dimensional complex manifold C*
```

To fully construct the Riemann sphere, we declare that it is the union of U and V:

```
sage: M.declare_union(U,V)
```

and we provide the transition map between the two charts as $w = 1/z$ on on $A = U \cap V$:

```
sage: Z_to_W = Z.transition_map(W, 1/z, intersection_name='A',
....:                           restrictions1= z!=0, restrictions2= w!=0)
sage: Z_to_W
Change of coordinates from Chart (A, (z,)) to Chart (A, (w,))
sage: Z_to_W.display()
w = 1/z
sage: Z_to_W.inverse()
Change of coordinates from Chart (A, (w,)) to Chart (A, (z,))
sage: Z_to_W.inverse().display()
z = 1/w
```

Let consider the complex number i as a point of the Riemann sphere:

```
sage: i = M((I,), chart=Z, name='i'); i
Point i on the 1-dimensional complex manifold C*
```

Its coordinates with respect to the charts `Z` and `W` are:

```
sage: Z(i)
(I,)
sage: W(i)
(-I,)
```

and we have:

```
sage: i in U
True
sage: i in V
True
```

The following subsets and charts have been defined:

```
sage: M.list_of_subsets()
[Open subset A of the 1-dimensional complex manifold C*,
 1-dimensional complex manifold C*,
 Open subset U of the 1-dimensional complex manifold C*,
 Open subset V of the 1-dimensional complex manifold C*]
sage: M.atlas()
[Chart (U, (z,)), Chart (V, (w,)), Chart (A, (z,)), Chart (A, (w,))]
```

A constant map $\mathbf{C}^* \to \mathbf{C}$:

```
sage: f = M.constant_scalar_field(3+2*I, name='f'); f
Scalar field f on the 1-dimensional complex manifold C*
sage: f.display()
f: C* --> C
on U: z |--> 2*I + 3
on V: w |--> 2*I + 3
sage: f(O)
2*I + 3
sage: f(i)
2*I + 3
sage: f(inf)
2*I + 3
sage: f.parent()
Algebra of differentiable scalar fields on the 1-dimensional complex
 manifold C*
sage: f.parent().category()
Category of commutative algebras over Symbolic Ring
```

A vector field on the Riemann sphere:

```
sage: v = M.vector_field(name='v')
sage: v[Z.frame(), 0] = z^2
sage: v.add_comp_by_continuation(W.frame(), U.intersection(V), W)
sage: v.display(Z.frame())
v = z^2 d/dz
sage: v.display(W.frame())
v = -d/dw
sage: v.parent()
Module X(C*) of vector fields on the 1-dimensional complex manifold C*
```

The vector field v acting on the scalar field f:

```
sage: v(f)
Scalar field v(f) on the 1-dimensional complex manifold C*
```

Since f is constant, $v(f)$ is vanishing:

```
sage: v(f).display()
v(f): C* --> C
on U: z |--> 0
on V: w |--> 0
```

The value of the vector field v at the point ∞ is a vector tangent to the Riemann sphere:

```
sage: v.at(inf)
Tangent vector v at Point inf on the 1-dimensional complex manifold C*
sage: v.at(inf).display()
v = -d/dw
sage: v.at(inf).parent()
Tangent space at Point inf on the 1-dimensional complex manifold C*
```

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [?]
- [?]
- [?]
- [?]
- [?]
- [?]

class sage.manifolds.differentiable.manifold.**DifferentiableManifold**(*n*, *name*, *field*, *structure*, *base_manifold=None*, *diff_degree=+Infinity*, *latex_name=None*, *start_index=0*, *category=None*, *unique_tag=None*)

Bases: *sage.manifolds.manifold.TopologicalManifold*

Differentiable manifold over a topological field K.

Given a non-discrete topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$; see however [?] for $K = \mathbf{Q}_p$ and [?] for other fields), a *differentiable manifold over* K is a topological manifold M over K equipped with an atlas whose transitions maps are of class C^k (i.e. k-times continuously differentiable) for a fixed positive integer k (possibly $k = \infty$). M is then called a C^k*-manifold over* K.

Note that

- if the mention of K is omitted, then $K = \mathbf{R}$ is assumed;
- if $K = \mathbf{C}$, any C^k-manifold with $k \geq 1$ is actually a C^∞-manifold (even an analytic manifold);
- if $K = \mathbf{R}$, any C^k-manifold with $k \geq 1$ admits a compatible C^∞-structure (Whitney's smoothing theorem).

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `field` – field K on which the manifold is defined; allowed values are
  - `'real'` or an object of type `RealField` (e.g., `RR`) for a manifold over $\mathbf{R}$
  - `'complex'` or an object of type `ComplexField` (e.g., `CC`) for a manifold over $\mathbf{C}$
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of manifolds
- `structure` – manifold structure (see *DifferentialStructure* or *RealDifferentialStructure*)
- `ambient` – (default: `None`) if not `None`, must be a differentiable manifold; the created object is then an open subset of `ambient`
- `diff_degree` – (default: `infinity`) degree k of differentiability

- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none is provided, it is set to `name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g. coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(field).Differentiable()` (or `Manifolds(field).Smooth()` if `diff_degree = infinity`) is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from *ManifoldSubset*, via *TopologicalManifold*, would return the previously constructed object corresponding to these arguments).

EXAMPLES:

A 4-dimensional differentiable manifold (over $\mathbf{R}$):

```
sage: M = Manifold(4, 'M', latex_name=r'\mathcal{M}'); M
4-dimensional differentiable manifold M
sage: type(M)
<class 'sage.manifolds.differentiable.manifold.DifferentiableManifold_with_
↪category'>
sage: latex(M)
\mathcal{M}
sage: dim(M)
4
```

Since the base field has not been specified, $\mathbf{R}$ has been assumed:

```
sage: M.base_field()
Real Field with 53 bits of precision
```

Since the degree of differentiability has not been specified, the default value, C^∞, has been assumed:

```
sage: M.diff_degree()
+Infinity
```

The input parameter `start_index` defines the range of indices on the manifold:

```
sage: M = Manifold(4, 'M')
sage: list(M.irange())
[0, 1, 2, 3]
sage: M = Manifold(4, 'M', start_index=1)
sage: list(M.irange())
[1, 2, 3, 4]
sage: list(Manifold(4, 'M', start_index=-2).irange())
[-2, -1, 0, 1]
```

A complex manifold:

```
sage: N = Manifold(3, 'N', field='complex'); N
3-dimensional complex manifold N
```

A differentiable manifold over $\mathbf{Q}_5$, the field of 5-adic numbers:

```
sage: N = Manifold(2, 'N', field=Qp(5)); N
2-dimensional differentiable manifold N over the 5-adic Field with
 capped relative precision 20
```

A differentiable manifold is of course a topological manifold:

```
sage: isinstance(M, sage.manifolds.manifold.TopologicalManifold)
True
sage: isinstance(N, sage.manifolds.manifold.TopologicalManifold)
True
```

A differentiable manifold is a Sage *parent* object, in the category of differentiable (here smooth) manifolds over a given topological field (see `Manifolds`):

```
sage: isinstance(M, Parent)
True
sage: M.category()
Category of smooth manifolds over Real Field with 53 bits of precision
sage: from sage.categories.manifolds import Manifolds
sage: M.category() is Manifolds(RR).Smooth()
True
sage: M.category() is Manifolds(M.base_field()).Smooth()
True
sage: M in Manifolds(RR).Smooth()
True
sage: N in Manifolds(Qp(5)).Smooth()
True
```

The corresponding Sage *elements* are points:

```
sage: X.<t, x, y, z> = M.chart()
sage: p = M.an_element(); p
Point on the 4-dimensional differentiable manifold M
sage: p.parent()
4-dimensional differentiable manifold M
sage: M.is_parent_of(p)
True
sage: p in M
True
```

The manifold's points are instances of class `ManifoldPoint`:

```
sage: isinstance(p, sage.manifolds.point.ManifoldPoint)
True
```

Since an open subset of a differentiable manifold M is itself a differentiable manifold, open subsets of M have all attributes of manifolds:

```
sage: U = M.open_subset('U', coord_def={X: t>0}); U
Open subset U of the 4-dimensional differentiable manifold M
sage: U.category()
Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision
sage: U.base_field() == M.base_field()
True
sage: dim(U) == dim(M)
True
```

The manifold passes all the tests of the test suite relative to its category:

```
sage: TestSuite(M).run()
```

affine_connection (*name*, *latex_name=None*)
Define an affine connection on the manifold.

See *AffineConnection* for a complete documentation.

INPUT:

- name – name given to the affine connection
- latex_name – (default: None) LaTeX symbol to denote the affine connection

OUTPUT:

- the affine connection, as an instance of *AffineConnection*

EXAMPLES:

Affine connection on an open subset of a 3-dimensional smooth manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: A = M.open_subset('A', latex_name=r'\mathcal{A}')
sage: nab = A.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the Open subset A of the 3-dimensional
 differentiable manifold M
```

See also:

AffineConnection for more examples.

automorphism_field (**comp*, ***kwargs*)
Define a field of automorphisms (invertible endomorphisms in each tangent space) on self.

Via the argument dest_map, it is possible to let the field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi : M \to N$ a differentiable map, a *field of automorphisms along* M *with values on* N is a differentiable map

$$t : M \longrightarrow T^{(1,1)}N$$

($T^{(1,1)}N$ being the tensor bundle of type $(1,1)$ over N) such that

$$\forall p \in M, \ t(p) \in \mathrm{GL}\left(T_{\Phi(p)}N\right),$$

where $\mathrm{GL}\left(T_{\Phi(p)}N\right)$ is the general linear group of the tangent space $T_{\Phi(p)}N$.

The standard case of a field of automorphisms *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

See also:

AutomorphismField and *AutomorphismFieldParal* for a complete documentation.

INPUT:

- comp – (optional) either the components of the field of automorphisms with respect to the vector frame specified by the argument frame or a dictionary of components, the keys of which are vector frames or pairs (f, c) where f is a vector frame and c the chart in which the components are expressed
- frame – (default: None; unused if comp is not given or is a dictionary) vector frame in which the components are given; if None, the default vector frame of self is assumed
- chart – (default: None; unused if comp is not given or is a dictionary) coordinate chart in which the components are expressed; if None, the default chart on the domain of frame is assumed
- name – (default: None) name given to the field

- `latex_name` – (default: `None`) LaTeX symbol to denote the field; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a field of automorphisms *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `AutomorphismField` (or if N is parallelizable, a `AutomorphismFieldParal`) representing the defined field of automorphisms

EXAMPLES:

A field of automorphisms on a 2-dimensional manifold:

```
sage: M = Manifold(2,'M')
sage: X.<x,y> = M.chart()
sage: a = M.automorphism_field([[1+x^2, 0], [0, 1+y^2]], name='A')
sage: a
Field of tangent-space automorphisms A on the 2-dimensional
 differentiable manifold M
sage: a.parent()
General linear group of the Free module X(M) of vector fields on
 the 2-dimensional differentiable manifold M
sage: a(X.frame()[0]).display()
A(d/dx) = (x^2 + 1) d/dx
sage: a(X.frame()[1]).display()
A(d/dy) = (y^2 + 1) d/dy
```

For more examples, see `AutomorphismField` and `AutomorphismFieldParal`.

automorphism_field_group(*dest_map=None*)

Return the group of tangent-space automorphism fields defined on `self`, possibly with values in another manifold, as a module over the algebra of scalar fields defined on `self`.

If M is the current manifold and Φ a differentiable map $\Phi : M \to N$, where N is a differentiable manifold, this method called with `dest_map` being Φ returns the general linear group $\mathrm{GL}(\mathfrak{X}(M, \Phi))$ of the module $\mathfrak{X}(M, \Phi)$ of vector fields along M with values in $\Phi(M) \subset N$.

INPUT:

- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi : M \to N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map, otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `AutomorphismFieldParalGroup` (if N is parallelizable) or a `AutomorphismFieldGroup` (if N is not parallelizable) representing $\mathrm{GL}(\mathfrak{X}(U, \Phi))$

EXAMPLES:

Group of tangent-space automorphism fields of a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: M.automorphism_field_group()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
sage: M.automorphism_field_group().category()
Category of groups
```

See also:

For more examples, see *AutomorphismFieldParalGroup* and *AutomorphismFieldGroup*.

change_of_frame (*frame1*, *frame2*)

Return a change of vector frames defined on `self`.

INPUT:

- `frame1` – vector frame 1
- `frame2` – vector frame 2

OUTPUT:

- a *AutomorphismField* representing, at each point, the vector space automorphism P that relates frame 1, (e_i) say, to frame 2, (n_i) say, according to $n_i = P(e_i)$

EXAMPLES:

Change of vector frames induced by a change of coordinates:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: c_uv.<u,v> = M.chart()
sage: c_xy.transition_map(c_uv, (x+y, x-y))
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: M.change_of_frame(c_xy.frame(), c_uv.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: M.change_of_frame(c_xy.frame(), c_uv.frame())[:]
[ 1/2  1/2]
[ 1/2 -1/2]
sage: M.change_of_frame(c_uv.frame(), c_xy.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: M.change_of_frame(c_uv.frame(), c_xy.frame())[:]
[ 1  1]
[ 1 -1]
sage: M.change_of_frame(c_uv.frame(), c_xy.frame()) == \
....:       M.change_of_frame(c_xy.frame(), c_uv.frame()).inverse()
True
```

In the present example, the manifold M is parallelizable, so that the module $X(M)$ of vector fields on M is free. A change of frame on M is then identical to a change of basis in $X(M)$:

```
sage: XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.print_bases()
Bases defined on the Free module X(M) of vector fields on the
 2-dimensional differentiable manifold M:
 - (M, (d/dx,d/dy)) (default basis)
 - (M, (d/du,d/dv))
sage: XM.change_of_basis(c_xy.frame(), c_uv.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: M.change_of_frame(c_xy.frame(), c_uv.frame()) is \
....:  XM.change_of_basis(c_xy.frame(), c_uv.frame())
True
```

changes_of_frame()

Return all the changes of vector frames defined on `self`.

OUTPUT:

- dictionary of fields of tangent-space automorphisms representing the changes of frames, the keys being the pair of frames

EXAMPLES:

Let us consider a first vector frame on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: e = X.frame(); e
Coordinate frame (M, (d/dx,d/dy))
```

At this stage, the dictionary of changes of frame is empty:

```
sage: M.changes_of_frame()
{}
```

We introduce a second frame on the manifold, relating it to frame `e` by a field of tangent space automorphisms:

```
sage: a = M.automorphism_field(name='a')
sage: a[:] = [[-y, x], [1, 2]]
sage: f = e.new_frame(a, 'f'); f
Vector frame (M, (f_0,f_1))
```

Then we have:

```
sage: M.changes_of_frame()  # random (dictionary output)
{(Coordinate frame (M, (d/dx,d/dy)),
  Vector frame (M, (f_0,f_1))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M,
 (Vector frame (M, (f_0,f_1)),
  Coordinate frame (M, (d/dx,d/dy))): Field of tangent-space
   automorphisms on the 2-dimensional differentiable manifold M}
```

Some checks:

```
sage: M.changes_of_frame()[(e,f)] == a
True
sage: M.changes_of_frame()[(f,e)] == a^(-1)
True
```

coframes()

Return the list of coframes defined on open subsets of `self`.

OUTPUT:

- list of coframes defined on open subsets of `self`

EXAMPLES:

Coframes on subsets of $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
```

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```
sage: M.coframes()
[Coordinate coframe (R^2, (dx,dy))]
sage: e = M.vector_frame('e')
sage: M.coframes()
[Coordinate coframe (R^2, (dx,dy)), Coframe (R^2, (e^0,e^1))]
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1}) # unit disk
sage: U.coframes()
[Coordinate coframe (U, (dx,dy))]
sage: e.restrict(U)
Vector frame (U, (e_0,e_1))
sage: U.coframes()
[Coordinate coframe (U, (dx,dy)), Coframe (U, (e^0,e^1))]
sage: M.coframes()
[Coordinate coframe (R^2, (dx,dy)),
 Coframe (R^2, (e^0,e^1)),
 Coordinate coframe (U, (dx,dy)),
 Coframe (U, (e^0,e^1))]
```

curve (*coord_expression*, *param*, *chart=None*, *name=None*, *latex_name=None*)

Define a differentiable curve in the manifold.

See also:

`DifferentiableCurve` for details.

INPUT:

- `coord_expression` – either
  - (i) a dictionary whose keys are charts on the manifold and values the coordinate expressions (as lists or tuples) of the curve in the given chart
  - (ii) a single coordinate expression in a given chart on the manifold, the latter being provided by the argument `chart`

  in both cases, if the dimension of the manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element
- `param` – a tuple of the type `(t, t_min, t_max)`, where
  - `t` is the curve parameter used in `coord_expression`;
  - `t_min` is its minimal value;
  - `t_max` its maximal value;

  if `t_min=-Infinity` and `t_max=+Infinity`, they can be omitted and `t` can be passed for `param` instead of the tuple `(t, t_min, t_max)`
- `chart` – (default: `None`) chart on the manifold used for case (ii) above; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

OUTPUT:

- `DifferentiableCurve`

EXAMPLES:

The lemniscate of Gerono in the 2-dimensional Euclidean plane:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = RealLine()
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
```

The same definition with the coordinate expression passed as a dictionary:

```
sage: c = M.curve({X: [sin(t), sin(2*t)/2]}, (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
```

An example of definition with `t_min` and `t_max` omitted: a helix in $\mathbf{R}^3$:

```
sage: R3 = Manifold(3, 'R^3')
sage: X.<x,y,z> = R3.chart()
sage: c = R3.curve([cos(t), sin(t), t], t, name='c') ; c
Curve c in the 3-dimensional differentiable manifold R^3
sage: c.domain() # check that t is unbounded
Real number line R
```

See also:

`DifferentiableCurve` for more examples, including plots.

default_frame()

Return the default vector frame defined on `self`.

By *vector frame*, it is meant a field on the manifold that provides, at each point p, a vector basis of the tangent space at p.

Unless changed via `set_default_frame()`, the default frame is the first one defined on the manifold, usually implicitely as the coordinate basis associated with the first chart defined on the manifold.

OUTPUT:

- a `VectorFrame` representing the default vector frame

EXAMPLES:

The default vector frame is often the coordinate frame associated with the first chart defined on the manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: M.default_frame()
Coordinate frame (M, (d/dx,d/dy))
```

diff_degree()

Return the manifold's degree of differentiability.

The degree of differentiability is the integer k (possibly $k = \infty$) such that the manifold is a C^k-manifold over its base field.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: M.diff_degree()
+Infinity
sage: M = Manifold(2, 'M', structure='differentiable', diff_degree=3)
sage: M.diff_degree()
3
```

diff_form (**args*, ***kwargs*)

Define a differential form on `self`.

Via the argument `dest_map`, it is possible to let the differential form take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold, $\Phi: M \to N$ a differentiable map and p a non-negative integer, a *differential form of degree* p (or p*-form*) *along* M *with values on* N is a differentiable map

$$t: M \longrightarrow T^{(0,p)}N$$

($T^{(0,p)}N$ being the tensor bundle of type $(0,p)$ over N) such that

$$\forall x \in M, \quad t(x) \in \Lambda^p(T^*_{\Phi(x)}N),$$

where $\Lambda^p(T^*_{\Phi(x)}N)$ is the p-th exterior power of the dual of the tangent space $T_{\Phi(x)}N$.

The standard case of a differential form *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

For $p = 1$, one can use the method `one_form()` instead.

See also:

`DiffForm` and `DiffFormParal` for a complete documentation.

INPUT:

- `degree` – the degree p of the differential form (i.e. its tensor rank)
- `comp` – (optional) either the components of the differential form with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the differential form
- `latex_name` – (default: `None`) LaTeX symbol to denote the differential form; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi: M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a differential form *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- the p-form as a `DiffForm` (or if N is parallelizable, a `DiffFormParal`)

EXAMPLES:

A 2-form on a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: f = M.diff_form(2, name='F'); f
2-form F on the 3-dimensional differentiable manifold M
sage: f[0,1], f[1,2] = x+y, x*z
sage: f.display()
F = (x + y) dx/\dy + x*z dy/\dz
```

For more examples, see `DiffForm` and `DiffFormParal`.

diff_form_module(*degree*, *dest_map=None*)

Return the set of differential forms of a given degree defined on `self`, possibly with values in another manifold, as a module over the algebra of scalar fields defined on `self`.

See also:

`DiffFormModule` for complete documentation.

INPUT:

- `degree` – positive integer; the degree p of the differential forms
- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi : M \to N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of differential forms *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `DiffFormModule` (or if N is parallelizable, a `DiffFormFreeModule`) representing the module $\Omega^p(M, \Phi)$ of p-forms on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Module of 2-forms on a 3-dimensional parallelizable manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: M.diff_form_module(2)
Free module Omega^2(M) of 2-forms on the 3-dimensional
 differentiable manifold M
sage: M.diff_form_module(2).category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: M.diff_form_module(2).base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: M.diff_form_module(2).rank()
3
```

The outcome is cached:

```
sage: M.diff_form_module(2) is M.diff_form_module(2)
True
```

diff_map(*codomain*, *coord_functions=None*, *chart1=None*, *chart2=None*, *name=None*, *latex_name=None*)

Define a differentiable map between the current differentiable manifold and a differentiable manifold over the same topological field.

See `DiffMap` for a complete documentation.

INPUT:

- `codomain` – the map codomain (a differentiable manifold over the same topological field as the current differentiable manifold)
- `coord_functions` – (default: `None`) if not `None`, must be either

  - (i) a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts (chart1, chart2) as keys (chart1 being a chart on the current manifold and chart2 a chart on codomain)
  - (ii) a single coordinate expression in a given pair of charts, the latter being provided by the arguments chart1 and chart2

  In both cases, if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element

- chart1 – (default: None; used only in case (ii) above) chart on the current manifold defining the start coordinates involved in coord_functions for case (ii); if none is provided, the coordinates are assumed to refer to the manifold's default chart
- chart2 – (default: None; used only in case (ii) above) chart on codomain defining the arrival coordinates involved in coord_functions for case (ii); if none is provided, the coordinates are assumed to refer to the default chart of codomain
- name – (default: None) name given to the differentiable map
- latex_name – (default: None) LaTeX symbol to denote the differentiable map; if none is provided, the LaTeX symbol is set to name

OUTPUT:

- the differentiable map, as an instance of *DiffMap*

EXAMPLES:

A differentiable map between an open subset of S^2 covered by regular spherical coordinates and $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2')
sage: U = M.open_subset('U')
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: N = Manifold(3, 'R^3', r'\RR^3')
sage: c_cart.<x,y,z> = N.chart()  # Cartesian coord. on R^3
sage: Phi = U.diff_map(N, (sin(th)*cos(ph), sin(th)*sin(ph), cos(th)),
....:                  name='Phi', latex_name=r'\Phi')
sage: Phi
Differentiable map Phi from the Open subset U of the 2-dimensional
 differentiable manifold S^2 to the 3-dimensional differentiable
 manifold R^3
```

The same definition, but with a dictionary with pairs of charts as keys (case (i) above):

```
sage: Phi1 = U.diff_map(N,
....:        {(c_spher, c_cart): (sin(th)*cos(ph), sin(th)*sin(ph),
....:         cos(th))}, name='Phi', latex_name=r'\Phi')
sage: Phi1 == Phi
True
```

The differentiable map acting on a point:

```
sage: p = U.point((pi/2, pi)) ; p
Point on the 2-dimensional differentiable manifold S^2
sage: Phi(p)
Point on the 3-dimensional differentiable manifold R^3
sage: Phi(p).coord(c_cart)
(-1, 0, 0)
sage: Phi1(p) == Phi(p)
True
```

See the documentation of class *DiffMap* for more examples.

diffeomorphism(*codomain*, *coord_functions=None*, *chart1=None*, *chart2=None*, *name=None*, *latex_name=None*)

Define a diffeomorphism between the current manifold and another one.

See *DiffMap* for a complete documentation.

INPUT:

- `codomain` – codomain of the diffeomorphism (the arrival manifold or some subset of it)
- `coord_functions` – (default: `None`) if not `None`, must be either
  - (i) a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts (chart1, chart2) as keys (chart1 being a chart on the current manifold and chart2 a chart on `codomain`)
  - (ii) a single coordinate expression in a given pair of charts, the latter being provided by the arguments `chart1` and `chart2`

  In both cases, if the dimension of the arrival manifold is 1, a single coordinate expression can be passed instead of a tuple with a single element
- `chart1` – (default: `None`; used only in case (ii) above) chart on the current manifold defining the start coordinates involved in `coord_functions` for case (ii); if none is provided, the coordinates are assumed to refer to the manifold's default chart
- `chart2` – (default: `None`; used only in case (ii) above) chart on `codomain` defining the arrival coordinates involved in `coord_functions` for case (ii); if none is provided, the coordinates are assumed to refer to the default chart of `codomain`
- `name` – (default: `None`) name given to the diffeomorphism
- `latex_name` – (default: `None`) LaTeX symbol to denote the diffeomorphism; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- the diffeomorphism, as an instance of *DiffMap*

EXAMPLES:

Diffeomorphism between the open unit disk in $\mathbf{R}^2$ and $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'M')  # the open unit disk
sage: forget()  # for doctests only
sage: c_xy.<x,y> = M.chart('x:(-1,1) y:(-1,1)')  # Cartesian coord on M
sage: c_xy.add_restrictions(x^2+y^2<1)
sage: N = Manifold(2, 'N')  # R^2
sage: c_XY.<X,Y> = N.chart()  # canonical coordinates on R^2
sage: Phi = M.diffeomorphism(N, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                        name='Phi', latex_name=r'\Phi')
sage: Phi
Diffeomorphism Phi from the 2-dimensional differentiable manifold M
 to the 2-dimensional differentiable manifold N
sage: Phi.display()
Phi: M --> N
   (x, y) |--> (X, Y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The inverse diffeomorphism:

```
sage: Phi^(-1)
Diffeomorphism Phi^(-1) from the 2-dimensional differentiable
 manifold N to the 2-dimensional differentiable manifold M
sage: (Phi^(-1)).display()
Phi^(-1): N --> M
   (X, Y) |--> (x, y) = (X/sqrt(X^2 + Y^2 + 1), Y/sqrt(X^2 + Y^2 + 1))
```

See the documentation of class `DiffMap` for more examples.

frames()

Return the list of vector frames defined on open subsets of `self`.

OUTPUT:

- list of vector frames defined on open subsets of `self`

EXAMPLES:

Vector frames on subsets of $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: M.frames()
[Coordinate frame (R^2, (d/dx,d/dy))]
sage: e = M.vector_frame('e')
sage: M.frames()
[Coordinate frame (R^2, (d/dx,d/dy)),
 Vector frame (R^2, (e_0,e_1))]
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1}) # unit disk
sage: U.frames()
[Coordinate frame (U, (d/dx,d/dy))]
sage: M.frames()
[Coordinate frame (R^2, (d/dx,d/dy)),
 Vector frame (R^2, (e_0,e_1)),
 Coordinate frame (U, (d/dx,d/dy))]
```

integrated_autoparallel_curve(*affine_connection*, *curve_param*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Construct an autoparallel curve on the manifold with respect to a given affine connection.

See also:

`IntegratedAutoparallelCurve` for details.

INPUT:

- `affine_connection` – `AffineConnection`; affine connection with respect to which the curve is autoparallel
- `curve_param` – a tuple of the type `(t, t_min, t_max)`, where
  - `t` is the symbolic variable to be used as the parameter of the curve (the equations defining an instance of `IntegratedAutoparallelCurve` are such that `t` will actually be an affine parameter of the curve);
  - `t_min` is its minimal (finite) value;
  - `t_max` its maximal (finite) value.
- `initial_tangent_vector` – `TangentVector`; initial tangent vector of the curve

- `chart` – (default: `None`) chart on the manifold in which the equations are given ; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

OUTPUT:

- *IntegratedAutoparallelCurve*

EXAMPLES:

Autoparallel curves associated with the Mercator projection of the 2-sphere $\mathbb{S}^2$:

```
sage: S2 = Manifold(2, 'S^2', start_index=1)
sage: polar.<th,ph> = S2.chart('th ph')
sage: epolar = polar.frame()
sage: ch_basis = S2.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2] = 1, 1/sin(th)
sage: epolar_ON=S2.default_frame().new_frame(ch_basis,'epolar_ON')
```

Set the affine connection associated with Mercator projection; it is metric compatible but it has non-vanishing torsion:

```
sage: nab = S2.affine_connection('nab')
sage: nab.set_coef(epolar_ON)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
sage: g = S2.metric('g')
sage: g[1,1], g[2,2] = 1, (sin(th))^2
sage: nab(g)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
sage: nab.torsion()[:]
[[[0, 0], [0, 0]], [[0, cos(th)/sin(th)], [-cos(th)/sin(th), 0]]]
```

Declare an integrated autoparallel curve with respect to this connection:

```
sage: p = S2.point((pi/4, 0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((1,1), basis=epolar_ON.at(p))
sage: t = var('t')
sage: c = S2.integrated_autoparallel_curve(nab, (t, 0, 6),
....:                                 v, chart=polar, name='c')
sage: sys = c.system(verbose=True)
Autoparallel curve c in the 2-dimensional differentiable
 manifold S^2 equipped with Affine connection nab on the
 2-dimensional differentiable manifold S^2, and integrated
 over the Real interval (0, 6) as a solution to the
 following equations, written with respect to
 Chart (S^2, (th, ph)):

Initial point: Point p on the 2-dimensional differentiable
 manifold S^2 with coordinates [1/4*pi, 0] with respect to
 Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
 2-dimensional differentiable manifold S^2 with
 components [1, sqrt(2)] with respect to
 Chart (S^2, (th, ph))
```

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```
d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = 0
d(Dph)/dt = -Dph*Dth*cos(th)/sin(th)

sage: sol = c.solve()
sage: interp = c.interpolate()
sage: p = c(1.3, verbose=True)
Evaluating point coordinates from the interpolation
 associated with the key 'cubic spline-interp-rk4_maxima'
 by default...
sage: p
Point on the 2-dimensional differentiable manifold S^2
sage: p.coordinates()     # abs tol 1e-12
(2.085398163397449, 1.4203172015958863)
sage: tgt_vec = c.tangent_vector_eval_at(3.7, verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'cubic spline-interp-rk4_maxima'
 by default...
sage: tgt_vec[:]     # abs tol 1e-12
[0.9999999999999732, -1.016513736236512]
```

integrated_curve(*equations_rhs*, *velocities*, *curve_param*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Construct a curve defined by a system of second order differential equations in the coordinate functions.

See also:

IntegratedCurve for details.

INPUT:

- `equations_rhs` – list of the right-hand sides of the equations on the velocities only
- `velocities` – list of the symbolic expressions used in `equations_rhs` to denote the velocities
- `curve_param` – a tuple of the type `(t, t_min, t_max)`, where
  - `t` is the symbolic variable used in `equations_rhs` to denote the parameter of the curve;
  - `t_min` is its minimal (finite) value;
  - `t_max` its maximal (finite) value.
- `initial_tangent_vector` – *TangentVector*; initial tangent vector of the curve
- `chart` – (default: `None`) chart on the manifold in which the equations are given; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

OUTPUT:

- *IntegratedCurve*

EXAMPLES:

Trajectory of a particle of unit mass and unit charge in a unit, uniform, stationary magnetic field:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: t = var('t')
sage: D = X.symbolic_velocities()
sage: eqns = [D[1], -D[0], SR(0)]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,6), v, name='c'); c
Integrated curve c in the 3-dimensional differentiable
 manifold M
sage: sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 6) as a solution to
 the following system, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = Dx2
d(Dx2)/dt = -Dx1
d(Dx3)/dt = 0

sage: sol = c.solve()
sage: interp = c.interpolate()
sage: p = c(1.3, verbose=True)
Evaluating point coordinates from the interpolation
 associated with the key 'cubic spline-interp-rk4_maxima'
 by default...
sage: p
Point on the 3-dimensional differentiable manifold M
sage: p.coordinates()     # abs tol 1e-12
(0.9635581155730744, -0.7325010457963622, 1.3)
sage: tgt_vec = c.tangent_vector_eval_at(3.7, verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'cubic spline-interp-rk4_maxima'
 by default...
sage: tgt_vec[:]     # abs tol 1e-12
[-0.8481008455360024, 0.5298346120470748, 1.0000000000000007]
```

integrated_geodesic(*metric*, *curve_param*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Construct a geodesic on the manifold with respect to a given metric.

See also:

IntegratedGeodesic for details.

INPUT:

- `metric` – *PseudoRiemannianMetric* metric with respect to which the curve is a geodesic

- `curve_param` – a tuple of the type `(t, t_min, t_max)`, where
  - `t` is the symbolic variable to be used as the parameter of the curve (the equations defining an instance of `IntegratedGeodesic` are such that `t` will actually be an affine parameter of the curve);
  - `t_min` is its minimal (finite) value;
  - `t_max` its maximal (finite) value.
- `initial_tangent_vector` – `TangentVector`; initial tangent vector of the curve
- `chart` – (default: `None`) chart on the manifold in which the equations are given; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

OUTPUT:

- `IntegratedGeodesic`

EXAMPLES:

Geodesics of the unit 2-sphere $\mathbb{S}^2$:

```
sage: S2 = Manifold(2, 'S^2', start_index=1)
sage: polar.<th,ph> = S2.chart('th ph')
sage: epolar = polar.frame()
```

Set the standard metric tensor g on $\mathbb{S}^2$:

```
sage: g = S2.metric('g')
sage: g[1,1], g[2,2] = 1, (sin(th))^2
```

Declare an integrated geodesic with respect to this metric:

```
sage: p = S2.point((pi/4, 0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((1, 1), basis=epolar.at(p))
sage: t = var('t')
sage: c = S2.integrated_geodesic(g, (t, 0, 6), v,
....:                                 chart=polar, name='c')
sage: sys = c.system(verbose=True)
Geodesic c in the 2-dimensional differentiable manifold S^2
 equipped with Riemannian metric g on the 2-dimensional
 differentiable manifold S^2, and integrated over the Real
 interval (0, 6) as a solution to the following geodesic
 equations, written with respect to Chart (S^2, (th, ph)):

Initial point: Point p on the 2-dimensional differentiable
manifold S^2 with coordinates [1/4*pi, 0] with respect to
Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
2-dimensional differentiable manifold S^2 with
components [1, 1] with respect to Chart (S^2, (th, ph))

d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = Dph^2*cos(th)*sin(th)
```

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```
d(Dph)/dt = -2*Dph*Dth*cos(th)/sin(th)

sage: sol = c.solve()
sage: interp = c.interpolate()
sage: p = c(1.3, verbose=True)
Evaluating point coordinates from the interpolation
 associated with the key 'cubic spline-interp-rk4_maxima'
 by default...
sage: p
Point on the 2-dimensional differentiable manifold S^2
sage: p.coordinates()     # abs tol 1e-12
(2.2047444794514663, 0.7986609561213334)
sage: tgt_vec = c.tangent_vector_eval_at(3.7, verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'cubic spline-interp-rk4_maxima'
 by default...
sage: tgt_vec[:]     # abs tol 1e-12
[-1.090742147346732, 0.620568327518154]
```

is_manifestly_parallelizable()

Return `True` if `self` is known to be a parallelizable and `False` otherwise.

If `False` is returned, either the manifold is not parallelizable or no vector frame has been defined on it yet.

EXAMPLES:

A just created manifold is a priori not manifestly parallelizable:

```
sage: M = Manifold(2, 'M')
sage: M.is_manifestly_parallelizable()
False
```

Defining a vector frame on it makes it parallelizable:

```
sage: e = M.vector_frame('e')
sage: M.is_manifestly_parallelizable()
True
```

Defining a coordinate chart on the whole manifold also makes it parallelizable:

```
sage: N = Manifold(4, 'N')
sage: X.<t,x,y,z> = N.chart()
sage: N.is_manifestly_parallelizable()
True
```

lorentzian_metric(*name*, *signature='positive'*, *latex_name=None*, *dest_map=None*)

Define a Lorentzian metric on the manifold.

A *Lorentzian metric* is a field of nondegenerate symmetric bilinear forms acting in the tangent spaces, with signature $(-, +, \cdots, +)$ or $(+, -, \cdots, -)$.

See `PseudoRiemannianMetric` for a complete documentation.

INPUT:

- `name` – name given to the metric
- `signature` – (default: 'positive') sign of the metric signature:

  - if set to 'positive', the signature is n-2, where n is the manifold's dimension, i.e. $(-, +, \cdots, +)$
  - if set to 'negative', the signature is -n+2, i.e. $(+, -, \cdots, -)$

- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`) instance of class `DiffMap` representing the destination map $\Phi : U \to M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of `PseudoRiemannianMetric` representing the defined Lorentzian metric.

EXAMPLES:

Metric of Minkowski spacetime:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: g = M.lorentzian_metric('g'); g
Lorentzian metric g on the 4-dimensional differentiable manifold M
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: g.display()
g = -dt*dt + dx*dx + dy*dy + dz*dz
sage: g.signature()
2
```

Choice of a negative signature:

```
sage: g = M.lorentzian_metric('g', signature='negative'); g
Lorentzian metric g on the 4-dimensional differentiable manifold M
sage: g[0,0], g[1,1], g[2,2], g[3,3] = 1, -1, -1, -1
sage: g.display()
g = dt*dt - dx*dx - dy*dy - dz*dz
sage: g.signature()
-2
```

metric(*name*, *signature=None*, *latex_name=None*, *dest_map=None*)

Define a pseudo-Riemannian metric on the manifold.

A *pseudo-Riemannian metric* is a field of nondegenerate symmetric bilinear forms acting in the tangent spaces. See `PseudoRiemannianMetric` for a complete documentation.

INPUT:

- `name` – name given to the metric
- `signature` – (default: `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`) instance of class `DiffMap` representing the destination map $\Phi : U \to M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of `PseudoRiemannianMetric` representing the defined pseudo-Riemannian metric.

EXAMPLES:

Metric on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: g = M.metric('g'); g
Riemannian metric g on the 3-dimensional differentiable manifold M
```

See also:

`PseudoRiemannianMetric` for more examples.

mixed_form(*name=None*, *latex_name=None*, *dest_map=None*, *comp=None*)

Define a mixed form on `self`.

Via the argument `dest_map`, it is possible to let the mixed form take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold, $\Phi : M \to N$ a differentiable map, a *mixed form along* Φ can be considered as a differentiable map

$$a : M \longrightarrow \bigoplus_{k=0}^{n} T^{(0,k)}N$$

($T^{(0,k)}N$ being the tensor bundle of type $(0, k)$ over N, $\oplus$ being the Whitney sum and n being the dimension of N) such that

$$\forall x \in M, \quad a(x) \in \bigoplus_{k=0}^{n} \Lambda^k(T^*_{\Phi(x)}N),$$

where $\Lambda^k(T^*_{\Phi(x)}N)$ is the k-th exterior power of the dual of the tangent space $T_{\Phi(x)}N$.

The standard case of a mixed form *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$.

See also:

`MixedForm` for complete documentation.

INPUT:

- `name` – (default: `None`) name given to the differential form
- `latex_name` – (default: `None`) LaTeX symbol to denote the differential form; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a differential form *on* M), otherwise `dest_map` must be a `DiffMap`
- `comp` – (default: `None`) homogeneous components of the mixed form as a list; if none is provided, the components are set to innocent unnamed differential forms

OUTPUT:

- the mixed form as a `MixedForm`

EXAMPLES:

A mixed form on an open subset of a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U', latex_name=r'\mathcal{U}'); U
Open subset U of the 3-dimensional differentiable manifold M
sage: c_xyz.<x,y,z> = U.chart()
sage: f = U.mixed_form(name='F'); f
Mixed differential form F on the Open subset U of the 3-dimensional
 differentiable manifold M
```

See the documentation of class *MixedForm* for more examples.

mixed_form_algebra(*dest_map=None*)
Return the set of mixed forms defined on `self`, possibly with values in another manifold, as a graded algebra.

See also:

MixedFormAlgebra for complete documentation.

INPUT:

- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi: M \rightarrow N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of mixed forms *on* M), otherwise `dest_map` must be a *DiffMap*

OUTPUT:

- a *MixedFormAlgebra* representing the graded algebra $\Omega^*(M, \Phi)$ of mixed forms on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Graded algebra of mixed forms on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: M.mixed_form_algebra()
Graded algebra Omega^*(M) of mixed differential forms on the
 2-dimensional differentiable manifold M
sage: M.mixed_form_algebra().category()
Category of graded algebras over Symbolic Ring
sage: M.mixed_form_algebra().base_ring()
Symbolic Ring
```

The outcome is cached:

```
sage: M.mixed_form_algebra() is M.mixed_form_algebra()
True
```

multivector_field(**args*, ***kwargs*)
Define a multivector field on `self`.

Via the argument `dest_map`, it is possible to let the multivector field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold, $\Phi: M \rightarrow N$ a differentiable map and p a non-negative integer, a *multivector field of degree* p (or p*-vector field*) *along* M *with values on* N is a differentiable map

$$t: M \longrightarrow T^{(p,0)}N$$

($T^{(p,0)}N$ being the tensor bundle of type $(p, 0)$ over N) such that

$$\forall x \in M, \quad t(x) \in \Lambda^p(T_{\Phi(x)}N),$$

where $\Lambda^p(T_{\Phi(x)}N)$ is the p-th exterior power of the tangent vector space $T_{\Phi(x)}N$.

The standard case of a p-vector field *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

For $p = 1$, one can use the method `vector_field()` instead.

See also:

`MultivectorField` and `MultivectorFieldParal` for a complete documentation.

INPUT:

- `degree` – the degree p of the multivector field (i.e. its tensor rank)
- `comp` – (optional) either the components of the multivector field with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the multivector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the multivector field; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a multivector field *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- the p-vector field as a `MultivectorField` (or if N is parallelizable, a `MultivectorFieldParal`)

EXAMPLES:

A 2-vector field on a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: h = M.multivector_field(2, name='H'); h
2-vector field H on the 3-dimensional differentiable manifold M
sage: h[0,1], h[0,2], h[1,2] = x+y, x*z, -3
sage: h.display()
H = (x + y) d/dx/\d/dy + x*z d/dx/\d/dz - 3 d/dy/\d/dz
```

For more examples, see `MultivectorField` and `MultivectorFieldParal`.

multivector_module(*degree*, *dest_map=None*)

Return the set of multivector fields of a given degree defined on `self`, possibly with values in another manifold, as a module over the algebra of scalar fields defined on `self`.

See also:

`MultivectorModule` for complete documentation.

INPUT:

- `degree` – positive integer; the degree p of the multivector fields

- dest_map – (default: None) destination map, i.e. a differentiable map $\Phi : M \to N$, where M is the current manifold and N a differentiable manifold; if None, it is assumed that $N = M$ and that Φ is the identity map (case of multivector fields *on* M), otherwise dest_map must be a DiffMap

OUTPUT:

- a MultivectorModule (or if N is parallelizable, a MultivectorFreeModule) representing the module $\Omega^p(M, \Phi)$ of p-forms on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Module of 2-vector fields on a 3-dimensional parallelizable manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: M.multivector_module(2)
Free module A^2(M) of 2-vector fields on the 3-dimensional
 differentiable manifold M
sage: M.multivector_module(2).category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: M.multivector_module(2).base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: M.multivector_module(2).rank()
3
```

The outcome is cached:

```
sage: M.multivector_module(2) is M.multivector_module(2)
True
```

one_form(**comp*, ***kwargs*)

Define a 1-form on the manifold.

Via the argument dest_map, it is possible to let the 1-form take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi : M \to N$ a differentiable map, a *1-form along* M *with values on* N is a differentiable map

$$t : M \longrightarrow T^*N$$

(T^*N being the cotangent bundle of N) such that

$$\forall p \in M, \quad t(p) \in T^*_{\Phi(p)}N,$$

where $T^*_{\Phi(p)}$ is the dual of the tangent space $T_{\Phi(p)}N$.

The standard case of a 1-form *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

See also:

DiffForm and DiffFormParal for a complete documentation.

INPUT:

- comp – (optional) either the components of 1-form with respect to the vector frame specified by the argument frame or a dictionary of components, the keys of which are vector frames or pairs (f, c) where f is a vector frame and c the chart in which the components are expressed

- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the 1-form
- `latex_name` – (default: `None`) LaTeX symbol to denote the 1-form; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a 1-form *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- the 1-form as a `DiffForm` (or if N is parallelizable, a `DiffFormParal`)

EXAMPLES:

A 1-form on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: om = M.one_form(-y, 2+x, name='omega', latex_name=r'\omega')
sage: om
1-form omega on the 2-dimensional differentiable manifold M
sage: om.display()
omega = -y dx + (x + 2) dy
sage: om.parent()
Free module Omega^1(M) of 1-forms on the 2-dimensional
 differentiable manifold M
```

For more examples, see `DiffForm` and `DiffFormParal`.

open_subset (*name*, *latex_name=None*, *coord_def={}*)

Create an open subset of the manifold.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a differentiable manifold by itself. Hence the returned object is an instance of `DifferentiableManifold`.

INPUT:

- `name` – name given to the open subset
- `latex_name` – (default: `None`) LaTeX symbol to denote the subset; if none is provided, it is set to `name`
- `coord_def` – (default: {}) definition of the subset in terms of coordinates; `coord_def` must a be dictionary with keys charts in the manifold's atlas and values the symbolic expressions formed by the coordinates to define the subset.

OUTPUT:

- the open subset, as an instance of `DifferentiableManifold`

EXAMPLES:

Creating an open subset of a differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: A = M.open_subset('A'); A
Open subset A of the 2-dimensional differentiable manifold M
```

As an open subset of a differentiable manifold, A is itself a differentiable manifold, on the same topological field and of the same dimension as M:

```
sage: A.category()
Join of Category of subobjects of sets and Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: A.base_field() == M.base_field()
True
sage: dim(A) == dim(M)
True
```

Creating an open subset of A:

```
sage: B = A.open_subset('B'); B
Open subset B of the 2-dimensional differentiable manifold M
```

We have then:

```
sage: A.list_of_subsets()
[Open subset A of the 2-dimensional differentiable manifold M,
 Open subset B of the 2-dimensional differentiable manifold M]
sage: B.is_subset(A)
True
sage: B.is_subset(M)
True
```

Defining an open subset by some coordinate restrictions: the open unit disk in of the Euclidean plane:

```
sage: X.<x,y> = M.chart() # Cartesian coordinates on M
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1}); U
Open subset U of the 2-dimensional differentiable manifold M
```

Since the argument coord_def has been set, U is automatically endowed with a chart, which is the restriction of X to U:

```
sage: U.atlas()
[Chart (U, (x, y))]
sage: U.default_chart()
Chart (U, (x, y))
sage: U.default_chart() is X.restrict(U)
True
```

An point in U:

```
sage: p = U.an_element(); p
Point on the 2-dimensional differentiable manifold M
sage: X(p)   # the coordinates (x,y) of p
(0, 0)
sage: p in U
True
```

Checking whether various points, defined by their coordinates with respect to chart X, are in U:

```
sage: M((0,1/2)) in U
True
sage: M((0,1)) in U
False
sage: M((1/2,1)) in U
```

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```
False
sage: M((-1/2,1/3)) in U
True
```

riemannian_metric(*name*, *latex_name=None*, *dest_map=None*)

Define a Riemannian metric on the manifold.

A *Riemannian metric* is a field of positive definite symmetric bilinear forms acting in the tangent spaces.

See `PseudoRiemannianMetric` for a complete documentation.

INPUT:

- `name` – name given to the metric
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`) instance of class `DiffMap` representing the destination map $\Phi : U \to M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of `PseudoRiemannianMetric` representing the defined Riemannian metric.

EXAMPLES:

Metric of the hyperbolic plane H^2:

```
sage: H2 = Manifold(2, 'H^2', start_index=1)
sage: X.<x,y> = H2.chart('x y:(0,+oo)')  # Poincaré half-plane coord.
sage: g = H2.riemannian_metric('g')
sage: g[1,1], g[2,2] = 1/y^2, 1/y^2
sage: g
Riemannian metric g on the 2-dimensional differentiable manifold H^2
sage: g.display()
g = y^(-2) dx*dx + y^(-2) dy*dy
sage: g.signature()
2
```

See also:

`PseudoRiemannianMetric` for more examples.

set_change_of_frame(*frame1*, *frame2*, *change_of_frame*, *compute_inverse=True*)

Relate two vector frames by an automorphism.

This updates the internal dictionary `self._frame_changes`.

INPUT:

- `frame1` – frame 1, denoted (e_i) below
- `frame2` – frame 2, denoted (f_i) below
- `change_of_frame` – instance of class `AutomorphismFieldParal` describing the automorphism P that relates the basis (e_i) to the basis (f_i) according to $f_i = P(e_i)$
- `compute_inverse` (default: True) – if set to True, the inverse automorphism is computed and the change from basis (f_i) to (e_i) is set to it in the internal dictionary `self._frame_changes`

EXAMPLES:

Connecting two vector frames on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: e = M.vector_frame('e')
sage: f = M.vector_frame('f')
sage: a = M.automorphism_field()
sage: a[e,:] = [[1,2],[0,3]]
sage: M.set_change_of_frame(e, f, a)
sage: f[0].display(e)
f_0 = e_0
sage: f[1].display(e)
f_1 = 2 e_0 + 3 e_1
sage: e[0].display(f)
e_0 = f_0
sage: e[1].display(f)
e_1 = -2/3 f_0 + 1/3 f_1
sage: M.change_of_frame(e,f)[e,:]
[1 2]
[0 3]
```

set_default_frame (*frame*)

Changing the default vector frame on `self`.

INPUT:

- `frame` – *VectorFrame* a vector frame defined on the manifold

EXAMPLES:

Changing the default frame on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: e = M.vector_frame('e')
sage: M.default_frame()
Coordinate frame (M, (d/dx,d/dy))
sage: M.set_default_frame(e)
sage: M.default_frame()
Vector frame (M, (e_0,e_1))
```

sym_bilin_form_field (**comp*, ***kwargs*)

Define a field of symmetric bilinear forms on `self`.

Via the argument `dest_map`, it is possible to let the field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi: M \to N$ a differentiable map, a *field of symmetric bilinear forms along* M *with values on* N is a differentiable map

$$t: M \longrightarrow T^{(0,2)}N$$

($T^{(0,2)}N$ being the tensor bundle of type $(0,2)$ over N) such that

$$\forall p \in M, \ t(p) \in S(T_{\Phi(p)}N),$$

where $S(T_{\Phi(p)}N)$ is the space of symmetric bilinear forms on the tangent space $T_{\Phi(p)}N$.

The standard case of fields of symmetric bilinear forms *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

INPUT:

- `comp` – (optional) either the components of the field of symmetric bilinear forms with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the field
- `latex_name` – (default: `None`) LaTeX symbol to denote the field; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a field *on* M), otherwise `dest_map` must be an instance of instance of class `DiffMap`

OUTPUT:

- a `TensorField` (or if N is parallelizable, a `TensorFieldParal`) of tensor type $(0, 2)$ and symmetric representing the defined field of symmetric bilinear forms

EXAMPLES:

A field of symmetric bilinear forms on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = M.sym_bilin_form_field(name='T'); t
Field of symmetric bilinear forms T on the 2-dimensional
 differentiable manifold M
```

Such a object is a tensor field of rank 2 and type $(0, 2)$:

```
sage: t.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 2-dimensional differentiable manifold M
sage: t.tensor_rank()
2
sage: t.tensor_type()
(0, 2)
```

The LaTeX symbol is deduced from the name or can be specified when creating the object:

```
sage: latex(t)
T
sage: om = M.sym_bilin_form_field(name='Omega', latex_name=r'\Omega')
sage: latex(om)
\Omega
```

Setting the components in the manifold's default vector frame:

```
sage: t[0,0], t[0,1], t[1,1] = -1, x, x*y
```

The unset components are either zero or deduced by symmetry:

```
sage: t[1, 0]
x
sage: t[:]
[ -1   x]
[  x x*y]
```

One can also set the components while defining the field of symmetric bilinear forms:

```
sage: t = M.sym_bilin_form_field([[-1, x], [x, x*y]], name='T')
```

A symmetric bilinear form acts on vector pairs:

```
sage: v1 = M.vector_field(y, x, name='V_1')
sage: v2 = M.vector_field(x+y, 2, name='V_2')
sage: s = t(v1,v2) ; s
Scalar field T(V_1,V_2) on the 2-dimensional differentiable
 manifold M
sage: s.expr()
x^3 + (3*x^2 + x)*y - y^2
sage: s.expr() - t[0,0]*v1[0]*v2[0] - \
....: t[0,1]*(v1[0]*v2[1]+v1[1]*v2[0]) - t[1,1]*v1[1]*v2[1]
0
sage: latex(s)
T\left(V_1,V_2\right)
```

Adding two symmetric bilinear forms results in another symmetric bilinear form:

```
sage: a = M.sym_bilin_form_field([[1, 2], [2, 3]])
sage: b = M.sym_bilin_form_field([[-1, 4], [4, 5]])
sage: s = a + b ; s
Field of symmetric bilinear forms on the 2-dimensional
 differentiable manifold M
sage: s[:]
[0 6]
[6 8]
```

But adding a symmetric bilinear from with a non-symmetric bilinear form results in a generic type $(0, 2)$ tensor:

```
sage: c = M.tensor_field(0, 2, [[-2, -3], [1,7]])
sage: s1 = a + c ; s1
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: s1[:]
[-1 -1]
[ 3 10]
sage: s2 = c + a ; s2
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: s2[:]
[-1 -1]
[ 3 10]
```

tangent_identity_field(*name='Id'*, *latex_name=None*, *dest_map=None*)
Return the field of identity maps in the tangent spaces on `self`.

Via the argument `dest_map`, it is possible to let the field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi: M \to N$ a differentiable map, a *field of identity maps along M with values on N* is a differentiable map

$$t: M \longrightarrow T^{(1,1)}N$$

($T^{(1,1)}N$ being the tensor bundle of type $(1,1)$ over N) such that

$$\forall p \in M, \ t(p) = \mathrm{Id}_{T_{\Phi(p)}N},$$

where $\mathrm{Id}_{T_{\Phi(p)}N}$ is the identity map of the tangent space $T_{\Phi(p)}N$.

The standard case of a field of identity maps *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

INPUT:

- `name` – (string; default: 'Id') name given to the field of identity maps
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the field of identity map; if none is provided, the LaTeX symbol is set to 'mathrm{Id}' if `name` is 'Id' and to `name` otherwise
- `dest_map` – (default: `None`) the destination map $\Phi : M \rightarrow N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a field of identity maps *on* M), otherwise `dest_map` must be a `DiffMap`

OUTPUT:

- a `AutomorphismField` (or if N is parallelizable, a `AutomorphismFieldParal`) representing the field of identity maps

EXAMPLES:

Field of tangent-space identity maps on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: a = M.tangent_identity_field(); a
Field of tangent-space identity maps on the 3-dimensional
 differentiable manifold M
sage: a.comp()
Kronecker delta of size 3x3
```

For more examples, see `AutomorphismField`.

tangent_space (*point*)

Tangent space to `self` at a given point.

INPUT:

- `point` – `ManifoldPoint`; point p on the manifold

OUTPUT:

- `TangentSpace` representing the tangent vector space T_pM, where M is the current manifold

EXAMPLES:

A tangent space to a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((2, -3), name='p')
sage: Tp = M.tangent_space(p); Tp
Tangent space at Point p on the 2-dimensional differentiable
```

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```
 manifold M
sage: Tp.category()
Category of finite dimensional vector spaces over Symbolic Ring
sage: dim(Tp)
2
```

See also:

TangentSpace for more examples.

tensor_field(**args*, ***kwargs*)

Define a tensor field on `self`.

Via the argument `dest_map`, it is possible to let the tensor field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold, $\Phi : M \to N$ a differentiable map and (k, l) a pair of non-negative integers, a *tensor field of type* (k, l) *along* M *with values on* N is a differentiable map

$$t : M \longrightarrow T^{(k,l)}N$$

($T^{(k,l)}N$ being the tensor bundle of type (k, l) over N) such that

$$\forall p \in M, \ t(p) \in T^{(k,l)}(T_{\Phi(p)}N),$$

where $T^{(k,l)}(T_{\Phi(p)}N)$ is the space of tensors of type (k, l) on the tangent space $T_{\Phi(p)}N$.

The standard case of tensor fields *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

See also:

TensorField and *TensorFieldParal* for a complete documentation.

INPUT:

- `k` – the contravariant rank k, the tensor type being (k, l)
- `l` – the covariant rank l, the tensor type being (k, l)
- `comp` – (optional) either the components of the tensor field with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the tensor field
- `latex_name` – (default: `None`) LaTeX symbol to denote the tensor field; if `None`, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention `position=0` for the first argument; for instance:
  - `sym = (0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym = [(0,2), (1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments

- antisym – (default: None) antisymmetry or list of antisymmetries among the arguments, with the same convention as for sym
- dest_map – (default: None) the destination map $\Phi : M \to N$; if None, it is assumed that $N = M$ and that Φ is the identity map (case of a tensor field *on* M), otherwise dest_map must be a *DiffMap*

OUTPUT:

- a *TensorField* (or if N is parallelizable, a *TensorFieldParal*) representing the defined tensor field

EXAMPLES:

A tensor field of type $(2, 0)$ on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = M.tensor_field(2, 0, [[1+x, -y], [0, x*y]], name='T'); t
Tensor field T of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: t.display()
T = (x + 1) d/dx*d/dx - y d/dx*d/dy + x*y d/dy*d/dy
```

The type $(2, 0)$ tensor fields on M form the set $\mathcal{T}^{(2,0)}(M)$, which is a module over the algebra $C^k(M)$ of differentiable scalar fields on M:

```
sage: t.parent()
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 2-dimensional differentiable manifold M
sage: t in M.tensor_field_module((2,0))
True
```

For more examples, see *TensorField* and *TensorFieldParal*.

tensor_field_module (*tensor_type*, *dest_map=None*)

Return the set of tensor fields of a given type defined on self, possibly with values in another manifold, as a module over the algebra of scalar fields defined on self.

See also:

TensorFieldModule for a complete documentation.

INPUT:

- tensor_type – pair (k, l) with k being the contravariant rank and l the covariant rank
- dest_map – (default: None) destination map, i.e. a differentiable map $\Phi : M \to N$, where M is the current manifold and N a differentiable manifold; if None, it is assumed that $N = M$ and that Φ is the identity map (case of tensor fields *on* M), otherwise dest_map must be a *DiffMap*

OUTPUT:

- a *TensorFieldModule* (or if N is parallelizable, a *TensorFieldFreeModule*) representing the module $\mathcal{T}^{(k,l)}(M, \Phi)$ of type-(k, l) tensor fields on M taking values on $\Phi(M) \subset M$

EXAMPLES:

Module of type-$(2, 1)$ tensor fields on a 3-dimensional open subset of a differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U')
```

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```
sage: c_xyz.<x,y,z> = U.chart()
sage: TU = U.tensor_field_module((2,1)) ; TU
Free module T^(2,1)(U) of type-(2,1) tensors fields on the Open
 subset U of the 3-dimensional differentiable manifold M
sage: TU.category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the Open subset U of the
 3-dimensional differentiable manifold M
sage: TU.base_ring()
Algebra of differentiable scalar fields on the Open subset U of
 the 3-dimensional differentiable manifold M
sage: TU.base_ring() is U.scalar_field_algebra()
True
sage: TU.an_element()
Tensor field of type (2,1) on the Open subset U of the
 3-dimensional differentiable manifold M
sage: TU.an_element().display()
2 d/dx*d/dx*dx
```

vector_field(**comp*, ***kwargs*)

Define a vector field on `self`.

Via the argument `dest_map`, it is possible to let the vector field take its values on another manifold. More precisely, if M is the current manifold, N a differentiable manifold and $\Phi : M \to N$ a differentiable map, a *vector field along M with values on N* is a differentiable map

$$v : M \longrightarrow TN$$

(TN being the tangent bundle of N) such that

$$\forall p \in M, \ v(p) \in T_{\Phi(p)}N,$$

where $T_{\Phi(p)}N$ is the tangent space to N at the point $\Phi(p)$.

The standard case of vector fields *on* M corresponds to $N = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in N (M is then an open interval of $\mathbf{R}$).

See also:

VectorField and *VectorFieldParal* for a complete documentation.

INPUT:

- `comp` – (optional) either the components of the vector field with respect to the vector frame specified by the argument `frame` or a dictionary of components, the keys of which are vector frames or pairs `(f, c)` where `f` is a vector frame and `c` the chart in which the components are expressed
- `frame` – (default: `None`; unused if `comp` is not given or is a dictionary) vector frame in which the components are given; if `None`, the default vector frame of `self` is assumed
- `chart` – (default: `None`; unused if `comp` is not given or is a dictionary) coordinate chart in which the components are expressed; if `None`, the default chart on the domain of `frame` is assumed
- `name` – (default: `None`) name given to the vector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the vector field; if none is provided, the LaTeX symbol is set to `name`
- `dest_map` – (default: `None`) the destination map $\Phi : M \to N$; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of a vector field *on* M), otherwise `dest_map` must be a *DiffMap*

OUTPUT:

- a *VectorField* (or if N is parallelizable, a *VectorFieldParal*) representing the defined vector field

EXAMPLES:

A vector field on a open subset of a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: U = M.open_subset('U')
sage: c_xyz.<x,y,z> = U.chart()
sage: v = U.vector_field(y, -x*z, 1+y, name='v'); v
Vector field v on the Open subset U of the 3-dimensional
 differentiable manifold M
sage: v.display()
v = y d/dx - x*z d/dy + (y + 1) d/dz
```

The vector fields on U form the set $\mathfrak{X}(U)$, which is a module over the algebra $C^k(U)$ of differentiable scalar fields on U:

```
sage: v.parent()
Free module X(U) of vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
sage: v in U.vector_field_module()
True
```

For more examples, see *VectorField* and *VectorFieldParal*.

vector_field_module(*dest_map=None*, *force_free=False*)

Return the set of vector fields defined on `self`, possibly with values in another differentiable manifold, as a module over the algebra of scalar fields defined on the manifold.

See *VectorFieldModule* for a complete documentation.

INPUT:

- `dest_map` – (default: `None`) destination map, i.e. a differentiable map $\Phi : M \to N$, where M is the current manifold and N a differentiable manifold; if `None`, it is assumed that $N = M$ and that Φ is the identity map (case of vector fields *on* M), otherwise `dest_map` must be a *DiffMap*
- `force_free` – (default: `False`) if set to `True`, force the construction of a *free* module (this implies that N is parallelizable)

OUTPUT:

- a *VectorFieldModule* (or if N is parallelizable, a *VectorFieldFreeModule*) representing the module $\mathfrak{X}(M, \Phi)$ of vector fields on M taking values on $\Phi(M) \subset N$

EXAMPLES:

Vector field module $\mathfrak{X}(U) := \mathfrak{X}(U, \mathrm{Id}_U)$ of the complement U of the two poles on the sphere $\mathbb{S}^2$:

```
sage: S2 = Manifold(2, 'S^2')
sage: U = S2.open_subset('U')  # the complement of the two poles
sage: spher_coord.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi') #
↪spherical coordinates
sage: XU = U.vector_field_module() ; XU
Free module X(U) of vector fields on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: XU.category()
```

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```
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: XU.base_ring()
Algebra of differentiable scalar fields on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: XU.base_ring() is U.scalar_field_algebra()
True
```

$\mathfrak{X}(U)$ is a free module because U is parallelizable (being a chart domain):

```
sage: U.is_manifestly_parallelizable()
True
```

Its rank is the manifold's dimension:

```
sage: XU.rank()
2
```

The elements of $\mathfrak{X}(U)$ are vector fields on U:

```
sage: XU.an_element()
Vector field on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: XU.an_element().display()
2 d/dth + 2 d/dph
```

Vector field module $\mathfrak{X}(U, \Phi)$ of the $\mathbf{R}^3$-valued vector fields along U, associated with the embedding Φ of $\mathbb{S}^2$ into $\mathbf{R}^3$:

```
sage: R3 = Manifold(3, 'R^3')
sage: cart_coord.<x, y, z> = R3.chart()
sage: Phi = U.diff_map(R3,
....:      [sin(th)*cos(ph), sin(th)*sin(ph), cos(th)], name='Phi')
sage: XU_R3 = U.vector_field_module(dest_map=Phi) ; XU_R3
Free module X(U,Phi) of vector fields along the Open subset U of
 the 2-dimensional differentiable manifold S^2 mapped into the
 3-dimensional differentiable manifold R^3
sage: XU_R3.base_ring()
Algebra of differentiable scalar fields on the Open subset U of the
 2-dimensional differentiable manifold S^2
```

$\mathfrak{X}(U, \Phi)$ is a free module because $\mathbf{R}^3$ is parallelizable and its rank is 3:

```
sage: XU_R3.rank()
3
```

vector_frame(*symbol=None*, *latex_symbol=None*, *dest_map=None*, *from_frame=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Define a vector frame on `self`.

A *vector frame* is a field on the manifold that provides, at each point p of the manifold, a vector basis of the tangent space at p (or at $\Phi(p)$ when `dest_map` is not `None`, see below).

See also:

`VectorFrame` for complete documentation.

INPUT:

- `symbol` – (default: `None`) either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual symbols of the vector fields; can be `None` only if `from_frame` is not `None` (see below)
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `dest_map` – (default: `None`) *`DiffMap`*; destination map $\Phi : U \to M$, where U is `self` and M is a differentiable manifold; for each $p \in U$, the vector frame evaluated at p is a basis of the tangent space $T_{\Phi(p)}M$; if `dest_map` is `None`, the identity is assumed (case of a vector frame *on* U)
- `from_frame` – (default: `None`) vector frame $\tilde{e}$ on the codomain M of the destination map Φ; the returned frame e is then such that for all $p \in U$, we have $e(p) = \tilde{e}(\Phi(p))$
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe

OUTPUT:

- a *`VectorFrame`* representing the defined vector frame

EXAMPLES:

Setting a vector frame on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: e = M.vector_frame('e'); e
Vector frame (M, (e_0,e_1,e_2))
sage: e[0]
Vector field e_0 on the 3-dimensional differentiable manifold M
```

See also:

For more options, in particular for the choice of symbols and indices, see *`VectorFrame`*.

2.2 Coordinate Charts on Differentiable Manifolds

The class *`DiffChart`* implements coordinate charts on a differentiable manifold over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$).

The subclass *`RealDiffChart`* is devoted to the case $K = \mathbf{R}$, for which the concept of coordinate range is meaningful. Moreover, *`RealDiffChart`* is endowed with some plotting capabilities (cf. method *`plot()`*).

Transition maps between charts are implemented via the class *`DiffCoordChange`*.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version

REFERENCES:

- Chap. 1 of [?]

class sage.manifolds.differentiable.chart.**DiffChart**(*domain*, *coordinates=''*, *names=None*, *calc_method=None*)

Bases: *sage.manifolds.chart.Chart*

Chart on a differentiable manifold.

Given a differentiable manifold M of dimension n over a topological field K, a *chart* is a member (U, φ) of the manifold's differentiable atlas; U is then an open subset of M and $\varphi : U \to V \subset K^n$ is a homeomorphism from U to an open subset V of K^n.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in K^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

INPUT:

- `domain` – open subset U on which the chart is defined
- `coordinates` – (default: '' (empty string)) single string defining the coordinate symbols, with `' '` (whitespace) as a separator; each item has at most three fields, separated by a colon (`:`):
  1. the coordinate symbol (a letter or a few letters)
  2. (optional) the period of the coordinate if the coordinate is periodic; the period field must be written as `period=T`, where `T` is the period (see examples below)
  3. (optional) the LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used

  The order of fields 2 and 3 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of LaTeX's backslash character (see examples below). If no period and no LaTeX spelling are to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).
- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used).
- `calc_method` – (default: `None`) string defining the calculus method for computations involving coordinates of the chart; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the default of *CalculusMethod* will be used

EXAMPLES:

A chart on a complex 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M', field='complex')
sage: X = M.chart('x y'); X
Chart (M, (x, y))
sage: latex(X)
\left(M,(x, y)\right)
sage: type(X)
<class 'sage.manifolds.differentiable.chart.DiffChart'>
```

To manipulate the coordinates (x, y) as global variables, one has to set:

```
sage: x,y = X[:]
```

However, a shortcut is to use the declarator `<x,y>` in the left-hand side of the chart declaration (there is then no need to pass the string `'x y'` to `chart()`):

```
sage: M = Manifold(2, 'M', field='complex')
sage: X.<x,y> = M.chart(); X
Chart (M, (x, y))
```

The coordinates are then immediately accessible:

```
sage: y
y
sage: x is X[0] and y is X[1]
True
```

The trick is performed by Sage preparser:

```
sage: preparse("X.<x,y> = M.chart()")
"X = M.chart(names=('x', 'y',)); (x, y,) = X._first_ngens(2)"
```

Note that `x` and `y` declared in `<x,y>` are mere Python variable names and do not have to coincide with the coordinate symbols; for instance, one may write:

```
sage: M = Manifold(2, 'M', field='complex')
sage: X.<x1,y1> = M.chart('x y'); X
Chart (M, (x, y))
```

Then `y` is not known as a global Python variable and the coordinate y is accessible only through the global variable `y1`:

```
sage: y1
y
sage: latex(y1)
y
sage: y1 is X[1]
True
```

However, having the name of the Python variable coincide with the coordinate symbol is quite convenient; so it is recommended to declare:

```
sage: M = Manifold(2, 'M', field='complex')
sage: X.<x,y> = M.chart()
```

In the above example, the chart X covers entirely the manifold M:

```
sage: X.domain()
2-dimensional complex manifold M
```

Of course, one may declare a chart only on an open subset of M:

```
sage: U = M.open_subset('U')
sage: Y.<z1, z2> = U.chart(r'z1:\zeta_1 z2:\zeta_2'); Y
Chart (U, (z1, z2))
sage: Y.domain()
Open subset U of the 2-dimensional complex manifold M
```

In the above declaration, we have also specified some LaTeX writing of the coordinates different from the text one:

```
sage: latex(z1)
{\zeta_1}
```

Note the prefix `r` in front of the string `r'z1:\zeta_1 z2:\zeta_2'`; it makes sure that the backslash character is treated as an ordinary character, to be passed to the LaTeX interpreter.

Periodic coordinates are declared through the keyword `period=` in the coordinate field:

```
sage: N = Manifold(2, 'N', field='complex')
sage: XN.<Z1,Z2> = N.chart('Z1:period=1+2*I Z2')
sage: XN.periods()
{0: 2*I + 1}
```

Coordinates are Sage symbolic variables (see `sage.symbolic.expression`):

```
sage: type(z1)
<type 'sage.symbolic.expression.Expression'>
```

In addition to the Python variable name provided in the operator `<., .>`, the coordinates are accessible by their indices:

```
sage: Y[0], Y[1]
(z1, z2)
```

The index range is that declared during the creation of the manifold. By default, it starts at 0, but this can be changed via the parameter `start_index`:

```
sage: M1 = Manifold(2, 'M_1', field='complex', start_index=1)
sage: Z.<u,v> = M1.chart()
sage: Z[1], Z[2]
(u, v)
```

The full set of coordinates is obtained by means of the operator `[:]`:

```
sage: Y[:]
(z1, z2)
```

Each constructed chart is automatically added to the manifold's user atlas:

```
sage: M.atlas()
[Chart (M, (x, y)), Chart (U, (z1, z2))]
```

and to the atlas of the chart's domain:

```
sage: U.atlas()
[Chart (U, (z1, z2))]
```

Manifold subsets have a *default chart*, which, unless changed via the method `set_default_chart()`, is the first defined chart on the subset (or on a open subset of it):

```
sage: M.default_chart()
Chart (M, (x, y))
sage: U.default_chart()
Chart (U, (z1, z2))
```

The default charts are not privileged charts on the manifold, but rather charts whose name can be skipped in the argument list of functions having an optional `chart=` argument.

The action of the chart map φ on a point is obtained by means of the call operator, i.e. the operator `()`:

```
sage: p = M.point((1+i, 2), chart=X); p
Point on the 2-dimensional complex manifold M
sage: X(p)
(I + 1, 2)
sage: X(p) == p.coord(X)
True
```

A vector frame is naturally associated to each chart:

```
sage: X.frame()
Coordinate frame (M, (d/dx,d/dy))
sage: Y.frame()
Coordinate frame (U, (d/dz1,d/dz2))
```

as well as a dual frame (basis of 1-forms):

```
sage: X.coframe()
Coordinate coframe (M, (dx,dy))
sage: Y.coframe()
Coordinate coframe (U, (dz1,dz2))
```

See also:

RealDiffChart for charts on differentiable manifolds over $\mathbf{R}$.

coframe()

Return the coframe (basis of coordinate differentials) associated with `self`.

OUTPUT:

- a *CoordCoFrame* representing the coframe

EXAMPLES:

Coordinate coframe associated with some chart on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: c_xy.coframe()
Coordinate coframe (M, (dx,dy))
sage: type(c_xy.coframe())
<class 'sage.manifolds.differentiable.vectorframe.CoordCoFrame'>
```

Check that `c_xy.coframe()` is indeed the coordinate coframe associated with the coordinates (x, y):

```
sage: dx = c_xy.coframe()[0] ; dx
1-form dx on the 2-dimensional differentiable manifold M
sage: dy = c_xy.coframe()[1] ; dy
1-form dy on the 2-dimensional differentiable manifold M
sage: ex = c_xy.frame()[0] ; ex
Vector field d/dx on the 2-dimensional differentiable manifold M
sage: ey = c_xy.frame()[1] ; ey
Vector field d/dy on the 2-dimensional differentiable manifold M
sage: dx(ex).display()
 dx(d/dx): M --> R
```

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```
    (x, y) |--> 1
sage: dx(ey).display()
dx(d/dy): M --> R
    (x, y) |--> 0
sage: dy(ex).display()
dy(d/dx): M --> R
    (x, y) |--> 0
sage: dy(ey).display()
dy(d/dy): M --> R
    (x, y) |--> 1
```

frame()

Return the vector frame (coordinate frame) associated with `self`.

OUTPUT:

- a *CoordFrame* representing the coordinate frame

EXAMPLES:

Coordinate frame associated with some chart on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: c_xy.frame()
Coordinate frame (M, (d/dx,d/dy))
sage: type(c_xy.frame())
<class 'sage.manifolds.differentiable.vectorframe.CoordFrame'>
```

Check that `c_xy.frame()` is indeed the coordinate frame associated with the coordinates (x, y):

```
sage: ex = c_xy.frame()[0] ; ex
Vector field d/dx on the 2-dimensional differentiable manifold M
sage: ey = c_xy.frame()[1] ; ey
Vector field d/dy on the 2-dimensional differentiable manifold M
sage: ex(M.scalar_field(x)).display()
M --> R
(x, y) |--> 1
sage: ex(M.scalar_field(y)).display()
M --> R
(x, y) |--> 0
sage: ey(M.scalar_field(x)).display()
M --> R
(x, y) |--> 0
sage: ey(M.scalar_field(y)).display()
M --> R
(x, y) |--> 1
```

restrict(*subset*, *restrictions=None*)

Return the restriction of `self` to some subset.

If the current chart is (U, φ), a *restriction* (or *subchart*) is a chart (V, ψ) such that $V \subset U$ and $\psi = \varphi|_V$.

If such subchart has not been defined yet, it is constructed here.

The coordinates of the subchart bare the same names as the coordinates of the original chart.

INPUT:

- `subset` – open subset V of the chart domain U

- `restrictions` – (default: `None`) list of coordinate restrictions defining the subset V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- a `DiffChart` (V, ψ)

EXAMPLES:

Coordinates on the unit open ball of $\mathbf{C}^2$ as a subchart of the global coordinates of $\mathbf{C}^2$:

```
sage: M = Manifold(2, 'C^2', field='complex')
sage: X.<z1, z2> = M.chart()
sage: B = M.open_subset('B')
sage: X_B = X.restrict(B, abs(z1)^2 + abs(z2)^2 < 1); X_B
Chart (B, (z1, z2))
```

symbolic_velocities(*left='D'*, *right=None*)

Return a list of symbolic variables ready to be used by the user as the derivatives of the coordinate functions with respect to a curve parameter (i.e. the velocities along the curve). It may actually serve to denote anything else than velocities, with a name including the coordinate functions. The choice of strings provided as 'left' and 'right' arguments is not entirely free since it must comply with Python prescriptions.

INPUT:

- `left` – (default: `D`) string to concatenate to the left of each coordinate functions of the chart
- `right` – (default: `None`) string to concatenate to the right of each coordinate functions of the chart

OUTPUT:

- a list of symbolic expressions with the desired names

EXAMPLES:

Symbolic derivatives of the Cartesian coordinates of the 3-dimensional Euclidean space:

```
sage: R3 = Manifold(3, 'R3', start_index=1)
sage: cart.<X,Y,Z> = R3.chart()
sage: D = cart.symbolic_velocities(); D
[DX, DY, DZ]
sage: D = cart.symbolic_velocities(left='d', right="/dt"); D
Traceback (most recent call last):
...
ValueError: The name "dX/dt" is not a valid Python
 identifier.
sage: D = cart.symbolic_velocities(left='d', right="_dt"); D
[dX_dt, dY_dt, dZ_dt]
sage: D = cart.symbolic_velocities(left='', right="'"); D
Traceback (most recent call last):
...
```

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```
ValueError: The name "X'" is not a valid Python
 identifier.
sage: D = cart.symbolic_velocities(left='', right="_dot"); D
[X_dot, Y_dot, Z_dot]
sage: R.<t> = RealLine()
sage: canon_chart = R.default_chart()
sage: D = canon_chart.symbolic_velocities() ; D
[Dt]
```

transition_map(*other*, *transformations*, *intersection_name=None*, *restrictions1=None*, *restrictions2=None*)

Construct the transition map between the current chart, (U,φ) say, and another one, (V,ψ) say.

If n is the manifold's dimension, the *transition map* is the map

$$\psi\circ\varphi^{-1}:\varphi(U\cap V)\subset K^n\to\psi(U\cap V)\subset K^n,$$

where K is the manifold's base field. In other words, the transition map expresses the coordinates $(y^1,\dots,y^n)$ of (V,ψ) in terms of the coordinates $(x^1,\dots,x^n)$ of (U,φ) on the open subset where the two charts intersect, i.e. on $U\cap V$.

By definition, the transition map $\psi\circ\varphi^{-1}$ must be of classe C^k, where k is the degree of differentiability of the manifold (cf. `diff_degree()`).

INPUT:

- `other` – the chart (V,ψ)
- `transformations` – tuple (or list) $(Y_1,\dots,Y_2)$, where Y_i is the symbolic expression of the coordinate y^i in terms of the coordinates $(x^1,\dots,x^n)$
- `intersection_name` – (default: `None`) name to be given to the subset $U\cap V$ if the latter differs from U or V
- `restrictions1` – (default: `None`) list of conditions on the coordinates of the current chart that define $U\cap V$ if the latter differs from U. `restrictions1` must be a list of of symbolic equalities or inequalities involving the coordinates, such as x>y or x^2+y^2 != 0. The items of the list `restrictions1` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions1`. For example, `restrictions1` = [x>y, (x!=0, y!=0), z^2<x] means (x>y) and ((x!=0) or (y!=0)) and (z^2<x). If the list `restrictions1` contains only one item, this item can be passed as such, i.e. writing x>y instead of the single-element list [x>y].
- `restrictions2` – (default: `None`) list of conditions on the coordinates of the chart (V,ψ) that define $U\cap V$ if the latter differs from V (see `restrictions1` for the syntax)

OUTPUT:

- The transition map $\psi\circ\varphi^{-1}$ defined on $U\cap V$, as an instance of `DiffCoordChange`.

EXAMPLES:

Transition map between two stereographic charts on the circle S^1:

```
sage: M = Manifold(1, 'S^1')
sage: U = M.open_subset('U') # Complement of the North pole
sage: cU.<x> = U.chart() # Stereographic chart from the North pole
sage: V = M.open_subset('V') # Complement of the South pole
sage: cV.<y> = V.chart() # Stereographic chart from the South pole
sage: M.declare_union(U,V)   # S^1 is the union of U and V
```

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```
sage: trans = cU.transition_map(cV, 1/x, intersection_name='W',
....:                          restrictions1= x!=0, restrictions2 = y!=0)
sage: trans
Change of coordinates from Chart (W, (x,)) to Chart (W, (y,))
sage: trans.display()
y = 1/x
```

The subset W, intersection of U and V, has been created by `transition_map()`:

```
sage: M.list_of_subsets()
[1-dimensional differentiable manifold S^1,
 Open subset U of the 1-dimensional differentiable manifold S^1,
 Open subset V of the 1-dimensional differentiable manifold S^1,
 Open subset W of the 1-dimensional differentiable manifold S^1]
sage: W = M.list_of_subsets()[3]
sage: W is U.intersection(V)
True
sage: M.atlas()
[Chart (U, (x,)), Chart (V, (y,)), Chart (W, (x,)), Chart (W, (y,))]
```

Transition map between the polar chart and the Cartesian one on $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart()
sage: U = M.open_subset('U') # the complement of the half line {y=0, x >= 0}
sage: c_spher.<r,phi> = U.chart(r'r:(0,+oo) phi:(0,2*pi):\phi')
sage: trans = c_spher.transition_map(c_cart, (r*cos(phi), r*sin(phi)),
....:                               restrictions2=(y!=0, x<0))
sage: trans
Change of coordinates from Chart (U, (r, phi)) to Chart (U, (x, y))
sage: trans.display()
x = r*cos(phi)
y = r*sin(phi)
```

In this case, no new subset has been created since $U \cap M = U$:

```
sage: M.list_of_subsets()
[2-dimensional differentiable manifold R^2,
 Open subset U of the 2-dimensional differentiable manifold R^2]
```

but a new chart has been created: $(U, (x, y))$:

```
sage: M.atlas()
[Chart (R^2, (x, y)), Chart (U, (r, phi)), Chart (U, (x, y))]
```

class sage.manifolds.differentiable.chart.**DiffCoordChange**(*chart1*, *chart2*, **transformations*)

Bases: *sage.manifolds.chart.CoordChange*

Transition map between two charts of a differentiable manifold.

Giving two coordinate charts (U, φ) and (V, ψ) on a differentiable manifold M of dimension n over a topological field K, the *transition map from* (U, φ) *to* (V, ψ) is the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset K^n \to \psi(U \cap V) \subset K^n,$$

In other words, the transition map $\psi \circ \varphi^{-1}$ expresses the coordinates $(y^1, \dots, y^n)$ of (V, ψ) in terms of the coordinates $(x^1, \dots, x^n)$ of (U, φ) on the open subset where the two charts intersect, i.e. on $U \cap V$.

By definition, the transition map $\psi \circ \varphi^{-1}$ must be of classe C^k, where k is the degree of differentiability of the manifold (cf. `diff_degree()`).

INPUT:

- `chart1` – chart (U, φ)
- `chart2` – chart (V, ψ)
- `transformations` – tuple (or list) $(Y_1, \ldots, Y_2)$, where Y_i is the symbolic expression of the coordinate y^i in terms of the coordinates $(x^1, \ldots, x^n)$

EXAMPLES:

Transition map on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
sage: X_to_Y
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: type(X_to_Y)
<class 'sage.manifolds.differentiable.chart.DiffCoordChange'>
sage: X_to_Y.display()
u = x + y
v = x - y
```

jacobian()

Return the Jacobian matrix of `self`.

If `self` corresponds to the change of coordinates

$$y^i = Y^i(x^1, \ldots, x^n) \qquad 1 \leq i \leq n$$

the Jacobian matrix J is given by

$$J_{ij} = \frac{\partial Y^i}{\partial x^j}$$

where i is the row index and j the column one.

OUTPUT:

- Jacobian matrix J, the elements J_{ij} of which being coordinate functions (cf. `ChartFunction`)

EXAMPLES:

Jacobian matrix of a 2-dimensional transition map:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y^2, 3*x-y])
sage: X_to_Y.jacobian()
[  1 2*y]
[  3  -1]
```

Each element of the Jacobian matrix is a coordinate function:

```
sage: parent(X_to_Y.jacobian()[0,0])
Ring of chart functions on Chart (M, (x, y))
```

jacobian_det()

Return the Jacobian determinant of `self`.

The Jacobian determinant is the determinant of the Jacobian matrix (see `jacobian()`).

OUTPUT:

- determinant of the Jacobian matrix J as a coordinate function (cf. `ChartFunction`)

EXAMPLES:

Jacobian determinant of a 2-dimensional transition map:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y^2, 3*x-y])
sage: X_to_Y.jacobian_det()
-6*y - 1
sage: X_to_Y.jacobian_det() == det(X_to_Y.jacobian())
True
```

The Jacobian determinant is a coordinate function:

```
sage: parent(X_to_Y.jacobian_det())
Ring of chart functions on Chart (M, (x, y))
```

class sage.manifolds.differentiable.chart.**RealDiffChart**(*domain*, *coordinates=''*, *names=None*, *calc_method=None*)

Bases: `sage.manifolds.differentiable.chart.DiffChart`, `sage.manifolds.chart.RealChart`

Chart on a differentiable manifold over $\mathbf{R}$.

Given a differentiable manifold M of dimension n over $\mathbf{R}$, a *chart* is a member (U, φ) of the manifold's differentiable atlas; U is then an open subset of M and $\varphi : U \to V \subset \mathbf{R}^n$ is a homeomorphism from U to an open subset V of $\mathbf{R}^n$.

The components $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p)) \in \mathbf{R}^n$ for any point $p \in U$, are called the *coordinates* of the chart (U, φ).

INPUT:

- `domain` – open subset U on which the chart is defined
- `coordinates` – (default: '' (empty string)) single string defining the coordinate symbols, with `' '` (whitespace) as a separator; each item has at most four fields, separated by a colon (`:`):
  1. the coordinate symbol (a letter or a few letters)
  2. (optional) the interval I defining the coordinate range: if not provided, the coordinate is assumed to span all $\mathbf{R}$; otherwise I must be provided in the form `(a,b)` (or equivalently `]a,b[`); the bounds `a` and `b` can be `+/-Infinity`, `Inf`, `infinity`, `inf` or `oo`; for *singular* coordinates, non-open intervals such as `[a,b]` and `(a,b]` (or equivalently `]a,b]`) are allowed; note that the interval declaration must not contain any whitespace
  3. (optional) indicator of the periodic character of the coordinate, either as `period=T`, where `T` is the period, or as the keyword `periodic` (the value of the period is then deduced from the interval I declared in field 2; see examples below)
  4. (optional) the LaTeX spelling of the coordinate; if not provided the coordinate symbol given in the first field will be used

The order of fields 2 to 4 does not matter and each of them can be omitted. If it contains any LaTeX expression, the string `coordinates` must be declared with the prefix 'r' (for "raw") to allow for a proper treatment of LaTeX's backslash character (see examples below). If interval range, no period and no LaTeX spelling are to be set for any coordinate, the argument `coordinates` can be omitted when the shortcut operator `<,>` is used to declare the chart (see examples below).

- `names` – (default: `None`) unused argument, except if `coordinates` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used).
- `calc_method` – (default: `None`) string defining the calculus method for computations involving coordinates of the chart; must be one of
  - `'SR'`: Sage's default symbolic engine (Symbolic Ring)
  - `'sympy'`: SymPy
  - `None`: the default of `CalculusMethod` will be used

EXAMPLES:

Cartesian coordinates on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
sage: type(c_cart)
<class 'sage.manifolds.differentiable.chart.RealDiffChart'>
```

To have the coordinates accessible as global variables, one has to set:

```
sage: (x,y,z) = c_cart[:]
```

However, a shortcut is to use the declarator `<x,y,z>` in the left-hand side of the chart declaration (there is then no need to pass the string `'x y z'` to `chart()`):

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = M.chart(); c_cart
Chart (R^3, (x, y, z))
```

The coordinates are then immediately accessible:

```
sage: y
y
sage: y is c_cart[2]
True
```

The trick is performed by Sage preparser:

```
sage: preparse("c_cart.<x,y,z> = M.chart()")
"c_cart = M.chart(names=('x', 'y', 'z',)); (x, y, z,) = c_cart._first_ngens(3)"
```

Note that `x, y, z` declared in `<x,y,z>` are mere Python variable names and do not have to coincide with the coordinate symbols; for instance, one may write:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart.<x1,y1,z1> = M.chart('x y z'); c_cart
Chart (R^3, (x, y, z))
```

Then `y` is not known as a global variable and the coordinate y is accessible only through the global variable `y1`:

```
sage: y1
y
sage: y1 is c_cart[2]
True
```

However, having the name of the Python variable coincide with the coordinate symbol is quite convenient; so it is recommended to declare:

```
sage: forget()    # for doctests only
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = M.chart()
```

Spherical coordinates on the subset U of $\mathbf{R}^3$ that is the complement of the half-plane $\{y = 0, x \geq 0\}$:

```
sage: U = M.open_subset('U')
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: c_spher
Chart (U, (r, th, ph))
```

Note the prefix 'r' for the string defining the coordinates in the arguments of `chart`.

Coordinates are Sage symbolic variables (see `sage.symbolic.expression`):

```
sage: type(th)
<type 'sage.symbolic.expression.Expression'>
sage: latex(th)
{\theta}
sage: assumptions(th)
[th is real, th > 0, th < pi]
```

Coordinate are also accessible by their indices:

```
sage: x1 = c_spher[1]; x2 = c_spher[2]; x3 = c_spher[3]
sage: [x1, x2, x3]
[r, th, ph]
sage: (x1, x2, x3) == (r, th, ph)
True
```

The full set of coordinates is obtained by means of the operator [:]:

```
sage: c_cart[:]
(x, y, z)
sage: c_spher[:]
(r, th, ph)
```

Let us check that the declared coordinate ranges have been taken into account:

```
sage: c_cart.coord_range()
x: (-oo, +oo); y: (-oo, +oo); z: (-oo, +oo)
sage: c_spher.coord_range()
r: (0, +oo); th: (0, pi); ph: (0, 2*pi)
sage: bool(th>0 and th<pi)
True
sage: assumptions()  # list all current symbolic assumptions
[x is real, y is real, z is real, r is real, r > 0, th is real,
 th > 0, th < pi, ph is real, ph > 0, ph < 2*pi]
```

The coordinate ranges are used for simplifications:

```
sage: simplify(abs(r)) # r has been declared to lie in the interval (0,+oo)
r
sage: simplify(abs(x)) # no positive range has been declared for x
abs(x)
```

A coordinate can be declared periodic by adding the keyword `periodic` to its range:

```
sage: V = M.open_subset('V')
sage: c_spher1.<r,th,ph1> = \
....: V.chart(r'r:(0,+oo) th:(0,pi):\theta ph1:(0,2*pi):periodic:\phi_1')
sage: c_spher1.periods()
{3: 2*pi}
sage: c_spher1.coord_range()
r: (0, +oo); th: (0, pi); ph1: [0, 2*pi] (periodic)
```

It is equivalent to give the period as `period=2*pi`, skipping the coordinate range:

```
sage: c_spher2.<r,th,ph2> = \
....: V.chart(r'r:(0,+oo) th:(0,pi):\theta ph2:period=2*pi:\phi_2')
sage: c_spher2.periods()
{3: 2*pi}
sage: c_spher2.coord_range()
r: (0, +oo); th: (0, pi); ph2: [0, 2*pi] (periodic)
```

Each constructed chart is automatically added to the manifold's user atlas:

```
sage: M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2))]
```

and to the atlas of its domain:

```
sage: U.atlas()
[Chart (U, (r, th, ph))]
```

Manifold subsets have a *default chart*, which, unless changed via the method *set_default_chart()*, is the first defined chart on the subset (or on a open subset of it):

```
sage: M.default_chart()
Chart (R^3, (x, y, z))
sage: U.default_chart()
Chart (U, (r, th, ph))
```

The default charts are not privileged charts on the manifold, but rather charts whose name can be skipped in the argument list of functions having an optional `chart=` argument.

The action of the chart map φ on a point is obtained by means of the call operator, i.e. the operator `()`:

```
sage: p = M.point((1,0,-2)); p
Point on the 3-dimensional differentiable manifold R^3
sage: c_cart(p)
(1, 0, -2)
sage: c_cart(p) == p.coord(c_cart)
True
sage: q = M.point((2,pi/2,pi/3), chart=c_spher) # point defined by its spherical␣
↪coordinates
sage: c_spher(q)
```

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```
(2, 1/2*pi, 1/3*pi)
sage: c_spher(q) == q.coord(c_spher)
True
sage: a = U.point((1,pi/2,pi)) # the default coordinates on U are the spherical␣
↪ones
sage: c_spher(a)
(1, 1/2*pi, pi)
sage: c_spher(a) == a.coord(c_spher)
True
```

Cartesian coordinates on U as an example of chart construction with coordinate restrictions: since U is the complement of the half-plane $\{y = 0, x \geq 0\}$, we must have $y \neq 0$ or $x < 0$ on U. Accordingly, we set:

```
sage: c_cartU.<x,y,z> = U.chart()
sage: c_cartU.add_restrictions((y!=0, x<0)) # the tuple (y!=0, x<0) means y!=0 or␣
↪x<0
sage: # c_cartU.add_restrictions([y!=0, x<0]) would have meant y!=0 AND x<0
sage: U.atlas()
[Chart (U, (r, th, ph)), Chart (U, (x, y, z))]
sage: M.atlas()
[Chart (R^3, (x, y, z)), Chart (U, (r, th, ph)),
 Chart (V, (r, th, ph1)), Chart (V, (r, th, ph2)),
 Chart (U, (x, y, z))]
sage: c_cartU.valid_coordinates(-1,0,2)
True
sage: c_cartU.valid_coordinates(1,0,2)
False
sage: c_cart.valid_coordinates(1,0,2)
True
```

A vector frame is naturally associated to each chart:

```
sage: c_cart.frame()
Coordinate frame (R^3, (d/dx,d/dy,d/dz))
sage: c_spher.frame()
Coordinate frame (U, (d/dr,d/dth,d/dph))
```

as well as a dual frame (basis of 1-forms):

```
sage: c_cart.coframe()
Coordinate coframe (R^3, (dx,dy,dz))
sage: c_spher.coframe()
Coordinate coframe (U, (dr,dth,dph))
```

Chart grids can be drawn in 2D or 3D graphics thanks to the method `plot()`.

restrict(*subset*, *restrictions=None*)

Return the restriction of the chart to some subset.

If the current chart is (U, φ), a *restriction* (or *subchart*) is a chart (V, ψ) such that $V \subset U$ and $\psi = \varphi|_V$.

If such subchart has not been defined yet, it is constructed here.

The coordinates of the subchart bare the same names as the coordinates of the original chart.

INPUT:

- `subset` – open subset V of the chart domain U
- `restrictions` – (default: `None`) list of coordinate restrictions defining the subset V

A restriction can be any symbolic equality or inequality involving the coordinates, such as `x > y` or `x^2 + y^2 != 0`. The items of the list `restrictions` are combined with the `and` operator; if some restrictions are to be combined with the `or` operator instead, they have to be passed as a tuple in some single item of the list `restrictions`. For example:

```
restrictions = [x > y, (x != 0, y != 0), z^2 < x]
```

means `(x > y) and ((x != 0) or (y != 0)) and (z^2 < x)`. If the list `restrictions` contains only one item, this item can be passed as such, i.e. writing `x > y` instead of the single element list `[x > y]`.

OUTPUT:

- a `RealDiffChart` (V, ψ)

EXAMPLES:

Cartesian coordinates on the unit open disc in $\mathbf{R}^2$ as a subchart of the global Cartesian coordinates:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: D = M.open_subset('D') # the unit open disc
sage: c_cart_D = c_cart.restrict(D, x^2+y^2<1)
sage: p = M.point((1/2, 0))
sage: p in D
True
sage: q = M.point((1, 2))
sage: q in D
False
```

Cartesian coordinates on the annulus $1 < \sqrt{x^2 + y^2} < 2$:

```
sage: A = M.open_subset('A')
sage: c_cart_A = c_cart.restrict(A, [x^2+y^2>1, x^2+y^2<4])
sage: p in A, q in A
(False, False)
sage: a = M.point((3/2,0))
sage: a in A
True
```

2.3 The Real Line and Open Intervals

The class `OpenInterval` implement open intervals as 1-dimensional differentiable manifolds over $\mathbf{R}$. The derived class `RealLine` is devoted to $\mathbf{R}$ itself, as the open interval $(-\infty, +\infty)$.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [?]

class sage.manifolds.differentiable.real_line.**OpenInterval**(*lower*, *upper*, *ambient_interval=None*, *name=None*, *latex_name=None*, *coordinate=None*, *names=None*, *start_index=0*)

Bases: *sage.manifolds.differentiable.manifold.DifferentiableManifold*

Open interval as a 1-dimensional differentiable manifold over $\mathbf{R}$.

INPUT:

- `lower` – lower bound of the interval (possibly `-Infinity`)
- `upper` – upper bound of the interval (possibly `+Infinity`)
- `ambient_interval` – (default: `None`) another open interval, to which the constructed interval is a subset of
- `name` – (default: `None`) string; name (symbol) given to the interval; if `None`, the name is constructed from `lower` and `upper`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the interval; if `None`, the LaTeX symbol is constructed from `lower` and `upper` if `name` is `None`, otherwise, it is set to `name`
- `coordinate` – (default: `None`) string defining the symbol of the canonical coordinate set on the interval; if none is provided and `names` is `None`, the symbol 't' is used
- `names` – (default: `None`) used only when `coordinate` is `None`: it must be a single-element tuple containing the canonical coordinate symbol (this is guaranteed if the shortcut `<names>` is used, see examples below)
- `start_index` – (default: 0) unique value of the index for vectors and forms on the interval manifold

EXAMPLES:

The interval $(0, \pi)$:

```
sage: I = OpenInterval(0, pi); I
Real interval (0, pi)
sage: latex(I)
\left(0, \pi\right)
```

`I` is a 1-dimensional smooth manifold over $\mathbf{R}$:

```
sage: I.category()
Category of smooth manifolds over Real Field with 53 bits of precision
sage: I.base_field()
Real Field with 53 bits of precision
sage: dim(I)
1
```

It is infinitely differentiable (smooth manifold):

```
sage: I.diff_degree()
+Infinity
```

The instance is unique (as long as the constructor arguments are the same):

```
sage: I is OpenInterval(0, pi)
True
sage: I is OpenInterval(0, pi, name='I')
False
```

The display of the interval can be customized:

```
sage: I  # default display
Real interval (0, pi)
sage: latex(I)  # default LaTeX display
\left(0, \pi\right)
sage: I1 = OpenInterval(0, pi, name='I'); I1
Real interval I
sage: latex(I1)
I
sage: I2 = OpenInterval(0, pi, name='I', latex_name=r'\mathcal{I}'); I2
Real interval I
sage: latex(I2)
\mathcal{I}
```

`I` is endowed with a canonical chart:

```
sage: I.canonical_chart()
Chart ((0, pi), (t,))
sage: I.canonical_chart() is I.default_chart()
True
sage: I.atlas()
[Chart ((0, pi), (t,))]
```

The canonical coordinate is returned by the method `canonical_coordinate()`:

```
sage: I.canonical_coordinate()
t
sage: t = I.canonical_coordinate()
sage: type(t)
<type 'sage.symbolic.expression.Expression'>
```

However, it can be obtained in the same step as the interval construction by means of the shortcut `I.<names>`:

```
sage: I.<t> = OpenInterval(0, pi)
sage: t
t
sage: type(t)
<type 'sage.symbolic.expression.Expression'>
```

The trick is performed by the Sage preparser:

```
sage: preparse("I.<t> = OpenInterval(0, pi)")
"I = OpenInterval(Integer(0), pi, names=('t',)); (t,) = I._first_ngens(1)"
```

In particular the shortcut can be used to set a canonical coordinate symbol different from `'t'`:

```
sage: J.<x> = OpenInterval(0, pi)
sage: J.canonical_chart()
Chart ((0, pi), (x,))
sage: J.canonical_coordinate()
x
```

The LaTeX symbol of the canonical coordinate can be adjusted via the same syntax as a chart declaration (see *RealChart*):

```
sage: J.<x> = OpenInterval(0, pi, coordinate=r'x:\xi')
sage: latex(x)
{\xi}
sage: latex(J.canonical_chart())
\left(\left(0, \pi\right),({\xi})\right)
```

An element of the open interval `I`:

```
sage: x = I.an_element(); x
Point on the Real interval (0, pi)
sage: x.coord() # coordinates in the default chart = canonical chart
(1/2*pi,)
```

As for any manifold subset, a specific element of `I` can be created by providing a tuple containing its coordinate(s) in a given chart:

```
sage: x = I((2,)) # (2,) = tuple of coordinates in the canonical chart
sage: x
Point on the Real interval (0, pi)
```

But for convenience, it can also be created directly from the coordinate:

```
sage: x = I(2); x
Point on the Real interval (0, pi)
sage: x.coord()
(2,)
sage: I(2) == I((2,))
True
```

By default, the coordinates passed for the element `x` are those relative to the canonical chart:

```
sage: I(2) ==  I((2,), chart=I.canonical_chart())
True
```

The lower and upper bounds of the interval `I`:

```
sage: I.lower_bound()
0
sage: I.upper_bound()
pi
```

One of the endpoint can be infinite:

```
sage: J = OpenInterval(1, +oo); J
Real interval (1, +Infinity)
sage: J.an_element().coord()
(2,)
```

The construction of a subinterval can be performed via the argument `ambient_interval` of `OpenInterval`:

```
sage: J = OpenInterval(0, 1, ambient_interval=I); J
Real interval (0, 1)
```

However, it is recommended to use the method *open_interval()* instead:

```
sage: J = I.open_interval(0, 1); J
Real interval (0, 1)
sage: J.is_subset(I)
True
sage: J.manifold() is I
True
```

A subinterval of a subinterval:

```
sage: K = J.open_interval(1/2, 1); K
Real interval (1/2, 1)
sage: K.is_subset(J)
True
sage: K.is_subset(I)
True
sage: K.manifold() is I
True
```

We have:

```
sage: I.list_of_subsets()
[Real interval (0, 1), Real interval (0, pi), Real interval (1/2, 1)]
sage: J.list_of_subsets()
[Real interval (0, 1), Real interval (1/2, 1)]
sage: K.list_of_subsets()
[Real interval (1/2, 1)]
```

As any open subset of a manifold, open subintervals are created in a category of subobjects of smooth manifolds:

```
sage: J.category()
Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision
sage: K.category()
Join of Category of subobjects of sets and Category of smooth manifolds
 over Real Field with 53 bits of precision
```

On the contrary, `I`, which has not been created as a subinterval, is in the category of smooth manifolds (see `Manifolds`):

```
sage: I.category()
Category of smooth manifolds over Real Field with 53 bits of precision
```

and we have:

```
sage: J.category() is I.category().Subobjects()
True
```

All intervals are parents:

```
sage: x = J(1/2); x
Point on the Real interval (0, pi)
sage: x.parent() is J
True
sage: y = K(3/4); y
Point on the Real interval (0, pi)
sage: y.parent() is K
True
```

We have:

```
sage: x in I, x in J, x in K
(True, True, False)
sage: y in I, y in J, y in K
(True, True, True)
```

The canonical chart of subintervals is inherited from the canonical chart of the parent interval:

```
sage: XI = I.canonical_chart(); XI
Chart ((0, pi), (t,))
sage: XI.coord_range()
t: (0, pi)
sage: XJ = J.canonical_chart(); XJ
Chart ((0, 1), (t,))
sage: XJ.coord_range()
t: (0, 1)
sage: XK = K.canonical_chart(); XK
Chart ((1/2, 1), (t,))
sage: XK.coord_range()
t: (1/2, 1)
```

canonical_chart()

Return the canonical chart defined on `self`.

OUTPUT:

- *RealDiffChart*

EXAMPLES:

Canonical chart on the interval $(0, \pi)$:

```
sage: I = OpenInterval(0, pi)
sage: I.canonical_chart()
Chart ((0, pi), (t,))
sage: I.canonical_chart().coord_range()
t: (0, pi)
```

The symbol used for the coordinate of the canonical chart is that defined during the construction of the interval:

```
sage: I.<x> = OpenInterval(0, pi)
sage: I.canonical_chart()
Chart ((0, pi), (x,))
```

canonical_coordinate()

Return the canonical coordinate defined on the interval.

OUTPUT:

- the symbolic variable representing the canonical coordinate

EXAMPLES:

Canonical coordinate on the interval $(0, \pi)$:

```
sage: I = OpenInterval(0, pi)
sage: I.canonical_coordinate()
t
```

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```
sage: type(I.canonical_coordinate())
<type 'sage.symbolic.expression.Expression'>
sage: I.canonical_coordinate().is_real()
True
```

The canonical coordinate is the first (unique) coordinate of the canonical chart:

```
sage: I.canonical_coordinate() is I.canonical_chart()[0]
True
```

Its default symbol is t; but it can be customized during the creation of the interval:

```
sage: I = OpenInterval(0, pi, coordinate='x')
sage: I.canonical_coordinate()
x
sage: I.<x> = OpenInterval(0, pi)
sage: I.canonical_coordinate()
x
```

inf()

Return the lower bound (infimum) of the interval.

EXAMPLES:

```
sage: I = OpenInterval(1/4, 3)
sage: I.lower_bound()
1/4
sage: J = OpenInterval(-oo, 2)
sage: J.lower_bound()
-Infinity
```

An alias of `lower_bound()` is `inf()`:

```
sage: I.inf()
1/4
sage: J.inf()
-Infinity
```

lower_bound()

Return the lower bound (infimum) of the interval.

EXAMPLES:

```
sage: I = OpenInterval(1/4, 3)
sage: I.lower_bound()
1/4
sage: J = OpenInterval(-oo, 2)
sage: J.lower_bound()
-Infinity
```

An alias of `lower_bound()` is `inf()`:

```
sage: I.inf()
1/4
sage: J.inf()
-Infinity
```

open_interval (*lower*, *upper*, *name=None*, *latex_name=None*)

Define an open subinterval of `self`.

INPUT:

- `lower` – lower bound of the subinterval (possibly `-Infinity`)
- `upper` – upper bound of the subinterval (possibly `+Infinity`)
- `name` – (default: `None`) string; name (symbol) given to the subinterval; if `None`, the name is constructed from `lower` and `upper`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the subinterval; if `None`, the LaTeX symbol is constructed from `lower` and `upper` if `name` is `None`, otherwise, it is set to `name`

OUTPUT:

- *OpenInterval* representing the open interval (`lower`, `upper`)

EXAMPLES:

The interval $(0, \pi)$ as a subinterval of $(-4, 4)$:

```
sage: I = OpenInterval(-4, 4)
sage: J = I.open_interval(0, pi); J
Real interval (0, pi)
sage: J.is_subset(I)
True
sage: I.list_of_subsets()
[Real interval (-4, 4), Real interval (0, pi)]
```

`J` is considered as an open submanifold of `I`:

```
sage: J.manifold() is I
True
```

The subinterval $(-4, 4)$ is `I` itself:

```
sage: I.open_interval(-4, 4) is I
True
```

sup ()

Return the upper bound (supremum) of the interval.

EXAMPLES:

```
sage: I = OpenInterval(1/4, 3)
sage: I.upper_bound()
3
sage: J = OpenInterval(1, +oo)
sage: J.upper_bound()
+Infinity
```

An alias of *upper_bound()* is *sup()*:

```
sage: I.sup()
3
sage: J.sup()
+Infinity
```

upper_bound ()

Return the upper bound (supremum) of the interval.

EXAMPLES:

```
sage: I = OpenInterval(1/4, 3)
sage: I.upper_bound()
3
sage: J = OpenInterval(1, +oo)
sage: J.upper_bound()
+Infinity
```

An alias of `upper_bound()` is `sup()`:

```
sage: I.sup()
3
sage: J.sup()
+Infinity
```

class sage.manifolds.differentiable.real_line.**RealLine**(*name='R'*, *latex_name='\Bold{R}'*, *coordinate=None*, *names=None*, *start_index=0*)

Bases: `sage.manifolds.differentiable.real_line.OpenInterval`

Field of real numbers, as a differentiable manifold of dimension 1 (real line) with a canonical coordinate chart.

INPUT:

- `name` – (default: `'R'`) string; name (symbol) given to the real line
- `latex_name` – (default: `r'\Bold{R}'`) string; LaTeX symbol to denote the real line
- `coordinate` – (default: `None`) string defining the symbol of the canonical coordinate set on the real line; if none is provided and `names` is `None`, the symbol 't' is used
- `names` – (default: `None`) used only when `coordinate` is `None`: it must be a single-element tuple containing the canonical coordinate symbol (this is guaranteed if the shortcut `<names>` is used, see examples below)
- `start_index` – (default: 0) unique value of the index for vectors and forms on the real line manifold

EXAMPLES:

Constructing the real line without any argument:

```
sage: R = RealLine() ; R
Real number line R
sage: latex(R)
\Bold{R}
```

`R` is a 1-dimensional real smooth manifold:

```
sage: R.category()
Category of smooth manifolds over Real Field with 53 bits of precision
sage: isinstance(R, sage.manifolds.differentiable.manifold.DifferentiableManifold)
True
sage: dim(R)
1
```

It is endowed with a canonical chart:

```
sage: R.canonical_chart()
Chart (R, (t,))
sage: R.canonical_chart() is R.default_chart()
True
sage: R.atlas()
[Chart (R, (t,))]
```

The instance is unique (as long as the constructor arguments are the same):

```
sage: R is RealLine()
True
sage: R is RealLine(latex_name='R')
False
```

The canonical coordinate is returned by the method `canonical_coordinate()`:

```
sage: R.canonical_coordinate()
t
sage: t = R.canonical_coordinate()
sage: type(t)
<type 'sage.symbolic.expression.Expression'>
```

However, it can be obtained in the same step as the real line construction by means of the shortcut `R.<names>`:

```
sage: R.<t> = RealLine()
sage: t
t
sage: type(t)
<type 'sage.symbolic.expression.Expression'>
```

The trick is performed by Sage preparser:

```
sage: preparse("R.<t> = RealLine()")
"R = RealLine(names=('t',)); (t,) = R._first_ngens(1)"
```

In particular the shortcut is to be used to set a canonical coordinate symbol different from 't':

```
sage: R.<x> = RealLine()
sage: R.canonical_chart()
Chart (R, (x,))
sage: R.atlas()
[Chart (R, (x,))]
sage: R.canonical_coordinate()
x
```

The LaTeX symbol of the canonical coordinate can be adjusted via the same syntax as a chart declaration (see `RealChart`):

```
sage: R.<x> = RealLine(coordinate=r'x:\xi')
sage: latex(x)
{\xi}
sage: latex(R.canonical_chart())
\left(\Bold{R},({\xi})\right)
```

The LaTeX symbol of the real line itself can also be customized:

```
sage: R.<x> = RealLine(latex_name=r'\mathbb{R}')
sage: latex(R)
\mathbb{R}
```

Elements of the real line can be constructed directly from a number:

```
sage: p = R(2) ; p
Point on the Real number line R
sage: p.coord()
(2,)
sage: p = R(1.742) ; p
Point on the Real number line R
sage: p.coord()
(1.74200000000000,)
```

Symbolic variables can also be used:

```
sage: p = R(pi, name='pi') ; p
Point pi on the Real number line R
sage: p.coord()
(pi,)
sage: a = var('a')
sage: p = R(a) ; p
Point on the Real number line R
sage: p.coord()
(a,)
```

The real line is considered as the open interval $(-\infty, +\infty)$:

```
sage: isinstance(R, sage.manifolds.differentiable.real_line.OpenInterval)
True
sage: R.lower_bound()
-Infinity
sage: R.upper_bound()
+Infinity
```

A real interval can be created from `R` means of the method `open_interval()`:

```
sage: I = R.open_interval(0, 1); I
Real interval (0, 1)
sage: I.manifold()
Real number line R
sage: R.list_of_subsets()
[Real interval (0, 1), Real number line R]
```

2.4 Scalar Fields

2.4.1 Algebra of Differentiable Scalar Fields

The class `DiffScalarFieldAlgebra` implements the commutative algebra $C^k(M)$ of differentiable scalar fields on a differentiable manifold M of class C^k over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$). By *differentiable scalar field*, it is meant a function $M \to K$ that is k-times continuously differentiable. $C^k(M)$ is an algebra over K, whose ring product is the pointwise multiplication of K-valued functions, which is clearly commutative.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version

REFERENCES:

- [?]
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class sage.manifolds.differentiable.scalarfield_algebra.**DiffScalarFieldAlgebra**(*domain*)

Bases: *sage.manifolds.scalarfield_algebra.ScalarFieldAlgebra*

Commutative algebra of differentiable scalar fields on a differentiable manifold.

If M is a differentiable manifold of class C^k over a topological field K, the *commutative algebra of scalar fields on* M is the set $C^k(M)$ of all k-times continuously differentiable maps $M \to K$. The set $C^k(M)$ is an algebra over K, whose ring product is the pointwise multiplication of K-valued functions, which is clearly commutative.

If $K = \mathbf{R}$ or $K = \mathbf{C}$, the field K over which the algebra $C^k(M)$ is constructed is represented by Sage's Symbolic Ring SR, since there is no exact representation of $\mathbf{R}$ nor $\mathbf{C}$ in Sage.

Via its base class *ScalarFieldAlgebra*, the class *DiffScalarFieldAlgebra* inherits from Parent, with the category set to CommutativeAlgebras. The corresponding *element* class is *DiffScalarField*.

INPUT:

- domain – the differentiable manifold M on which the scalar fields are defined (must be an instance of class *DifferentiableManifold*)

EXAMPLES:

Algebras of scalar fields on the sphere S^2 and on some open subset of it:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: W = U.intersection(V)  # S^2 minus the two poles
sage: CW = W.scalar_field_algebra() ; CW
Algebra of differentiable scalar fields on the Open subset W of the
 2-dimensional differentiable manifold M
```

$C^k(M)$ and $C^k(W)$ belong to the category of commutative algebras over $\mathbf{R}$ (represented here by Sage's Symbolic Ring):

```
sage: CM.category()
Category of commutative algebras over Symbolic Ring
sage: CM.base_ring()
Symbolic Ring
```

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```
sage: CW.category()
Category of commutative algebras over Symbolic Ring
sage: CW.base_ring()
Symbolic Ring
```

The elements of $C^k(M)$ are scalar fields on M:

```
sage: CM.an_element()
Scalar field on the 2-dimensional differentiable manifold M
sage: CM.an_element().display()  # this sample element is a constant field
M --> R
on U: (x, y) |--> 2
on V: (u, v) |--> 2
```

Those of $C^k(W)$ are scalar fields on W:

```
sage: CW.an_element()
Scalar field on the Open subset W of the 2-dimensional differentiable
 manifold M
sage: CW.an_element().display()  # this sample element is a constant field
W --> R
(x, y) |--> 2
(u, v) |--> 2
```

The zero element:

```
sage: CM.zero()
Scalar field zero on the 2-dimensional differentiable manifold M
sage: CM.zero().display()
zero: M --> R
on U: (x, y) |--> 0
on V: (u, v) |--> 0
```

```
sage: CW.zero()
Scalar field zero on the Open subset W of the 2-dimensional
 differentiable manifold M
sage: CW.zero().display()
zero: W --> R
   (x, y) |--> 0
   (u, v) |--> 0
```

The unit element:

```
sage: CM.one()
Scalar field 1 on the 2-dimensional differentiable manifold M
sage: CM.one().display()
1: M --> R
on U: (x, y) |--> 1
on V: (u, v) |--> 1
```

```
sage: CW.one()
Scalar field 1 on the Open subset W of the 2-dimensional differentiable
 manifold M
sage: CW.one().display()
1: W --> R
```

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```
(x, y) |--> 1
(u, v) |--> 1
```

A generic element can be constructed as for any parent in Sage, namely by means of the `__call__` operator on the parent (here with the dictionary of the coordinate expressions defining the scalar field):

```
sage: f = CM({c_xy: atan(x^2+y^2), c_uv: pi/2 - atan(u^2+v^2)}); f
Scalar field on the 2-dimensional differentiable manifold M
sage: f.display()
M --> R
on U: (x, y) |--> arctan(x^2 + y^2)
on V: (u, v) |--> 1/2*pi - arctan(u^2 + v^2)
sage: f.parent()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
```

Specific elements can also be constructed in this way:

```
sage: CM(0) == CM.zero()
True
sage: CM(1) == CM.one()
True
```

Note that the zero scalar field is cached:

```
sage: CM(0) is CM.zero()
True
```

Elements can also be constructed by means of the method `scalar_field()` acting on the domain (this allows one to set the name of the scalar field at the construction):

```
sage: f1 = M.scalar_field({c_xy: atan(x^2+y^2), c_uv: pi/2 - atan(u^2+v^2)},
....:                      name='f')
sage: f1.parent()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: f1 == f
True
sage: M.scalar_field(0, chart='all') == CM.zero()
True
```

The algebra $C^k(M)$ coerces to $C^k(W)$ since W is an open subset of M:

```
sage: CW.has_coerce_map_from(CM)
True
```

The reverse is of course false:

```
sage: CM.has_coerce_map_from(CW)
False
```

The coercion map is nothing but the restriction to W of scalar fields on M:

```
sage: fW = CW(f) ; fW
Scalar field on the Open subset W of the 2-dimensional differentiable
 manifold M
```

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```
sage: fW.display()
W --> R
(x, y) |--> arctan(x^2 + y^2)
(u, v) |--> 1/2*pi - arctan(u^2 + v^2)
```

```
sage: CW(CM.one()) == CW.one()
True
```

The coercion map allows for the addition of elements of $C^k(W)$ with elements of $C^k(M)$, the result being an element of $C^k(W)$:

```
sage: s = fW + f
sage: s.parent()
Algebra of differentiable scalar fields on the Open subset W of the
 2-dimensional differentiable manifold M
sage: s.display()
W --> R
(x, y) |--> 2*arctan(x^2 + y^2)
(u, v) |--> pi - 2*arctan(u^2 + v^2)
```

Another coercion is that from the Symbolic Ring, the parent of all symbolic expressions (cf. `SymbolicRing`). Since the Symbolic Ring is the base ring for the algebra `CM`, the coercion of a symbolic expression `s` is performed by the operation `s*CM.one()`, which invokes the reflected multiplication operator `sage.manifolds.scalarfield.ScalarField._rmul_()`. If the symbolic expression does not involve any chart coordinate, the outcome is a constant scalar field:

```
sage: h = CM(pi*sqrt(2)) ; h
Scalar field on the 2-dimensional differentiable manifold M
sage: h.display()
M --> R
on U: (x, y) |--> sqrt(2)*pi
on V: (u, v) |--> sqrt(2)*pi
sage: a = var('a')
sage: h = CM(a); h.display()
M --> R
on U: (x, y) |--> a
on V: (u, v) |--> a
```

If the symbolic expression involves some coordinate of one of the manifold's charts, the outcome is initialized only on the chart domain:

```
sage: h = CM(a+x); h.display()
M --> R
on U: (x, y) |--> a + x
sage: h = CM(a+u); h.display()
M --> R
on V: (u, v) |--> a + u
```

If the symbolic expression involves coordinates of different charts, the scalar field is created as a Python object, but is not initialized, in order to avoid any ambiguity:

```
sage: h = CM(x+u); h.display()
M --> R
```

TESTS OF THE ALGEBRA LAWS:

Ring laws:

```
sage: h = CM(pi*sqrt(2))
sage: s = f + h ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> sqrt(2)*pi + arctan(x^2 + y^2)
on V: (u, v) |--> 1/2*pi*(2*sqrt(2) + 1) - arctan(u^2 + v^2)
```

```
sage: s = f - h ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> -sqrt(2)*pi + arctan(x^2 + y^2)
on V: (u, v) |--> -1/2*pi*(2*sqrt(2) - 1) - arctan(u^2 + v^2)
```

```
sage: s = f*h ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> sqrt(2)*pi*arctan(x^2 + y^2)
on V: (u, v) |--> 1/2*sqrt(2)*(pi^2 - 2*pi*arctan(u^2 + v^2))
```

```
sage: s = f/h ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> 1/2*sqrt(2)*arctan(x^2 + y^2)/pi
on V: (u, v) |--> 1/4*sqrt(2)*(pi - 2*arctan(u^2 + v^2))/pi
```

```
sage: f*(h+f) == f*h + f*f
True
```

Ring laws with coercion:

```
sage: f - fW == CW.zero()
True
sage: f/fW == CW.one()
True
sage: s = f*fW ; s
Scalar field on the Open subset W of the 2-dimensional differentiable
 manifold M
sage: s.display()
W --> R
(x, y) |--> arctan(x^2 + y^2)^2
(u, v) |--> 1/4*pi^2 - pi*arctan(u^2 + v^2) + arctan(u^2 + v^2)^2
sage: s/f == fW
True
```

Multiplication by a number:

```
sage: s = 2*f ; s
Scalar field on the 2-dimensional differentiable manifold M
```

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```
sage: s.display()
M --> R
on U: (x, y) |--> 2*arctan(x^2 + y^2)
on V: (u, v) |--> pi - 2*arctan(u^2 + v^2)
```

```
sage: 0*f == CM.zero()
True
sage: 1*f == f
True
sage: 2*(f/2) == f
True
sage: (f+2*f)/3 == f
True
sage: 1/3*(f+2*f) == f
True
```

The Sage test suite for algebras is passed:

```
sage: TestSuite(CM).run()
```

It is passed also for $C^k(W)$:

```
sage: TestSuite(CW).run()
```

Element
alias of *sage.manifolds.differentiable.scalarfield.DiffScalarField*

2.4.2 Differentiable Scalar Fields

Given a differentiable manifold M of class C^k over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a *differentiable scalar field* on M is a map

$$f : M \longrightarrow K$$

of class C^k.

Differentiable scalar fields are implemented by the class *DiffScalarField*.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Eric Gourgoulhon (2018): operators gradient, Laplacian and d'Alembertian

REFERENCES:

- [?]
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class sage.manifolds.differentiable.scalarfield.**DiffScalarField**(*parent*, *coord_expression=None*, *chart=None*, *name=None*, *latex_name=None*)

Bases: *sage.manifolds.scalarfield.ScalarField*

Differentiable scalar field on a differentiable manifold.

Given a differentiable manifold M of class C^k over a topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a *differentiable scalar field* defined on M is a map

$$f : M \longrightarrow K$$

that is k-times continuously differentiable.

The class `DiffScalarField` is a Sage *element* class, whose *parent* class is `DiffScalarFieldAlgebra`. It inherits from the class `ScalarField` devoted to generic continuous scalar fields on topological manifolds.

INPUT:

- `parent` – the algebra of scalar fields containing the scalar field (must be an instance of class `DiffScalarFieldAlgebra`)
- `coord_expression` – (default: `None`) coordinate expression(s) of the scalar field; this can be either
  - a dictionary of coordinate expressions in various charts on the domain, with the charts as keys;
  - a single coordinate expression; if the argument `chart` is `'all'`, this expression is set to all the charts defined on the open set; otherwise, the expression is set in the specific chart provided by the argument `chart`

  NB: If `coord_expression` is `None` or incomplete, coordinate expressions can be added after the creation of the object, by means of the methods `add_expr()`, `add_expr_by_continuation()` and `set_expr()`
- `chart` – (default: `None`) chart defining the coordinates used in `coord_expression` when the latter is a single coordinate expression; if none is provided (default), the default chart of the open set is assumed. If `chart=='all'`, `coord_expression` is assumed to be independent of the chart (constant scalar field).
- `name` – (default: `None`) string; name (symbol) given to the scalar field
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the scalar field; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A scalar field on the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                intersection_name='W',
....:                                restrictions1= x^2+y^2!=0,
....:                                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                    name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

For scalar fields defined by a single coordinate expression, the latter can be passed instead of the dictionary over the charts:

```
sage: g = U.scalar_field(x*y, chart=c_xy, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional differentiable
 manifold M
```

The above is indeed equivalent to:

```
sage: g = U.scalar_field({c_xy: x*y}, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional differentiable
 manifold M
```

Since `c_xy` is the default chart of `U`, the argument `chart` can be skipped:

```
sage: g = U.scalar_field(x*y, name='g') ; g
Scalar field g on the Open subset U of the 2-dimensional differentiable
 manifold M
```

The scalar field g is defined on U and has an expression in terms of the coordinates (u, v) on $W = U \cap V$:

```
sage: g.display()
g: U --> R
   (x, y) |--> x*y
on W: (u, v) |--> u*v/(u^4 + 2*u^2*v^2 + v^4)
```

Scalar fields on M can also be declared with a single chart:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
```

Their definition must then be completed by providing the expressions on other charts, via the method *add_expr()*, to get a global cover of the manifold:

```
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

We can even first declare the scalar field without any coordinate expression and provide them subsequently:

```
sage: f = M.scalar_field(name='f')
sage: f.add_expr(1/(1+x^2+y^2), chart=c_xy)
sage: f.add_expr((u^2+v^2)/(1+u^2+v^2), chart=c_uv)
sage: f.display()
f: M --> R
on U: (x, y) |--> 1/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

We may also use the method *add_expr_by_continuation()* to complete the coordinate definition using the analytic continuation from domains in which charts overlap:

```
sage: f = M.scalar_field(1/(1+x^2+y^2), chart=c_xy, name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
sage: f.add_expr_by_continuation(c_uv, U.intersection(V))
sage: f.display()
```

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```
f: M --> R
on U: (x, y) |--> 1/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

A scalar field can also be defined by some unspecified function of the coordinates:

```
sage: h = U.scalar_field(function('H')(x, y), name='h') ; h
Scalar field h on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: h.display()
h: U --> R
   (x, y) |--> H(x, y)
on W: (u, v) |--> H(u/(u^2 + v^2), v/(u^2 + v^2))
```

We may use the argument `latex_name` to specify the LaTeX symbol denoting the scalar field if the latter is different from `name`:

```
sage: latex(f)
f
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2), c_uv: (u^2+v^2)/(1+u^2+v^2)},
....:                    name='f', latex_name=r'\mathcal{F}')
sage: latex(f)
\mathcal{F}
```

The coordinate expression in a given chart is obtained via the method *`expr()`*, which returns a symbolic expression:

```
sage: f.expr(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
sage: type(f.expr(c_uv))
<type 'sage.symbolic.expression.Expression'>
```

The method *`coord_function()`* returns instead a function of the chart coordinates, i.e. an instance of *`ChartFunction`*:

```
sage: f.coord_function(c_uv)
(u^2 + v^2)/(u^2 + v^2 + 1)
sage: type(f.coord_function(c_uv))
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: f.coord_function(c_uv).display()
(u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

The value returned by the method *`expr()`* is actually the coordinate expression of the chart function:

```
sage: f.expr(c_uv) is f.coord_function(c_uv).expr()
True
```

A constant scalar field is declared by setting the argument `chart` to `'all'`:

```
sage: c = M.scalar_field(2, chart='all', name='c') ; c
Scalar field c on the 2-dimensional differentiable manifold M
sage: c.display()
c: M --> R
on U: (x, y) |--> 2
on V: (u, v) |--> 2
```

A shortcut is to use the method *`constant_scalar_field()`*:

```
sage: c == M.constant_scalar_field(2)
True
```

The constant value can be some unspecified parameter:

```
sage: var('a')
a
sage: c = M.constant_scalar_field(a, name='c') ; c
Scalar field c on the 2-dimensional differentiable manifold M
sage: c.display()
c: M --> R
on U: (x, y) |--> a
on V: (u, v) |--> a
```

A special case of constant field is the zero scalar field:

```
sage: zer = M.constant_scalar_field(0) ; zer
Scalar field zero on the 2-dimensional differentiable manifold M
sage: zer.display()
zero: M --> R
on U: (x, y) |--> 0
on V: (u, v) |--> 0
```

It can be obtained directly by means of the function *zero_scalar_field()*:

```
sage: zer is M.zero_scalar_field()
True
```

A third way is to get it as the zero element of the algebra $C^k(M)$ of scalar fields on M (see below):

```
sage: zer is M.scalar_field_algebra().zero()
True
```

By definition, a scalar field acts on the manifold's points, sending them to elements of the manifold's base field (real numbers in the present case):

```
sage: N = M.point((0,0), chart=c_uv) # the North pole
sage: S = M.point((0,0), chart=c_xy) # the South pole
sage: E = M.point((1,0), chart=c_xy) # a point at the equator
sage: f(N)
0
sage: f(S)
1
sage: f(E)
1/2
sage: h(E)
H(1, 0)
sage: c(E)
a
sage: zer(E)
0
```

A scalar field can be compared to another scalar field:

```
sage: f == g
False
```

. . . to a symbolic expression:

```
sage: f == x*y
False
sage: g == x*y
True
sage: c == a
True
```

. . . to a number:

```
sage: f == 2
False
sage: zer == 0
True
```

. . . to anything else:

```
sage: f == M
False
```

Standard mathematical functions are implemented:

```
sage: sqrt(f)
Scalar field sqrt(f) on the 2-dimensional differentiable manifold M
sage: sqrt(f).display()
sqrt(f): M --> R
on U: (x, y) |--> 1/sqrt(x^2 + y^2 + 1)
on V: (u, v) |--> sqrt(u^2 + v^2)/sqrt(u^2 + v^2 + 1)
```

```
sage: tan(f)
Scalar field tan(f) on the 2-dimensional differentiable manifold M
sage: tan(f).display()
tan(f): M --> R
on U: (x, y) |--> sin(1/(x^2 + y^2 + 1))/cos(1/(x^2 + y^2 + 1))
on V: (u, v) |--> sin((u^2 + v^2)/(u^2 + v^2 + 1))/cos((u^2 + v^2)/(u^2 + v^2 +
↪1))
```

Arithmetics of scalar fields

Scalar fields on M (resp. U) belong to the algebra $C^k(M)$ (resp. $C^k(U)$):

```
sage: f.parent()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: f.parent() is M.scalar_field_algebra()
True
sage: g.parent()
Algebra of differentiable scalar fields on the Open subset U of the
 2-dimensional differentiable manifold M
sage: g.parent() is U.scalar_field_algebra()
True
```

Consequently, scalar fields can be added:

```
sage: s = f + c ; s
Scalar field f+c on the 2-dimensional differentiable manifold M
```

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```
sage: s.display()
f+c: M --> R
on U: (x, y) |--> (a*x^2 + a*y^2 + a + 1)/(x^2 + y^2 + 1)
on V: (u, v) |--> ((a + 1)*u^2 + (a + 1)*v^2 + a)/(u^2 + v^2 + 1)
```

and subtracted:

```
sage: s = f - c ; s
Scalar field f-c on the 2-dimensional differentiable manifold M
sage: s.display()
f-c: M --> R
on U: (x, y) |--> -(a*x^2 + a*y^2 + a - 1)/(x^2 + y^2 + 1)
on V: (u, v) |--> -((a - 1)*u^2 + (a - 1)*v^2 + a)/(u^2 + v^2 + 1)
```

Some tests:

```
sage: f + zer == f
True
sage: f - f == zer
True
sage: f + (-f) == zer
True
sage: (f+c)-f == c
True
sage: (f-c)+c == f
True
```

We may add a number (interpreted as a constant scalar field) to a scalar field:

```
sage: s = f + 1 ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
on V: (u, v) |--> (2*u^2 + 2*v^2 + 1)/(u^2 + v^2 + 1)
sage: (f+1)-1 == f
True
```

The number can represented by a symbolic variable:

```
sage: s = a + f ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s == c + f
True
```

However if the symbolic variable is a chart coordinate, the addition is performed only on the chart domain:

```
sage: s = f + x; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> (x^3 + x*y^2 + x + 1)/(x^2 + y^2 + 1)
sage: s = f + u; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on V: (u, v) |--> (u^3 + (u + 1)*v^2 + u^2 + u)/(u^2 + v^2 + 1)
```

The addition of two scalar fields with different domains is possible if the domain of one of them is a subset of the domain of the other; the domain of the result is then this subset:

```
sage: f.domain()
2-dimensional differentiable manifold M
sage: g.domain()
Open subset U of the 2-dimensional differentiable manifold M
sage: s = f + g ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.domain()
Open subset U of the 2-dimensional differentiable manifold M
sage: s.display()
U --> R
(x, y) |--> (x*y^3 + (x^3 + x)*y + 1)/(x^2 + y^2 + 1)
on W: (u, v) |--> (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6 + u*v^3
 + (u^3 + u)*v)/(u^6 + v^6 + (3*u^2 + 1)*v^4 + u^4 + (3*u^4 + 2*u^2)*v^2)
```

The operation actually performed is $f|_U + g$:

```
sage: s == f.restrict(U) + g
True
```

In Sage framework, the addition of f and g is permitted because there is a *coercion* of the parent of f, namely $C^k(M)$, to the parent of g, namely $C^k(U)$ (see `DiffScalarFieldAlgebra`):

```
sage: CM = M.scalar_field_algebra()
sage: CU = U.scalar_field_algebra()
sage: CU.has_coerce_map_from(CM)
True
```

The coercion map is nothing but the restriction to domain U:

```
sage: CU.coerce(f) == f.restrict(U)
True
```

Since the algebra $C^k(M)$ is a vector space over $\mathbf{R}$, scalar fields can be multiplied by a number, either an explicit one:

```
sage: s = 2*f ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> 2/(x^2 + y^2 + 1)
on V: (u, v) |--> 2*(u^2 + v^2)/(u^2 + v^2 + 1)
```

or a symbolic one:

```
sage: s = a*f ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> a/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)*a/(u^2 + v^2 + 1)
```

However, if the symbolic variable is a chart coordinate, the multiplication is performed only in the corresponding chart:

```
sage: s = x*f; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> x/(x^2 + y^2 + 1)
sage: s = u*f; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on V: (u, v) |--> (u^2 + v^2)*u/(u^2 + v^2 + 1)
```

Some tests:

```
sage: 0*f == 0
True
sage: 0*f == zer
True
sage: 1*f == f
True
sage: (-2)*f == - f - f
True
```

The ring multiplication of the algebras $C^k(M)$ and $C^k(U)$ is the pointwise multiplication of functions:

```
sage: s = f*f ; s
Scalar field f*f on the 2-dimensional differentiable manifold M
sage: s.display()
f*f: M --> R
on U: (x, y) |--> 1/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) |--> (u^4 + 2*u^2*v^2 + v^4)/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 +␣
↪1)
sage: s = g*h ; s
Scalar field g*h on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s.display()
g*h: U --> R
   (x, y) |--> x*y*H(x, y)
on W: (u, v) |--> u*v*H(u/(u^2 + v^2), v/(u^2 + v^2))/(u^4 + 2*u^2*v^2 + v^4)
```

Thanks to the coercion $C^k(M) \to C^k(U)$ mentionned above, it is possible to multiply a scalar field defined on M by a scalar field defined on U, the result being a scalar field defined on U:

```
sage: f.domain(), g.domain()
(2-dimensional differentiable manifold M,
 Open subset U of the 2-dimensional differentiable manifold M)
sage: s = f*g ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U --> R
(x, y) |--> x*y/(x^2 + y^2 + 1)
on W: (u, v) |--> u*v/(u^4 + v^4 + (2*u^2 + 1)*v^2 + u^2)
sage: s == f.restrict(U)*g
True
```

Scalar fields can be divided (pointwise division):

```
sage: s = f/c ; s
Scalar field f/c on the 2-dimensional differentiable manifold M
sage: s.display()
f/c: M --> R
on U: (x, y) |--> 1/(a*x^2 + a*y^2 + a)
on V: (u, v) |--> (u^2 + v^2)/(a*u^2 + a*v^2 + a)
sage: s = g/h ; s
Scalar field g/h on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s.display()
g/h: U --> R
   (x, y) |--> x*y/H(x, y)
on W: (u, v) |--> u*v/((u^4 + 2*u^2*v^2 + v^4)*H(u/(u^2 + v^2), v/(u^2 + v^2)))
sage: s = f/g ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U --> R
(x, y) |--> 1/(x*y^3 + (x^3 + x)*y)
on W: (u, v) |--> (u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)/(u*v^3 + (u^3 + u)*v)
sage: s == f.restrict(U)/g
True
```

For scalar fields defined on a single chart domain, we may perform some arithmetics with symbolic expressions involving the chart coordinates:

```
sage: s = g + x^2 - y ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U --> R
(x, y) |--> x^2 + (x - 1)*y
on W: (u, v) |--> -(v^3 - u^2 + (u^2 - u)*v)/(u^4 + 2*u^2*v^2 + v^4)
```

```
sage: s = g*x ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U --> R
(x, y) |--> x^2*y
on W: (u, v) |--> u^2*v/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6)
```

```
sage: s = g/x ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U --> R
(x, y) |--> y
on W: (u, v) |--> v/(u^2 + v^2)
sage: s = x/g ; s
Scalar field on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: s.display()
U --> R
(x, y) |--> 1/y
on W: (u, v) |--> (u^2 + v^2)/v
```

The test suite is passed:

```
sage: TestSuite(f).run()
sage: TestSuite(zer).run()
```

bracket (*other*)

Return the Schouten-Nijenhuis bracket of `self`, considered as a multivector field of degree 0, with a multivector field.

See `bracket()` for details.

INPUT:

- `other` – a multivector field of degree p

OUTPUT:

- if $p = 0$, a zero scalar field
- if $p = 1$, an instance of `DiffScalarField` representing the Schouten-Nijenhuis bracket `[self, other]`
- if $p \geq 2$, an instance of `MultivectorField` representing the Schouten-Nijenhuis bracket `[self,other]`

EXAMPLES:

The Schouten-Nijenhuis bracket of two scalar fields is identically zero:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y^2}, name='f')
sage: g = M.scalar_field({X: y-x}, name='g')
sage: s = f.bracket(g); s
Scalar field zero on the 2-dimensional differentiable manifold M
sage: s.display()
zero: M --> R
   (x, y) |--> 0
```

while the Schouten-Nijenhuis bracket of a scalar field f with a multivector field a is equal to minus the interior product of the differential of f with a:

```
sage: a = M.multivector_field(2, name='a')
sage: a[0,1] = x*y ; a.display()
a = x*y d/dx/\d/dy
sage: s = f.bracket(a); s
Vector field -i_df a on the 2-dimensional differentiable manifold M
sage: s.display()
-i_df a = 2*x*y^2 d/dx - x*y d/dy
```

See `bracket()` for other examples.

dalembertian (*metric=None*)

Return the d'Alembertian of `self` with respect to a given Lorentzian metric.

The *d'Alembertian* of a scalar field f with respect to a Lorentzian metric g is nothing but the Laplacian (see `laplacian()`) of f with respect to that metric:

$$\square f = g^{ij} \nabla_i \nabla_j f = \nabla_i \nabla^i f$$

where ∇ is the Levi-Civita connection of g.

Note: If the metric g is not Lorentzian, the name *d'Alembertian* is not appropriate and one should use `laplacian()` instead.

INPUT:

- `metric` – (default: `None`) the Lorentzian metric g involved in the definition of the d'Alembertian; if none is provided, the domain of `self` is supposed to be endowed with a default Lorentzian metric (i.e. is supposed to be Lorentzian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the d'Alembertian

OUTPUT:

- instance of `DiffScalarField` representing the d'Alembertian of `self`

EXAMPLES:

d'Alembertian of a scalar field in Minkowski spacetime:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: f = M.scalar_field(t + x^2 + t^2*y^3 - x*z^4, name='f')
sage: s = f.dalembertian(); s
Scalar field Box(f) on the 4-dimensional Lorentzian manifold M
sage: s.display()
Box(f): M --> R
   (t, x, y, z) |--> 6*t^2*y - 2*y^3 - 12*x*z^2 + 2
```

The function `dalembertian()` from the `operators` module can be used instead of the method `dalembertian()`:

```
sage: from sage.manifolds.operators import dalembertian
sage: dalembertian(f) == s
True
```

degree()

Return the degree of `self`, considered as a differential form or a multivector field, i.e. zero.

This trivial method is provided for consistency with the exterior calculus scheme, cf. the methods `degree()` (differential forms) and `degree()` (multivector fields).

OUTPUT:

- 0

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y^2})
sage: f.degree()
0
```

differential()

Return the differential of `self`.

OUTPUT:

- a *DiffForm* (or of *DiffFormParal* if the scalar field's domain is parallelizable) representing the 1-form that is the differential of the scalar field

EXAMPLES:

Differential of a scalar field on a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()
sage: f = M.scalar_field(cos(x)*z^3 + exp(y)*z^2, name='f')
sage: df = f.differential() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
sage: df.parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional
 differentiable manifold M
```

The result is cached, i.e. is not recomputed unless `f` is changed:

```
sage: f.differential() is df
True
```

Since the exterior derivative of a scalar field (considered a 0-form) is nothing but its differential, `exterior_derivative()` is an alias of `differential()`:

```
sage: df = f.exterior_derivative() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
```

One may also use the function *exterior_derivative()* or its alias *xder()* instead of the method `exterior_derivative()`:

```
sage: from sage.manifolds.utilities import xder
sage: xder(f) is f.exterior_derivative()
True
```

Differential computed on a chart that is not the default one:

```
sage: c_uvw.<u,v,w> = M.chart()
sage: g = M.scalar_field(u*v^2*w^3, c_uvw, name='g')
sage: dg = g.differential() ; dg
1-form dg on the 3-dimensional differentiable manifold M
sage: dg._components
{Coordinate frame (M, (d/du,d/dv,d/dw)): 1-index components w.r.t.
 Coordinate frame (M, (d/du,d/dv,d/dw))}
sage: dg.comp(c_uvw.frame())[:, c_uvw]
[v^2*w^3, 2*u*v*w^3, 3*u*v^2*w^2]
sage: dg.display(c_uvw)
dg = v^2*w^3 du + 2*u*v*w^3 dv + 3*u*v^2*w^2 dw
```

The exterior derivative is nilpotent:

```
sage: ddf = df.exterior_derivative() ; ddf
2-form ddf on the 3-dimensional differentiable manifold M
sage: ddf == 0
True
sage: ddf[:] # for the incredule
[0 0 0]
[0 0 0]
[0 0 0]
sage: ddg = dg.exterior_derivative() ; ddg
2-form ddg on the 3-dimensional differentiable manifold M
sage: ddg == 0
True
```

exterior_derivative()

Return the differential of `self`.

OUTPUT:

- a *DiffForm* (or of *DiffFormParal* if the scalar field's domain is parallelizable) representing the 1-form that is the differential of the scalar field

EXAMPLES:

Differential of a scalar field on a 3-dimensional differentiable manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()
sage: f = M.scalar_field(cos(x)*z^3 + exp(y)*z^2, name='f')
sage: df = f.differential() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
sage: df.parent()
Free module Omega^1(M) of 1-forms on the 3-dimensional
 differentiable manifold M
```

The result is cached, i.e. is not recomputed unless `f` is changed:

```
sage: f.differential() is df
True
```

Since the exterior derivative of a scalar field (considered a 0-form) is nothing but its differential, `exterior_derivative()` is an alias of `differential()`:

```
sage: df = f.exterior_derivative() ; df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = -z^3*sin(x) dx + z^2*e^y dy + (3*z^2*cos(x) + 2*z*e^y) dz
sage: latex(df)
\mathrm{d}f
```

One may also use the function *exterior_derivative()* or its alias *xder()* instead of the method `exterior_derivative()`:

```
sage: from sage.manifolds.utilities import xder
sage: xder(f) is f.exterior_derivative()
True
```

Differential computed on a chart that is not the default one:

```
sage: c_uvw.<u,v,w> = M.chart()
sage: g = M.scalar_field(u*v^2*w^3, c_uvw, name='g')
sage: dg = g.differential() ; dg
1-form dg on the 3-dimensional differentiable manifold M
sage: dg._components
{Coordinate frame (M, (d/du,d/dv,d/dw)): 1-index components w.r.t.
 Coordinate frame (M, (d/du,d/dv,d/dw))}
sage: dg.comp(c_uvw.frame())[:, c_uvw]
[v^2*w^3, 2*u*v*w^3, 3*u*v^2*w^2]
sage: dg.display(c_uvw)
dg = v^2*w^3 du + 2*u*v*w^3 dv + 3*u*v^2*w^2 dw
```

The exterior derivative is nilpotent:

```
sage: ddf = df.exterior_derivative() ; ddf
2-form ddf on the 3-dimensional differentiable manifold M
sage: ddf == 0
True
sage: ddf[:] # for the incredule
[0 0 0]
[0 0 0]
[0 0 0]
sage: ddg = dg.exterior_derivative() ; ddg
2-form ddg on the 3-dimensional differentiable manifold M
sage: ddg == 0
True
```

gradient (*metric=None*)

Return the gradient of `self` (with respect to a given metric).

The *gradient* of a scalar field f with respect to a metric g is the vector field $\mathrm{grad}\, f$ whose components in any coordinate frame are

$$(\mathrm{grad}\, f)^i = g^{ij} \frac{\partial F}{\partial x^j}$$

where the x^j's are the coordinates with respect to which the frame is defined and F is the chart function representing f in these coordinates: $f(p) = F(x^1(p), \ldots, x^n(p))$ for any point p in the chart domain. In other words, the gradient of f is the vector field that is the g-dual of the differential of f.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the gradient; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the gradient

OUTPUT:

- instance of *VectorField* representing the gradient of `self`

EXAMPLES:

Gradient of a scalar field in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: f = M.scalar_field(cos(x*y), name='f')
sage: v = f.gradient(); v
```

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```
Vector field grad(f) on the Euclidean plane E^2
sage: v.display()
grad(f) = -y*sin(x*y) e_x - x*sin(x*y) e_y
sage: v[:]
[-y*sin(x*y), -x*sin(x*y)]
```

Gradient in polar coordinates:

```
sage: M.<r,phi> = EuclideanSpace(coordinates='polar')
sage: f = M.scalar_field(r*cos(phi), name='f')
sage: f.gradient().display()
grad(f) = cos(phi) e_r - sin(phi) e_phi
sage: f.gradient()[:]
[cos(phi), -sin(phi)]
```

Note that `(e_r, e_phi)` is the orthonormal vector frame associated with polar coordinates (see `polar_frame()`); the gradient expressed in the coordinate frame is:

```
sage: f.gradient().display(M.polar_coordinates().frame())
grad(f) = cos(phi) d/dr - sin(phi)/r d/dphi
```

The function `grad()` from the `operators` module can be used instead of the method `gradient()`:

```
sage: from sage.manifolds.operators import grad
sage: grad(f) == f.gradient()
True
```

The gradient can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1/(1+r^2)
sage: h.display(M.polar_coordinates().frame())
h = -dr*dr + r^2/(r^2 + 1) dphi*dphi
sage: v = f.gradient(h); v
Vector field grad_h(f) on the Euclidean plane E^2
sage: v.display()
grad_h(f) = -cos(phi) e_r + (-r^2*sin(phi) - sin(phi)) e_phi
```

hodge_dual(*metric*)

Compute the Hodge dual of the scalar field with respect to some metric.

If M is the domain of the scalar field (denoted by f), n is the dimension of M and g is a pseudo-Riemannian metric on M, the *Hodge dual* of f w.r.t. g is the n-form $*f$ defined by

$$*f = f\epsilon,$$

where ϵ is the volume n-form associated with g (see `volume_form()`).

INPUT:

- `metric` – a pseudo-Riemannian metric defined on the same manifold as the current scalar field; must be an instance of `PseudoRiemannianMetric`

OUTPUT:

- the n-form $*f$

EXAMPLES:

Hodge dual of a scalar field in the Euclidean space R^3:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: f = M.scalar_field(function('F')(x,y,z), name='f')
sage: sf = f.hodge_dual(g) ; sf
3-form *f on the 3-dimensional differentiable manifold M
sage: sf.display()
*f = F(x, y, z) dx/\dy/\dz
sage: ssf = sf.hodge_dual(g) ; ssf
Scalar field **f on the 3-dimensional differentiable manifold M
sage: ssf.display()
**f: M --> R
   (x, y, z) |--> F(x, y, z)
sage: ssf == f # must hold for a Riemannian metric
True
```

Instead of calling the method `hodge_dual()` on the scalar field, one can invoke the method `hodge_star()` of the metric:

```
sage: f.hodge_dual(g) == g.hodge_star(f)
True
```

laplacian(*metric=None*)

Return the Laplacian of `self` with respect to a given metric (Laplace-Beltrami operator).

The *Laplacian* of a scalar field f with respect to a metric g is the scalar field

$$\Delta f = g^{ij} \nabla_i \nabla_j f = \nabla_i \nabla^i f$$

where ∇ is the Levi-Civita connection of g. Δ is also called the *Laplace-Beltrami operator*.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the Laplacian; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the Laplacian

OUTPUT:

- instance of `DiffScalarField` representing the Laplacian of `self`

EXAMPLES:

Laplacian of a scalar field on the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: f = M.scalar_field(function('F')(x,y), name='f')
sage: s = f.laplacian(); s
Scalar field Delta(f) on the Euclidean plane E^2
sage: s.display()
Delta(f): E^2 --> R
   (x, y) |--> d^2(F)/dx^2 + d^2(F)/dy^2
```

The function `laplacian()` from the `operators` module can be used instead of the method `laplacian()`:

```
sage: from sage.manifolds.operators import laplacian
sage: laplacian(f) == s
True
```

The Laplacian can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1/(1+x^2+y^2)
sage: s = f.laplacian(h); s
Scalar field Delta_h(f) on the Euclidean plane E^2
sage: s.display()
Delta_h(f): E^2 --> R
   (x, y) |--> (y^4*d^2(F)/dy^2 + y^3*d(F)/dy
   + (2*(x^2 + 1)*d^2(F)/dy^2 - d^2(F)/dx^2)*y^2
   + (x^2 + 1)*y*d(F)/dy + x*d(F)/dx - (x^2 + 1)*d^2(F)/dx^2
   + (x^4 + 2*x^2 + 1)*d^2(F)/dy^2)/(x^2 + y^2 + 1)
```

The Laplacian of f is equal to the divergence of the gradient of f:

$$\Delta f = \mathrm{div}(\mathrm{grad}\, f)$$

Let us check this formula:

```
sage: s == f.gradient(h).div(h)
True
```

lie_der(*vector*)

Compute the Lie derivative with respect to a vector field.

In the present case (scalar field), the Lie derivative is equal to the scalar field resulting from the action of the vector field on the scalar field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the scalar field that is the Lie derivative of the scalar field with respect to `vector`

EXAMPLES:

Lie derivative on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x^2*cos(y))
sage: v = M.vector_field(name='v')
sage: v[:] = (-y, x)
sage: f.lie_derivative(v)
Scalar field on the 2-dimensional differentiable manifold M
sage: f.lie_derivative(v).expr()
-x^3*sin(y) - 2*x*y*cos(y)
```

The result is cached:

```
sage: f.lie_derivative(v) is f.lie_derivative(v)
True
```

An alias is `lie_der`:

```
sage: f.lie_der(v) is f.lie_derivative(v)
True
```

Alternative expressions of the Lie derivative of a scalar field:

```
sage: f.lie_der(v) == v(f)   # the vector acting on f
True
sage: f.lie_der(v) == f.differential()(v)   # the differential of f acting on␣
↪the vector
True
```

A vanishing Lie derivative:

```
sage: f.set_expr(x^2 + y^2)
sage: f.lie_der(v).display()
M --> R
(x, y) |--> 0
```

lie_derivative(*vector*)

Compute the Lie derivative with respect to a vector field.

In the present case (scalar field), the Lie derivative is equal to the scalar field resulting from the action of the vector field on the scalar field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the scalar field that is the Lie derivative of the scalar field with respect to `vector`

EXAMPLES:

Lie derivative on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x^2*cos(y))
sage: v = M.vector_field(name='v')
sage: v[:] = (-y, x)
sage: f.lie_derivative(v)
Scalar field on the 2-dimensional differentiable manifold M
sage: f.lie_derivative(v).expr()
-x^3*sin(y) - 2*x*y*cos(y)
```

The result is cached:

```
sage: f.lie_derivative(v) is f.lie_derivative(v)
True
```

An alias is `lie_der`:

```
sage: f.lie_der(v) is f.lie_derivative(v)
True
```

Alternative expressions of the Lie derivative of a scalar field:

```
sage: f.lie_der(v) == v(f)  # the vector acting on f
True
sage: f.lie_der(v) == f.differential()(v)  # the differential of f acting on␣
↪the vector
True
```

A vanishing Lie derivative:

```
sage: f.set_expr(x^2 + y^2)
sage: f.lie_der(v).display()
M --> R
(x, y) |--> 0
```

tensor_type()

Return the tensor type of `self`, when the latter is considered as a tensor field on the manifold. This is always $(0, 0)$.

OUTPUT:

- always $(0, 0)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: f = M.scalar_field(x+2*y)
sage: f.tensor_type()
(0, 0)
```

wedge(*other*)

Return the exterior product of `self`, considered as a differential form of degree 0 or a multivector field of degree 0, with `other`.

See `wedge()` (exterior product of differential forms) or `wedge()` (exterior product of multivector fields) for details.

For a scalar field f and a p-form (or p-vector field) a, the exterior product reduces to the standard product on the left by an element of the base ring of the module of p-forms (or p-vector fields): $f \wedge a = fa$.

INPUT:

- `other` – a differential form or a multivector field a

OUTPUT:

- the product fa, where f is `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: f = M.scalar_field({X: x+y^2}, name='f')
sage: a = M.diff_form(2, name='a')
sage: a[0,1] = x*y
sage: s = f.wedge(a); s
2-form on the 2-dimensional differentiable manifold M
sage: s.display()
(x*y^3 + x^2*y) dx/\dy
```

2.5 Differentiable Maps and Curves

2.5.1 Sets of Morphisms between Differentiable Manifolds

The class *DifferentiableManifoldHomset* implements sets of morphisms between two differentiable manifolds over the same topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$), a morphism being a *differentiable map* for the category of differentiable manifolds.

The subclass *DifferentiableCurveSet* is devoted to the specific case of differential curves, i.e. morphisms whose domain is an open interval of $\mathbf{R}$.

The subclass *IntegratedCurveSet* is devoted to differentiable curves that are defined as a solution to a system of second order differential equations.

The subclass *IntegratedAutoparallelCurveSet* is devoted to differentiable curves that are defined as autoparallel curves with respect to a certain affine connection.

The subclass *IntegratedGeodesicSet* is devoted to differentiable curves that are defined as geodesics with respect to to a certain metric.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks
- Karim Van Aelst (2017): sets of integrated curves

REFERENCES:

- [?]
- [?]

class sage.manifolds.differentiable.manifold_homset.**DifferentiableCurveSet**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: *sage.manifolds.differentiable.manifold_homset.DifferentiableManifoldHomset*

Set of differentiable curves in a differentiable manifold.

Given an open interval I of $\mathbf{R}$ (possibly $I = \mathbf{R}$) and a differentiable manifold M over $\mathbf{R}$, this is the set $\mathrm{Hom}(I, M)$ of morphisms (i.e. differentiable curves) $I \to M$.

INPUT:

- `domain` – *OpenInterval* if an open interval $I \subset \mathbf{R}$ (domain of the morphisms), or *RealLine* if $I = \mathbf{R}$
- `codomain` – *DifferentiableManifold*; differentiable manifold M (codomain of the morphisms)
- `name` – (default: `None`) string; name given to the set of curves; if `None`, `Hom(I, M)` will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the set of curves; if `None`, $\mathrm{Hom}(I, M)$ will be used

EXAMPLES:

Set of curves $\mathbf{R} \longrightarrow M$, where M is a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = RealLine() ; R
Real number line R
sage: H = Hom(R, M) ; H
Set of Morphisms from Real number line R to 2-dimensional
 differentiable manifold M in Category of smooth manifolds over Real
 Field with 53 bits of precision
sage: H.category()
Category of homsets of topological spaces
sage: latex(H)
\mathrm{Hom}\left(\Bold{R},M\right)
sage: H.domain()
Real number line R
sage: H.codomain()
2-dimensional differentiable manifold M
```

An element of H is a curve in M:

```
sage: c = H.an_element(); c
Curve in the 2-dimensional differentiable manifold M
sage: c.display()
R --> M
   t |--> (x, y) = (1/(t^2 + 1) - 1/2, 0)
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

The set of curves $(0,1) \longrightarrow U$, where U is an open subset of M:

```
sage: I = R.open_interval(0, 1) ; I
Real interval (0, 1)
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1}) ; U
Open subset U of the 2-dimensional differentiable manifold M
sage: H = Hom(I, U) ; H
Set of Morphisms from Real interval (0, 1) to Open subset U of the
 2-dimensional differentiable manifold M in Join of Category of
 subobjects of sets and Category of smooth manifolds over Real Field
 with 53 bits of precision
```

An element of H is a curve in U:

```
sage: c = H.an_element() ; c
Curve in the Open subset U of the 2-dimensional differentiable
 manifold M
sage: c.display()
(0, 1) --> U
   t |--> (x, y) = (1/(t^2 + 1) - 1/2, 0)
```

The set of curves $\mathbf{R} \longrightarrow \mathbf{R}$ is a set of (manifold) endomorphisms:

```
sage: E = Hom(R, R) ; E
Set of Morphisms from Real number line R to Real number line R in
 Category of smooth manifolds over Real Field with 53 bits of precision
sage: E.category()
Category of endsets of topological spaces
```

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```
sage: E.is_endomorphism_set()
True
sage: E is End(R)
True
```

It is a monoid for the law of morphism composition:

```
sage: E in Monoids()
True
```

The identity element of the monoid is the identity map of $\mathbf{R}$:

```
sage: E.one()
Identity map Id_R of the Real number line R
sage: E.one() is R.identity_map()
True
sage: E.one().display()
Id_R: R --> R
   t |--> t
```

A "typical" element of the monoid:

```
sage: E.an_element().display()
R --> R
   t |--> 1/(t^2 + 1) - 1/2
```

The test suite is passed by `E`:

```
sage: TestSuite(E).run()
```

Similarly, the set of curves $I \longrightarrow I$ is a monoid, whose elements are (manifold) endomorphisms:

```
sage: EI = Hom(I, I) ; EI
Set of Morphisms from Real interval (0, 1) to Real interval (0, 1) in
 Join of Category of subobjects of sets and
     Category of smooth manifolds over Real Field with 53 bits of precision
sage: EI.category()
Category of endsets of subobjects of sets and topological spaces
sage: EI is End(I)
True
sage: EI in Monoids()
True
```

The identity element and a "typical" element of this monoid:

```
sage: EI.one()
Identity map Id_(0, 1) of the Real interval (0, 1)
sage: EI.one().display()
Id_(0, 1): (0, 1) --> (0, 1)
   t |--> t
sage: EI.an_element().display()
(0, 1) --> (0, 1)
   t |--> 1/2/(t^2 + 1) + 1/4
```

The test suite is passed by `EI`:

```
sage: TestSuite(EI).run()
```

Element
alias of `sage.manifolds.differentiable.curve.DifferentiableCurve`

class sage.manifolds.differentiable.manifold_homset.**DifferentiableManifoldHomset**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: `sage.manifolds.manifold_homset.TopologicalManifoldHomset`

Set of differentiable maps between two differentiable manifolds.

Given two differentiable manifolds M and N over a topological field K, the class `DifferentiableManifoldHomset` implements the set $\mathrm{Hom}(M, N)$ of morphisms (i.e. differentiable maps) $M \to N$.

This is a Sage *parent* class, whose *element* class is `DiffMap`.

INPUT:

- `domain` – differentiable manifold M (domain of the morphisms), as an instance of `DifferentiableManifold`
- `codomain` – differentiable manifold N (codomain of the morphisms), as an instance of `DifferentiableManifold`
- `name` – (default: `None`) string; name given to the homset; if `None`, Hom(M,N) will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the homset; if `None`, $\mathrm{Hom}(M, N)$ will be used

EXAMPLES:

Set of differentiable maps between a 2-dimensional differentiable manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: N = Manifold(3, 'N')
sage: Y.<u,v,w> = N.chart()
sage: H = Hom(M, N) ; H
Set of Morphisms from 2-dimensional differentiable manifold M to
 3-dimensional differentiable manifold N in Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: type(H)
<class 'sage.manifolds.differentiable.manifold_homset.
↪DifferentiableManifoldHomset_with_category'>
sage: H.category()
Category of homsets of topological spaces
sage: latex(H)
\mathrm{Hom}\left(M,N\right)
sage: H.domain()
2-dimensional differentiable manifold M
sage: H.codomain()
3-dimensional differentiable manifold N
```

An element of `H` is a differentiable map from `M` to `N`:

```
sage: H.Element
<class 'sage.manifolds.differentiable.diff_map.DiffMap'>
sage: f = H.an_element() ; f
Differentiable map from the 2-dimensional differentiable manifold M to the
 3-dimensional differentiable manifold N
sage: f.display()
M --> N
   (x, y) |--> (u, v, w) = (0, 0, 0)
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

When the codomain coincides with the domain, the homset is a set of *endomorphisms* in the category of differentiable manifolds:

```
sage: E = Hom(M, M) ; E
Set of Morphisms from 2-dimensional differentiable manifold M to
 2-dimensional differentiable manifold M in Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: E.category()
Category of endsets of topological spaces
sage: E.is_endomorphism_set()
True
sage: E is End(M)
True
```

In this case, the homset is a monoid for the law of morphism composition:

```
sage: E in Monoids()
True
```

This was of course not the case for `H = Hom(M, N)`:

```
sage: H in Monoids()
False
```

The identity element of the monoid is of course the identity map of `M`:

```
sage: E.one()
Identity map Id_M of the 2-dimensional differentiable manifold M
sage: E.one() is M.identity_map()
True
sage: E.one().display()
Id_M: M --> M
   (x, y) |--> (x, y)
```

The test suite is passed by `E`:

```
sage: TestSuite(E).run()
```

This test suite includes more tests than in the case of `H`, since `E` has some extra structure (monoid).

Element
alias of *sage.manifolds.differentiable.diff_map.DiffMap*

class sage.manifolds.differentiable.manifold_homset.**IntegratedAutoparallelCurveSet**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: *sage.manifolds.differentiable.manifold_homset.IntegratedCurveSet*

Set of integrated autoparallel curves in a differentiable manifold.

INPUT:

- domain – *OpenInterval* open interval $I \subset \mathbf{R}$ with finite boundaries (domain of the morphisms)
- codomain – *DifferentiableManifold*; differentiable manifold M (codomain of the morphisms)
- name – (default: None) string; name given to the set of integrated autoparallel curves; if None, Hom_autoparallel(I, M) will be used
- latex_name – (default: None) string; LaTeX symbol to denote the set of integrated autoparallel curves; if None, $\mathrm{Hom}_{\mathrm{autoparallel}}(I, M)$ will be used

EXAMPLES:

This parent class needs to be imported:

```
sage: from sage.manifolds.differentiable.manifold_homset import␣
↪IntegratedAutoparallelCurveSet
```

Integrated autoparallel curves are only allowed to be defined on an interval with finite bounds. This forbids to define an instance of this parent class whose domain has infinite bounds:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = RealLine()
sage: H = IntegratedAutoparallelCurveSet(R, M)
Traceback (most recent call last):
...
ValueError: both boundaries of the interval defining the domain
 of a Homset of integrated autoparallel curves need to be finite
```

An instance whose domain is an interval with finite bounds allows to build a curve that is autoparallel with respect to a connection defined on the codomain:

```
sage: I = R.open_interval(-1, 2)
sage: H = IntegratedAutoparallelCurveSet(I, M) ; H
Set of Morphisms from Real interval (-1, 2) to 2-dimensional
 differentiable manifold M in Category of homsets of subobjects
 of sets and topological spaces which actually are integrated
 autoparallel curves with respect to a certain affine connection
sage: nab = M.affine_connection('nabla')
sage: nab[0,1,0], nab[0,0,1] = 1,2
sage: nab.torsion()[:]
[[[0, -1], [1, 0]], [[0, 0], [0, 0]]]
sage: t = var('t')
sage: p = M.point((3,4))
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,2))
sage: c = H(nab, t, v, name='c') ; c
Integrated autoparallel curve c in the 2-dimensional
 differentiable manifold M
```

A "typical" element of `H` is an autoparallel curve in `M`:

```
sage: d = H.an_element(); d
Integrated autoparallel curve in the 2-dimensional
 differentiable manifold M
sage: sys = d.system(verbose=True)
Autoparallel curve in the 2-dimensional differentiable manifold
 M equipped with Affine connection nab on the 2-dimensional
 differentiable manifold M, and integrated over the Real
 interval (-1, 2) as a solution to the following equations,
 written with respect to Chart (M, (x, y)):

Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, -1/2] with respect to
 Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [-1/6/(e^(-1) - 1), 1/3] with respect to Chart (M, (x, y))

d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -Dx*Dy
d(Dy)/dt = 0
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

For any open interval J with finite bounds (a, b), all curves are autoparallel with respect to any connection. Therefore, the set of autoparallel curves $J \longrightarrow J$ is a set of numerical (manifold) endomorphisms that is a monoid for the law of morphism composition:

```
sage: [a,b] = var('a b')
sage: J = R.open_interval(a, b)
sage: H = IntegratedAutoparallelCurveSet(J, J); H
Set of Morphisms from Real interval (a, b) to Real interval
 (a, b) in Category of endsets of subobjects of sets and
 topological spaces which actually are integrated autoparallel
 curves with respect to a certain affine connection
sage: H.category()
Category of endsets of subobjects of sets and topological spaces
sage: H in Monoids()
True
```

Although it is a monoid, no identity map is implemented via the `one` method of this class or its subclass devoted to geodesics. This is justified by the lack of relevance of the identity map within the framework of this parent class and its subclass, whose purpose is mainly devoted to numerical issues (therefore, the user is left free to set a numerical version of the identity if needed):

```
sage: H.one()
Traceback (most recent call last):
...
ValueError: the identity is not implemented for integrated
 curves and associated subclasses
```

A "typical" element of the monoid:

```
sage: g = H.an_element() ; g
Integrated autoparallel curve in the Real interval (a, b)
sage: sys = g.system(verbose=True)
Autoparallel curve in the Real interval (a, b) equipped with
 Affine connection nab on the Real interval (a, b), and
 integrated over the Real interval (a, b) as a solution to the
 following equations, written with respect to Chart ((a, b), (t,)):

Initial point: Point on the Real number line R with coordinates
 [0] with respect to Chart ((a, b), (t,))
Initial tangent vector: Tangent vector at Point on the Real
 number line R with components
 [-(e^(1/2) - 1)/(a - b)] with respect to
 Chart ((a, b), (t,))

d(t)/ds = Dt
d(Dt)/ds = -Dt^2
```

The test suite is passed, tests `_test_one` and `_test_prod` being skipped for reasons mentioned above:

```
sage: TestSuite(H).run(skip=["_test_one", "_test_prod"])
```

Element

alias of `sage.manifolds.differentiable.integrated_curve.IntegratedAutoparallelCurve`

class `sage.manifolds.differentiable.manifold_homset.`**`IntegratedCurveSet`**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: `sage.manifolds.differentiable.manifold_homset.DifferentiableCurveSet`

Set of integrated curves in a differentiable manifold.

INPUT:

- `domain` – `OpenInterval` open interval $I \subset \mathbf{R}$ with finite boundaries (domain of the morphisms)
- `codomain` – `DifferentiableManifold`; differentiable manifold M (codomain of the morphisms)
- `name` – (default: `None`) string; name given to the set of integrated curves; if `None`, `Hom_integrated(I, M)` will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the set of integrated curves; if `None`, $\mathrm{Hom_{integrated}}(I, M)$ will be used

EXAMPLES:

This parent class needs to be imported:

```
sage: from sage.manifolds.differentiable.manifold_homset import IntegratedCurveSet
```

Integrated curves are only allowed to be defined on an interval with finite bounds. This forbids to define an instance of this parent class whose domain has infinite bounds:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = RealLine()
sage: H = IntegratedCurveSet(R, M)
```

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```
Traceback (most recent call last):
...
ValueError: both boundaries of the interval defining the domain
 of a Homset of integrated curves need to be finite
```

An instance whose domain is an interval with finite bounds allows to build an integrated curve defined on the interval:

```
sage: I = R.open_interval(-1, 2)
sage: H = IntegratedCurveSet(I, M) ; H
Set of Morphisms from Real interval (-1, 2) to 2-dimensional
 differentiable manifold M in Category of homsets of subobjects
 of sets and topological spaces which actually are integrated
 curves
sage: eqns_rhs = [1,1]
sage: vels = X.symbolic_velocities()
sage: t = var('t')
sage: p = M.point((3,4))
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,2))
sage: c = H(eqns_rhs, vels, t, v, name='c') ; c
Integrated curve c in the 2-dimensional differentiable
 manifold M
```

A "typical" element of `H` is a curve in `M`:

```
sage: d = H.an_element(); d
Integrated curve in the 2-dimensional differentiable manifold M
sage: sys = d.system(verbose=True)
Curve in the 2-dimensional differentiable manifold M integrated
 over the Real interval (-1, 2) as a solution to the following
 system, written with respect to Chart (M, (x, y)):

Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, 0] with respect to Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [1/4, 0] with respect to Chart (M, (x, y))

d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -1/4*sin(t + 1)
d(Dy)/dt = 0
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

More generally, an instance of this class may be defined with abstract bounds (a, b):

```
sage: [a,b] = var('a b')
sage: J = R.open_interval(a, b)
sage: H = IntegratedCurveSet(J, M) ; H
Set of Morphisms from Real interval (a, b) to 2-dimensional
 differentiable manifold M in Category of homsets of subobjects
 of sets and topological spaces which actually are integrated
 curves
```

A "typical" element of H is a curve in M:

```
sage: f = H.an_element(); f
Integrated curve in the 2-dimensional differentiable manifold M
sage: sys = f.system(verbose=True)
Curve in the 2-dimensional differentiable manifold M integrated
 over the Real interval (a, b) as a solution to the following
 system, written with respect to Chart (M, (x, y)):

Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, 0] with respect to Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [1/4, 0] with respect to Chart (M, (x, y))

d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -1/4*sin(-a + t)
d(Dy)/dt = 0
```

Yet, even in the case of abstract bounds, considering any of them to be infinite is still prohibited since no numerical integration could be performed:

```
sage: f.solve(parameters_values={a:-1, b:+oo})
Traceback (most recent call last):
...
ValueError: both boundaries of the interval need to be finite
```

The set of integrated curves $J \longrightarrow J$ is a set of numerical (manifold) endomorphisms:

```
sage: H = IntegratedCurveSet(J, J); H
Set of Morphisms from Real interval (a, b) to Real interval
 (a, b) in Category of endsets of subobjects of sets and
 topological spaces which actually are integrated curves
sage: H.category()
Category of endsets of subobjects of sets and topological spaces
```

It is a monoid for the law of morphism composition:

```
sage: H in Monoids()
True
```

Although it is a monoid, no identity map is implemented via the one method of this class or any of its subclasses. This is justified by the lack of relevance of the identity map within the framework of this parent class and its subclasses, whose purpose is mainly devoted to numerical issues (therefore, the user is left free to set a numerical version of the identity if needed):

```
sage: H.one()
Traceback (most recent call last):
...
ValueError: the identity is not implemented for integrated
 curves and associated subclasses
```

A "typical" element of the monoid:

```
sage: g = H.an_element() ; g
Integrated curve in the Real interval (a, b)
```

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```
sage: sys = g.system(verbose=True)
Curve in the Real interval (a, b) integrated over the Real
 interval (a, b) as a solution to the following system, written
 with respect to Chart ((a, b), (t,)):

Initial point: Point on the Real number line R with coordinates
 [0] with respect to Chart ((a, b), (t,))
Initial tangent vector: Tangent vector at Point on the Real
 number line R with components [1/4] with respect to
 Chart ((a, b), (t,))

d(t)/ds = Dt
d(Dt)/ds = -1/4*sin(-a + s)
```

The test suite is passed, tests `_test_one` and `_test_prod` being skipped for reasons mentioned above:

```
sage: TestSuite(H).run(skip=["_test_one", "_test_prod"])
```

Element
alias of *sage.manifolds.differentiable.integrated_curve.IntegratedCurve*

one()
Raise an error refusing to provide the identity element. This overrides the `one` method of class *TopologicalManifoldHomset*, which would actually raise an error as well, due to lack of option `is_identity` in `element_constructor` method of `self`.

class sage.manifolds.differentiable.manifold_homset.**IntegratedGeodesicSet**(*domain*, *codomain*, *name=None*, *latex_name=None*)

Bases: *sage.manifolds.differentiable.manifold_homset.IntegratedAutoparallelCurveSet*

Set of integrated geodesic in a differentiable manifold.

INPUT:

- `domain` – *OpenInterval* open interval $I \subset \mathbf{R}$ with finite boundaries (domain of the morphisms)
- `codomain` – *DifferentiableManifold*; differentiable manifold M (codomain of the morphisms)
- `name` – (default: `None`) string; name given to the set of integrated geodesics; if `None`, `Hom_geodesic(I, M)` will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the set of integrated geodesics; if `None`, $\mathrm{Hom}_{\mathrm{geodesic}}(I, M)$ will be used

EXAMPLES:

This parent class needs to be imported:

```
sage: from sage.manifolds.differentiable.manifold_homset import␣
↪IntegratedGeodesicSet
```

Integrated geodesics are only allowed to be defined on an interval with finite bounds. This forbids to define an instance of this parent class whose domain has infinite bounds:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R.<t> = RealLine()
sage: H = IntegratedGeodesicSet(R, M)
Traceback (most recent call last):
...
ValueError: both boundaries of the interval defining the domain
 of a Homset of integrated geodesics need to be finite
```

An instance whose domain is an interval with finite bounds allows to build a geodesic with respect to a metric defined on the codomain:

```
sage: I = R.open_interval(-1, 2)
sage: H = IntegratedGeodesicSet(I, M) ; H
Set of Morphisms from Real interval (-1, 2) to 2-dimensional
 differentiable manifold M in Category of homsets of subobjects
 of sets and topological spaces which actually are integrated
 geodesics with respect to a certain metric
sage: g = M.metric('g')
sage: g[0,0], g[1,1], g[0,1] = 1, 1, 2
sage: t = var('t')
sage: p = M.point((3,4))
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,2))
sage: c = H(g, t, v, name='c') ; c
Integrated geodesic c in the 2-dimensional differentiable
 manifold M
```

A "typical" element of `H` is a geodesic in `M`:

```
sage: d = H.an_element(); d
Integrated geodesic in the 2-dimensional differentiable
 manifold M
sage: sys = d.system(verbose=True)
Geodesic in the 2-dimensional differentiable manifold M equipped
 with Riemannian metric g on the 2-dimensional differentiable
 manifold M, and integrated over the Real interval (-1, 2) as a
 solution to the following geodesic equations, written
 with respect to Chart (M, (x, y)):

Initial point: Point on the 2-dimensional differentiable
 manifold M with coordinates [0, 0] with respect to
 Chart (M, (x, y))
Initial tangent vector: Tangent vector at Point on the
 2-dimensional differentiable manifold M with components
 [1/3*e^(1/2) - 1/3, 0] with respect to Chart (M, (x, y))

d(x)/dt = Dx
d(y)/dt = Dy
d(Dx)/dt = -Dx^2
d(Dy)/dt = 0
```

The test suite is passed:

```
sage: TestSuite(H).run()
```

For any open interval J with finite bounds (a, b), all curves are geodesics with respect to any metric. Therefore, the set of geodesics $J \longrightarrow J$ is a set of numerical (manifold) endomorphisms that is a monoid for the law of morphism composition:

```
sage: [a,b] = var('a b')
sage: J = R.open_interval(a, b)
sage: H = IntegratedGeodesicSet(J, J); H
Set of Morphisms from Real interval (a, b) to Real interval
 (a, b) in Category of endsets of subobjects of sets and
 topological spaces which actually are integrated geodesics
 with respect to a certain metric
sage: H.category()
Category of endsets of subobjects of sets and topological spaces
sage: H in Monoids()
True
```

Although it is a monoid, no identity map is implemented via the `one` method of this class. This is justified by the lack of relevance of the identity map within the framework of this parent class, whose purpose is mainly devoted to numerical issues (therefore, the user is left free to set a numerical version of the identity if needed):

```
sage: H.one()
Traceback (most recent call last):
...
ValueError: the identity is not implemented for integrated
 curves and associated subclasses
```

A "typical" element of the monoid:

```
sage: g = H.an_element() ; g
Integrated geodesic in the Real interval (a, b)
sage: sys = g.system(verbose=True)
Geodesic in the Real interval (a, b) equipped with Riemannian
 metric g on the Real interval (a, b), and integrated over the
 Real interval (a, b) as a solution to the following geodesic
 equations, written with respect to Chart ((a, b), (t,)):

Initial point: Point on the Real number line R with coordinates
 [0] with respect to Chart ((a, b), (t,))
Initial tangent vector: Tangent vector at Point on the Real
 number line R with components [-(e^(1/2) - 1)/(a - b)]
 with respect to Chart ((a, b), (t,))

d(t)/ds = Dt
d(Dt)/ds = -Dt^2
```

The test suite is passed, tests `_test_one` and `_test_prod` being skipped for reasons mentioned above:

```
sage: TestSuite(H).run(skip=["_test_one", "_test_prod"])
```

Element

alias of *sage.manifolds.differentiable.integrated_curve.IntegratedGeodesic*

2.5.2 Differentiable Maps between Differentiable Manifolds

The class *DiffMap* implements differentiable maps from a differentiable manifold M to a differentiable manifold N over the same topological field K as M (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$):

$$\Phi : M \longrightarrow N$$

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version

REFERENCES:

- Chap. 1 of [**?**]
- Chaps. 2 and 3 of [**?**]

class sage.manifolds.differentiable.diff_map.**DiffMap**(*parent*, *coord_functions=None*, *name=None*, *latex_name=None*, *is_isomorphism=False*, *is_identity=False*)

Bases: *sage.manifolds.continuous_map.ContinuousMap*

Differentiable map between two differentiable manifolds.

This class implements differentiable maps of the type

$$\Phi : M \longrightarrow N$$

where M and N are differentiable manifolds over the same topological field K (in most applications, $K = \mathbf{R}$ or $K = \mathbf{C}$).

Differentiable maps are the *morphisms* of the *category* of differentiable manifolds. The set of all differentiable maps from M to N is therefore the homset between M and N, which is denoted by $\mathrm{Hom}(M, N)$.

The class *DiffMap* is a Sage *element* class, whose *parent* class is *DifferentiableManifoldHomset*. It inherits from the class *ContinuousMap* since a differentiable map is obviously a continuous one.

INPUT:

- `parent` – homset $\mathrm{Hom}(M, N)$ to which the differentiable map belongs
- `coord_functions` – (default: `None`) if not `None`, must be a dictionary of the coordinate expressions (as lists (or tuples) of the coordinates of the image expressed in terms of the coordinates of the considered point) with the pairs of charts (chart1, chart2) as keys (chart1 being a chart on M and chart2 a chart on N). If the dimension of the map's codomain is 1, a single coordinate expression can be passed instead of a tuple with a single element
- `name` – (default: `None`) name given to the differentiable map
- `latex_name` – (default: `None`) LaTeX symbol to denote the differentiable map; if `None`, the LaTeX symbol is set to `name`
- `is_isomorphism` – (default: `False`) determines whether the constructed object is a isomorphism (i.e. a diffeomorphism); if set to `True`, then the manifolds M and N must have the same dimension.
- `is_identity` – (default: `False`) determines whether the constructed object is the identity map; if set to `True`, then N must be M and the entry `coord_functions` is not used.

Note: If the information passed by means of the argument `coord_functions` is not sufficient to fully specify the differentiable map, further coordinate expressions, in other charts, can be subsequently added by means of the method *add_expr()*

EXAMPLES:

The standard embedding of the sphere S^2 into $\mathbf{R}^3$:

```
sage: M = Manifold(2, 'S^2') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                  intersection_name='W', restrictions1= x^2+y^2!=0,
....:                  restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: N = Manifold(3, 'R^3', r'\RR^3')  # R^3
sage: c_cart.<X,Y,Z> = N.chart()  # Cartesian coordinates on R^3
sage: Phi = M.diff_map(N,
....: {(c_xy, c_cart): [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2), (x^2+y^2-1)/(1+x^2+y^
↪2)],
....:  (c_uv, c_cart): [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2), (1-u^2-v^2)/(1+u^2+v^
↪2)]},
....: name='Phi', latex_name=r'\Phi')
sage: Phi
Differentiable map Phi from the 2-dimensional differentiable manifold
 S^2 to the 3-dimensional differentiable manifold R^3
sage: Phi.parent()
Set of Morphisms from 2-dimensional differentiable manifold S^2 to
 3-dimensional differentiable manifold R^3 in Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(M, N)
True
sage: type(Phi)
<class 'sage.manifolds.differentiable.manifold_homset.
↪DifferentiableManifoldHomset_with_category.element_class'>
sage: Phi.display()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1),
 (x^2 + y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1),
 -(u^2 + v^2 - 1)/(u^2 + v^2 + 1))
```

It is possible to create the map via the method *diff_map()* only in a single pair of charts: the argument `coord_functions` is then a mere list of coordinate expressions (and not a dictionary) and the arguments `chart1` and `chart2` have to be provided if the charts differ from the default ones on the domain and/or the codomain:

```
sage: Phi1 = M.diff_map(N, [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2),
....:                        (x^2+y^2-1)/(1+x^2+y^2)],
....:                   chart1=c_xy, chart2=c_cart, name='Phi',
....:                   latex_name=r'\Phi')
```

Since `c_xy` and `c_cart` are the default charts on respectively `M` and `N`, they can be omitted, so that the above declaration is equivalent to:

```
sage: Phi1 = M.diff_map(N, [2*x/(1+x^2+y^2), 2*y/(1+x^2+y^2),
....:                        (x^2+y^2-1)/(1+x^2+y^2)],
....:                   name='Phi', latex_name=r'\Phi')
```

With such a declaration, the differentiable map is only partially defined on the manifold S^2, being known in only one chart:

```
sage: Phi1.display()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1),
 (x^2 + y^2 - 1)/(x^2 + y^2 + 1))
```

The definition can be completed by means of the method *add_expr()*:

```
sage: Phi1.add_expr(c_uv, c_cart, [2*u/(1+u^2+v^2), 2*v/(1+u^2+v^2),
....:                               (1-u^2-v^2)/(1+u^2+v^2)])
sage: Phi1.display()
Phi: S^2 --> R^3
on U: (x, y) |--> (X, Y, Z) = (2*x/(x^2 + y^2 + 1), 2*y/(x^2 + y^2 + 1),
 (x^2 + y^2 - 1)/(x^2 + y^2 + 1))
on V: (u, v) |--> (X, Y, Z) = (2*u/(u^2 + v^2 + 1), 2*v/(u^2 + v^2 + 1),
 -(u^2 + v^2 - 1)/(u^2 + v^2 + 1))
```

At this stage, `Phi1` and `Phi` are fully equivalent:

```
sage: Phi1 == Phi
True
```

The test suite is passed:

```
sage: TestSuite(Phi).run()
sage: TestSuite(Phi1).run()
```

The map acts on points:

```
sage: np = M.point((0,0), chart=c_uv, name='N')  # the North pole
sage: Phi(np)
Point Phi(N) on the 3-dimensional differentiable manifold R^3
sage: Phi(np).coord() # Cartesian coordinates
(0, 0, 1)
sage: sp = M.point((0,0), chart=c_xy, name='S')  # the South pole
sage: Phi(sp).coord() # Cartesian coordinates
(0, 0, -1)
```

The differential $\mathrm{d}\Phi$ of the map Φ at the North pole and at the South pole:

```
sage: Phi.differential(np)
Generic morphism:
  From: Tangent space at Point N on the 2-dimensional differentiable manifold S^2
  To:   Tangent space at Point Phi(N) on the 3-dimensional differentiable␣
↪manifold R^3
sage: Phi.differential(sp)
Generic morphism:
  From: Tangent space at Point S on the 2-dimensional differentiable manifold S^2
  To:   Tangent space at Point Phi(S) on the 3-dimensional differentiable␣
↪manifold R^3
```

The matrix of the linear map $\mathrm{d}\Phi_N$ with respect to the default bases of $T_N S^2$ and $T_{\Phi(N)}\mathbf{R}^3$:

```
sage: Phi.differential(np).matrix()
[2 0]
[0 2]
[0 0]
```

the default bases being:

```
sage: Phi.differential(np).domain().default_basis()
Basis (d/du,d/dv) on the Tangent space at Point N on the 2-dimensional
 differentiable manifold S^2
sage: Phi.differential(np).codomain().default_basis()
Basis (d/dX,d/dY,d/dZ) on the Tangent space at Point Phi(N) on the
 3-dimensional differentiable manifold R^3
```

Differentiable maps can be composed by means of the operator $*$: let us introduce the map $\mathbf{R}^3 \to \mathbf{R}^2$ corresponding to the projection from the point $(X, Y, Z) = (0, 0, 1)$ onto the equatorial plane $Z = 0$:

```
sage: P = Manifold(2, 'R^2', r'\RR^2') # R^2 (equatorial plane)
sage: cP.<xP, yP> = P.chart()
sage: Psi = N.diff_map(P, (X/(1-Z), Y/(1-Z)), name='Psi',
....:                  latex_name=r'\Psi')
sage: Psi
Differentiable map Psi from the 3-dimensional differentiable manifold
 R^3 to the 2-dimensional differentiable manifold R^2
sage: Psi.display()
Psi: R^3 --> R^2
   (X, Y, Z) |--> (xP, yP) = (-X/(Z - 1), -Y/(Z - 1))
```

Then we compose `Psi` with `Phi`, thereby getting a map $S^2 \to \mathbf{R}^2$:

```
sage: ster = Psi*Phi ; ster
Differentiable map from the 2-dimensional differentiable manifold S^2
 to the 2-dimensional differentiable manifold R^2
```

Let us test on the South pole (`sp`) that `ster` is indeed the composite of `Psi` and `Phi`:

```
sage: ster(sp) == Psi(Phi(sp))
True
```

Actually `ster` is the stereographic projection from the North pole, as its coordinate expression reveals:

```
sage: ster.display()
S^2 --> R^2
on U: (x, y) |--> (xP, yP) = (x, y)
on V: (u, v) |--> (xP, yP) = (u/(u^2 + v^2), v/(u^2 + v^2))
```

If its codomain is 1-dimensional, a differentiable map must be defined by a single symbolic expression for each pair of charts, and not by a list/tuple with a single element:

```
sage: N = Manifold(1, 'N')
sage: c_N = N.chart('X')
sage: Phi = M.diff_map(N, {(c_xy, c_N): x^2+y^2,
....: (c_uv, c_N): 1/(u^2+v^2)})  # not ...[1/(u^2+v^2)] or (1/(u^2+v^2),)
```

An example of differentiable map $\mathbf{R} \to \mathbf{R}^2$:

```
sage: R = Manifold(1, 'R')  # field R
sage: T.<t> = R.chart()  # canonical chart on R
sage: R2 = Manifold(2, 'R^2')  # R^2
sage: c_xy.<x,y> = R2.chart() # Cartesian coordinates on R^2
sage: Phi = R.diff_map(R2, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 2-dimensional differentiable manifold R^2
sage: Phi.parent()
```

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```
Set of Morphisms from 1-dimensional differentiable manifold R to
 2-dimensional differentiable manifold R^2 in Category of smooth
 manifolds over Real Field with 53 bits of precision
sage: Phi.parent() is Hom(R, R2)
True
sage: Phi.display()
Phi: R --> R^2
   t |--> (x, y) = (cos(t), sin(t))
```

An example of diffeomorphism between the unit open disk and the Euclidean plane $\mathbf{R}^2$:

```
sage: D = R2.open_subset('D', coord_def={c_xy: x^2+y^2<1}) # the open unit disk
sage: Phi = D.diffeomorphism(R2, [x/sqrt(1-x^2-y^2), y/sqrt(1-x^2-y^2)],
....:                              name='Phi', latex_name=r'\Phi')
sage: Phi
Diffeomorphism Phi from the Open subset D of the 2-dimensional
 differentiable manifold R^2 to the 2-dimensional differentiable
 manifold R^2
sage: Phi.parent()
Set of Morphisms from Open subset D of the 2-dimensional differentiable
 manifold R^2 to 2-dimensional differentiable manifold R^2 in Join of
 Category of subobjects of sets and Category of smooth manifolds over
 Real Field with 53 bits of precision
sage: Phi.parent() is Hom(D, R2)
True
sage: Phi.display()
Phi: D --> R^2
   (x, y) |--> (x, y) = (x/sqrt(-x^2 - y^2 + 1), y/sqrt(-x^2 - y^2 + 1))
```

The image of a point:

```
sage: p = D.point((1/2,0))
sage: q = Phi(p) ; q
Point on the 2-dimensional differentiable manifold R^2
sage: q.coord()
(1/3*sqrt(3), 0)
```

The inverse diffeomorphism is computed by means of the method `inverse()`:

```
sage: Phi.inverse()
Diffeomorphism Phi^(-1) from the 2-dimensional differentiable manifold R^2
 to the Open subset D of the 2-dimensional differentiable manifold R^2
sage: Phi.inverse().display()
Phi^(-1): R^2 --> D
   (x, y) |--> (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

Equivalently, one may use the notations `^(-1)` or `~` to get the inverse:

```
sage: Phi^(-1) is Phi.inverse()
True
sage: ~Phi is Phi.inverse()
True
```

Check that `~Phi` is indeed the inverse of `Phi`:

```
sage: (~Phi)(q) == p
True
```

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```
sage: Phi * ~Phi == R2.identity_map()
True
sage: ~Phi * Phi == D.identity_map()
True
```

The coordinate expression of the inverse diffeomorphism:

```
sage: (~Phi).display()
Phi^(-1): R^2 --> D
   (x, y) |--> (x, y) = (x/sqrt(x^2 + y^2 + 1), y/sqrt(x^2 + y^2 + 1))
```

A special case of diffeomorphism: the identity map of the open unit disk:

```
sage: id = D.identity_map() ; id
Identity map Id_D of the Open subset D of the 2-dimensional
 differentiable manifold R^2
sage: latex(id)
\mathrm{Id}_{D}
sage: id.parent()
Set of Morphisms from Open subset D of the 2-dimensional differentiable
 manifold R^2 to Open subset D of the 2-dimensional differentiable
 manifold R^2 in Join of Category of subobjects of sets and Category of
 smooth manifolds over Real Field with 53 bits of precision
sage: id.parent() is Hom(D, D)
True
sage: id is Hom(D,D).one()   # the identity element of the monoid Hom(D,D)
True
```

The identity map acting on a point:

```
sage: id(p)
Point on the 2-dimensional differentiable manifold R^2
sage: id(p) == p
True
sage: id(p) is p
True
```

The coordinate expression of the identity map:

```
sage: id.display()
Id_D: D --> D
   (x, y) |--> (x, y)
```

The identity map is its own inverse:

```
sage: id^(-1) is id
True
sage: ~id is id
True
```

differential(*point*)

Return the differential of `self` at a given point.

If the differentiable map `self` is

$$\Phi : M \longrightarrow N,$$

where M and N are differentiable manifolds, the *differential* of Φ at a point $p \in M$ is the tangent space linear map:

$$\mathrm{d}\Phi_p : T_pM \longrightarrow T_{\Phi(p)}N$$

defined by

$$\forall v \in T_pM, \quad \mathrm{d}\Phi_p(v) : \begin{array}{ccc} C^k(N) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & v(f \circ \Phi) \end{array}$$

INPUT:

- `point` – point p in the domain M of the differentiable map Φ

OUTPUT:

- $\mathrm{d}\Phi_p$, the differential of Φ at p, as a `FiniteRankFreeModuleMorphism`

EXAMPLES:

Differential of a differentiable map between a 2-dimensional manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: N = Manifold(3, 'N')
sage: Y.<u,v,w> = N.chart()
sage: Phi = M.diff_map(N, {(X,Y): (x-2*y, x*y, x^2-y^3)}, name='Phi',
....:                  latex_name = r'\Phi')
sage: p = M.point((2,-1), name='p')
sage: dPhip = Phi.differential(p) ; dPhip
Generic morphism:
  From: Tangent space at Point p on the 2-dimensional differentiable manifold
↪M
  To:   Tangent space at Point Phi(p) on the 3-dimensional differentiable
↪manifold N
sage: latex(dPhip)
{\mathrm{d}\Phi}_{p}
sage: dPhip.parent()
Set of Morphisms from Tangent space at Point p on the 2-dimensional
 differentiable manifold M to Tangent space at Point Phi(p) on the
 3-dimensional differentiable manifold N in Category of finite
 dimensional vector spaces over Symbolic Ring
```

The matrix of $\mathrm{d}\Phi_p$ w.r.t. to the default bases of T_pM and $T_{\Phi(p)}N$:

```
sage: dPhip.matrix()
[ 1 -2]
[-1  2]
[ 4 -3]
```

differential_functions(*chart1=None*, *chart2=None*)

Return the coordinate expression of the differential of the differentiable map with respect to a pair of charts.

If the differentiable map is

$$\Phi : M \longrightarrow N,$$

where M and N are differentiable manifolds, the *differential* of Φ at a point $p \in M$ is the tangent space linear map:

$$\mathrm{d}\Phi_p : T_pM \longrightarrow T_{\Phi(p)}N$$

defined by

$$\forall v \in T_pM, \quad \mathrm{d}\Phi_p(v): \begin{array}{ccc} C^k(N) & \longrightarrow & \mathbb{R}, \\ f & \longmapsto & v(f\circ\Phi). \end{array}$$

If the coordinate expression of Φ is

$$y^i = Y^i(x^1,\ldots,x^n), \quad 1\leq i\leq m,$$

where $(x^1,\ldots,x^n)$ are coordinates of a chart on M and $(y^1,\ldots,y^m)$ are coordinates of a chart on $\Phi(M)$, the expression of the differential of Φ with respect to these coordinates is

$$J_{ij} = \frac{\partial Y^i}{\partial x^j} \quad 1\leq i\leq m, \qquad 1\leq j\leq n.$$

$\left.J_{ij}\right|_p$ is then the matrix of the linear map $\mathrm{d}\Phi_p$ with respect to the bases of T_pM and $T_{\Phi(p)}N$ associated to the above charts:

$$\mathrm{d}\Phi_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.J_{ij}\right|_p \left.\frac{\partial}{\partial y^i}\right|_{\Phi(p)}.$$

INPUT:

- `chart1` – (default: `None`) chart on the domain M of Φ (coordinates denoted by (x^j) above); if `None`, the domain's default chart is assumed
- `chart2` – (default: `None`) chart on the codomain of Φ (coordinates denoted by (y^i) above); if `None`, the codomain's default chart is assumed

OUTPUT:

- the functions J_{ij} as a double array, J_{ij} being the element `[i][j]` represented by a *`ChartFunction`*

To get symbolic expressions, use the method *`jacobian_matrix()`* instead.

EXAMPLES:

Differential functions of a map between a 2-dimensional manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: N = Manifold(3, 'N')
sage: Y.<u,v,w> = N.chart()
sage: Phi = M.diff_map(N, {(X,Y): (x-2*y, x*y, x^2-y^3)}, name='Phi',
....:                 latex_name = r'\Phi')
sage: J = Phi.differential_functions(X, Y) ; J
[      1      -2]
[      y       x]
[    2*x -3*y^2]
```

The result is cached:

```
sage: Phi.differential_functions(X, Y) is J
True
```

The elements of `J` are functions of the coordinates of the chart `X`:

```
sage: J[2][0]
2*x
sage: type(J[2][0])
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_
↪class'>

sage: J[2][0].display()
(x, y) |--> 2*x
```

In contrast, the method *jacobian_matrix()* leads directly to symbolic expressions:

```
sage: JJ = Phi.jacobian_matrix(X,Y) ; JJ
[      1      -2]
[      y       x]
[   2*x -3*y^2]
sage: JJ[2,0]
2*x
sage: type(JJ[2,0])
<type 'sage.symbolic.expression.Expression'>
sage: bool( JJ[2,0] == J[2][0].expr() )
True
```

jacobian_matrix(*chart1=None*, *chart2=None*)

Return the Jacobian matrix resulting from the coordinate expression of the differentiable map with respect to a pair of charts.

If Φ is the current differentiable map and its coordinate expression is

$$y^i = Y^i(x^1, \ldots, x^n), \quad 1 \leq i \leq m,$$

where $(x^1, \ldots, x^n)$ are coordinates of a chart X on the domain of Φ and $(y^1, \ldots, y^m)$ are coordinates of a chart Y on the codomain of Φ, the *Jacobian matrix* of the differentiable map Φ w.r.t. to charts X and Y is

$$J = \left(\frac{\partial Y^i}{\partial x^j} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}},$$

where i is the row index and j the column one.

INPUT:

- `chart1` – (default: `None`) chart X on the domain of Φ; if none is provided, the domain's default chart is assumed
- `chart2` – (default: `None`) chart Y on the codomain of Φ; if none is provided, the codomain's default chart is assumed

OUTPUT:

- the matrix J defined above

EXAMPLES:

Jacobian matrix of a map between a 2-dimensional manifold and a 3-dimensional one:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: N = Manifold(3, 'N')
sage: Y.<u,v,w> = N.chart()
```

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```
sage: Phi = M.diff_map(N, {(X,Y): (x-2*y, x*y, x^2-y^3)}, name='Phi',
....:                       latex_name = r'\Phi')
sage: Phi.display()
Phi: M --> N
   (x, y) |--> (u, v, w) = (x - 2*y, x*y, -y^3 + x^2)
sage: J = Phi.jacobian_matrix(X, Y) ; J
[     1      -2]
[     y       x]
[   2*x -3*y^2]
sage: J.parent()
Full MatrixSpace of 3 by 2 dense matrices over Symbolic Ring
```

pullback (*tensor*)

Pullback operator associated with `self`.

In what follows, let Φ denote a differentiable map, M its domain and N its codomain.

INPUT:

- `tensor` – *TensorField*; a fully covariant tensor field T on N, i.e. a tensor field of type $(0, p)$, with p a positive or zero integer; the case $p = 0$ corresponds to a scalar field

OUTPUT:

- a *TensorField* representing a fully covariant tensor field on M that is the pullback of T by Φ

EXAMPLES:

Pullback on S^2 of a scalar field defined on R^3:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi') #␣
↪spherical coord. on U
sage: N = Manifold(3, 'R^3', r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = N.chart() # Cartesian coord. on R^3
sage: Phi = U.diff_map(N, (sin(th)*cos(ph), sin(th)*sin(ph), cos(th)),
....:                       name='Phi', latex_name=r'\Phi')
sage: f = N.scalar_field(x*y*z, name='f') ; f
Scalar field f on the 3-dimensional differentiable manifold R^3
sage: f.display()
f: R^3 --> R
   (x, y, z) |--> x*y*z
sage: pf = Phi.pullback(f) ; pf
Scalar field Phi_*(f) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: pf.display()
Phi_*(f): U --> R
   (th, ph) |--> cos(ph)*cos(th)*sin(ph)*sin(th)^2
```

Pullback on S^2 of the standard Euclidean metric on R^3:

```
sage: g = N.sym_bilin_form_field(name='g')
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: g.display()
g = dx*dx + dy*dy + dz*dz
sage: pg = Phi.pullback(g) ; pg
Field of symmetric bilinear forms Phi_*(g) on the Open subset U of
```

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```
 the 2-dimensional differentiable manifold S^2
sage: pg.display()
Phi_*(g) = dth*dth + sin(th)^2 dph*dph
```

Parallel computation:

```
sage: Parallelism().set('tensor', nproc=2)
sage: pg = Phi.pullback(g) ; pg
Field of symmetric bilinear forms Phi_*(g) on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: pg.display()
Phi_*(g) = dth*dth + sin(th)^2 dph*dph
sage: Parallelism().set('tensor', nproc=1)  # switch off parallelization
```

Pullback on S^2 of a 3-form on R^3:

```
sage: a = N.diff_form(3, name='A')
sage: a[1,2,3] = f
sage: a.display()
A = x*y*z dx/\dy/\dz
sage: pa = Phi.pullback(a) ; pa
3-form Phi_*(A) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: pa.display() # should be zero (as any 3-form on a 2-dimensional␣
↪manifold)
Phi_*(A) = 0
```

pushforward(*tensor*)

Pushforward operator associated with `self`.

In what follows, let Φ denote the differentiable map, M its domain and N its codomain.

INPUT:

- `tensor` – *TensorField*; a fully contrariant tensor field T on M, i.e. a tensor field of type $(p, 0)$, with p a positive integer

OUTPUT:

- a *TensorField* representing a fully contravariant tensor field along M with values in N, which is the pushforward of T by Φ

EXAMPLES:

Pushforward of a vector field on the 2-sphere S^2 to the Euclidean 3-space $\mathbf{R}^3$, via the standard embedding of S^2:

```
sage: S2 = Manifold(2, 'S^2', start_index=1)
sage: U = S2.open_subset('U')  # domain of spherical coordinates
sage: spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: R3 = Manifold(3, 'R^3', start_index=1)
sage: cart.<x,y,z> = R3.chart()
sage: Phi = U.diff_map(R3, {(spher, cart): [sin(th)*cos(ph),
....:   sin(th)*sin(ph), cos(th)]}, name='Phi', latex_name=r'\Phi')
sage: v = U.vector_field(name='v')
sage: v[:] = 0, 1
sage: v.display()
v = d/dph
sage: pv = Phi.pushforward(v); pv
```

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```
Vector field Phi^*(v) along the Open subset U of the 2-dimensional
 differentiable manifold S^2 with values on the 3-dimensional
 differentiable manifold R^3
sage: pv.display()
Phi^*(v) = -sin(ph)*sin(th) d/dx + cos(ph)*sin(th) d/dy
```

Pushforward of a vector field on the real line to the $\mathbf{R}^3$, via a helix embedding:

```
sage: R.<t> = RealLine()
sage: Psi = R.diff_map(R3, [cos(t), sin(t), t], name='Psi',
....:                  latex_name=r'\Psi')
sage: u = R.vector_field(name='u')
sage: u[0] = 1
sage: u.display()
u = d/dt
sage: pu = Psi.pushforward(u); pu
Vector field Psi^*(u) along the Real number line R with values on
 the 3-dimensional differentiable manifold R^3
sage: pu.display()
Psi^*(u) = -sin(t) d/dx + cos(t) d/dy + d/dz
```

2.5.3 Curves in Manifolds

Given a differentiable manifold M, a *differentiable curve* in M is a differentiable mapping

$$\gamma : I \longrightarrow M,$$

where I is an interval of $\mathbf{R}$.

Differentiable curves are implemented by `DifferentiableCurve`.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- Chap. 1 of [**?**]
- Chap. 3 of [**?**]

class `sage.manifolds.differentiable.curve.`**`DifferentiableCurve`**(*parent*, *coord_expression=None*, *name=None*, *latex_name=None*, *is_isomorphism=False*, *is_identity=False*)

Bases: `sage.manifolds.differentiable.diff_map.DiffMap`

Curve in a differentiable manifold.

Given a differentiable manifold M, a *differentiable curve* in M is a differentiable map

$$\gamma : I \longrightarrow M,$$

where I is an interval of $\mathbf{R}$.

INPUT:

- parent – *DifferentiableCurveSet* the set of curves $\mathrm{Hom}(I, M)$ to which the curve belongs
- coord_expression – (default: None) dictionary (possibly empty) of the functions of the curve parameter t expressing the curve in various charts of M, the keys of the dictionary being the charts and the values being lists or tuples of n symbolic expressions of t, where n is the dimension of M
- name – (default: None) string; symbol given to the curve
- latex_name – (default: None) string; LaTeX symbol to denote the curve; if none is provided, name will be used
- is_isomorphism – (default: False) determines whether the constructed object is a diffeomorphism; if set to True, then M must have dimension one
- is_identity – (default: False) determines whether the constructed object is the identity map; if set to True, then M must be the interval I

EXAMPLES:

The lemniscate of Gerono in the 2-dimensional Euclidean plane:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = var('t')
sage: c = M.curve({X: [sin(t), sin(2*t)/2]}, (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
sage: type(c)
<class 'sage.manifolds.differentiable.manifold_homset.DifferentiableCurveSet_with_
↪category.element_class'>
```

Instead of declaring the parameter t as a symbolic variable by means of var('t'), it is equivalent to get it as the canonical coordinate of the real number line (see *RealLine*):

```
sage: R.<t> = RealLine()
sage: c = M.curve({X: [sin(t), sin(2*t)/2]}, (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
```

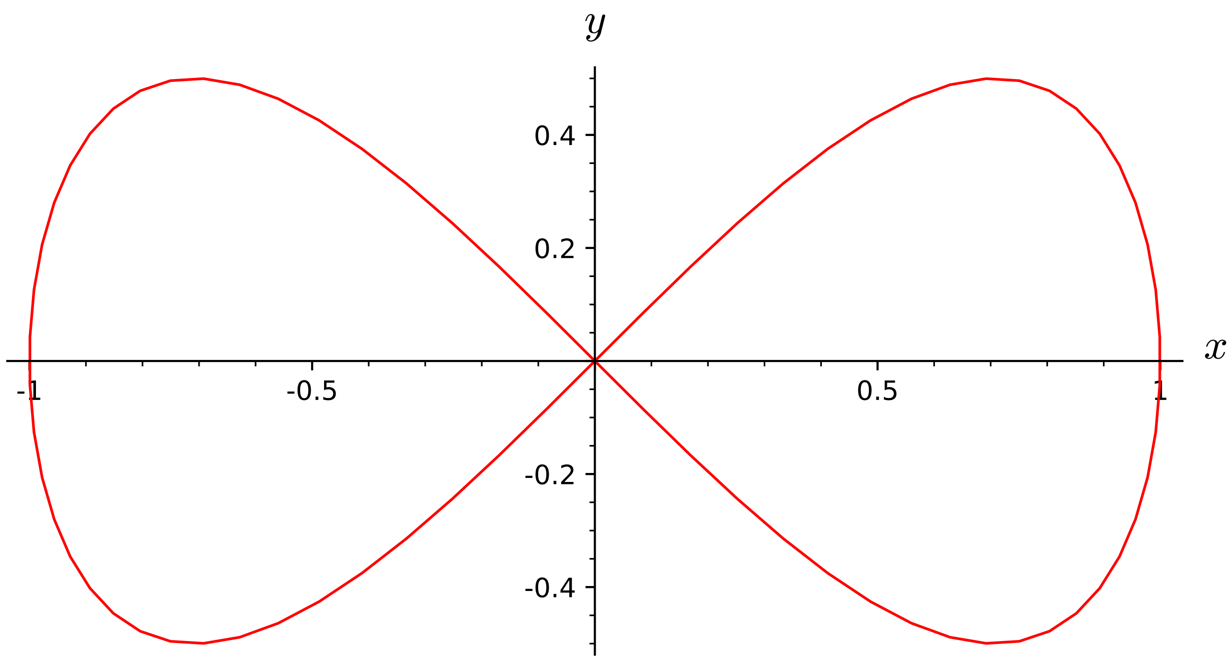
A graphical view of the curve is provided by the method *plot()*:

```
sage: c.plot(aspect_ratio=1)
Graphics object consisting of 1 graphics primitive
```

Curves are considered as (manifold) morphisms from real intervals to differentiable manifolds:

```
sage: c.parent()
Set of Morphisms from Real interval (0, 2*pi) to 2-dimensional
 differentiable manifold M in Join of Category of subobjects of sets
 and Category of smooth manifolds over Real Field with 53 bits of
 precision
sage: I = R.open_interval(0, 2*pi)
sage: c.parent() is Hom(I, M)
True
sage: c.domain()
Real interval (0, 2*pi)
sage: c.domain() is I
True
sage: c.codomain()
2-dimensional differentiable manifold M
```

Accordingly, all methods of *DiffMap* are available for them. In particular, the method *display()* shows the coordinate representations in various charts of manifold M:


y
0.4
0.2
-0.2
-0.4
x
-1
-0.5
0.5
1

```
sage: c.display()
c: (0, 2*pi) --> M
   t |--> (x, y) = (sin(t), 1/2*sin(2*t))
```

Another map method is using the usual call syntax, which returns the image of a point in the curve's domain:

```
sage: t0 = pi/2
sage: I(t0)
Point on the Real number line R
sage: c(I(t0))
Point on the 2-dimensional differentiable manifold M
sage: c(I(t0)).coord(X)
(1, 0)
```

For curves, the value of the parameter, instead of the corresponding point in the real line manifold, can be passed directly:

```
sage: c(t0)
Point c(1/2*pi) on the 2-dimensional differentiable manifold M
sage: c(t0).coord(X)
(1, 0)
sage: c(t0) == c(I(t0))
True
```

Instead of a dictionary of coordinate expressions, the curve can be defined by a single coordinate expression in a given chart:

```
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), chart=X, name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
sage: c.display()
c: (0, 2*pi) --> M
   t |--> (x, y) = (sin(t), 1/2*sin(2*t))
```

Since `X` is the default chart on `M`, it can be omitted:

```
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
sage: c.display()
c: (0, 2*pi) --> M
   t |--> (x, y) = (sin(t), 1/2*sin(2*t))
```

Note that a curve in M can also be created as a differentiable map $I \to M$:

```
sage: c1 = I.diff_map(M, coord_functions={X: [sin(t), sin(2*t)/2]},
....:                 name='c') ; c1
Curve c in the 2-dimensional differentiable manifold M
sage: c1.parent() is c.parent()
True
sage: c1 == c
True
```

LaTeX symbols representing a curve:

```
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi))
sage: latex(c)
\mbox{Curve in the 2-dimensional differentiable manifold M}
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c')
```

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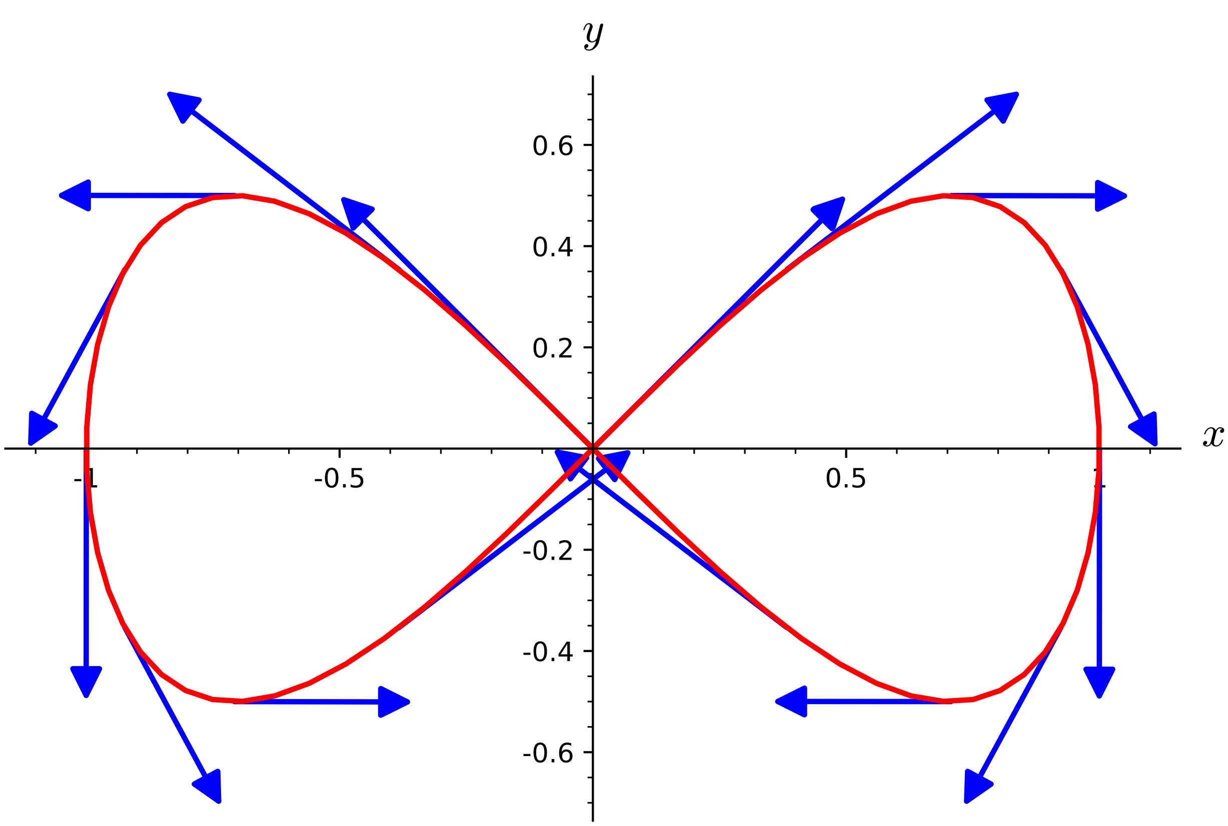
```
sage: latex(c)
c
sage: c = M.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c',
....:             latex_name=r'\gamma')
sage: latex(c)
\gamma
```

The curve's tangent vector field (velocity vector):

```
sage: v = c.tangent_vector_field() ; v
Vector field c' along the Real interval (0, 2*pi) with values on the
 2-dimensional differentiable manifold M
sage: v.display()
c' = cos(t) d/dx + (2*cos(t)^2 - 1) d/dy
```

Plot of the curve and its tangent vector field:

```
sage: show(c.plot(thickness=2, aspect_ratio=1) +
....:      v.plot(chart=X, number_values=17, scale=0.5))
```



Value of the tangent vector field at $t = \pi$:

```
sage: v.at(R(pi))
Tangent vector c' at Point on the 2-dimensional differentiable
```

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```
 manifold M
sage: v.at(R(pi)) in M.tangent_space(c(R(pi)))
True
sage: v.at(R(pi)).display()
c' = -d/dx + d/dy
```

Curves $\mathbf{R} \to \mathbf{R}$ can be composed: the operator $\circ$ is given by $\star$:

```
sage: f = R.curve(t^2, (t,-oo,+oo))
sage: g = R.curve(cos(t), (t,-oo,+oo))
sage: s = g*f ; s
Differentiable map from the Real number line R to itself
sage: s.display()
R --> R
   t |--> cos(t^2)
sage: s = f*g ; s
Differentiable map from the Real number line R to itself
sage: s.display()
R --> R
   t |--> cos(t)^2
```

coord_expr(*chart=None*)

Return the coordinate functions expressing the curve in a given chart.

INPUT:

- `chart` – (default: `None`) chart on the curve's codomain; if `None`, the codomain's default chart is assumed

OUTPUT:

- symbolic expression representing the curve in the above chart

EXAMPLES:

Cartesian and polar expression of a curve in the Euclidean plane:

```
sage: M = Manifold(2, 'R^2', r'\RR^2')  # the Euclidean plane R^2
sage: c_xy.<x,y> = M.chart() # Cartesian coordinate on R^2
sage: U = M.open_subset('U', coord_def={c_xy: (y!=0, x<0)}) # the complement␣
↪of the segment y=0 and x>0
sage: c_cart = c_xy.restrict(U) # Cartesian coordinates on U
sage: c_spher.<r,ph> = U.chart(r'r:(0,+oo) ph:(0,2*pi):\phi') # spherical␣
↪coordinates on U
sage: # Links between spherical coordinates and Cartesian ones:
sage: ch_cart_spher = c_cart.transition_map(c_spher, [sqrt(x*x+y*y), atan2(y,
↪x)])
sage: ch_cart_spher.set_inverse(r*cos(ph), r*sin(ph))
sage: R.<t> = RealLine()
sage: c = U.curve({c_spher: (1,t)}, (t, 0, 2*pi), name='c')
sage: c.coord_expr(c_spher)
(1, t)
sage: c.coord_expr(c_cart)
(cos(t), sin(t))
```

Since `c_cart` is the default chart on `U`, it can be omitted:

```
sage: c.coord_expr()
(cos(t), sin(t))
```

Cartesian expression of a cardiod:

```
sage: c = U.curve({c_spher: (2*(1+cos(t)), t)}, (t, 0, 2*pi), name='c')
sage: c.coord_expr(c_cart)
(2*cos(t)^2 + 2*cos(t), 2*(cos(t) + 1)*sin(t))
```

plot(*chart=None*, *ambient_coords=None*, *mapping=None*, *prange=None*, *include_end_point=(True, True)*, *end_point_offset=(0.001, 0.001)*, *parameters=None*, *color='red'*, *style='-'*, *label_axes=True*, *aspect_ratio='automatic'*, *max_range=8*, *plot_points=75*, *thickness=1*, ***kwds*)

Plot the current curve in a Cartesian graph based on the coordinates of some ambient chart.

The curve is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of a given chart, called hereafter the *ambient chart*. The ambient chart's domain must overlap with the curve's codomain or with the codomain of the composite curve $\Phi \circ c$, where c is the current curve and Φ some manifold differential map (argument `mapping` below).

INPUT:

- `chart` – (default: `None`) the ambient chart (see above); if `None`, the default chart of the codomain of the curve (or of the curve composed with Φ) is used
- `ambient_coords` – (default: `None`) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if `None`, all the coordinates of the ambient chart are considered
- `mapping` – (default: `None`) differentiable mapping Φ (instance of `DiffMap`) providing the link between the curve and the ambient chart `chart` (cf. above); if `None`, the ambient chart is supposed to be defined on the codomain of the curve.
- `prange` – (default: `None`) range of the curve parameter for the plot; if `None`, the entire parameter range declared during the curve construction is considered (with -Infinity replaced by `-max_range` and +Infinity by `max_range`)
- `include_end_point` – (default: `(True, True)`) determines whether the end points of `prange` are included in the plot
- `end_point_offset` – (default: `(0.001, 0.001)`) offsets from the end points when they are not included in the plot: if `include_end_point[0] == False`, the minimal value of the curve parameter used for the plot is `prange[0] + end_point_offset[0]`, while if `include_end_point[1] == False`, the maximal value is `prange[1] - end_point_offset[1]`.
- `max_range` – (default: 8) numerical value substituted to +Infinity if the latter is the upper bound of the parameter range; similarly `-max_range` is the numerical valued substituted for -Infinity
- `parameters` – (default: `None`) dictionary giving the numerical values of the parameters that may appear in the coordinate expression of the curve
- `color` – (default: 'red') color of the drawn curve
- `style` – (default: '-') color of the drawn curve; NB: `style` is effective only for 2D plots
- `thickness` – (default: 1) thickness of the drawn curve
- `plot_points` – (default: 75) number of points to plot the curve
- `label_axes` – (default: `True`) boolean determining whether the labels of the coordinate axes of `chart` shall be added to the graph; can be set to `False` if the graph is 3D and must be superposed with another graph.
- `aspect_ratio` – (default: `'automatic'`) aspect ratio of the plot; the default value (`'automatic'`) applies only for 2D plots; for 3D plots, the default value is `1` instead

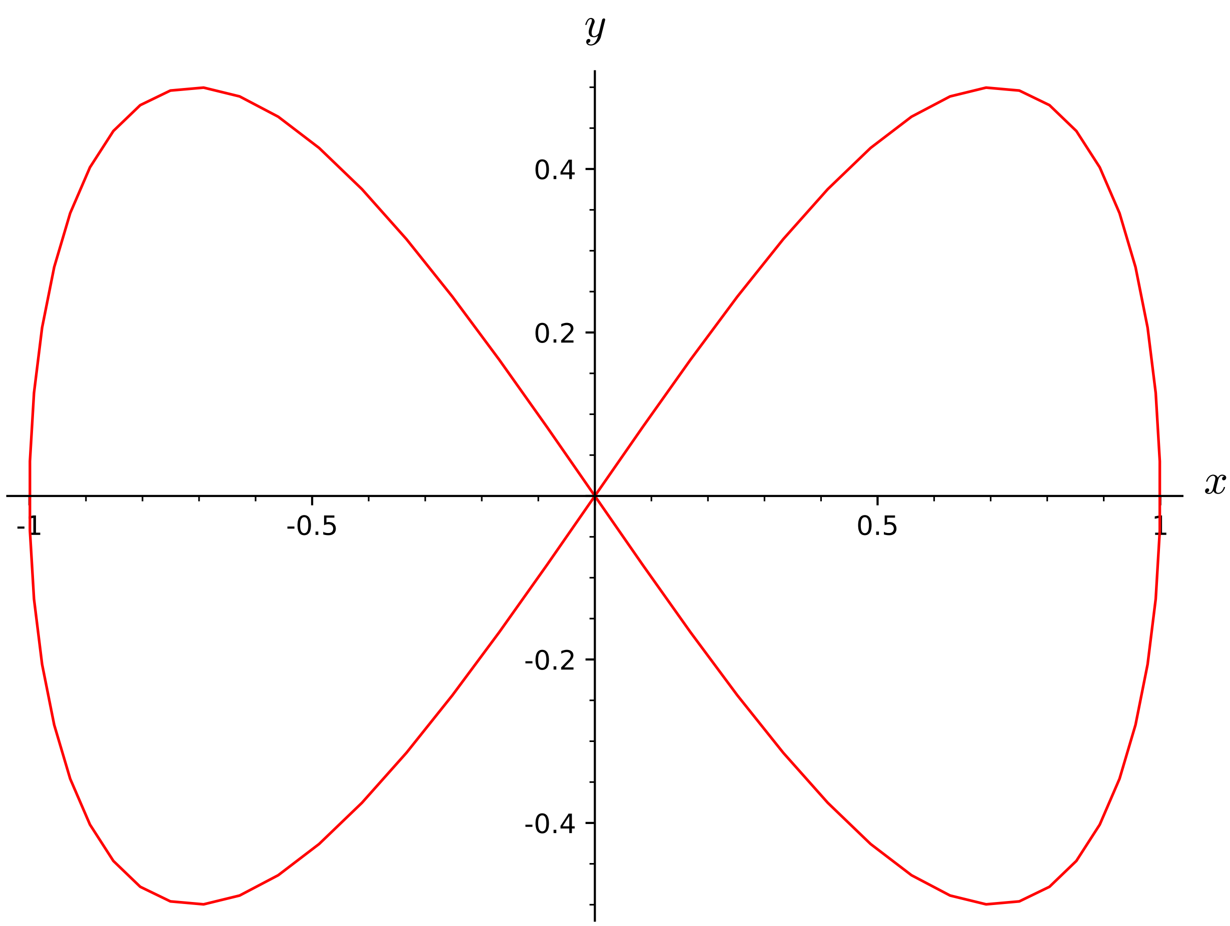
OUTPUT:

- a graphic object, either an instance of `Graphics` for a 2D plot (i.e. based on 2 coordinates of `chart`) or an instance of `Graphics3d` for a 3D plot (i.e. based on 3 coordinates of `chart`)

EXAMPLES:

Plot of the lemniscate of Gerono:

```
sage: R2 = Manifold(2, 'R^2')
sage: X.<x,y> = R2.chart()
sage: R.<t> = RealLine()
sage: c = R2.curve([sin(t), sin(2*t)/2], (t, 0, 2*pi), name='c')
sage: c.plot()  # 2D plot
Graphics object consisting of 1 graphics primitive
```



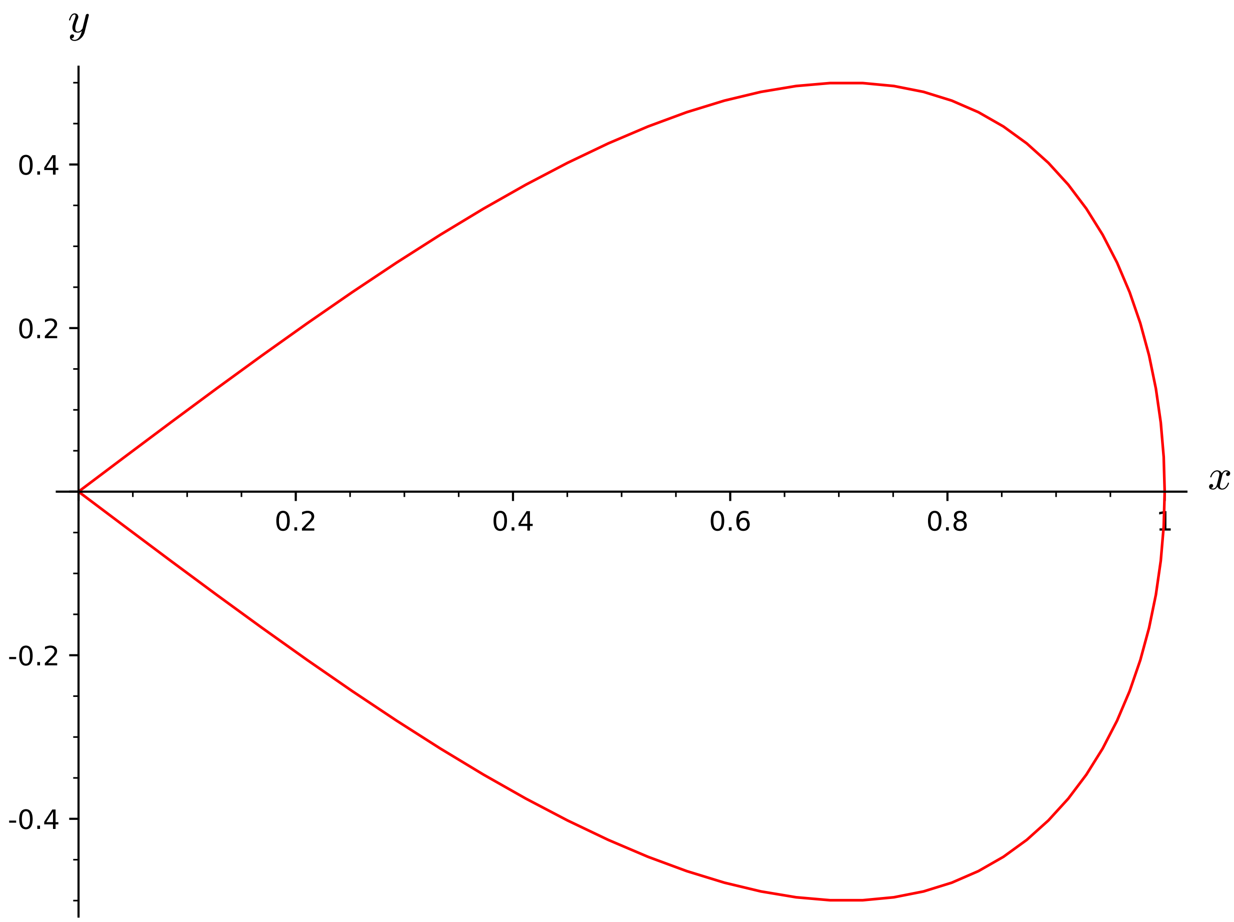
Plot for a subinterval of the curve's domain:

```
sage: c.plot(prange=(0,pi))
Graphics object consisting of 1 graphics primitive
```

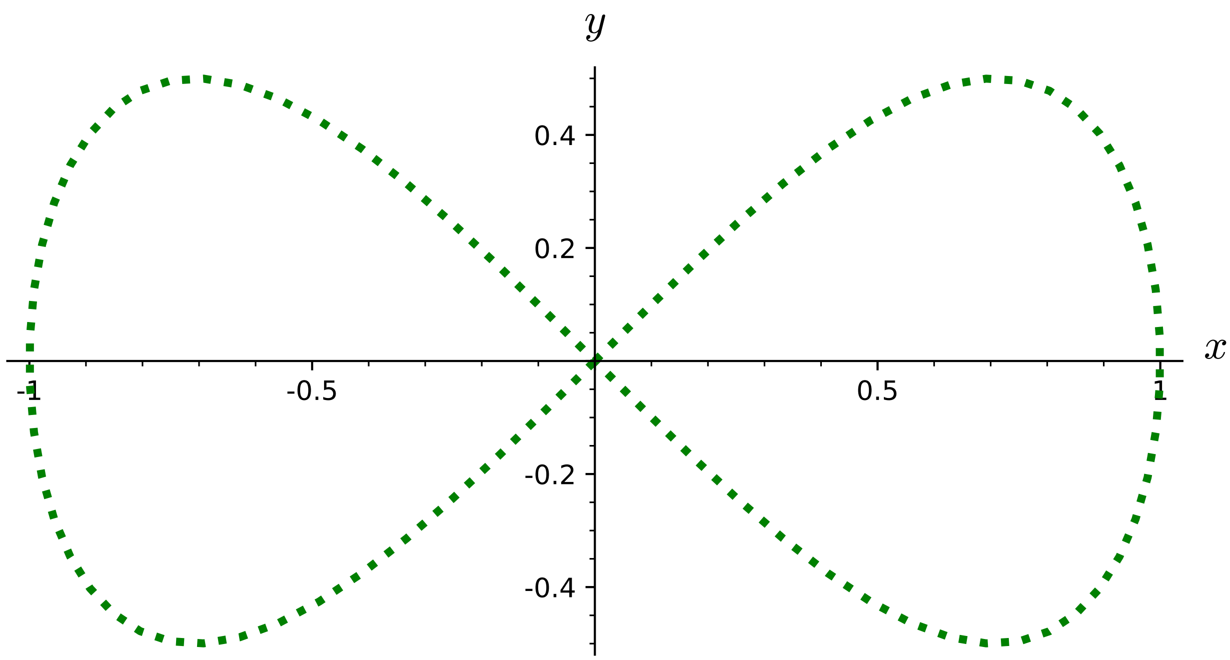
Plot with various options:

```
sage: c.plot(color='green', style=':', thickness=3, aspect_ratio=1)
Graphics object consisting of 1 graphics primitive
```

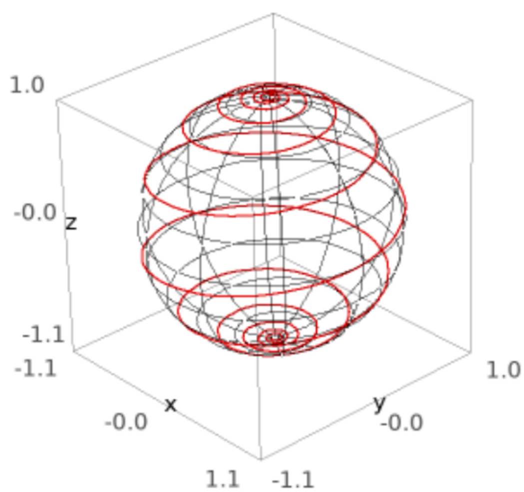
Plot via a mapping to another manifold: loxodrome of a sphere viewed in $\mathbf{R}^3$:



y
0.4
0.2
-0.2
-0.4
x
0.2
0.4
0.6
0.8
1


y
0.4
0.2
x
-1
-0.5
0.5
1
-0.2
-0.4

```
sage: S2 = Manifold(2, 'S^2')
sage: U = S2.open_subset('U')
sage: XS.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: R3 = Manifold(3, 'R^3')
sage: X3.<x,y,z> = R3.chart()
sage: F = S2.diff_map(R3, {(XS, X3): [sin(th)*cos(ph),
....:                       sin(th)*sin(ph), cos(th)]}, name='F')
sage: F.display()
F: S^2 --> R^3
on U: (th, ph) |--> (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
sage: c = S2.curve([2*atan(exp(-t/10)), t], (t, -oo, +oo), name='c')
sage: graph_c = c.plot(mapping=F, max_range=40,
....:                  plot_points=200, thickness=2, label_axes=False)  # 3D
↪plot
sage: graph_S2 = XS.plot(X3, mapping=F, number_values=11, color='black') #
↪plot of the sphere
sage: show(graph_c + graph_S2) # the loxodrome + the sphere
```

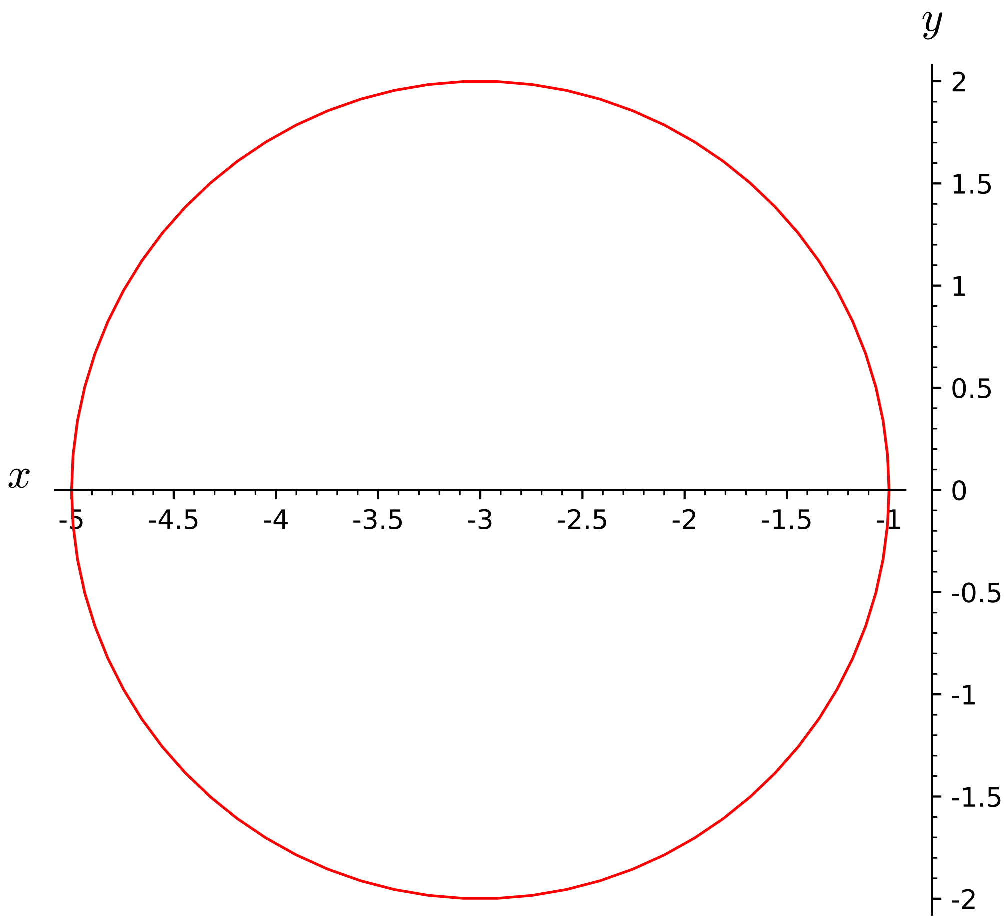


Example of use of the argument `parameters`: we define a curve with some symbolic parameters `a` and `b`:

```
sage: a, b = var('a b')
sage: c = R2.curve([a*cos(t) + b, a*sin(t)], (t, 0, 2*pi), name='c')
```

To make a plot, we set specific values for `a` and `b` by means of the Python dictionary `parameters`:

```
sage: c.plot(parameters={a: 2, b: -3}, aspect_ratio=1)
Graphics object consisting of 1 graphics primitive
```



tangent_vector_field(*name=None*, *latex_name=None*)

Return the tangent vector field to the curve (velocity vector).

INPUT:

- `name` – (default: `None`) string; symbol given to the tangent vector field; if none is provided, the primed curve symbol (if any) will be used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the tangent vector field; if `None` then (i) if `name` is `None` as well, the primed curve LaTeX symbol (if any) will be used or (ii) if `name` is not `None`, `name` will be used

OUTPUT:

- the tangent vector field, as an instance of *VectorField*

EXAMPLES:

Tangent vector field to a circle curve in $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: X.<x,y> = M.chart()
sage: R.<t> = RealLine()
```

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```
sage: c = M.curve([cos(t), sin(t)], (t, 0, 2*pi), name='c')
sage: v = c.tangent_vector_field() ; v
Vector field c' along the Real interval (0, 2*pi) with values on
 the 2-dimensional differentiable manifold R^2
sage: v.display()
c' = -sin(t) d/dx + cos(t) d/dy
sage: latex(v)
{c'}
sage: v.parent()
Free module X((0, 2*pi),c) of vector fields along the Real interval
 (0, 2*pi) mapped into the 2-dimensional differentiable manifold R^2
```

Value of the tangent vector field for some specific value of the curve parameter ($t = \pi$):

```
sage: R(pi) in c.domain()  # pi in (0, 2*pi)
True
sage: vp = v.at(R(pi)) ; vp
Tangent vector c' at Point on the 2-dimensional differentiable
 manifold R^2
sage: vp.parent() is M.tangent_space(c(R(pi)))
True
sage: vp.display()
c' = -d/dy
```

Tangent vector field to a curve in a non-parallelizable manifold (the 2-sphere S^2): first, we introduce the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                    intersection_name='W', restrictions1= x^2+y^2!=0,
....:                    restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: A = W.open_subset('A', coord_def={c_xy.restrict(W): (y!=0, x<0)})
sage: c_spher.<th,ph> = A.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi') #␣
↪spherical coordinates
sage: spher_to_xy = c_spher.transition_map(c_xy.restrict(A),
....:              (sin(th)*cos(ph)/(1-cos(th)), sin(th)*sin(ph)/(1-cos(th))) )
sage: spher_to_xy.set_inverse(2*atan(1/sqrt(x^2+y^2)), atan2(y, x),␣
↪check=False)
```

Then we define a curve (a loxodrome) by its expression in terms of spherical coordinates and evaluate the tangent vector field:

```
sage: R.<t> = RealLine()
sage: c = M.curve({c_spher: [2*atan(exp(-t/10)), t]}, (t, -oo, +oo),
....:             name='c') ; c
Curve c in the 2-dimensional differentiable manifold M
sage: vc = c.tangent_vector_field() ; vc
Vector field c' along the Real number line R with values on
 the 2-dimensional differentiable manifold M
```

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```
sage: vc.parent()
Module X(R,c) of vector fields along the Real number line R
 mapped into the 2-dimensional differentiable manifold M
sage: vc.display(c_spher.frame().along(c.restrict(R,A)))
c' = -1/5*e^(1/10*t)/(e^(1/5*t) + 1) d/dth + d/dph
```

2.5.4 Integrated Curves and Geodesics in Manifolds

Given a differentiable manifold M, an *integrated curve* in M is a differentiable curve constructed as a solution to a system of second order differential equations.

Integrated curves are implemented by `IntegratedCurve`, which the classes `IntegratedAutoparallelCurve` and `IntegratedGeodesic` inherit.

Examples: A geodesic in hyperbolic Poincaré half-plane

First declare a chart over the Poincaré half-plane:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart('x y:(0,+oo)')
```

Then declare the hyperbolic Poincaré metric:

```
sage: g = M.metric('g')
sage: g[0,0], g[1,1] = 1/y^2, 1/y^2
sage: g.display()
g = y^(-2) dx*dx + y^(-2) dy*dy
```

Pick an initial point and an initial tangent vector:

```
sage: p = M((0,1), name='p')
sage: v = M.tangent_space(p)((1,3/2), name='v')
sage: v.display()
v = d/dx + 3/2 d/dy
```

Declare a geodesic with such initial conditions, denoting `t` the corresponding affine parameter:

```
sage: t = var('t')
sage: c = M.integrated_geodesic(g, (t, 0, 10), v, name='c')
```

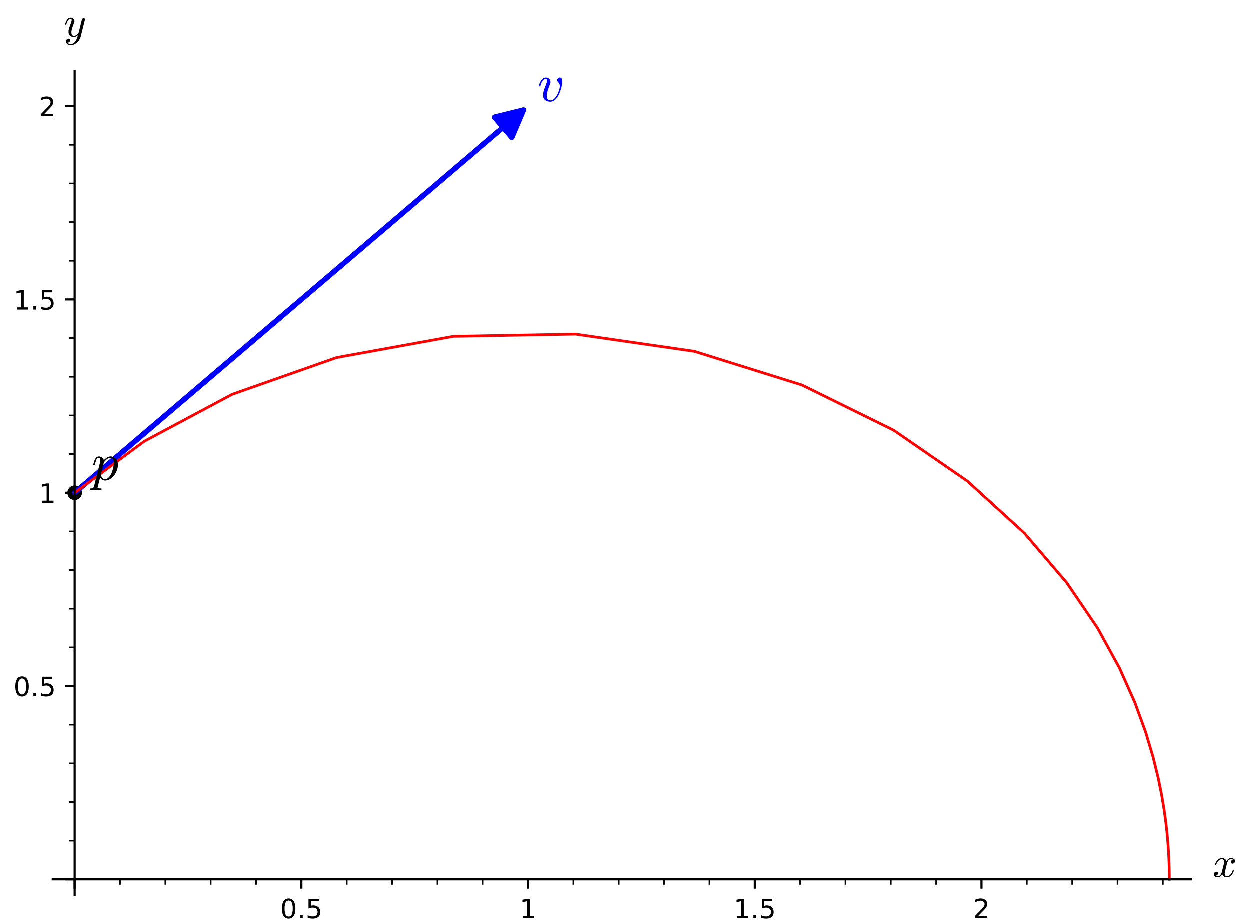
Numerically integrate the geodesic:

```
sage: sol = c.solve()
```

Plot the geodesic after interpolating the solution `sol`, since it is required to plot:

```
sage: interp = c.interpolate()
sage: graph = c.plot_integrated()
sage: p_plot = p.plot(size=30, label_offset=0.07, fontsize=20)
sage: v_plot = v.plot(label_offset=0.05, fontsize=20)
sage: graph + p_plot + v_plot
Graphics object consisting of 5 graphics primitives
```

AUTHORS:



y
x
v
p
2
1.5
1
0.5
0.5
1
1.5
2

- Karim Van Aelst (2017): initial version
- Florentin Jaffredo (2018): integration over multiple charts, use of `fast_callable` to improve the computation speed

class `sage.manifolds.differentiable.integrated_curve.`**`IntegratedAutoparallelCurve`**(*parent, affine_connection, curve_parameter, initial_tangent_vector, chart=None, name=None, latex_name=None, verbose=False, across_charts*

Bases: `sage.manifolds.differentiable.integrated_curve.IntegratedCurve`

Autoparallel curve on the manifold with respect to a given affine connection.

INPUT:

- `parent` – `IntegratedAutoparallelCurveSet` the set of curves $\mathrm{Hom_{autoparallel}}(I, M)$ to which the curve belongs
- `affine_connection` – `AffineConnection` affine connection with respect to which the curve is autoparallel
- `curve_parameter` – symbolic expression to be used as the parameter of the curve (the equations defining an instance of IntegratedAutoparallelCurve are such that `t` will actually be an affine parameter of the curve)
- `initial_tangent_vector` – `TangentVector` initial tangent vector of the curve
- `chart` – (default: `None`) chart on the manifold in terms of which the equations are expressed; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

EXAMPLES:

Autoparallel curves associated with the Mercator projection of the unit 2-sphere $\mathbb{S}^2$.

See also:

https://idontgetoutmuch.wordpress.com/2016/11/24/mercator-a-connection-with-torsion/ for more details about Mercator projection.

On the Mercator projection, the lines of longitude all appear vertical and then all parallel with respect to each other. Likewise, all the lines of latitude appear horizontal and parallel with respect to each other. These curves may be recovered as autoparallel curves of a certain connection ∇ to be made explicit.

Start with declaring the standard polar coordinates (θ, ϕ) on $\mathbb{S}^2$ and the corresponding coordinate frame (e_θ, e_ϕ):

```
sage: S2 = Manifold(2, 'S^2', start_index=1)
sage: polar.<th,ph>=S2.chart()
sage: epolar = polar.frame()
```

Normalizing e_ϕ provides an orthonormal basis:

```
sage: ch_basis = S2.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2] = 1, 1/sin(th)
sage: epolar_ON = epolar.new_frame(ch_basis,'epolar_ON')
```

Denote $(\hat{e}_\theta, \hat{e}_\phi)$ such an orthonormal frame field. In any point, the vector field $\hat{e}_\theta$ is normalized and tangent to the line of longitude through the point. Likewise, $\hat{e}_\phi$ is normalized and tangent to the line of latitude.

Now, set an affine connection with respect to such fields that are parallely transported in all directions, that is: $\nabla\hat{e}_\theta = \nabla\hat{e}_\phi = 0$. This is equivalent to setting all the connection coefficients to zero with respect to this frame:

```
sage: nab = S2.affine_connection('nab')
sage: nab.set_coef(frame=epolar_ON)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
```

This connection is such that two vectors are parallel if their angles to a given meridian are the same. Check that this connection is compatible with the Euclidean metric tensor g induced on $\mathbb{S}^2$:

```
sage: g = S2.metric('g')
sage: g[1,1], g[2,2] = 1, (sin(th))^2
sage: nab(g)[:]
[[[0, 0], [0, 0]], [[0, 0], [0, 0]]]
```

Yet, this connection is not the Levi-Civita connection, which implies that it has non-vanishing torsion:

```
sage: nab.torsion()[:]
[[[0, 0], [0, 0]], [[0, cos(th)/sin(th)], [-cos(th)/sin(th), 0]]]
```

Set generic initial conditions for the autoparallel curves to compute:

```
sage: [th0, ph0, v_th0, v_ph0] = var('th0 ph0 v_th0 v_ph0')
sage: p = S2.point((th0, ph0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((v_th0, v_ph0), basis=epolar_ON.at(p))
```

Note here that the components `(v_th0, v_ph0)` of the initial tangent vector `v` refer to the basis `epolar_ON` $= (\hat{e}_\theta, \hat{e}_\phi)$ and not the coordinate basis `epolar` $= (e_\theta, e_\phi)$. This is merely to help picture the aspect of the tangent vector in the usual embedding of $\mathbb{S}^2$ in $\mathbb{R}^3$ thanks to using an orthonormal frame, since providing the components with respect to the coordinate basis would require multiplying the second component (i.e. the ϕ component) in order to picture the vector in the same way. This subtlety will need to be taken into account later when the numerical curve will be compared to the analytical solution.

Now, declare the corresponding integrated autoparallel curve and display the differential system it satisfies:

```
sage: [t, tmin, tmax] = var('t tmin tmax')
sage: c = S2.integrated_autoparallel_curve(nab, (t, tmin, tmax),
....:                                   v, chart=polar, name='c')
sage: sys = c.system(verbose=True)
Autoparallel curve c in the 2-dimensional differentiable
 manifold S^2 equipped with Affine connection nab on the
 2-dimensional differentiable manifold S^2, and integrated over
 the Real interval (tmin, tmax) as a solution to the following
 equations, written with respect to Chart (S^2, (th, ph)):

Initial point: Point p on the 2-dimensional differentiable
 manifold S^2 with coordinates [th0, ph0] with respect to
 Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
```

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```
 2-dimensional differentiable manifold S^2 with
 components [v_th0, v_ph0/sin(th0)] with respect to Chart (S^2, (th, ph))

d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = 0
d(Dph)/dt = -Dph*Dth*cos(th)/sin(th)
```

Set a dictionary providing the parameter range and the initial conditions for a line of latitude and a line of longitude:

```
sage: dict_params={'latit':{tmin:0,tmax:3,th0:pi/4,ph0:0.1,v_th0:0,v_ph0:1},
....:   'longi':{tmin:0,tmax:3,th0:0.1,ph0:0.1,v_th0:1,v_ph0:0}}
```

Declare the Mercator coordinates (ξ, ζ) and the corresponding coordinate change from the polar coordinates:

```
sage: mercator.<xi,ze> = S2.chart(r'xi:(-oo,oo):\xi ze:(0,2*pi):\zeta')
sage: polar.transition_map(mercator, (log(tan(th/2)), ph))
Change of coordinates from Chart (S^2, (th, ph)) to Chart
 (S^2, (xi, ze))
```

Ask for the identity map in terms of these charts in order to add this coordinate change to its dictionary of expressions. This is required to plot the curve with respect to the Mercator chart:

```
sage: identity = S2.identity_map()
sage: identity.coord_functions(polar, mercator)
Coordinate functions (log(sin(1/2*th)/cos(1/2*th)), ph) on the
 Chart (S^2, (th, ph))
```

Solve, interpolate and prepare the plot for the solutions corresponding to the two initial conditions previously set:

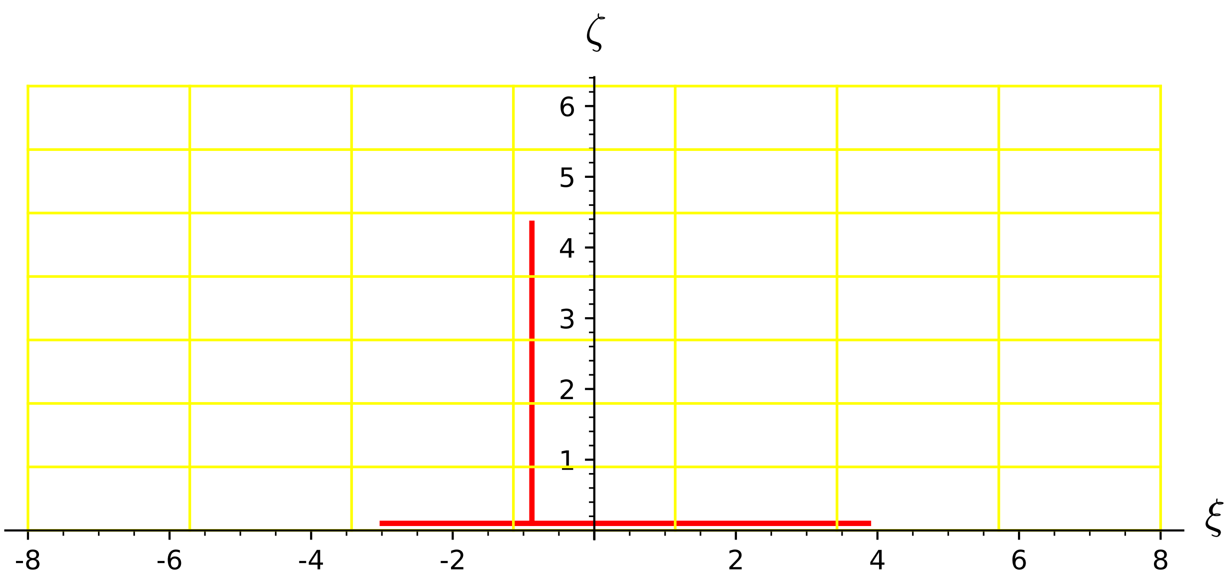
```
sage: graph2D_mercator = Graphics()
sage: for key in dict_params:
....:     sol = c.solve(solution_key='sol-'+key,
....:                        parameters_values=dict_params[key])
....:     interp = c.interpolate(solution_key='sol-'+key,
....:                           interpolation_key='interp-'+key)
....:     graph2D_mercator+=c.plot_integrated(interpolation_key='interp-'+key,
....:                             chart=mercator, thickness=2)
```

Prepare a grid of Mercator coordinates lines, and plot the curves over it:

```
sage: graph2D_mercator_coords=mercator.plot(chart=mercator,
....:                             number_values=8,color='yellow')
sage: graph2D_mercator + graph2D_mercator_coords
Graphics object consisting of 18 graphics primitives
```

The resulting curves are horizontal and vertical as expected. It is easier to check that these are latitude and longitude lines respectively when plotting them on $\mathbb{S}^2$. To do so, use $\mathbb{R}^3$ as the codomain of the standard map embedding $(\mathbb{S}^2, (\theta, \phi))$ in the 3-dimensional Euclidean space:

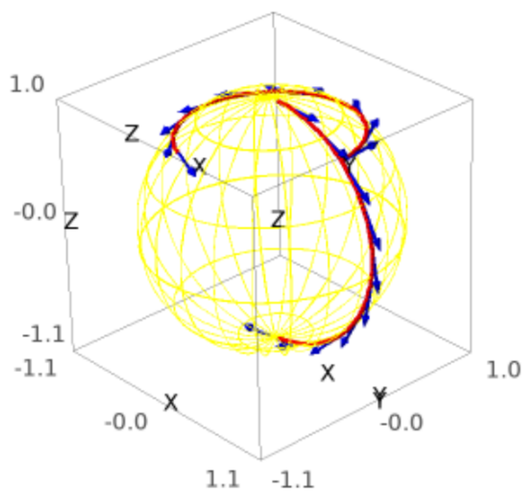
```
sage: R3 = Manifold(3, 'R3', start_index=1)
sage: cart.<X,Y,Z> = R3.chart()
sage: euclid_embedding = S2.diff_map(R3,
....:  {(polar, cart):[sin(th)*cos(ph),sin(th)*sin(ph),cos(th)]})
```



ζ
ξ
6
5
4
3
2
1
-8
-6
-4
-2
2
4
6
8

Plot the resulting curves on the grid of polar coordinates lines on $\mathbb{S}^2$:

```
sage: graph3D_embedded_curves = Graphics()
sage: for key in dict_params:
....:     graph3D_embedded_curves += c.plot_integrated(interpolation_key='interp-
↪'+key,
....:             mapping=euclid_embedding, thickness=5,
....:             display_tangent=True, scale=0.4, width_tangent=0.5)
sage: graph3D_embedded_polar_coords = polar.plot(chart=cart,
....:                           mapping=euclid_embedding,
....:                           number_values=15, color='yellow')
sage: graph3D_embedded_curves + graph3D_embedded_polar_coords
Graphics3d Object
```

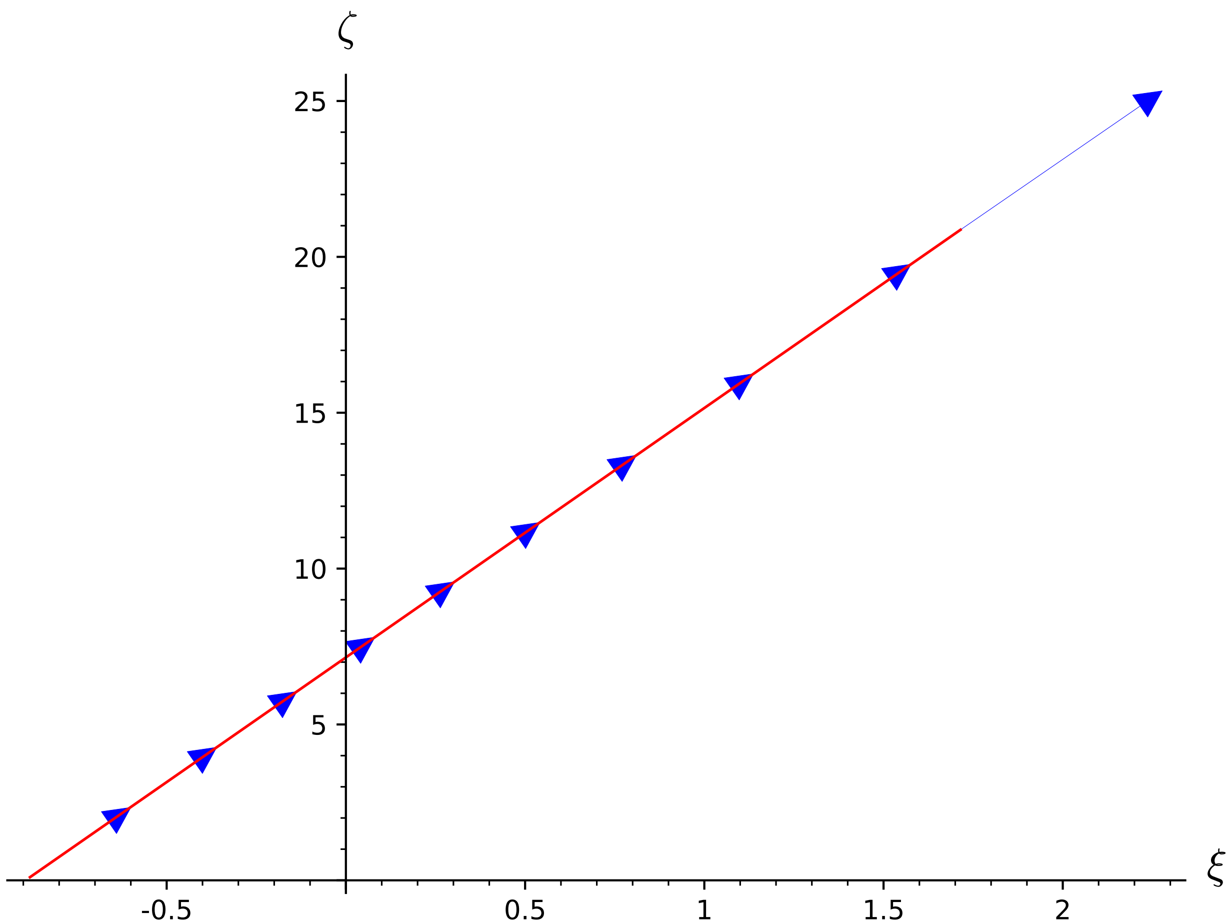


Finally, one may plot a general autoparallel curve with respect to ∇ that is neither a line of latitude or longitude. The vectors tangent to such a curve make an angle different from 0 or $\pi/2$ with the lines of latitude and longitude. Then, compute a curve such that both components of its initial tangent vectors are non zero:

```
sage: sol = c.solve(solution_key='sol-angle',
....:  parameters_values={tmin:0,tmax:2,th0:pi/4,ph0:0.1,v_th0:1,v_ph0:8})
sage: interp = c.interpolate(solution_key='sol-angle',
....:                           interpolation_key='interp-angle')
```

Plot the resulting curve in the Mercator plane. This generates a straight line, as expected:

```
sage: c.plot_integrated(interpolation_key='interp-angle',
....:           chart=mercator, thickness=1, display_tangent=True,
....:           scale=0.2, width_tangent=0.2)
Graphics object consisting of 11 graphics primitives
```



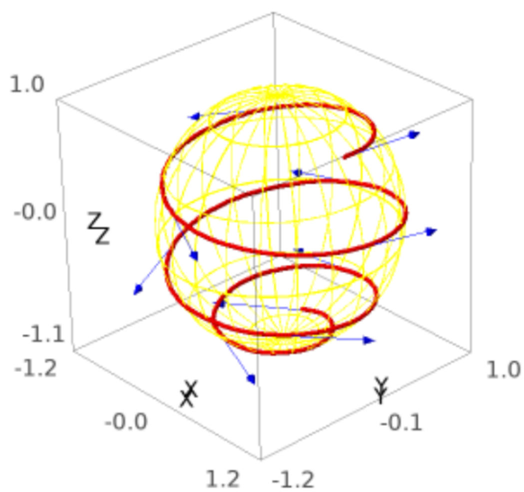
One may eventually plot such a curve on $\mathbb{S}^2$:

```
sage: graph3D_embedded_angle_curve=c.plot_integrated(interpolation_key='interp-
↪angle',
....:         mapping=euclid_embedding, thickness=5,
....:         display_tangent=True, scale=0.1, width_tangent=0.5)
sage: graph3D_embedded_angle_curve + graph3D_embedded_polar_coords
Graphics3d Object
```

All the curves presented are loxodromes, and the differential system defining them (displayed above) may be solved analytically, providing the following expressions:

$$\theta(t) = \theta_0 + \dot{\theta}_0(t - t_0),$$
$$\phi(t) = \phi_0 - \frac{1}{\tan\alpha}\left(\ln\tan\frac{\theta_0 + \dot{\theta}_0(t - t_0)}{2} - \ln\tan\frac{\theta_0}{2}\right),$$

where α is the angle between the curve and any latitude line it crosses; then, one finds $\tan\alpha = -\dot{\theta}_0/(\dot{\phi}_0 \sin\theta_0)$ (then $\tan\alpha \leq 0$ when the initial tangent vector points towards the southeast).


1.0
-0.0
Z
-1.1
-1.2
X
-0.0
1.2
-1.2
Y
-0.1
1.0

In order to use these expressions to compare with the result provided by the numerical integration, remember that the components `(v_th0, v_ph0)` of the initial tangent vector `v` refer to the basis `epolar_ON` $= (\hat{e}_\theta, \hat{e}_\phi)$ and not the coordinate basis `epolar` $= (e_\theta, e_\phi)$. Therefore, the following relations hold: `v_ph0` $= \dot{\phi}_0 \sin\theta_0$ (and not merely $\dot{\phi}_0$), while `v_th0` clearly is $\dot{\theta}_0$.

With this in mind, plot an analytical curve to compare with a numerical solution:

```
sage: graph2D_mercator_angle_curve=c.plot_integrated(interpolation_key='interp-
↪angle',
....:                                    chart=mercator, thickness=1)
sage: expr_ph = ph0+v_ph0/v_th0*(ln(tan((v_th0*t+th0)/2))-ln(tan(th0/2)))
sage: c_loxo = S2.curve({polar:[th0+v_th0*t, expr_ph]}, (t,0,2),
....:                                                   name='c_loxo')
```

Ask for the expression of the loxodrome in terms of the Mercator chart in order to add it to its dictionary of expressions. It is a particularly long expression, and there is no particular need to diplay it, which is why it may simply be affected to an arbitrary variable `expr_mercator`, which will never be used again. But adding the expression to the dictionary is required to plot the curve with respect to the Mercator chart:

```
sage: expr_mercator = c_loxo.expression(chart2=mercator)
```

Plot the curves (for clarity, set a 2 degrees shift in the initial value of θ_0 so that the curves do not overlap):

```
sage: graph2D_mercator_loxo = c_loxo.plot(chart=mercator,
....:  parameters={th0:pi/4+2*pi/180, ph0:0.1, v_th0:1, v_ph0:8},
....:  thickness=1, color='blue')
sage: graph2D_mercator_angle_curve + graph2D_mercator_loxo
Graphics object consisting of 2 graphics primitives
```

Both curves do have the same aspect. One may eventually compare these curves on $\mathbb{S}^2$:

```
sage: graph3D_embedded_angle_curve=c.plot_integrated(interpolation_key='interp-
↪angle',
....:                          mapping=euclid_embedding, thickness=3)
sage: graph3D_embedded_loxo = c_loxo.plot(mapping=euclid_embedding,
....:  parameters={th0:pi/4+2*pi/180, ph0:0.1, v_th0:1, v_ph0:8},
....:  thickness=3, color = 'blue')
sage: (graph3D_embedded_angle_curve + graph3D_embedded_loxo
....:  + graph3D_embedded_polar_coords)
Graphics3d Object
```

system(*verbose=False*)

Provide a detailed description of the system defining the autoparallel curve and returns the system defining it: chart, equations and initial conditions.

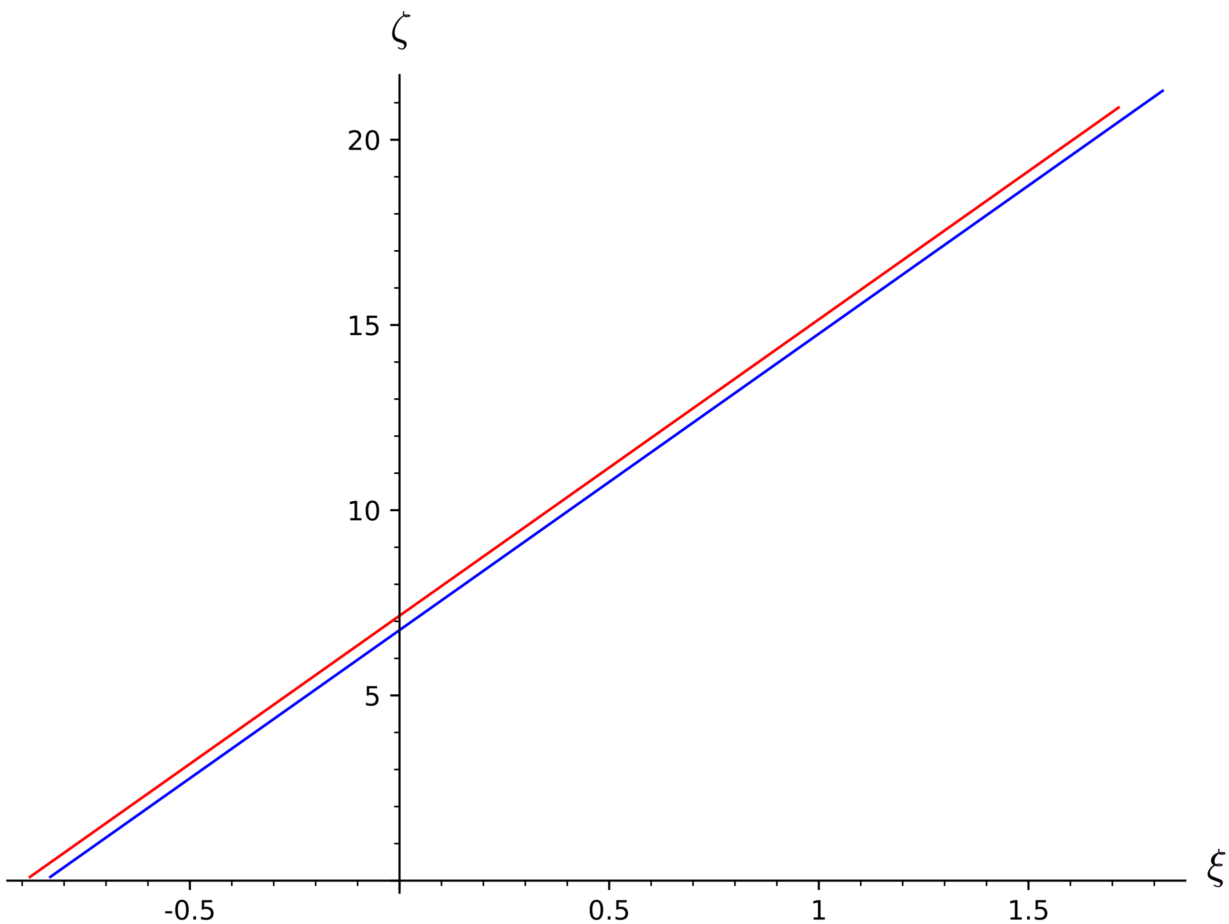
INPUT:

- `verbose` – (default: `False`) prints a detailed description of the curve

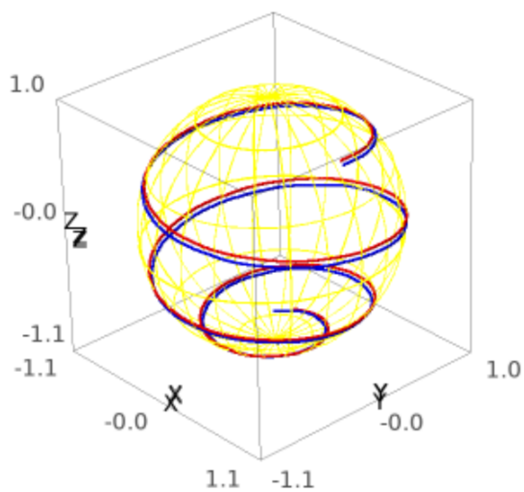
OUTPUT:

- list containing the
  - the equations
  - the initial conditions
  - the chart

EXAMPLES:



ζ
20
15
10
5
ξ
-0.5
0.5
1
1.5


1.0
-0.0
Z
-1.1
-1.1
-0.0
X
1.1
-1.1
Y
-0.0
1.0

System defining an autoparallel curve:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, A, B] = var('t A B')
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[X.frame(),0,0,1],nab[X.frame(),2,1,2]=A*x1^2,B*x2*x3
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_autoparallel_curve(nab, (t, 0, 5), v)
sage: sys = c.system(verbose=True)
Autoparallel curve in the 3-dimensional differentiable
 manifold M equipped with Affine connection nabla on the
 3-dimensional differentiable manifold M, and integrated
 over the Real interval (0, 5) as a solution to the
 following equations, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = -A*Dx1*Dx2*x1^2
d(Dx2)/dt = 0
d(Dx3)/dt = -B*Dx2*Dx3*x2*x3

sage: sys_bis = c.system()
sage: sys_bis == sys
True
```

class `sage.manifolds.differentiable.integrated_curve.`**`IntegratedCurve`**(*parent*, *equations_rhs*, *velocities*, *curve_parameter*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Bases: *sage.manifolds.differentiable.curve.DifferentiableCurve*

Given a chart with coordinates denoted $(x_1, \ldots, x_n)$, an instance of *IntegratedCurve* is a curve $t \mapsto (x_1(t), \ldots, x_n(t))$ constructed as a solution to a system of second order differential equations satisfied by the coordinate curves $t \mapsto x_i(t)$.

INPUT:

- `parent` – *IntegratedCurveSet* the set of curves $\mathrm{Hom_{integrated}}(I, M)$ to which the curve belongs
- `equations_rhs` – list of the right-hand sides of the equations on the velocities only (the term *velocity* referring to the derivatives dx_i/dt of the coordinate curves)
- `velocities` – list of the symbolic expressions used in `equations_rhs` to denote the velocities
- `curve_parameter` – symbolic expression used in `equations_rhs` to denote the parameter of the curve (denoted t in the descriptions above)
- `initial_tangent_vector` – *TangentVector* initial tangent vector of the curve
- `chart` – (default: `None`) chart on the manifold in which the equations are given; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

EXAMPLES:

Motion of a charged particle in an axial magnetic field linearly increasing in time and exponentially decreasing in space:

$$\mathbf{B}(t, \mathbf{x}) = \frac{B_0 t}{T} \exp\left(-\frac{x_1^2 + x_2^2}{L^2}\right) \mathbf{e_3}.$$

Equations of motion are:

$$\begin{aligned}
\ddot{x}_1(t) &= \frac{qB(t, \mathbf{x}(t))}{m} \dot{x}_2(t), \\
\ddot{x}_2(t) &= -\frac{qB(t, \mathbf{x}(t))}{m} \dot{x}_1(t), \\
\ddot{x}_3(t) &= 0.
\end{aligned}$$

Start with declaring a chart on a 3-dimensional manifold and the symbolic expressions denoting the velocities and the various parameters:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x1,x2,x3> = M.chart()
sage: var('t B_0 m q L T')
(t, B_0, m, q, L, T)
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities(); D
[Dx1, Dx2, Dx3]
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
```

Set the initial conditions:

```
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
```

Declare an integrated curve and display information relative to it:

```
sage: c = M.integrated_curve(eqns, D, (t, 0, 5), v, name='c',
....:                                            verbose=True)
The curve was correctly set.
```

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```
Parameters appearing in the differential system defining the
 curve are [B_0, L, T, m, q].
sage: c
Integrated curve c in the 3-dimensional differentiable
 manifold M
sage: sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 5) as a solution to the
 following system, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on
 the 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = B_0*Dx2*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx2)/dt = -B_0*Dx1*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx3)/dt = 0
```

Generate a solution of the system and an interpolation of this solution:

```
sage: sol = c.solve(step=0.2,
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1},
....:         solution_key='carac time 1', verbose=True)
Performing numerical integration with method 'rk4_maxima'...
Numerical integration completed.

Checking all points are in the chart domain...
All points are in the chart domain.

The resulting list of points was associated with the key
 'carac time 1' (if this key already referred to a former
 numerical solution, such a solution was erased).
sage: interp = c.interpolate(solution_key='carac time 1',
....:                 interpolation_key='interp 1', verbose=True)
Performing cubic spline interpolation by default...
Interpolation completed and associated with the key 'interp 1'
 (if this key already referred to a former interpolation,
 such an interpolation was erased).
```

Such an interpolation is required to evaluate the curve and the vector tangent to the curve for any value of the curve parameter:

```
sage: p = c(1.9, verbose=True)
Evaluating point coordinates from the interpolation associated
 with the key 'interp 1' by default...
sage: p
Point on the 3-dimensional differentiable manifold M
sage: p.coordinates()     # abs tol 1e-12
(1.3776707219621374, -0.9000776970132945, 1.9)
```

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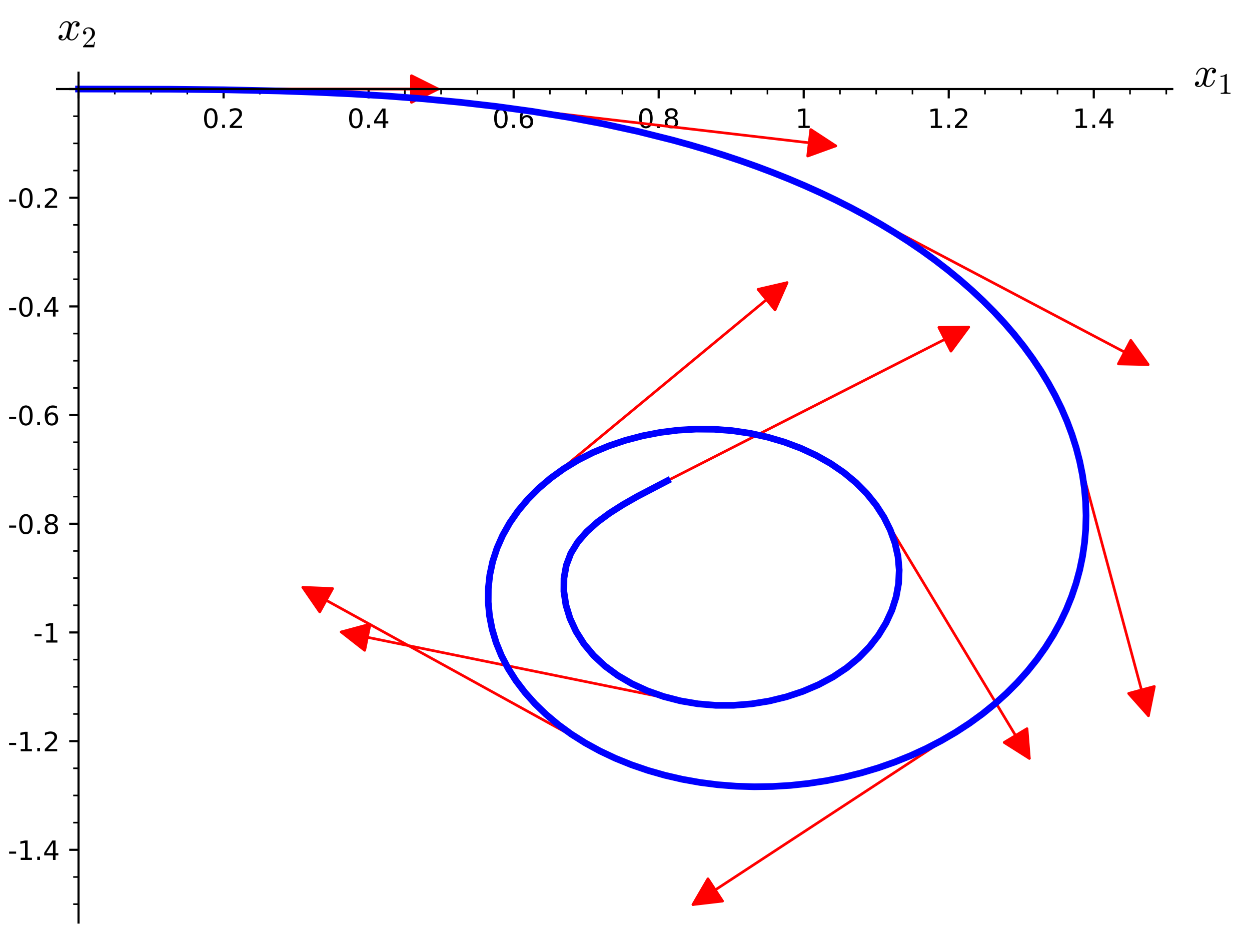
```
sage: v2 = c.tangent_vector_eval_at(4.3, verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'interp 1' by default...
sage: v2
Tangent vector at Point on the 3-dimensional differentiable
 manifold M
sage: v2[:]      # abs tol 1e-12
[-0.9303968397216424, -0.3408080563014475, 1.0000000000000004]
```

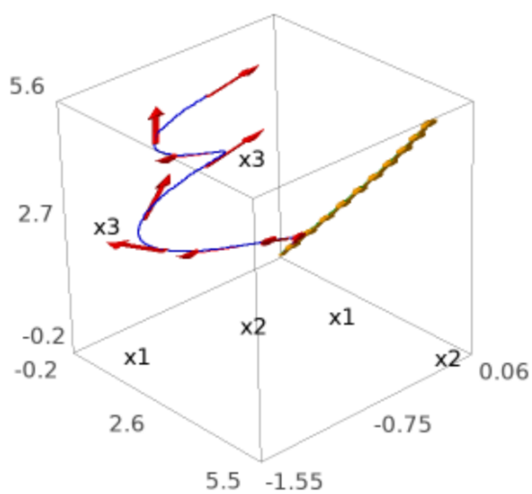
Plotting a numerical solution (with or without its tangent vector field) also requires the solution to be interpolated at least once:

```
sage: c_plot_2d_1 = c.plot_integrated(ambient_coords=[x1, x2],
....:                  interpolation_key='interp 1', thickness=2.5,
....:                  display_tangent=True, plot_points=200,
....:                  plot_points_tangent=10, scale=0.5,
....:                  color='blue', color_tangent='red',
....:                  verbose=True)
A tiny final offset equal to 0.000251256281407035 was introduced
 for the last point in order to safely compute it from the
 interpolation.
sage: c_plot_2d_1
Graphics object consisting of 11 graphics primitives
```



An instance of *IntegratedCurve* may store several numerical solutions and interpolations:

```
sage: sol = c.solve(step=0.2,
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:100},
....:         solution_key='carac time 100')
sage: interp = c.interpolate(solution_key='carac time 100',
....:                         interpolation_key='interp 100')
sage: c_plot_3d_100 = c.plot_integrated(interpolation_key='interp 100',
....:                     thickness=2.5, display_tangent=True,
....:                     plot_points=200, plot_points_tangent=10,
....:                     scale=0.5, color='green',
....:                     color_tangent='orange')
sage: c_plot_3d_1 = c.plot_integrated(interpolation_key='interp 1',
....:                     thickness=2.5, display_tangent=True,
....:                     plot_points=200, plot_points_tangent=10,
....:                     scale=0.5, color='blue',
....:                     color_tangent='red')
sage: c_plot_3d_1 + c_plot_3d_100
Graphics3d Object
```



interpolate (*solution_key=None*, *method=None*, *interpolation_key=None*, *verbose=False*)
Interpolate the chosen numerical solution using the given interpolation method.

INPUT:

- `solution_key` – (default: `None`) key which the numerical solution to interpolate is associated to ;

a default value is chosen if none is provided

- `method` – (default: `None`) interpolation scheme to use; algorithms available are
  - `'cubic spline'`, which makes use of `GSL` via `Spline`
- `interpolation_key` – (default: `None`) key which the resulting interpolation will be associated to ; a default value is given if none is provided
- `verbose` – (default: `False`) prints information about the interpolation in progress

OUTPUT:

- built interpolation object

EXAMPLES:

Interpolating a numerical solution previously computed:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sol = c.solve(method='rk4_maxima',
....:        solution_key='sol_T1',
....:        parameters_values={B_0:1, m:1, q:1, L:10, T:1})
sage: interp = c.interpolate(solution_key='my solution')
Traceback (most recent call last):
...
ValueError: no existing key 'my solution' referring to any
 numerical solution
sage: interp = c.interpolate(solution_key='sol_T1',
....:                        method='my method')
Traceback (most recent call last):
...
ValueError: no available method of interpolation referred to
 as 'my method'
sage: interp = c.interpolate(method='cubic spline',
....:                        solution_key='sol_T1',
....:                        interpolation_key='interp_T1',
....:                        verbose=True)
Interpolation completed and associated with the key
 'interp_T1' (if this key already referred to a former
 interpolation, such an interpolation was erased).
sage: interp = c.interpolate(verbose=True)
Interpolating the numerical solution associated with the
 key 'sol_T1' by default...
Performing cubic spline interpolation by default...
Resulting interpolation will be associated with the key
 'cubic spline-interp-sol_T1' by default.
Interpolation completed and associated with the key
 'cubic spline-interp-sol_T1' (if this key already referred
 to a former interpolation, such an interpolation was
 erased).
```

interpolation(*interpolation_key=None*, *verbose=False*)

Return the interpolation object associated with the given key.

INPUT:

- `interpolation_key` – (default: `None`) key which the requested interpolation is associated to; a default value is chosen if none is provided
- `verbose` – (default: `False`) prints information about the interpolation object returned

OUTPUT:

- requested interpolation object

EXAMPLES:

Requesting an interpolation object previously computed:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sol = c.solve(method='rk4_maxima',
....:         solution_key='sol_T1',
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1})
sage: interp = c.interpolate(method='cubic spline',
....:                          solution_key='sol_T1',
....:                          interpolation_key='interp_T1')
sage: c.interpolation(interpolation_key='my interp')
Traceback (most recent call last):
...
ValueError: no existing key 'my interp' referring to any
 interpolation
sage: default_interp = c.interpolation(verbose=True)
Returning the interpolation associated with the key
 'interp_T1' by default...
sage: default_interp == interp
True
sage: interp_mute = c.interpolation()
sage: interp_mute == interp
True
```

plot_integrated(*chart=None*, *ambient_coords=None*, *mapping=None*, *prange=None*, *interpolation_key=None*, *include_end_point=(True, True)*, *end_point_offset=(0.001, 0.001)*, *verbose=False*, *color='red'*, *style='-'*, *label_axes=True*, *display_tangent=False*, *color_tangent='blue'*, *across_charts=False*, *scale=1*, *width_tangent=1*, *plot_points=75*, *thickness=1*, *plot_points_tangent=10*, *aspect_ratio='automatic'*, ***kwds*)

Plot the 2D or 3D projection of `self` onto the space of the chosen two or three ambient coordinates, based on the interpolation of a numerical solution previously computed.

See also:

`plot` for complete information about the input.

ADDITIONAL INPUT:

- `interpolation_key` – (default: `None`) key associated to the interpolation object used for the plot; a default value is chosen if none is provided
- `verbose` – (default: `False`) prints information about the interpolation object used and the plotting in progress
- `display_tangent` – (default: `False`) determines whether some tangent vectors should also be plotted
- `color_tangent` – (default: `blue`) color of the tangent vectors when these are plotted
- `plot_points_tangent` – (default: 10) number of tangent vectors to display when these are plotted
- `width_tangent` – (default: 1) sets the width of the arrows representing the tangent vectors
- `scale` – (default: 1) scale applied to the tangent vectors before displaying them

EXAMPLES:

Trajectory of a particle of unit mass and unit charge in an unit, axial, uniform, stationary magnetic field:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: var('t')
t
sage: D = X.symbolic_velocities()
sage: eqns = [D[1], -D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,6), v, name='c')
sage: sol = c.solve()
sage: interp = c.interpolate()
sage: c_plot_2d = c.plot_integrated(ambient_coords=[x1, x2],
....:                 thickness=2.5,
....:                 display_tangent=True, plot_points=200,
....:                 plot_points_tangent=10, scale=0.5,
....:                 color='blue', color_tangent='red',
....:                 verbose=True)
Plotting from the interpolation associated with the key
 'cubic spline-interp-rk4_maxima' by default...
A tiny final offset equal to 0.000301507537688442 was
 introduced for the last point in order to safely compute it
 from the interpolation.
sage: c_plot_2d
Graphics object consisting of 11 graphics primitives
```

solution(*solution_key=None*, *verbose=False*)

Return the solution (list of points) associated with the given key.

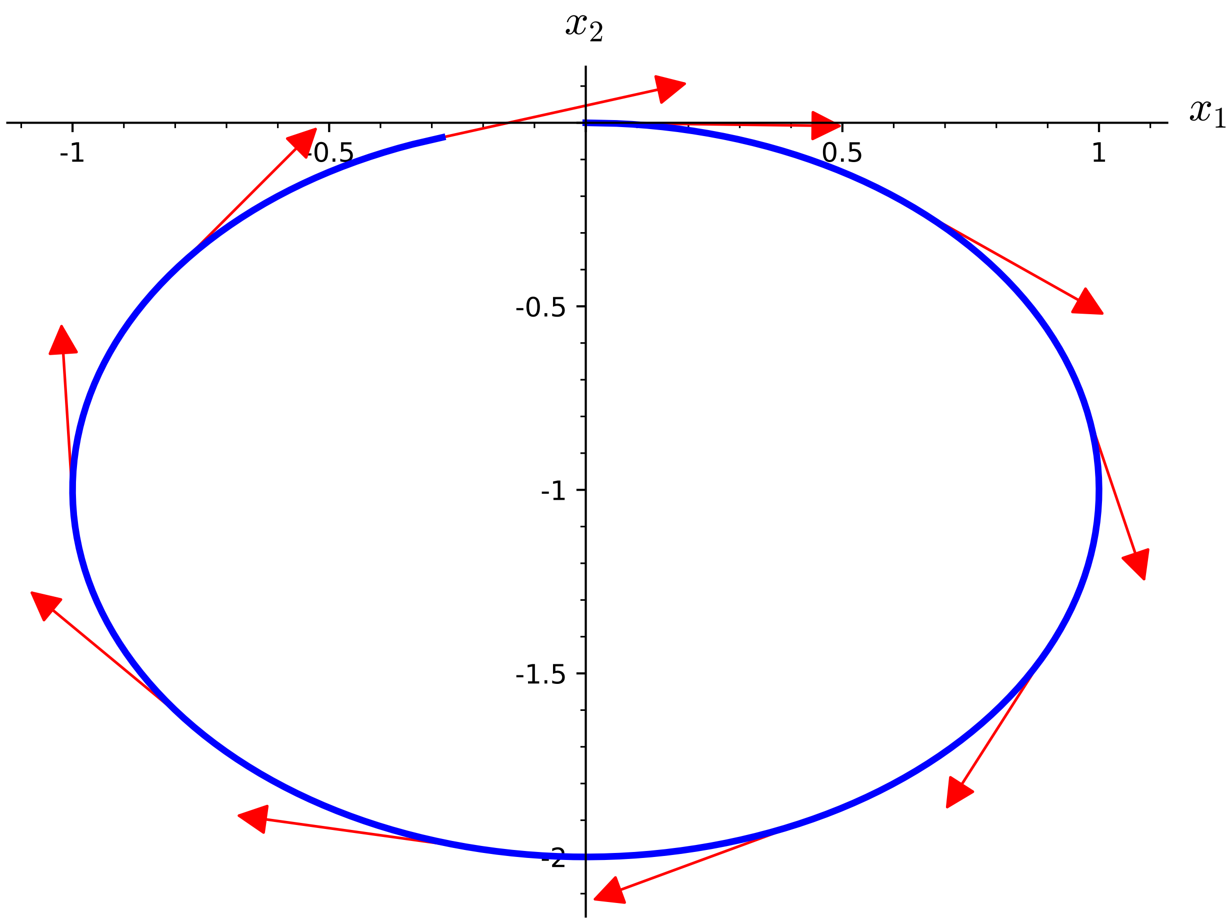
INPUT:

- `solution_key` – (default: `None`) key which the requested numerical solution is associated to; a default value is chosen if none is provided
- `verbose` – (default: `False`) prints information about the solution returned

OUTPUT:

- list of the numerical points of the solution requested

EXAMPLES:


x_2
x_1
-1
-0.5
0.5
1
-0.5
-1
-1.5
-2

Requesting a numerical solution previously computed:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sol = c.solve(solution_key='sol_T1',
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1})
sage: sol_bis = c.solution(verbose=True)
Returning the numerical solution associated with the key
 'sol_T1' by default...
sage: sol_bis == sol
True
sage: sol_ter = c.solution(solution_key='sol_T1')
sage: sol_ter == sol
True
sage: sol_mute = c.solution()
sage: sol_mute == sol
True
```

solve (*step=None*, *method='rk4_maxima'*, *solution_key=None*, *parameters_values=None*, *verbose=False*)

Integrate the curve numerically over the domain of integration.

INPUT:

- `step` – (default: `None`) step of integration; default value is a hundredth of the domain of integration if none is provided
- `method` – (default: `'rk4_maxima'`) numerical scheme to use for the integration of the curve; algorithms available are:
  - `'rk4_maxima'` - 4th order classical Runge-Kutta, which makes use of Maxima's dynamics package via Sage solver `desolve_system_rk4`
  - `'ode_int'` - makes use of `odeint` from `scipy.integrate` module via Sage solver `desolve_odeint`

and those provided by `GSL` via Sage class `ode_solver`:

- `'rk2'` - embedded Runge-Kutta (2,3)
- `'rk4'` - 4th order classical Runge-Kutta
- `'rkf45'` - Runge-Kutta-Felhberg (4,5)
- `'rkck'` - embedded Runge-Kutta-Cash-Karp (4,5)
- `'rk8pd'` - Runge-Kutta prince-dormand (8,9)
- `'rk2imp'` - implicit 2nd order Runge-Kutta at Gaussian points
- `'rk4imp'` - implicit 4th order Runge-Kutta at Gaussian points
- `'gear1'` - $M = 1$ implicit Gear
- `'gear2'` - $M = 2$ implicit Gear

- 'bsimp' - implicit Bulirsch-Stoer (requires Jacobian)
- solution_key – (default: None) key which the resulting numerical solution will be associated to; a default value is given if none is provided
- parameters_values – (default: None) list of numerical values of the parameters present in the system defining the curve, to be substituted in the equations before integration
- verbose – (default: False) prints information about the computation in progress

OUTPUT:

- list of the numerical points of the solution computed

EXAMPLES:

Computing a numerical solution:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sol = c.solve(parameters_values={m:1, q:1, L:10, T:1})
Traceback (most recent call last):
...
ValueError: numerical values should be provided for each of
 the parameters [B_0, L, T, m, q]
sage: sol = c.solve(method='my method',
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1})
Traceback (most recent call last):
...
ValueError: no available method of integration referred to
 as 'my method'
sage: sol = c.solve(
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1},
....:         verbose=True)
Performing numerical integration with method 'rk4_maxima'...
Resulting list of points will be associated with the key
 'rk4_maxima' by default.
Numerical integration completed.

Checking all points are in the chart domain...
All points are in the chart domain.

The resulting list of points was associated with the key
 'rk4_maxima' (if this key already referred to a former
 numerical solution, such a solution was erased).
sage: sol_mute = c.solve(
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1})
sage: sol_mute == sol
True
```

solve_across_charts(*charts=None*, *step=None*, *solution_key=None*, *parameters_values=None*, *verbose=False*)

Integrate the curve numerically over the domain of integration, with the ability to switch chart mid-integration.

The only supported solver is `scipy.integrate.ode`, because it supports basic event handling, needed to detect when the curve is reaching the frontier of the chart. This is an adaptive step solver. So the `step` is not the step of integration but instead the step used to peak at the current chart, and switch if needed.

INPUT:

- `step` – (default: `None`) step of chart checking; default value is a hundredth of the domain of integration if none is provided. If your curve can't find a new frame on exiting the current frame, consider reducing this parameter.
- `charts` – (default: `None`) list of chart allowed. The integration stops once it leaves those charts. By default the whole atlas is taken (only the top-charts).
- `solution_key` – (default: `None`) key which the resulting numerical solution will be associated to; a default value is given if none is provided
- `parameters_values` – (default: `None`) list of numerical values of the parameters present in the system defining the curve, to be substituted in the equations before integration
- `verbose` – (default: `False`) prints information about the computation in progress

OUTPUT:

- list of the numerical points of the solution computed

EXAMPLES:

This example illustrates the use of the function `solve_across_charts()` to integrate a geodesic of the Euclidean plane (a straight line) in polar coordinates.

In pure polar coordinates (r, θ), artefacts can appear near the origin because of the fast variation of θ, resulting in the direction of the geodesic being different before and after getting close to the origin.

The solution to this problem is to switch to Cartesian coordinates near $(0, 0)$ to avoid any singularity.

First let's declare the plane as a 2-dimensional manifold, with two charts P en C (for "Polar" and "Cartesian") and their transition maps:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: C.<x,y> = M.chart()
sage: P.<r,th> = M.chart()
sage: P_to_C = P.transition_map(C,(r*cos(th), r*sin(th)))
sage: C_to_P = C.transition_map(P,(sqrt(x**2+y**2), atan2(y,x)))
```

Let us also add restrictions on those charts, to avoid any singularity. We have to make sure that the charts still intersect. Here the intersection is the donut region $2 < r < 3$:

```
sage: P.add_restrictions(r > 2)
sage: C.add_restrictions(x**2+y**2 < 3**2)
```

We still have to define the metric. This is done in the Cartesian frame. The metric in the polar frame is computed automatically:

```
sage: g = M.metric()
sage: g[0,0,C]=1
sage: g[1,1,C]=1
sage: g[P.frame(), : ,P]
[  1   0]
[  0 r^2]
```

To visualize our manifold, let's declare a mapping between every chart and the Cartesian chart, and then plot each chart in term of this mapping:

```
sage: phi = M.diff_map(M, {(C,C): [x, y], (P,C): [r*cos(th), r*sin(th)]})
sage: fig = P.plot(number_values=9, chart=C, mapping=phi,
....:               color='grey', ranges= {r:(2, 6), th:(0,2*pi)})
sage: fig += C.plot(number_values=13, chart=C, mapping=phi,
....:                color='grey', ranges= {x:(-3, 3), y:(-3, 3)})
```

There is a clear non-empty intersection between the two charts. This is the key point to successfully switch chart during the integration. Indeed, at least 2 points must fall in the intersection.

Geodesic integration

Let's define the time as t, the initial point as p, and the initial velocity vector as v (define as a member of the tangent space T_p). The chosen geodesic should enter the central region from the left and leave it to the right:

```
sage: t = var('t')
sage: p = M((5,pi+0.3), P)
sage: Tp = M.tangent_space(p)
sage: v = Tp((-1,-0.03), P.frame().at(p))
```

While creating the integrated geodesic, we need to specify the optional argument `across_chart=True`, to prepare the compiled version of the changes of charts:

```
sage: c = M.integrated_geodesic(g, (t, 0, 10), v, across_charts=True)
```

The integration is done as usual, but using the method *`solve_across_charts()`* instead of *`solve()`*. This forces the use of `scipy.integrate.ode` as the solver, because of event handling support.

The argument `verbose=True` will cause the solver to write a small message each time it is switching chart:

```
sage: sol = c.solve_across_charts(step=0.1, verbose=True)
Performing numerical integration with method 'ode'.
Integration will take place on the whole manifold domain.
Resulting list of points will be associated with the key 'ode_multichart' by
↪default.
   ...
Exiting chart, trying to switch to another chart.
New chart found. Resuming integration.
Exiting chart, trying to switch to another chart.
New chart found. Resuming integration.
Integration successful.
```

As expected, two changes of chart occur.

The returned solution is a list of pairs `(chart, solution)`, where each solution is given on a unique chart, and the last point of a solution is the first of the next.

The following code prints the corresponding charts:

```
sage: for chart, solution in sol:
....:     print(chart)
Chart (M, (r, th))
```

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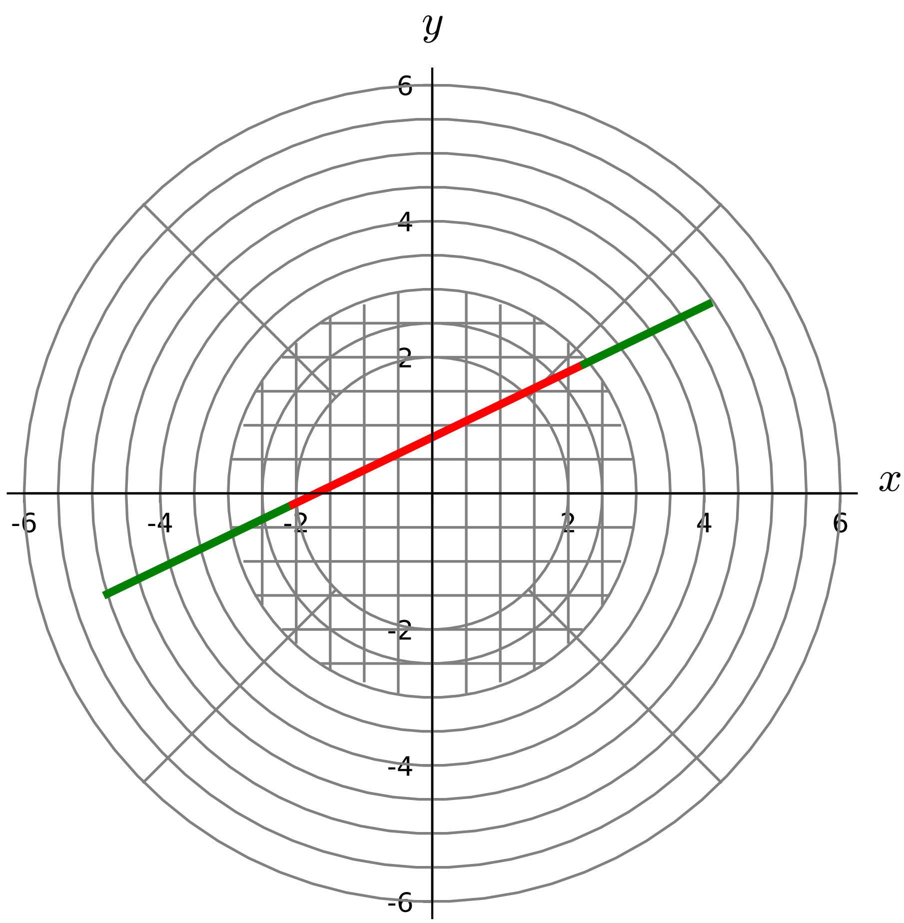
```
Chart (M, (x, y))
Chart (M, (r, th))
```

The interpolation is done as usual:

```
sage: interp = c.interpolate()
```

To plot the result, you must first be sure that the mapping encompasses all the chart, which is the case here. You must also specify `across_charts=True` in order to call `plot_integrated()` again on each part. Finally, `color` can be a list, which will be cycled through:

```
sage: fig += c.plot_integrated(mapping=phi, color=["green","red"],
....: thickness=3, plot_points=100, across_charts=True)
sage: fig
Graphics object consisting of 43 graphics primitives
```



solve_analytical(*verbose=False*)

Solve the differential system defining `self` analytically.

Solve analytically the differential system defining a curve using Maxima via Sage solver `desolve_system`. In case of success, the analytical expressions are added to the dictionary of expressions representing the curve. Pay attention to the fact that `desolve_system` only considers initial conditions given at an initial parameter value equal to zero, although the parameter range may not contain zero. Yet, assuming that it does, values of the coordinates functions at such zero initial parameter value are denoted by the name of the coordinate function followed by the string "_0".

OUTPUT:

- list of the analytical expressions of the coordinate functions (when the differential system could be solved analytically), or boolean False (in case the differential system could not be solved analytically)

EXAMPLES:

Analytical expression of the trajectory of a charged particle in a uniform, stationary magnetic field:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q] = var('t B_0 m q')
sage: D = X.symbolic_velocities()
sage: eqns = [q*B_0/m*D[1], -q*B_0/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 5) as a solution to
 the following system, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with components
 [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = B_0*Dx2*q/m
d(Dx2)/dt = -B_0*Dx1*q/m
d(Dx3)/dt = 0

sage: sol = c.solve_analytical()
sage: c.expr()
((B_0*q*x1_0 - Dx2_0*m*cos(B_0*q*t/m) +
   Dx1_0*m*sin(B_0*q*t/m) + Dx2_0*m)/(B_0*q),
 (B_0*q*x2_0 + Dx1_0*m*cos(B_0*q*t/m) +
  Dx2_0*m*sin(B_0*q*t/m) - Dx1_0*m)/(B_0*q),
 Dx3_0*t + x3_0)
```

system(*verbose=False*)

Provide a detailed description of the system defining the curve and return the system defining it: chart, equations and initial conditions.

INPUT:

- verbose – (default: False) prints a detailed description of the curve

OUTPUT:

- list containing
  - the equations

– the initial conditions

– the chart

EXAMPLES:

System defining an integrated curve:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sys = c.system(verbose=True)
Curve c in the 3-dimensional differentiable manifold M
 integrated over the Real interval (0, 5) as a solution to
 the following system, written with respect to
 Chart (M, (x1, x2, x3)):

Initial point: Point p on the 3-dimensional differentiable
 manifold M with coordinates [0, 0, 0] with respect to
 Chart (M, (x1, x2, x3))
Initial tangent vector: Tangent vector at Point p on the
 3-dimensional differentiable manifold M with
 components [1, 0, 1] with respect to Chart (M, (x1, x2, x3))

d(x1)/dt = Dx1
d(x2)/dt = Dx2
d(x3)/dt = Dx3
d(Dx1)/dt = B_0*Dx2*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx2)/dt = -B_0*Dx1*q*t*e^(-(x1^2 + x2^2)/L^2)/(T*m)
d(Dx3)/dt = 0

sage: sys_mute = c.system()
sage: sys_mute == sys
True
```

tangent_vector_eval_at(*t*, *interpolation_key=None*, *verbose=False*)

Return the vector tangent to `self` at the given curve parameter with components evaluated from the given interpolation.

INPUT:

- `t` – curve parameter value at which the tangent vector is evaluated
- `interpolation_key` – (default: `None`) key which the interpolation requested to compute the tangent vector is associated to; a default value is chosen if none is provided
- `verbose` – (default: `False`) prints information about the interpolation used

OUTPUT:

- *`TangentVector`* tangent vector with numerical components

EXAMPLES:

Evaluating a vector tangent to the curve:

```
sage: M = Manifold(3, 'M')
sage: X.<x1,x2,x3> = M.chart()
sage: [t, B_0, m, q, L, T] = var('t B_0 m q L T')
sage: B = B_0*t/T*exp(-(x1^2 + x2^2)/L^2)
sage: D = X.symbolic_velocities()
sage: eqns = [q*B/m*D[1], -q*B/m*D[0], 0]
sage: p = M.point((0,0,0), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((1,0,1))
sage: c = M.integrated_curve(eqns, D, (t,0,5), v, name='c')
sage: sol = c.solve(method='rk4_maxima',
....:         solution_key='sol_T1',
....:         parameters_values={B_0:1, m:1, q:1, L:10, T:1})
sage: interp = c.interpolate(method='cubic spline',
....:                          solution_key='sol_T1',
....:                          interpolation_key='interp_T1')
sage: tg_vec = c.tangent_vector_eval_at(1.22,
....:                          interpolation_key='my interp')
Traceback (most recent call last):
...
ValueError: no existing key 'my interp' referring to any
 interpolation
sage: tg_vec = c.tangent_vector_eval_at(1.22, verbose=True)
Evaluating tangent vector components from the interpolation
 associated with the key 'interp_T1' by default...
sage: tg_vec
Tangent vector at Point on the 3-dimensional differentiable
 manifold M
sage: tg_vec[:]      # abs tol 1e-12
[0.7392639473853356, -0.6734182305341726, 1.0000000000000007]
sage: tg_vec_mute = c.tangent_vector_eval_at(1.22,
....:                          interpolation_key='interp_T1')
sage: tg_vec_mute == tg_vec
True
```

class sage.manifolds.differentiable.integrated_curve.**IntegratedGeodesic**(*parent*, *metric*, *curve_parameter*, *initial_tangent_vector*, *chart=None*, *name=None*, *latex_name=None*, *verbose=False*, *across_charts=False*)

Bases: *sage.manifolds.differentiable.integrated_curve.IntegratedAutoparallelCurve*

Geodesic on the manifold with respect to a given metric.

INPUT:

- parent – *IntegratedGeodesicSet* the set of curves $\mathrm{Hom}_{\mathrm{geodesic}}(I, M)$ to which the curve belongs

- `metric` – *PseudoRiemannianMetric* metric with respect to which the curve is a geodesic
- `curve_parameter` – symbolic expression to be used as the parameter of the curve (the equations defining an instance of IntegratedGeodesic are such that `t` will actually be an affine parameter of the curve);
- `initial_tangent_vector` – *TangentVector* initial tangent vector of the curve
- `chart` – (default: `None`) chart on the manifold in terms of which the equations are expressed; if `None` the default chart of the manifold is assumed
- `name` – (default: `None`) string; symbol given to the curve
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the curve; if none is provided, `name` will be used

EXAMPLES:

Geodesics of the unit 2-sphere $\mathbb{S}^2$. Start with declaring the standard polar coordinates (θ, ϕ) on $\mathbb{S}^2$ and the corresponding coordinate frame (e_θ, e_ϕ):

```
sage: S2 = Manifold(2, 'S^2', start_index=1)
sage: polar.<th,ph>=S2.chart('th ph')
sage: epolar = polar.frame()
```

Set the Euclidean metric tensor g induced on $\mathbb{S}^2$:

```
sage: g = S2.metric('g')
sage: g[1,1], g[2,2] = 1, (sin(th))^2
```

Set generic initial conditions for the geodesics to compute:

```
sage: [th0, ph0, v_th0, v_ph0] = var('th0 ph0 v_th0 v_ph0')
sage: p = S2.point((th0, ph0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((v_th0, v_ph0), basis=epolar.at(p))
```

Declare the corresponding integrated geodesic and display the differential system it satisfies:

```
sage: [t, tmin, tmax] = var('t tmin tmax')
sage: c = S2.integrated_geodesic(g, (t, tmin, tmax), v,
....:                               chart=polar, name='c')
sage: sys = c.system(verbose=True)
Geodesic c in the 2-dimensional differentiable manifold S^2
 equipped with Riemannian metric g on the 2-dimensional
 differentiable manifold S^2, and integrated over the Real
 interval (tmin, tmax) as a solution to the following geodesic
 equations, written with respect to Chart (S^2, (th, ph)):

Initial point: Point p on the 2-dimensional differentiable
manifold S^2 with coordinates [th0, ph0] with respect to
Chart (S^2, (th, ph))
Initial tangent vector: Tangent vector at Point p on the
2-dimensional differentiable manifold S^2 with
components [v_th0, v_ph0] with respect to Chart (S^2, (th, ph))

d(th)/dt = Dth
d(ph)/dt = Dph
d(Dth)/dt = Dph^2*cos(th)*sin(th)
d(Dph)/dt = -2*Dph*Dth*cos(th)/sin(th)
```

Set a dictionary providing the parameter range and the initial conditions for various geodesics:

```
sage: dict_params={'equat':{tmin:0,tmax:3,th0:pi/2,ph0:0.1,v_th0:0,v_ph0:1},
....:   'longi':{tmin:0,tmax:3,th0:0.1,ph0:0.1,v_th0:1,v_ph0:0},
....:   'angle':{tmin:0,tmax:3,th0:pi/4,ph0:0.1,v_th0:1,v_ph0:1}}
```

Use $\mathbb{R}^3$ as the codomain of the standard map embedding $(\mathbb{S}^2, (\theta, \phi))$ in the 3-dimensional Euclidean space:

```
sage: R3 = Manifold(3, 'R3', start_index=1)
sage: cart.<X,Y,Z> = R3.chart()
sage: euclid_embedding = S2.diff_map(R3,
....:  {(polar, cart):[sin(th)*cos(ph),sin(th)*sin(ph),cos(th)]})
```

Solve, interpolate and prepare the plot for the solutions corresponding to the three initial conditions previously set:

```
sage: graph3D_embedded_geods = Graphics()
sage: for key in dict_params:
....:     sol = c.solve(solution_key='sol-'+key,
....:                        parameters_values=dict_params[key])
....:     interp = c.interpolate(solution_key='sol-'+key,
....:                           interpolation_key='interp-'+key)
....:     graph3D_embedded_geods += c.plot_integrated(interpolation_key='interp-
↪'+key,
....:                     mapping=euclid_embedding, thickness=5,
....:                     display_tangent=True, scale=0.3,
....:                     width_tangent=0.5)
```

Plot the resulting geodesics on the grid of polar coordinates lines on $\mathbb{S}^2$ and check that these are great circles:

```
sage: graph3D_embedded_polar_coords = polar.plot(chart=cart,
....:                          mapping=euclid_embedding,
....:                          number_values=15, color='yellow')
sage: graph3D_embedded_geods + graph3D_embedded_polar_coords
Graphics3d Object
```

system(*verbose=False*)

Return the system defining the geodesic: chart, equations and initial conditions.

INPUT:

- `verbose` – (default: `False`) prints a detailed description of the curve

OUTPUT:

- list containing
  - the equations
  - the initial equations
  - the chart

EXAMPLES:

System defining a geodesic:

```
sage: S2 = Manifold(2, 'S^2')
sage: X.<theta,phi> = S2.chart()
sage: [t, A] = var('t A')
sage: g = S2.metric('g')
```

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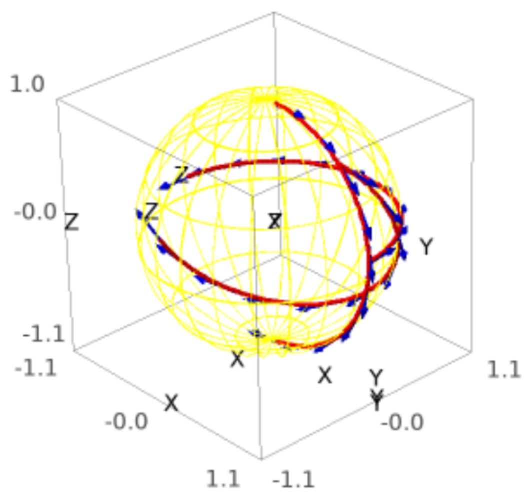
1.0
-0.0
-1.1
-1.1
-0.0
1.1
-1.1
-0.0
1.1
Z
X
Y

(continued from previous page)

```
sage: g[0,0] = A
sage: g[1,0] = 0
sage: g[1,1] = A*sin(theta)^2
sage: p = S2.point((pi/2,0), name='p')
sage: Tp = S2.tangent_space(p)
sage: v = Tp((1/sqrt(2),1/sqrt(2)))
sage: c = S2.integrated_geodesic(g, (t, 0, pi), v, name='c')
sage: sys = c.system(verbose=True)
Geodesic c in the 2-dimensional differentiable manifold S^2
 equipped with Riemannian metric g on the 2-dimensional
 differentiable manifold S^2, and integrated over the Real
 interval (0, pi) as a solution to the following geodesic
 equations, written with respect to Chart (S^2, (theta, phi)):

Initial point: Point p on the 2-dimensional differentiable
 manifold S^2 with coordinates [1/2*pi, 0] with respect to
 Chart (S^2, (theta, phi))
Initial tangent vector: Tangent vector at Point p on the
 2-dimensional differentiable manifold S^2 with
 components [1/2*sqrt(2), 1/2*sqrt(2)] with respect to
 Chart (S^2, (theta, phi))

d(theta)/dt = Dtheta
d(phi)/dt = Dphi
d(Dtheta)/dt = Dphi^2*cos(theta)*sin(theta)
d(Dphi)/dt = -2*Dphi*Dtheta*cos(theta)/sin(theta)

sage: sys_bis = c.system()
sage: sys_bis == sys
True
```

2.6 Tangent Spaces

2.6.1 Tangent Spaces

The class *TangentSpace* implements tangent vector spaces to a differentiable manifold.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- Chap. 3 of [**?**]

class sage.manifolds.differentiable.tangent_space.**TangentSpace**(*point*)

Bases: sage.tensor.modules.finite_rank_free_module.FiniteRankFreeModule

Tangent space to a differentiable manifold at a given point.

Let M be a differentiable manifold of dimension n over a topological field K and $p \in M$. The tangent space T_pM is an n-dimensional vector space over K (without a distinguished basis).

INPUT:

- point – *ManifoldPoint*; point p at which the tangent space is defined

EXAMPLES:

Tangent space on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: p = M.point((-1,2), name='p')
sage: Tp = M.tangent_space(p) ; Tp
Tangent space at Point p on the 2-dimensional differentiable manifold M
```

Tangent spaces are free modules of finite rank over `SymbolicRing` (actually vector spaces of finite dimension over the manifold base field K, with $K = \mathbf{R}$ here):

```
sage: Tp.base_ring()
Symbolic Ring
sage: Tp.category()
Category of finite dimensional vector spaces over Symbolic Ring
sage: Tp.rank()
2
sage: dim(Tp)
2
```

The tangent space is automatically endowed with bases deduced from the vector frames around the point:

```
sage: Tp.bases()
[Basis (d/dx,d/dy) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M]
sage: M.frames()
[Coordinate frame (M, (d/dx,d/dy))]
```

At this stage, only one basis has been defined in the tangent space, but new bases can be added from vector frames on the manifold by means of the method `at()`, for instance, from the frame associated with some new coordinates:

```
sage: c_uv.<u,v> = M.chart()
sage: c_uv.frame().at(p)
Basis (d/du,d/dv) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: Tp.bases()
[Basis (d/dx,d/dy) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M,
 Basis (d/du,d/dv) on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M]
```

All the bases defined on `Tp` are on the same footing. Accordingly the tangent space is not in the category of modules with a distinguished basis:

```
sage: Tp in ModulesWithBasis(SR)
False
```

It is simply in the category of modules:

```
sage: Tp in Modules(SR)
True
```

Since the base ring is a field, it is actually in the category of vector spaces:

```
sage: Tp in VectorSpaces(SR)
True
```

A typical element:

```
sage: v = Tp.an_element() ; v
Tangent vector at Point p on the
 2-dimensional differentiable manifold M
sage: v.display()
d/dx + 2 d/dy
sage: v.parent()
Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

The zero vector:

```
sage: Tp.zero()
Tangent vector zero at Point p on the
 2-dimensional differentiable manifold M
sage: Tp.zero().display()
zero = 0
sage: Tp.zero().parent()
Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

Tangent spaces are unique:

```
sage: M.tangent_space(p) is Tp
True
sage: p1 = M.point((-1,2))
sage: M.tangent_space(p1) is Tp
True
```

even if points are not:

```
sage: p1 is p
False
```

Actually `p1` and `p` share the same tangent space because they compare equal:

```
sage: p1 == p
True
```

The tangent-space uniqueness holds even if the points are created in different coordinate systems:

```
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y))
sage: uv_to_xv = xy_to_uv.inverse()
sage: p2 = M.point((1, -3), chart=c_uv, name='p_2')
sage: p2 is p
False
sage: M.tangent_space(p2) is Tp
True
sage: p2 == p
True
```

See also:

`FiniteRankFreeModule` for more documentation.

Element
alias of *sage.manifolds.differentiable.tangent_vector.TangentVector*

base_point()
Return the manifold point at which `self` is defined.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((1,-2), name='p')
sage: Tp = M.tangent_space(p)
sage: Tp.base_point()
Point p on the 2-dimensional differentiable manifold M
sage: Tp.base_point() is p
True
```

dim()
Return the vector space dimension of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((1,-2), name='p')
sage: Tp = M.tangent_space(p)
sage: Tp.dimension()
2
```

A shortcut is `dim()`:

```
sage: Tp.dim()
2
```

One can also use the global function `dim`:

```
sage: dim(Tp)
2
```

dimension()
Return the vector space dimension of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((1,-2), name='p')
sage: Tp = M.tangent_space(p)
sage: Tp.dimension()
2
```

A shortcut is `dim()`:

```
sage: Tp.dim()
2
```

One can also use the global function `dim`:

```
sage: dim(Tp)
2
```

2.6.2 Tangent Vectors

The class *TangentVector* implements tangent vectors to a differentiable manifold.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- Chap. 3 of [**?**]

class sage.manifolds.differentiable.tangent_vector.**TangentVector**(*parent*, *name=None*, *latex_name=None*)

Bases: sage.tensor.modules.free_module_element.FiniteRankFreeModuleElement

Tangent vector to a differentiable manifold at a given point.

INPUT:

- parent – *TangentSpace*; the tangent space to which the vector belongs
- name – (default: None) string; symbol given to the vector
- latex_name – (default: None) string; LaTeX symbol to denote the vector; if None, name will be used

EXAMPLES:

A tangent vector v on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((2,3), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((-2,1), name='v') ; v
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
sage: v.display()
v = -2 d/dx + d/dy
sage: v.parent()
Tangent space at Point p on the 2-dimensional differentiable manifold M
sage: v in Tp
True
```

By definition, a tangent vector at $p \in M$ is a *derivation at* p on the space $C^\infty(M)$ of smooth scalar fields on M. Indeed let us consider a generic scalar field f:

```
sage: f = M.scalar_field(function('F')(x,y), name='f')
sage: f.display()
f: M --> R
   (x, y) |--> F(x, y)
```

The tangent vector v maps f to the real number $v^i \left.\frac{\partial F}{\partial x^i}\right|_p$:

```
sage: v(f)
-2*D[0](F)(2, 3) + D[1](F)(2, 3)
sage: vdf(x, y) = v[0]*diff(f.expr(), x) + v[1]*diff(f.expr(), y)
sage: X(p)
(2, 3)
sage: bool( v(f) == vdf(*X(p)) )
True
```

and if g is a second scalar field on M:

```
sage: g = M.scalar_field(function('G')(x,y), name='g')
```

then the product fg is also a scalar field on M:

```
sage: (f*g).display()
f*g: M --> R
   (x, y) |--> F(x, y)*G(x, y)
```

and we have the derivation law $v(fg) = v(f)g(p) + f(p)v(g)$:

```
sage: bool( v(f*g) == v(f)*g(p) + f(p)*v(g) )
True
```

See also:

`FiniteRankFreeModuleElement` for more documentation.

plot(*chart=None*, *ambient_coords=None*, *mapping=None*, *color='blue'*, *print_label=True*, *label=None*, *label_color=None*, *fontsize=10*, *label_offset=0.1*, *parameters=None*, *scale=1*, ***extra_options*)

Plot the vector in a Cartesian graph based on the coordinates of some ambient chart.

The vector is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of a given chart, called hereafter the *ambient chart*. The vector's base point p (or its image $\Phi(p)$ by some differentiable mapping Φ) must lie in the ambient chart's domain. If Φ is different from the identity mapping, the vector actually depicted is $\mathrm{d}\Phi_p(v)$, where v is the current vector (`self`) (see the example of a vector tangent to the 2-sphere below, where $\Phi : S^2 \to \mathbf{R}^3$).

INPUT:

- `chart` – (default: `None`) the ambient chart (see above); if `None`, it is set to the default chart of the open set containing the point at which the vector (or the vector image via the differential $\mathrm{d}\Phi_p$ of `mapping`) is defined
- `ambient_coords` – (default: `None`) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if `None`, all the coordinates of the ambient chart are considered
- `mapping` – (default: `None`) `DiffMap`; differentiable mapping Φ providing the link between the point p at which the vector is defined and the ambient chart `chart`: the domain of `chart` must contain $\Phi(p)$; if `None`, the identity mapping is assumed
- `scale` – (default: 1) value by which the length of the arrow representing the vector is multiplied
- `color` – (default: 'blue') color of the arrow representing the vector
- `print_label` – (boolean; default: `True`) determines whether a label is printed next to the arrow representing the vector
- `label` – (string; default: `None`) label printed next to the arrow representing the vector; if `None`, the vector's symbol is used, if any

- `label_color` – (default: `None`) color to print the label; if `None`, the value of `color` is used
- `fontsize` – (default: 10) size of the font used to print the label
- `label_offset` – (default: 0.1) determines the separation between the vector arrow and the label
- `parameters` – (default: `None`) dictionary giving the numerical values of the parameters that may appear in the coordinate expression of `self` (see example below)
- `**extra_options` – extra options for the arrow plot, like `linestyle`, `width` or `arrowsize` (see `arrow2d()` and `arrow3d()` for details)

OUTPUT:

- a graphic object, either an instance of `Graphics` for a 2D plot (i.e. based on 2 coordinates of `chart`) or an instance of `Graphics3d` for a 3D plot (i.e. based on 3 coordinates of `chart`)

EXAMPLES:

Vector tangent to a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M((2,2), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((2, 1), name='v') ; v
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
```

Plot of the vector alone (arrow + label):

```
sage: v.plot()
Graphics object consisting of 2 graphics primitives
```

Plot atop of the chart grid:

```
sage: X.plot() + v.plot()
Graphics object consisting of 20 graphics primitives
```

Plots with various options:

```
sage: X.plot() + v.plot(color='green', scale=2, label='V')
Graphics object consisting of 20 graphics primitives
```

```
sage: X.plot() + v.plot(print_label=False)
Graphics object consisting of 19 graphics primitives
```

```
sage: X.plot() + v.plot(color='green', label_color='black',
....:                   fontsize=20, label_offset=0.2)
Graphics object consisting of 20 graphics primitives
```

```
sage: X.plot() + v.plot(linestyle=':', width=4, arrowsize=8,
....:                   fontsize=20)
Graphics object consisting of 20 graphics primitives
```

Plot with specific values of some free parameters:

```
sage: var('a b')
(a, b)
```

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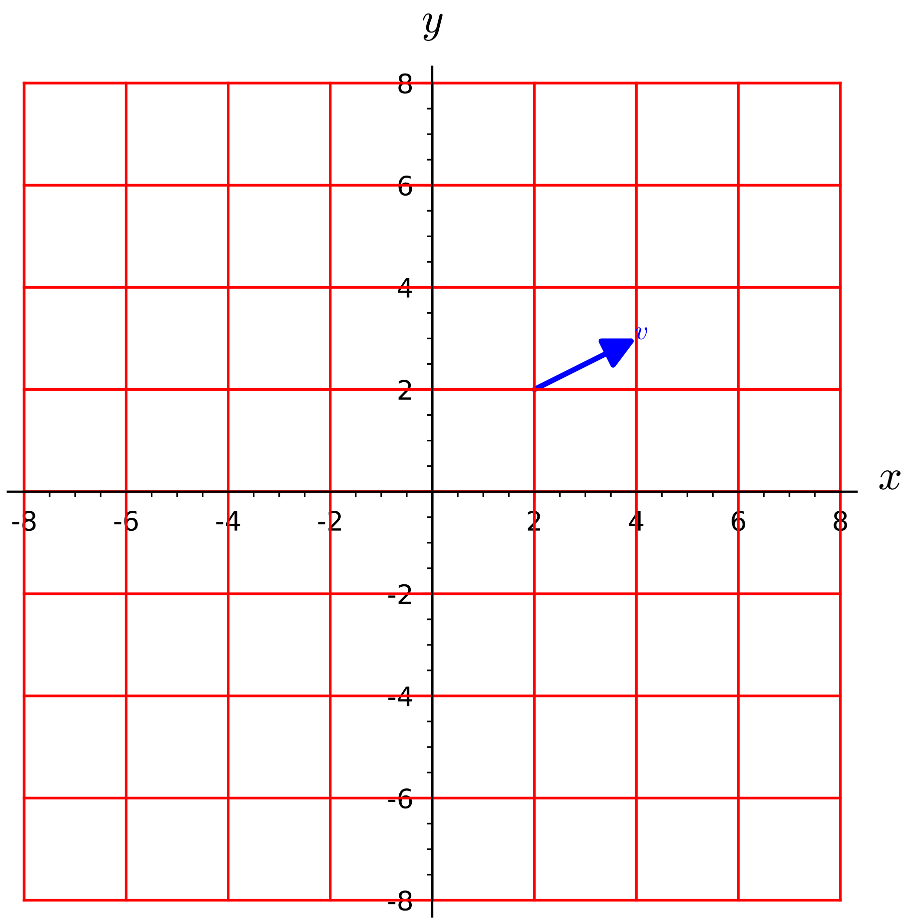
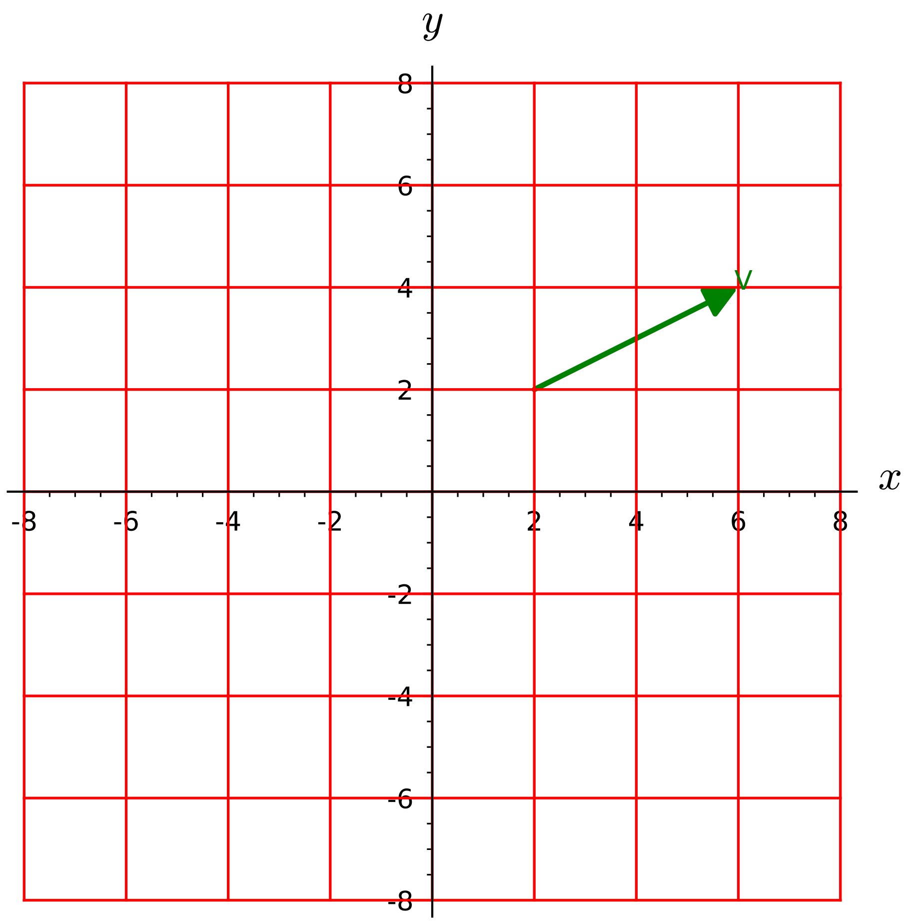
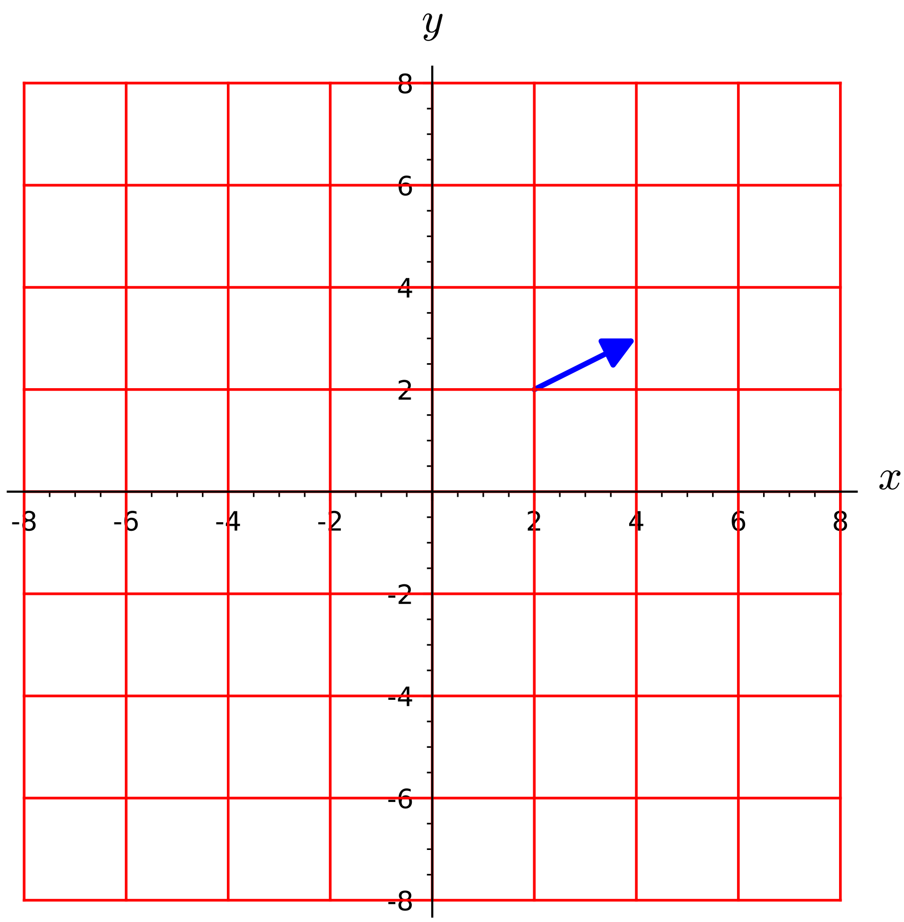
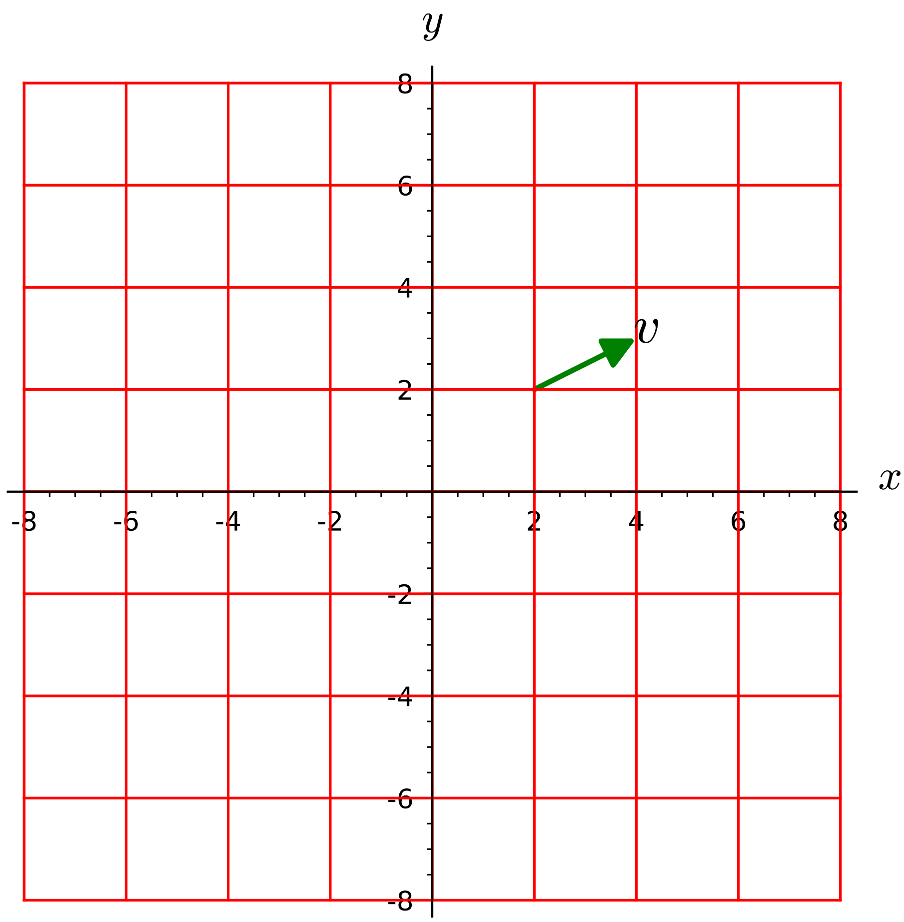


y
x
v
8
6
4
2
-2
-4
-6
-8

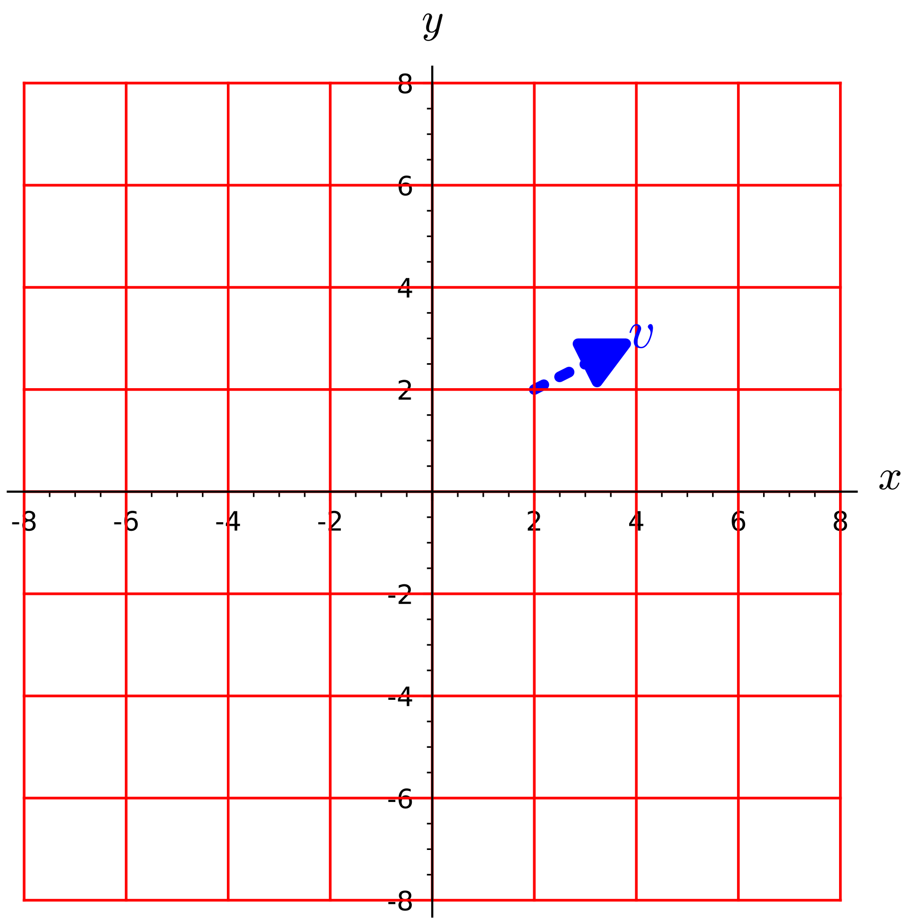

y
x
v
8
6
4
2
-2
-4
-6
-8
-8
-6
-4
-2
2
4
6
8



y
x
8
6
4
2
-2
-4
-6
-8
-8
-6
-4
-2
2
4
6
8



y
x
v
8
6
4
2
-2
-4
-6
-8
-8
-6
-4
-2
2
4
6
8



y
8
6
4
2
-2
-4
-6
-8
-8
-6
-4
-2
2
4
6
8
x
v

(continued from previous page)

```
sage: v = Tp((1+a, -b^2), name='v') ; v.display()
v = (a + 1) d/dx - b^2 d/dy
sage: X.plot() + v.plot(parameters={a: -2, b: 3})
Graphics object consisting of 20 graphics primitives
```

Special case of the zero vector:

```
sage: v = Tp.zero() ; v
Tangent vector zero at Point p on the 2-dimensional differentiable
 manifold M
sage: X.plot() + v.plot()
Graphics object consisting of 19 graphics primitives
```

Vector tangent to a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: p = M((0,1,2,3), name='p')
sage: Tp = M.tangent_space(p)
sage: v = Tp((5,4,3,2), name='v') ; v
Tangent vector v at Point p on the 4-dimensional differentiable
 manifold M
```

We cannot make a 4D plot directly:

```
sage: v.plot()
Traceback (most recent call last):
...
ValueError: the number of coordinates involved in the plot must
 be either 2 or 3, not 4
```

Rather, we have to select some chart coordinates for the plot, via the argument `ambient_coords`. For instance, for a 2-dimensional plot in terms of the coordinates (x, y):

```
sage: v.plot(ambient_coords=(x,y))
Graphics object consisting of 2 graphics primitives
```

This plot involves only the components v^x and v^y of v. Similarly, for a 3-dimensional plot in terms of the coordinates (t, x, y):

```
sage: g = v.plot(ambient_coords=(t,x,z))
sage: print(g)
Graphics3d Object
```

This plot involves only the components v^t, v^x and v^z of v. A nice 3D view atop the coordinate grid is obtained via:

```
sage: (X.plot(ambient_coords=(t,x,z))  # long time
....:  + v.plot(ambient_coords=(t,x,z),
....:           label_offset=0.5, width=6))
Graphics3d Object
```

An example of plot via a differential mapping: plot of a vector tangent to a 2-sphere viewed in $\mathbf{R}^3$:

```
sage: S2 = Manifold(2, 'S^2')
sage: U = S2.open_subset('U') # the open set covered by spherical coord.
```

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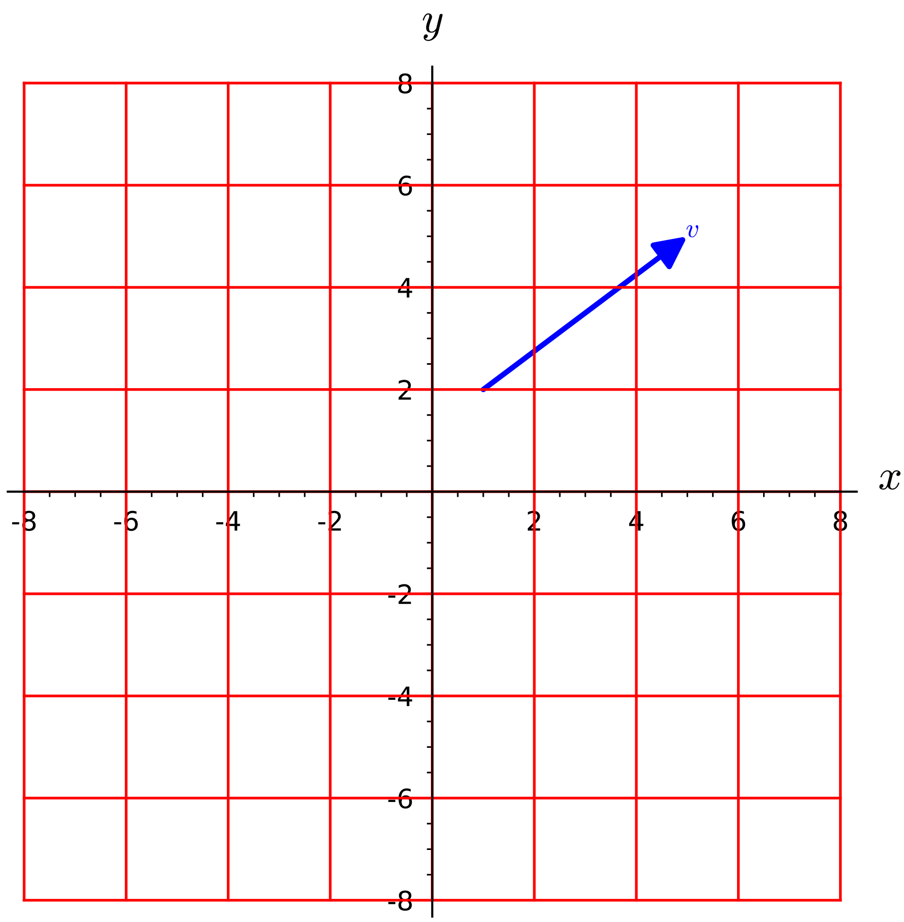
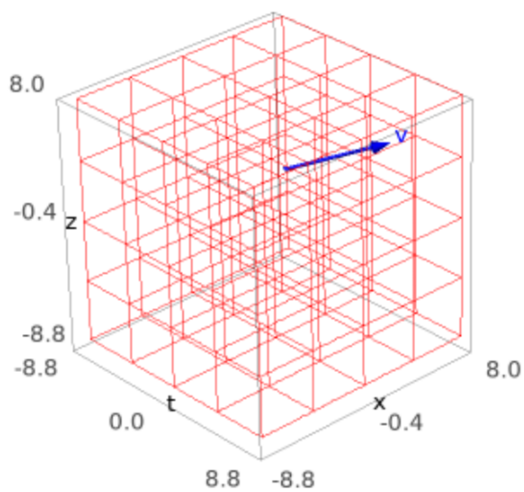
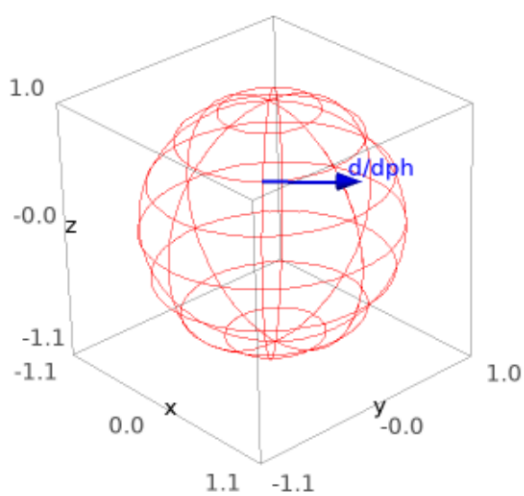


y
x
v
8
6
4
2
-2
-4
-6
-8


8.0
-0.4
z
-8.8
-8.8
t
0.0
8.8
-8.8
x
-0.4
8.0
v

(continued from previous page)

```
sage: XS.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: R3 = Manifold(3, 'R^3')
sage: X3.<x,y,z> = R3.chart()
sage: F = S2.diff_map(R3, {(XS, X3): [sin(th)*cos(ph),
....:                                  sin(th)*sin(ph),
....:                                  cos(th)]}, name='F')
sage: F.display() # the standard embedding of S^2 into R^3
F: S^2 --> R^3
on U: (th, ph) |--> (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
sage: p = U.point((pi/4, 7*pi/4), name='p')
sage: v = XS.frame()[1].at(p) ; v  # the coordinate vector d/dphi at p
Tangent vector d/dph at Point p on the 2-dimensional differentiable
 manifold S^2
sage: graph_v = v.plot(mapping=F)
sage: graph_S2 = XS.plot(chart=X3, mapping=F, number_values=9)  # long time
sage: graph_v + graph_S2  # long time
Graphics3d Object
```



2.7 Vector Fields

2.7.1 Vector Field Modules

The set of vector fields along a differentiable manifold U with values on a differentiable manifold M via a differentiable map $\Phi : U \to M$ (possibly $U = M$ and $\Phi = \mathrm{Id}_M$) is a module over the algebra $C^k(U)$ of differentiable scalar fields on U. If Φ is the identity map, this module is considered a Lie algebroid under the Lie bracket $[\ ,\ ]$ (cf. Wikipedia article Lie_algebroid). It is a free module if and only if M is parallelizable. Accordingly, there are two classes for vector field modules:

- *VectorFieldModule* for vector fields with values on a generic (in practice, not parallelizable) differentiable manifold M.
- *VectorFieldFreeModule* for vector fields with values on a parallelizable manifold M.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): structure of Lie algebroid (trac ticket #20771)

REFERENCES:

- [?]
- [?]
- [?]

class sage.manifolds.differentiable.vectorfield_module.**VectorFieldFreeModule**(*domain*, *dest_map=None*)

Bases: sage.tensor.modules.finite_rank_free_module.FiniteRankFreeModule

Free module of vector fields along a differentiable manifold U with values on a parallelizable manifold M, via a differentiable map $U \to M$.

Given a differentiable map

$$\Phi:\ U \longrightarrow M$$

the *vector field module* $\mathfrak{X}(U, \Phi)$ is the set of all vector fields of the type

$$v:\ U \longrightarrow TM$$

(where TM is the tangent bundle of M) such that

$$\forall p \in U,\ v(p) \in T_{\Phi(p)}M,$$

where $T_{\Phi(p)}M$ is the tangent space to M at the point $\Phi(p)$.

Since M is parallelizable, the set $\mathfrak{X}(U, \Phi)$ is a free module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U (see *DiffScalarFieldAlgebra*). In fact, it carries the structure of a finite-dimensional Lie algebroid (cf. Wikipedia article Lie_algebroid).

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$; we then denote $\mathfrak{X}(M, \mathrm{Id}_M)$ by merely $\mathfrak{X}(M)$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is not parallelizable, the class *VectorFieldModule* should be used instead, for $\mathfrak{X}(U, \Phi)$ is no longer a free module.

INPUT:

- `domain` – differentiable manifold U along which the vector fields are defined
- `dest_map` – (default: `None`) destination map $\Phi : U \to M$ (type: `DiffMap`); if `None`, it is assumed that $U = M$ and Φ is the identity map of M (case of vector fields *on* M)

EXAMPLES:

Module of vector fields on $\mathbf{R}^2$:

```
sage: M = Manifold(2, 'R^2')
sage: cart.<x,y> = M.chart()   # Cartesian coordinates on R^2
sage: XM = M.vector_field_module() ; XM
Free module X(R^2) of vector fields on the 2-dimensional differentiable
 manifold R^2
sage: XM.category()
Category of finite dimensional modules
 over Algebra of differentiable scalar fields
 on the 2-dimensional differentiable manifold R^2
sage: XM.base_ring() is M.scalar_field_algebra()
True
```

Since $\mathbf{R}^2$ is obviously parallelizable, `XM` is a free module:

```
sage: isinstance(XM, FiniteRankFreeModule)
True
```

Some elements:

```
sage: XM.an_element().display()
2 d/dx + 2 d/dy
sage: XM.zero().display()
zero = 0
sage: v = XM([-y,x]) ; v
Vector field on the 2-dimensional differentiable manifold R^2
sage: v.display()
-y d/dx + x d/dy
```

An example of module of vector fields with a destination map Φ different from the identity map, namely a mapping $\Phi : I \to \mathbf{R}^2$, where I is an open interval of $\mathbf{R}$:

```
sage: I = Manifold(1, 'I')
sage: canon.<t> = I.chart('t:(0,2*pi)')
sage: Phi = I.diff_map(M, coord_functions=[cos(t), sin(t)], name='Phi',
....:                  latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold
 I to the 2-dimensional differentiable manifold R^2
sage: Phi.display()
Phi: I --> R^2
   t |--> (x, y) = (cos(t), sin(t))
sage: XIM = I.vector_field_module(dest_map=Phi) ; XIM
Free module X(I,Phi) of vector fields along the 1-dimensional
 differentiable manifold I mapped into the 2-dimensional differentiable
 manifold R^2
sage: XIM.category()
Category of finite dimensional modules
 over Algebra of differentiable scalar fields
 on the 1-dimensional differentiable manifold I
```

The rank of the free module $\mathfrak{X}(I, \Phi)$ is the dimension of the manifold $\mathbf{R}^2$, namely two:

```
sage: XIM.rank()
2
```

A basis of it is induced by the coordinate vector frame of $\mathbf{R}^2$:

```
sage: XIM.bases()
[Vector frame (I, (d/dx,d/dy)) with values on the 2-dimensional
 differentiable manifold R^2]
```

Some elements of this module:

```
sage: XIM.an_element().display()
2 d/dx + 2 d/dy
sage: v = XIM([t, t^2]) ; v
Vector field along the 1-dimensional differentiable manifold I with
 values on the 2-dimensional differentiable manifold R^2
sage: v.display()
t d/dx + t^2 d/dy
```

The test suite is passed:

```
sage: TestSuite(XIM).run()
```

Let us introduce an open subset of $J \subset I$ and the vector field module corresponding to the restriction of Φ to it:

```
sage: J = I.open_subset('J', coord_def= {canon: t<pi})
sage: XJM = J.vector_field_module(dest_map=Phi.restrict(J)); XJM
Free module X(J,Phi) of vector fields along the Open subset J of the
 1-dimensional differentiable manifold I mapped into the 2-dimensional
 differentiable manifold R^2
```

We have then:

```
sage: XJM.default_basis()
Vector frame (J, (d/dx,d/dy)) with values on the 2-dimensional
 differentiable manifold R^2
sage: XJM.default_basis() is XIM.default_basis().restrict(J)
True
sage: v.restrict(J)
Vector field along the Open subset J of the 1-dimensional
 differentiable manifold I with values on the 2-dimensional
 differentiable manifold R^2
sage: v.restrict(J).display()
t d/dx + t^2 d/dy
```

Let us now consider the module of vector fields on the circle S^1; we start by constructing the S^1 manifold:

```
sage: M = Manifold(1, 'S^1')
sage: U = M.open_subset('U')  # the complement of one point
sage: c_t.<t> =  U.chart('t:(0,2*pi)') # the standard angle coordinate
sage: V = M.open_subset('V') # the complement of the point t=pi
sage: M.declare_union(U,V)   # S^1 is the union of U and V
sage: c_u.<u> = V.chart('u:(0,2*pi)') # the angle t-pi
sage: t_to_u = c_t.transition_map(c_u, (t-pi,), intersection_name='W',
....:                      restrictions1 = t!=pi, restrictions2 = u!=pi)
```

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```
sage: u_to_t = t_to_u.inverse()
sage: W = U.intersection(V)
```

S^1 cannot be covered by a single chart, so it cannot be covered by a coordinate frame. It is however parallelizable and we introduce a global vector frame as follows. We notice that on their common subdomain, W, the coordinate vectors $\partial/\partial t$ and $\partial/\partial u$ coincide, as we can check explicitly:

```
sage: c_t.frame()[0].display(c_u.frame().restrict(W))
d/dt = d/du
```

Therefore, we can extend $\partial/\partial t$ to all V and hence to all S^1, to form a vector field on S^1 whose components w.r.t. both $\partial/\partial t$ and $\partial/\partial u$ are 1:

```
sage: e = M.vector_frame('e')
sage: U.set_change_of_frame(e.restrict(U), c_t.frame(),
....:                       U.tangent_identity_field())
sage: V.set_change_of_frame(e.restrict(V), c_u.frame(),
....:                       V.tangent_identity_field())
sage: e[0].display(c_t.frame())
e_0 = d/dt
sage: e[0].display(c_u.frame())
e_0 = d/du
```

Equipped with the frame e, the manifold S^1 is manifestly parallelizable:

```
sage: M.is_manifestly_parallelizable()
True
```

Consequently, the module of vector fields on S^1 is a free module:

```
sage: XM = M.vector_field_module() ; XM
Free module X(S^1) of vector fields on the 1-dimensional differentiable
 manifold S^1
sage: isinstance(XM, FiniteRankFreeModule)
True
sage: XM.category()
Category of finite dimensional modules
 over Algebra of differentiable scalar fields
 on the 1-dimensional differentiable manifold S^1
sage: XM.base_ring() is M.scalar_field_algebra()
True
```

The zero element:

```
sage: z = XM.zero() ; z
Vector field zero on the 1-dimensional differentiable manifold S^1
sage: z.display()
zero = 0
sage: z.display(c_t.frame())
zero = 0
```

The module $\mathfrak{X}(S^1)$ coerces to any module of vector fields defined on a subdomain of S^1, for instance $\mathfrak{X}(U)$:

```
sage: XU = U.vector_field_module() ; XU
Free module X(U) of vector fields on the Open subset U of the
 1-dimensional differentiable manifold S^1
```

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```
sage: XU.has_coerce_map_from(XM)
True
sage: XU.coerce_map_from(XM)
Coercion map:
  From: Free module X(S^1) of vector fields on the 1-dimensional
   differentiable manifold S^1
  To:   Free module X(U) of vector fields on the Open subset U of the
   1-dimensional differentiable manifold S^1
```

The conversion map is actually the restriction of vector fields defined on S^1 to U.

The Sage test suite for modules is passed:

```
sage: TestSuite(XM).run()
```

Element

alias of *sage.manifolds.differentiable.vectorfield.VectorFieldParal*

ambient_domain()

Return the manifold in which the vector fields of `self` take their values.

If the module is $\mathfrak{X}(U, \Phi)$, returns the codomain M of Φ.

OUTPUT:

- a *DifferentiableManifold* representing the manifold in which the vector fields of `self` take their values

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.ambient_domain()
3-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Y.<u,v> = U.chart()
sage: Phi = U.diff_map(M, {(Y,X): [u+v, u-v, u*v]}, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.ambient_domain()
3-dimensional differentiable manifold M
```

basis(*symbol=None*, *latex_symbol=None*, *from_frame=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Define a basis of `self`.

A basis of the vector field module is actually a vector frame along the differentiable manifold U over which the vector field module is defined.

If the basis specified by the given symbol already exists, it is simply returned. If no argument is provided the module's default basis is returned.

INPUT:

- `symbol` – (default: `None`) either a string, to be used as a common base for the symbols of the elements of the basis, or a tuple of strings, representing the individual symbols of the elements of the basis
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the elements of the basis, or a tuple of strings, representing the individual LaTeX symbols of the elements of the basis; if `None`, `symbol` is used in place of `latex_symbol`

- `from_frame` – (default: `None`) vector frame $\tilde{e}$ on the codomain M of the destination map Φ of `self`; the returned basis e is then such that for all $p \in U$, we have $e(p) = \tilde{e}(\Phi(p))$
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the elements of the basis; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the elements of the basis; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual basis; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual basis
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual basis

OUTPUT:

- a `VectorFrame` representing a basis on `self`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: e = XM.basis('e'); e
Vector frame (M, (e_0,e_1))
```

See `VectorFrame` for more examples and documentation.

destination_map()

Return the differential map associated to `self`.

The differential map associated to this module is the map

$$\Phi : U \longrightarrow M$$

such that this module is the set $\mathfrak{X}(U, \Phi)$ of all vector fields of the type

$$v : U \longrightarrow TM$$

(where TM is the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M,$$

where $T_{\Phi(p)}M$ is the tangent space to M at the point $\Phi(p)$.

OUTPUT:

- a `DiffMap` representing the differential map Φ

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.destination_map()
Identity map Id_M of the 3-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Y.<u,v> = U.chart()
sage: Phi = U.diff_map(M, {(Y,X): [u+v, u-v, u*v]}, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.destination_map()
Differentiable map Phi from the 2-dimensional differentiable
 manifold U to the 3-dimensional differentiable manifold M
```

domain()

Return the domain of the vector fields in `self`.

If the module is $\mathfrak{X}(U,\Phi)$, returns the domain U of Φ.

OUTPUT:

- a *DifferentiableManifold* representing the domain of the vector fields that belong to this module

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.domain()
3-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Y.<u,v> = U.chart()
sage: Phi = U.diff_map(M, {(Y,X): [u+v, u-v, u*v]}, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.domain()
2-dimensional differentiable manifold U
```

dual_exterior_power(p)

Return the p-th exterior power of the dual of `self`.

If the vector field module `self` is $\mathfrak{X}(U,\Phi)$, the p-th exterior power of its dual is the set $\Omega^p(U,\Phi)$ of p-forms along U with values on $\Phi(U)$. It is a free module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U.

INPUT:

- `p` – non-negative integer

OUTPUT:

- for $p = 0$, the base ring, i.e. $C^k(U)$
- for $p \geq 1$, a *DiffFormFreeModule* representing the module $\Omega^p(U,\Phi)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.dual_exterior_power(2)
Free module Omega^2(M) of 2-forms on the 2-dimensional
 differentiable manifold M
sage: XM.dual_exterior_power(1)
Free module Omega^1(M) of 1-forms on the 2-dimensional
 differentiable manifold M
sage: XM.dual_exterior_power(1) is XM.dual()
True
sage: XM.dual_exterior_power(0)
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: XM.dual_exterior_power(0) is M.scalar_field_algebra()
True
```

See also:

DiffFormFreeModule for more examples and documentation.

exterior_power (p)

Return the p-th exterior power of `self`.

If the vector field module `self` is $\mathfrak{X}(U,\Phi)$, its p-th exterior power is the set $A^p(U,\Phi)$ of p-vector fields along U with values on $\Phi(U)$. It is a free module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U.

INPUT:

- `p` – non-negative integer

OUTPUT:

- for $p = 0$, the base ring, i.e. $C^k(U)$
- for $p = 1$, the vector field free module `self`, since $A^1(U,\Phi) = \mathfrak{X}(U,\Phi)$
- for $p \geq 2$, instance of `MultivectorFreeModule` representing the module $A^p(U,\Phi)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()   # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.exterior_power(2)
Free module A^2(M) of 2-vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(1)
Free module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(1) is XM
True
sage: XM.exterior_power(0)
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(0) is M.scalar_field_algebra()
True
```

See also:

`MultivectorFreeModule` for more examples and documentation.

general_linear_group ()

Return the general linear group of `self`.

If the vector field module is $\mathfrak{X}(U,\Phi)$, the *general linear group* is the group $\mathrm{GL}(\mathfrak{X}(U,\Phi))$ of automorphisms of $\mathfrak{X}(U,\Phi)$. Note that an automorphism of $\mathfrak{X}(U,\Phi)$ can also be viewed as a *field* along U of automorphisms of the tangent spaces of $V = \Phi(U)$.

OUTPUT:

- a `AutomorphismFieldParalGroup` representing $\mathrm{GL}(\mathfrak{X}(U,\Phi))$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()   # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.general_linear_group()
General linear group of the Free module X(M) of vector fields on
 the 2-dimensional differentiable manifold M
```

See also:

AutomorphismFieldParalGroup for more examples and documentation.

metric(*name*, *signature=None*, *latex_name=None*)

Construct a pseudo-Riemannian metric (nondegenerate symmetric bilinear form) on the current vector field module.

A pseudo-Riemannian metric of the vector field module is actually a field of tangent-space non-degenerate symmetric bilinear forms along the manifold U on which the vector field module is defined.

INPUT:

- `name` – (string) name given to the metric
- `signature` – (integer; default: `None`) signature S of the metric: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

OUTPUT:

- instance of *PseudoRiemannianMetricParal* representing the defined pseudo-Riemannian metric.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.metric('g')
Riemannian metric g on the 2-dimensional differentiable manifold M
sage: XM.metric('g', signature=0)
Lorentzian metric g on the 2-dimensional differentiable manifold M
```

See also:

PseudoRiemannianMetricParal for more documentation.

sym_bilinear_form(*name=None*, *latex_name=None*)

Construct a symmetric bilinear form on `self`.

A symmetric bilinear form on the vector field module is actually a field of tangent-space symmetric bilinear forms along the differentiable manifold U over which the vector field module is defined.

INPUT:

- `name` – string (default: `None`); name given to the symmetric bilinear bilinear form
- `latex_name` – string (default: `None`); LaTeX symbol to denote the symmetric bilinear form; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- a *TensorFieldParal* of tensor type $(0, 2)$ and symmetric

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
```

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```
sage: XM.sym_bilinear_form(name='a')
Field of symmetric bilinear forms a on the 2-dimensional
 differentiable manifold M
```

See also:

TensorFieldParal for more examples and documentation.

tensor(*tensor_type*, *name=None*, *latex_name=None*, *sym=None*, *antisym=None*, *specific_type=None*)
Construct a tensor on `self`.

The tensor is actually a tensor field along the differentiable manifold U over which `self` is defined.

INPUT:

- `tensor_type` – pair (k,l) with k being the contravariant rank and l the covariant rank
- `name` – (string; default: `None`) name given to the tensor
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the tensor; if none is provided, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention position=0 for the first argument; for instance:
  - `sym = (0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym = [(0,2), (1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`
- `specific_type` – (default: `None`) specific subclass of *TensorFieldParal* for the output

OUTPUT:

- a *TensorFieldParal* representing the tensor defined on `self` with the provided characteristics

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()   # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.tensor((1,2), name='t')
Tensor field t of type (1,2) on the 2-dimensional
 differentiable manifold M
sage: XM.tensor((1,0), name='a')
Vector field a on the 2-dimensional differentiable
 manifold M
sage: XM.tensor((0,2), name='a', antisym=(0,1))
2-form a on the 2-dimensional differentiable manifold M
sage: XM.tensor((2,0), name='a', antisym=(0,1))
2-vector field a on the 2-dimensional differentiable
 manifold M
```

See *TensorFieldParal* for more examples and documentation.

tensor_from_comp(*tensor_type*, *comp*, *name=None*, *latex_name=None*)
Construct a tensor on `self` from a set of components.

The tensor is actually a tensor field along the differentiable manifold U over which the vector field module is defined. The tensor symmetries are deduced from those of the components.

INPUT:

- `tensor_type` – pair (k, l) with k being the contravariant rank and l the covariant rank
- `comp` – `Components`; the tensor components in a given basis
- `name` – string (default: `None`); name given to the tensor
- `latex_name` – string (default: `None`); LaTeX symbol to denote the tensor; if `None`, the LaTeX symbol is set to `name`

OUTPUT:

- a *TensorFieldParal* representing the tensor defined on the vector field module with the provided characteristics

EXAMPLES:

A 2-dimensional set of components transformed into a type-$(1, 1)$ tensor field:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: XM = M.vector_field_module()
sage: from sage.tensor.modules.comp import Components
sage: comp = Components(M.scalar_field_algebra(), X.frame(), 2,
....:                   output_formatter=XM._output_formatter)
sage: comp[:] = [[1+x, -y], [x*y, 2-y^2]]
sage: t = XM.tensor_from_comp((1,1), comp, name='t'); t
Tensor field t of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: t.display()
t = (x + 1) d/dx*dx - y d/dx*dy + x*y d/dy*dx + (-y^2 + 2) d/dy*dy
```

The same set of components transformed into a type-$(0, 2)$ tensor field:

```
sage: t = XM.tensor_from_comp((0,2), comp, name='t'); t
Tensor field t of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: t.display()
t = (x + 1) dx*dx - y dx*dy + x*y dy*dx + (-y^2 + 2) dy*dy
```

tensor_module (k, l)

Return the free module of all tensors of type (k, l) defined on `self`.

INPUT:

- `k` – non-negative integer; the contravariant rank, the tensor type being (k, l)
- `l` – non-negative integer; the covariant rank, the tensor type being (k, l)

OUTPUT:

- a *TensorFieldFreeModule* representing the free module of type-(k, l) tensors on the vector field module

EXAMPLES:

A tensor field module on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()  # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.tensor_module(1,2)
Free module T^(1,2)(M) of type-(1,2) tensors fields on the
 2-dimensional differentiable manifold M
```

The special case of tensor fields of type (1,0):

```
sage: XM.tensor_module(1,0)
Free module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
```

The result is cached:

```
sage: XM.tensor_module(1,2) is XM.tensor_module(1,2)
True
sage: XM.tensor_module(1,0) is XM
True
```

See also:

TensorFieldFreeModule for more examples and documentation.

class sage.manifolds.differentiable.vectorfield_module.**VectorFieldModule**(*domain*, *dest_map=None*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

Module of vector fields along a differentiable manifold U with values on a differentiable manifold M, via a differentiable map $U \to M$.

Given a differentiable map

$$\Phi: U \longrightarrow M,$$

the *vector field module* $\mathfrak{X}(U, \Phi)$ is the set of all vector fields of the type

$$v: U \longrightarrow TM$$

(where TM is the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M,$$

where $T_{\Phi(p)}M$ is the tangent space to M at the point $\Phi(p)$.

The set $\mathfrak{X}(U, \Phi)$ is a module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U (see *DiffScalarFieldAlgebra*). Furthermore, it is a Lie algebroid under the Lie bracket (cf. Wikipedia article Lie_algebroid)

$$[X, Y] = X \circ Y - Y \circ X$$

over the scalarfields if Φ is the identity map. That is to say the Lie bracket is antisymmetric, bilinear over the base field, satisfies the Jacobi identity, and $[X, fY] = X(f)Y + f[X, Y]$.

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$; we then denote $\mathfrak{X}(M, \mathrm{Id}_M)$ by merely $\mathfrak{X}(M)$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is parallelizable, the class *VectorFieldFreeModule* should be used instead.

INPUT:

- `domain` – differentiable manifold U along which the vector fields are defined
- `dest_map` – (default: `None`) destination map $\Phi : U \to M$ (type: *DiffMap*); if `None`, it is assumed that $U = M$ and Φ is the identity map of M (case of vector fields *on* M)

EXAMPLES:

Module of vector fields on the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
```

$\mathfrak{X}(M)$ is a module over the algebra $C^k(M)$:

```
sage: XM.category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold M
sage: XM.base_ring() is M.scalar_field_algebra()
True
```

$\mathfrak{X}(M)$ is not a free module:

```
sage: isinstance(XM, FiniteRankFreeModule)
False
```

because $M = S^2$ is not parallelizable:

```
sage: M.is_manifestly_parallelizable()
False
```

On the contrary, the module of vector fields on U is a free module, since U is parallelizable (being a coordinate domain):

```
sage: XU = U.vector_field_module()
sage: isinstance(XU, FiniteRankFreeModule)
True
sage: U.is_manifestly_parallelizable()
True
```

The zero element of the module:

```
sage: z = XM.zero() ; z
Vector field zero on the 2-dimensional differentiable manifold M
sage: z.display(c_xy.frame())
zero = 0
sage: z.display(c_uv.frame())
zero = 0
```

The module $\mathfrak{X}(M)$ coerces to any module of vector fields defined on a subdomain of M, for instance $\mathfrak{X}(U)$:

```
sage: XU.has_coerce_map_from(XM)
True
sage: XU.coerce_map_from(XM)
Coercion map:
  From: Module X(M) of vector fields on the 2-dimensional
   differentiable manifold M
  To:   Free module X(U) of vector fields on the Open subset U of the
   2-dimensional differentiable manifold M
```

The conversion map is actually the restriction of vector fields defined on M to U.

Element
alias of *sage.manifolds.differentiable.vectorfield.VectorField*

alternating_contravariant_tensor(*degree*, *name=None*, *latex_name=None*)
Construct an alternating contravariant tensor on the vector field module `self`.

An alternating contravariant tensor on `self` is actually a multivector field along the differentiable manifold U over which `self` is defined.

INPUT:

- `degree` – degree of the alternating contravariant tensor (i.e. its tensor rank)
- `name` – (default: `None`) string; name given to the alternating contravariant tensor
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the alternating contravariant tensor; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of *MultivectorField*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.alternating_contravariant_tensor(2, name='a')
2-vector field a on the 2-dimensional differentiable
 manifold M
```

An alternating contravariant tensor of degree 1 is simply a vector field:

```
sage: XM.alternating_contravariant_tensor(1, name='a')
Vector field a on the 2-dimensional differentiable
 manifold M
```

See also:

MultivectorField for more examples and documentation.

alternating_form(*degree*, *name=None*, *latex_name=None*)
Construct an alternating form on the vector field module `self`.

An alternating form on `self` is actually a differential form along the differentiable manifold U over which `self` is defined.

INPUT:

- `degree` – the degree of the alternating form (i.e. its tensor rank)
- `name` – (string; optional) name given to the alternating form
- `latex_name` – (string; optional) LaTeX symbol to denote the alternating form; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of `DiffForm`

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.alternating_form(2, name='a')
2-form a on the 2-dimensional differentiable manifold M
sage: XM.alternating_form(1, name='a')
1-form a on the 2-dimensional differentiable manifold M
```

See also:

`DiffForm` for more examples and documentation.

ambient_domain()

Return the manifold in which the vector fields of this module take their values.

If the module is $\mathfrak{X}(U, \Phi)$, returns the codomain M of Φ.

OUTPUT:

- instance of `DifferentiableManifold` representing the manifold in which the vector fields of this module take their values

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: XM = M.vector_field_module()
sage: XM.ambient_domain()
5-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Phi = U.diff_map(M, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.ambient_domain()
5-dimensional differentiable manifold M
```

automorphism(*name=None*, *latex_name=None*)

Construct an automorphism of the vector field module.

An automorphism of the vector field module is actually a field of tangent-space automorphisms along the differentiable manifold U over which the vector field module is defined.

INPUT:

- `name` – (string; optional) name given to the automorphism
- `latex_name` – (string; optional) LaTeX symbol to denote the automorphism; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of *AutomorphismField*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.automorphism()
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: XM.automorphism(name='a')
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold M
```

See also:

AutomorphismField for more examples and documentation.

destination_map()

Return the differential map associated to this module.

The differential map associated to this module is the map

$$\Phi : U \longrightarrow M$$

such that this module is the set $\mathfrak{X}(U, \Phi)$ of all vector fields of the type

$$v : U \longrightarrow TM$$

(where TM is the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M,$$

where $T_{\Phi(p)}M$ is the tangent space to M at the point $\Phi(p)$.

OUTPUT:

- instance of *DiffMap* representing the differential map Φ

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: XM = M.vector_field_module()
sage: XM.destination_map()
Identity map Id_M of the 5-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Phi = U.diff_map(M, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.destination_map()
Differentiable map Phi from the 2-dimensional differentiable
 manifold U to the 5-dimensional differentiable manifold M
```

domain()

Return the domain of the vector fields in this module.

If the module is $\mathfrak{X}(U, \Phi)$, returns the domain U of Φ.

OUTPUT:

- instance of *DifferentiableManifold* representing the domain of the vector fields that belong to this module

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: XM = M.vector_field_module()
sage: XM.domain()
5-dimensional differentiable manifold M
sage: U = Manifold(2, 'U')
sage: Phi = U.diff_map(M, name='Phi')
sage: XU = U.vector_field_module(dest_map=Phi)
sage: XU.domain()
2-dimensional differentiable manifold U
```

dual()

Return the dual module.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.dual()
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
```

dual_exterior_power(p)

Return the p-th exterior power of the dual of the vector field module.

If the vector field module is $\mathfrak{X}(U, \Phi)$, the p-th exterior power of its dual is the set $\Omega^p(U, \Phi)$ of p-forms along U with values on $\Phi(U)$. It is a module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U.

INPUT:

- `p` – non-negative integer

OUTPUT:

- for $p = 0$, the base ring, i.e. $C^k(U)$
- for $p \geq 1$, instance of *DiffFormModule* representing the module $\Omega^p(U, \Phi)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.dual_exterior_power(2)
Module Omega^2(M) of 2-forms on the 2-dimensional differentiable
 manifold M
sage: XM.dual_exterior_power(1)
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
sage: XM.dual_exterior_power(1) is XM.dual()
True
sage: XM.dual_exterior_power(0)
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: XM.dual_exterior_power(0) is M.scalar_field_algebra()
True
```

See also:

DiffFormModule for more examples and documentation.

exterior_power(p)

Return the p-th exterior power of `self`.

If the vector field module `self` is $\mathfrak{X}(U,\Phi)$, its p-th exterior power is the set $A^p(U,\Phi)$ of p-vector fields along U with values on $\Phi(U)$. It is a module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U.

INPUT:

- `p` – non-negative integer

OUTPUT:

- for $p=0$, the base ring, i.e. $C^k(U)$
- for $p=1$, the vector field module `self`, since $A^1(U,\Phi)=\mathfrak{X}(U,\Phi)$
- for $p\geq 2$, instance of `MultivectorModule` representing the module $A^p(U,\Phi)$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.exterior_power(2)
Module A^2(M) of 2-vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(1)
Module X(M) of vector fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(1) is XM
True
sage: XM.exterior_power(0)
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: XM.exterior_power(0) is M.scalar_field_algebra()
True
```

See also:

`MultivectorModule` for more examples and documentation.

general_linear_group()

Return the general linear group of `self`.

If the vector field module is $\mathfrak{X}(U,\Phi)$, the *general linear group* is the group $\mathrm{GL}(\mathfrak{X}(U,\Phi))$ of automorphisms of $\mathfrak{X}(U,\Phi)$. Note that an automorphism of $\mathfrak{X}(U,\Phi)$ can also be viewed as a *field* along U of automorphisms of the tangent spaces of $M\supset\Phi(U)$.

OUTPUT:

- instance of class `AutomorphismFieldGroup` representing $\mathrm{GL}(\mathfrak{X}(U,\Phi))$

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.general_linear_group()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
```

See also:

`AutomorphismFieldGroup` for more examples and documentation.

identity_map(*name='Id'*, *latex_name=None*)

Construct the identity map on the vector field module.

The identity map on the vector field module is actually a field of tangent-space identity maps along the differentiable manifold U over which the vector field module is defined.

INPUT:

- `name` – (string; default: `'Id'`) name given to the identity map
- `latex_name` – (string; optional) LaTeX symbol to denote the identity map; if none is provided, the LaTeX symbol is set to `'\mathrm{Id}'` if `name` is `'Id'` and to `name` otherwise

OUTPUT:

- instance of *AutomorphismField*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.identity_map()
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
```

linear_form(*name=None*, *latex_name=None*)

Construct a linear form on the vector field module.

A linear form on the vector field module is actually a field of linear forms (i.e. a 1-form) along the differentiable manifold U over which the vector field module is defined.

INPUT:

- `name` – (string; optional) name given to the linear form
- `latex_name` – (string; optional) LaTeX symbol to denote the linear form; if none is provided, the LaTeX symbol is set to `name`

OUTPUT:

- instance of *DiffForm*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.linear_form()
1-form on the 2-dimensional differentiable manifold M
sage: XM.linear_form(name='a')
1-form a on the 2-dimensional differentiable manifold M
```

See also:

DiffForm for more examples and documentation.

metric(*name*, *signature=None*, *latex_name=None*)

Construct a pseudo-Riemannian metric (nondegenerate symmetric bilinear form) on the current vector field module.

A pseudo-Riemannian metric of the vector field module is actually a field of tangent-space non-degenerate symmetric bilinear forms along the manifold U on which the vector field module is defined.

INPUT:

- `name` – (string) name given to the metric

- `signature` – (integer; default: `None`) signature S of the metric: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

OUTPUT:

- instance of `PseudoRiemannianMetric` representing the defined pseudo-Riemannian metric.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.metric('g')
Riemannian metric g on the 2-dimensional differentiable manifold M
sage: XM.metric('g', signature=0)
Lorentzian metric g on the 2-dimensional differentiable manifold M
```

See also:

`PseudoRiemannianMetric` for more documentation.

tensor(*tensor_type*, *name=None*, *latex_name=None*, *sym=None*, *antisym=None*, *specific_type=None*)
Construct a tensor on `self`.

The tensor is actually a tensor field on the domain of the vector field module.

INPUT:

- `tensor_type` – pair (k,l) with k being the contravariant rank and l the covariant rank
- `name` – (string; default: `None`) name given to the tensor
- `latex_name` – (string; default: `None`) LaTeX symbol to denote the tensor; if none is provided, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention position=0 for the first argument; for instance:
  - `sym=(0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym=[(0,2),(1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`
- `specific_type` – (default: `None`) specific subclass of `TensorField` for the output

OUTPUT:

- instance of `TensorField` representing the tensor defined on the vector field module with the provided characteristics

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.tensor((1,2), name='t')
Tensor field t of type (1,2) on the 2-dimensional differentiable
```

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```
 manifold M
sage: XM.tensor((1,0), name='a')
Vector field a on the 2-dimensional differentiable manifold M
sage: XM.tensor((0,2), name='a', antisym=(0,1))
2-form a on the 2-dimensional differentiable manifold M
```

See also:

TensorField for more examples and documentation.

tensor_module(k, l)

Return the module of type-(k, l) tensors on `self`.

INPUT:

- `k` – non-negative integer; the contravariant rank, the tensor type being (k, l)
- `l` – non-negative integer; the covariant rank, the tensor type being (k, l)

OUTPUT:

- instance of *TensorFieldModule* representing the module $T^{(k,l)}(U, \Phi)$ of type-(k, l) tensors on the vector field module

EXAMPLES:

A tensor field module on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: XM = M.vector_field_module()
sage: XM.tensor_module(1,2)
Module T^(1,2)(M) of type-(1,2) tensors fields on the 2-dimensional
 differentiable manifold M
```

The special case of tensor fields of type (1,0):

```
sage: XM.tensor_module(1,0)
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
```

The result is cached:

```
sage: XM.tensor_module(1,2) is XM.tensor_module(1,2)
True
sage: XM.tensor_module(1,0) is XM
True
```

See *TensorFieldModule* for more examples and documentation.

zero()

Return the zero of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()   # makes M parallelizable
sage: XM = M.vector_field_module()
sage: XM.zero()
Vector field zero on the 2-dimensional differentiable
 manifold M
```

2.7.2 Vector Fields

Given two differentiable manifolds U and M over the same topological field K and a differentiable map

$$\Phi : U \longrightarrow M,$$

we define a *vector field along* U *with values on* M to be a differentiable map

$$v : U \longrightarrow TM$$

(TM being the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M.$$

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Vector fields are implemented via two classes: `VectorFieldParal` and `VectorField`, depending respectively whether the manifold M is parallelizable or not, i.e. whether the bundle TM is trivial or not.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Marco Mancini (2015): parallelization of vector field plots
- Travis Scrimshaw (2016): review tweaks
- Eric Gourgoulhon (2017): vector fields inherit from multivector fields
- Eric Gourgoulhon (2018): dot and cross products, operators norm and curl

REFERENCES:

- [**?**]
- [**?**]
- [**?**]
- [**?**]

class sage.manifolds.differentiable.vectorfield.**VectorField**(*vector_field_module*, *name=None*, *latex_name=None*)

Bases: `sage.manifolds.differentiable.multivectorfield.MultivectorField`

Vector field along a differentiable manifold.

An instance of this class is a vector field along a differentiable manifold U with values on a differentiable manifold M, via a differentiable map $U \rightarrow M$. More precisely, given a differentiable map

$$\Phi : U \longrightarrow M,$$

a *vector field along* U *with values on* M is a differentiable map

$$v : U \longrightarrow TM$$

(TM being the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M.$$

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is parallelizable, then `VectorFieldParal` *must* be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $M \supset \Phi(U)$
- `name` – (default: `None`) name given to the vector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the vector field; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A vector field on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_tu.<t,u> = V.chart()
sage: transf = c_xy.transition_map(c_tu, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= t+u>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_tu.frame()
sage: c_tuW = c_tu.restrict(W) ; eVW = c_tuW.frame()
sage: v = M.vector_field(name='v') ; v
Vector field v on the 2-dimensional differentiable manifold M
sage: v.parent()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
```

The vector field is first defined on the domain U by means of its components with respect to the frame `eU`:

```
sage: v[eU,:] = [-y, 1+x]
```

The components with respect to the frame `eV` are then deduced by continuation of the components with respect to the frame `eVW` on the domain $W = U \cap V$, expressed in terms on the coordinates covering V:

```
sage: v[eV,0] = v[eVW,0,c_tuW].expr()
sage: v[eV,1] = v[eVW,1,c_tuW].expr()
```

At this stage, the vector field is fully defined on the whole manifold:

```
sage: v.display(eU)
v = -y d/dx + (x + 1) d/dy
sage: v.display(eV)
v = (u + 1) d/dt + (-t - 1) d/du
```

The vector field acting on scalar fields:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_tu: t^2}, name='f')
sage: s = v(f) ; s
Scalar field v(f) on the 2-dimensional differentiable manifold M
sage: s.display()
v(f): M --> R
on U: (x, y) |--> 2*x^2 - 2*y^2 + 2*x + 2*y
on V: (t, u) |--> 2*t*u + 2*t
```

Some checks:

```
sage: v(f) == f.differential()(v)
True
sage: v(f) == f.lie_der(v)
True
```

The result is defined on the intersection of the vector field's domain and the scalar field's one:

```
sage: s = v(f.restrict(U)) ; s
Scalar field v(f) on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s == v(f).restrict(U)
True
sage: s = v(f.restrict(W)) ; s
Scalar field v(f) on the Open subset W of the 2-dimensional
 differentiable manifold M
sage: s.display()
v(f): W --> R
   (x, y) |--> 2*x^2 - 2*y^2 + 2*x + 2*y
   (t, u) |--> 2*t*u + 2*t
sage: s = v.restrict(U)(f) ; s
Scalar field v(f) on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: s.display()
v(f): U --> R
   (x, y) |--> 2*x^2 - 2*y^2 + 2*x + 2*y
on W: (t, u) |--> 2*t*u + 2*t
sage: s = v.restrict(U)(f.restrict(V)) ; s
Scalar field v(f) on the Open subset W of the 2-dimensional
 differentiable manifold M
sage: s.display()
v(f): W --> R
   (x, y) |--> 2*x^2 - 2*y^2 + 2*x + 2*y
   (t, u) |--> 2*t*u + 2*t
```

bracket (*other*)

Return the Lie bracket `[self, other]`.

INPUT:

- `other` – a *VectorField*

OUTPUT:

- the *VectorField* `[self, other]`

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: v = -X.frame()[0] + 2*X.frame()[1] - (x^2 - y)*X.frame()[2]
sage: w = (z + y) * X.frame()[1] - X.frame()[2]
sage: vw = v.bracket(w); vw
Vector field on the 3-dimensional differentiable manifold M
sage: vw.display()
(-x^2 + y + 2) d/dy + (-y - z) d/dz
```

Some checks:

```
sage: vw == - w.bracket(v)
True
sage: f = M.scalar_field({X: x+y*z})
sage: vw(f) == v(w(f)) - w(v(f))
True
sage: vw == w.lie_derivative(v)
True
```

cross(*other*, *metric=None*)

Return the cross product of `self` with another vector field (with respect to a given metric), assuming that the domain of `self` is 3-dimensional.

If `self` is a vector field u on a 3-dimensional differentiable orientable manifold M and `other` is a vector field v on M, the *cross product* (also called *vector product*) *of* u *by* v with respect to a pseudo-Riemannian metric g on M is the vector field $w = u \times v$ defined by

$$w^i = \epsilon^i{}_{jk} u^j v^k = g^{il} \epsilon_{ljk} u^j v^k$$

where ϵ is the volume 3-form (Levi-Civita tensor) of g (cf. *volume_form()*)

Note: The method `cross_product` is meaningful only if for vector fields on a 3-dimensional manifold.

INPUT:

- `other` – a vector field, defined on the same domain as `self`
- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the cross product; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the cross product

OUTPUT:

- instance of *VectorField* representing the cross product of `self` by `other`.

EXAMPLES:

Cross product in the Euclidean 3-space:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: u = M.vector_field(-y, x, 0, name='u')
sage: v = M.vector_field(x, y, 0, name='v')
sage: w = u.cross_product(v); w
Vector field u x v on the Euclidean space E^3
sage: w.display()
u x v = (-x^2 - y^2) e_z
```

A shortcut alias of `cross_product` is `cross`:

```
sage: u.cross(v) == w
True
```

The cross product of a vector field with itself is zero:

```
sage: u.cross_product(u).display()
u x u = 0
```

Cross product with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2], h[3,3] = 1/(1+y^2), 1/(1+z^2), 1/(1+x^2)
sage: w = u.cross_product(v, metric=h); w
Vector field on the Euclidean space E^3
sage: w.display()
-(x^2 + y^2)*sqrt(x^2 + 1)/(sqrt(y^2 + 1)*sqrt(z^2 + 1)) e_z
```

cross_product(*other*, *metric=None*)

Return the cross product of `self` with another vector field (with respect to a given metric), assuming that the domain of `self` is 3-dimensional.

If `self` is a vector field u on a 3-dimensional differentiable orientable manifold M and `other` is a vector field v on M, the *cross product* (also called *vector product*) *of* u *by* v with respect to a pseudo-Riemannian metric g on M is the vector field $w = u \times v$ defined by

$$w^i = \epsilon^i{}_{jk} u^j v^k = g^{il} \epsilon_{ljk} u^j v^k$$

where ϵ is the volume 3-form (Levi-Civita tensor) of g (cf. `volume_form()`)

Note: The method `cross_product` is meaningful only if for vector fields on a 3-dimensional manifold.

INPUT:

- `other` – a vector field, defined on the same domain as `self`
- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the cross product; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the cross product

OUTPUT:

- instance of `VectorField` representing the cross product of `self` by `other`.

EXAMPLES:

Cross product in the Euclidean 3-space:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: u = M.vector_field(-y, x, 0, name='u')
sage: v = M.vector_field(x, y, 0, name='v')
sage: w = u.cross_product(v); w
Vector field u x v on the Euclidean space E^3
sage: w.display()
u x v = (-x^2 - y^2) e_z
```

A shortcut alias of `cross_product` is `cross`:

```
sage: u.cross(v) == w
True
```

The cross product of a vector field with itself is zero:

```
sage: u.cross_product(u).display()
u x u = 0
```

Cross product with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2], h[3,3] = 1/(1+y^2), 1/(1+z^2), 1/(1+x^2)
sage: w = u.cross_product(v, metric=h); w
Vector field on the Euclidean space E^3
sage: w.display()
-(x^2 + y^2)*sqrt(x^2 + 1)/(sqrt(y^2 + 1)*sqrt(z^2 + 1)) e_z
```

curl (*metric=None*)

Return the curl of `self` with respect to a given metric, assuming that the domain of `self` is 3-dimensional.

If `self` is a vector field v on a 3-dimensional differentiable orientable manifold M, the curl of v with respect to a metric g on M is the vector field defined by

$$\operatorname{curl} v = (*(\mathrm{d}v^\flat))^\sharp$$

where $v^\flat$ is the 1-form associated to v by the metric g (see `down()`), $*(\mathrm{d}v^\flat)$ is the Hodge dual with respect to g of the 2-form $\mathrm{d}v^\flat$ (exterior derivative of $v^\flat$) (see `hodge_dual()`) and $(*(\mathrm{d}v^\flat))^\sharp$ is corresponding vector field by g-duality (see `up()`).

An alternative expression of the curl is

$$(\operatorname{curl} v)^i = \epsilon^{ijk} \nabla_j v_k$$

where ∇ is the Levi-Civita connection of g (cf. `LeviCivitaConnection`) and ϵ the volume 3-form (Levi-Civita tensor) of g (cf. `volume_form()`)

Note: The method `curl` is meaningful only if `self` is a vector field on a 3-dimensional manifold.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the curl; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the curl

OUTPUT:

- instance of `VectorField` representing the curl of `self`

EXAMPLES:

Curl of a vector field in the Euclidean 3-space:

```
sage: M.<x,y,z> = EuclideanSpace()
sage: v = M.vector_field(-y, x, 0, name='v')
sage: v.display()
v = -y e_x + x e_y
sage: s = v.curl(); s
Vector field curl(v) on the Euclidean space E^3
sage: s.display()
curl(v) = 2 e_z
```

The function `curl()` from the `operators` module can be used instead of the method `curl()`:

```
sage: from sage.manifolds.operators import curl
sage: curl(v) == s
True
```

If one prefers the notation rot over curl, it suffices to do:

```
sage: from sage.manifolds.operators import curl as rot
sage: rot(v) == s
True
```

The curl of a gradient vanishes identically:

```
sage: f = M.scalar_field(function('F')(x,y,z))
sage: gradf = f.gradient()
sage: gradf.display()
d(F)/dx e_x + d(F)/dy e_y + d(F)/dz e_z
sage: s = curl(gradf); s
Vector field on the Euclidean space E^3
sage: s.display()
0
```

dot (*other*, *metric=None*)

Return the scalar product of self with another vector field (with respect to a given metric).

If self is the vector field u and other is the vector field v, the *scalar product of* u *by* v with respect to a given pseudo-Riemannian metric g is the scalar field s defined by

$$s = u \cdot v = g(u, v) = g_{ij} u^i v^j$$

INPUT:

- other – a vector field, defined on the same domain as self
- metric – (default: None) the pseudo-Riemannian metric g involved in the definition of the scalar product; if none is provided, the domain of self is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the scalar product

OUTPUT:

- instance of *DiffScalarField* representing the scalar product of self by other.

EXAMPLES:

Scalar product in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: u = M.vector_field(x, y, name='u')
sage: v = M.vector_field(y, x, name='v')
sage: s = u.dot_product(v); s
Scalar field u.v on the Euclidean plane E^2
sage: s.display()
u.v: E^2 --> R
   (x, y) |--> 2*x*y
```

A shortcut alias of dot_product is dot:

```
sage: u.dot(v) == s
True
```

A test of orthogonality:

```
sage: v[:] = -y, x
sage: u.dot_product(v) == 0
True
```

Scalar product with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2] = 1/(1+y^2), 1/(1+x^2)
sage: s = u.dot_product(v, metric=h); s
Scalar field h(u,v) on the Euclidean plane E^2
sage: s.display()
h(u,v): E^2 --> R
   (x, y) |--> -(x^3*y - x*y^3)/((x^2 + 1)*y^2 + x^2 + 1)
```

dot_product(*other*, *metric=None*)

Return the scalar product of `self` with another vector field (with respect to a given metric).

If `self` is the vector field u and other is the vector field v, the *scalar product of* u *by* v with respect to a given pseudo-Riemannian metric g is the scalar field s defined by

$$s = u \cdot v = g(u, v) = g_{ij} u^i v^j$$

INPUT:

- `other` – a vector field, defined on the same domain as `self`
- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the scalar product; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the scalar product

OUTPUT:

- instance of *DiffScalarField* representing the scalar product of `self` by `other`.

EXAMPLES:

Scalar product in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: u = M.vector_field(x, y, name='u')
sage: v = M.vector_field(y, x, name='v')
sage: s = u.dot_product(v); s
Scalar field u.v on the Euclidean plane E^2
sage: s.display()
u.v: E^2 --> R
   (x, y) |--> 2*x*y
```

A shortcut alias of `dot_product` is `dot`:

```
sage: u.dot(v) == s
True
```

A test of orthogonality:

```
sage: v[:] = -y, x
sage: u.dot_product(v) == 0
True
```

Scalar product with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2] = 1/(1+y^2), 1/(1+x^2)
sage: s = u.dot_product(v, metric=h); s
```

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```
Scalar field h(u,v) on the Euclidean plane E^2
sage: s.display()
h(u,v): E^2 --> R
   (x, y) |--> -(x^3*y - x*y^3)/((x^2 + 1)*y^2 + x^2 + 1)
```

norm(*metric=None*)

Return the norm of `self` (with respect to a given metric).

The *norm* of a vector field v with respect to a given pseudo-Riemannian metric g is the scalar field $\|v\|$ defined by

$$\|v\| = \sqrt{g(v, v)}$$

Note: If the metric g is not positive definite, it may be that $\|v\|$ takes imaginary values.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the norm; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the norm

OUTPUT:

- instance of *DiffScalarField* representing the norm of `self`.

EXAMPLES:

Norm in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: v = M.vector_field(-y, x, name='v')
sage: s = v.norm(); s
Scalar field |v| on the Euclidean plane E^2
sage: s.display()
|v|: E^2 --> R
   (x, y) |--> sqrt(x^2 + y^2)
```

The global function `norm()` can be used instead of the method `norm()`:

```
sage: norm(v) == s
True
```

Norm with respect to a metric that is not the default one:

```
sage: h = M.riemannian_metric('h')
sage: h[1,1], h[2,2] = 1/(1+y^2), 1/(1+x^2)
sage: s = v.norm(metric=h); s
Scalar field |v|_h on the Euclidean plane E^2
sage: s.display()
|v|_h: E^2 --> R
   (x, y) |--> sqrt((2*x^2 + 1)*y^2 + x^2)/(sqrt(x^2 + 1)*sqrt(y^2 + 1))
```

plot(*chart=None*, *ambient_coords=None*, *mapping=None*, *chart_domain=None*, *fixed_coords=None*, *ranges=None*, *number_values=None*, *steps=None*, *parameters=None*, *label_axes=True*, *color='blue'*, *max_range=8*, *scale=1*, ***extra_options*)

Plot the vector field in a Cartesian graph based on the coordinates of some ambient chart.

The vector field is drawn in terms of two (2D graphics) or three (3D graphics) coordinates of a given chart, called hereafter the *ambient chart*. The vector field's base points p (or their images $\Phi(p)$ by some differentiable mapping Φ) must lie in the ambient chart's domain.

INPUT:

- `chart` – (default: `None`) the ambient chart (see above); if `None`, the default chart of the vector field's domain is used
- `ambient_coords` – (default: `None`) tuple containing the 2 or 3 coordinates of the ambient chart in terms of which the plot is performed; if `None`, all the coordinates of the ambient chart are considered
- `mapping` – `DiffMap` (default: `None`); differentiable map Φ providing the link between the vector field's domain and the ambient chart `chart`; if `None`, the identity map is assumed
- `chart_domain` – (default: `None`) chart on the vector field's domain to define the points at which vector arrows are to be plotted; if `None`, the default chart of the vector field's domain is used
- `fixed_coords` – (default: `None`) dictionary with keys the coordinates of `chart_domain` that are kept fixed and with values the value of these coordinates; if `None`, all the coordinates of `chart_domain` are used
- `ranges` – (default: `None`) dictionary with keys the coordinates of `chart_domain` to be used and values tuples `(x_min, x_max)` specifying the coordinate range for the plot; if `None`, the entire coordinate range declared during the construction of `chart_domain` is considered (with `-Infinity` replaced by `-max_range` and `+Infinity` by `max_range`)
- `number_values` – (default: `None`) either an integer or a dictionary with keys the coordinates of `chart_domain` to be used and values the number of values of the coordinate for sampling the part of the vector field's domain involved in the plot ; if `number_values` is a single integer, it represents the number of values for all coordinates; if `number_values` is `None`, it is set to 9 for a 2D plot and to 5 for a 3D plot
- `steps` – (default: `None`) dictionary with keys the coordinates of `chart_domain` to be used and values the step between each constant value of the coordinate; if `None`, the step is computed from the coordinate range (specified in `ranges`) and `number_values`; on the contrary, if the step is provided for some coordinate, the corresponding number of values is deduced from it and the coordinate range
- `parameters` – (default: `None`) dictionary giving the numerical values of the parameters that may appear in the coordinate expression of the vector field (see example below)
- `label_axes` – (default: `True`) boolean determining whether the labels of the coordinate axes of `chart` shall be added to the graph; can be set to `False` if the graph is 3D and must be superposed with another graph
- `color` – (default: 'blue') color of the arrows representing the vectors
- `max_range` – (default: 8) numerical value substituted to `+Infinity` if the latter is the upper bound of the range of a coordinate for which the plot is performed over the entire coordinate range (i.e. for which no specific plot range has been set in `ranges`); similarly `-max_range` is the numerical valued substituted for `-Infinity`
- `scale` – (default: 1) value by which the lengths of the arrows representing the vectors is multiplied
- `**extra_options` – extra options for the arrow plot, like `linestyle`, `width` or `arrowsize` (see `arrow2d()` and `arrow3d()` for details)

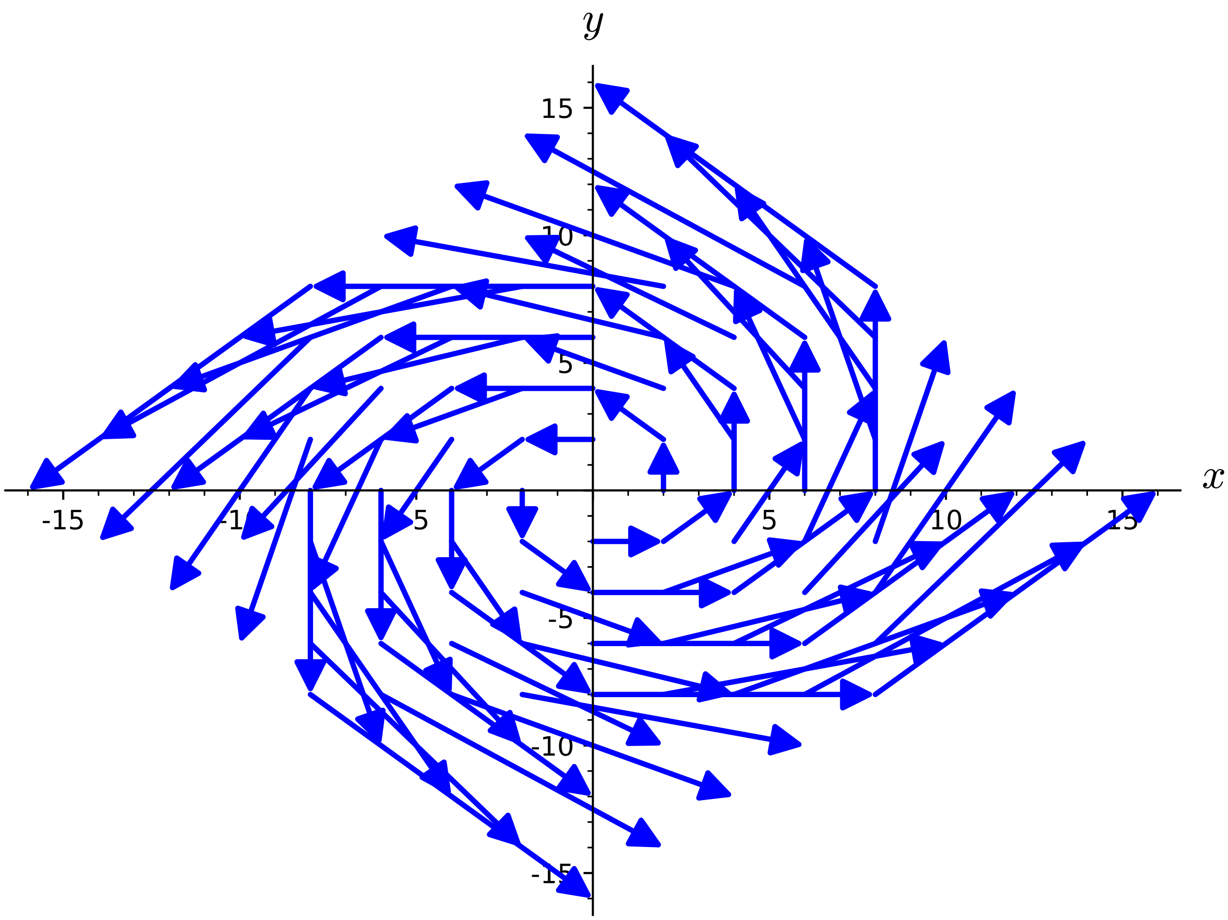
OUTPUT:

- a graphic object, either an instance of `Graphics` for a 2D plot (i.e. based on 2 coordinates of `chart`) or an instance of `Graphics3d` for a 3D plot (i.e. based on 3 coordinates of `chart`)

EXAMPLES:

Plot of a vector field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: v = M.vector_field(-y, x, name='v')
sage: v.display()
v = -y d/dx + x d/dy
sage: v.plot()
Graphics object consisting of 80 graphics primitives
```



Plot with various options:

```
sage: v.plot(scale=0.5, color='green', linestyle='--', width=1,
....:        arrowsize=6)
Graphics object consisting of 80 graphics primitives
```

```
sage: v.plot(max_range=4, number_values=5, scale=0.5)
Graphics object consisting of 24 graphics primitives
```

Plot using parallel computation:

```
sage: Parallelism().set(nproc=2)
sage: v.plot(scale=0.5,  number_values=10, linestyle='--', width=1,
```

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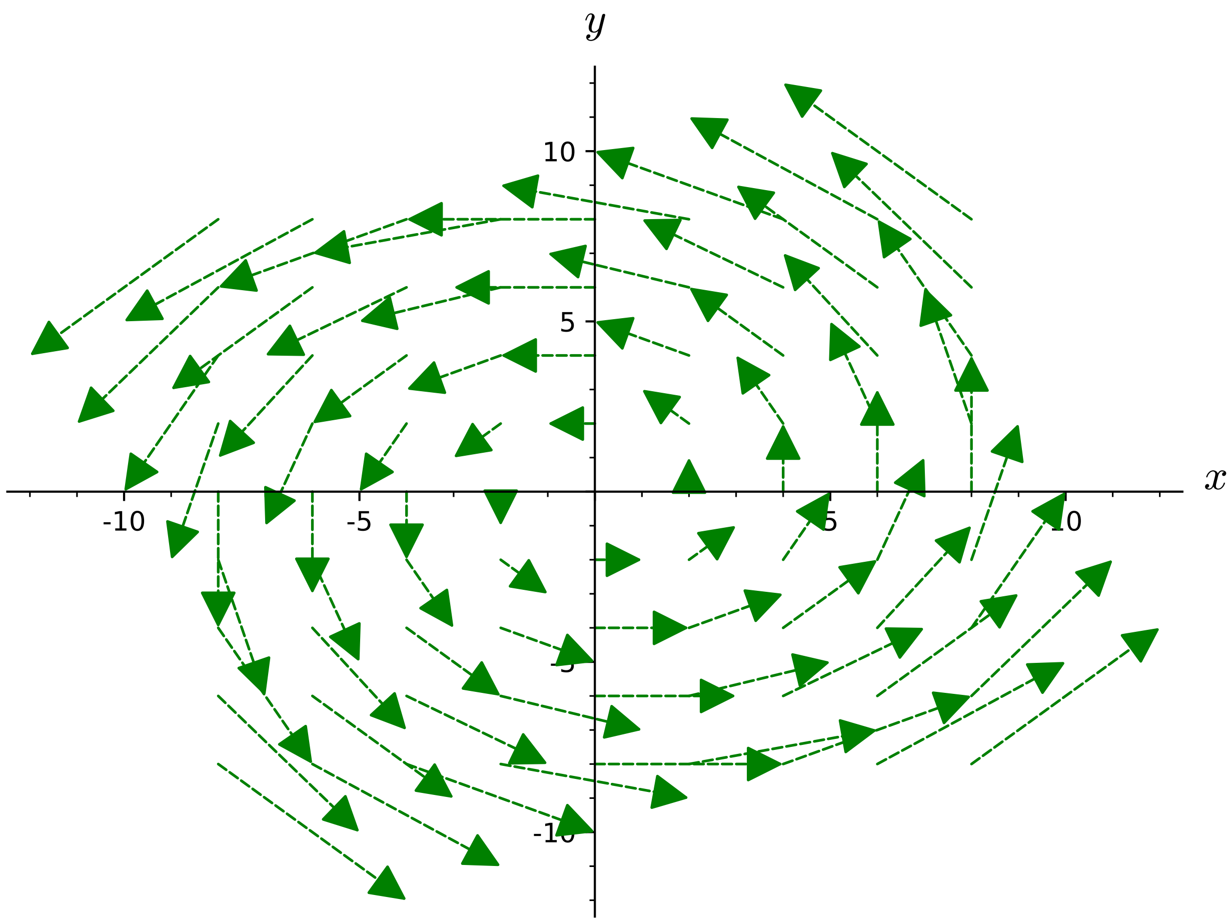
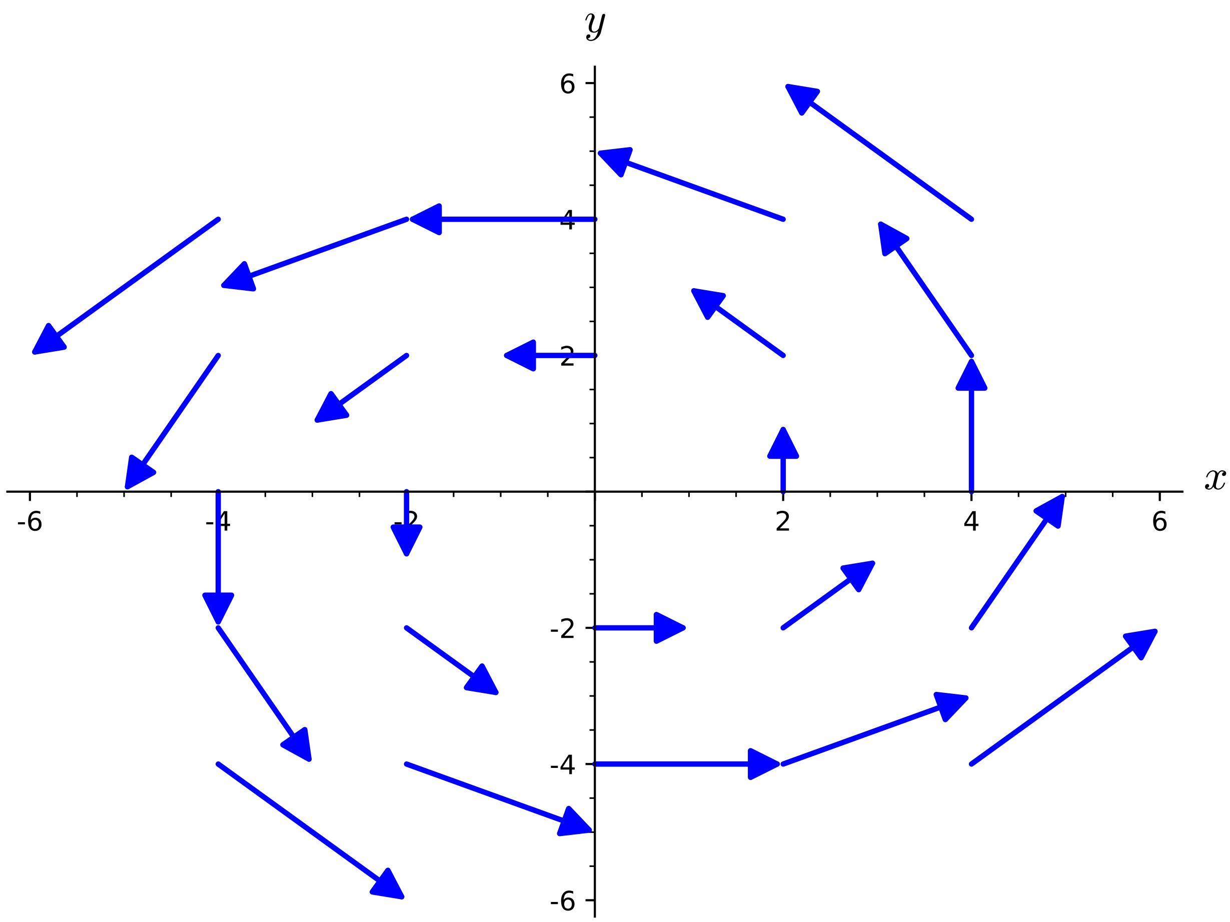


y
x
10
5
-5
-10
-10
-5
5
10



y
x
6
4
2
-2
-4
-6
-6
-4
-2
2
4
6

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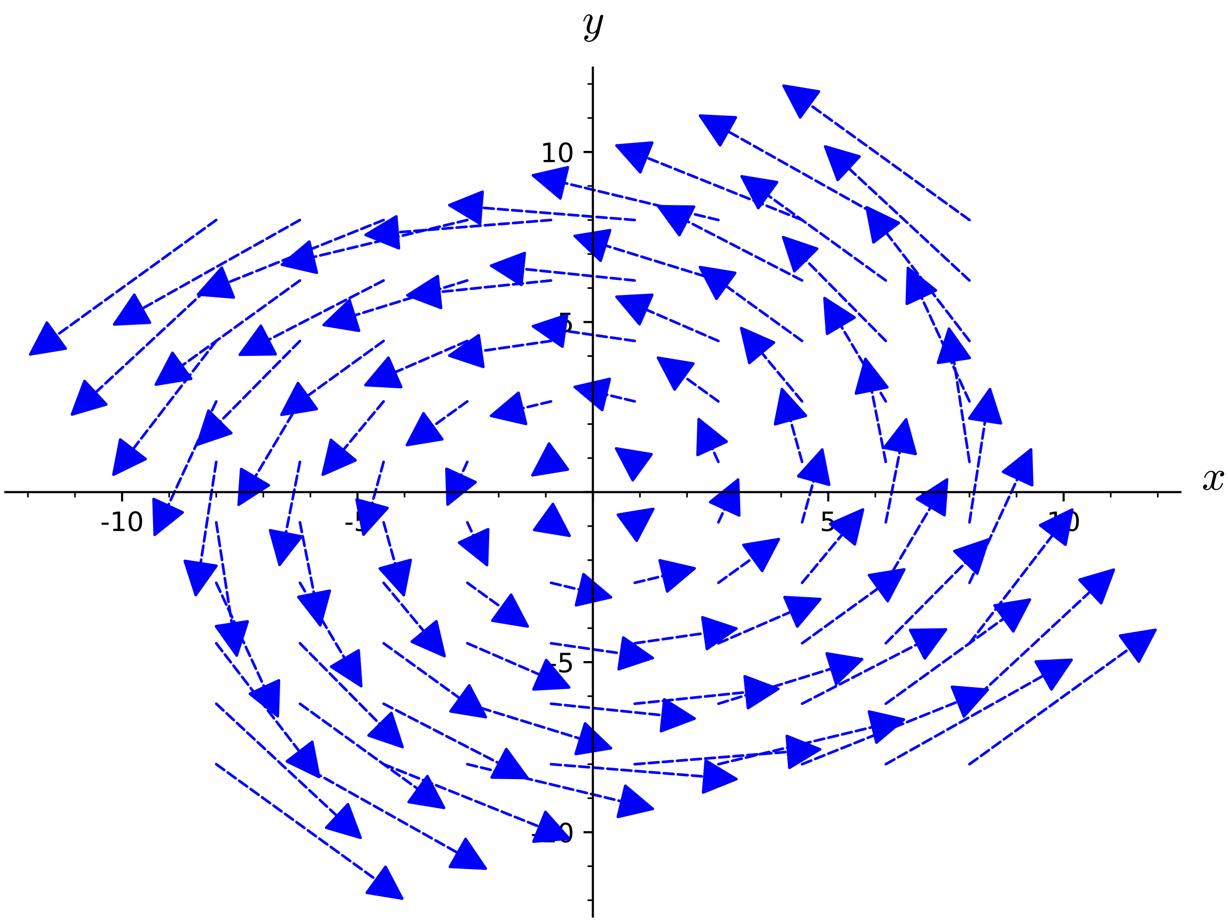
```
....:          arrowsize=6)
Graphics object consisting of 100 graphics primitives
```




```
sage: Parallelism().set(nproc=1)   # switch off parallelization
```

Plots along a line of fixed coordinate:

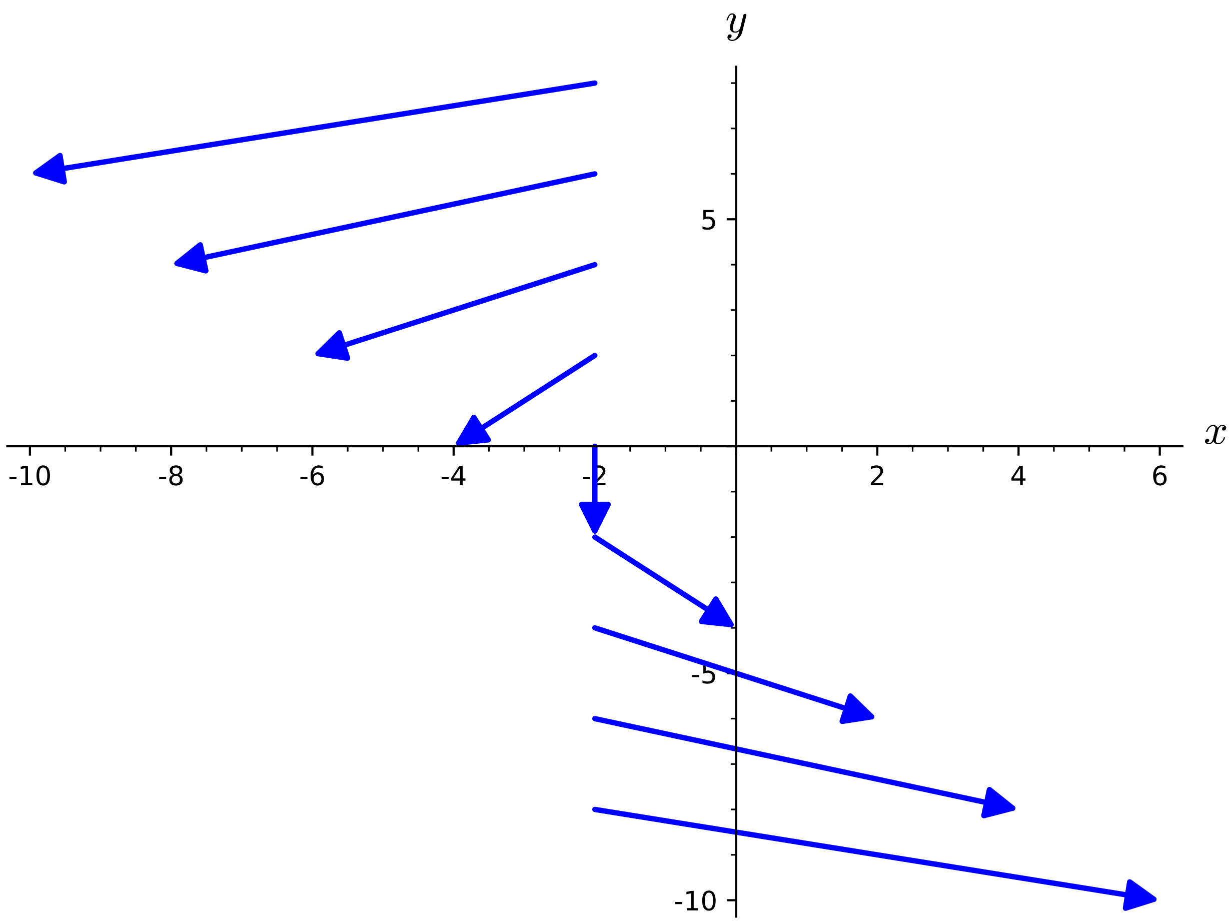
```
sage: v.plot(fixed_coords={x: -2})
Graphics object consisting of 9 graphics primitives
```

```
sage: v.plot(fixed_coords={y: 1})
Graphics object consisting of 9 graphics primitives
```

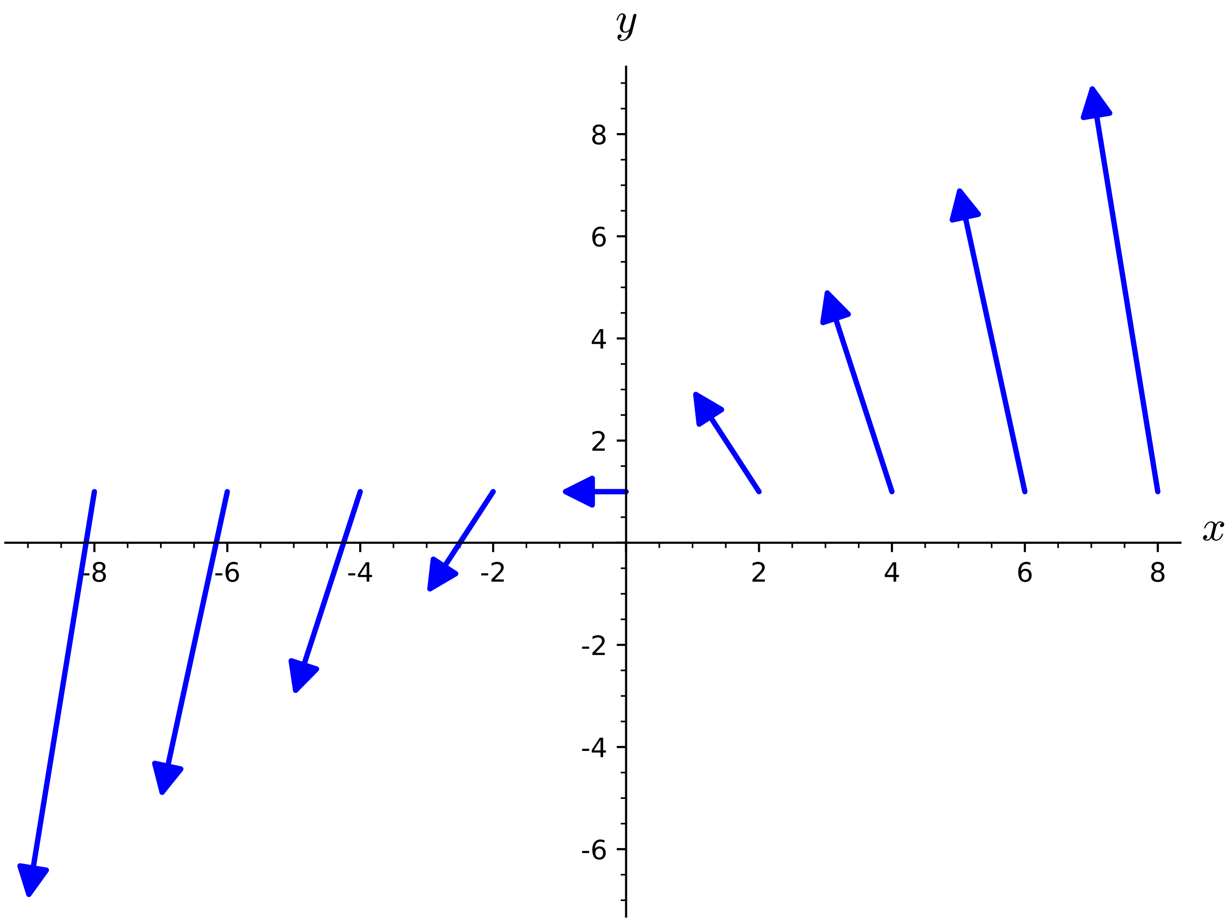
Let us now consider a vector field on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: v = M.vector_field((t/8)^2, -t*y/4, t*x/4, t*z/4, name='v')
sage: v.display()
v = 1/64*t^2 d/dt - 1/4*t*y d/dx + 1/4*t*x d/dy + 1/4*t*z d/dz
```

We cannot make a 4D plot directly:



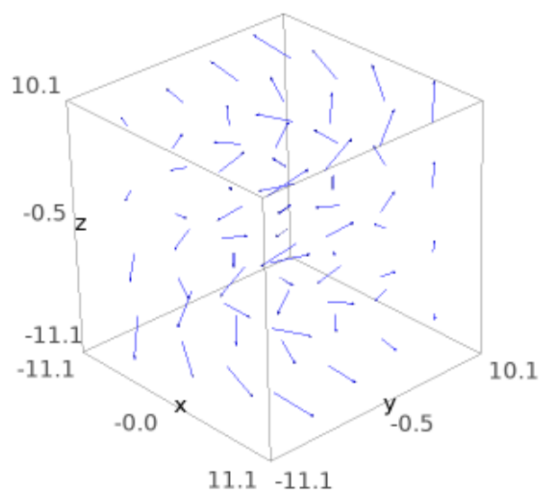
y
x
5
-5
-10
-10
-8
-6
-4
-2
2
4
6


y
8
6
4
2
-2
-4
-6
x
-8
-6
-4
-2
2
4
6
8

```
sage: v.plot()
Traceback (most recent call last):
...
ValueError: the number of ambient coordinates must be either 2 or 3, not 4
```

Rather, we have to select some coordinates for the plot, via the argument `ambient_coords`. For instance, for a 3D plot:

```
sage: v.plot(ambient_coords=(x, y, z), fixed_coords={t: 1},  # long time
....:         number_values=4)
Graphics3d Object
```

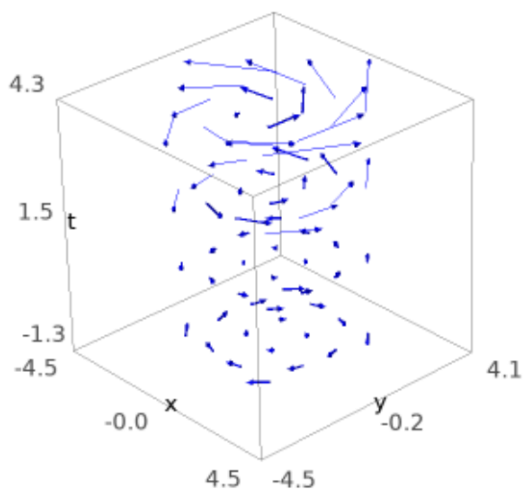


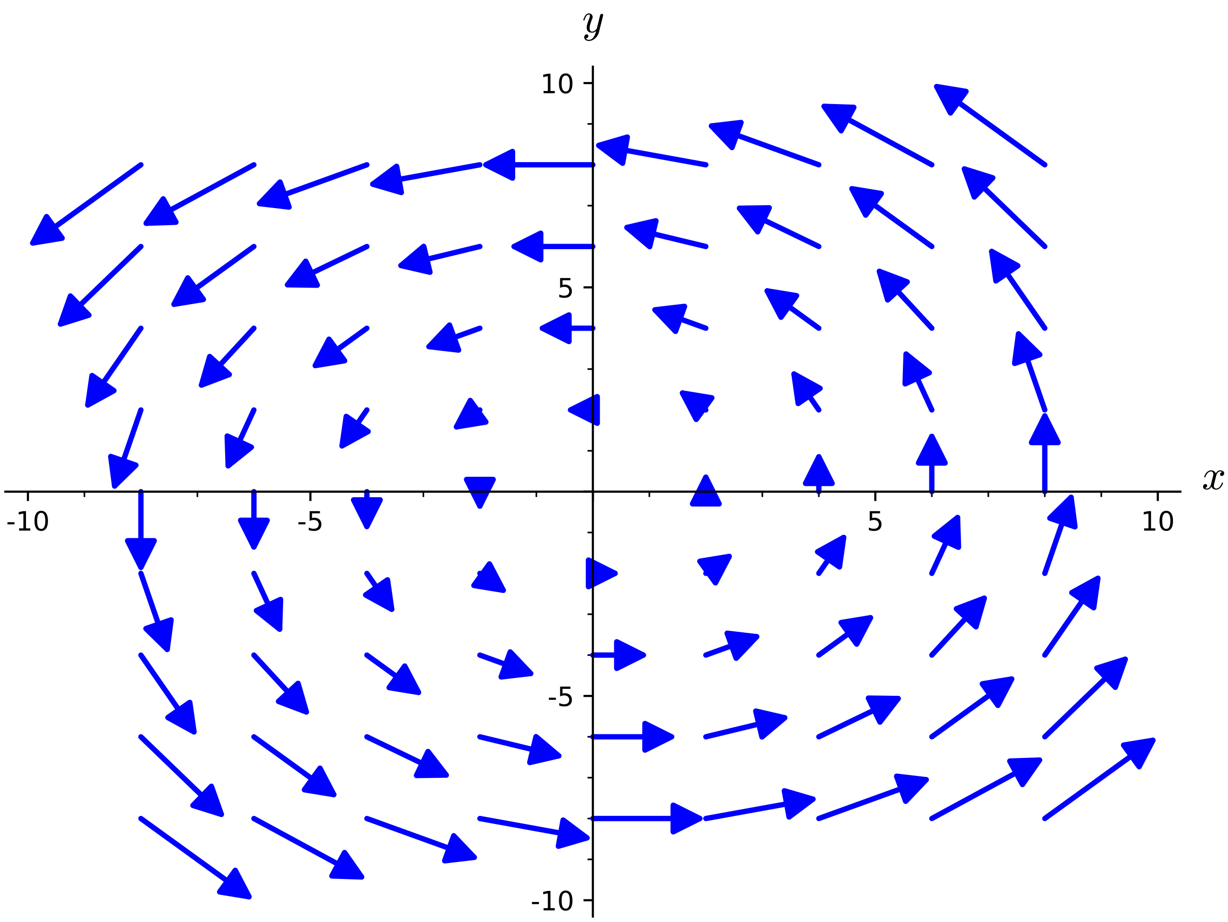

```
sage: v.plot(ambient_coords=(x, y, t), fixed_coords={z: 0},  # long time
....:         ranges={x: (-2,2), y: (-2,2), t: (-1, 4)},
....:         number_values=4)
Graphics3d Object
```

or, for a 2D plot:

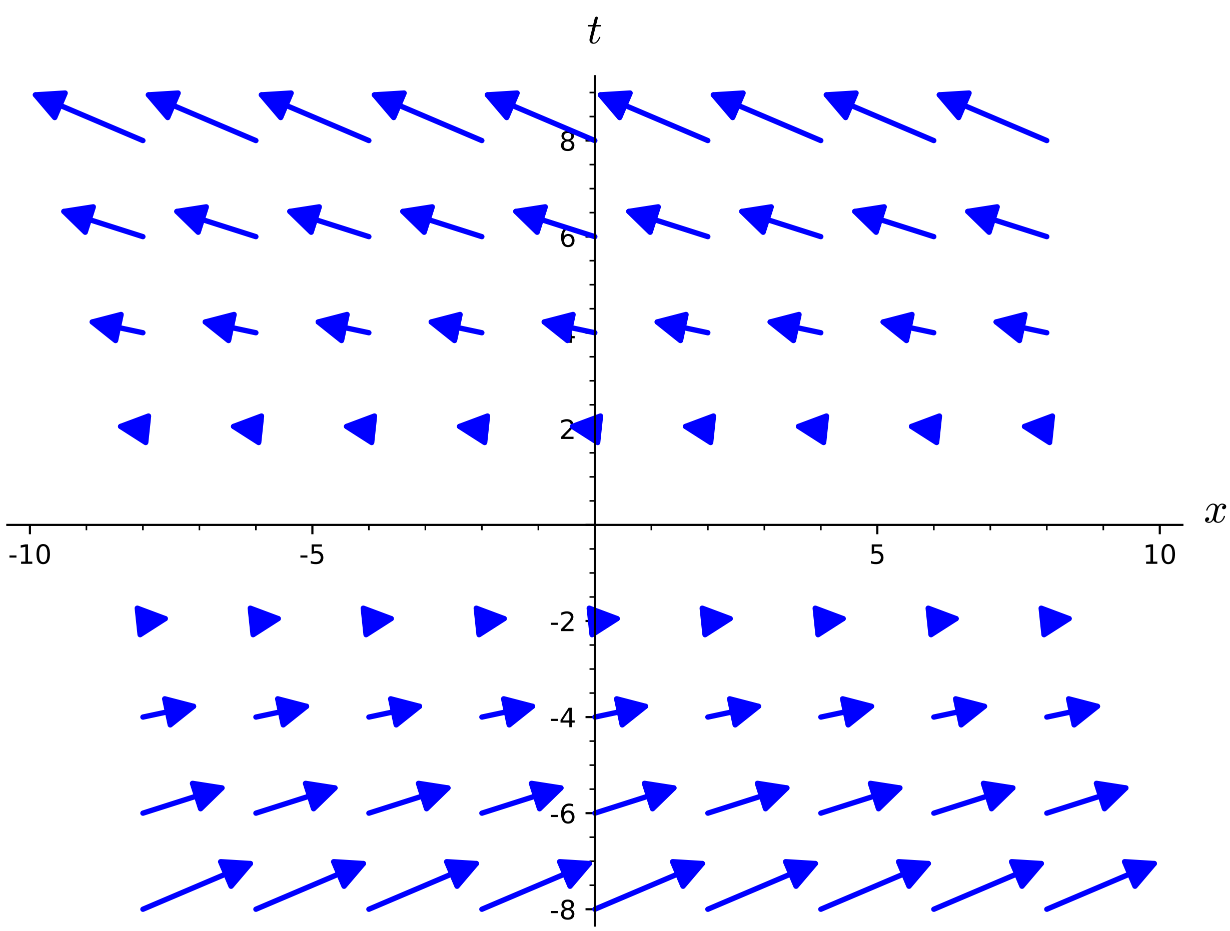
```
sage: v.plot(ambient_coords=(x, y), fixed_coords={t: 1, z: 0})  # long time
Graphics object consisting of 80 graphics primitives
```

```
sage: v.plot(ambient_coords=(x, t), fixed_coords={y: 1, z: 0})  # long time
Graphics object consisting of 72 graphics primitives
```


4.3
1.5
t
-1.3
-4.5
-0.0
x
4.5
-4.5
y
-0.2
4.1

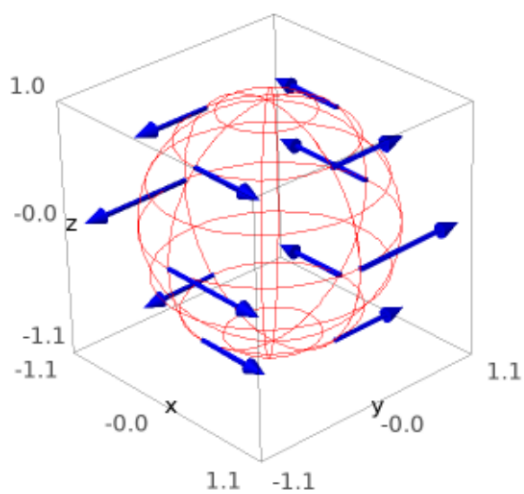


y
x
10
5
-5
-10
-10
-5
5
10


t
8
6
4
2
x
-10
-5
5
10
-2
-4
-6
-8

An example of plot via a differential mapping: plot of a vector field tangent to a 2-sphere viewed in $\mathbf{R}^3$:

```
sage: S2 = Manifold(2, 'S^2')
sage: U = S2.open_subset('U') # the open set covered by spherical coord.
sage: XS.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: R3 = Manifold(3, 'R^3')
sage: X3.<x,y,z> = R3.chart()
sage: F = S2.diff_map(R3, {(XS, X3): [sin(th)*cos(ph),
....:                          sin(th)*sin(ph), cos(th)]}, name='F')
sage: F.display() # the standard embedding of S^2 into R^3
F: S^2 --> R^3
on U: (th, ph) |--> (x, y, z) = (cos(ph)*sin(th), sin(ph)*sin(th), cos(th))
sage: v = XS.frame()[1] ; v  # the coordinate vector d/dphi
Vector field d/dph on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: graph_v = v.plot(chart=X3, mapping=F, label_axes=False)
sage: graph_S2 = XS.plot(chart=X3, mapping=F, number_values=9)
sage: graph_v + graph_S2
Graphics3d Object
```



Note that the default values of some arguments of the method `plot` are stored in the dictionary `plot.options`:

```
sage: v.plot.options  # random (dictionary output)
{'color': 'blue', 'max_range': 8, 'scale': 1}
```

so that they can be adjusted by the user:

```
sage: v.plot.options['color'] = 'red'
```

From now on, all plots of vector fields will use red as the default color. To restore the original default options, it suffices to type:

```
sage: v.plot.reset()
```

class sage.manifolds.differentiable.vectorfield.**VectorFieldParal**(*vector_field_module*, *name=None*, *latex_name=None*)

Bases: sage.tensor.modules.free_module_element.FiniteRankFreeModuleElement, *sage.manifolds.differentiable.multivectorfield.MultivectorFieldParal*, *sage.manifolds.differentiable.vectorfield.VectorField*

Vector field along a differentiable manifold, with values on a parallelizable manifold.

An instance of this class is a vector field along a differentiable manifold U with values on a parallelizable manifold M, via a differentiable map $\Phi : U \to M$. More precisely, given a differentiable map

$$\Phi : U \longrightarrow M,$$

a *vector field along* U *with values on* M is a differentiable map

$$v : U \longrightarrow TM$$

(TM being the tangent bundle of M) such that

$$\forall p \in U, \ v(p) \in T_{\Phi(p)}M.$$

The standard case of vector fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is not parallelizable, then *VectorField* *must* be used instead.

INPUT:

- vector_field_module – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $M \supset \Phi(U)$
- name – (default: None) name given to the vector field
- latex_name – (default: None) LaTeX symbol to denote the vector field; if none is provided, the LaTeX symbol is set to name

EXAMPLES:

A vector field on a parallelizable 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()
sage: v = M.vector_field(name='V') ; v
Vector field V on the 3-dimensional differentiable manifold M
sage: latex(v)
V
```

Vector fields are considered as elements of a module over the ring (algebra) of scalar fields on M:

```
sage: v.parent()
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
sage: v.parent().base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: v.parent() is M.vector_field_module()
True
```

A vector field is a tensor field of rank 1 and of type $(1,0)$:

```
sage: v.tensor_rank()
1
sage: v.tensor_type()
(1, 0)
```

Components of a vector field with respect to a given frame:

```
sage: e = M.vector_frame('e') ; M.set_default_frame(e)
sage: v[0], v[1], v[2] = (1+y, 4*x*z, 9)  # components on M's default frame (e)
sage: v.comp()
1-index components w.r.t. Vector frame (M, (e_0,e_1,e_2))
```

The totality of the components are accessed via the operator `[:]`:

```
sage: v[:] = (1+y, 4*x*z, 9)
sage: v[:]
[y + 1, 4*x*z, 9]
```

The components are also read on the expansion on the frame `e`, as provided by the method `display()`:

```
sage: v.display()  # expansion in the default frame
V = (y + 1) e_0 + 4*x*z e_1 + 9 e_2
```

A subset of the components can be accessed by using slice notation:

```
sage: v[1:] = (-2, -x*y)
sage: v[:]
[y + 1, -2, -x*y]
sage: v[:2]
[y + 1, -2]
```

Components in another frame:

```
sage: f = M.vector_frame('f')
sage: for i in range(3):
....:     v.set_comp(f)[i] = (i+1)**3 * c_xyz[i]
....:
sage: v.comp(f)[2]
27*z
sage: v[f, 2]  # equivalent to above
27*z
sage: v.display(f)
V = x f_0 + 8*y f_1 + 27*z f_2
```

One can set the components at the vector definition:

```
sage: v = M.vector_field(1+y, 4*x*z, 9, name='V')
sage: v.display()
V = (y + 1) e_0 + 4*x*z e_1 + 9 e_2
```

If the components regard a vector frame different from the default one, the vector frame has to be specified via the argument `frame`:

```
sage: v = M.vector_field(x, 8*y, 27*z, frame=f, name='V')
sage: v.display(f)
V = x f_0 + 8*y f_1 + 27*z f_2
```

For providing the components in various frames, one may use a dictionary:

```
sage: v = M.vector_field({e: [1+y, -2, -x*y], f: [x, 8*y, 27*z]},
....:                      name='V')
sage: v.display(e)
V = (y + 1) e_0 - 2 e_1 - x*y e_2
sage: v.display(f)
V = x f_0 + 8*y f_1 + 27*z f_2
```

It is also possible to construct a vector field from a vector of symbolic expressions (or any other iterable):

```
sage: v = M.vector_field(vector([1+y, 4*x*z, 9]), name='V')
sage: v.display()
V = (y + 1) e_0 + 4*x*z e_1 + 9 e_2
```

The range of the indices depends on the convention set for the manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: e = M.vector_frame('e') ; M.set_default_frame(e)
sage: v = M.vector_field(1+y, 4*x*z, 9, name='V')
sage: v[0]
Traceback (most recent call last):
...
IndexError: index out of range: 0 not in [1, 3]
sage: v[1]  # OK
y + 1
```

A vector field acts on scalar fields (derivation along the vector field):

```
sage: M = Manifold(2, 'M')
sage: c_cart.<x,y> = M.chart()
sage: f = M.scalar_field(x*y^2, name='f')
sage: v = M.vector_field(-y, x, name='v')
sage: v.display()
v = -y d/dx + x d/dy
sage: v(f)
Scalar field v(f) on the 2-dimensional differentiable manifold M
sage: v(f).expr()
2*x^2*y - y^3
sage: latex(v(f))
v\left(f\right)
```

Example of a vector field associated with a non-trivial map Φ; a vector field along a curve in M:

```
sage: R = Manifold(1, 'R')
sage: T.<t> = R.chart()   # canonical chart on R
sage: Phi = R.diff_map(M, [cos(t), sin(t)], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 2-dimensional differentiable manifold M
sage: Phi.display()
Phi: R --> M
   t |--> (x, y) = (cos(t), sin(t))
sage: w = R.vector_field(-sin(t), cos(t), dest_map=Phi, name='w') ; w
Vector field w along the 1-dimensional differentiable manifold R with
 values on the 2-dimensional differentiable manifold M
sage: w.parent()
Free module X(R,Phi) of vector fields along the 1-dimensional
 differentiable manifold R mapped into the 2-dimensional differentiable
 manifold M
sage: w.display()
w = -sin(t) d/dx + cos(t) d/dy
```

Value at a given point:

```
sage: p = R((0,), name='p') ; p
Point p on the 1-dimensional differentiable manifold R
sage: w.at(p)
Tangent vector w at Point Phi(p) on the 2-dimensional differentiable
 manifold M
sage: w.at(p).display()
w = d/dy
sage: w.at(p) == v.at(Phi(p))
True
```

2.7.3 Vector Frames

The class `VectorFrame` implements vector frames on differentiable manifolds. By *vector frame*, it is meant a field e on some differentiable manifold U endowed with a differentiable map $\Phi : U \to M$ to a differentiable manifold M such that for each $p \in U$, $e(p)$ is a vector basis of the tangent space $T_{\Phi(p)}M$.

The standard case of a vector frame *on* U corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

A derived class of `VectorFrame` is `CoordFrame`; it regards the vector frames associated with a chart, i.e. the so-called *coordinate bases*.

The vector frame duals, i.e. the coframes, are implemented via the class `CoFrame`. The derived class `CoordCoFrame` is devoted to coframes deriving from a chart.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015): initial version
- Travis Scrimshaw (2016): review tweaks
- Eric Gourgoulhon (2018): some refactoring and more functionalities in the choice of symbols for vector frame elements (trac ticket #24792)

REFERENCES:

- [**?**]

EXAMPLES:

Defining a vector frame on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: e = M.vector_frame('e') ; e
Vector frame (M, (e_0,e_1,e_2))
sage: latex(e)
\left(M, \left(e_{0},e_{1},e_{2}\right)\right)
```

The first frame defined on a manifold is its default frame; in the present case it is the coordinate frame defined when introducing the chart X:

```
sage: M.default_frame()
Coordinate frame (M, (d/dx,d/dy,d/dz))
```

The default frame can be changed via the method *set_default_frame()*:

```
sage: M.set_default_frame(e)
sage: M.default_frame()
Vector frame (M, (e_0,e_1,e_2))
```

The elements of a vector frame are vector fields on the manifold:

```
sage: for vec in e:
....:     print(vec)
....:
Vector field e_0 on the 3-dimensional differentiable manifold M
Vector field e_1 on the 3-dimensional differentiable manifold M
Vector field e_2 on the 3-dimensional differentiable manifold M
```

Each element of a vector frame can be accessed by its index:

```
sage: e[0]
Vector field e_0 on the 3-dimensional differentiable manifold M
```

The slice operator : can be used to access to more than one element:

```
sage: e[0:2]
(Vector field e_0 on the 3-dimensional differentiable manifold M,
 Vector field e_1 on the 3-dimensional differentiable manifold M)
sage: e[:]
(Vector field e_0 on the 3-dimensional differentiable manifold M,
 Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M)
```

The index range depends on the starting index defined on the manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: e = M.vector_frame('e')
sage: [e[i] for i in M.irange()]
[Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M,
 Vector field e_3 on the 3-dimensional differentiable manifold M]
sage: e[1], e[2], e[3]
(Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M,
 Vector field e_3 on the 3-dimensional differentiable manifold M)
```

Let us check that the vector fields `e[i]` are the frame vectors from their components with respect to the frame e:

```
sage: e[1].comp(e)[:]
[1, 0, 0]
sage: e[2].comp(e)[:]
[0, 1, 0]
sage: e[3].comp(e)[:]
[0, 0, 1]
```

Defining a vector frame on a manifold automatically creates the dual coframe, which, by default, bares the same name (here e):

```
sage: M.coframes()
[Coordinate coframe (M, (dx,dy,dz)), Coframe (M, (e^1,e^2,e^3))]
sage: f = M.coframes()[1] ; f
Coframe (M, (e^1,e^2,e^3))
sage: f is e.coframe()
True
```

Each element of the coframe is a 1-form:

```
sage: f[1], f[2], f[3]
(1-form e^1 on the 3-dimensional differentiable manifold M,
1-form e^2 on the 3-dimensional differentiable manifold M,
1-form e^3 on the 3-dimensional differentiable manifold M)
sage: latex(f[1]), latex(f[2]), latex(f[3])
(e^{1}, e^{2}, e^{3})
```

Let us check that the coframe (e^i) is indeed the dual of the vector frame (e_i):

```
sage: f[1](e[1]) # the 1-form e^1 applied to the vector field e_1
Scalar field e^1(e_1) on the 3-dimensional differentiable manifold M
sage: f[1](e[1]).expr() # the explicit expression of e^1(e_1)
1
sage: f[1](e[1]).expr(), f[1](e[2]).expr(), f[1](e[3]).expr()
(1, 0, 0)
sage: f[2](e[1]).expr(), f[2](e[2]).expr(), f[2](e[3]).expr()
(0, 1, 0)
sage: f[3](e[1]).expr(), f[3](e[2]).expr(), f[3](e[3]).expr()
(0, 0, 1)
```

The coordinate frame associated to spherical coordinates of the sphere S^2:

```
sage: M = Manifold(2, 'S^2', start_index=1) # Part of S^2 covered by spherical coord.
sage: c_spher.<th,ph> = M.chart(r'th:[0,pi]:\theta ph:[0,2*pi):\phi')
sage: b = M.default_frame() ; b
Coordinate frame (S^2, (d/dth,d/dph))
sage: b[1]
Vector field d/dth on the 2-dimensional differentiable manifold S^2
sage: b[2]
Vector field d/dph on the 2-dimensional differentiable manifold S^2
```

The orthonormal frame constructed from the coordinate frame:

```
sage: change_frame = M.automorphism_field()
sage: change_frame[:] = [[1,0], [0, 1/sin(th)]]
sage: e = b.new_frame(change_frame, 'e') ; e
Vector frame (S^2, (e_1,e_2))
```

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```
sage: e[1][:]
[1, 0]
sage: e[2][:]
[0, 1/sin(th)]
```

The change-of-frame automorphisms and their matrices:

```
sage: M.change_of_frame(c_spher.frame(), e)
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold S^2
sage: M.change_of_frame(c_spher.frame(), e)[:]
[        1         0]
[        0 1/sin(th)]
sage: M.change_of_frame(e, c_spher.frame())
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold S^2
sage: M.change_of_frame(e, c_spher.frame())[:]
[      1       0]
[      0 sin(th)]
```

class `sage.manifolds.differentiable.vectorframe.`**`CoFrame`**(*frame*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*)

Bases: `sage.tensor.modules.free_module_basis.FreeModuleCoBasis`

Coframe on a differentiable manifold.

By *coframe*, it is meant a field f on some differentiable manifold U endowed with a differentiable map $\Phi : U \to M$ to a differentiable manifold M such that for each $p \in U$, $f(p)$ is a basis of the vector space $T^*_{\Phi(p)}M$ (the dual to the tangent space $T_{\Phi(p)}M$).

The standard case of a coframe *on* U corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

INPUT:

- `frame` – the vector frame dual to the coframe
- `symbol` – either a string, to be used as a common base for the symbols of the 1-forms constituting the coframe, or a tuple of strings, representing the individual symbols of the 1-forms
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the 1-forms constituting the coframe, or a tuple of strings, representing the individual LaTeX symbols of the 1-forms; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the 1-forms of the coframe; if `None`, the indices will be generated as integers within the range declared on the coframe's domain
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the 1-forms of the coframe; if `None`, `indices` is used instead

EXAMPLES:

Coframe on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: v = M.vector_frame('v')
```

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```
sage: from sage.manifolds.differentiable.vectorframe import CoFrame
sage: e = CoFrame(v, 'e') ; e
Coframe (M, (e^1,e^2,e^3))
```

Instead of importing CoFrame in the global namespace, the coframe can be obtained by means of the method `dual_basis()`; the symbol is then the same as that of the frame:

```
sage: a = v.dual_basis() ; a
Coframe (M, (v^1,v^2,v^3))
sage: a[1] == e[1]
True
sage: a[1] is e[1]
False
sage: e[1].display(v)
e^1 = v^1
```

The 1-forms composing the coframe are obtained via the operator `[]`:

```
sage: e[1], e[2], e[3]
(1-form e^1 on the 3-dimensional differentiable manifold M,
 1-form e^2 on the 3-dimensional differentiable manifold M,
 1-form e^3 on the 3-dimensional differentiable manifold M)
```

Checking that e is the dual of v:

```
sage: e[1](v[1]).expr(), e[1](v[2]).expr(), e[1](v[3]).expr()
(1, 0, 0)
sage: e[2](v[1]).expr(), e[2](v[2]).expr(), e[2](v[3]).expr()
(0, 1, 0)
sage: e[3](v[1]).expr(), e[3](v[2]).expr(), e[3](v[3]).expr()
(0, 0, 1)
```

at (*point*)

Return the value of `self` at a given point on the manifold, this value being a basis of the dual of the tangent space at the point.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain U of the coframe (denoted f hereafter)

OUTPUT:

- `FreeModuleCoBasis` representing the basis $f(p)$ of the vector space $T^*_{\Phi(p)}M$, dual to the tangent space $T_{\Phi(p)}M$, where $\Phi : U \to M$ is the differentiable map associated with f (possibly $\Phi = \mathrm{Id}_U$)

EXAMPLES:

Cobasis of a tangent space on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: p = M.point((-1,2), name='p')
sage: f = X.coframe() ; f
Coordinate coframe (M, (dx,dy))
sage: fp = f.at(p) ; fp
Dual basis (dx,dy) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: type(fp)
```

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```
<class 'sage.tensor.modules.free_module_basis.FreeModuleCoBasis'>
sage: fp[0]
Linear form dx on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: fp[1]
Linear form dy on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: fp is X.frame().at(p).dual_basis()
True
```

set_name (*symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *index_position='up'*, *include_domain=True*)

Set (or change) the text name and LaTeX name of `self`.

INPUT:

- `symbol` – either a string, to be used as a common base for the symbols of the 1-forms constituting the coframe, or a list/tuple of strings, representing the individual symbols of the 1-forms
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the 1-forms constituting the coframe, or a list/tuple of strings, representing the individual LaTeX symbols of the 1-forms; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the 1-forms of the coframe; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the 1-forms; if `None`, `indices` is used instead
- `index_position` – (default: `'up'`) determines the position of the indices labelling the 1-forms of the coframe; can be either `'down'` or `'up'`
- `include_domain` – (default: `True`) boolean determining whether the name of the domain is included in the beginning of the coframe name

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e').coframe(); e
Coframe (M, (e^0,e^1))
sage: e.set_name('f'); e
Coframe (M, (f^0,f^1))
sage: e.set_name('e', latex_symbol=r'\epsilon')
sage: latex(e)
\left(M, \left(\epsilon^{0},\epsilon^{1}\right)\right)
sage: e.set_name('e', include_domain=False); e
Coframe (e^0,e^1)
sage: e.set_name(['a', 'b'], latex_symbol=[r'\alpha', r'\beta']); e
Coframe (M, (a,b))
sage: latex(e)
\left(M, \left(\alpha,\beta\right)\right)
sage: e.set_name('e', indices=['x','y'],
....:            latex_indices=[r'\xi', r'\zeta']); e
Coframe (M, (e^x,e^y))
sage: latex(e)
\left(M, \left(e^{\xi},e^{\zeta}\right)\right)
```

class sage.manifolds.differentiable.vectorframe.**CoordCoFrame**(*coord_frame*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*)

Bases: *sage.manifolds.differentiable.vectorframe.CoFrame*

Coordinate coframe on a differentiable manifold.

By *coordinate coframe*, it is meant the n-tuple of the differentials of the coordinates of some chart on the manifold, with n being the manifold's dimension.

INPUT:

- `coord_frame` – coordinate frame dual to the coordinate coframe
- `symbol` – either a string, to be used as a common base for the symbols of the 1-forms constituting the coframe, or a tuple of strings, representing the individual symbols of the 1-forms
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the 1-forms constituting the coframe, or a tuple of strings, representing the individual LaTeX symbols of the 1-forms; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the 1-forms of the coframe; if `None`, the indices will be generated as integers within the range declared on the vector frame's domain
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the 1-forms of the coframe; if `None`, `indices` is used instead

EXAMPLES:

Coordinate coframe on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: M.frames()
[Coordinate frame (M, (d/dx,d/dy,d/dz))]
sage: M.coframes()
[Coordinate coframe (M, (dx,dy,dz))]
sage: dX = M.coframes()[0] ; dX
Coordinate coframe (M, (dx,dy,dz))
```

The 1-forms composing the coframe are obtained via the operator `[]`:

```
sage: dX[1]
1-form dx on the 3-dimensional differentiable manifold M
sage: dX[2]
1-form dy on the 3-dimensional differentiable manifold M
sage: dX[3]
1-form dz on the 3-dimensional differentiable manifold M
sage: dX[1][:]
[1, 0, 0]
sage: dX[2][:]
[0, 1, 0]
sage: dX[3][:]
[0, 0, 1]
```

The coframe is the dual of the coordinate frame:

```
sage: e = X.frame() ; e
Coordinate frame (M, (d/dx,d/dy,d/dz))
sage: dX[1](e[1]).expr(), dX[1](e[2]).expr(), dX[1](e[3]).expr()
(1, 0, 0)
sage: dX[2](e[1]).expr(), dX[2](e[2]).expr(), dX[2](e[3]).expr()
(0, 1, 0)
sage: dX[3](e[1]).expr(), dX[3](e[2]).expr(), dX[3](e[3]).expr()
(0, 0, 1)
```

Each 1-form of a coordinate coframe is closed:

```
sage: dX[1].exterior_derivative()
2-form ddx on the 3-dimensional differentiable manifold M
sage: dX[1].exterior_derivative() == 0
True
```

class `sage.manifolds.differentiable.vectorframe.`**`CoordFrame`**(*chart*)

Bases: *`sage.manifolds.differentiable.vectorframe.VectorFrame`*

Coordinate frame on a differentiable manifold.

By *coordinate frame*, it is meant a vector frame on a differentiable manifold M that is associated to a coordinate chart on M.

INPUT:

- `chart` – the chart defining the coordinates

EXAMPLES:

The coordinate frame associated to spherical coordinates of the sphere S^2:

```
sage: M = Manifold(2, 'S^2', start_index=1)  # Part of S^2 covered by spherical␣
↪coord.
sage: M.chart(r'th:[0,pi]:\theta ph:[0,2*pi):\phi')
Chart (S^2, (th, ph))
sage: b = M.default_frame()
sage: b
Coordinate frame (S^2, (d/dth,d/dph))
sage: b[1]
Vector field d/dth on the 2-dimensional differentiable manifold S^2
sage: b[2]
Vector field d/dph on the 2-dimensional differentiable manifold S^2
sage: latex(b)
\left(S^2, \left(\frac{\partial}{\partial {\theta} },\frac{\partial}{\partial
↪{\phi} }\right)\right)
```

chart()

Return the chart defining this coordinate frame.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: e = X.frame()
sage: e.chart()
Chart (M, (x, y))
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: e.restrict(U).chart()
Chart (U, (x, y))
```

structure_coeff()

Return the structure coefficients associated to `self`.

n being the manifold's dimension, the structure coefficients of the frame (e_i) are the n^3 scalar fields $C^k_{\ ij}$ defined by

$$[e_i, e_j] = C^k_{\ ij} e_k.$$

In the present case, since (e_i) is a coordinate frame, $C^k_{\ ij} = 0$.

OUTPUT:

- the structure coefficients $C^k_{\ ij}$, as a vanishing instance of `CompWithSym` with 3 indices ordered as (k, i, j)

EXAMPLES:

Structure coefficients of the coordinate frame associated to spherical coordinates in the Euclidean space $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)  # Part of R^3 covered by spherical coord.
sage: c_spher = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: b = M.default_frame() ; b
Coordinate frame (R^3, (d/dr,d/dth,d/dph))
sage: c = b.structure_coeff() ; c
3-indices components w.r.t. Coordinate frame
 (R^3, (d/dr,d/dth,d/dph)), with antisymmetry on the index
 positions (1, 2)
sage: c == 0
True
```

class sage.manifolds.differentiable.vectorframe.**VectorFrame**(*vector_field_module*, *symbol*, *latex_symbol=None*, *from_frame=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Bases: `sage.tensor.modules.free_module_basis.FreeModuleBasis`

Vector frame on a differentiable manifold.

By *vector frame*, it is meant a field e on some differentiable manifold U endowed with a differentiable map $\Phi : U \to M$ to a differentiable manifold M such that for each $p \in U$, $e(p)$ is a vector basis of the tangent space $T_{\Phi(p)}M$.

The standard case of a vector frame *on* U corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

For each instanciation of a vector frame, a coframe is automatically created, as an instance of the class *CoFrame*. It is returned by the method *coframe()*.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $M \supset \Phi(U)$
- `symbol` – either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a tuple of strings, representing the individual symbols of the vector fields

- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `from_frame` – (default: `None`) vector frame $\tilde{e}$ on the codomain M of the destination map Φ; the constructed frame e is then such that $\forall p \in U, e(p) = \tilde{e}(\Phi(p))$
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on the vector frame's domain
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe

EXAMPLES:

Defining a vector frame on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: e = M.vector_frame('e') ; e
Vector frame (M, (e_1,e_2,e_3))
sage: latex(e)
\left(M, \left(e_{1},e_{2},e_{3}\right)\right)
```

The individual elements of the vector frame are accessed via square brackets, with the possibility to invoke the slice operator '`:`' to get more than a single element:

```
sage: e[2]
Vector field e_2 on the 3-dimensional differentiable manifold M
sage: e[1:3]
(Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M)
sage: e[:]
(Vector field e_1 on the 3-dimensional differentiable manifold M,
 Vector field e_2 on the 3-dimensional differentiable manifold M,
 Vector field e_3 on the 3-dimensional differentiable manifold M)
```

The LaTeX symbol can be specified:

```
sage: E = M.vector_frame('E', latex_symbol=r"\epsilon")
sage: latex(E)
\left(M, \left(\epsilon_{1},\epsilon_{2},\epsilon_{3}\right)\right)
```

By default, the elements of the vector frame are labelled by integers within the range specified at the manifold declaration. It is however possible to fully customize the labels, via the argument `indices`:

```
sage: u = M.vector_frame('u', indices=('x', 'y', 'z')) ; u
Vector frame (M, (u_x,u_y,u_z))
sage: u[1]
Vector field u_x on the 3-dimensional differentiable manifold M
sage: u.coframe()
Coframe (M, (u^x,u^y,u^z))
```

The LaTeX format of the indices can be adjusted:

```
sage: v = M.vector_frame('v', indices=('a', 'b', 'c'),
....:                    latex_indices=(r'\alpha', r'\beta', r'\gamma'))
sage: v
Vector frame (M, (v_a,v_b,v_c))
sage: latex(v)
\left(M, \left(v_{\alpha},v_{\beta},v_{\gamma}\right)\right)
sage: latex(v.coframe())
\left(M, \left(v^{\alpha},v^{\beta},v^{\gamma}\right)\right)
```

The symbol of each element of the vector frame can also be freely chosen, by providing a tuple of symbols as the first argument of `vector_frame`; it is then mandatory to specify as well some symbols for the dual coframe:

```
sage: h = M.vector_frame(('a', 'b', 'c'), symbol_dual=('A', 'B', 'C'))
sage: h
Vector frame (M, (a,b,c))
sage: h[1]
Vector field a on the 3-dimensional differentiable manifold M
sage: h.coframe()
Coframe (M, (A,B,C))
sage: h.coframe()[1]
1-form A on the 3-dimensional differentiable manifold M
```

Example with a non-trivial map Φ (see above); a vector frame along a curve:

```
sage: U = Manifold(1, 'U')  # open interval (-1,1) as a 1-dimensional manifold
sage: T.<t> = U.chart('t:(-1,1)')  # canonical chart on U
sage: Phi = U.diff_map(M, [cos(t), sin(t), t], name='Phi',
....:                  latex_name=r'\Phi')
sage: Phi
Differentiable map Phi from the 1-dimensional differentiable manifold U
 to the 3-dimensional differentiable manifold M
sage: f = U.vector_frame('f', dest_map=Phi) ; f
Vector frame (U, (f_1,f_2,f_3)) with values on the 3-dimensional
 differentiable manifold M
sage: f.domain()
1-dimensional differentiable manifold U
sage: f.ambient_domain()
3-dimensional differentiable manifold M
```

The value of the vector frame at a given point is a basis of the corresponding tangent space:

```
sage: p = U((0,), name='p') ; p
Point p on the 1-dimensional differentiable manifold U
sage: f.at(p)
Basis (f_1,f_2,f_3) on the Tangent space at Point Phi(p) on the
 3-dimensional differentiable manifold M
```

Vector frames are bases of free modules formed by vector fields:

```
sage: e.module()
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
sage: e.module().base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: e.module() is M.vector_field_module()
```

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```
True
sage: e in M.vector_field_module().bases()
True
```

```
sage: f.module()
Free module X(U,Phi) of vector fields along the 1-dimensional
 differentiable manifold U mapped into the 3-dimensional differentiable
 manifold M
sage: f.module().base_ring()
Algebra of differentiable scalar fields on the 1-dimensional
 differentiable manifold U
sage: f.module() is U.vector_field_module(dest_map=Phi)
True
sage: f in U.vector_field_module(dest_map=Phi).bases()
True
```

along (*mapping*)

Return the vector frame deduced from the current frame via a differentiable map, the codomain of which is included in the domain of of the current frame.

If e is the current vector frame, V its domain and if $\Phi : U \to V$ is a differentiable map from some differentiable manifold U to V, the returned object is a vector frame $\tilde{e}$ along U with values on V such that

$$\forall p \in U, \ \tilde{e}(p) = e(\Phi(p)).$$

INPUT:

- `mapping` – differentiable map $\Phi : U \to V$

OUTPUT:

- vector frame $\tilde{e}$ along U defined above.

EXAMPLES:

Vector frame along a curve:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: R = Manifold(1, 'R')  # R as a 1-dimensional manifold
sage: T.<t> = R.chart()  # canonical chart on R
sage: Phi = R.diff_map(M, {(T,X): [cos(t), t]}, name='Phi',
....:                  latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold R to the 2-dimensional differentiable manifold M
sage: e = X.frame() ; e
Coordinate frame (M, (d/dx,d/dy))
sage: te = e.along(Phi) ; te
Vector frame (R, (d/dx,d/dy)) with values on the 2-dimensional
 differentiable manifold M
```

Check of the formula $\tilde{e}(p) = e(\Phi(p))$:

```
sage: p = R((pi,)) ; p
Point on the 1-dimensional differentiable manifold R
sage: te[0].at(p) == e[0].at(Phi(p))
True
sage: te[1].at(p) == e[1].at(Phi(p))
True
```

The result is cached:

```
sage: te is e.along(Phi)
True
```

ambient_domain()

Return the differentiable manifold in which `self` takes its values.

The ambient domain is the codomain M of the differentiable map $\Phi : U \to M$ associated with the frame.

OUTPUT:

- a *DifferentiableManifold* representing M

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e')
sage: e.ambient_domain()
2-dimensional differentiable manifold M
```

In the present case, since Φ is the identity map:

```
sage: e.ambient_domain() == e.domain()
True
```

An example with a non trivial map Φ:

```
sage: U = Manifold(1, 'U')
sage: T.<t> = U.chart()
sage: X.<x,y> = M.chart()
sage: Phi = U.diff_map(M, {(T,X): [cos(t), t]}, name='Phi',
....:                  latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold U to the 2-dimensional differentiable manifold M
sage: f = U.vector_frame('f', dest_map=Phi); f
Vector frame (U, (f_0,f_1)) with values on the 2-dimensional
 differentiable manifold M
sage: f.ambient_domain()
2-dimensional differentiable manifold M
sage: f.domain()
1-dimensional differentiable manifold U
```

at(*point*)

Return the value of `self` at a given point, this value being a basis of the tangent vector space at the point.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain U of the vector frame (denoted e hereafter)

OUTPUT:

- `FreeModuleBasis` representing the basis $e(p)$ of the tangent vector space $T_{\Phi(p)}M$, where $\Phi : U \to M$ is the differentiable map associated with e (possibly $\Phi = \mathrm{Id}_U$)

EXAMPLES:

Basis of a tangent space to a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
```

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```
sage: p = M.point((-1,2), name='p')
sage: e = X.frame() ; e
Coordinate frame (M, (d/dx,d/dy))
sage: ep = e.at(p) ; ep
Basis (d/dx,d/dy) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: type(ep)
<class 'sage.tensor.modules.free_module_basis.FreeModuleBasis'>
sage: ep[0]
Tangent vector d/dx at Point p on the 2-dimensional differentiable
 manifold M
sage: ep[1]
Tangent vector d/dy at Point p on the 2-dimensional differentiable
 manifold M
```

Note that the symbols used to denote the vectors are same as those for the vector fields of the frame. At this stage, `ep` is the unique basis on the tangent space at `p`:

```
sage: Tp = M.tangent_space(p)
sage: Tp.bases()
[Basis (d/dx,d/dy) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M]
```

Let us consider a vector frame that is a not a coordinate one:

```
sage: aut = M.automorphism_field()
sage: aut[:] = [[1+y^2, 0], [0, 2]]
sage: f = e.new_frame(aut, 'f') ; f
Vector frame (M, (f_0,f_1))
sage: fp = f.at(p) ; fp
Basis (f_0,f_1) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

There are now two bases on the tangent space:

```
sage: Tp.bases()
[Basis (d/dx,d/dy) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M,
 Basis (f_0,f_1) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M]
```

Moreover, the changes of bases in the tangent space have been computed from the known relation between the frames `e` and `f` (field of automorphisms `aut` defined above):

```
sage: Tp.change_of_basis(ep, fp)
Automorphism of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: Tp.change_of_basis(ep, fp).display()
5 d/dx*dx + 2 d/dy*dy
sage: Tp.change_of_basis(fp, ep)
Automorphism of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: Tp.change_of_basis(fp, ep).display()
1/5 d/dx*dx + 1/2 d/dy*dy
```

The dual bases:

```
sage: e.coframe()
Coordinate coframe (M, (dx,dy))
sage: ep.dual_basis()
Dual basis (dx,dy) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: ep.dual_basis() is e.coframe().at(p)
True
sage: f.coframe()
Coframe (M, (f^0,f^1))
sage: fp.dual_basis()
Dual basis (f^0,f^1) on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: fp.dual_basis() is f.coframe().at(p)
True
```

coframe()

Return the coframe of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e')
sage: e.coframe()
Coframe (M, (e^0,e^1))
sage: X.<x,y> = M.chart()
sage: X.frame().coframe()
Coordinate coframe (M, (dx,dy))
```

destination_map()

Return the differential map associated to this vector frame.

Let e denote the vector frame; the differential map associated to it is the map $\Phi : U \to M$ such that for each $p \in U$, $e(p)$ is a vector basis of the tangent space $T_{\Phi(p)}M$.

OUTPUT:

- a *`DiffMap`* representing the differential map Φ

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e')
sage: e.destination_map()
Identity map Id_M of the 2-dimensional differentiable manifold M
```

An example with a non trivial map Φ:

```
sage: U = Manifold(1, 'U')
sage: T.<t> = U.chart()
sage: X.<x,y> = M.chart()
sage: Phi = U.diff_map(M, {(T,X): [cos(t), t]}, name='Phi',
....:                  latex_name=r'\Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable
 manifold U to the 2-dimensional differentiable manifold M
sage: f = U.vector_frame('f', dest_map=Phi); f
Vector frame (U, (f_0,f_1)) with values on the 2-dimensional
 differentiable manifold M
sage: f.destination_map()
```

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```
Differentiable map Phi from the 1-dimensional differentiable
 manifold U to the 2-dimensional differentiable manifold M
```

domain()

Return the domain on which `self` is defined.

OUTPUT:

- a *DifferentiableManifold*; representing the domain of the vector frame

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e')
sage: e.domain()
2-dimensional differentiable manifold M
sage: U = M.open_subset('U')
sage: f = e.restrict(U)
sage: f.domain()
Open subset U of the 2-dimensional differentiable manifold M
```

new_frame(*change_of_frame*, *symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *symbol_dual=None*, *latex_symbol_dual=None*)

Define a new vector frame from `self`.

The new vector frame is defined from a field of tangent-space automorphisms; its domain is the same as that of the current frame.

INPUT:

- `change_of_frame` – *AutomorphismFieldParal*; the field of tangent space automorphisms P that relates the current frame (e_i) to the new frame (n_i) according to $n_i = P(e_i)$
- `symbol` – either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual symbols of the vector fields
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `symbol_dual` – (default: `None`) same as `symbol` but for the dual coframe; if `None`, `symbol` must be a string and is used for the common base of the symbols of the elements of the dual coframe
- `latex_symbol_dual` – (default: `None`) same as `latex_symbol` but for the dual coframe

OUTPUT:

- the new frame (n_i), as an instance of *VectorFrame*

EXAMPLES:

Frame resulting from a $\pi/3$-rotation in the Euclidean plane:

```
sage: M = Manifold(2, 'R^2')
sage: X.<x,y> = M.chart()
sage: e = M.vector_frame('e') ; M.set_default_frame(e)
sage: M._frame_changes
{}
sage: rot = M.automorphism_field()
sage: rot[:] = [[sqrt(3)/2, -1/2], [1/2, sqrt(3)/2]]
sage: n = e.new_frame(rot, 'n')
sage: n[0][:]
[1/2*sqrt(3), 1/2]
sage: n[1][:]
[-1/2, 1/2*sqrt(3)]
sage: a =  M.change_of_frame(e,n)
sage: a[:]
[1/2*sqrt(3)         -1/2]
[        1/2 1/2*sqrt(3)]
sage: a == rot
True
sage: a is rot
False
sage: a._components # random (dictionary output)
{Vector frame (R^2, (e_0,e_1)): 2-indices components w.r.t.
 Vector frame (R^2, (e_0,e_1)),
 Vector frame (R^2, (n_0,n_1)): 2-indices components w.r.t.
 Vector frame (R^2, (n_0,n_1))}
sage: a.comp(n)[:]
[1/2*sqrt(3)         -1/2]
[        1/2 1/2*sqrt(3)]
sage: a1 = M.change_of_frame(n,e)
sage: a1[:]
[1/2*sqrt(3)          1/2]
[       -1/2 1/2*sqrt(3)]
sage: a1 == rot.inverse()
True
sage: a1 is rot.inverse()
False
sage: e[0].comp(n)[:]
[1/2*sqrt(3), -1/2]
sage: e[1].comp(n)[:]
[1/2, 1/2*sqrt(3)]
```

restrict (*subdomain*)

Return the restriction of `self` to some open subset of its domain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset V of the current frame domain U

OUTPUT:

- the restriction of the current frame to V as a `VectorFrame`

EXAMPLES:

Restriction of a frame defined on $\mathbf{R}^2$ to the unit disk:

```
sage: M = Manifold(2, 'R^2', start_index=1)
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
```

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```
sage: a = M.automorphism_field()
sage: a[:] = [[1-y^2,0], [1+x^2, 2]]
sage: e = c_cart.frame().new_frame(a, 'e') ; e
Vector frame (R^2, (e_1,e_2))
sage: U = M.open_subset('U', coord_def={c_cart: x^2+y^2<1})
sage: e_U = e.restrict(U) ; e_U
Vector frame (U, (e_1,e_2))
```

The vectors of the restriction have the same symbols as those of the original frame:

```
sage: e_U[1].display()
e_1 = (-y^2 + 1) d/dx + (x^2 + 1) d/dy
sage: e_U[2].display()
e_2 = 2 d/dy
```

They are actually the restrictions of the original frame vectors:

```
sage: e_U[1] is e[1].restrict(U)
True
sage: e_U[2] is e[2].restrict(U)
True
```

set_name (*symbol*, *latex_symbol=None*, *indices=None*, *latex_indices=None*, *index_position='down'*, *include_domain=True*)

Set (or change) the text name and LaTeX name of `self`.

INPUT:

- `symbol` – either a string, to be used as a common base for the symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual symbols of the vector fields
- `latex_symbol` – (default: `None`) either a string, to be used as a common base for the LaTeX symbols of the vector fields constituting the vector frame, or a list/tuple of strings, representing the individual LaTeX symbols of the vector fields; if `None`, `symbol` is used in place of `latex_symbol`
- `indices` – (default: `None`; used only if `symbol` is a single string) tuple of strings representing the indices labelling the vector fields of the frame; if `None`, the indices will be generated as integers within the range declared on `self`
- `latex_indices` – (default: `None`) tuple of strings representing the indices for the LaTeX symbols of the vector fields; if `None`, `indices` is used instead
- `index_position` – (default: `'down'`) determines the position of the indices labelling the vector fields of the frame; can be either `'down'` or `'up'`
- `include_domain` – (default: `True`) boolean determining whether the name of the domain is included in the beginning of the vector frame name

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: e = M.vector_frame('e'); e
Vector frame (M, (e_0,e_1))
sage: e.set_name('f'); e
Vector frame (M, (f_0,f_1))
sage: e.set_name('e', include_domain=False); e
Vector frame (e_0,e_1)
sage: e.set_name(['a', 'b']); e
Vector frame (M, (a,b))
```

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```
sage: e.set_name('e', indices=['x', 'y']); e
Vector frame (M, (e_x,e_y))
sage: e.set_name('e', latex_symbol=r'\epsilon')
sage: latex(e)
\left(M, \left(\epsilon_{0},\epsilon_{1}\right)\right)
sage: e.set_name('e', latex_symbol=[r'\alpha', r'\beta'])
sage: latex(e)
\left(M, \left(\alpha,\beta\right)\right)
sage: e.set_name('e', latex_symbol='E',
....:            latex_indices=[r'\alpha', r'\beta'])
sage: latex(e)
\left(M, \left(E_{\alpha},E_{\beta}\right)\right)
```

structure_coeff()

Evaluate the structure coefficients associated to `self`.

n being the manifold's dimension, the structure coefficients of the vector frame (e_i) are the n^3 scalar fields $C^k_{\ ij}$ defined by

$$[e_i, e_j] = C^k_{\ ij} e_k$$

OUTPUT:

- the structure coefficients $C^k_{\ ij}$, as an instance of `CompWithSym` with 3 indices ordered as (k, i, j).

EXAMPLES:

Structure coefficients of the orthonormal frame associated to spherical coordinates in the Euclidean space $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', r'\RR^3', start_index=1)  # Part of R^3 covered␣
↪by spherical coordinates
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):\phi')
sage: ch_frame = M.automorphism_field()
sage: ch_frame[1,1], ch_frame[2,2], ch_frame[3,3] = 1, 1/r, 1/(r*sin(th))
sage: M.frames()
[Coordinate frame (R^3, (d/dr,d/dth,d/dph))]
sage: e = c_spher.frame().new_frame(ch_frame, 'e')
sage: e[1][:]  # components of e_1 in the manifold's default frame (d/dr, d/
↪dth, d/dth)
[1, 0, 0]
sage: e[2][:]
[0, 1/r, 0]
sage: e[3][:]
[0, 0, 1/(r*sin(th))]
sage: c = e.structure_coeff() ; c
3-indices components w.r.t. Vector frame (R^3, (e_1,e_2,e_3)), with
 antisymmetry on the index positions (1, 2)
sage: c[:]
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
 [[0, -1/r, 0], [1/r, 0, 0], [0, 0, 0]],
 [[0, 0, -1/r], [0, 0, -cos(th)/(r*sin(th))], [1/r, cos(th)/(r*sin(th)), 0]]]
sage: c[2,1,2]  # C^2_{12}
-1/r
sage: c[3,1,3]  # C^3_{13}
-1/r
sage: c[3,2,3]  # C^3_{23}
-cos(th)/(r*sin(th))
```

2.7.4 Group of Tangent-Space Automorphism Fields

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), the *group of tangent-space automorphism fields* associated with U and Φ is the general linear group $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ of the module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $M \supset \Phi(U)$ (see *VectorFieldModule*). Note that $\mathfrak{X}(U, \Phi)$ is a module over $C^k(U)$, the algebra of differentiable scalar fields on U. Elements of $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ are fields along U of automorphisms of tangent spaces to M.

Two classes implement $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ depending whether M is parallelizable or not: *AutomorphismFieldParalGroup* and *AutomorphismFieldGroup*.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

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class sage.manifolds.differentiable.automorphismfield_group.**AutomorphismFieldGroup**(*vector_field*

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

General linear group of the module of vector fields along a differentiable manifold U with values on a differentiable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), the *group of tangent-space automorphism fields* associated with U and Φ is the general linear group $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ of the module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $M \supset \Phi(U)$ (see *VectorFieldModule*). Note that $\mathfrak{X}(U, \Phi)$ is a module over $C^k(U)$, the algebra of differentiable scalar fields on U. Elements of $\mathrm{GL}(\mathfrak{X}(U, \Phi))$ are fields along U of automorphisms of tangent spaces to M.

Note: If M is parallelizable, then *AutomorphismFieldParalGroup* *must* be used instead.

INPUT:

- vector_field_module – *VectorFieldModule*; module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M

EXAMPLES:

Group of tangent-space automorphism fields of the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:              intersection_name='W',
....:              restrictions1= x^2+y^2!=0, restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: G = M.automorphism_field_group() ; G
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
```

G is the general linear group of the vector field module $\mathfrak{X}(M)$:

```
sage: XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: G is XM.general_linear_group()
True
```

G is a non-abelian group:

```
sage: G.category()
Category of groups
sage: G in Groups()
True
sage: G in CommutativeAdditiveGroups()
False
```

The elements of G are tangent-space automorphisms:

```
sage: a = G.an_element(); a
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: a.parent() is G
True
sage: a.restrict(U).display()
2 d/dx*dx + 2 d/dy*dy
sage: a.restrict(V).display()
2 d/du*du + 2 d/dv*dv
```

The identity element of the group G:

```
sage: e = G.one() ; e
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
sage: eU = U.default_frame() ; eU
Coordinate frame (U, (d/dx,d/dy))
sage: eV = V.default_frame() ; eV
Coordinate frame (V, (d/du,d/dv))
sage: e.display(eU)
Id = d/dx*dx + d/dy*dy
sage: e.display(eV)
Id = d/du*du + d/dv*dv
```

Element
alias of *sage.manifolds.differentiable.automorphismfield.AutomorphismField*

base_module()
Return the vector-field module of which `self` is the general linear group.

OUTPUT:

- *VectorFieldModule*

EXAMPLES:

Base module of the group of tangent-space automorphism fields of the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
```

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```
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                 intersection_name='W', restrictions1= x^
↪2+y^2!=0,
....:                                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: G = M.automorphism_field_group()
sage: G.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: G.base_module() is M.vector_field_module()
True
```

one()

Return identity element of `self`.

The group identity element is the field of tangent-space identity maps.

OUTPUT:

- *AutomorphismField* representing the identity element

EXAMPLES:

Identity element of the group of tangent-space automorphism fields of the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                 intersection_name='W', restrictions1= x^
↪2+y^2!=0,
....:                                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: G = M.automorphism_field_group()
sage: G.one()
Field of tangent-space identity maps on the 2-dimensional differentiable␣
↪manifold M
sage: G.one().restrict(U)[:]
[1 0]
[0 1]
sage: G.one().restrict(V)[:]
[1 0]
[0 1]
```

class sage.manifolds.differentiable.automorphismfield_group.**AutomorphismFieldParalGroup**(*vec*

Bases: `sage.tensor.modules.free_module_linear_group.FreeModuleLinearGroup`

General linear group of the module of vector fields along a differentiable manifold U with values on a parallelizable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a parallelizable manifold M (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), the *group of tangent-space automorphism fields* associated with U and Φ is the general linear group $\mathrm{GL}(\mathfrak{X}(U,\Phi))$ of the module $\mathfrak{X}(U,\Phi)$ of vector fields along U with values on $M \supset \Phi(U)$ (see

VectorFieldFreeModule). Note that $\mathfrak{X}(U,\Phi)$ is a free module over $C^k(U)$, the algebra of differentiable scalar fields on U. Elements of $\mathrm{GL}(\mathfrak{X}(U,\Phi))$ are fields along U of automorphisms of tangent spaces to M.

Note: If M is not parallelizable, the class *AutomorphismFieldGroup* must be used instead.

INPUT:

- `vector_field_module` – *VectorFieldFreeModule*; free module $\mathfrak{X}(U,\Phi)$ of vector fields along U with values on M

EXAMPLES:

Group of tangent-space automorphism fields of a 2-dimensional parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: G = M.automorphism_field_group(); G
General linear group of the Free module X(M) of vector fields on the
 2-dimensional differentiable manifold M
sage: latex(G)
\mathrm{GL}\left( \mathfrak{X}\left(M\right) \right)
```

`G` is nothing but the general linear group of the module $\mathfrak{X}(M)$:

```
sage: G is XM.general_linear_group()
True
```

`G` is a group:

```
sage: G.category()
Category of groups
sage: G in Groups()
True
```

It is not an abelian group:

```
sage: G in CommutativeAdditiveGroups()
False
```

The elements of `G` are tangent-space automorphisms:

```
sage: G.Element
<class 'sage.manifolds.differentiable.automorphismfield.AutomorphismFieldParal'>
sage: a = G.an_element() ; a
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: a.parent() is G
True
```

As automorphisms of $\mathfrak{X}(M)$, the elements of `G` map a vector field to a vector field:

```
sage: v = XM.an_element() ; v
Vector field on the 2-dimensional differentiable manifold M
sage: v.display()
```

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```
2 d/dx + 2 d/dy
sage: a(v)
Vector field on the 2-dimensional differentiable manifold M
sage: a(v).display()
2 d/dx - 2 d/dy
```

Indeed the matrix of a with respect to the frame (∂_x, ∂_y) is:

```
sage: a[X.frame(),:]
[ 1  0]
[ 0 -1]
```

The elements of G can also be considered as tensor fields of type $(1, 1)$:

```
sage: a.tensor_type()
(1, 1)
sage: a.tensor_rank()
2
sage: a.domain()
2-dimensional differentiable manifold M
sage: a.display()
d/dx*dx - d/dy*dy
```

The identity element of the group G is:

```
sage: id = G.one() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
sage: id*a == a
True
sage: a*id == a
True
sage: a*a^(-1) == id
True
sage: a^(-1)*a == id
True
```

Construction of an element by providing its components with respect to the manifold's default frame (frame associated to the coordinates (x, y)):

```
sage: b = G([[1+x^2,0], [0,1+y^2]]) ; b
Field of tangent-space automorphisms on the 2-dimensional
 differentiable manifold M
sage: b.display()
(x^2 + 1) d/dx*dx + (y^2 + 1) d/dy*dy
sage: (~b).display()  # the inverse automorphism
1/(x^2 + 1) d/dx*dx + 1/(y^2 + 1) d/dy*dy
```

We check the group law on these elements:

```
sage: (a*b)^(-1) == b^(-1) * a^(-1)
True
```

Invertible tensor fields of type $(1, 1)$ can be converted to elements of G:

```
sage: t = M.tensor_field(1, 1, name='t')
sage: t[:] = [[1+exp(y), x*y], [0, 1+x^2]]
sage: t1 = G(t) ; t1
Field of tangent-space automorphisms t on the 2-dimensional
 differentiable manifold M
sage: t1 in G
True
sage: t1.display()
t = (e^y + 1) d/dx*dx + x*y d/dx*dy + (x^2 + 1) d/dy*dy
sage: t1^(-1)
Field of tangent-space automorphisms t^(-1) on the 2-dimensional
 differentiable manifold M
sage: (t1^(-1)).display()
t^(-1) = 1/(e^y + 1) d/dx*dx - x*y/(x^2 + (x^2 + 1)*e^y + 1) d/dx*dy
 + 1/(x^2 + 1) d/dy*dy
```

Since any automorphism field can be considered as a tensor field of type-$(1, 1)$ on M, there is a coercion map from G to the module $T^{(1,1)}(M)$ of type-$(1, 1)$ tensor fields:

```
sage: T11 = M.tensor_field_module((1,1)) ; T11
Free module T^(1,1)(M) of type-(1,1) tensors fields on the
 2-dimensional differentiable manifold M
sage: T11.has_coerce_map_from(G)
True
```

An explicit call of this coercion map is:

```
sage: tt = T11(t1) ; tt
Tensor field t of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: tt == t
True
```

An implicit call of the coercion map is performed to subtract an element of G from an element of $T^{(1,1)}(M)$:

```
sage: s = t - t1 ; s
Tensor field t-t of type (1,1) on
 the 2-dimensional differentiable manifold M
sage: s.parent() is T11
True
sage: s.display()
t-t = 0
```

as well as for the reverse operation:

```
sage: s = t1 - t ; s
Tensor field t-t of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: s.display()
t-t = 0
```

Element

alias of *sage.manifolds.differentiable.automorphismfield.AutomorphismFieldParal*

2.7.5 Tangent-Space Automorphism Fields

The class `AutomorphismField` implements fields of automorphisms of tangent spaces to a generic (a priori not parallelizable) differentiable manifold, while the class `AutomorphismFieldParal` is devoted to fields of automorphisms of tangent spaces to a parallelizable manifold. The latter play the important role of transitions between vector frames sharing the same domain on a differentiable manifold.

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

class sage.manifolds.differentiable.automorphismfield.**AutomorphismField**(*vector_field_module*, *name=None*, *latex_name=None*, *is_identity=False*)

Bases: `sage.manifolds.differentiable.tensorfield.TensorField`

Field of automorphisms of tangent spaces to a generic (a priori not parallelizable) differentiable manifold.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M, a *field of tangent-space automorphisms along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(1,1)}M,$$

with $T^{(1,1)}M$ being the tensor bundle of type $(1,1)$ over M, such that

$$\forall p \in U,\ a(p) \in \mathrm{Aut}(T_{\Phi(p)}M),$$

i.e. $a(p)$ is an automorphism of the tangent space to M at the point $\Phi(p)$.

The standard case of a field of tangent-space automorphisms *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is parallelizable, then `AutomorphismFieldParal` *must* be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map Φ
- `name` – (default: `None`) name given to the field
- `latex_name` – (default: `None`) LaTeX symbol to denote the field; if none is provided, the LaTeX symbol is set to `name`
- `is_identity` – (default: `False`) determines whether the constructed object is a field of identity automorphisms

EXAMPLES:

Field of tangent-space automorphisms on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
```

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```
....:                                   restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: a = M.automorphism_field(name='a') ; a
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold M
sage: a.parent()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
```

We first define the components of a with respect to the coordinate frame on U:

```
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a[eU,:] = [[1,x], [0,2]]
```

It is equivalent to pass the components while defining a:

```
sage: a = M.automorphism_field({eU: [[1,x], [0,2]]}, name='a')
```

We then set the components with respect to the coordinate frame on V by extending the expressions of the components in the corresponding subframe on $W = U \cap V$:

```
sage: W = U.intersection(V)
sage: a.add_comp_by_continuation(eV, W, c_uv)
```

At this stage, the automorphism field a is fully defined:

```
sage: a.display(eU)
a = d/dx*dx + x d/dx*dy + 2 d/dy*dy
sage: a.display(eV)
a = (1/4*u + 1/4*v + 3/2) d/du*du + (-1/4*u - 1/4*v - 1/2) d/du*dv
 + (1/4*u + 1/4*v - 1/2) d/dv*du + (-1/4*u - 1/4*v + 3/2) d/dv*dv
```

In particular, we may ask for its inverse on the whole manifold M:

```
sage: ia = a.inverse() ; ia
Field of tangent-space automorphisms a^(-1) on the 2-dimensional
 differentiable manifold M
sage: ia.display(eU)
a^(-1) = d/dx*dx - 1/2*x d/dx*dy + 1/2 d/dy*dy
sage: ia.display(eV)
a^(-1) = (-1/8*u - 1/8*v + 3/4) d/du*du + (1/8*u + 1/8*v + 1/4) d/du*dv
 + (-1/8*u - 1/8*v + 1/4) d/dv*du + (1/8*u + 1/8*v + 3/4) d/dv*dv
```

Equivalently, one can use the power minus one to get the inverse:

```
sage: ia is a^(-1)
True
```

or the operator ~:

```
sage: ia is ~a
True
```

inverse()

Return the inverse automorphism of `self`.

EXAMPLES:

Inverse of a field of tangent-space automorphisms on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: W = U.intersection(V)
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:    intersection_name='W', restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.automorphism_field({eU: [[1,x], [0,2]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: ia = a.inverse() ; ia
Field of tangent-space automorphisms a^(-1) on the 2-dimensional
 differentiable manifold M
sage: a[eU,:], ia[eU,:]
(
[1 x]  [     1 -1/2*x]
[0 2], [     0    1/2]
)
sage: a[eV,:], ia[eV,:]
(
[ 1/4*u + 1/4*v + 3/2 -1/4*u - 1/4*v - 1/2]
[ 1/4*u + 1/4*v - 1/2 -1/4*u - 1/4*v + 3/2],
[-1/8*u - 1/8*v + 3/4  1/8*u + 1/8*v + 1/4]
[-1/8*u - 1/8*v + 1/4  1/8*u + 1/8*v + 3/4]
)
```

Let us check that ia is indeed the inverse of a:

```
sage: s = a.contract(ia)
sage: s[eU,:], s[eV,:]
(
[1 0]  [1 0]
[0 1], [0 1]
)
sage: s = ia.contract(a)
sage: s[eU,:], s[eV,:]
(
[1 0]  [1 0]
[0 1], [0 1]
)
```

The result is cached:

```
sage: a.inverse() is ia
True
```

Instead of `inverse()`, one can use the power minus one to get the inverse:

```
sage: ia is a^(-1)
True
```

or the operator ~:

```
sage: ia is ~a
True
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subdomain.

This is a redefinition of `sage.manifolds.differentiable.tensorfield.TensorField.restrict()` to take into account the identity map.

INPUT:

- `subdomain` – `DifferentiableManifold` open subset V of `self._domain`
- `dest_map` – (default: `None`) `DiffMap`; destination map $\Phi : V \to N$, where N is a subdomain of `self._codomain`; if `None`, the restriction of `self.base_module().destination_map()` to V is used

OUTPUT:

- a `AutomorphismField` representing the restriction

EXAMPLES:

Restrictions of an automorphism field on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of the North pole
sage: stereoN.<x,y> = U.chart()  # stereographic coordinates from the North␣
↪pole
sage: eN = stereoN.frame() # the associated vector frame
sage: V =  M.open_subset('V') # the complement of the South pole
sage: stereoS.<u,v> = V.chart()  # stereographic coordinates from the South␣
↪pole
sage: eS = stereoS.frame() # the associated vector frame
sage: transf = stereoN.transition_map(stereoS, (x/(x^2+y^2), y/(x^2+y^2)),
....:                                    intersection_name='W',
....:                                    restrictions1= x^2+y^2!=0,
....:                                    restrictions2= u^2+v^2!=0)
sage: inv = transf.inverse() # transformation from stereoS to stereoN
sage: W = U.intersection(V) # the complement of the North and South poles
sage: stereoN_W = W.atlas()[0]  # restriction of stereo. coord. from North␣
↪pole to W
sage: stereoS_W = W.atlas()[1]  # restriction of stereo. coord. from South␣
↪pole to W
sage: eN_W = stereoN_W.frame() ; eS_W = stereoS_W.frame()
sage: a = M.automorphism_field({eN: [[1, atan(x^2+y^2)], [0,3]]},
....:                          name='a')
sage: a.add_comp_by_continuation(eS, W, chart=stereoS); a
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold S^2
sage: a.restrict(U)
Field of tangent-space automorphisms a on the Open subset U of the
 2-dimensional differentiable manifold S^2
sage: a.restrict(U)[eN,:]
[               1 arctan(x^2 + y^2)]
[               0                 3]
sage: a.restrict(V)
Field of tangent-space automorphisms a on the Open subset V of the
 2-dimensional differentiable manifold S^2
sage: a.restrict(V)[eS,:]
[   (u^4 + 10*u^2*v^2 + v^4 + 2*(u^3*v - u*v^3)*arctan(1/(u^2 + v^2)))/(u^4 +␣
↪2*u^2*v^2 + v^4)  -(4*u^3*v - 4*u*v^3 + (u^4 - 2*u^2*v^2 + v^4)*arctan(1/(u^
↪2 + v^2)))/(u^4 + 2*u^2*v^2 + v^4)]
[                    4*(u^2*v^2*arctan(1/(u^2 + v^2)) - u^3*v + u*v^3)/(u^4 +␣
↪2*u^2*v^2 + v^4) (3*u^4 - 2*u^2*v^2 + 3*v^4 - 2*(u^3*v - u*v^3)*arctan(1/(u^
↪2 + v^2)))/(u^4 + 2*u^2*v^2 + v^4)]
```

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```
sage: a.restrict(W)
Field of tangent-space automorphisms a on the Open subset W of the
 2-dimensional differentiable manifold S^2
sage: a.restrict(W)[eN_W,:]
[                1 arctan(x^2 + y^2)]
[                0                 3]
```

Restrictions of the field of tangent-space identity maps:

```
sage: id = M.tangent_identity_field() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold S^2
sage: id.restrict(U)
Field of tangent-space identity maps on the Open subset U of the
 2-dimensional differentiable manifold S^2
sage: id.restrict(U)[eN,:]
[1 0]
[0 1]
sage: id.restrict(V)
Field of tangent-space identity maps on the Open subset V of the
 2-dimensional differentiable manifold S^2
sage: id.restrict(V)[eS,:]
[1 0]
[0 1]
sage: id.restrict(W)[eN_W,:]
[1 0]
[0 1]
sage: id.restrict(W)[eS_W,:]
[1 0]
[0 1]
```

class sage.manifolds.differentiable.automorphismfield.**AutomorphismFieldParal**(*vector_field_module, name=None, latex_name=None, is_identity=False*)

Bases: sage.tensor.modules.free_module_automorphism.FreeModuleAutomorphism, *sage.manifolds.differentiable.tensorfield_paral.TensorFieldParal*

Field of tangent-space automorphisms with values on a parallelizable manifold.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a parallelizable manifold M, a *field of tangent-space automorphisms along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(1,1)}M$$

($T^{(1,1)}M$ being the tensor bundle of type $(1,1)$ over M) such that

$$\forall p \in U, \ a(p) \in \mathrm{Aut}(T_{\Phi(p)}M)$$

i.e. $a(p)$ is an automorphism of the tangent space to M at the point $\Phi(p)$.

The standard case of a field of tangent-space automorphisms *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is not parallelizable, the class *AutomorphismField* *must* be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map Φ
- `name` – (default: `None`) name given to the field
- `latex_name` – (default: `None`) LaTeX symbol to denote the field; if none is provided, the LaTeX symbol is set to `name`
- `is_identity` – (default: `False`) determines whether the constructed object is a field of identity automorphisms

EXAMPLES:

A $\pi/3$-rotation in the Euclidean 2-plane:

```
sage: M = Manifold(2, 'R^2')
sage: c_xy.<x,y> = M.chart()
sage: rot = M.automorphism_field([[sqrt(3)/2, -1/2], [1/2, sqrt(3)/2]],
....:                                 name='R'); rot
Field of tangent-space automorphisms R on the 2-dimensional
 differentiable manifold R^2
sage: rot.parent()
General linear group of the Free module X(R^2) of vector fields on the
 2-dimensional differentiable manifold R^2
```

The inverse automorphism is obtained via the method `inverse()`:

```
sage: inv = rot.inverse() ; inv
Field of tangent-space automorphisms R^(-1) on the 2-dimensional
 differentiable manifold R^2
sage: latex(inv)
R^{-1}
sage: inv[:]
[1/2*sqrt(3)         1/2]
[       -1/2 1/2*sqrt(3)]
sage: rot[:]
[1/2*sqrt(3)        -1/2]
[        1/2 1/2*sqrt(3)]
sage: inv[:] * rot[:]  # check
[1 0]
[0 1]
```

Equivalently, one can use the power minus one to get the inverse:

```
sage: inv is rot^(-1)
True
```

or the operator ~:

```
sage: inv is ~rot
True
```

at (*point*)

Value of `self` at a given point.

If the current field of tangent-space automorphisms is

$$a:\ U \longrightarrow T^{(1,1)}M$$

associated with the differentiable map

$$\Phi : U \longrightarrow M,$$

where U and M are two manifolds (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), then for any point $p \in U$, $a(p)$ is an automorphism of the tangent space $T_{\Phi(p)}M$.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain of the field of automorphisms a

OUTPUT:

- the automorphism $a(p)$ of the tangent vector space $T_{\Phi(p)}M$

EXAMPLES:

Automorphism at some point of a tangent space of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: a = M.automorphism_field([[1+exp(y), x*y], [0, 1+x^2]],
....:                          name='a')
sage: a.display()
a = (e^y + 1) d/dx*dx + x*y d/dx*dy + (x^2 + 1) d/dy*dy
sage: p = M.point((-2,3), name='p') ; p
Point p on the 2-dimensional differentiable manifold M
sage: ap = a.at(p) ; ap
Automorphism a of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: ap.display()
a = (e^3 + 1) d/dx*dx - 6 d/dx*dy + 5 d/dy*dy
sage: ap.parent()
General linear group of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
```

The identity map of the tangent space at point `p`:

```
sage: id = M.tangent_identity_field() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
sage: idp = id.at(p) ; idp
Identity map of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: idp is M.tangent_space(p).identity_map()
True
sage: idp.display()
Id = d/dx*dx + d/dy*dy
sage: idp.parent()
General linear group of the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: idp * ap == ap
True
```

inverse()

Return the inverse automorphism of `self`.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: a = M.automorphism_field([[0, 2], [-1, 0]], name='a')
sage: b = a.inverse(); b
Field of tangent-space automorphisms a^(-1) on the 2-dimensional
 differentiable manifold M
sage: b[:]
[  0  -1]
[1/2   0]
sage: a[:]
[ 0  2]
[-1  0]
```

The result is cached:

```
sage: a.inverse() is b
True
```

Instead of `inverse()`, one can use the power minus one to get the inverse:

```
sage: b is a^(-1)
True
```

or the operator ~:

```
sage: b is ~a
True
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subset of its domain.

If such restriction has not been defined yet, it is constructed here.

This is a redefinition of *sage.manifolds.differentiable.tensorfield_paral.TensorFieldParal.restrict()* to take into account the identity map.

INPUT:

- `subdomain` – *DifferentiableManifold*; open subset V of `self._domain`
- `dest_map` – (default: `None`) *DiffMap* destination map $\Phi : V \rightarrow N$, where N is a subset of `self._codomain`; if `None`, the restriction of `self.base_module().destination_map()` to V is used

OUTPUT:

- a *AutomorphismFieldParal* representing the restriction

EXAMPLES:

Restriction of an automorphism field defined on $\mathbf{R}^2$ to a disk:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: D = M.open_subset('D') # the unit open disc
sage: c_cart_D = c_cart.restrict(D, x^2+y^2<1)
sage: a = M.automorphism_field([[1, x*y], [0, 3]], name='a'); a
Field of tangent-space automorphisms a on the 2-dimensional
 differentiable manifold R^2
sage: a.restrict(D)
```

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```
Field of tangent-space automorphisms a on the Open subset D of the
 2-dimensional differentiable manifold R^2
sage: a.restrict(D)[:]
[  1 x*y]
[  0   3]
```

Restriction to the disk of the field of tangent-space identity maps:

```
sage: id = M.tangent_identity_field() ; id
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold R^2
sage: id.restrict(D)
Field of tangent-space identity maps on the Open subset D of the
 2-dimensional differentiable manifold R^2
sage: id.restrict(D)[:]
[1 0]
[0 1]
sage: id.restrict(D) == D.tangent_identity_field()
True
```

2.8 Tensor Fields

2.8.1 Tensor Field Modules

The set of tensor fields along a differentiable manifold U with values on a differentiable manifold M via a differentiable map $\Phi : U \to M$ (possibly $U = M$ and $\Phi = \mathrm{Id}_M$) is a module over the algebra $C^k(U)$ of differentiable scalar fields on U. It is a free module if and only if M is parallelizable. Accordingly, two classes are devoted to tensor field modules:

- *TensorFieldModule* for tensor fields with values on a generic (in practice, not parallelizable) differentiable manifold M,
- *TensorFieldFreeModule* for tensor fields with values on a parallelizable manifold M.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2014-2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [?]
- [?]
- [?]

class sage.manifolds.differentiable.tensorfield_module.**TensorFieldFreeModule**(*vector_field_module*, *tensor_type*)

Bases: sage.tensor.modules.tensor_free_module.TensorFreeModule

Free module of tensor fields of a given type (k, l) along a differentiable manifold U with values on a parallelizable manifold M, via a differentiable map $U \to M$.

Given two non-negative integers k and l and a differentiable map

$$\Phi : U \longrightarrow M,$$

the *tensor field module* $T^{(k,l)}(U, \Phi)$ is the set of all tensor fields of the type

$$t : U \longrightarrow T^{(k,l)} M$$

(where $T^{(k,l)} M$ is the tensor bundle of type (k, l) over M) such that

$$t(p) \in T^{(k,l)}(T_{\Phi(p)} M)$$

for all $p \in U$, i.e. $t(p)$ is a tensor of type (k, l) on the tangent vector space $T_{\Phi(p)} M$. Since M is parallelizable, the set $T^{(k,l)}(U, \Phi)$ is a free module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U (see `DiffScalarFieldAlgebra`).

The standard case of tensor fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$; we then denote $T^{(k,l)}(M, \mathrm{Id}_M)$ by merely $T^{(k,l)}(M)$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is not parallelizable, the class `TensorFieldModule` should be used instead, for $T^{(k,l)}(U, \Phi)$ is no longer a free module.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to M$
- `tensor_type` – pair (k, l) with k being the contravariant rank and l the covariant rank

EXAMPLES:

Module of type-$(2, 0)$ tensor fields on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3')
sage: c_xyz.<x,y,z> = M.chart()   # Cartesian coordinates
sage: T20 = M.tensor_field_module((2,0)) ; T20
Free module T^(2,0)(R^3) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold R^3
```

$T^{(2,0)}(\mathbf{R}^3)$ is a module over the algebra $C^k(\mathbf{R}^3)$:

```
sage: T20.category()
Category of finite dimensional modules over Algebra of differentiable
 scalar fields on the 3-dimensional differentiable manifold R^3
sage: T20.base_ring() is M.scalar_field_algebra()
True
```

$T^{(2,0)}(\mathbf{R}^3)$ is a free module:

```
sage: isinstance(T20, FiniteRankFreeModule)
True
```

because $M = \mathbf{R}^3$ is parallelizable:

```
sage: M.is_manifestly_parallelizable()
True
```

The zero element:

```
sage: z = T20.zero() ; z
Tensor field zero of type (2,0) on the 3-dimensional differentiable
 manifold R^3
sage: z[:]
[0 0 0]
[0 0 0]
[0 0 0]
```

A random element:

```
sage: t = T20.an_element() ; t
Tensor field of type (2,0) on the 3-dimensional differentiable
 manifold R^3
sage: t[:]
[2 0 0]
[0 0 0]
[0 0 0]
```

The module $T^{(2,0)}(\mathbf{R}^3)$ coerces to any module of type-$(2,0)$ tensor fields defined on some subdomain of $\mathbf{R}^3$:

```
sage: U = M.open_subset('U', coord_def={c_xyz: x>0})
sage: T20U = U.tensor_field_module((2,0))
sage: T20U.has_coerce_map_from(T20)
True
sage: T20.has_coerce_map_from(T20U)  # the reverse is not true
False
sage: T20U.coerce_map_from(T20)
Coercion map:
  From: Free module T^(2,0)(R^3) of type-(2,0) tensors fields on the 3-
↪dimensional differentiable manifold R^3
  To:   Free module T^(2,0)(U) of type-(2,0) tensors fields on the Open subset U␣
↪of the 3-dimensional differentiable manifold R^3
```

The coercion map is actually the *restriction* of tensor fields defined on $\mathbf{R}^3$ to U.

There is also a coercion map from fields of tangent-space automorphisms to tensor fields of type $(1,1)$:

```
sage: T11 = M.tensor_field_module((1,1)) ; T11
Free module T^(1,1)(R^3) of type-(1,1) tensors fields on the
 3-dimensional differentiable manifold R^3
sage: GL = M.automorphism_field_group() ; GL
General linear group of the Free module X(R^3) of vector fields on the
 3-dimensional differentiable manifold R^3
sage: T11.has_coerce_map_from(GL)
True
```

An explicit call to this coercion map is:

```
sage: id = GL.one() ; id
Field of tangent-space identity maps on the 3-dimensional
 differentiable manifold R^3
sage: tid = T11(id) ; tid
Tensor field Id of type (1,1) on the 3-dimensional differentiable
 manifold R^3
sage: tid[:]
[1 0 0]
```

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```
[0 1 0]
[0 0 1]
```

Element
alias of `sage.manifolds.differentiable.tensorfield_paral.TensorFieldParal`

class sage.manifolds.differentiable.tensorfield_module.**TensorFieldModule**(*vector_field_module*, *tensor_type*)

Bases: `sage.structure.unique_representation.UniqueRepresentation`, `sage.structure.parent.Parent`

Module of tensor fields of a given type (k, l) along a differentiable manifold U with values on a differentiable manifold M, via a differentiable map $U \to M$.

Given two non-negative integers k and l and a differentiable map

$$\Phi : U \longrightarrow M,$$

the *tensor field module* $T^{(k,l)}(U, \Phi)$ is the set of all tensor fields of the type

$$t : U \longrightarrow T^{(k,l)}M$$

(where $T^{(k,l)}M$ is the tensor bundle of type (k, l) over M) such that

$$t(p) \in T^{(k,l)}(T_{\Phi(p)}M)$$

for all $p \in U$, i.e. $t(p)$ is a tensor of type (k, l) on the tangent vector space $T_{\Phi(p)}M$. The set $T^{(k,l)}(U, \Phi)$ is a module over $C^k(U)$, the ring (algebra) of differentiable scalar fields on U (see `DiffScalarFieldAlgebra`).

The standard case of tensor fields *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$; we then denote $T^{(k,l)}(M, \mathrm{Id}_M)$ by merely $T^{(k,l)}(M)$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is parallelizable, the class `TensorFieldFreeModule` should be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to M$
- `tensor_type` – pair (k, l) with k being the contravariant rank and l the covariant rank

EXAMPLES:

Module of type-$(2, 0)$ tensor fields on the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
```

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```
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: T20 = M.tensor_field_module((2,0)); T20
Module T^(2,0)(M) of type-(2,0) tensors fields on the 2-dimensional
 differentiable manifold M
```

$T^{(2,0)}(M)$ is a module over the algebra $C^k(M)$:

```
sage: T20.category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold M
sage: T20.base_ring() is M.scalar_field_algebra()
True
```

$T^{(2,0)}(M)$ is not a free module:

```
sage: isinstance(T20, FiniteRankFreeModule)
False
```

because $M = S^2$ is not parallelizable:

```
sage: M.is_manifestly_parallelizable()
False
```

On the contrary, the module of type-$(2,0)$ tensor fields on U is a free module, since U is parallelizable (being a coordinate domain):

```
sage: T20U = U.tensor_field_module((2,0))
sage: isinstance(T20U, FiniteRankFreeModule)
True
sage: U.is_manifestly_parallelizable()
True
```

The zero element:

```
sage: z = T20.zero() ; z
Tensor field zero of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: z is T20(0)
True
sage: z[c_xy.frame(),:]
[0 0]
[0 0]
sage: z[c_uv.frame(),:]
[0 0]
[0 0]
```

The module $T^{(2,0)}(M)$ coerces to any module of type-$(2,0)$ tensor fields defined on some subdomain of M, for instance $T^{(2,0)}(U)$:

```
sage: T20U.has_coerce_map_from(T20)
True
```

The reverse is not true:

```
sage: T20.has_coerce_map_from(T20U)
False
```

The coercion:

```
sage: T20U.coerce_map_from(T20)
Coercion map:
  From: Module T^(2,0)(M) of type-(2,0) tensors fields on the 2-dimensional
↪differentiable manifold M
  To:   Free module T^(2,0)(U) of type-(2,0) tensors fields on the Open subset U
↪of the 2-dimensional differentiable manifold M
```

The coercion map is actually the *restriction* of tensor fields defined on M to U:

```
sage: t = M.tensor_field(2,0, name='t')
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: t[eU,:] = [[2,0], [0,-3]]
sage: t.add_comp_by_continuation(eV, W, chart=c_uv)
sage: T20U(t)  # the conversion map in action
Tensor field t of type (2,0) on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: T20U(t) is t.restrict(U)
True
```

There is also a coercion map from fields of tangent-space automorphisms to tensor fields of type-$(1,1)$:

```
sage: T11 = M.tensor_field_module((1,1)) ; T11
Module T^(1,1)(M) of type-(1,1) tensors fields on the 2-dimensional
 differentiable manifold M
sage: GL = M.automorphism_field_group() ; GL
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
sage: T11.has_coerce_map_from(GL)
True
```

Explicit call to the coercion map:

```
sage: a = GL.one() ; a
Field of tangent-space identity maps on the 2-dimensional
 differentiable manifold M
sage: a.parent()
General linear group of the Module X(M) of vector fields on the
 2-dimensional differentiable manifold M
sage: ta = T11.coerce(a) ; ta
Tensor field Id of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: ta.parent()
Module T^(1,1)(M) of type-(1,1) tensors fields on the 2-dimensional
 differentiable manifold M
sage: ta[eU,:]  # ta on U
[1 0]
[0 1]
sage: ta[eV,:]  # ta on V
[1 0]
[0 1]
```

Element

alias of *sage.manifolds.differentiable.tensorfield.TensorField*

base_module()

Return the vector field module on which `self` is constructed.

OUTPUT:

- a `VectorFieldModule` representing the module on which `self` is defined

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: T13 = M.tensor_field_module((1,3))
sage: T13.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: T13.base_module() is M.vector_field_module()
True
sage: T13.base_module().base_ring()
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
```

tensor_type()

Return the tensor type of `self`.

OUTPUT:

- pair (k, l) of non-negative integers such that the tensor fields belonging to this module are of type (k, l)

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: T13 = M.tensor_field_module((1,3))
sage: T13.tensor_type()
(1, 3)
sage: T20 = M.tensor_field_module((2,0))
sage: T20.tensor_type()
(2, 0)
```

zero()

Return the zero of `self`.

2.8.2 Tensor Fields

The class `TensorField` implements tensor fields on differentiable manifolds. The derived class `TensorFieldParal` is devoted to tensor fields with values on parallelizable manifolds.

Various derived classes of `TensorField` are devoted to specific tensor fields:

- `VectorField` for vector fields (rank-1 contravariant tensor fields)
- `AutomorphismField` for fields of tangent-space automorphisms
- `DiffForm` for differential forms (fully antisymmetric covariant tensor fields)
- `MultivectorField` for multivector fields (fully antisymmetric contravariant tensor fields)

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Travis Scrimshaw (2016): review tweaks
- Eric Gourgoulhon (2018): operators divergence, Laplacian and d'Alembertian; method `TensorField.along()`

- Florentin Jaffredo (2018) : series expansion with respect to a given parameter

REFERENCES:

- [?]
- [?]
- [?]

class sage.manifolds.differentiable.tensorfield.**TensorField**(*vector_field_module, tensor_type, name=None, latex_name=None, sym=None, antisym=None, parent=None*)

Bases: sage.structure.element.ModuleElement

Tensor field along a differentiable manifold.

An instance of this class is a tensor field along a differentiable manifold U with values on a differentiable manifold M, via a differentiable map $\Phi : U \to M$. More precisely, given two non-negative integers k and l and a differentiable map

$$\Phi : U \longrightarrow M,$$

a *tensor field of type* (k, l) *along* U *with values on* M is a differentiable map

$$t : U \longrightarrow T^{(k,l)}M$$

(where $T^{(k,l)}M$ is the tensor bundle of type (k, l) over M) such that

$$\forall p \in U, \ t(p) \in T^{(k,l)}(T_qM)$$

i.e. $t(p)$ is a tensor of type (k, l) on the tangent space T_qM at the point $q = \Phi(p)$, that is to say a multilinear map

$$t(p) : \underbrace{T_q^*M \times \cdots \times T_q^*M}_{k \text{ times}} \times \underbrace{T_qM \times \cdots \times T_qM}_{l \text{ times}} \longrightarrow K,$$

where T_q^*M is the dual vector space to T_qM and K is the topological field over which the manifold M is defined. The integer $k + l$ is called the *tensor rank*.

The standard case of a tensor field *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

If M is parallelizable, the class *TensorFieldParal* should be used instead.

This is a Sage *element* class, the corresponding *parent* class being *TensorFieldModule*.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to M$ (cf. *VectorFieldModule*)
- `tensor_type` – pair (k, l) with k being the contravariant rank and l the covariant rank
- `name` – (default: `None`) name given to the tensor field
- `latex_name` – (default: `None`) LaTeX symbol to denote the tensor field; if none is provided, the LaTeX symbol is set to `name`

- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention `position = 0` for the first argument; for instance:
  - `sym = (0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym = [(0,2), (1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments.
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`
- `parent` – (default: `None`) some specific parent (e.g. exterior power for differential forms); if `None`, `vector_field_module.tensor_module(k,l)` is used

EXAMPLES:

Tensor field of type (0,2) on the sphere S^2:

```
sage: M = Manifold(2, 'S^2') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: t = M.tensor_field(0,2, name='t') ; t
Tensor field t of type (0,2) on the 2-dimensional differentiable
 manifold S^2
sage: t.parent()
Module T^(0,2)(S^2) of type-(0,2) tensors fields on the 2-dimensional
 differentiable manifold S^2
sage: t.parent().category()
Category of modules over Algebra of differentiable scalar fields on the
 2-dimensional differentiable manifold S^2
```

The parent of t is not a free module, for the sphere S^2 is not parallelizable:

```
sage: isinstance(t.parent(), FiniteRankFreeModule)
False
```

To fully define t, we have to specify its components in some vector frames defined on subsets of S^2; let us start by the open subset U:

```
sage: eU = c_xy.frame()
sage: t[eU,:] = [[1,0], [-2,3]]
sage: t.display(eU)
t = dx*dx - 2 dy*dx + 3 dy*dy
```

To set the components of t on V consistently, we copy the expressions of the components in the common subset W:

```
sage: eV = c_uv.frame()
sage: eVW = eV.restrict(W)
sage: c_uvW = c_uv.restrict(W)
```

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```
sage: t[eV,0,0] = t[eVW,0,0,c_uvW].expr()  # long time
sage: t[eV,0,1] = t[eVW,0,1,c_uvW].expr()  # long time
sage: t[eV,1,0] = t[eVW,1,0,c_uvW].expr()  # long time
sage: t[eV,1,1] = t[eVW,1,1,c_uvW].expr()  # long time
```

Actually, the above operation can by performed in a single line by means of the method `add_comp_by_continuation()`:

```
sage: t.add_comp_by_continuation(eV, W, chart=c_uv)  # long time
```

At this stage, t is fully defined, having components in frames eU and eV and the union of the domains of eU and eV being the whole manifold:

```
sage: t.display(eV)  # long time
t = (u^4 - 4*u^3*v + 10*u^2*v^2 + 4*u*v^3 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 +
↪4*u^2*v^6 + v^8) du*du
 - 4*(u^3*v + 2*u^2*v^2 - u*v^3)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8)
↪du*dv
 + 2*(u^4 - 2*u^3*v - 2*u^2*v^2 + 2*u*v^3 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 +
↪4*u^2*v^6 + v^8) dv*du
 + (3*u^4 + 4*u^3*v - 2*u^2*v^2 - 4*u*v^3 + 3*v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 +
↪4*u^2*v^6 + v^8) dv*dv
```

Let us consider two vector fields, a and b, on S^2:

```
sage: a = M.vector_field({eU: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a.display(eV)
a = -v d/du + u d/dv
sage: b = M.vector_field({eU: [y, -1]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b.display(eV)
b = ((2*u + 1)*v^3 + (2*u^3 - u^2)*v)/(u^2 + v^2) d/du
 - (u^4 - v^4 + 2*u*v^2)/(u^2 + v^2) d/dv
```

As a tensor field of type $(0, 2)$, t acts on the pair (a, b), resulting in a scalar field:

```
sage: f = t(a,b); f
Scalar field t(a,b) on the 2-dimensional differentiable manifold S^2
sage: f.display()  # long time
t(a,b): S^2 --> R
on U: (x, y) |--> -2*x*y - y^2 - 3*x
on V: (u, v) |--> -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
```

The vectors can be defined only on subsets of S^2, the domain of the result is then the common subset:

```
sage: s = t(a.restrict(U), b) ; s  # long time
Scalar field t(a,b) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: s.display()  # long time
t(a,b): U --> R
   (x, y) |--> -2*x*y - y^2 - 3*x
on W: (u, v) |--> -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
sage: s = t(a.restrict(U), b.restrict(W)) ; s  # long time
Scalar field t(a,b) on the Open subset W of the 2-dimensional
 differentiable manifold S^2
```

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```
sage: s.display()  # long time
t(a,b): W --> R
   (x, y) |--> -2*x*y - y^2 - 3*x
   (u, v) |--> -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
```

The tensor itself can be defined only on some open subset of S^2, yielding a result whose domain is this subset:

```
sage: s = t.restrict(V)(a,b); s  # long time
Scalar field t(a,b) on the Open subset V of the 2-dimensional
 differentiable manifold S^2
sage: s.display()  # long time
t(a,b): V --> R
   (u, v) |--> -(3*u^3 + (3*u + 1)*v^2 + 2*u*v)/(u^4 + 2*u^2*v^2 + v^4)
on W: (x, y) |--> -2*x*y - y^2 - 3*x
```

Tests regarding the multiplication by a scalar field:

```
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2),
....:                    c_uv: (u^2 + v^2)/(u^2 + v^2 + 1)}, name='f')
sage: t.parent().base_ring() is f.parent()
True
sage: s = f*t; s  # long time
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold S^2
sage: s[[0,0]] == f*t[[0,0]]  # long time
True
sage: s.restrict(U) == f.restrict(U) * t.restrict(U)  # long time
True
sage: s = f*t.restrict(U); s
Tensor field of type (0,2) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: s.restrict(U) == f.restrict(U) * t.restrict(U)
True
```

Same examples with SymPy as the symbolic engine

From now on, we ask that all symbolic calculus on manifold M are performed by SymPy:

```
sage: M.set_calculus_method('sympy')
```

We define the tensor t as above:

```
sage: t = M.tensor_field(0, 2, {eU:  [[1,0], [-2,3]]}, name='t')
sage: t.display(eU)
t = dx*dx - 2 dy*dx + 3 dy*dy
sage: t.add_comp_by_continuation(eV, W, chart=c_uv)  # long time
sage: t.display(eV)  # long time
t = (u**4 - 4*u**3*v + 10*u**2*v**2 + 4*u*v**3 + v**4)/(u**8 +
 4*u**6*v**2 + 6*u**4*v**4 + 4*u**2*v**6 + v**8) du*du +
 4*u*v*(-u**2 - 2*u*v + v**2)/(u**8 + 4*u**6*v**2 + 6*u**4*v**4
 + 4*u**2*v**6 + v**8) du*dv + 2*(u**4 - 2*u**3*v - 2*u**2*v**2
 + 2*u*v**3 + v**4)/(u**8 + 4*u**6*v**2 + 6*u**4*v**4 +
 4*u**2*v**6 + v**8) dv*du + (3*u**4 + 4*u**3*v - 2*u**2*v**2 -
 4*u*v**3 + 3*v**4)/(u**8 + 4*u**6*v**2 + 6*u**4*v**4 +
 4*u**2*v**6 + v**8) dv*dv
```

The default coordinate representations of tensor components are now SymPy objects:

```
sage: t[eV,1,1,c_uv].expr() # long time
(3*u**4 + 4*u**3*v - 2*u**2*v**2 - 4*u*v**3 + 3*v**4)/(u**8 +
 4*u**6*v**2 + 6*u**4*v**4 + 4*u**2*v**6 + v**8)
sage: type(t[eV,1,1,c_uv].expr()) # long time
<class 'sympy.core.mul.Mul'>
```

Let us consider two vector fields, a and b, on S^2:

```
sage: a = M.vector_field({eU: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a.display(eV)
a = -v d/du + u d/dv
sage: b = M.vector_field({eU: [y, -1]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b.display(eV)
b = v*(2*u**3 - u**2 + 2*u*v**2 + v**2)/(u**2 + v**2) d/du
    + (-u**4 - 2*u*v**2 + v**4)/(u**2 + v**2) d/dv
```

As a tensor field of type $(0, 2)$, t acts on the pair (a, b), resulting in a scalar field:

```
sage: f = t(a,b)
sage: f.display()  # long time
t(a,b): S^2 --> R
on U: (x, y) |--> -2*x*y - 3*x - y**2
on V: (u, v) |--> -(3*u**3 + 3*u*v**2 + 2*u*v + v**2)/(u**4 + 2*u**2*v**2 + v**4)
```

The vectors can be defined only on subsets of S^2, the domain of the result is then the common subset:

```
sage: s = t(a.restrict(U), b)
sage: s.display()  # long time
t(a,b): U --> R
   (x, y) |--> -2*x*y - 3*x - y**2
on W: (u, v) |--> -(3*u**3 + 3*u*v**2 + 2*u*v + v**2)/(u**4 + 2*u**2*v**2 + v**4)
sage: s = t(a.restrict(U), b.restrict(W))  # long time
sage: s.display()  # long time
t(a,b): W --> R
   (x, y) |--> -2*x*y - 3*x - y**2
   (u, v) |--> -(3*u**3 + 3*u*v**2 + 2*u*v + v**2)/(u**4 + 2*u**2*v**2 + v**4)
```

The tensor itself can be defined only on some open subset of S^2, yielding a result whose domain is this subset:

```
sage: s = t.restrict(V)(a,b)  # long time
sage: s.display()  # long time
t(a,b): V --> R
   (u, v) |--> -(3*u**3 + 3*u*v**2 + 2*u*v + v**2)/(u**4 + 2*u**2*v**2 + v**4)
on W: (x, y) |--> -2*x*y - 3*x - y**2
```

Tests regarding the multiplication by a scalar field:

```
sage: f = M.scalar_field({c_xy: 1/(1+x^2+y^2),
....:                     c_uv: (u^2 + v^2)/(u^2 + v^2 + 1)}, name='f')
sage: s = f*t # long time
sage: s[[0,0]] == f*t[[0,0]]  # long time
True
sage: s.restrict(U) == f.restrict(U) * t.restrict(U)  # long time
True
```

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```
sage: s = f*t.restrict(U)
sage: s.restrict(U) == f.restrict(U) * t.restrict(U)
True
```

add_comp(*basis=None*)

Return the components of `self` in a given vector frame for assignment.

The components with respect to other frames having the same domain as the provided vector frame are kept. To delete them, use the method `set_comp()` instead.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are defined; if `None`, the components are assumed to refer to the tensor field domain's default frame

OUTPUT:

- components in the given frame, as a `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 2, name='t')
sage: t.add_comp(e_uv)
3-indices components w.r.t. Coordinate frame (V, (d/du,d/dv))
sage: t.add_comp(e_uv)[1,0,1] = u+v
sage: t.display(e_uv)
t = (u + v) d/dv*du*dv
```

Setting the components in a new frame:

```
sage: e = V.vector_frame('e')
sage: t.add_comp(e)
3-indices components w.r.t. Vector frame (V, (e_0,e_1))
sage: t.add_comp(e)[0,1,1] = u*v
sage: t.display(e)
t = u*v e_0*e^1*e^1
```

The components with respect to `e_uv` are kept:

```
sage: t.display(e_uv)
t = (u + v) d/dv*du*dv
```

add_comp_by_continuation(*frame, subdomain, chart=None*)

Set components with respect to a vector frame by continuation of the coordinate expression of the components in a subframe.

The continuation is performed by demanding that the components have the same coordinate expression as those on the restriction of the frame to a given subdomain.

INPUT:

- `frame` – vector frame e in which the components are to be set

- `subdomain` – open subset of e's domain in which the components are known or can be evaluated from other components
- `chart` – (default: `None`) coordinate chart on e's domain in which the extension of the expression of the components is to be performed; if `None`, the default's chart of e's domain is assumed

EXAMPLES:

Components of a vector field on the sphere S^2:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: # The two open subsets covered by stereographic coordinates (North and␣
↪South):
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic␣
↪coordinates
sage: transf = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:             intersection_name='W', restrictions1= x^2+y^2!=0,
....:             restrictions2= u^2+v^2!=0)
sage: inv = transf.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.vector_field({eU: [x, 2+y]}, name='a')
```

At this stage, the vector field has been defined only on the open subset `U` (through its components in the frame `eU`):

```
sage: a.display(eU)
a = x d/dx + (y + 2) d/dy
```

The components with respect to the restriction of `eV` to the common subdomain `W`, in terms of the `(u,v)` coordinates, are obtained by a change-of-frame formula on `W`:

```
sage: a.display(eV.restrict(W), c_uv.restrict(W))
a = (-4*u*v - u) d/du + (2*u^2 - 2*v^2 - v) d/dv
```

The continuation consists in extending the definition of the vector field to the whole open subset `V` by demanding that the components in the frame eV have the same coordinate expression as the above one:

```
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
```

We have then:

```
sage: a.display(eV)
a = (-4*u*v - u) d/du + (2*u^2 - 2*v^2 - v) d/dv
```

and a is defined on the entire manifold S^2.

add_expr_from_subdomain(*frame*, *subdomain*)

Add an expression to an existing component from a subdomain.

INPUT:

- `frame` – vector frame e in which the components are to be set
- `subdomain` – open subset of e's domain in which the components have additional expressions.

EXAMPLES:

We are going to consider a vector field in $\mathbf{R}^3$ along the 2-sphere:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: S = Manifold(2, 'S', structure="Riemannian")
sage: E.<X,Y,Z> = M.chart()
```

Let us define S in terms of stereographic charts:

```
sage: U = S.open_subset('U')
sage: V = S.open_subset('V')
sage: S.declare_union(U,V)
sage: stereoN.<x,y> = U.chart()
sage: stereoS.<xp,yp> = V.chart("xp:x' yp:y'")
sage: stereoN_to_S = stereoN.transition_map(stereoS,
....:                                  (x/(x^2+y^2), y/(x^2+y^2)),
....:                                  intersection_name='W',
....:                                  restrictions1= x^2+y^2!=0,
....:                                  restrictions2= xp^2+yp^2!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: W = U.intersection(V)
sage: stereoN_W = stereoN.restrict(W)
sage: stereoS_W = stereoS.restrict(W)
```

The embedding of S^2 in $\mathbf{R}^3$:

```
sage: phi = S.diff_map(M, {(stereoN, E): [2*x/(1+x^2+y^2),
....:                                     2*y/(1+x^2+y^2),
....:                                     (x^2+y^2-1)/(1+x^2+y^2)],
....:                      (stereoS, E): [2*xp/(1+xp^2+yp^2),
....:                                     2*yp/(1+xp^2+yp^2),
....:                               (1-xp^2-yp^2)/(1+xp^2+yp^2)]},
....:                  name='Phi', latex_name=r'\Phi')
```

To define a vector field v along S taking its values in M, we first set the components on U:

```
sage: v = M.vector_field(name='v').along(phi)
sage: vU = v.restrict(U)
sage: vU[:] = [x,y,x**2+y**2]
```

But because M is parallelizable, these components can be extended to S itself:

```
sage: v.add_comp_by_continuation(E.frame().along(phi), U)
```

One can see that v is not yet fully defined: the components (scalar fields) do not have values on the whole manifold:

```
sage: sorted(v._components.values())[0]._comp[(0,)].display()
S --> R
on U: (x, y) |--> x
```

To fix that, we first extend the components from W to V using *add_comp_by_continuation()*:

```
sage: v.add_comp_by_continuation(E.frame().along(phi).restrict(V),
....:                            W, stereoS)
```

Then, the expression on the subdomain V is added to the already known components on S by:

```
sage: v.add_expr_from_subdomain(E.frame().along(phi), V)
```

The definition of v is now complete:

```
sage: sorted(v._components.values())[0]._comp[(2,)].display()
S --> R
on U: (x, y) |--> x^2 + y^2
on V: (xp, yp) |--> 1/(xp^2 + yp^2)
```

along(*mapping*)

Return the tensor field deduced from `self` via a differentiable map, the codomain of which is included in the domain of `self`.

More precisely, if `self` is a tensor field t on M and if $\Phi : U \to M$ is a differentiable map from some differentiable manifold U to M, the returned object is a tensor field $\tilde{t}$ along U with values on M such that

$$\forall p \in U, \ \tilde{t}(p) = t(\Phi(p)).$$

INPUT:

- `mapping` – differentiable map $\Phi : U \to M$

OUTPUT:

- tensor field $\tilde{t}$ along U defined above.

EXAMPLES:

Let us consider the 2-dimensional sphere S^2:

```
sage: M = Manifold(2, 'S^2') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
```

and the following map from the open interval $(0, 5\pi/2)$ to S^2, the image of it being the great circle $x = 0$, $u = 0$, which goes through the North and South poles:

```
sage: I.<t> = OpenInterval(0, 5*pi/2)
sage: J = I.open_interval(0, 3*pi/2)
sage: K = I.open_interval(pi, 5*pi/2)
sage: c_J = J.canonical_chart(); c_K = K.canonical_chart()
sage: Phi = I.diff_map(M, {(c_J, c_xy):
....:                      (0, sgn(pi-t)*sqrt((1+cos(t))/(1-cos(t)))),
....:                      (c_K, c_uv):
....:                      (0,  sgn(t-2*pi)*sqrt((1-cos(t))/(1+cos(t))))},
....:                 name='Phi')
```

Let us consider a vector field on S^2:

```
sage: eU = c_xy.frame(); eV = c_uv.frame()
sage: w = M.vector_field(name='w')
sage: w[eU,0] = 1
sage: w.add_comp_by_continuation(eV, W, chart=c_uv)
sage: w.display(eU)
w = d/dx
```

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```
sage: w.display(eV)
w = (-u^2 + v^2) d/du - 2*u*v d/dv
```

We have then:

```
sage: wa = w.along(Phi); wa
Vector field w along the Real interval (0, 5/2*pi) with values on
 the 2-dimensional differentiable manifold S^2
sage: wa.display(eU.along(Phi))
w = d/dx
sage: wa.display(eV.along(Phi))
w = -(cos(t) - 1)*sgn(-2*pi + t)^2/(cos(t) + 1) d/du
```

Some tests:

```
sage: p = K.an_element()
sage: wa.at(p) == w.at(Phi(p))
True
sage: wa.at(J(4*pi/3)) == wa.at(K(4*pi/3))
True
sage: wa.at(I(4*pi/3)) == wa.at(K(4*pi/3))
True
sage: wa.at(K(7*pi/4)) == eU[0].at(Phi(I(7*pi/4))) # since eU[0]=d/dx
True
```

antisymmetrize(**pos*)

Antisymmetrization over some arguments.

INPUT:

- `pos` – (default: `None`) list of argument positions involved in the antisymmetrization (with the convention `position=0` for the first argument); if `None`, the antisymmetrization is performed over all the arguments

OUTPUT:

- the antisymmetrized tensor field (instance of `TensorField`)

EXAMPLES:

Antisymmetrization of a type-$(0, 2)$ tensor field on a 2-dimensional non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.tensor_field(0,2, {eU: [[1,x], [2,y]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a[eV,:]
[ 1/4*u + 3/4 -1/4*u + 3/4]
[ 1/4*v - 1/4 -1/4*v - 1/4]
sage: s = a.antisymmetrize() ; s
2-form on the 2-dimensional differentiable manifold M
```

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```
sage: s[eU,:]
[          0  1/2*x - 1]
[-1/2*x + 1          0]
sage: s[eV,:]
[                   0 -1/8*u - 1/8*v + 1/2]
[ 1/8*u + 1/8*v - 1/2                    0]
sage: s == a.antisymmetrize(0,1)  # explicit positions
True
sage: s == a.antisymmetrize(1,0)  # the order of positions does not matter
True
```

See also:

For more details and examples, see `sage.tensor.modules.free_module_tensor.FreeModuleTensor.antisymmetrize()`.

at (*point*)

Value of `self` at a point of its domain.

If the current tensor field is

$$t:\ U \longrightarrow T^{(k,l)}M$$

associated with the differentiable map

$$\Phi:\ U \longrightarrow M,$$

where U and M are two manifolds (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), then for any point $p \in U$, $t(p)$ is a tensor on the tangent space to M at the point $\Phi(p)$.

INPUT:

- `point` – *ManifoldPoint*; point p in the domain of the tensor field U

OUTPUT:

- `FreeModuleTensor` representing the tensor $t(p)$ on the tangent vector space $T_{\Phi(p)}M$

EXAMPLES:

Tensor on a tangent space of a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.tensor_field(1, 1, {eU: [[1+y,x], [0,x+y]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a.display(eU)
a = (y + 1) d/dx*dx + x d/dx*dy + (x + y) d/dy*dy
sage: a.display(eV)
a = (u + 1/2) d/du*du + (-1/2*u - 1/2*v + 1/2) d/du*dv
 + 1/2 d/dv*du + (1/2*u - 1/2*v + 1/2) d/dv*dv
```

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```
sage: p = M.point((2,3), chart=c_xy, name='p')
sage: ap = a.at(p) ; ap
Type-(1,1) tensor a on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: ap.parent()
Free module of type-(1,1) tensors on the Tangent space at Point p
 on the 2-dimensional differentiable manifold M
sage: ap.display(eU.at(p))
a = 4 d/dx*dx + 2 d/dx*dy + 5 d/dy*dy
sage: ap.display(eV.at(p))
a = 11/2 d/du*du - 3/2 d/du*dv + 1/2 d/dv*du + 7/2 d/dv*dv
sage: p.coord(c_uv) # to check the above expression
(5, -1)
```

base_module()

Return the vector field module on which `self` acts as a tensor.

OUTPUT:

- instance of *VectorFieldModule*

EXAMPLES:

The module of vector fields on the 2-sphere as a "base module":

```
sage: M = Manifold(2, 'S^2')
sage: t = M.tensor_field(0,2)
sage: t.base_module()
Module X(S^2) of vector fields on the 2-dimensional differentiable
 manifold S^2
sage: t.base_module() is M.vector_field_module()
True
sage: XM = M.vector_field_module()
sage: XM.an_element().base_module() is XM
True
```

comp(*basis=None*, *from_basis=None*)

Return the components in a given vector frame.

If the components are not known already, they are computed by the tensor change-of-basis formula from components in another vector frame.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are required; if none is provided, the components are assumed to refer to the tensor field domain's default frame
- `from_basis` – (default: `None`) vector frame from which the required components are computed, via the tensor change-of-basis formula, if they are not known already in the basis `basis`

OUTPUT:

- components in the vector frame `basis`, as a `Components`

EXAMPLES:

Components of a type-$(1,1)$ tensor field defined on two open subsets:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U')
```

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```
sage: c_xy.<x, y> = U.chart()
sage: e = U.default_frame() ; e
Coordinate frame (U, (d/dx,d/dy))
sage: V = M.open_subset('V')
sage: c_uv.<u, v> = V.chart()
sage: f = V.default_frame() ; f
Coordinate frame (V, (d/du,d/dv))
sage: M.declare_union(U,V)   # M is the union of U and V
sage: t = M.tensor_field(1,1, name='t')
sage: t[e,0,0] = - x + y^3
sage: t[e,0,1] = 2+x
sage: t[f,1,1] = - u*v
sage: t.comp(e)
2-indices components w.r.t. Coordinate frame (U, (d/dx,d/dy))
sage: t.comp(e)[:]
[y^3 - x    x + 2]
[      0        0]
sage: t.comp(f)
2-indices components w.r.t. Coordinate frame (V, (d/du,d/dv))
sage: t.comp(f)[:]
[   0    0]
[   0 -u*v]
```

Since `e` is `M`'s default frame, the argument `e` can be omitted:

```
sage: e is M.default_frame()
True
sage: t.comp() is t.comp(e)
True
```

Example of computation of the components via a change of frame:

```
sage: a = V.automorphism_field()
sage: a[:] = [[1+v, -u^2], [0, 1-u]]
sage: h = f.new_frame(a, 'h')
sage: t.comp(h)
2-indices components w.r.t. Vector frame (V, (h_0,h_1))
sage: t.comp(h)[:]
[              0 -u^3*v/(v + 1)]
[              0           -u*v]
```

contract(**args*)

Contraction of `self` with another tensor field on one or more indices.

INPUT:

- `pos1` – positions of the indices in the current tensor field involved in the contraction; `pos1` must be a sequence of integers, with 0 standing for the first index position, 1 for the second one, etc.; if `pos1` is not provided, a single contraction on the last index position of the tensor field is assumed
- `other` – the tensor field to contract with
- `pos2` – positions of the indices in `other` involved in the contraction, with the same conventions as for `pos1`; if `pos2` is not provided, a single contraction on the first index position of `other` is assumed

OUTPUT:

- tensor field resulting from the contraction at the positions `pos1` and `pos2` of the tensor field with `other`

EXAMPLES:

Contractions of a type-$(1,1)$ tensor field with a type-$(2,0)$ one on a 2-dimensional non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.tensor_field(1, 1, {eU: [[1, x], [0, 2]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b = M.tensor_field(2, 0, {eU: [[y, -1], [x+y, 2]]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: s = a.contract(b) ; s   # contraction on last index of a and first one␣
↪of b
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
```

Check 1: components with respect to the manifold's default frame (`eU`):

```
sage: all([bool(s[i,j] == sum(a[i,k]*b[k,j] for k in M.irange()))
....:      for j in M.irange()] for i in M.irange())
True
```

Check 2: components with respect to the frame `eV`:

```
sage: all([bool(s[eV,i,j] == sum(a[eV,i,k]*b[eV,k,j]
....:                          for k in M.irange()))
....:      for j in M.irange()] for i in M.irange())
True
```

Instead of the explicit call to the method `contract()`, one may use the index notation with Einstein convention (summation over repeated indices); it suffices to pass the indices as a string inside square brackets:

```
sage: a['^i_k']*b['^kj'] == s
True
```

Indices not involved in the contraction may be replaced by dots:

```
sage: a['^._k']*b['^k.'] == s
True
```

LaTeX notation may be used:

```
sage: a['^{i}_{k}']*b['^{kj}'] == s
True
```

Contraction on the last index of `a` and last index of `b`:

```
sage: s = a.contract(b, 1) ; s
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: a['^i_k']*b['^jk'] == s
True
```

Contraction on the first index of `b` and the last index of `a`:

```
sage: s = b.contract(0,a,1) ; s
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: b['^ki']*a['^j_k'] == s
True
```

The domain of the result is the intersection of the domains of the two tensor fields:

```
sage: aU = a.restrict(U) ; bV = b.restrict(V)
sage: s = aU.contract(b) ; s
Tensor field of type (2,0) on the Open subset U of the
 2-dimensional differentiable manifold M
sage: s = a.contract(bV) ; s
Tensor field of type (2,0) on the Open subset V of the
 2-dimensional differentiable manifold M
sage: s = aU.contract(bV) ; s
Tensor field of type (2,0) on the Open subset W of the
 2-dimensional differentiable manifold M
sage: s0 = a.contract(b)
sage: s == s0.restrict(W)
True
```

The contraction can be performed on more than one index: `c` being a type-$(2,2)$ tensor, contracting the indices in positions 2 and 3 of `c` with respectively those in positions 0 and 1 of `b` is:

```
sage: c = a*a ; c
Tensor field of type (2,2) on the 2-dimensional differentiable
 manifold M
sage: s = c.contract(2,3, b, 0,1) ; s  # long time
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
```

The same double contraction using index notation:

```
sage: s == c['^.._kl']*b['^kl']  # long time
True
```

The symmetries are either conserved or destroyed by the contraction:

```
sage: c = c.symmetrize(0,1).antisymmetrize(2,3)
sage: c.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
sage: s = b.contract(0, c, 2) ; s
Tensor field of type (3,1) on the 2-dimensional differentiable
 manifold M
sage: s.symmetries()
symmetry: (1, 2);  no antisymmetry
```

Case of a scalar field result:

```
sage: a = M.one_form({eU: [y, 1+x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b = M.vector_field({eU: [x, y^2]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a.display(eU)
a = y dx + (x + 1) dy
sage: b.display(eU)
b = x d/dx + y^2 d/dy
sage: s = a.contract(b) ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> (x + 1)*y^2 + x*y
on V: (u, v) |--> 1/8*u^3 - 1/8*u*v^2 + 1/8*v^3 + 1/2*u^2 - 1/8*(u^2 + 4*u)*v
sage: s == a['_i']*b['^i'] # use of index notation
True
sage: s == b.contract(a)
True
```

Case of a vanishing scalar field result:

```
sage: b = M.vector_field({eU: [1+x, -y]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: s = a.contract(b) ; s
Scalar field zero on the 2-dimensional differentiable manifold M
sage: s.display()
zero: M --> R
on U: (x, y) |--> 0
on V: (u, v) |--> 0
```

copy()

Return an exact copy of `self`.

Note: The name and the derived quantities are not copied.

EXAMPLES:

Copy of a type-$(1,1)$ tensor field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 1, name='t')
sage: t[e_xy,:] = [[x+y, 0], [2, 1-y]]
sage: t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: s = t.copy(); s
Tensor field of type (1,1) on
 the 2-dimensional differentiable manifold M
sage: s.display(e_xy)
(x + y) d/dx*dx + 2 d/dy*dx + (-y + 1) d/dy*dy
```

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```
sage: s == t
True
```

If the original tensor field is modified, the copy is not:

```
sage: t[e_xy,0,0] = -1
sage: t.display(e_xy)
t = -d/dx*dx + 2 d/dy*dx + (-y + 1) d/dy*dy
sage: s.display(e_xy)
(x + y) d/dx*dx + 2 d/dy*dx + (-y + 1) d/dy*dy
sage: s == t
False
```

dalembertian(*metric=None*)

Return the d'Alembertian of `self` with respect to a given Lorentzian metric.

The *d'Alembertian* of a tensor field t with respect to a Lorentzian metric g is nothing but the Laplace-Beltrami operator of g applied to t (see `laplacian()`); if `self` a tensor field t of type (k, l), the d'Alembertian of t with respect to g is then the tensor field of type (k, l) defined by

$$(\Box t)^{a_1 \ldots a_k}{}_{b_1 \ldots b_k} = \nabla_i \nabla^i t^{a_1 \ldots a_k}{}_{b_1 \ldots b_k},$$

where ∇ is the Levi-Civita connection of g (cf. `LeviCivitaConnection`) and $\nabla^i := g^{ij} \nabla_j$.

Note: If the metric g is not Lorentzian, the name *d'Alembertian* is not appropriate and one should use `laplacian()` instead.

INPUT:

- `metric` – (default: `None`) the Lorentzian metric g involved in the definition of the d'Alembertian; if none is provided, the domain of `self` is supposed to be endowed with a default Lorentzian metric (i.e. is supposed to be Lorentzian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the d'Alembertian

OUTPUT:

- instance of `TensorField` representing the d'Alembertian of `self`

EXAMPLES:

d'Alembertian of a vector field in Minkowski spacetime, representing the electric field of a simple plane electromagnetic wave:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: X.<t,x,y,z> = M.chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: e = M.vector_field(name='e')
sage: e[1] = cos(t-z)
sage: e.display()  # plane wave propagating in the z direction
e = cos(t - z) d/dx
sage: De = e.dalembertian(); De
Vector field Box(e) on the 4-dimensional Lorentzian manifold M
```

The function `dalembertian()` from the `operators` module can be used instead of the method `dalembertian()`:

```
sage: from sage.manifolds.operators import dalembertian
sage: dalembertian(e) == De
True
```

We check that the electric field obeys the wave equation:

```
sage: De.display()
Box(e) = 0
```

disp (*frame=None*, *chart=None*)

Display the tensor field in terms of its expansion with respect to a given vector frame.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the tensor is expanded; if `frame` is `None` and `chart` is not `None`, the coordinate frame associated with `chart` is assumed; if both `frame` and `chart` are `None`, the default frame of the domain of definition of the tensor field is assumed
- `chart` – (default: `None`) chart with respect to which the components of the tensor field in the selected frame are expressed; if `None`, the default chart of the vector frame domain is assumed

EXAMPLES:

Display of a type-$(1,1)$ tensor field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1,1, name='t')
sage: t[e_xy,:] = [[x, 1], [y, 0]]
sage: t.add_comp_by_continuation(e_uv, W, c_uv)
sage: t.display(e_xy)
t = x d/dx*dx + d/dx*dy + y d/dy*dx
sage: t.display(e_uv)
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

Since `e_xy` is `M`'s default frame, the argument `e_xy` can be omitted:

```
sage: e_xy is M.default_frame()
True
sage: t.display()
t = x d/dx*dx + d/dx*dy + y d/dy*dx
```

Similarly, since `e_uv` is `V`'s default frame, the argument `e_uv` can be omitted when considering the restriction of `t` to `V`:

```
sage: t.restrict(V).display()
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

If the coordinate expression of the components are to be displayed in a chart distinct from the default one on the considered domain, then the chart has to be passed as the second argument of `display`. For instance, on $W = U \cap V$, two charts are available: `c_xy.restrict(W)` (the default one) and `c_uv.restrict(W)`. Accordingly, one can have two views of the expansion of `t` in the *same* vector frame `e_uv.restrict(W)`:

```
sage: t.display(e_uv.restrict(W))  # W's default chart assumed
t = (1/2*x + 1/2*y + 1/2) d/du*du + (1/2*x + 1/2*y - 1/2) d/du*dv
  + (1/2*x - 1/2*y + 1/2) d/dv*du + (1/2*x - 1/2*y - 1/2) d/dv*dv
sage: t.display(e_uv.restrict(W), c_uv.restrict(W))
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

As a shortcut, one can pass just a chart to `display`. It is then understood that the expansion is to be performed with respect to the coordinate frame associated with this chart. Therefore the above command can be abridged to:

```
sage: t.display(c_uv.restrict(W))
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

and one has:

```
sage: t.display(c_xy)
t = x d/dx*dx + d/dx*dy + y d/dy*dx
sage: t.display(c_uv)
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
sage: t.display(c_xy.restrict(W))
t = x d/dx*dx + d/dx*dy + y d/dy*dx
sage: t.restrict(W).display(c_uv.restrict(W))
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

One can ask for the display with respect to a frame in which `t` has not been initialized yet (this will automatically trigger the use of the change-of-frame formula for tensors):

```
sage: a = V.automorphism_field()
sage: a[:] = [[1+v, -u^2], [0, 1-u]]
sage: f = e_uv.new_frame(a, 'f')
sage: [f[i].display() for i in M.irange()]
[f_0 = (v + 1) d/du, f_1 = -u^2 d/du + (-u + 1) d/dv]
sage: t.display(f)
t = -1/2*(u^2*v + 1)/(u - 1) f_0*f^0
  - 1/2*(2*u^3 - 5*u^2 - (u^4 + u^3 - u^2)*v + 3*u - 1)/((u - 1)*v + u - 1) f_
↪0*f^1
  - 1/2*(v^2 + 2*v + 1)/(u - 1) f_1*f^0
  + 1/2*(u^2 + (u^2 + u - 1)*v - u + 1)/(u - 1) f_1*f^1
```

A shortcut of `display()` is `disp()`:

```
sage: t.disp(e_uv)
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

display (*frame=None*, *chart=None*)

Display the tensor field in terms of its expansion with respect to a given vector frame.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the tensor is expanded; if `frame` is `None` and `chart` is not `None`, the coordinate frame associated with `chart` is assumed; if both `frame` and `chart` are `None`, the default frame of the domain of definition of the tensor field is assumed
- `chart` – (default: `None`) chart with respect to which the components of the tensor field in the selected frame are expressed; if `None`, the default chart of the vector frame domain is assumed

EXAMPLES:

Display of a type-$(1,1)$ tensor field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1,1, name='t')
sage: t[e_xy,:] = [[x, 1], [y, 0]]
sage: t.add_comp_by_continuation(e_uv, W, c_uv)
sage: t.display(e_xy)
t = x d/dx*dx + d/dx*dy + y d/dy*dx
sage: t.display(e_uv)
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

Since `e_xy` is `M`'s default frame, the argument `e_xy` can be omitted:

```
sage: e_xy is M.default_frame()
True
sage: t.display()
t = x d/dx*dx + d/dx*dy + y d/dy*dx
```

Similarly, since `e_uv` is `V`'s default frame, the argument `e_uv` can be omitted when considering the restriction of `t` to `V`:

```
sage: t.restrict(V).display()
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

If the coordinate expression of the components are to be displayed in a chart distinct from the default one on the considered domain, then the chart has to be passed as the second argument of `display`. For instance, on $W = U \cap V$, two charts are available: `c_xy.restrict(W)` (the default one) and `c_uv.restrict(W)`. Accordingly, one can have two views of the expansion of `t` in the *same* vector frame `e_uv.restrict(W)`:

```
sage: t.display(e_uv.restrict(W))  # W's default chart assumed
t = (1/2*x + 1/2*y + 1/2) d/du*du + (1/2*x + 1/2*y - 1/2) d/du*dv
  + (1/2*x - 1/2*y + 1/2) d/dv*du + (1/2*x - 1/2*y - 1/2) d/dv*dv
sage: t.display(e_uv.restrict(W), c_uv.restrict(W))
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

As a shortcut, one can pass just a chart to `display`. It is then understood that the expansion is to be performed with respect to the coordinate frame associated with this chart. Therefore the above command can be abridged to:

```
sage: t.display(c_uv.restrict(W))
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

and one has:

```
sage: t.display(c_xy)
t = x d/dx*dx + d/dx*dy + y d/dy*dx
sage: t.display(c_uv)
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
sage: t.display(c_xy.restrict(W))
t = x d/dx*dx + d/dx*dy + y d/dy*dx
sage: t.restrict(W).display(c_uv.restrict(W))
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

One can ask for the display with respect to a frame in which `t` has not been initialized yet (this will automatically trigger the use of the change-of-frame formula for tensors):

```
sage: a = V.automorphism_field()
sage: a[:] = [[1+v, -u^2], [0, 1-u]]
sage: f = e_uv.new_frame(a, 'f')
sage: [f[i].display() for i in M.irange()]
[f_0 = (v + 1) d/du, f_1 = -u^2 d/du + (-u + 1) d/dv]
sage: t.display(f)
t = -1/2*(u^2*v + 1)/(u - 1) f_0*f^0
  - 1/2*(2*u^3 - 5*u^2 - (u^4 + u^3 - u^2)*v + 3*u - 1)/((u - 1)*v + u - 1) f_
↪0*f^1
  - 1/2*(v^2 + 2*v + 1)/(u - 1) f_1*f^0
  + 1/2*(u^2 + (u^2 + u - 1)*v - u + 1)/(u - 1) f_1*f^1
```

A shortcut of `display()` is `disp()`:

```
sage: t.disp(e_uv)
t = (1/2*u + 1/2) d/du*du + (1/2*u - 1/2) d/du*dv
  + (1/2*v + 1/2) d/dv*du + (1/2*v - 1/2) d/dv*dv
```

display_comp(*frame=None, chart=None, coordinate_labels=True, only_nonzero=True, only_nonredundant=False*)

Display the tensor components with respect to a given frame, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the tensor field components are defined; if `None`, then
  - if `chart` is not `None`, the coordinate frame associated to `chart` is used
  - otherwise, the default basis of the vector field module on which the tensor field is defined is used
- `chart` – (default: `None`) chart specifying the coordinate expression of the components; if `None`, the default chart of the tensor field domain is used
- `coordinate_labels` – (default: `True`) boolean; if `True`, coordinate symbols are used by default (instead of integers) as index labels whenever `frame` is a coordinate frame
- `only_nonzero` – (default: `True`) boolean; if `True`, only nonzero components are displayed
- `only_nonredundant` – (default: `False`) boolean; if `True`, only nonredundant components are displayed in case of symmetries

EXAMPLES:

Display of the components of a type-$(1,1)$ tensor field defined on two open subsets:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U')
sage: c_xy.<x, y> = U.chart()
sage: e = U.default_frame()
sage: V = M.open_subset('V')
sage: c_uv.<u, v> = V.chart()
sage: f = V.default_frame()
sage: M.declare_union(U,V)   # M is the union of U and V
sage: t = M.tensor_field(1,1, name='t')
sage: t[e,0,0] = - x + y^3
sage: t[e,0,1] = 2+x
sage: t[f,1,1] = - u*v
sage: t.display_comp(e)
t^x_x = y^3 - x
t^x_y = x + 2
sage: t.display_comp(f)
t^v_v = -u*v
```

Components in a chart frame:

```
sage: t.display_comp(chart=c_xy)
t^x_x = y^3 - x
t^x_y = x + 2
sage: t.display_comp(chart=c_uv)
t^v_v = -u*v
```

See documentation of *sage.manifolds.differentiable.tensorfield_paral.TensorFieldParal.display_comp()* for more options.

div (*metric=None*)

Return the divergence of `self` (with respect to a given metric).

The divergence is taken on the *last* index: if `self` is a tensor field t of type $(k,0)$ with $k \geq 1$, the divergence of t with respect to the metric g is the tensor field of type $(k-1,0)$ defined by

$$(\operatorname{div} t)^{a_1\ldots a_{k-1}} = \nabla_i t^{a_1\ldots a_{k-1}i} = (\nabla t)^{a_1\ldots a_{k-1}i}{}_i,$$

where ∇ is the Levi-Civita connection of g (cf. *LeviCivitaConnection*).

This definition is extended to tensor fields of type (k,l) with $k \geq 0$ and $l \geq 1$, by raising the last index with the metric g: $\operatorname{div} t$ is then the tensor field of type $(k,l-1)$ defined by

$$(\operatorname{div} t)^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1}} = \nabla_i (g^{ij} t^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1}j}) = (\nabla t^\sharp)^{a_1\ldots a_k i}{}_{b_1\ldots b_{l-1}i},$$

where $t^\sharp$ is the tensor field deduced from t by raising the last index with the metric g (see `up()`).

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the divergence; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see `PseudoRiemannianManifold`) and the latter is used to define the divergence.

OUTPUT:

- instance of either `DiffScalarField` if $(k,l) = (1,0)$ (`self` is a vector field) or $(k,l) = (0,1)$ (`self` is a 1-form) or of `TensorField` if $k + l \geq 2$ representing the divergence of `self` with respect to `metric`

EXAMPLES:

Divergence of a vector field in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: v = M.vector_field(x, y, name='v')
sage: s = v.divergence(); s
Scalar field div(v) on the Euclidean plane E^2
sage: s.display()
div(v): E^2 --> R
   (x, y) |--> 2
```

A shortcut alias of `divergence` is `div`:

```
sage: v.div() == s
True
```

The function `div()` from the `operators` module can be used instead of the method `divergence()`:

```
sage: from sage.manifolds.operators import div
sage: div(v) == s
True
```

The divergence can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1/(1+x^2+y^2)
sage: s = v.div(h); s
Scalar field div_h(v) on the Euclidean plane E^2
sage: s.display()
div_h(v): E^2 --> R
   (x, y) |--> (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
```

The standard formula

$$\operatorname{div}_h v = \frac{1}{\sqrt{|\det h|}} \frac{\partial}{\partial x^i} \left(\sqrt{|\det h|}\, v^i \right)$$

is checked as follows:

```
sage: sqrth = h.sqrt_abs_det().expr(); sqrth
1/sqrt(x^2 + y^2 + 1)
sage: s == 1/sqrth * sum( (sqrth*v[i]).diff(i) for i in M.irange())
True
```

A divergence-free vector:

```
sage: w = M.vector_field(-y, x, name='w')
sage: w.div().display()
div(w): E^2 --> R
   (x, y) |--> 0
sage: w.div(h).display()
div_h(w): E^2 --> R
   (x, y) |--> 0
```

Divergence of a type- `(2,0)` tensor field:

```
sage: t = v*w; t
Tensor field v*w of type (2,0) on the Euclidean plane E^2
sage: s = t.div(); s
Vector field div(v*w) on the Euclidean plane E^2
sage: s.display()
div(v*w) = -y e_x + x e_y
```

divergence(*metric=None*)

Return the divergence of `self` (with respect to a given metric).

The divergence is taken on the *last* index: if `self` is a tensor field t of type $(k,0)$ with $k \geq 1$, the divergence of t with respect to the metric g is the tensor field of type $(k-1,0)$ defined by

$$(\mathrm{div}\, t)^{a_1\ldots a_{k-1}} = \nabla_i t^{a_1\ldots a_{k-1} i} = (\nabla t)^{a_1\ldots a_{k-1} i}{}_i,$$

where ∇ is the Levi-Civita connection of g (cf. *LeviCivitaConnection*).

This definition is extended to tensor fields of type (k,l) with $k \geq 0$ and $l \geq 1$, by raising the last index with the metric g: $\mathrm{div}\, t$ is then the tensor field of type $(k,l-1)$ defined by

$$(\mathrm{div}\, t)^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1}} = \nabla_i (g^{ij} t^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1} j}) = (\nabla t^\sharp)^{a_1\ldots a_k i}{}_{b_1\ldots b_{l-1} i},$$

where $t^\sharp$ is the tensor field deduced from t by raising the last index with the metric g (see *up()*).

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the divergence; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the divergence.

OUTPUT:

- instance of either *DiffScalarField* if $(k,l) = (1,0)$ (`self` is a vector field) or $(k,l) = (0,1)$ (`self` is a 1-form) or of *TensorField* if $k + l \geq 2$ representing the divergence of `self` with respect to `metric`

EXAMPLES:

Divergence of a vector field in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: v = M.vector_field(x, y, name='v')
sage: s = v.divergence(); s
Scalar field div(v) on the Euclidean plane E^2
sage: s.display()
div(v): E^2 --> R
   (x, y) |--> 2
```

A shortcut alias of `divergence` is `div`:

```
sage: v.div() == s
True
```

The function `div()` from the `operators` module can be used instead of the method `divergence()`:

```
sage: from sage.manifolds.operators import div
sage: div(v) == s
True
```

The divergence can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1/(1+x^2+y^2)
sage: s = v.div(h); s
Scalar field div_h(v) on the Euclidean plane E^2
sage: s.display()
div_h(v): E^2 --> R
   (x, y) |--> (x^2 + y^2 + 2)/(x^2 + y^2 + 1)
```

The standard formula

$$\mathrm{div}_h\, v = \frac{1}{\sqrt{|\det h|}} \frac{\partial}{\partial x^i} \left(\sqrt{|\det h|}\, v^i \right)$$

is checked as follows:

```
sage: sqrth = h.sqrt_abs_det().expr(); sqrth
1/sqrt(x^2 + y^2 + 1)
sage: s == 1/sqrth * sum( (sqrth*v[i]).diff(i) for i in M.irange())
True
```

A divergence-free vector:

```
sage: w = M.vector_field(-y, x, name='w')
sage: w.div().display()
div(w): E^2 --> R
   (x, y) |--> 0
sage: w.div(h).display()
div_h(w): E^2 --> R
   (x, y) |--> 0
```

Divergence of a type- `(2,0)` tensor field:

```
sage: t = v*w; t
Tensor field v*w of type (2,0) on the Euclidean plane E^2
sage: s = t.div(); s
Vector field div(v*w) on the Euclidean plane E^2
```

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```
sage: s.display()
div(v*w) = -y e_x + x e_y
```

domain()

Return the manifold on which `self` is defined.

OUTPUT:

- instance of class *DifferentiableManifold*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: t = M.tensor_field(1,2)
sage: t.domain()
2-dimensional differentiable manifold M
sage: U = M.open_subset('U', coord_def={c_xy: x<0})
sage: h = t.restrict(U)
sage: h.domain()
Open subset U of the 2-dimensional differentiable manifold M
```

down(*metric*, *pos=None*)

Compute a metric dual of the tensor field by lowering some index with a given metric.

If T is the tensor field, (k,l) its type and p the position of a contravariant index (i.e. $0 \leq p < k$), this method called with `pos` $= p$ yields the tensor field $T^\flat$ of type $(k-1,l+1)$ whose components are

$$(T^\flat)^{a_1\ldots a_{k-1}}{}_{b_1\ldots b_{l+1}} = g_{b_1 i}\, T^{a_1\ldots a_p\, i\, a_{p+1}\ldots a_{k-1}}{}_{b_2\ldots b_{l+1}},$$

g_{ab} being the components of the metric tensor.

The reverse operation is *TensorField.up()*.

INPUT:

- `metric` – metric g, as an instance of *PseudoRiemannianMetric*
- `pos` – (default: `None`) position of the index (with the convention `pos=0` for the first index); if `None`, the lowering is performed over all the contravariant indices, starting from the last one

OUTPUT:

- the tensor field $T^\flat$ resulting from the index lowering operation

EXAMPLES:

Lowering the index of a vector field results in a 1-form:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-y
sage: v = M.vector_field(-1, 2)
sage: w = v.down(g) ; w
1-form on the 2-dimensional differentiable manifold M
sage: w.display()
(2*x*y - x - 1) dx + (-(x + 2)*y + 2) dy
```

Using the index notation instead of *down()*:

```
sage: w == g['_ab']*v['^b']
True
```

The reverse operation:

```
sage: v1 = w.up(g) ; v1
Vector field on the 2-dimensional differentiable manifold M
sage: v1 == v
True
```

Lowering the indices of a tensor field of type (2,0):

```
sage: t = M.tensor_field(2, 0, [[1,2], [3,4]])
sage: td0 = t.down(g, 0) ; td0  # lowering the first index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: td0 == g['_ac']*t['^cb'] # the same operation in index notation
True
sage: td0[:]
[  3*x*y + x + 1   (x - 3)*y + 3]
[4*x*y + 2*x + 2 2*(x - 2)*y + 4]
sage: tdd0 = td0.down(g) ; tdd0 # the two indices have been lowered, starting␣
↪from the first one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: tdd0 == g['_ac']*td0['^c_b'] # the same operation in index notation
True
sage: tdd0[:]
[      4*x^2*y^2 + x^2 + 5*(x^2 + x)*y + 2*x + 1 2*(x^2 - 2*x)*y^2 + (x^2 +␣
↪2*x - 3)*y + 3*x + 3]
[(3*x^2 - 4*x)*y^2 + (x^2 + 3*x - 2)*y + 2*x + 2           (x^2 - 5*x + 4)*y^
↪2 + (5*x - 8)*y + 4]
sage: td1 = t.down(g, 1) ; td1  # lowering the second index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: td1 == g['_ac']*t['^bc'] # the same operation in index notation
True
sage: td1[:]
[  2*x*y + x + 1   (x - 2)*y + 2]
[4*x*y + 3*x + 3 (3*x - 4)*y + 4]
sage: tdd1 = td1.down(g) ; tdd1 # the two indices have been lowered, starting␣
↪from the second one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: tdd1 == g['_ac']*td1['^c_b'] # the same operation in index notation
True
sage: tdd1[:]
[      4*x^2*y^2 + x^2 + 5*(x^2 + x)*y + 2*x + 1 (3*x^2 - 4*x)*y^2 + (x^2 +␣
↪3*x - 2)*y + 2*x + 2]
[2*(x^2 - 2*x)*y^2 + (x^2 + 2*x - 3)*y + 3*x + 3           (x^2 - 5*x + 4)*y^
↪2 + (5*x - 8)*y + 4]
sage: tdd1 == tdd0   # the order of index lowering is important
False
sage: tdd = t.down(g) ; tdd  # both indices are lowered, starting from the␣
↪last one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
```

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```
sage: tdd[:]
[      4*x^2*y^2 + x^2 + 5*(x^2 + x)*y + 2*x + 1 (3*x^2 - 4*x)*y^2 + (x^2 +␣
↪3*x - 2)*y + 2*x + 2]
[2*(x^2 - 2*x)*y^2 + (x^2 + 2*x - 3)*y + 3*x + 3           (x^2 - 5*x + 4)*y^
↪2 + (5*x - 8)*y + 4]
sage: tdd0 == tdd  # to get tdd0, indices have been lowered from the first␣
↪one, contrary to tdd
False
sage: tdd1 == tdd  # the same order for index lowering has been applied
True
sage: u0tdd = tdd.up(g, 0) ; u0tdd # the first index is raised again
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: uu0tdd = u0tdd.up(g) ; uu0tdd # the second index is then raised
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: u1tdd = tdd.up(g, 1) ; u1tdd  # raising operation, starting from the␣
↪last index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: uu1tdd = u1tdd.up(g) ; uu1tdd
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: uutdd = tdd.up(g) ; uutdd  # both indices are raised, starting from the␣
↪first one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: uutdd == t  # should be true
True
sage: uu0tdd == t # should be true
True
sage: uu1tdd == t # not true, because of the order of index raising to get␣
↪uu1tdd
False
```

laplacian(*metric=None*)

Return the Laplacian of `self` with respect to a given metric (Laplace-Beltrami operator).

If `self` is a tensor field t of type (k, l), the Laplacian of t with respect to the metric g is the tensor field of type (k, l) defined by

$$(\Delta t)^{a_1 \ldots a_k}{}_{b_1 \ldots b_k} = \nabla_i \nabla^i t^{a_1 \ldots a_k}{}_{b_1 \ldots b_k},$$

where ∇ is the Levi-Civita connection of g (cf. *LeviCivitaConnection*) and $\nabla^i := g^{ij} \nabla_j$. The operator $\Delta = \nabla_i \nabla^i$ is called the *Laplace-Beltrami operator* of metric g.

INPUT:

- `metric` – (default: `None`) the pseudo-Riemannian metric g involved in the definition of the Laplacian; if none is provided, the domain of `self` is supposed to be endowed with a default metric (i.e. is supposed to be pseudo-Riemannian manifold, see *PseudoRiemannianManifold*) and the latter is used to define the Laplacian

OUTPUT:

- instance of *TensorField* representing the Laplacian of `self`

EXAMPLES:

Laplacian of a vector field in the Euclidean plane:

```
sage: M.<x,y> = EuclideanSpace()
sage: v = M.vector_field(x^3 + y^2, x*y, name='v')
sage: Dv = v.laplacian(); Dv
Vector field Delta(v) on the Euclidean plane E^2
sage: Dv.display()
Delta(v) = (6*x + 2) e_x
```

The function `laplacian()` from the `operators` module can be used instead of the method `laplacian()`:

```
sage: from sage.manifolds.operators import laplacian
sage: laplacian(v) == Dv
True
```

In the present case (Euclidean metric and Cartesian coordinates), the components of the Laplacian are the Laplacians of the components:

```
sage: all(Dv[[i]] == laplacian(v[[i]]) for i in M.irange())
True
```

The Laplacian can be taken with respect to a metric tensor that is not the default one:

```
sage: h = M.lorentzian_metric('h')
sage: h[1,1], h[2,2] = -1, 1+x^2
sage: Dv = v.laplacian(h); Dv
Vector field Delta_h(v) on the Euclidean plane E^2
sage: Dv.display()
Delta_h(v) = -(8*x^5 - 2*x^4 - x^2*y^2 + 15*x^3 - 4*x^2 + 6*x
 - 2)/(x^4 + 2*x^2 + 1) e_x - 3*x^3*y/(x^4 + 2*x^2 + 1) e_y
```

lie_der(*vector*)

Lie derivative of `self` with respect to a vector field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the tensor field that is the Lie derivative of the current tensor field with respect to `vector`

EXAMPLES:

Lie derivative of a type-$(1,1)$ tensor field along a vector field on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 1, {e_xy: [[x, 1], [y, 0]]}, name='t')
sage: t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: w = M.vector_field({e_xy: [-y, x]}, name='w')
sage: w.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
```

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```
sage: lt = t.lie_derivative(w); lt
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: lt.display(e_xy)
d/dx*dx - x d/dx*dy + (-y - 1) d/dy*dy
sage: lt.display(e_uv)
-1/2*u d/du*du + (1/2*u + 1) d/du*dv + (-1/2*v + 1) d/dv*du + 1/2*v d/dv*dv
```

The result is cached:

```
sage: t.lie_derivative(w) is lt
True
```

An alias is `lie_der`:

```
sage: t.lie_der(w) is t.lie_derivative(w)
True
```

Lie derivative of a vector field:

```
sage: a = M.vector_field({e_xy: [1-x, x-y]}, name='a')
sage: a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.lie_der(w)
Vector field on the 2-dimensional differentiable manifold M
sage: a.lie_der(w).display(e_xy)
x d/dx + (-y - 1) d/dy
sage: a.lie_der(w).display(e_uv)
(v - 1) d/du + (u + 1) d/dv
```

The Lie derivative is antisymmetric:

```
sage: a.lie_der(w) == - w.lie_der(a)
True
```

and it coincides with the commutator of the two vector fields:

```
sage: f = M.scalar_field({c_xy: 3*x-1, c_uv:  3/2*(u+v)-1})
sage: a.lie_der(w)(f) == w(a(f)) - a(w(f))  # long time
True
```

lie_derivative(*vector*)

Lie derivative of `self` with respect to a vector field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the tensor field that is the Lie derivative of the current tensor field with respect to `vector`

EXAMPLES:

Lie derivative of a type-$(1,1)$ tensor field along a vector field on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
```

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```
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 1, {e_xy: [[x, 1], [y, 0]]}, name='t')
sage: t.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: w = M.vector_field({e_xy: [-y, x]}, name='w')
sage: w.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: lt = t.lie_derivative(w); lt
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: lt.display(e_xy)
d/dx*dx - x d/dx*dy + (-y - 1) d/dy*dy
sage: lt.display(e_uv)
-1/2*u d/du*du + (1/2*u + 1) d/du*dv + (-1/2*v + 1) d/dv*du + 1/2*v d/dv*dv
```

The result is cached:

```
sage: t.lie_derivative(w) is lt
True
```

An alias is `lie_der`:

```
sage: t.lie_der(w) is t.lie_derivative(w)
True
```

Lie derivative of a vector field:

```
sage: a = M.vector_field({e_xy: [1-x, x-y]}, name='a')
sage: a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.lie_der(w)
Vector field on the 2-dimensional differentiable manifold M
sage: a.lie_der(w).display(e_xy)
x d/dx + (-y - 1) d/dy
sage: a.lie_der(w).display(e_uv)
(v - 1) d/du + (u + 1) d/dv
```

The Lie derivative is antisymmetric:

```
sage: a.lie_der(w) == - w.lie_der(a)
True
```

and it coincides with the commutator of the two vector fields:

```
sage: f = M.scalar_field({c_xy: 3*x-1, c_uv:  3/2*(u+v)-1})
sage: a.lie_der(w)(f) == w(a(f)) - a(w(f))   # long time
True
```

restrict (*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- subdomain – *DifferentiableManifold*; open subset U of the tensor field domain S
- dest_map – *DiffMap* (default: None); destination map $\Psi: U \to V$, where V is an open subset of the manifold M where the tensor field takes it values; if None, the restriction of Φ to U is used, Φ being the differentiable map $S \to M$ associated with the tensor field

OUTPUT:

- *TensorField* representing the restriction

EXAMPLES:

Restrictions of a vector field on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of the North pole
sage: stereoN.<x,y> = U.chart()  # stereographic coordinates from the North␣
↪pole
sage: eN = stereoN.frame() # the associated vector frame
sage: V =  M.open_subset('V') # the complement of the South pole
sage: stereoS.<u,v> = V.chart()  # stereographic coordinates from the South␣
↪pole
sage: eS = stereoS.frame() # the associated vector frame
sage: transf = stereoN.transition_map(stereoS, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: inv = transf.inverse() # transformation from stereoS to stereoN
sage: W = U.intersection(V) # the complement of the North and South poles
sage: stereoN_W = W.atlas()[0]  # restriction of stereographic coord. from␣
↪North pole to W
sage: stereoS_W = W.atlas()[1]  # restriction of stereographic coord. from␣
↪South pole to W
sage: eN_W = stereoN_W.frame() ; eS_W = stereoS_W.frame()
sage: v = M.vector_field({eN: [1, 0]}, name='v')
sage: v.display()
v = d/dx
sage: vU = v.restrict(U) ; vU
Vector field v on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: vU.display()
v = d/dx
sage: vU == eN[1]
True
sage: vW = v.restrict(W) ; vW
Vector field v on the Open subset W of the 2-dimensional
 differentiable manifold S^2
sage: vW.display()
v = d/dx
sage: vW.display(eS_W, stereoS_W)
v = (-u^2 + v^2) d/du - 2*u*v d/dv
sage: vW == eN_W[1]
True
```

At this stage, defining the restriction of v to the open subset V fully specifies v:

```
sage: v.restrict(V)[1] = vW[eS_W, 1, stereoS_W].expr()  # note that eS is the␣
↪default frame on V
sage: v.restrict(V)[2] = vW[eS_W, 2, stereoS_W].expr()
sage: v.display(eS, stereoS)
v = (-u^2 + v^2) d/du - 2*u*v d/dv
```

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```
sage: v.restrict(U).display()
v = d/dx
sage: v.restrict(V).display()
v = (-u^2 + v^2) d/du - 2*u*v d/dv
```

The restriction of the vector field to its own domain is of course itself:

```
sage: v.restrict(M) is v
True
sage: vU.restrict(U) is vU
True
```

set_calc_order(*symbol*, *order*, *truncate=False*)

Trigger a series expansion with respect to a small parameter in computations involving the tensor field.

This property is propagated by usual operations. The internal representation must be `SR` for this to take effect.

If the small parameter is ϵ and T is `self`, the power series expansion to order n is

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n + O(\epsilon^{n+1}),$$

where $T_0, T_1, \ldots, T_n$ are $n + 1$ tensor fields of the same tensor type as `self` and do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the components of `self` are expanded in power series
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`
- `truncate` – (default: `False`) determines whether the components of `self` are replaced by their expansions to the given order

EXAMPLES:

Let us consider two vector fields depending on a small parameter h on a non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.vector_field()
sage: h = var('h', domain='real')
sage: a[eU,:] = (cos(h*x), -y)
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b = M.vector_field()
sage: b[eU,:] = (exp(h*x), exp(h*y))
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
```

If we set the calculus order on one of the vector fields, any operation involving both of them is performed to that order:

```
sage: a.set_calc_order(h, 2)
sage: s = a + b
sage: s[eU,:]
[h*x + 2, 1/2*h^2*y^2 + h*y - y + 1]
sage: s[eV,:]
[1/8*(u^2 - 2*u*v + v^2)*h^2 + h*u - 1/2*u + 1/2*v + 3,
 -1/8*(u^2 - 2*u*v + v^2)*h^2 + h*v + 1/2*u - 1/2*v + 1]
```

Note that the components of `a` have not been affected by the above call to `set_calc_order`:

```
sage: a[eU,:]
[cos(h*x), -y]
sage: a[eV,:]
[cos(1/2*h*u)*cos(1/2*h*v) - sin(1/2*h*u)*sin(1/2*h*v) - 1/2*u + 1/2*v,
 cos(1/2*h*u)*cos(1/2*h*v) - sin(1/2*h*u)*sin(1/2*h*v) + 1/2*u - 1/2*v]
```

To have `set_calc_order` act on them, set the optional argument `truncate` to `True`:

```
sage: a.set_calc_order(h, 2, truncate=True)
sage: a[eU,:]
[-1/2*h^2*x^2 + 1, -y]
sage: a[eV,:]
[-1/8*(u^2 + 2*u*v + v^2)*h^2 - 1/2*u + 1/2*v + 1,
 -1/8*(u^2 + 2*u*v + v^2)*h^2 + 1/2*u - 1/2*v + 1]
```

set_comp(*basis=None*)

Return the components of `self` in a given vector frame for assignment.

The components with respect to other frames having the same domain as the provided vector frame are deleted, in order to avoid any inconsistency. To keep them, use the method *add_comp()* instead.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are defined; if none is provided, the components are assumed to refer to the tensor field domain's default frame

OUTPUT:

- components in the given frame, as a `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: e_uv = c_uv.frame()
sage: t = M.tensor_field(1, 2, name='t')
sage: t.set_comp(e_uv)
3-indices components w.r.t. Coordinate frame (V, (d/du,d/dv))
sage: t.set_comp(e_uv)[1,0,1] = u+v
sage: t.display(e_uv)
t = (u + v) d/dv*du*dv
```

Setting the components in a new frame (`e`):

```
sage: e = V.vector_frame('e')
sage: t.set_comp(e)
3-indices components w.r.t. Vector frame (V, (e_0,e_1))
sage: t.set_comp(e)[0,1,1] = u*v
sage: t.display(e)
t = u*v e_0*e^1*e^1
```

Since the frames e and e_uv are defined on the same domain, the components w.r.t. e_uv have been erased:

```
sage: t.display(c_uv.frame())
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Coordinate frame (V, (d/du,d/dv))
```

set_name (*name=None*, *latex_name=None*)

Set (or change) the text name and LaTeX name of self.

INPUT:

- name – string (default: None); name given to the tensor field
- latex_name – string (default: None); LaTeX symbol to denote the tensor field; if None while name is provided, the LaTeX symbol is set to name

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: t = M.tensor_field(1, 3); t
Tensor field of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: t.set_name(name='t')
sage: t
Tensor field t of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: latex(t)
t
sage: t.set_name(latex_name=r'\tau')
sage: latex(t)
\tau
sage: t.set_name(name='a')
sage: t
Tensor field a of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: latex(t)
a
```

set_restriction (*rst*)

Define a restriction of self to some subdomain.

INPUT:

- rst – *TensorField* of the same type and symmetries as the current tensor field self, defined on a subdomain of the domain of self

EXAMPLES:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: t = M.tensor_field(1, 2, name='t')
sage: s = U.tensor_field(1, 2)
sage: s[0,0,1] = x+y
sage: t.set_restriction(s)
sage: t.display(c_xy.frame())
t = (x + y) d/dx*dx*dy
sage: t.restrict(U) == s
True
```

symmetries()

Print the list of symmetries and antisymmetries.

EXAMPLES:

```
sage: M = Manifold(2, 'S^2')
sage: t = M.tensor_field(1,2)
sage: t.symmetries()
no symmetry;  no antisymmetry
sage: t = M.tensor_field(1,2, sym=(1,2))
sage: t.symmetries()
symmetry: (1, 2);  no antisymmetry
sage: t = M.tensor_field(2,2, sym=(0,1), antisym=(2,3))
sage: t.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
sage: t = M.tensor_field(2,2, antisym=[(0,1),(2,3)])
sage: t.symmetries()
no symmetry;  antisymmetries: [(0, 1), (2, 3)]
```

symmetrize(**pos*)

Symmetrization over some arguments.

INPUT:

- `pos` – (default: `None`) list of argument positions involved in the symmetrization (with the convention `position=0` for the first argument); if `None`, the symmetrization is performed over all the arguments

OUTPUT:

- the symmetrized tensor field (instance of `TensorField`)

EXAMPLES:

Symmetrization of a type-$(0, 2)$ tensor field on a 2-dimensional non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
```

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```
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.tensor_field(0,2, {eU: [[1,x], [2,y]]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: a[eV,:]
[ 1/4*u + 3/4 -1/4*u + 3/4]
[ 1/4*v - 1/4 -1/4*v - 1/4]
sage: s = a.symmetrize() ; s
Field of symmetric bilinear forms on the 2-dimensional
 differentiable manifold M
sage: s[eU,:]
[        1 1/2*x + 1]
[1/2*x + 1         y]
sage: s[eV,:]
[         1/4*u + 3/4 -1/8*u + 1/8*v + 1/4]
[-1/8*u + 1/8*v + 1/4         -1/4*v - 1/4]
sage: s == a.symmetrize(0,1)  # explicit positions
True
```

See also:

For more details and examples, see `sage.tensor.modules.free_module_tensor.FreeModuleTensor.symmetrize()`.

tensor_rank()

Return the tensor rank of `self`.

OUTPUT:

- integer $k + l$, where k is the contravariant rank and l is the covariant rank

EXAMPLES:

```
sage: M = Manifold(2, 'S^2')
sage: t = M.tensor_field(1,2)
sage: t.tensor_rank()
3
sage: v = M.vector_field()
sage: v.tensor_rank()
1
```

tensor_type()

Return the tensor type of `self`.

OUTPUT:

- pair (k, l), where k is the contravariant rank and l is the covariant rank

EXAMPLES:

```
sage: M = Manifold(2, 'S^2')
sage: t = M.tensor_field(1,2)
sage: t.tensor_type()
(1, 2)
sage: v = M.vector_field()
sage: v.tensor_type()
(1, 0)
```

trace(*pos1=0*, *pos2=1*)

Trace (contraction) on two slots of the tensor field.

INPUT:

- pos1 – (default: 0) position of the first index for the contraction, with the convention pos1=0 for the first slot
- pos2 – (default: 1) position of the second index for the contraction, with the same convention as for pos1. The variance type of pos2 must be opposite to that of pos1

OUTPUT:

- tensor field resulting from the (pos1, pos2) contraction

EXAMPLES:

Trace of a type-$(1,1)$ tensor field on a 2-dimensional non-parallelizable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: W = U.intersection(V)
sage: a = M.tensor_field(1,1, name='a')
sage: a[e_xy,:] = [[1,x], [2,y]]
sage: a.add_comp_by_continuation(e_uv, W, chart=c_uv)
sage: s = a.trace() ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.display()
M --> R
on U: (x, y) |--> y + 1
on V: (u, v) |--> 1/2*u - 1/2*v + 1
sage: s == a.trace(0,1) # explicit mention of the positions
True
```

Instead of the explicit call to the method *trace()*, one may use the index notation with Einstein convention (summation over repeated indices); it suffices to pass the indices as a string inside square brackets:

```
sage: a['^i_i']
Scalar field on the 2-dimensional differentiable manifold M
sage: a['^i_i'] == s
True
```

Any letter can be used to denote the repeated index:

```
sage: a['^b_b'] == s
True
```

Trace of a type-$(1,2)$ tensor field:

```
sage: b = M.tensor_field(1,2, name='b') ; b
Tensor field b of type (1,2) on the 2-dimensional differentiable
 manifold M
sage: b[e_xy,:] = [[[0,x+y], [y,0]], [[0,2], [3*x,-2]]]
sage: b.add_comp_by_continuation(e_uv, W, chart=c_uv)  # long time
sage: s = b.trace(0,1) ; s # contraction on first and second slots
1-form on the 2-dimensional differentiable manifold M
```

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```
sage: s.display(e_xy)
3*x dx + (x + y - 2) dy
sage: s.display(e_uv)  # long time
(5/4*u + 3/4*v - 1) du + (1/4*u + 3/4*v + 1) dv
```

Use of the index notation:

```
sage: b['^k_ki']
1-form on the 2-dimensional differentiable manifold M
sage: b['^k_ki'] == s  # long time
True
```

Indices not involved in the contraction may be replaced by dots:

```
sage: b['^k_k.'] == s  # long time
True
```

The symbol ^ may be omitted:

```
sage: b['k_k.'] == s  # long time
True
```

LaTeX notations are allowed:

```
sage: b['^{k}_{ki}'] == s  # long time
True
```

Contraction on first and third slots:

```
sage: s = b.trace(0,2) ; s
1-form on the 2-dimensional differentiable manifold M
sage: s.display(e_xy)
2 dx + (y - 2) dy
sage: s.display(e_uv)  # long time
(1/4*u - 1/4*v) du + (-1/4*u + 1/4*v + 2) dv
```

Use of index notation:

```
sage: b['^k_.k'] == s  # long time
True
```

up (*metric*, *pos=None*)

Compute a metric dual of the tensor field by raising some index with a given metric.

If T is the tensor field, (k, l) its type and p the position of a covariant index (i.e. $k \leq p < k + l$), this method called with `pos` $= p$ yields the tensor field $T^\sharp$ of type $(k + 1, l - 1)$ whose components are

$$(T^\sharp)^{a_1 \ldots a_{k+1}}{}_{b_1 \ldots b_{l-1}} = g^{a_{k+1} i} \, T^{a_1 \ldots a_k}{}_{b_1 \ldots b_{p-k} \, i \, b_{p-k+1} \ldots b_{l-1}},$$

g^{ab} being the components of the inverse metric.

The reverse operation is `TensorField.down()`.

INPUT:

- `metric` – metric g, as an instance of `PseudoRiemannianMetric`
- `pos` – (default: `None`) position of the index (with the convention `pos=0` for the first index); if `None`, the raising is performed over all the covariant indices, starting from the first one

OUTPUT:

- the tensor field $T^\sharp$ resulting from the index raising operation

EXAMPLES:

Raising the index of a 1-form results in a vector field:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-y
sage: w = M.one_form(-1, 2)
sage: v = w.up(g) ; v
Vector field on the 2-dimensional differentiable manifold M
sage: v.display()
((2*x - 1)*y + 1)/(x^2*y^2 + (x + 1)*y - x - 1) d/dx
 - (x*y + 2*x + 2)/(x^2*y^2 + (x + 1)*y - x - 1) d/dy
sage: ig = g.inverse(); ig[:]
[ (y - 1)/(x^2*y^2 + (x + 1)*y - x - 1)      x*y/(x^2*y^2 + (x + 1)*y - x -␣
↪1)]
[     x*y/(x^2*y^2 + (x + 1)*y - x - 1) -(x + 1)/(x^2*y^2 + (x + 1)*y - x -␣
↪1)]
```

Using the index notation instead of `up()`:

```
sage: v == ig['^ab']*w['_b']
True
```

The reverse operation:

```
sage: w1 = v.down(g) ; w1
1-form on the 2-dimensional differentiable manifold M
sage: w1.display()
-dx + 2 dy
sage: w1 == w
True
```

The reverse operation in index notation:

```
sage: g['_ab']*v['^b'] == w
True
```

Raising the indices of a tensor field of type (0,2):

```
sage: t = M.tensor_field(0, 2, [[1,2], [3,4]])
sage: tu0 = t.up(g, 0) ; tu0  # raising the first index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: tu0[:]
[  ((3*x + 1)*y - 1)/(x^2*y^2 + (x + 1)*y - x - 1) 2*((2*x + 1)*y - 1)/(x^2*y^
↪2 + (x + 1)*y - x - 1)]
[     (x*y - 3*x - 3)/(x^2*y^2 + (x + 1)*y - x - 1)    2*(x*y - 2*x - 2)/(x^2*y^
↪2 + (x + 1)*y - x - 1)]
sage: tu0 == ig['^ac']*t['_cb'] # the same operation in index notation
True
sage: tuu0 = tu0.up(g) ; tuu0 # the two indices have been raised, starting␣
↪from the first one
Tensor field of type (2,0) on the 2-dimensional differentiable
```

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```
 manifold M
sage: tuu0 == tu0['^a_c']*ig['^cb'] # the same operation in index notation
True
sage: tu1 = t.up(g, 1) ; tu1 # raising the second index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: tu1 == ig['^ac']*t['_bc'] # the same operation in index notation
True
sage: tu1[:]
[((2*x + 1)*y - 1)/(x^2*y^2 + (x + 1)*y - x - 1) ((4*x + 3)*y - 3)/(x^2*y^2 +␣
↪(x + 1)*y - x - 1)]
[  (x*y - 2*x - 2)/(x^2*y^2 + (x + 1)*y - x - 1) (3*x*y - 4*x - 4)/(x^2*y^2 +␣
↪(x + 1)*y - x - 1)]
sage: tuu1 = tu1.up(g) ; tuu1 # the two indices have been raised, starting␣
↪from the second one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: tuu1 == tu1['^a_c']*ig['^cb'] # the same operation in index notation
True
sage: tuu0 == tuu1 # the order of index raising is important
False
sage: tuu = t.up(g) ; tuu # both indices are raised, starting from the first␣
↪one
Tensor field of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: tuu0 == tuu # the same order for index raising has been applied
True
sage: tuu1 == tuu # to get tuu1, indices have been raised from the last one,␣
↪contrary to tuu
False
sage: d0tuu = tuu.down(g, 0) ; d0tuu # the first index is lowered again
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: dd0tuu = d0tuu.down(g) ; dd0tuu  # the second index is then lowered
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: d1tuu = tuu.down(g, 1) ; d1tuu # lowering operation, starting from the␣
↪last index
Tensor field of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: dd1tuu = d1tuu.down(g) ; dd1tuu
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: ddtuu = tuu.down(g) ; ddtuu # both indices are lowered, starting from␣
↪the last one
Tensor field of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: ddtuu == t # should be true
True
sage: dd0tuu == t # not true, because of the order of index lowering to get␣
↪dd0tuu
False
sage: dd1tuu == t # should be true
True
```

2.8.3 Tensor Fields with Values on a Parallelizable Manifold

The class *TensorFieldParal* implements tensor fields along a differentiable manifolds with values on a parallelizable differentiable manifold. For non-parallelizable manifolds, see the class *TensorField*.

Various derived classes of *TensorFieldParal* are devoted to specific tensor fields:

- *VectorFieldParal* for vector fields (rank-1 contravariant tensor fields)
- *AutomorphismFieldParal* for fields of tangent-space automorphisms
- *DiffFormParal* for differential forms (fully antisymmetric covariant tensor fields)
- *MultivectorFieldParal* for multivector fields (fully antisymmetric contravariant tensor fields)

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Travis Scrimshaw (2016): review tweaks
- Eric Gourgoulhon (2018): method *TensorFieldParal.along()*
- Florentin Jaffredo (2018) : series expansion with respect to a given parameter

REFERENCES:

- [?]
- [?]
- [?]

EXAMPLES:

A tensor field of type $(1, 1)$ on a 2-dimensional differentiable manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: t = M.tensor_field(1, 1, name='T') ; t
Tensor field T of type (1,1) on the 2-dimensional differentiable manifold M
sage: t.tensor_type()
(1, 1)
sage: t.tensor_rank()
2
```

Components with respect to the manifold's default frame are created by providing the relevant indices inside square brackets:

```
sage: t[1,1] = x^2
```

Unset components are initialized to zero:

```
sage: t[:]  # list of components w.r.t. the manifold's default vector frame
[x^2   0]
[  0   0]
```

It is also possible to initialize the components at the tensor field construction:

```
sage: t = M.tensor_field(1, 1, [[x^2, 0], [0, 0]], name='T')
sage: t[:]
[x^2   0]
[  0   0]
```

The full set of components with respect to a given vector frame is returned by the method `comp()`:

```
sage: t.comp(c_xy.frame())
2-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy))
```

If no vector frame is mentioned in the argument of `comp()`, it is assumed to be the manifold's default frame:

```
sage: M.default_frame()
Coordinate frame (M, (d/dx,d/dy))
sage: t.comp() is t.comp(c_xy.frame())
True
```

Individual components with respect to the manifold's default frame are accessed by listing their indices inside double square brackets. They are `scalar fields` on the manifold:

```
sage: t[[1,1]]
Scalar field on the 2-dimensional differentiable manifold M
sage: t[[1,1]].display()
M --> R
(x, y) |--> x^2
sage: t[[1,2]]
Scalar field zero on the 2-dimensional differentiable manifold M
sage: t[[1,2]].display()
zero: M --> R
   (x, y) |--> 0
```

A direct access to the coordinate expression of some component is obtained via the single square brackets:

```
sage: t[1,1]
x^2
sage: t[1,1] is t[[1,1]].coord_function() # the coordinate function
True
sage: t[1,1] is t[[1,1]].coord_function(c_xy)
True
sage: t[1,1].expr() is t[[1,1]].expr() # the symbolic expression
True
```

Expressions in a chart different from the manifold's default one are obtained by specifying the chart as the last argument inside the single square brackets:

```
sage: c_uv.<u,v> = M.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y])
sage: uv_to_xy = xy_to_uv.inverse()
sage: t[1,1, c_uv]
1/4*u^2 + 1/2*u*v + 1/4*v^2
```

Note that `t[1,1, c_uv]` is the component of the tensor `t` with respect to the coordinate frame associated to the chart (x, y) expressed in terms of the coordinates (u, v). Indeed, `t[1,1, c_uv]` is a shortcut for `t.comp(c_xy.frame())[[1,1]].coord_function(c_uv)`:

```
sage: t[1,1, c_uv] is t.comp(c_xy.frame())[[1,1]].coord_function(c_uv)
True
```

Similarly, `t[1,1]` is a shortcut for `t.comp(c_xy.frame())[[1,1]].coord_function(c_xy)`:

```
sage: t[1,1] is t.comp(c_xy.frame())[[1,1]].coord_function(c_xy)
True
```

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```
sage: t[1,1] is t.comp()[[1,1]].coord_function()  # since c_xy.frame() and c_xy are␣
↪the manifold's default values
True
```

All the components can be set at once via `[:]`:

```
sage: t[:] = [[1, -x], [x*y, 2]]
sage: t[:]
[  1  -x]
[x*y   2]
```

To set the components in a vector frame different from the manifold's default one, the method *set_comp()* can be employed:

```
sage: e = M.vector_frame('e')
sage: t.set_comp(e)[1,1] = x+y
sage: t.set_comp(e)[2,1], t.set_comp(e)[2,2] = y, -3*x
```

but, as a shortcut, one may simply specify the frame as the first argument of the square brackets:

```
sage: t[e,1,1] = x+y
sage: t[e,2,1], t[e,2,2] = y, -3*x
sage: t.comp(e)
2-indices components w.r.t. Vector frame (M, (e_1,e_2))
sage: t.comp(e)[:]
[x + y      0]
[    y   -3*x]
sage: t[e,:]  # a shortcut of the above
[x + y      0]
[    y   -3*x]
```

All the components in some frame can be set at once, via the operator `[:]`:

```
sage: t[e,:] = [[x+y, 0], [y, -3*x]]
sage: t[e,:]  # same as above:
[x + y      0]
[    y   -3*x]
```

Equivalently, one can initialize the components in `e` at the tensor field construction:

```
sage: t = M.tensor_field(1, 1, [[x+y, 0], [y, -3*x]], frame=e, name='T')
sage: t[e,:]  # same as above:
[x + y      0]
[    y   -3*x]
```

To avoid any inconsistency between the various components, the method *set_comp()* clears the components in other frames. To keep the other components, one must use the method *add_comp()*:

```
sage: t = M.tensor_field(1, 1, name='T')  # Let us restart
sage: t[:] = [[1, -x], [x*y, 2]]  # by first setting the components in the frame c_xy.
↪frame()
sage: # We now set the components in the frame e with add_comp:
sage: t.add_comp(e)[:] = [[x+y, 0], [y, -3*x]]
```

The expansion of the tensor field in a given frame is obtained via the method `display`:

```
sage: t.display()  # expansion in the manifold's default frame
T = d/dx*dx - x d/dx*dy + x*y d/dy*dx + 2 d/dy*dy
sage: t.display(e)
T = (x + y) e_1*e^1 + y e_2*e^1 - 3*x e_2*e^2
```

See `display()` for more examples.

By definition, a tensor field acts as a multilinear map on 1-forms and vector fields; in the present case, `T` being of type $(1, 1)$, it acts on pairs (1-form, vector field):

```
sage: a = M.one_form(1, x, name='a')
sage: v = M.vector_field(y, 2, name='V')
sage: t(a,v)
Scalar field T(a,V) on the 2-dimensional differentiable manifold M
sage: t(a,v).display()
T(a,V): M --> R
   (x, y) |--> x^2*y^2 + 2*x + y
   (u, v) |--> 1/16*u^4 - 1/8*u^2*v^2 + 1/16*v^4 + 3/2*u + 1/2*v
sage: latex(t(a,v))
T\left(a,V\right)
```

Check by means of the component expression of `t(a,v)`:

```
sage: t(a,v).expr() - t[1,1]*a[1]*v[1] - t[1,2]*a[1]*v[2] \
....: - t[2,1]*a[2]*v[1] - t[2,2]*a[2]*v[2]
0
```

A scalar field (rank-0 tensor field):

```
sage: f = M.scalar_field(x*y + 2, name='f') ; f
Scalar field f on the 2-dimensional differentiable manifold M
sage: f.tensor_type()
(0, 0)
```

A scalar field acts on points on the manifold:

```
sage: p = M.point((1,2))
sage: f(p)
4
```

See `DiffScalarField` for more details on scalar fields.

A vector field (rank-1 contravariant tensor field):

```
sage: v = M.vector_field(-x, y, name='v') ; v
Vector field v on the 2-dimensional differentiable manifold M
sage: v.tensor_type()
(1, 0)
sage: v.display()
v = -x d/dx + y d/dy
```

A field of symmetric bilinear forms:

```
sage: q = M.sym_bilin_form_field(name='Q') ; q
Field of symmetric bilinear forms Q on the 2-dimensional differentiable
 manifold M
sage: q.tensor_type()
(0, 2)
```

The components of a symmetric bilinear form are dealt by the subclass `CompFullySym` of the class `Components`, which takes into account the symmetry between the two indices:

```
sage: q[1,1], q[1,2], q[2,2] = (0, -x, y) # no need to set the component (2,1)
sage: type(q.comp())
<class 'sage.tensor.modules.comp.CompFullySym'>
sage: q[:] # note that the component (2,1) is equal to the component (1,2)
[ 0 -x]
[-x  y]
sage: q.display()
Q = -x dx*dy - x dy*dx + y dy*dy
```

More generally, tensor symmetries or antisymmetries can be specified via the keywords `sym` and `antisym`. For instance a rank-4 covariant tensor symmetric with respect to its first two arguments (no. 0 and no. 1) and antisymmetric with respect to its last two ones (no. 2 and no. 3) is declared as follows:

```
sage: t = M.tensor_field(0, 4, name='T', sym=(0,1), antisym=(2,3))
sage: t[1,2,1,2] = 3
sage: t[2,1,1,2] # check of the symmetry with respect to the first 2 indices
3
sage: t[1,2,2,1] # check of the antisymmetry with respect to the last 2 indices
-3
```

class `sage.manifolds.differentiable.tensorfield_paral.`**`TensorFieldParal`**(*vector_field_module*, *tensor_type*, *name=None*, *latex_name=None*, *sym=None*, *antisym=None*)

Bases: `sage.tensor.modules.free_module_tensor.FreeModuleTensor`, *`sage.manifolds.differentiable.tensorfield.TensorField`*

Tensor field along a differentiable manifold, with values on a parallelizable manifold.

An instance of this class is a tensor field along a differentiable manifold U with values on a parallelizable manifold M, via a differentiable map $\Phi : U \to M$. More precisely, given two non-negative integers k and l and a differentiable map

$$\Phi : U \longrightarrow M,$$

a *tensor field of type* (k, l) *along* U *with values on* M is a differentiable map

$$t : U \longrightarrow T^{(k,l)}M$$

(where $T^{(k,l)}M$ is the tensor bundle of type (k, l) over M) such that

$$t(p) \in T^{(k,l)}(T_q M)$$

for all $p \in U$, i.e. $t(p)$ is a tensor of type (k, l) on the tangent space $T_q M$ at the point $q = \Phi(p)$. That is to say a multilinear map

$$t(p) : \underbrace{T_q^* M \times \cdots \times T_q^* M}_{k \text{ times}} \times \underbrace{T_q M \times \cdots \times T_q M}_{l \text{ times}} \longrightarrow K,$$

where $T_q^* M$ is the dual vector space to $T_q M$ and K is the topological field over which the manifold M is defined. The integer $k+l$ is called the *tensor rank*.

The standard case of a tensor field *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is not parallelizable, the class `TensorField` should be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to M$ (cf. `VectorFieldFreeModule`)
- `tensor_type` – pair (k, l) with k being the contravariant rank and l the covariant rank
- `name` – (default: `None`) name given to the tensor field
- `latex_name` – (default: `None`) LaTeX symbol to denote the tensor field; if none is provided, the LaTeX symbol is set to `name`
- `sym` – (default: `None`) a symmetry or a list of symmetries among the tensor arguments: each symmetry is described by a tuple containing the positions of the involved arguments, with the convention position=0 for the first argument; for instance:
  - `sym=(0,1)` for a symmetry between the 1st and 2nd arguments
  - `sym=[(0,2),(1,3,4)]` for a symmetry between the 1st and 3rd arguments and a symmetry between the 2nd, 4th and 5th arguments
- `antisym` – (default: `None`) antisymmetry or list of antisymmetries among the arguments, with the same convention as for `sym`

EXAMPLES:

A tensor field of type $(2,0)$ on a 3-dimensional parallelizable manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()   # makes M parallelizable
sage: t = M.tensor_field(2, 0, name='T') ; t
Tensor field T of type (2,0) on the 3-dimensional differentiable
 manifold M
```

Tensor fields are considered as elements of a module over the ring $C^k(M)$ of scalar fields on M:

```
sage: t.parent()
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold M
sage: t.parent().base_ring()
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
```

The components with respect to the manifold's default frame are set or read by means of square brackets:

```
sage: e = M.vector_frame('e') ; M.set_default_frame(e)
sage: for i in M.irange():
....:     for j in M.irange():
....:         t[i,j] = (i+1)**(j+1)
....:
sage: [[ t[i,j] for j in M.irange()] for i in M.irange()]
[[1, 1, 1], [2, 4, 8], [3, 9, 27]]
```

A shortcut for the above is using `[:]`:

```
sage: t[:]
[ 1  1  1]
[ 2  4  8]
[ 3  9 27]
```

The components with respect to another frame are set via the method *set_comp()* and read via the method *comp()*; both return an instance of `Components`:

```
sage: f = M.vector_frame('f')  # a new frame defined on M, in addition to e
sage: t.set_comp(f)[0,0] = -3
sage: t.comp(f)
2-indices components w.r.t. Vector frame (M, (f_0,f_1,f_2))
sage: t.comp(f)[0,0]
-3
sage: t.comp(f)[:]  # the full list of components
[-3  0  0]
[ 0  0  0]
[ 0  0  0]
```

To avoid any inconsistency between the various components, the method *set_comp()* deletes the components in other frames. Accordingly, the components in the frame `e` have been deleted:

```
sage: t._components
{Vector frame (M, (f_0,f_1,f_2)): 2-indices components w.r.t. Vector
 frame (M, (f_0,f_1,f_2))}
```

To keep the other components, one must use the method *add_comp()*:

```
sage: t = M.tensor_field(2, 0, name='T')  # let us restart
sage: t[0,0] = 2                          # sets the components in the frame e
sage: # We now set the components in the frame f with add_comp:
sage: t.add_comp(f)[0,0] = -3
sage: # The components w.r.t. frame e have been kept:
sage: t._components  # random (dictionary output)
{Vector frame (M, (e_0,e_1,e_2)): 2-indices components w.r.t. Vector frame (M, (e_
↪0,e_1,e_2)),
 Vector frame (M, (f_0,f_1,f_2)): 2-indices components w.r.t. Vector frame (M, (f_
↪0,f_1,f_2))}
```

The basic properties of a tensor field are:

```
sage: t.domain()
3-dimensional differentiable manifold M
sage: t.tensor_type()
(2, 0)
```

Symmetries and antisymmetries are declared via the keywords `sym` and `antisym`. For instance, a rank-6 covariant tensor that is symmetric with respect to its 1st and 3rd arguments and antisymmetric with respect to the 2nd, 5th and 6th arguments is set up as follows:

```
sage: a = M.tensor_field(0, 6, name='T', sym=(0,2), antisym=(1,4,5))
sage: a[0,0,1,0,1,2] = 3
sage: a[1,0,0,0,1,2] # check of the symmetry
3
sage: a[0,1,1,0,0,2], a[0,1,1,0,2,0] # check of the antisymmetry
(-3, 3)
```

Multiple symmetries or antisymmetries are allowed; they must then be declared as a list. For instance, a rank-4 covariant tensor that is antisymmetric with respect to its 1st and 2nd arguments and with respect to its 3rd and 4th argument must be declared as:

```
sage: r = M.tensor_field(0, 4, name='T', antisym=[(0,1), (2,3)])
sage: r[0,1,2,0] = 3
sage: r[1,0,2,0] # first antisymmetry
-3
sage: r[0,1,0,2] # second antisymmetry
-3
sage: r[1,0,0,2] # both antisymmetries acting
3
```

Tensor fields of the same type can be added and subtracted:

```
sage: a = M.tensor_field(2, 0)
sage: a[0,0], a[0,1], a[0,2] = (1,2,3)
sage: b = M.tensor_field(2, 0)
sage: b[0,0], b[1,1], b[2,2], b[0,2] = (4,5,6,7)
sage: s = a + 2*b ; s
Tensor field of type (2,0) on the 3-dimensional differentiable
 manifold M
sage: a[:], (2*b)[:], s[:]
(
[1 2 3]  [ 8  0 14]  [ 9  2 17]
[0 0 0]  [ 0 10  0]  [ 0 10  0]
[0 0 0], [ 0  0 12], [ 0  0 12]
)
sage: s = a - b ; s
Tensor field of type (2,0) on the 3-dimensional differentiable
 manifold M
sage: a[:], b[:], s[:]
(
[1 2 3]  [4 0 7]  [-3  2 -4]
[0 0 0]  [0 5 0]  [ 0 -5  0]
[0 0 0], [0 0 6], [ 0  0 -6]
)
```

Symmetries are preserved by the addition whenever it is possible:

```
sage: a = M.tensor_field(2, 0, sym=(0,1))
sage: a[0,0], a[0,1], a[0,2] = (1,2,3)
sage: s = a + b
sage: a[:], b[:], s[:]
(
[1 2 3]  [4 0 7]  [ 5  2 10]
[2 0 0]  [0 5 0]  [ 2  5  0]
[3 0 0], [0 0 6], [ 3  0  6]
)
sage: a.symmetries()
symmetry: (0, 1);  no antisymmetry
sage: b.symmetries()
no symmetry;  no antisymmetry
sage: s.symmetries()
no symmetry;  no antisymmetry
sage: # let us now make b symmetric:
sage: b = M.tensor_field(2, 0, sym=(0,1))
sage: b[0,0], b[1,1], b[2,2], b[0,2] = (4,5,6,7)
```

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```
sage: s = a + b
sage: a[:], b[:], s[:]
(
[1 2 3]  [4 0 7]  [ 5  2 10]
[2 0 0]  [0 5 0]  [ 2  5  0]
[3 0 0], [7 0 6], [10  0  6]
)
sage: s.symmetries()  # s is symmetric because both a and b are
symmetry: (0, 1);  no antisymmetry
```

The tensor product is taken with the operator $*$:

```
sage: c = a*b ; c
Tensor field of type (4,0) on the 3-dimensional differentiable
 manifold M
sage: c.symmetries()  # since a and b are both symmetric, a*b has two symmetries:
symmetries: [(0, 1), (2, 3)];  no antisymmetry
```

The tensor product of two fully contravariant tensors is not symmetric in general:

```
sage: a*b == b*a
False
```

The tensor product of a fully contravariant tensor by a fully covariant one is symmetric:

```
sage: d = M.diff_form(2)  # a fully covariant tensor field
sage: d[0,1], d[0,2], d[1,2] = (3, 2, 1)
sage: s = a*d ; s
Tensor field of type (2,2) on the 3-dimensional differentiable
 manifold M
sage: s.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
sage: s1 = d*a ; s1
Tensor field of type (2,2) on the 3-dimensional differentiable
 manifold M
sage: s1.symmetries()
symmetry: (0, 1);  antisymmetry: (2, 3)
sage: d*a == a*d
True
```

Example of tensor field associated with a non-trivial differentiable map Φ: tensor field along a curve in M:

```
sage: R = Manifold(1, 'R')  # R as a 1-dimensional manifold
sage: T.<t> = R.chart()  # canonical chart on R
sage: Phi = R.diff_map(M, [cos(t), sin(t), t], name='Phi') ; Phi
Differentiable map Phi from the 1-dimensional differentiable manifold R
 to the 3-dimensional differentiable manifold M
sage: h = R.tensor_field(2, 0, name='h', dest_map=Phi) ; h
Tensor field h of type (2,0) along the 1-dimensional differentiable
 manifold R with values on the 3-dimensional differentiable manifold M
sage: h.parent()
Free module T^(2,0)(R,Phi) of type-(2,0) tensors fields along the
 1-dimensional differentiable manifold R mapped into the 3-dimensional
 differentiable manifold M
sage: h[0,0], h[0,1], h[2,0] = 1+t, t^2, sin(t)
sage: h.display()
h = (t + 1) d/dx*d/dx + t^2 d/dx*d/dy + sin(t) d/dz*d/dx
```

add_comp(*basis=None*)

Return the components of the tensor field in a given vector frame for assignment.

The components with respect to other frames on the same domain are kept. To delete them, use the method `set_comp()` instead.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are defined; if none is provided, the components are assumed to refer to the tensor field domain's default frame

OUTPUT:

- components in the given frame, as an instance of the class `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: e_xy = X.frame()
sage: t = M.tensor_field(1,1, name='t')
sage: t.add_comp(e_xy)
2-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy))
sage: t.add_comp(e_xy)[1,0] = 2
sage: t.display(e_xy)
t = 2 d/dy*dx
```

Adding components with respect to a new frame (`e`):

```
sage: e = M.vector_frame('e')
sage: t.add_comp(e)
2-indices components w.r.t. Vector frame (M, (e_0,e_1))
sage: t.add_comp(e)[0,1] = x
sage: t.display(e)
t = x e_0*e^1
```

The components with respect to the frame `e_xy` are kept:

```
sage: t.display(e_xy)
t = 2 d/dy*dx
```

Adding components in a frame defined on a subdomain:

```
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: f = U.vector_frame('f')
sage: t.add_comp(f)
2-indices components w.r.t. Vector frame (U, (f_0,f_1))
sage: t.add_comp(f)[0,1] = 1+y
sage: t.display(f)
t = (y + 1) f_0*f^1
```

The components previously defined are kept:

```
sage: t.display(e_xy)
t = 2 d/dy*dx
sage: t.display(e)
t = x e_0*e^1
```

along (*mapping*)

Return the tensor field deduced from `self` via a differentiable map, the codomain of which is included in the domain of `self`.

More precisely, if `self` is a tensor field t on M and if $\Phi : U \to M$ is a differentiable map from some differentiable manifold U to M, the returned object is a tensor field $\tilde{t}$ along U with values on M such that

$$\forall p \in U, \ \tilde{t}(p) = t(\Phi(p)).$$

INPUT:

- `mapping` – differentiable map $\Phi : U \to M$

OUTPUT:

- tensor field $\tilde{t}$ along U defined above.

EXAMPLES:

Let us consider the map Φ between the interval $U = (0, 2\pi)$ and the Euclidean plane $M = \mathbf{R}^2$ defining the lemniscate of Gerono:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = var('t', domain='real')
sage: Phi = M.curve({X: [sin(t), sin(2*t)/2]}, (t, 0, 2*pi),
....:               name='Phi')
sage: U = Phi.domain(); U
Real interval (0, 2*pi)
```

and a vector field on M:

```
sage: v = M.vector_field(-y , x, name='v')
```

We have then:

```
sage: vU = v.along(Phi); vU
Vector field v along the Real interval (0, 2*pi) with values on
 the 2-dimensional differentiable manifold M
sage: vU.display()
v = -cos(t)*sin(t) d/dx + sin(t) d/dy
sage: vU.parent()
Free module X((0, 2*pi),Phi) of vector fields along the Real
 interval (0, 2*pi) mapped into the 2-dimensional differentiable
 manifold M
sage: vU.parent() is Phi.tangent_vector_field().parent()
True
```

We check that the defining relation $\tilde{t}(p) = t(\Phi(p))$ holds:

```
sage: p = U(t)  # a generic point of U
sage: vU.at(p) == v.at(Phi(p))
True
```

Case of a tensor field of type `(0,2)`:

```
sage: a = M.tensor_field(0, 2)
sage: a[0,0], a[0,1], a[1,1] = x+y, x*y, x^2-y^2
sage: aU = a.along(Phi); aU
Tensor field of type (0,2) along the Real interval (0, 2*pi) with
```

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```
 values on the 2-dimensional differentiable manifold M
sage: aU.display()
(cos(t) + 1)*sin(t) dx*dx + cos(t)*sin(t)^2 dx*dy + sin(t)^4 dy*dy
sage: aU.parent()
Free module T^(0,2)((0, 2*pi),Phi) of type-(0,2) tensors fields
 along the Real interval (0, 2*pi) mapped into the 2-dimensional
 differentiable manifold M
sage: aU.at(p) == a.at(Phi(p))
True
```

at (*point*)

Value of `self` at a point of its domain.

If the current tensor field is

$$t:\ U \longrightarrow T^{(k,l)}M$$

associated with the differentiable map

$$\Phi:\ U \longrightarrow M,$$

where U and M are two manifolds (possibly $U = M$ and $\Phi = \mathrm{Id}_M$), then for any point $p \in U$, $t(p)$ is a tensor on the tangent space to M at the point $\Phi(p)$.

INPUT:

- `point` – *ManifoldPoint* point p in the domain of the tensor field U

OUTPUT:

- `FreeModuleTensor` representing the tensor $t(p)$ on the tangent vector space $T_{\Phi(p)}M$

EXAMPLES:

Vector in a tangent space of a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: p = M.point((-2,3), name='p')
sage: v = M.vector_field(y, x^2, name='v')
sage: v.display()
v = y d/dx + x^2 d/dy
sage: vp = v.at(p) ; vp
Tangent vector v at Point p on the 2-dimensional differentiable
 manifold M
sage: vp.parent()
Tangent space at Point p on the 2-dimensional differentiable
 manifold M
sage: vp.display()
v = 3 d/dx + 4 d/dy
```

A 1-form gives birth to a linear form in the tangent space:

```
sage: w = M.one_form(-x, 1+y, name='w')
sage: w.display()
w = -x dx + (y + 1) dy
sage: wp = w.at(p) ; wp
Linear form w on the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
```

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```
sage: wp.parent()
Dual of the Tangent space at Point p on the 2-dimensional
 differentiable manifold M
sage: wp.display()
w = 2 dx + 4 dy
```

A tensor field of type $(1, 1)$ yields a tensor of type $(1, 1)$ in the tangent space:

```
sage: t = M.tensor_field(1, 1, name='t')
sage: t[0,0], t[0,1], t[1,1] = 1+x, x*y, 1-y
sage: t.display()
t = (x + 1) d/dx*dx + x*y d/dx*dy + (-y + 1) d/dy*dy
sage: tp = t.at(p) ; tp
Type-(1,1) tensor t on the Tangent space at Point p on the
 2-dimensional differentiable manifold M
sage: tp.parent()
Free module of type-(1,1) tensors on the Tangent space at Point p
 on the 2-dimensional differentiable manifold M
sage: tp.display()
t = -d/dx*dx - 6 d/dx*dy - 2 d/dy*dy
```

A 2-form yields an alternating form of degree 2 in the tangent space:

```
sage: a = M.diff_form(2, name='a')
sage: a[0,1] = x*y
sage: a.display()
a = x*y dx/\dy
sage: ap = a.at(p) ; ap
Alternating form a of degree 2 on the Tangent space at Point p on
 the 2-dimensional differentiable manifold M
sage: ap.parent()
2nd exterior power of the dual of the Tangent space at Point p on
 the 2-dimensional differentiable manifold M
sage: ap.display()
a = -6 dx/\dy
```

Example with a non trivial map Φ:

```
sage: U = Manifold(1, 'U')  # (0,2*pi) as a 1-dimensional manifold
sage: T.<t> = U.chart(r't:(0,2*pi)')  # canonical chart on U
sage: Phi = U.diff_map(M, [cos(t), sin(t)], name='Phi',
....:                  latex_name=r'\Phi')
sage: v = U.vector_field(1+t, t^2, name='v', dest_map=Phi) ; v
Vector field v along the 1-dimensional differentiable manifold U
 with values on the 2-dimensional differentiable manifold M
sage: v.display()
v = (t + 1) d/dx + t^2 d/dy
sage: p = U((pi/6,))
sage: vp = v.at(p) ; vp
Tangent vector v at Point on the 2-dimensional differentiable
 manifold M
sage: vp.parent() is M.tangent_space(Phi(p))
True
sage: vp.display()
v = (1/6*pi + 1) d/dx + 1/36*pi^2 d/dy
```

comp(*basis=None*, *from_basis=None*)

Return the components in a given vector frame.

If the components are not known already, they are computed by the tensor change-of-basis formula from components in another vector frame.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are required; if none is provided, the components are assumed to refer to the tensor field domain's default frame
- `from_basis` – (default: `None`) vector frame from which the required components are computed, via the tensor change-of-basis formula, if they are not known already in the basis `basis`

OUTPUT:

- components in the vector frame `basis`, as an instance of the class `Components`

EXAMPLES:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: t = M.tensor_field(1,2, name='t')
sage: t[1,2,1] = x*y
sage: t.comp(X.frame())
3-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy))
sage: t.comp()  # the default frame is X.frame()
3-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy))
sage: t.comp()[:]
[[[0, 0], [x*y, 0]], [[0, 0], [0, 0]]]
sage: e = M.vector_frame('e')
sage: t[e, 2,1,1] = x-3
sage: t.comp(e)
3-indices components w.r.t. Vector frame (M, (e_1,e_2))
sage: t.comp(e)[:]
[[[0, 0], [0, 0]], [[x - 3, 0], [0, 0]]]
```

contract(**args*)

Contraction with another tensor field, on one or more indices.

INPUT:

- `pos1` – positions of the indices in `self` involved in the contraction; `pos1` must be a sequence of integers, with 0 standing for the first index position, 1 for the second one, etc. If `pos1` is not provided, a single contraction on the last index position of `self` is assumed
- `other` – the tensor field to contract with
- `pos2` – positions of the indices in `other` involved in the contraction, with the same conventions as for `pos1`. If `pos2` is not provided, a single contraction on the first index position of `other` is assumed

OUTPUT:

- tensor field resulting from the contraction at the positions `pos1` and `pos2` of `self` with `other`

EXAMPLES:

Contraction of a tensor field of type $(2,0)$ with a tensor field of type $(1,1)$:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: a = M.tensor_field(2,0, [[1+x, 2], [y, -x^2]], name='a')
sage: b = M.tensor_field(1,1, [[-y, 1], [x, x+y]], name='b')
```

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```
sage: s = a.contract(0, b, 1); s
Tensor field of type (2,0) on the 2-dimensional differentiable manifold M
sage: s.display()
-x*y d/dx*d/dx + (x^2 + x*y + y^2 + x) d/dx*d/dy
 + (-x^2 - 2*y) d/dy*d/dx + (-x^3 - x^2*y + 2*x) d/dy*d/dy
```

Check:

```
sage: all(s[ind] == sum(a[k, ind[0]]*b[ind[1], k] for k in [0..1])
....:     for ind in M.index_generator(2))
True
```

The same contraction with repeated index notation:

```
sage: s == a['^ki']*b['^j_k']
True
```

Contraction on the second index of `a`:

```
sage: s = a.contract(1, b, 1); s
Tensor field of type (2,0) on the 2-dimensional differentiable manifold M
sage: s.display()
(-(x + 1)*y + 2) d/dx*d/dx + (x^2 + 3*x + 2*y) d/dx*d/dy
 + (-x^2 - y^2) d/dy*d/dx + (-x^3 - (x^2 - x)*y) d/dy*d/dy
```

Check:

```
sage: all(s[ind] == sum(a[ind[0], k]*b[ind[1], k] for k in [0..1])
....:     for ind in M.index_generator(2))
True
```

The same contraction with repeated index notation:

```
sage: s == a['^ik']*b['^j_k']
True
```

See also:

`sage.manifolds.differentiable.tensorfield.TensorField.contract()` for more examples.

`display_comp`(*frame=None*, *chart=None*, *coordinate_labels=True*, *only_nonzero=True*, *only_nonredundant=False*)

Display the tensor components with respect to a given frame, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the tensor field components are defined; if `None`, then
  - if `chart` is not `None`, the coordinate frame associated to `chart` is used
  - otherwise, the default basis of the vector field module on which the tensor field is defined is used
- `chart` – (default: `None`) chart specifying the coordinate expression of the components; if `None`, the default chart of the tensor field domain is used

- `coordinate_labels` – (default: `True`) boolean; if `True`, coordinate symbols are used by default (instead of integers) as index labels whenever `frame` is a coordinate frame
- `only_nonzero` – (default: `True`) boolean; if `True`, only nonzero components are displayed
- `only_nonredundant` – (default: `False`) boolean; if `True`, only nonredundant components are displayed in case of symmetries

EXAMPLES:

Display of the components of a type-$(2,1)$ tensor field on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: t = M.tensor_field(2, 1, name='t', sym=(0,1))
sage: t[0,0,0], t[0,1,0], t[1,1,1] = x+y, x*y, -3
sage: t.display_comp()
t^xx_x = x + y
t^xy_x = x*y
t^yx_x = x*y
t^yy_y = -3
```

By default, only the non-vanishing components are displayed; to see all the components, the argument `only_nonzero` must be set to `False`:

```
sage: t.display_comp(only_nonzero=False)
t^xx_x = x + y
t^xx_y = 0
t^xy_x = x*y
t^xy_y = 0
t^yx_x = x*y
t^yx_y = 0
t^yy_x = 0
t^yy_y = -3
```

`t` being symmetric with respect to its first two indices, one may ask to skip the components that can be deduced by symmetry:

```
sage: t.display_comp(only_nonredundant=True)
t^xx_x = x + y
t^xy_x = x*y
t^yy_y = -3
```

Instead of coordinate labels, one may ask for integers:

```
sage: t.display_comp(coordinate_labels=False)
t^00_0 = x + y
t^01_0 = x*y
t^10_0 = x*y
t^11_1 = -3
```

Display in a frame different from the default one (note that since `f` is not a coordinate frame, integer are used to label the indices):

```
sage: a = M.automorphism_field()
sage: a[:] = [[1+y^2, 0], [0, 2+x^2]]
sage: f = X.frame().new_frame(a, 'f')
sage: t.display_comp(frame=f)
t^00_0 = (x + y)/(y^2 + 1)
```

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```
t^01_0 = x*y/(x^2 + 2)
t^10_0 = x*y/(x^2 + 2)
t^11_1 = -3/(x^2 + 2)
```

Display with respect to a chart different from the default one:

```
sage: Y.<u,v> = M.chart()
sage: X_to_Y = X.transition_map(Y, [x+y, x-y])
sage: Y_to_X = X_to_Y.inverse()
sage: t.display_comp(chart=Y)
t^uu_u = 1/4*u^2 - 1/4*v^2 + 1/2*u - 3/2
t^uu_v = 1/4*u^2 - 1/4*v^2 + 1/2*u + 3/2
t^uv_u = 1/2*u + 3/2
t^uv_v = 1/2*u - 3/2
t^vu_u = 1/2*u + 3/2
t^vu_v = 1/2*u - 3/2
t^vv_u = -1/4*u^2 + 1/4*v^2 + 1/2*u - 3/2
t^vv_v = -1/4*u^2 + 1/4*v^2 + 1/2*u + 3/2
```

Note that the frame defining the components is the coordinate frame associated with chart `Y`, i.e. we have:

```
sage: str(t.display_comp(chart=Y)) == str(t.display_comp(frame=Y.frame(),␣
↪chart=Y))
True
```

Display of the components with respect to a specific frame, expressed in terms of a specific chart:

```
sage: t.display_comp(frame=f, chart=Y)
t^00_0 = 4*u/(u^2 - 2*u*v + v^2 + 4)
t^01_0 = (u^2 - v^2)/(u^2 + 2*u*v + v^2 + 8)
t^10_0 = (u^2 - v^2)/(u^2 + 2*u*v + v^2 + 8)
t^11_1 = -12/(u^2 + 2*u*v + v^2 + 8)
```

lie_der(*vector*)

Compute the Lie derivative with respect to a vector field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the tensor field that is the Lie derivative of `self` with respect to `vector`

EXAMPLES:

Lie derivative of a vector:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: v = M.vector_field(-y, x, name='v')
sage: w = M.vector_field(2*x+y, x*y)
sage: w.lie_derivative(v)
Vector field on the 2-dimensional differentiable manifold M
sage: w.lie_derivative(v).display()
((x - 2)*y + x) d/dx + (x^2 - y^2 - 2*x - y) d/dy
```

The result is cached:

```
sage: w.lie_derivative(v) is w.lie_derivative(v)
True
```

An alias is `lie_der`:

```
sage: w.lie_der(v) is w.lie_derivative(v)
True
```

The Lie derivative is antisymmetric:

```
sage: w.lie_der(v) == -v.lie_der(w)
True
```

For vectors, it coincides with the commutator:

```
sage: f = M.scalar_field(x^3 + x*y^2)
sage: w.lie_der(v)(f).display()
M --> R
(x, y) |--> -(x + 2)*y^3 + 3*x^3 - x*y^2 + 5*(x^3 - 2*x^2)*y
sage: w.lie_der(v)(f) == v(w(f)) - w(v(f))  # rhs = commutator [v,w] acting␣
↪on f
True
```

Lie derivative of a 1-form:

```
sage: om = M.one_form(y^2*sin(x), x^3*cos(y))
sage: om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
sage: om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
```

Parallel computation:

```
sage: Parallelism().set('tensor', nproc=2)
sage: om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
sage: om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
sage: Parallelism().set('tensor', nproc=1)  # switch off parallelization
```

Check of Cartan identity:

```
sage: om.lie_der(v) == (v.contract(0, om.exterior_derivative(), 0)
....:                   + om(v).exterior_derivative())
True
```

lie_derivative(*vector*)

Compute the Lie derivative with respect to a vector field.

INPUT:

- `vector` – vector field with respect to which the Lie derivative is to be taken

OUTPUT:

- the tensor field that is the Lie derivative of `self` with respect to `vector`

EXAMPLES:

Lie derivative of a vector:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: v = M.vector_field(-y, x, name='v')
sage: w = M.vector_field(2*x+y, x*y)
sage: w.lie_derivative(v)
Vector field on the 2-dimensional differentiable manifold M
sage: w.lie_derivative(v).display()
((x - 2)*y + x) d/dx + (x^2 - y^2 - 2*x - y) d/dy
```

The result is cached:

```
sage: w.lie_derivative(v) is w.lie_derivative(v)
True
```

An alias is `lie_der`:

```
sage: w.lie_der(v) is w.lie_derivative(v)
True
```

The Lie derivative is antisymmetric:

```
sage: w.lie_der(v) == -v.lie_der(w)
True
```

For vectors, it coincides with the commutator:

```
sage: f = M.scalar_field(x^3 + x*y^2)
sage: w.lie_der(v)(f).display()
M --> R
(x, y) |--> -(x + 2)*y^3 + 3*x^3 - x*y^2 + 5*(x^3 - 2*x^2)*y
sage: w.lie_der(v)(f) == v(w(f)) - w(v(f))   # rhs = commutator [v,w] acting␣
↪on f
True
```

Lie derivative of a 1-form:

```
sage: om = M.one_form(y^2*sin(x), x^3*cos(y))
sage: om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
sage: om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
```

Parallel computation:

```
sage: Parallelism().set('tensor', nproc=2)
sage: om.lie_der(v)
1-form on the 2-dimensional differentiable manifold M
sage: om.lie_der(v).display()
(-y^3*cos(x) + x^3*cos(y) + 2*x*y*sin(x)) dx
 + (-x^4*sin(y) - 3*x^2*y*cos(y) - y^2*sin(x)) dy
sage: Parallelism().set('tensor', nproc=1)  # switch off parallelization
```

Check of Cartan identity:

```
sage: om.lie_der(v) == (v.contract(0, om.exterior_derivative(), 0)
....:                   + om(v).exterior_derivative())
True
```

restrict (*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – *DifferentiableManifold*; open subset U of the tensor field domain S
- `dest_map` – *DiffMap* (default: `None`); destination map $\Psi : U \to V$, where V is an open subset of the manifold M where the tensor field takes it values; if `None`, the restriction of Φ to U is used, Φ being the differentiable map $S \to M$ associated with the tensor field

OUTPUT:

- instance of *TensorFieldParal* representing the restriction

EXAMPLES:

Restriction of a vector field defined on $\mathbf{R}^2$ to a disk:

```
sage: M = Manifold(2, 'R^2')
sage: c_cart.<x,y> = M.chart() # Cartesian coordinates on R^2
sage: v = M.vector_field(x+y, -1+x^2, name='v')
sage: D = M.open_subset('D') # the unit open disc
sage: c_cart_D = c_cart.restrict(D, x^2+y^2<1)
sage: v_D = v.restrict(D) ; v_D
Vector field v on the Open subset D of the 2-dimensional
 differentiable manifold R^2
sage: v_D.display()
v = (x + y) d/dx + (x^2 - 1) d/dy
```

The symbolic expressions of the components with respect to Cartesian coordinates are equal:

```
sage: bool( v_D[1].expr() == v[1].expr() )
True
```

but neither the chart functions representing the components (they are defined on different charts):

```
sage: v_D[1] == v[1]
False
```

nor the scalar fields representing the components (they are defined on different open subsets):

```
sage: v_D[[1]] == v[[1]]
False
```

The restriction of the vector field to its own domain is of course itself:

```
sage: v.restrict(M) is v
True
```

series_expansion (*symbol*, *order*)

Expand the tensor field in power series with respect to a small parameter.

If the small parameter is ϵ and T is `self`, the power series expansion to order n is

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n + O(\epsilon^{n+1}),$$

where $T_0, T_1, \ldots, T_n$ are $n + 1$ tensor fields of the same tensor type as `self` and do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the components of `self` are expanded in power series
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`

OUTPUT:

- list of the tensor fields T_i (size `order+1`)

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
sage: g = M.metric()
sage: h1 = M.tensor_field(0,2,sym=(0,1))
sage: h2 = M.tensor_field(0,2,sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h1[0, 1], h1[1, 2], h1[2, 3] = 1, 1, 1
sage: h2[0, 2], h2[1, 3] = 1, 1
sage: g.set(g + e*h1 + e^2*h2)
sage: g_ser = g.series_expansion(e, 2); g_ser
[Field of symmetric bilinear forms on the 4-dimensional Lorentzian manifold M,
 Field of symmetric bilinear forms on the 4-dimensional Lorentzian manifold M,
 Field of symmetric bilinear forms on the 4-dimensional Lorentzian manifold M]
sage: g_ser[0][:]
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]
sage: g_ser[1][:]
[0 1 0 0]
[1 0 1 0]
[0 1 0 1]
[0 0 1 0]
sage: g_ser[2][:]
[0 0 1 0]
[0 0 0 1]
[1 0 0 0]
[0 1 0 0]
sage: all([g_ser[1] == h1, g_ser[2] == h2])
True
```

set_calc_order(*symbol*, *order*, *truncate=False*)

Trigger a power series expansion with respect to a small parameter in computations involving the tensor field.

This property is propagated by usual operations. The internal representation must be `SR` for this to take effect.

If the small parameter is ϵ and T is `self`, the power series expansion to order n is

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n + O(\epsilon^{n+1}),$$

where $T_0, T_1, \ldots, T_n$ are $n + 1$ tensor fields of the same tensor type as `self` and do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the components of `self` are expanded in power series
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`
- `truncate` – (default: `False`) determines whether the components of `self` are replaced by their expansions to the given order

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
sage: g = M.metric()
sage: h1 = M.tensor_field(0, 2, sym=(0,1))
sage: h2 = M.tensor_field(0, 2, sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h1[0, 1], h1[1, 2], h1[2, 3] = 1, 1, 1
sage: h2[0, 2], h2[1, 3] = 1, 1
sage: g.set(g + e*h1 + e^2*h2)
sage: g.set_calc_order(e, 1)
sage: g[:]
[ -1    e e^2    0]
[  e    1    e e^2]
[e^2    e    1    e]
[  0 e^2    e    1]
sage: g.set_calc_order(e, 1, truncate=True)
sage: g[:]
[-1  e  0  0]
[ e  1  e  0]
[ 0  e  1  e]
[ 0  0  e  1]
```

set_comp(*basis=None*)

Return the components of the tensor field in a given vector frame for assignment.

The components with respect to other frames on the same domain are deleted, in order to avoid any inconsistency. To keep them, use the method *add_comp()* instead.

INPUT:

- `basis` – (default: `None`) vector frame in which the components are defined; if none is provided, the components are assumed to refer to the tensor field domain's default frame

OUTPUT:

- components in the given frame, as an instance of the class `Components`; if such components did not exist previously, they are created

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: e_xy = X.frame()
sage: t = M.tensor_field(1,1, name='t')
sage: t.set_comp(e_xy)
```

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```
2-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy))
sage: t.set_comp(e_xy)[1,0] = 2
sage: t.display(e_xy)
t = 2 d/dy*dx
```

Setting components in a new frame (e):

```
sage: e = M.vector_frame('e')
sage: t.set_comp(e)
2-indices components w.r.t. Vector frame (M, (e_0,e_1))
sage: t.set_comp(e)[0,1] = x
sage: t.display(e)
t = x e_0*e^1
```

The components with respect to the frame `e_xy` have be erased:

```
sage: t.display(e_xy)
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Coordinate frame (M, (d/dx,d/dy))
```

Setting components in a frame defined on a subdomain deletes previously defined components as well:

```
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: f = U.vector_frame('f')
sage: t.set_comp(f)
2-indices components w.r.t. Vector frame (U, (f_0,f_1))
sage: t.set_comp(f)[0,1] = 1+y
sage: t.display(f)
t = (y + 1) f_0*f^1
sage: t.display(e)
Traceback (most recent call last):
...
ValueError: no basis could be found for computing the components
 in the Vector frame (M, (e_0,e_1))
```

truncate(*symbol*, *order*)

Return the tensor field truncated at a given order in the power series expansion with respect to some small parameter.

If the small parameter is ϵ and T is `self`, the power series expansion to order n is

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n + O(\epsilon^{n+1}),$$

where $T_0, T_1, \ldots, T_n$ are $n + 1$ tensor fields of the same tensor type as `self` and do not depend upon ϵ.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the components of `self` are expanded in power series
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`

OUTPUT:

- the tensor field $T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots + \epsilon^n T_n$

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
sage: g = M.metric()
sage: h1 = M.tensor_field(0,2,sym=(0,1))
sage: h2 = M.tensor_field(0,2,sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h1[0, 1], h1[1, 2], h1[2, 3] = 1, 1, 1
sage: h2[0, 2], h2[1, 3] = 1, 1
sage: g.set(g + e*h1 + e^2*h2)
sage: g[:]
[ -1   e e^2   0]
[  e   1   e e^2]
[e^2   e   1   e]
[  0 e^2   e   1]
sage: g.truncate(e, 1)[:]
[-1  e  0  0]
[ e  1  e  0]
[ 0  e  1  e]
[ 0  0  e  1]
```

2.9 Differential Forms

2.9.1 Differential Form Modules

The set $\Omega^p(U,\Phi)$ of p-forms along a differentiable manifold U with values on a differentiable manifold M via a differentiable map $\Phi: U \to M$ (possibly $U = M$ and $\Phi = \mathrm{Id}_M$) is a module over the algebra $C^k(U)$ of differentiable scalar fields on U. It is a free module if and only if M is parallelizable. Accordingly, two classes implement $\Omega^p(U,\Phi)$:

- `DiffFormModule` for differential forms with values on a generic (in practice, not parallelizable) differentiable manifold M
- `DiffFormFreeModule` for differential forms with values on a parallelizable manifold M

AUTHORS:

- Eric Gourgoulhon (2015): initial version
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [?]
- [?]

class sage.manifolds.differentiable.diff_form_module.**DiffFormFreeModule**(*vector_field_module*, *degree*)

Bases: `sage.tensor.modules.ext_pow_free_module.ExtPowerDualFreeModule`

Free module of differential forms of a given degree p (p-forms) along a differentiable manifold U with values on a parallelizable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi: U \to M$ to a parallelizable manifold M of dimension n, the set $\Omega^p(U,\Phi)$ of p-forms along U with values on M is a free module of rank $\binom{n}{p}$ over $C^k(U)$, the commutative algebra of differentiable scalar fields on U (see `DiffScalarFieldAlgebra`).

The standard case of p-forms *on* a differentiable manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: This class implements $\Omega^p(U, \Phi)$ in the case where M is parallelizable; $\Omega^p(U, \Phi)$ is then a *free* module. If M is not parallelizable, the class `DiffFormModule` must be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \rightarrow V$
- `degree` – positive integer; the degree p of the differential forms

EXAMPLES:

Free module of 2-forms on a parallelizable 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
sage: A = M.diff_form_module(2) ; A
Free module Omega^2(M) of 2-forms on the 3-dimensional differentiable
 manifold M
sage: latex(A)
\Omega^{2}\left(M\right)
```

`A` is nothing but the second exterior power of the dual of `XM`, i.e. we have $\Omega^2(M) = \Lambda^2(\mathfrak{X}(M)^*)$ (see `ExtPowerDualFreeModule`):

```
sage: A is XM.dual_exterior_power(2)
True
```

$\Omega^2(M)$ is a module over the algebra $C^k(M)$ of (differentiable) scalar fields on M:

```
sage: A.category()
Category of finite dimensional modules over Algebra of differentiable
 scalar fields on the 3-dimensional differentiable manifold M
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: A in Modules(CM)
True
sage: A.base_ring()
Algebra of differentiable scalar fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module()
Free module X(M) of vector fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module() is XM
True
sage: A.rank()
3
```

Elements can be constructed from A. In particular, `0` yields the zero element of A:

```
sage: A(0)
2-form zero on the 3-dimensional differentiable manifold M
sage: A(0) is A.zero()
True
```

while non-zero elements are constructed by providing their components in a given vector frame:

```
sage: comp = [[0,3*x,-z],[-3*x,0,4],[z,-4,0]]
sage: a = A(comp, frame=X.frame(), name='a') ; a
2-form a on the 3-dimensional differentiable manifold M
sage: a.display()
a = 3*x dx/\dy - z dx/\dz + 4 dy/\dz
```

An alternative is to construct the 2-form from an empty list of components and to set the nonzero nonredundant components afterwards:

```
sage: a = A([], name='a')
sage: a[0,1] = 3*x  # component in the manifold's default frame
sage: a[0,2] = -z
sage: a[1,2] = 4
sage: a.display()
a = 3*x dx/\dy - z dx/\dz + 4 dy/\dz
```

The module $\Omega^1(M)$ is nothing but the dual of $\mathfrak{X}(M)$ (the free module of vector fields on M):

```
sage: L1 = M.diff_form_module(1) ; L1
Free module Omega^1(M) of 1-forms on the 3-dimensional differentiable
 manifold M
sage: L1 is XM.dual()
True
```

Since any tensor field of type $(0,1)$ is a 1-form, there is a coercion map from the set $T^{(0,1)}(M)$ of such tensors to $\Omega^1(M)$:

```
sage: T01 = M.tensor_field_module((0,1)) ; T01
Free module T^(0,1)(M) of type-(0,1) tensors fields on the
 3-dimensional differentiable manifold M
sage: L1.has_coerce_map_from(T01)
True
```

There is also a coercion map in the reverse direction:

```
sage: T01.has_coerce_map_from(L1)
True
```

For a degree $p \geq 2$, the coercion holds only in the direction $\Omega^p(M) \to T^{(0,p)}(M)$:

```
sage: T02 = M.tensor_field_module((0,2)); T02
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 3-dimensional differentiable manifold M
sage: T02.has_coerce_map_from(A)
True
sage: A.has_coerce_map_from(T02)
False
```

The coercion map $T^{(0,1)}(M) \to \Omega^1(M)$ in action:

```
sage: b = T01([-x,2,3*y], name='b'); b
Tensor field b of type (0,1) on the 3-dimensional differentiable
 manifold M
sage: b.display()
b = -x dx + 2 dy + 3*y dz
sage: lb = L1(b) ; lb
1-form b on the 3-dimensional differentiable manifold M
sage: lb.display()
b = -x dx + 2 dy + 3*y dz
```

The coercion map $\Omega^1(M) \to T^{(0,1)}(M)$ in action:

```
sage: tlb = T01(lb); tlb
Tensor field b of type (0,1) on
 the 3-dimensional differentiable manifold M
sage: tlb == b
True
```

The coercion map $\Omega^2(M) \to T^{(0,2)}(M)$ in action:

```
sage: T02 = M.tensor_field_module((0,2)) ; T02
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 3-dimensional differentiable manifold M
sage: ta = T02(a) ; ta
Tensor field a of type (0,2) on the 3-dimensional differentiable
 manifold M
sage: ta.display()
a = 3*x dx*dy - z dx*dz - 3*x dy*dx + 4 dy*dz + z dz*dx - 4 dz*dy
sage: a.display()
a = 3*x dx/\dy - z dx/\dz + 4 dy/\dz
sage: ta.symmetries()  # the antisymmetry is preserved
no symmetry;  antisymmetry: (0, 1)
```

There is also coercion to subdomains, which is nothing but the restriction of the differential form to some subset of its domain:

```
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1})
sage: B = U.diff_form_module(2) ; B
Free module Omega^2(U) of 2-forms on the Open subset U of the
 3-dimensional differentiable manifold M
sage: B.has_coerce_map_from(A)
True
sage: a_U = B(a) ; a_U
2-form a on the Open subset U of the 3-dimensional differentiable
 manifold M
sage: a_U.display()
a = 3*x dx/\dy - z dx/\dz + 4 dy/\dz
```

Element
alias of *sage.manifolds.differentiable.diff_form.DiffFormParal*

class sage.manifolds.differentiable.diff_form_module.**DiffFormModule**(*vector_field_module*, *degree*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

Module of differential forms of a given degree p (p-forms) along a differentiable manifold U with values on a differentiable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M, the set $\Omega^p(U,\Phi)$ of p-forms along U with values on M is a module over $C^k(U)$, the commutative algebra of differentiable scalar fields on U (see *DiffScalarFieldAlgebra*). The standard case of p-forms *on* a differentiable manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: This class implements $\Omega^p(U,\Phi)$ in the case where M is not assumed to be parallelizable; the module $\Omega^p(U,\Phi)$ is then not necessarily free. If M is parallelizable, the class *DiffFormFreeModule* must be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U,\Phi)$ of vector fields along U with values on M via the map $\Phi : U \to M$
- `degree` – positive integer; the degree p of the differential forms

EXAMPLES:

Module of 2-forms on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:  intersection_name='W', restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A = M.diff_form_module(2) ; A
Module Omega^2(M) of 2-forms on the 2-dimensional differentiable
 manifold M
sage: latex(A)
\Omega^{2}\left(M\right)
```

`A` is nothing but the second exterior power of the dual of `XM`, i.e. we have $\Omega^2(M) = \Lambda^2(\mathfrak{X}(M)^*)$:

```
sage: A is XM.dual_exterior_power(2)
True
```

Modules of differential forms are unique:

```
sage: A is M.diff_form_module(2)
True
```

$\Omega^2(M)$ is a module over the algebra $C^k(M)$ of (differentiable) scalar fields on M:

```
sage: A.category()
Category of modules over Algebra of differentiable scalar fields on
 the 2-dimensional differentiable manifold M
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
```

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```
sage: A in Modules(CM)
True
sage: A.base_ring() is CM
True
sage: A.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A.base_module() is XM
True
```

Elements can be constructed from `A()`. In particular, `0` yields the zero element of `A`:

```
sage: z = A(0) ; z
2-form zero on the 2-dimensional differentiable manifold M
sage: z.display(eU)
zero = 0
sage: z.display(eV)
zero = 0
sage: z is A.zero()
True
```

while non-zero elements are constructed by providing their components in a given vector frame:

```
sage: a = A([[0,3*x],[-3*x,0]], frame=eU, name='a') ; a
2-form a on the 2-dimensional differentiable manifold M
sage: a.add_comp_by_continuation(eV, W, c_uv) # finishes initializ. of a
sage: a.display(eU)
a = 3*x dx/\dy
sage: a.display(eV)
a = (-3/4*u - 3/4*v) du/\dv
```

An alternative is to construct the 2-form from an empty list of components and to set the nonzero nonredundant components afterwards:

```
sage: a = A([], name='a')
sage: a[eU,0,1] = 3*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = 3*x dx/\dy
sage: a.display(eV)
a = (-3/4*u - 3/4*v) du/\dv
```

The module $\Omega^1(M)$ is nothing but the dual of $\mathfrak{X}(M)$ (the module of vector fields on M):

```
sage: L1 = M.diff_form_module(1) ; L1
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
sage: L1 is XM.dual()
True
```

Since any tensor field of type $(0, 1)$ is a 1-form, there is a coercion map from the set $T^{(0,1)}(M)$ of such tensors to $\Omega^1(M)$:

```
sage: T01 = M.tensor_field_module((0,1)) ; T01
Module T^(0,1)(M) of type-(0,1) tensors fields on the 2-dimensional
 differentiable manifold M
```

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```
sage: L1.has_coerce_map_from(T01)
True
```

There is also a coercion map in the reverse direction:

```
sage: T01.has_coerce_map_from(L1)
True
```

For a degree $p \geq 2$, the coercion holds only in the direction $\Omega^p(M) \to T^{(0,p)}(M)$:

```
sage: T02 = M.tensor_field_module((0,2)) ; T02
Module T^(0,2)(M) of type-(0,2) tensors fields on the 2-dimensional
 differentiable manifold M
sage: T02.has_coerce_map_from(A)
True
sage: A.has_coerce_map_from(T02)
False
```

The coercion map $T^{(0,1)}(M) \to \Omega^1(M)$ in action:

```
sage: b = T01([y,x], frame=eU, name='b') ; b
Tensor field b of type (0,1) on the 2-dimensional differentiable
 manifold M
sage: b.add_comp_by_continuation(eV, W, c_uv)
sage: b.display(eU)
b = y dx + x dy
sage: b.display(eV)
b = 1/2*u du - 1/2*v dv
sage: lb = L1(b) ; lb
1-form b on the 2-dimensional differentiable manifold M
sage: lb.display(eU)
b = y dx + x dy
sage: lb.display(eV)
b = 1/2*u du - 1/2*v dv
```

The coercion map $\Omega^1(M) \to T^{(0,1)}(M)$ in action:

```
sage: tlb = T01(lb) ; tlb
Tensor field b of type (0,1) on the 2-dimensional differentiable
 manifold M
sage: tlb.display(eU)
b = y dx + x dy
sage: tlb.display(eV)
b = 1/2*u du - 1/2*v dv
sage: tlb == b
True
```

The coercion map $\Omega^2(M) \to T^{(0,2)}(M)$ in action:

```
sage: ta = T02(a) ; ta
Tensor field a of type (0,2) on the 2-dimensional differentiable
 manifold M
sage: ta.display(eU)
a = 3*x dx*dy - 3*x dy*dx
sage: a.display(eU)
a = 3*x dx/\dy
```

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```
sage: ta.display(eV)
a = (-3/4*u - 3/4*v) du*dv + (3/4*u + 3/4*v) dv*du
sage: a.display(eV)
a = (-3/4*u - 3/4*v) du/\dv
```

There is also coercion to subdomains, which is nothing but the restriction of the differential form to some subset of its domain:

```
sage: L2U = U.diff_form_module(2) ; L2U
Free module Omega^2(U) of 2-forms on the Open subset U of the
 2-dimensional differentiable manifold M
sage: L2U.has_coerce_map_from(A)
True
sage: a_U = L2U(a) ; a_U
2-form a on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: a_U.display(eU)
a = 3*x dx/\dy
```

Element
alias of *sage.manifolds.differentiable.diff_form.DiffForm*

base_module()
Return the vector field module on which the differential form module `self` is constructed.

OUTPUT:

- a *VectorFieldModule* representing the module on which `self` is defined

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A2 = M.diff_form_module(2) ; A2
Module Omega^2(M) of 2-forms on the 3-dimensional differentiable
 manifold M
sage: A2.base_module()
Module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
sage: A2.base_module() is M.vector_field_module()
True
sage: U = M.open_subset('U')
sage: A2U = U.diff_form_module(2) ; A2U
Module Omega^2(U) of 2-forms on the Open subset U of the
 3-dimensional differentiable manifold M
sage: A2U.base_module()
Module X(U) of vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
```

degree()
Return the degree of the differential forms in `self`.

OUTPUT:

- integer p such that `self` is a set of p-forms

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: M.diff_form_module(1).degree()
```

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```
1
sage: M.diff_form_module(2).degree()
2
sage: M.diff_form_module(3).degree()
3
```

zero()

Return the zero of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A2 = M.diff_form_module(2)
sage: A2.zero()
2-form zero on the 3-dimensional differentiable manifold M
```

2.9.2 Differential Forms

Let U and M be two differentiable manifolds. Given a positive integer p and a differentiable map $\Phi : U \to M$, a *differential form of degree* p, or p*-form*, *along* U *with values on* M is a field along U of alternating multilinear forms of degree p in the tangent spaces to M. The standard case of a differential form *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Two classes implement differential forms, depending whether the manifold M is parallelizable:

- `DiffFormParal` when M is parallelizable
- `DiffForm` when M is not assumed parallelizable.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013, 2014): initial version
- Joris Vankerschaver (2010): developed a previous class, `DifferentialForm` (cf. trac ticket #24444), which inspired the storage of the non-zero components as a dictionary whose keys are the indices.
- Travis Scrimshaw (2016): review tweaks

REFERENCES:

- [?]
- [?]

class sage.manifolds.differentiable.diff_form.**DiffForm**(*vector_field_module*, *degree*, *name=None*, *latex_name=None*)

Bases: `sage.manifolds.differentiable.tensorfield.TensorField`

Differential form with values on a generic (i.e. a priori not parallelizable) differentiable manifold.

Given a differentiable manifold U, a differentiable map $\Phi : U \to M$ to a differentiable manifold M and a positive integer p, a *differential form of degree* p (or p*-form*) *along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(0,p)}M$$

($T^{(0,p)}M$ being the tensor bundle of type $(0, p)$ over M) such that

$$\forall x \in U, \quad a(x) \in \Lambda^p(T^*_{\Phi(x)}M),$$

where $T^*_{\Phi(x)}M$ is the dual of the tangent space to M at $\Phi(x)$ and Λ^p stands for the exterior power of degree p (cf. ExtPowerDualFreeModule). In other words, $a(x)$ is an alternating multilinear form of degree p of the tangent vector space $T_{\Phi(x)}M$.

The standard case of a differential form *on* a manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is parallelizable, the class *DiffFormParal* must be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map Φ
- `degree` – the degree of the differential form (i.e. its tensor rank)
- `name` – (default: `None`) name given to the differential form
- `latex_name` – (default: `None`) LaTeX symbol to denote the differential form; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

Differential form of degree 2 on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                                  restrictions1= x>0, restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.diff_form(2, name='a') ; a
2-form a on the 2-dimensional differentiable manifold M
sage: a.parent()
Module Omega^2(M) of 2-forms on the 2-dimensional differentiable
 manifold M
sage: a.degree()
2
```

Setting the components of `a`:

```
sage: a[eU,0,1] = x*y^2 + 2*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = (x*y^2 + 2*x) dx/\dy
sage: a.display(eV)
a = (-1/16*u^3 + 1/16*u*v^2 - 1/16*v^3
 + 1/16*(u^2 - 8)*v - 1/2*u) du/\dv
```

A 1-form on `M`:

```
sage: a = M.one_form(name='a') ; a
1-form a on the 2-dimensional differentiable manifold M
sage: a.parent()
Module Omega^1(M) of 1-forms on the 2-dimensional differentiable
 manifold M
```

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```
sage: a.degree()
1
```

Setting the components of the 1-form in a consistent way:

```
sage: a[eU,:] = [-y, x]
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y dx + x dy
sage: a.display(eV)
a = 1/2*v du - 1/2*u dv
```

It is also possible to set the components at the 1-form definition, via a dictionary whose keys are the vector frames:

```
sage: a1 = M.one_form({eU: [-y, x], eV: [v/2, -u/2]}, name='a')
sage: a1 == a
True
```

The exterior derivative of the 1-form is a 2-form:

```
sage: da = a.exterior_derivative() ; da
2-form da on the 2-dimensional differentiable manifold M
sage: da.display(eU)
da = 2 dx/\dy
sage: da.display(eV)
da = -du/\dv
```

Another 1-form defined by its components in `eU`:

```
sage: b = M.one_form(1+x*y, x^2, frame=eU, name='b')
```

Since `eU` is the default vector frame on `M`, it can be omitted in the definition:

```
sage: b = M.one_form(1+x*y, x^2, name='b')
sage: b.add_comp_by_continuation(eV, W, c_uv)
```

Adding two 1-forms results in another 1-form:

```
sage: s = a + b ; s
1-form a+b on the 2-dimensional differentiable manifold M
sage: s.display(eU)
a+b = ((x - 1)*y + 1) dx + (x^2 + x) dy
sage: s.display(eV)
a+b = (1/4*u^2 + 1/4*(u + 2)*v + 1/2) du
 + (-1/4*u*v - 1/4*v^2 - 1/2*u + 1/2) dv
```

The exterior product of two 1-forms is a 2-form:

```
sage: s = a.wedge(b) ; s
2-form a/\b on the 2-dimensional differentiable manifold M
sage: s.display(eU)
a/\b = (-2*x^2*y - x) dx/\dy
sage: s.display(eV)
a/\b = (1/8*u^3 - 1/8*u*v^2 - 1/8*v^3 + 1/8*(u^2 + 2)*v + 1/4*u) du/\dv
```

Multiplying a 1-form by a scalar field results in another 1-form:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_uv: u^2}, name='f')
sage: s = f*a ; s
1-form on the 2-dimensional differentiable manifold M
sage: s.display(eU)
(-x^2*y - 2*x*y^2 - y^3) dx + (x^3 + 2*x^2*y + x*y^2) dy
sage: s.display(eV)
1/2*u^2*v du - 1/2*u^3 dv
```

Examples with SymPy as the symbolic engine

From now on, we ask that all symbolic calculus on manifold M are performed by SymPy:

```
sage: M.set_calculus_method('sympy')
```

We define a 2-form a as above:

```
sage: a = M.diff_form(2, name='a')
sage: a[eU,0,1] = x*y^2 + 2*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = (x*y**2 + 2*x) dx/\dy
sage: a.display(eV)
a = (-u**3/16 + u**2*v/16 + u*v**2/16 - u/2 - v**3/16 - v/2) du/\dv
```

A 1-form on `M`:

```
sage: a = M.one_form(-y, x, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y dx + x dy
sage: a.display(eV)
a = v/2 du - u/2 dv
```

The exterior derivative of `a`:

```
sage: da = a.exterior_derivative()
sage: da.display(eU)
da = 2 dx/\dy
sage: da.display(eV)
da = -du/\dv
```

Another 1-form:

```
sage: b = M.one_form(1+x*y, x^2, name='b')
sage: b.add_comp_by_continuation(eV, W, c_uv)
```

Adding two 1-forms:

```
sage: s = a + b
sage: s.display(eU)
a+b = (x*y - y + 1) dx + x*(x + 1) dy
sage: s.display(eV)
a+b = (u**2/4 + u*v/4 + v/2 + 1/2) du + (-u*v/4 - u/2 - v**2/4 + 1/2) dv
```

The exterior product of two 1-forms:

```
sage: s = a.wedge(b)
sage: s.display(eU)
a/\b = -x*(2*x*y + 1) dx/\dy
sage: s.display(eV)
a/\b = (u**3/8 + u**2*v/8 - u*v**2/8 + u/4 - v**3/8 + v/4) du/\dv
```

Multiplying a 1-form by a scalar field:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_uv: u^2}, name='f')
sage: s = f*a
sage: s.display(eU)
-y*(x**2 + 2*x*y + y**2) dx + x*(x**2 + 2*x*y + y**2) dy
sage: s.display(eV)
u**2*v/2 du - u**3/2 dv
```

degree()

Return the degree of `self`.

OUTPUT:

- integer p such that the differential form is a p-form

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: a = M.diff_form(2); a
2-form on the 3-dimensional differentiable manifold M
sage: a.degree()
2
sage: b = M.diff_form(1); b
1-form on the 3-dimensional differentiable manifold M
sage: b.degree()
1
```

exterior_derivative()

Compute the exterior derivative of `self`.

OUTPUT:

- instance of *DiffForm* representing the exterior derivative of the differential form

EXAMPLES:

Exterior derivative of a 1-form on the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

The 1-form:

```
sage: a = M.one_form({e_xy: [-y^2, x^2]}, name='a')
sage: a.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.display(e_xy)
a = -y^2 dx + x^2 dy
sage: a.display(e_uv)
a = -(2*u^3*v - u^2*v^2 + v^4)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^
↪8) du
 + (u^4 - u^2*v^2 + 2*u*v^3)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8)␣
↪dv
```

Its exterior derivative:

```
sage: da = a.exterior_derivative(); da
2-form da on the 2-dimensional differentiable manifold M
sage: da.display(e_xy)
da = (2*x + 2*y) dx/\dy
sage: da.display(e_uv)
da = -2*(u + v)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) du/\dv
```

The result is cached, i.e. is not recomputed unless `a` is changed:

```
sage: a.exterior_derivative() is da
True
```

Instead of invoking the method `exterior_derivative()`, one may use the global function `exterior_derivative()` or its alias `xder()`:

```
sage: from sage.manifolds.utilities import xder
sage: xder(a) is a.exterior_derivative()
True
```

Let us check Cartan's identity:

```
sage: v = M.vector_field({e_xy: [-y, x]}, name='v')
sage: v.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: a.lie_der(v) == v.contract(xder(a)) + xder(a(v))   # long time
True
```

hodge_dual (*metric*)

Compute the Hodge dual of the differential form with respect to some metric.

If the differential form is a p-form A, its *Hodge dual* with respect to a pseudo-Riemannian metric g is the $(n-p)$-form $*A$ defined by

$$*A_{i_1\ldots i_{n-p}} = \frac{1}{p!} A_{k_1\ldots k_p} \epsilon^{k_1\ldots k_p}{}_{i_1\ldots i_{n-p}}$$

where n is the manifold's dimension, ϵ is the volume n-form associated with g (see `volume_form()`) and the indices $k_1, \ldots, k_p$ are raised with g.

INPUT:

- `metric`: a pseudo-Riemannian metric defined on the same manifold as the current differential form; must be an instance of `PseudoRiemannianMetric`

OUTPUT:

- the $(n-p)$-form $*A$

EXAMPLES:

Hodge dual of a 1-form on the 2-sphere equipped with the standard metric: we first construct $\mathbb{S}^2$ and its metric g:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic coord.␣
↪(North and South)
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                  intersection_name='W', restrictions1= x^2+y^2!=0,
....:                  restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: g = M.metric('g')
sage: g[eU,1,1], g[eU,2,2] = 4/(1+x^2+y^2)^2, 4/(1+x^2+y^2)^2
sage: g[eV,1,1], g[eV,2,2] = 4/(1+u^2+v^2)^2, 4/(1+u^2+v^2)^2
```

Then we construct the 1-form and take its Hodge dual w.r.t. g:

```
sage: a = M.one_form({eU: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y dx + x dy
sage: a.display(eV)
a = -v/(u^4 + 2*u^2*v^2 + v^4) du + u/(u^4 + 2*u^2*v^2 + v^4) dv
sage: sa = a.hodge_dual(g); sa
1-form *a on the 2-dimensional differentiable manifold S^2
sage: sa.display(eU)
*a = -x dx - y dy
sage: sa.display(eV)
*a = -u/(u^4 + 2*u^2*v^2 + v^4) du - v/(u^4 + 2*u^2*v^2 + v^4) dv
```

Instead of calling the method `hodge_dual()` on the differential form, one can invoke the method `hodge_star()` of the metric:

```
sage: a.hodge_dual(g) == g.hodge_star(a)
True
```

For a 1-form and a Riemannian metric in dimension 2, the Hodge dual applied twice is minus the identity:

```
sage: ssa = sa.hodge_dual(g); ssa
1-form **a on the 2-dimensional differentiable manifold S^2
sage: ssa == -a
True
```

The Hodge dual of the metric volume 2-form is the constant scalar field 1 (considered as a 0-form):

```
sage: eps = g.volume_form(); eps
2-form eps_g on the 2-dimensional differentiable manifold S^2
sage: eps.display(eU)
eps_g = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx/\dy
sage: eps.display(eV)
eps_g = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du/\dv
sage: seps = eps.hodge_dual(g); seps
Scalar field *eps_g on the 2-dimensional differentiable manifold S^2
```

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```
sage: seps.display()
*eps_g: S^2 --> R
on U: (x, y) |--> 1
on V: (u, v) |--> 1
```

interior_product (*qvect*)

Interior product with a multivector field.

If `self` is a differential form A of degree p and B is a multivector field of degree $q \geq p$ on the same manifold, the interior product of A by B is the multivector field $\iota_A B$ of degree $q - p$ defined by

$$(\iota_A B)^{i_1 \ldots i_{q-p}} = A_{k_1 \ldots k_p} B^{k_1 \ldots k_p i_1 \ldots i_{q-p}}$$

Note: `A.interior_product(B)` yields the same result as `A.contract(0, ..., p-1, B, 0, ..., p-1)` (cf. *contract()*), but `interior_product` is more efficient, the alternating character of A being not used to reduce the computation in *contract()*

INPUT:

- `qvect` – multivector field B (instance of *MultivectorField*); the degree of B must be at least equal to the degree of `self`

OUTPUT:

- scalar field (case $p = q$) or *MultivectorField* (case $p < q$) representing the interior product $\iota_A B$, where A is `self`

See also:

interior_product() for the interior product of a multivector field with a differential form

EXAMPLES:

Interior product of a 1-form with a 2-vector field on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() # stereographic coord. North
sage: c_uv.<u,v> = V.chart() # stereographic coord. South
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W', restrictions1= x^2+y^2!=0,
....:                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.one_form({e_xy: [y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: b = M.multivector_field(2, name='b')
sage: b[e_xy,1,2] = x*y
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: s = a.interior_product(b); s
Vector field i_a b on the 2-dimensional differentiable manifold S^2
sage: s.display(e_xy)
i_a b = -x^2*y d/dx + x*y^2 d/dy
sage: s.display(e_uv)
```

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```
i_a b = (u^4*v - 3*u^2*v^3)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) d/du
 + (3*u^3*v^2 - u*v^4)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) d/dv
sage: s == a.contract(b)
True
```

Interior product of a 2-form with a 2-vector field:

```
sage: a = M.diff_form(2, name='a')
sage: a[e_xy,1,2] = 4/(x^2+y^2+1)^2    # the standard area 2-form
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: s = a.interior_product(b); s
Scalar field i_a b on the 2-dimensional differentiable manifold S^2
sage: s.display()
i_a b: S^2 --> R
on U: (x, y) |--> 8*x*y/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) |--> 8*u*v/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
```

Some checks:

```
sage: s == a.contract(0, 1, b, 0, 1)
True
sage: s.restrict(U) == 2 * a[[e_xy,1,2]] * b[[e_xy,1,2]]
True
sage: s.restrict(V) == 2 * a[[e_uv,1,2]] * b[[e_uv,1,2]]
True
```

wedge (*other*)

Exterior product with another differential form.

INPUT:

- `other` – another differential form (on the same manifold)

OUTPUT:

- instance of *DiffForm* representing the exterior product `self/\other`

EXAMPLES:

Exterior product of two 1-forms on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic coord.
↪(North and South)
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.one_form({e_xy: [y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: b = M.one_form({e_xy: [x^2 + y^2, y]}, name='b')
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: c = a.wedge(b); c
2-form a/\b on the 2-dimensional differentiable manifold S^2
sage: c.display(e_xy)
```

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```
a/\b = (-x^3 - (x - 1)*y^2) dx/\dy
sage: c.display(e_uv)
a/\b = -(v^2 - u)/(u^8 + 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8) du/\dv
```

If one of the two operands is unnamed, the result is unnamed too:

```
sage: b1 = M.diff_form(1)  # no name set
sage: b1[e_xy,:] = x^2 + y^2, y
sage: b1.add_comp_by_continuation(e_uv, W, c_uv)
sage: c1 = a.wedge(b1); c1
2-form on the 2-dimensional differentiable manifold S^2
sage: c1.display(e_xy)
(-x^3 - (x - 1)*y^2) dx/\dy
```

To give a name to the result, one shall use the method `set_name()`:

```
sage: c1.set_name('c');  c1
2-form c on the 2-dimensional differentiable manifold S^2
sage: c1.display(e_xy)
c = (-x^3 - (x - 1)*y^2) dx/\dy
```

class sage.manifolds.differentiable.diff_form.**DiffFormParal**(*vector_field_module*, *degree*, *name=None*, *latex_name=None*)

Bases: `sage.tensor.modules.free_module_alt_form.FreeModuleAltForm`, `sage.manifolds.differentiable.tensorfield_paral.TensorFieldParal`

Differential form with values on a parallelizable manifold.

Given a differentiable manifold U, a differentiable map $\Phi : U \to M$ to a parallelizable manifold M and a positive integer p, a *differential form of degree* p (or p*-form*) *along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(0,p)}M$$

($T^{(0,p)}M$ being the tensor bundle of type $(0,p)$ over M) such that

$$\forall x \in U, \quad a(x) \in \Lambda^p(T^*_{\Phi(x)}M),$$

where $T^*_{\Phi(x)}M$ is the dual of the tangent space to M at $\Phi(x)$ and Λ^p stands for the exterior power of degree p (cf. `ExtPowerDualFreeModule`). In other words, $a(x)$ is an alternating multilinear form of degree p of the tangent vector space $T_{\Phi(x)}M$.

The standard case of a differential form *on* a manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is not parallelizable, the class `DiffForm` must be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U,\Phi)$ of vector fields along U with values on M via the map Φ
- `degree` – the degree of the differential form (i.e. its tensor rank)
- `name` – (default: `None`) name given to the differential form

- `latex_name` – (default: `None`) LaTeX symbol to denote the differential form; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A 2-form on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: c_txyz.<t,x,y,z> = M.chart()
sage: a = M.diff_form(2, name='a') ; a
2-form a on the 4-dimensional differentiable manifold M
sage: a.parent()
Free module Omega^2(M) of 2-forms on the 4-dimensional differentiable
 manifold M
```

A differential form is a tensor field of purely covariant type:

```
sage: a.tensor_type()
(0, 2)
```

It is antisymmetric, its components being `CompFullyAntiSym`:

```
sage: a.symmetries()
no symmetry;  antisymmetry: (0, 1)
sage: a[0,1] = 2
sage: a[1,0]
-2
sage: a.comp()
Fully antisymmetric 2-indices components w.r.t. Coordinate frame (M, (d/dt,d/dx,d/
↪dy,d/dz))
sage: type(a.comp())
<class 'sage.tensor.modules.comp.CompFullyAntiSym'>
```

Setting a component with repeated indices to a non-zero value results in an error:

```
sage: a[1,1] = 3
Traceback (most recent call last):
...
ValueError: by antisymmetry, the component cannot have a nonzero value
 for the indices (1, 1)
sage: a[1,1] = 0   # OK, albeit useless
sage: a[1,2] = 3   # OK
```

The expansion of a differential form with respect to a given coframe is displayed via the method `display()`:

```
sage: a.display() # expansion with respect to the default coframe (dt, dx, dy, dz)
a = 2 dt/\dx + 3 dx/\dy
sage: latex(a.display()) # output for the notebook
a = 2 \mathrm{d} t\wedge \mathrm{d} x
 + 3 \mathrm{d} x\wedge \mathrm{d} y
```

Differential forms can be added or subtracted:

```
sage: b = M.diff_form(2)
sage: b[0,1], b[0,2], b[0,3] = (1,2,3)
sage: s = a + b ; s
2-form on the 4-dimensional differentiable manifold M
sage: a[:], b[:], s[:]
```

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```
(
[ 0  2  0  0]  [ 0  1  2  3]  [ 0  3  2  3]
[-2  0  3  0]  [-1  0  0  0]  [-3  0  3  0]
[ 0 -3  0  0]  [-2  0  0  0]  [-2 -3  0  0]
[ 0  0  0  0], [-3  0  0  0], [-3  0  0  0]
)
sage: s = a - b ; s
2-form on the 4-dimensional differentiable manifold M
sage: s[:]
[ 0  1 -2 -3]
[-1  0  3  0]
[ 2 -3  0  0]
[ 3  0  0  0]
```

An example of 3-form is the volume element on $\mathbf{R}^3$ in Cartesian coordinates:

```
sage: M = Manifold(3, 'R3', latex_name=r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = M.chart()
sage: eps = M.diff_form(3, name='epsilon', latex_name=r'\epsilon')
sage: eps[1,2,3] = 1  # the only independent component
sage: eps[:] # all the components are set from the previous line:
[[[0, 0, 0], [0, 0, 1], [0, -1, 0]], [[0, 0, -1], [0, 0, 0], [1, 0, 0]],
 [[0, 1, 0], [-1, 0, 0], [0, 0, 0]]]
sage: eps.display()
epsilon = dx/\dy/\dz
```

Spherical components of the volume element from the tensorial change-of-frame formula:

```
sage: c_spher.<r,th,ph> = M.chart(r'r:[0,+oo) th:[0,pi]:\theta ph:[0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart,
....:                 [r*sin(th)*cos(ph), r*sin(th)*sin(ph), r*cos(th)])
sage: cart_to_spher = spher_to_cart.set_inverse(sqrt(x^2+y^2+z^2),
....:                             atan2(sqrt(x^2+y^2),z), atan2(y, x))
sage: eps.comp(c_spher.frame()) # computation of the components in the spherical␣
↪frame
Fully antisymmetric 3-indices components w.r.t. Coordinate frame
 (R3, (d/dr,d/dth,d/dph))
sage: eps.comp(c_spher.frame())[1,2,3, c_spher]
r^2*sin(th)
sage: eps.display(c_spher.frame())
epsilon = sqrt(x^2 + y^2 + z^2)*sqrt(x^2 + y^2) dr/\dth/\dph
sage: eps.display(c_spher.frame(), c_spher)
epsilon = r^2*sin(th) dr/\dth/\dph
```

As a shortcut of the above command, on can pass just the chart `c_spher` to `display`, the vector frame being then assumed to be the coordinate frame associated with the chart:

```
sage: eps.display(c_spher)
epsilon = r^2*sin(th) dr/\dth/\dph
```

The exterior product of two differential forms is performed via the method `wedge()`:

```
sage: a = M.one_form(x*y*z, -z*x, y*z, name='A')
sage: b = M.one_form(cos(z), sin(x), cos(y), name='B')
sage: ab = a.wedge(b) ; ab
2-form A/\B on the 3-dimensional differentiable manifold R3
```

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```
sage: ab[:]
[                          0   x*y*z*sin(x) + x*z*cos(z)   x*y*z*cos(y) - y*z*cos(z)]
[-x*y*z*sin(x) - x*z*cos(z)                           0    -(x*cos(y) + y*sin(x))*z]
[-x*y*z*cos(y) + y*z*cos(z)    (x*cos(y) + y*sin(x))*z                            0]
sage: ab.display()
A/\B = (x*y*z*sin(x) + x*z*cos(z)) dx/\dy + (x*y*z*cos(y) - y*z*cos(z)) dx/\dz
 - (x*cos(y) + y*sin(x))*z dy/\dz
```

Let us check the formula relating the exterior product to the tensor product for 1-forms:

```
sage: a.wedge(b) == a*b - b*a
True
```

The tensor product of a 1-form and a 2-form is not a 3-form but a tensor field of type $(0,3)$ with less symmetries:

```
sage: c = a*ab ; c
Tensor field A*(A/\B) of type (0,3) on the 3-dimensional differentiable
 manifold R3
sage: c.symmetries()  # the antisymmetry is only w.r.t. the last 2 arguments:
no symmetry;  antisymmetry: (1, 2)
sage: d = ab*a ; d
Tensor field (A/\B)*A of type (0,3) on the 3-dimensional differentiable
 manifold R3
sage: d.symmetries()  # the antisymmetry is only w.r.t. the first 2 arguments:
no symmetry;  antisymmetry: (0, 1)
```

The exterior derivative of a differential form is obtained by means of the `exterior_derivative()`:

```
sage: da = a.exterior_derivative() ; da
2-form dA on the 3-dimensional differentiable manifold R3
sage: da.display()
dA = -(x + 1)*z dx/\dy - x*y dx/\dz + (x + z) dy/\dz
sage: db = b.exterior_derivative() ; db
2-form dB on the 3-dimensional differentiable manifold R3
sage: db.display()
dB = cos(x) dx/\dy + sin(z) dx/\dz - sin(y) dy/\dz
sage: dab = ab.exterior_derivative() ; dab
3-form d(A/\B) on the 3-dimensional differentiable manifold R3
```

As a 3-form over a 3-dimensional manifold, `d(A/\B)` is necessarily proportional to the volume 3-form:

```
sage: dab == dab[[1,2,3]]/eps[[1,2,3]]*eps
True
```

We may also check that the classical anti-derivation formula is fulfilled:

```
sage: dab == da.wedge(b) - a.wedge(db)
True
```

The Lie derivative of a 2-form is a 2-form:

```
sage: v = M.vector_field(y*z, -x*z, x*y, name='v')
sage: ab.lie_der(v)  # long time
2-form on the 3-dimensional differentiable manifold R3
```

Let us check Cartan formula, which expresses the Lie derivative in terms of exterior derivatives:

```
sage: ab.lie_der(v) == (v.contract(ab.exterior_derivative())  # long time
....:                   + v.contract(ab).exterior_derivative())
True
```

A 1-form on a $\mathbf{R}^3$:

```
sage: om = M.one_form(name='omega', latex_name=r'\omega'); om
1-form omega on the 3-dimensional differentiable manifold R3
```

A 1-form is of course a differential form:

```
sage: isinstance(om, sage.manifolds.differentiable.diff_form.DiffFormParal)
True
sage: om.parent()
Free module Omega^1(R3) of 1-forms on the 3-dimensional differentiable
 manifold R3
sage: om.tensor_type()
(0, 1)
```

Setting the components with respect to the manifold's default frame:

```
sage: om[:] = (2*z, x, x-y)
sage: om[:]
[2*z, x, x - y]
sage: om.display()
omega = 2*z dx + x dy + (x - y) dz
```

A 1-form acts on vector fields:

```
sage: v = M.vector_field(x, 2*y, 3*z, name='V')
sage: om(v)
Scalar field omega(V) on the 3-dimensional differentiable manifold R3
sage: om(v).display()
omega(V): R3 --> R
   (x, y, z) |--> 2*x*y + (5*x - 3*y)*z
   (r, th, ph) |--> 2*r^2*cos(ph)*sin(ph)*sin(th)^2 + r^2*(5*cos(ph)
                    - 3*sin(ph))*cos(th)*sin(th)
sage: latex(om(v))
\omega\left(V\right)
```

The tensor product of two 1-forms is a tensor field of type $(0, 2)$:

```
sage: a = M.one_form(1, 2, 3, name='A')
sage: b = M.one_form(6, 5, 4, name='B')
sage: c = a*b ; c
Tensor field A*B of type (0,2) on the 3-dimensional differentiable
 manifold R3
sage: c[:]
[ 6  5  4]
[12 10  8]
[18 15 12]
sage: c.symmetries()    # c has no symmetries:
no symmetry;  no antisymmetry
```

exterior_derivative()

Compute the exterior derivative of `self`.

OUTPUT:

- a `DiffFormParal` representing the exterior derivative of the differential form

EXAMPLES:

Exterior derivative of a 1-form on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: c_txyz.<t,x,y,z> = M.chart()
sage: a = M.one_form(t*x*y*z, z*y**2, x*z**2, x**2 + y**2, name='A')
sage: da = a.exterior_derivative() ; da
2-form dA on the 4-dimensional differentiable manifold M
sage: da.display()
dA = -t*y*z dt/\dx - t*x*z dt/\dy - t*x*y dt/\dz
 + (-2*y*z + z^2) dx/\dy + (-y^2 + 2*x) dx/\dz
 + (-2*x*z + 2*y) dy/\dz
sage: latex(da)
\mathrm{d}A
```

The result is cached, i.e. is not recomputed unless `a` is changed:

```
sage: a.exterior_derivative() is da
True
```

Instead of invoking the method `exterior_derivative()`, one may use the global function `exterior_derivative()` or its alias `xder()`:

```
sage: from sage.manifolds.utilities import xder
sage: xder(a) is a.exterior_derivative()
True
```

The exterior derivative is nilpotent:

```
sage: dda = da.exterior_derivative() ; dda
3-form ddA on the 4-dimensional differentiable manifold M
sage: dda.display()
ddA = 0
sage: dda == 0
True
```

Let us check Cartan's identity:

```
sage: v = M.vector_field(-y, x, t, z, name='v')
sage: a.lie_der(v) == v.contract(xder(a)) + xder(a(v)) # long time
True
```

hodge_dual (*metric*)

Compute the Hodge dual of the differential form with respect to some metric.

If the differential form is a p-form A, its *Hodge dual* with respect to a pseudo-Riemannian metric g is the $(n-p)$-form $*A$ defined by

$$*A_{i_1 \ldots i_{n-p}} = \frac{1}{p!} A_{k_1 \ldots k_p} \epsilon^{k_1 \ldots k_p}{}_{i_1 \ldots i_{n-p}}$$

where n is the manifold's dimension, ϵ is the volume n-form associated with g (see `volume_form()`) and the indices $k_1, \ldots, k_p$ are raised with g.

INPUT:

- `metric`: a pseudo-Riemannian metric defined on the same manifold as the current differential form; must be an instance of `PseudoRiemannianMetric`

OUTPUT:

- the $(n-p)$-form $*A$

EXAMPLES:

Hodge dual of a 1-form in the Euclidean space R^3:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric('g')  # the Euclidean metric
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: var('Ax Ay Az')
(Ax, Ay, Az)
sage: a = M.one_form(Ax, Ay, Az, name='A')
sage: sa = a.hodge_dual(g) ; sa
2-form *A on the 3-dimensional differentiable manifold M
sage: sa.display()
*A = Az dx/\dy - Ay dx/\dz + Ax dy/\dz
sage: ssa = sa.hodge_dual(g) ; ssa
1-form **A on the 3-dimensional differentiable manifold M
sage: ssa.display()
**A = Ax dx + Ay dy + Az dz
sage: ssa == a  # must hold for a Riemannian metric in dimension 3
True
```

Instead of calling the method `hodge_dual()` on the differential form, one can invoke the method `hodge_star()` of the metric:

```
sage: a.hodge_dual(g) == g.hodge_star(a)
True
```

See the documentation of `hodge_star()` for more examples.

interior_product (*qvect*)

Interior product with a multivector field.

If `self` is a differential form A of degree p and B is a multivector field of degree $q \geq p$ on the same manifold, the interior product of A by B is the multivector field $\iota_A B$ of degree $q-p$ defined by

$$(\iota_A B)^{i_1 \ldots i_{q-p}} = A_{k_1 \ldots k_p} B^{k_1 \ldots k_p i_1 \ldots i_{q-p}}$$

Note: `A.interior_product(B)` yields the same result as `A.contract(0,..., p-1, B, 0,..., p-1)` (cf. `contract()`), but `interior_product` is more efficient, the alternating character of A being not used to reduce the computation in `contract()`

INPUT:

- `qvect` – multivector field B (instance of `MultivectorFieldParal`); the degree of B must be at least equal to the degree of `self`

OUTPUT:

- scalar field (case $p=q$) or `MultivectorFieldParal` (case $p<q$) representing the interior product $\iota_A B$, where A is `self`

See also:

`interior_product()` for the interior product of a multivector field with a differential form

EXAMPLES:

Interior product of a 1-form with a 2-vector field on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: a = M.one_form(2, 1+x, y*z, name='a')
sage: b = M.multivector_field(2, name='b')
sage: b[1,2], b[1,3], b[2,3] = y^2, z+x, -z^2
sage: s = a.interior_product(b); s
Vector field i_a b on the 3-dimensional differentiable
 manifold M
sage: s.display()
i_a b = (-(x + 1)*y^2 - x*y*z - y*z^2) d/dx
 + (y*z^3 + 2*y^2) d/dy + (-(x + 1)*z^2 + 2*x + 2*z) d/dz
sage: s == a.contract(b)
True
```

Interior product of a 2-form with a 2-vector field:

```
sage: a = M.diff_form(2, name='a')
sage: a[1,2], a[1,3], a[2,3] = x*y, -3, z
sage: s = a.interior_product(b); s
Scalar field i_a b on the 3-dimensional differentiable manifold M
sage: s.display()
i_a b: M --> R
   (x, y, z) |--> 2*x*y^3 - 2*z^3 - 6*x - 6*z
sage: s == a.contract(0,1,b,0,1)
True
```

wedge (*other*)

Exterior product of `self` with another differential form.

INPUT:

- `other` – another differential form

OUTPUT:

- instance of *DiffFormParal* representing the exterior product `self/\other`

EXAMPLES:

Exterior product of a 1-form and a 2-form on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: a = M.one_form(2, 1+x, y*z, name='a')
sage: b = M.diff_form(2, name='b')
sage: b[1,2], b[1,3], b[2,3] = y^2, z+x, z^2
sage: a.display()
a = 2 dx + (x + 1) dy + y*z dz
sage: b.display()
b = y^2 dx/\dy + (x + z) dx/\dz + z^2 dy/\dz
sage: s = a.wedge(b); s
3-form a/\b on the 3-dimensional differentiable manifold M
sage: s.display()
a/\b = (-x^2 + (y^3 - x - 1)*z + 2*z^2 - x) dx/\dy/\dz
```

Check:

```
sage: s[1,2,3] == a[1]*b[2,3] + a[2]*b[3,1] + a[3]*b[1,2]
True
```

2.10 Mixed Differential Forms

2.10.1 Graded Algebra of Mixed Differential Forms

Let M and N be differentiable manifolds and $\varphi : M \to N$ a differentiable map. The space of *mixed differential forms along* φ, denoted by $\Omega^*(M,\varphi)$, is given by the direct sum $\bigoplus_{j=0}^{n} \Omega^j(M,\varphi)$ of differential form modules, where $n = \dim(N)$. With the wedge product, $\Omega^*(M,\varphi)$ inherits the structure of a graded algebra.

AUTHORS:

- Michael Jung (2019) : initial version

class sage.manifolds.differentiable.mixed_form_algebra.**MixedFormAlgebra**(*vector_field_module*)

Bases: sage.structure.parent.Parent, sage.structure.unique_representation.UniqueRepresentation

An instance of this class represents the graded algebra of mixed form. That is, if $\varphi : M \to N$ is a differentiable map between two differentiable manifolds M and N, the *graded algebra of mixed forms* $\Omega^*(M,\varphi)$ *along* φ is defined via the direct sum $\bigoplus_{j=0}^{n} \Omega^j(M,\varphi)$ consisting of differential form modules (cf. DiffFormModule), where n is the dimension of N. Hence, $\Omega^*(M,\varphi)$ is a module over $C^k(M)$ and a vector space over $\mathbf{R}$ or $\mathbf{C}$. Furthermore notice, that

$$\Omega^*(M,\varphi) \cong C^k\left(\bigoplus_{j=0}^{n} \Lambda^j(\varphi^* T^* N)\right),$$

where C^k denotes the global section functor for differentiable sections of order k here.

The wedge product induces a multiplication on $\Omega^*(M,\varphi)$ and gives it the structure of a graded algebra since

$$\Omega^k(M,\varphi) \wedge \Omega^l(M,\varphi) \subset \Omega^{k+l}(M,\varphi).$$

INPUT:

- vector_field_module – module $\mathfrak{X}(M,\varphi)$ of vector fields along M associated with the map $\varphi : M \to N$

EXAMPLES:

Graded algebra of mixed forms on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: Omega = M.mixed_form_algebra(); Omega
Graded algebra Omega^*(M) of mixed differential forms on the
 3-dimensional differentiable manifold M
sage: Omega.category()
Category of graded algebras over Symbolic Ring
sage: Omega.base_ring()
Symbolic Ring
sage: Omega.vector_field_module()
Free module X(M) of vector fields on the 3-dimensional differentiable
 manifold M
```

Elements can be created from scratch:

```
sage: A = Omega(0); A
Mixed differential form zero on the 3-dimensional differentiable
 manifold M
sage: A is Omega.zero()
True
sage: B = Omega(1); B
Mixed differential form one on the 3-dimensional differentiable
 manifold M
sage: B is Omega.one()
True
sage: C = Omega([2,0,0,0]); C
Mixed differential form on the 3-dimensional differentiable manifold M
```

There are some important coercions implemented:

```
sage: Omega0 = M.scalar_field_algebra(); Omega0
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: Omega.has_coerce_map_from(Omega0)
True
sage: Omega2 = M.diff_form_module(2); Omega2
Free module Omega^2(M) of 2-forms on the 3-dimensional differentiable
 manifold M
sage: Omega.has_coerce_map_from(Omega2)
True
```

Restrictions induce coercions as well:

```
sage: U = M.open_subset('U'); U
Open subset U of the 3-dimensional differentiable manifold M
sage: OmegaU = U.mixed_form_algebra(); OmegaU
Graded algebra Omega^*(U) of mixed differential forms on the Open subset
 U of the 3-dimensional differentiable manifold M
sage: OmegaU.has_coerce_map_from(Omega)
True
```

Element
alias of *sage.manifolds.differentiable.mixed_form.MixedForm*

one()
Return the one of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A = M.mixed_form_algebra()
sage: A.one()
Mixed differential form one on the 3-dimensional differentiable
 manifold M
```

vector_field_module()
Return the underlying vector field module.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: N = Manifold(3, 'N')
```

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```
sage: Phi = M.diff_map(N, name='Phi'); Phi
Differentiable map Phi from the 2-dimensional differentiable manifold M
 to the 3-dimensional differentiable manifold N
sage: A = M.mixed_form_algebra(Phi); A
Graded algebra Omega^*(M,Phi) of mixed differential forms along the
 2-dimensional differentiable manifold M mapped into the 3-dimensional
 differentiable manifold N via Phi
sage: A.vector_field_module()
Module X(M,Phi) of vector fields along the 2-dimensional differentiable
 manifold M mapped into the 3-dimensional differentiable manifold N
```

zero()

Return the zero of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A = M.mixed_form_algebra()
sage: A.zero()
Mixed differential form zero on the 3-dimensional differentiable
 manifold M
```

2.10.2 Mixed Differential Forms

Let M and N be differentiable manifolds and $\varphi : M \longrightarrow N$ a differentiable map. A *mixed differential form along* φ is an element of the graded algebra represented by *MixedFormAlgebra*. Its homogeneous components consist of differential forms along φ. Mixed forms are useful to represent characteristic classes and perform computations of such.

AUTHORS:

- Michael Jung (2019) : initial version

class sage.manifolds.differentiable.mixed_form.**MixedForm**(*parent*, *comp=None*, *name=None*, *latex_name=None*)

Bases: `sage.structure.element.AlgebraElement`

An instance of this class is a mixed form along some differentiable map $\varphi : M \to N$ between two differentiable manifolds M and N. More precisely, a mixed form a along $\varphi : M \to N$ can be considered as a differentiable map

$$a : M \longrightarrow \bigoplus_{k=0}^{n} T^{(0,k)}N,$$

where $T^{(0,k)}$ denotes the tensor bundle of type $(0, k)$, $\bigoplus$ the Whitney sum and n the dimension of N, such that

$$\forall x \in M, \quad a(x) \in \bigoplus_{k=0}^{n} \Lambda^k \left(T^*_{\varphi(x)}N\right),$$

where $\Lambda^k(T^*_{\varphi(x)}N)$ is the k-th exterior power of the dual of the tangent space $T_{\varphi(x)}N$.

The standard case of a mixed form *on* M corresponds to $M = N$ with $\varphi = \mathrm{Id}_M$.

INPUT:

- `parent` – graded algebra of mixed forms represented by *MixedFormAlgebra* where the mixed form `self` shall belong to

- `comp` – (default: `None`) homogeneous components of the mixed form as a list; if none is provided, the components are set to innocent unnamed differential forms
- `name` – (default: `None`) name given to the mixed form
- `latex_name` – (default: `None`) LaTeX symbol to denote the mixed form; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

Initialize a mixed form on a 2-dimensional parallelizable differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: c_xy.<x,y> = M.chart()
sage: F = M.mixed_form(name='F'); F
Mixed differential form F on the 2-dimensional differentiable manifold M
sage: F.parent()
Graded algebra Omega^*(M) of mixed differential forms on the
 2-dimensional differentiable manifold M
```

To define the homogenous components of a mixed form, it is convenient to define some differential forms first:

```
sage: f = M.scalar_field(x, name='f'); f
Scalar field f on the 2-dimensional differentiable manifold M
sage: omega = M.diff_form(1, name='omega', latex_name=r'\omega'); omega
1-form omega on the 2-dimensional differentiable manifold M
sage: omega[c_xy.frame(),0] = y*x; omega.disp()
omega = x*y dx
sage: eta = M.diff_form(2, name='eta', latex_name=r'\eta'); eta
2-form eta on the 2-dimensional differentiable manifold M
sage: eta[c_xy.frame(),0,1] = y^2*x; eta.disp()
eta = x*y^2 dx/\dy
```

The components of the mixed form `F` can be manipulated very easily:

```
sage: F[:] = [f, omega, eta]; F.disp() # display names
F = f + omega + eta
sage: F.disp(c_xy.frame()) # display in coordinates
F = [x] + [x*y dx] + [x*y^2 dx/\dy]
sage: F[0]
Scalar field f on the 2-dimensional differentiable manifold M
sage: F[0] is f
True
sage: F[1]
1-form omega on the 2-dimensional differentiable manifold M
sage: F[1] is omega
True
sage: F[2]
2-form eta on the 2-dimensional differentiable manifold M
sage: F[2] is eta
True
```

Alternatively, the components can be determined from scratch:

```
sage: G = M.mixed_form(name='G', comp=[f, omega, eta])
sage: G == F
True
```

Mixed forms are elements of an algebra, so they can be added, and multiplied via the wedge product:

```
sage: xF = x*F; xF
Mixed differential form x/\F on the 2-dimensional differentiable
 manifold M
sage: xF.disp(c_xy.frame())
x/\F = [x^2] + [x^2*y dx] + [x^2*y^2 dx/\dy]
sage: FpxF = F+xF; FpxF
Mixed differential form F+x/\F on the 2-dimensional differentiable
 manifold M
sage: FpxF.disp(c_xy.frame())
F+x/\F = [x^2 + x] + [(x^2 + x)*y dx] + [(x^2 + x)*y^2 dx/\dy]
sage: FxF = F*xF; FxF
Mixed differential form F/\(x/\F) on the 2-dimensional differentiable
 manifold M
sage: FxF.disp(c_xy.frame())
F/\(x/\F) = [x^3] + [2*x^3*y dx] + [2*x^3*y^2 dx/\dy]
```

Coercions are fully implemented:

```
sage: omegaF = omega*F
sage: omegaF.disp(c_xy.frame())
omega/\F = [0] + [x^2*y dx] + [0]
sage: omegapF = omega+F
sage: omegapF.disp(c_xy.frame())
omega+F = [x] + [2*x*y dx] + [x*y^2 dx/\dy]
```

Moreover, it is possible to compute the exterior derivative of a mixed form:

```
sage: dF = F.exterior_derivative(); dF.disp()
dF = zero + df + domega
sage: dF.disp(c_xy.frame())
dF = [0] + [dx] + [-x dx/\dy]
```

Initialize a mixed form on a 2-dimensional non-parallelizable differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)    # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:                   intersection_name='W', restrictions1= x>0,
....:                   restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
sage: omega = M.diff_form(1, name='omega', latex_name=r'\omega')
sage: omega[c_xy.frame(),0] = y*x; omega.disp(e_xy)
omega = x*y dx
sage: eta = M.diff_form(2, name='eta', latex_name=r'\eta')
sage: eta[c_uv.frame(),0,1] = u*v^2; eta.disp(e_uv)
eta = u*v^2 du/\dv
sage: F = M.mixed_form(name='F', comp=[x, omega, eta]); F
Mixed differential form F on the 2-dimensional differentiable manifold M
sage: F.add_comp_by_continuation(e_uv, V.intersection(U), c_uv)
sage: F.disp(e_uv)
F = [1/2*u + 1/2*v] + [(1/8*u^2 - 1/8*v^2) du + (1/8*u^2 - 1/8*v^2) dv]
 + [u*v^2 du/\dv]
sage: F.add_comp_by_continuation(e_xy, V.intersection(U), c_xy)
sage: F.disp(e_xy)
F = [x] + [x*y dx] + [(-2*x^3 + 2*x^2*y + 2*x*y^2 - 2*y^3) dx/\dy]
```

add_comp_by_continuation(*frame*, *subdomain*, *chart=None*)

Set components with respect to a vector frame by continuation of the coordinate expression of the components in a subframe.

The continuation is performed by demanding that the components have the same coordinate expression as those on the restriction of the frame to a given subdomain.

INPUT:

- `frame` – vector frame e in which the components are to be set
- `subdomain` – open subset of e's domain in which the components are known or can be evaluated from other components
- `chart` – (default: `None`) coordinate chart on e's domain in which the extension of the expression of the components is to be performed; if `None`, the default's chart of e's domain is assumed

EXAMPLES:

Mixed form defined by differential forms with components on different parts of the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: f = M.scalar_field(x, name='f')
sage: omega = M.diff_form(1, name='omega')
sage: omega[e_xy,0] = x
sage: eta = M.diff_form(2, name='eta')
sage: eta[e_uv,0,1] = u*v
sage: F = M.mixed_form(name='F', comp=[f, omega, eta])
sage: F.add_comp_by_continuation(e_xy, U.intersection(V), c_xy)
sage: F.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: F.disp(e_xy)
F = [x] + [x dx] + [-x*y/(x^8 + 4*x^6*y^2 + 6*x^4*y^4 + 4*x^2*y^6 +
 y^8) dx/\dy]
sage: F.disp(e_uv)
F = [u/(u^2 + v^2)] + [-(u^3 - u*v^2)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 +
 v^6) du - 2*u^2*v/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) dv] +
 [u*v du/\dv]
```

copy()

Return an exact copy of `self`.

Note: The name and names of the components are not copied.

EXAMPLES:

Initialize a 2-dimensional manifold and differential forms:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
```

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```
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
sage: f = M.scalar_field(x, name='f', chart=c_xy); f.disp()
f: M --> R
    on U: (x, y) |--> x
sage: omega = M.diff_form(1, name='omega')
sage: omega[e_xy,0] = x; omega.disp()
omega = x dx
sage: A = M.mixed_form(name='A', comp=[f, omega, 0]); A.disp()
A = f + omega + zero
sage: A.add_comp_by_continuation(e_uv, U.intersection(V), c_uv)
sage: A.disp(e_uv)
A = [1/2*u + 1/2*v] + [(1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv] +
 [0]
```

An exact copy is made. The copy is an entirely new instance and has a different name, but has the very same values:

```
sage: B = A.copy(); B.disp()
f + (unnamed 1-form) + (unnamed 2-form)
sage: B.disp(e_uv)
[1/2*u + 1/2*v] + [(1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv] + [0]
sage: A == B
True
sage: A is B
False
```

Notice, that changes in the differential forms usually cause changes in the original instance. But for the copy of a mixed form, the components are copied as well:

```
sage: omega[e_xy,0] = y; omega.disp()
omega = y dx
sage: A.disp(e_xy)
A = [x] + [y dx] + [0]
sage: B.disp(e_xy)
[x] + [x dx] + [0]
```

disp(*basis=None*, *chart=None*, *from_chart=None*)

Display the components of mixed forms.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- basis – (default: None) vector frame with respect to which the mixed form is expanded; if None, only the names of the components are displayed
- chart – (default: None) chart with respect to which the components of the mixed form in the selected frame are expressed; if None, the default chart of the vector frame domain is assumed

EXAMPLES:

Display a mixed form on a 2-dimensional non-parallelizable differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x-y, x+y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
sage: omega = M.diff_form(1, name='omega', latex_name=r'\omega')
sage: omega[c_xy.frame(),0] = x; omega.disp(e_xy)
omega = x dx
sage: eta = M.diff_form(2, name='eta', latex_name=r'\eta')
sage: eta[c_uv.frame(),0,1] = u*v; eta.disp(e_uv)
eta = u*v du/\dv
sage: F = M.mixed_form(name='F', comp=[0, omega, eta]); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
sage: F.disp() # display names of homogenous components
F = zero + omega + eta
sage: F.add_comp_by_continuation(e_uv, V.intersection(U), c_uv)
sage: F.disp(e_uv)
F = [0] + [(1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv] + [u*v du/\dv]
sage: F.add_comp_by_continuation(e_xy, V.intersection(U), c_xy)
sage: F.disp(e_xy)
F = [0] + [x dx] + [(2*x^2 - 2*y^2) dx/\dy]
```

display(*basis=None*, *chart=None*, *from_chart=None*)

Display the components of mixed forms.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `basis` – (default: `None`) vector frame with respect to which the mixed form is expanded; if `None`, only the names of the components are displayed
- `chart` – (default: `None`) chart with respect to which the components of the mixed form in the selected frame are expressed; if `None`, the default chart of the vector frame domain is assumed

EXAMPLES:

Display a mixed form on a 2-dimensional non-parallelizable differentiable manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x-y, x+y),
....:                    intersection_name='W', restrictions1= x>0,
....:                    restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame() # define frames
sage: omega = M.diff_form(1, name='omega', latex_name=r'\omega')
sage: omega[c_xy.frame(),0] = x; omega.disp(e_xy)
omega = x dx
sage: eta = M.diff_form(2, name='eta', latex_name=r'\eta')
sage: eta[c_uv.frame(),0,1] = u*v; eta.disp(e_uv)
eta = u*v du/\dv
```

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```
sage: F = M.mixed_form(name='F', comp=[0, omega, eta]); F
Mixed differential form F on the 2-dimensional differentiable
 manifold M
sage: F.disp() # display names of homogenous components
F = zero + omega + eta
sage: F.add_comp_by_continuation(e_uv, V.intersection(U), c_uv)
sage: F.disp(e_uv)
F = [0] + [(1/4*u + 1/4*v) du + (1/4*u + 1/4*v) dv] + [u*v du/\dv]
sage: F.add_comp_by_continuation(e_xy, V.intersection(U), c_xy)
sage: F.disp(e_xy)
F = [0] + [x dx] + [(2*x^2 - 2*y^2) dx/\dy]
```

exterior_derivative()

Compute the exterior derivative of `self`.

More precisely, the *exterior derivative* on $\Omega^k(M,\varphi)$ is a linear map

$$\mathrm{d}_k : \Omega^k(M,\varphi) \to \Omega^{k+1}(M,\varphi),$$

where $\Omega^k(M,\varphi)$ denotes the space of differential forms of degree k along φ (see *exterior_derivative()* for further information). By linear extension, this induces a map on $\Omega^*(M,\varphi)$:

$$\mathrm{d} : \Omega^*(M,\varphi) \to \Omega^*(M,\varphi).$$

OUTPUT:

- a *MixedForm* representing the exterior derivative of the mixed form

EXAMPLES:

Exterior derivative of a mixed form on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: f = M.scalar_field(z^2, name='f')
sage: f.disp()
f: M --> R
    (x, y, z) |--> z^2
sage: a = M.diff_form(2, 'a')
sage: a[1,2], a[1,3], a[2,3] = z+y^2, z+x, x^2
sage: a.disp()
a = (y^2 + z) dx/\dy + (x + z) dx/\dz + x^2 dy/\dz
sage: F = M.mixed_form(name='F', comp=[f, 0, a, 0]); F.disp()
F = f + zero + a + zero
sage: dF = F.exterior_derivative()
sage: dF.disp()
dF = zero + df + dzero + da
sage: dF = F.exterior_derivative()
sage: dF.disp(c_xyz.frame())
dF = [0] + [2*z dz] + [0] + [(2*x + 1) dx/\dy/\dz]
```

Due to long calculation times, the result is cached, i.e. is not recomputed unless `F` is changed:

```
sage: F.exterior_derivative() is dF
True
```

restrict(*subdomain*, *dest_map=None*)

Return the restriction of `self` to some subdomain.

INPUT:

- `subdomain` – *DifferentiableManifold*; open subset U of the domain of `self`
- `dest_map` – *DiffMap* (default: `None`); destination map $\Psi : U \to V$, where V is an open subset of the manifold N where the mixed form takes it values; if `None`, the restriction of Φ to U is used, Φ being the differentiable map $S \to M$ associated with the mixed form

OUTPUT:

- *MixedForm* representing the restriction

EXAMPLES:

Initialize the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W', restrictions1= x^2+y^2!=0,
....:                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

And predefine some forms:

```
sage: f = M.scalar_field(x^2, name='f', chart=c_xy)
sage: omega = M.diff_form(1, name='omega')
sage: omega[e_xy,0] = y^2
sage: eta = M.diff_form(2, name='eta')
sage: eta[e_xy,0,1] = x^2*y^2
```

Now, a mixed form can be restricted to some subdomain:

```
sage: F = M.mixed_form(name='F', comp=[f, omega, eta])
sage: F.add_comp_by_continuation(e_uv, V.intersection(U), c_uv)
sage: FV = F.restrict(V); FV
Mixed differential form F on the Open subset V of the 2-dimensional
 differentiable manifold M
sage: FV[:]
[Scalar field f on the Open subset V of the 2-dimensional
 differentiable manifold M,
 1-form omega on the Open subset V of the 2-dimensional
 differentiable manifold M,
 2-form eta on the Open subset V of the 2-dimensional
 differentiable manifold M]
sage: FV.disp(e_uv)
F = [u^2/(u^4 + 2*u^2*v^2 + v^4)] + [-(u^2*v^2 - v^4)/(u^8 +
 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8) du - 2*u*v^3/(u^8 +
 4*u^6*v^2 + 6*u^4*v^4 + 4*u^2*v^6 + v^8) dv] + [-u^2*v^2/(u^12 +
 6*u^10*v^2 + 15*u^8*v^4 + 20*u^6*v^6 + 15*u^4*v^8 + 6*u^2*v^10 +
 v^12) du/\dv]
```

set_name (*name=None*, *latex_name=None*)

Redefine the string and LaTeX representation of the object.

INPUT:

- name – (default: None) name given to the mixed form
- latex_name – (default: None) LaTeX symbol to denote the mixed form; if none is provided, the LaTeX symbol is set to name

EXAMPLES:

```
sage: M = Manifold(4, 'M')
sage: F = M.mixed_form(name='dummy', latex_name=r'\ugly'); F
Mixed differential form dummy on the 4-dimensional differentiable
 manifold M
sage: latex(F)
\ugly
sage: F.set_name(name='fancy', latex_name=r'\eta'); F
Mixed differential form fancy on the 4-dimensional differentiable
 manifold M
sage: latex(F)
\eta
```

set_restriction(*rst*)

Set a (component-wise) restriction of self to some subdomain.

INPUT:

- rst – *MixedForm* of the same type as self, defined on a subdomain of the domain of self

EXAMPLES:

Initialize the 2-sphere:

```
sage: M = Manifold(2, 'M') # the 2-dimensional sphere S^2
sage: U = M.open_subset('U') # complement of the North pole
sage: c_xy.<x,y> = U.chart() # stereographic coordinates from the North pole
sage: V = M.open_subset('V') # complement of the South pole
sage: c_uv.<u,v> = V.chart() # stereographic coordinates from the South pole
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                intersection_name='W', restrictions1= x^2+y^2!=0,
....:                restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: e_xy = c_xy.frame(); e_uv = c_uv.frame()
```

And define some forms on the subset U:

```
sage: f = U.scalar_field(x, name='f', chart=c_xy)
sage: omega = U.diff_form(1, name='omega')
sage: omega[e_xy,0] = y
sage: AU = U.mixed_form(name='A', comp=[f, omega, 0]); AU
Mixed differential form A on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: AU.disp(e_xy)
A = [x] + [y dx] + [0]
```

A mixed form on M can be specified by some mixed form on a subset:

```
sage: A = M.mixed_form(name='A'); A
Mixed differential form A on the 2-dimensional differentiable
 manifold M
sage: A.set_restriction(AU)
sage: A.disp(e_xy)
```

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```
A = [x] + [y dx] + [0]
sage: A.add_comp_by_continuation(e_uv, V.intersection(U), c_uv)
sage: A.disp(e_uv)
A = [u/(u^2 + v^2)] + [-(u^2*v - v^3)/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 +
 v^6) du - 2*u*v^2/(u^6 + 3*u^4*v^2 + 3*u^2*v^4 + v^6) dv] + [0]
sage: A.restrict(U) == AU
True
```

wedge (*other*)

Wedge product on the graded algebra of mixed forms.

More precisely, the wedge product is a bilinear map

$$\wedge : \Omega^k(M,\varphi) \times \Omega^l(M,\varphi) \to \Omega^{k+l}(M,\varphi),$$

where $\Omega^k(M,\varphi)$ denotes the space of differential forms of degree k along φ. By bilinear extension, this induces a map

$$\wedge : \Omega^*(M,\varphi) \times \Omega^*(M,\varphi) \to \Omega^*(M,\varphi)\text{``}$$

and equips $\Omega^*(M,\varphi)$ with a multiplication such that it becomes a graded algebra.

INPUT:

- `other` – mixed form in the same algebra as `self`

OUTPUT:

- the mixed form resulting from the wedge product of `self` with `other`

EXAMPLES:

Initialize a mixed form on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: c_xyz.<x,y,z> = M.chart()
sage: f = M.scalar_field(x, name='f')
sage: f.disp()
f: M --> R
   (x, y, z) |--> x
sage: g = M.scalar_field(y, name='g')
sage: g.disp()
g: M --> R
   (x, y, z) |--> y
sage: omega = M.diff_form(1, name='omega')
sage: omega[c_xyz.frame(),0] = x
sage: omega.disp()
omega = x dx
sage: eta = M.diff_form(1, name='eta')
sage: eta[c_xyz.frame(),1] = y
sage: eta.disp()
eta = y dy
sage: mu = M.diff_form(2, name='mu')
sage: mu[c_xyz.frame(),[0,2]] = z
sage: mu.disp()
mu = z dx/\dz
sage: A = M.mixed_form(name='A', comp=[f, omega, mu, 0])
sage: A.disp(c_xyz.frame())
A = [x] + [x dx] + [z dx/\dz] + [0]
```

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```
sage: B = M.mixed_form(name='B', comp=[g, eta, mu, 0])
sage: B.disp(c_xyz.frame())
B = [y] + [y dy] + [z dx/\dz] + [0]
```

The wedge product of A and B yields:

```
sage: AwB = A.wedge(B); AwB
Mixed differential form A/\B on the 3-dimensional differentiable
 manifold M
sage: AwB.disp(c_xyz.frame())
A/\B = [x*y] + [x*y dx + x*y dy] + [x*y dx/\dy + (x + y)*z dx/\dz] +
 [-y*z dx/\dy/\dz]
sage: BwA = B.wedge(A); BwA # Don't even try, it's not commutative!
Mixed differential form B/\A on the 3-dimensional differentiable
 manifold M
sage: BwA.disp(c_xyz.frame()) # I told you so!
B/\A = [x*y] + [x*y dx + x*y dy] + [-x*y dx/\dy + (x + y)*z dx/\dz]
 + [-y*z dx/\dy/\dz]
```

The multiplication symbol may be used instead as well:

```
sage: A*B
Mixed differential form A/\B on the 3-dimensional differentiable
 manifold M
sage: A*B == AwB
True
```

Yet, the multiplication includes coercions:

```
sage: xA = x*A; xA.disp(c_xyz.frame())
x/\A = [x^2] + [x^2 dx] + [x*z dx/\dz] + [0]
sage: Aeta = A*eta; Aeta.disp(c_xyz.frame())
A/\eta = [0] + [x*y dy] + [x*y dx/\dy] + [-y*z dx/\dy/\dz]
```

2.11 Alternating Multivector Fields

2.11.1 Multivector Field Modules

The set $A^p(U, \Phi)$ of p-vector fields along a differentiable manifold U with values on a differentiable manifold M via a differentiable map $\Phi : U \to M$ (possibly $U = M$ and $\Phi = \mathrm{Id}_M$) is a module over the algebra $C^k(U)$ of differentiable scalar fields on U. It is a free module if and only if M is parallelizable. Accordingly, two classes implement $A^p(U, \Phi)$:

- *MultivectorModule* for p-vector fields with values on a generic (in practice, not parallelizable) differentiable manifold M
- *MultivectorFreeModule* for p-vector fields with values on a parallelizable manifold M

AUTHORS:

- Eric Gourgoulhon (2017): initial version

REFERENCES:

- R. L. Bishop and S. L. Goldberg (1980) [**?**]
- C.-M. Marle (1997) [**?**]

class sage.manifolds.differentiable.multivector_module.**MultivectorFreeModule**(*vector_field_module*, *degree*)

Bases: sage.tensor.modules.ext_pow_free_module.ExtPowerFreeModule

Free module of multivector fields of a given degree p (p-vector fields) along a differentiable manifold U with values on a parallelizable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a parallelizable manifold M of dimension n, the set $A^p(U, \Phi)$ of p-vector fields (i.e. alternating tensor fields of type $(p, 0)$) along U with values on M is a free module of rank $\binom{n}{p}$ over $C^k(U)$, the commutative algebra of differentiable scalar fields on U (see *DiffScalarFieldAlgebra*). The standard case of p-vector fields *on* a differentiable manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: This class implements $A^p(U, \Phi)$ in the case where M is parallelizable; $A^p(U, \Phi)$ is then a *free* module. If M is not parallelizable, the class *MultivectorModule* must be used instead.

INPUT:

- vector_field_module – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U associated with the map $\Phi : U \to V$
- degree – positive integer; the degree p of the multivector fields

EXAMPLES:

Free module of 2-vector fields on a parallelizable 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: XM = M.vector_field_module() ; XM
Free module X(M) of vector fields on the 3-dimensional
 differentiable manifold M
sage: A = M.multivector_module(2) ; A
Free module A^2(M) of 2-vector fields on the 3-dimensional
 differentiable manifold M
sage: latex(A)
A^{2}\left(M\right)
```

A is nothing but the second exterior power of XM, i.e. we have $A^2(M) = \Lambda^2(\mathfrak{X}(M))$ (see ExtPowerFreeModule):

```
sage: A is XM.exterior_power(2)
True
```

$A^2(M)$ is a module over the algebra $C^k(M)$ of (differentiable) scalar fields on M:

```
sage: A.category()
Category of finite dimensional modules over Algebra of
 differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 3-dimensional
 differentiable manifold M
sage: A in Modules(CM)
True
sage: A.base_ring()
```

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```
Algebra of differentiable scalar fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module()
Free module X(M) of vector fields on
 the 3-dimensional differentiable manifold M
sage: A.base_module() is XM
True
sage: A.rank()
3
```

Elements can be constructed from A. In particular, `0` yields the zero element of A:

```
sage: A(0)
2-vector field zero on the 3-dimensional differentiable
 manifold M
sage: A(0) is A.zero()
True
```

while non-zero elements are constructed by providing their components in a given vector frame:

```
sage: comp = [[0,3*x,-z],[-3*x,0,4],[z,-4,0]]
sage: a = A(comp, frame=X.frame(), name='a') ; a
2-vector field a on the 3-dimensional differentiable manifold M
sage: a.display()
a = 3*x d/dx/\d/dy - z d/dx/\d/dz + 4 d/dy/\d/dz
```

An alternative is to construct the 2-vector field from an empty list of components and to set the nonzero nonredundant components afterwards:

```
sage: a = A([], name='a')
sage: a[0,1] = 3*x  # component in the manifold's default frame
sage: a[0,2] = -z
sage: a[1,2] = 4
sage: a.display()
a = 3*x d/dx/\d/dy - z d/dx/\d/dz + 4 d/dy/\d/dz
```

The module $A^1(M)$ is nothing but $\mathfrak{X}(M)$ (the free module of vector fields on M):

```
sage: A1 = M.multivector_module(1) ; A1
Free module X(M) of vector fields on the 3-dimensional
 differentiable manifold M
sage: A1 is XM
True
```

There is a coercion map $A^p(M) \to T^{(p,0)}(M)$:

```
sage: T20 = M.tensor_field_module((2,0)); T20
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold M
sage: T20.has_coerce_map_from(A)
True
```

but of course not in the reverse direction, since not all contravariant tensor field is alternating:

```
sage: A.has_coerce_map_from(T20)
False
```

The coercion map $A^2(M) \to T^{(2,0)}(M)$ in action:

```
sage: T20 = M.tensor_field_module((2,0)) ; T20
Free module T^(2,0)(M) of type-(2,0) tensors fields on the
 3-dimensional differentiable manifold M
sage: ta = T20(a) ; ta
Tensor field a of type (2,0) on the 3-dimensional differentiable
 manifold M
sage: ta.display()
a = 3*x d/dx*d/dy - z d/dx*d/dz - 3*x d/dy*d/dx + 4 d/dy*d/dz
 + z d/dz*d/dx - 4 d/dz*d/dy
sage: a.display()
a = 3*x d/dx/\d/dy - z d/dx/\d/dz + 4 d/dy/\d/dz
sage: ta.symmetries()  # the antisymmetry is preserved
no symmetry;  antisymmetry: (0, 1)
```

There is also coercion to subdomains, which is nothing but the restriction of the multivector field to some subset of its domain:

```
sage: U = M.open_subset('U', coord_def={X: x^2+y^2<1})
sage: B = U.multivector_module(2) ; B
Free module A^2(U) of 2-vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
sage: B.has_coerce_map_from(A)
True
sage: a_U = B(a) ; a_U
2-vector field a on the Open subset U of the 3-dimensional
 differentiable manifold M
sage: a_U.display()
a = 3*x d/dx/\d/dy - z d/dx/\d/dz + 4 d/dy/\d/dz
```

Element
alias of *sage.manifolds.differentiable.multivectorfield.MultivectorFieldParal*

class sage.manifolds.differentiable.multivector_module.**MultivectorModule**(*vector_field_module*, *degree*)

Bases: sage.structure.unique_representation.UniqueRepresentation, sage.structure.parent.Parent

Module of multivector fields of a given degree p (p-vector fields) along a differentiable manifold U with values on a differentiable manifold M.

Given a differentiable manifold U and a differentiable map $\Phi : U \to M$ to a differentiable manifold M, the set $A^p(U, \Phi)$ of p-vector fields (i.e. alternating tensor fields of type $(p, 0)$) along U with values on M is a module over $C^k(U)$, the commutative algebra of differentiable scalar fields on U (see *DiffScalarFieldAlgebra*). The standard case of p-vector fields *on* a differentiable manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: This class implements $A^p(U, \Phi)$ in the case where M is not assumed to be parallelizable; the module $A^p(U, \Phi)$ is then not necessarily free. If M is parallelizable, the class *MultivectorFreeModule* must be used instead.

INPUT:

- vector_field_module – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map $\Phi : U \to M$

- `degree` – positive integer; the degree p of the multivector fields

EXAMPLES:

Module of 2-vector fields on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y),
....:  intersection_name='W', restrictions1= x>0,
....:  restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: XM = M.vector_field_module() ; XM
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A = M.multivector_module(2) ; A
Module A^2(M) of 2-vector fields on the 2-dimensional
 differentiable manifold M
sage: latex(A)
A^{2}\left(M\right)
```

`A` is nothing but the second exterior power of of `XM`, i.e. we have $A^2(M) = \Lambda^2(\mathfrak{X}(M))$:

```
sage: A is XM.exterior_power(2)
True
```

Modules of multivector fields are unique:

```
sage: A is M.multivector_module(2)
True
```

$A^2(M)$ is a module over the algebra $C^k(M)$ of (differentiable) scalar fields on M:

```
sage: A.category()
Category of modules over Algebra of differentiable scalar fields
 on the 2-dimensional differentiable manifold M
sage: CM = M.scalar_field_algebra() ; CM
Algebra of differentiable scalar fields on the 2-dimensional
 differentiable manifold M
sage: A in Modules(CM)
True
sage: A.base_ring() is CM
True
sage: A.base_module()
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A.base_module() is XM
True
```

Elements can be constructed from `A()`. In particular, `0` yields the zero element of `A`:

```
sage: z = A(0) ; z
2-vector field zero on the 2-dimensional differentiable
 manifold M
sage: z.display(eU)
```

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```
zero = 0
sage: z.display(eV)
zero = 0
sage: z is A.zero()
True
```

while non-zero elements are constructed by providing their components in a given vector frame:

```
sage: a = A([[0,3*x],[-3*x,0]], frame=eU, name='a') ; a
2-vector field a on the 2-dimensional differentiable manifold M
sage: a.add_comp_by_continuation(eV, W, c_uv) # finishes initializ. of a
sage: a.display(eU)
a = 3*x d/dx/\d/dy
sage: a.display(eV)
a = (-3*u - 3*v) d/du/\d/dv
```

An alternative is to construct the 2-vector field from an empty list of components and to set the nonzero nonredundant components afterwards:

```
sage: a = A([], name='a')
sage: a[eU,0,1] = 3*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = 3*x d/dx/\d/dy
sage: a.display(eV)
a = (-3*u - 3*v) d/du/\d/dv
```

The module $A^1(M)$ is nothing but the dual of $\mathfrak{X}(M)$ (the module of vector fields on M):

```
sage: A1 = M.multivector_module(1) ; A1
Module X(M) of vector fields on the 2-dimensional differentiable
 manifold M
sage: A1 is XM
True
```

There is a coercion map $A^p(M) \to T^{(p,0)}(M)$:

```
sage: T20 = M.tensor_field_module((2,0)) ; T20
Module T^(2,0)(M) of type-(2,0) tensors fields on the
 2-dimensional differentiable manifold M
sage: T20.has_coerce_map_from(A)
True
```

but of course not in the reverse direction, since not all contravariant tensor field is alternating:

```
sage: A.has_coerce_map_from(T20)
False
```

The coercion map $A^2(M) \to T^{(2,0)}(M)$ in action:

```
sage: ta = T20(a) ; ta
Tensor field a of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: ta.display(eU)
a = 3*x d/dx*d/dy - 3*x d/dy*d/dx
sage: a.display(eU)
```

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```
a = 3*x d/dx/\d/dy
sage: ta.display(eV)
a = (-3*u - 3*v) d/du*d/dv + (3*u + 3*v) d/dv*d/du
sage: a.display(eV)
a = (-3*u - 3*v) d/du/\d/dv
```

There is also coercion to subdomains, which is nothing but the restriction of the multivector field to some subset of its domain:

```
sage: A2U = U.multivector_module(2) ; A2U
Free module A^2(U) of 2-vector fields on the Open subset U of
 the 2-dimensional differentiable manifold M
sage: A2U.has_coerce_map_from(A)
True
sage: a_U = A2U(a) ; a_U
2-vector field a on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: a_U.display(eU)
a = 3*x d/dx/\d/dy
```

Element
alias of *sage.manifolds.differentiable.multivectorfield.MultivectorField*

base_module()
Return the vector field module on which the multivector field module `self` is constructed.

OUTPUT:

- a *VectorFieldModule* representing the module on which `self` is defined

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A2 = M.multivector_module(2) ; A2
Module A^2(M) of 2-vector fields on the 3-dimensional
 differentiable manifold M
sage: A2.base_module()
Module X(M) of vector fields on the 3-dimensional
 differentiable manifold M
sage: A2.base_module() is M.vector_field_module()
True
sage: U = M.open_subset('U')
sage: A2U = U.multivector_module(2) ; A2U
Module A^2(U) of 2-vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
sage: A2U.base_module()
Module X(U) of vector fields on the Open subset U of the
 3-dimensional differentiable manifold M
```

degree()
Return the degree of the multivector fields in `self`.

OUTPUT:

- integer p such that `self` is a set of p-vector fields

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: M.multivector_module(2).degree()
2
sage: M.multivector_module(3).degree()
3
```

zero()

Return the zero of `self`.

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: A2 = M.multivector_module(2)
sage: A2.zero()
2-vector field zero on the 3-dimensional differentiable
 manifold M
```

2.11.2 Multivector Fields

Let U and M be two differentiable manifolds. Given a positive integer p and a differentiable map $\Phi : U \rightarrow M$, a *multivector field of degree* p, or p*-vector field*, *along* U *with values on* M is a field along U of alternating contravariant tensors of rank p in the tangent spaces to M. The standard case of a multivector field *on* a differentiable manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Two classes implement multivector fields, depending whether the manifold M is parallelizable:

- *MultivectorFieldParal* when M is parallelizable
- *MultivectorField* when M is not assumed parallelizable.

AUTHORS:

- Eric Gourgoulhon (2017): initial version

REFERENCES:

- R. L. Bishop and S. L. Goldberg (1980) [**?**]
- C.-M. Marle (1997) [**?**]

class sage.manifolds.differentiable.multivectorfield.**MultivectorField**(*vector_field_module*, *degree*, *name=None*, *latex_name=None*)

Bases: *sage.manifolds.differentiable.tensorfield.TensorField*

Multivector field with values on a generic (i.e. a priori not parallelizable) differentiable manifold.

Given a differentiable manifold U, a differentiable map $\Phi : U \rightarrow M$ to a differentiable manifold M and a positive integer p, a *multivector field of degree* p (or p*-vector field*) *along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a : U \longrightarrow T^{(p,0)}M$$

($T^{(p,0)}M$ being the tensor bundle of type $(p, 0)$ over M) such that

$$\forall x \in U, \quad a(x) \in \Lambda^p(T_{\Phi(x)}M),$$

where $T_{\Phi(x)}M$ is the vector space tangent to M at $\Phi(x)$ and Λ^p stands for the exterior power of degree p (cf. `ExtPowerFreeModule`). In other words, $a(x)$ is an alternating contravariant tensor of degree p of the tangent vector space $T_{\Phi(x)}M$.

The standard case of a multivector field *on* a manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is parallelizable, the class `MultivectorFieldParal` must be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on M via the map Φ
- `degree` – the degree of the multivector field (i.e. its tensor rank)
- `name` – (default: `None`) name given to the multivector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the multivector field; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

Multivector field of degree 2 on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y),
....:                        intersection_name='W',
....:                        restrictions1= x>0, restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: a = M.multivector_field(2, name='a') ; a
2-vector field a on the 2-dimensional differentiable manifold M
sage: a.parent()
Module A^2(M) of 2-vector fields on the 2-dimensional differentiable
 manifold M
sage: a.degree()
2
```

Setting the components of `a`:

```
sage: a[eU,0,1] = x*y^2 + 2*x
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = (x*y^2 + 2*x) d/dx/\d/dy
sage: a.display(eV)
a = (-1/4*u^3 + 1/4*u*v^2 - 1/4*v^3 + 1/4*(u^2 - 8)*v - 2*u) d/du/\d/dv
```

It is also possible to set the components while defining the 2-vector field definition, via a dictionary whose keys are the vector frames:

```
sage: a1 = M.multivector_field(2, {eU: [[0, x*y^2 + 2*x],
....:                          [-x*y^2 - 2*x, 0]]}, name='a')
sage: a1.add_comp_by_continuation(eV, W, c_uv)
sage: a1 == a
True
```

The exterior product of two vector fields is a 2-vector field:

```
sage: a = M.vector_field({eU: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: b = M.vector_field({eU: [1+x*y, x^2]}, name='b')
sage: b.add_comp_by_continuation(eV, W, c_uv)
sage: s = a.wedge(b) ; s
2-vector field a/\b on the 2-dimensional differentiable manifold M
sage: s.display(eU)
a/\b = (-2*x^2*y - x) d/dx/\d/dy
sage: s.display(eV)
a/\b = (1/2*u^3 - 1/2*u*v^2 - 1/2*v^3 + 1/2*(u^2 + 2)*v + u) d/du/\d/dv
```

Multiplying a 2-vector field by a scalar field results in another 2-vector field:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_uv: u^2}, name='f')
sage: s = f*s ; s
2-vector field on the 2-dimensional differentiable manifold M
sage: s.display(eU)
(-2*x^2*y^3 - x^3 - (4*x^3 + x)*y^2 - 2*(x^4 + x^2)*y) d/dx/\d/dy
sage: s.display(eV)
(1/2*u^5 - 1/2*u^3*v^2 - 1/2*u^2*v^3 + u^3 + 1/2*(u^4 + 2*u^2)*v)
  d/du/\d/dv
```

bracket (*other*)

Return the Schouten-Nijenhuis bracket of `self` with another multivector field.

The Schouten-Nijenhuis bracket extends the Lie bracket of vector fields (cf. `bracket()`) to multivector fields.

Denoting by $A^p(M)$ the $C^k(M)$-module of p-vector fields on the C^k-differentiable manifold M over the field K (cf. `MultivectorModule`), the *Schouten-Nijenhuis bracket* is a K-bilinear map

$$\begin{array}{ccc} A^p(M) \times A^q(M) & \longrightarrow & A^{p+q-1}(M) \\ (a,b) & \longmapsto & [a,b] \end{array}$$

which obeys the following properties:

- if $p = 0$ and $q = 0$, (i.e. a and b are two scalar fields), $[a, b] = 0$
- if $p = 0$ (i.e. a is a scalar field) and $q \geq 1$, $[a, b] = -\iota_{\mathrm{d}a} b$ (minus the interior product of the differential of a by b)
- if $p = 1$ (i.e. a is a vector field), $[a, b] = \mathcal{L}_a b$ (the Lie derivative of b along a)
- $[a, b] = -(-1)^{(p-1)(q-1)}[b, a]$
- for any multivector field c and $(a, b) \in A^p(M) \times A^q(M)$, $[a, .]$ obeys the *graded Leibniz rule*

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(p-1)q} b \wedge [a, c]$$

- for $(a, b, c) \in A^p(M) \times A^q(M) \times A^r(M)$, the *graded Jacobi identity* holds:

$$(-1)^{(p-1)(r-1)}[a, [b, c]] + (-1)^{(q-1)(p-1)}[b, [c, a]] + (-1)^{(r-1)(q-1)}[c, [a, b]] = 0$$

Note: There are two definitions of the Schouten-Nijenhuis bracket in the literature, which differ from each other when p is even by an overall sign. The definition adopted here is that of [**?**], [**?**] and Wikipedia article Schouten-Nijenhuis_bracket. The other definition, adopted e.g. by [**?**], [**?**] and [**?**], is $[a, b]' = (-1)^{p+1}[a, b]$.

INPUT:

- other – a multivector field

OUTPUT:

- instance of *MultivectorField* (or of *DiffScalarField* if $p = 1$ and $q = 0$) representing the Schouten-Nijenhuis bracket $[a, b]$, where a is self and b is other

EXAMPLES:

Bracket of two vector fields on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() # stereographic coord. North
sage: c_uv.<u,v> = V.chart() # stereographic coord. South
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                  intersection_name='W', restrictions1= x^2+y^2!=0,
....:                  restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.vector_field({e_xy: [y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: b = M.vector_field({e_xy: [x*y, x-y]}, name='b')
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: s = a.bracket(b); s
Vector field [a,b] on the 2-dimensional differentiable manifold S^2
sage: s.display(e_xy)
[a,b] = (x^2 + y^2 - x + y) d/dx + (-(x - 1)*y - x) d/dy
```

For two vector fields, the bracket coincides with the Lie derivative:

```
sage: s == b.lie_derivative(a)
True
```

Schouten-Nijenhuis bracket of a 2-vector field and a 1-vector field:

```
sage: c = a.wedge(b); c
2-vector field a/\b on the 2-dimensional differentiable
 manifold S^2
sage: s = c.bracket(a); s
2-vector field [a/\b,a] on the 2-dimensional differentiable
 manifold S^2
sage: s.display(e_xy)
[a/\b,a] = (x^3 + (2*x - 1)*y^2 - x^2 + 2*x*y) d/dx/\d/dy
```

Since a is a vector field, we have in this case:

```
sage: s == - c.lie_derivative(a)
True
```

See also:

MultivectorFieldParal.bracket() for more examples and check of standards identities involving the Schouten-Nijenhuis bracket

degree()

Return the degree of self.

OUTPUT:

- integer p such that `self` is a p-vector field

EXAMPLES:

```
sage: M = Manifold(3, 'M')
sage: a = M.multivector_field(2); a
2-vector field on the 3-dimensional differentiable manifold M
sage: a.degree()
2
sage: b = M.vector_field(); b
Vector field on the 3-dimensional differentiable manifold M
sage: b.degree()
1
```

interior_product (*form*)

Interior product with a differential form.

If `self` is a multivector field A of degree p and B is a differential form of degree $q \geq p$ on the same manifold as A, the interior product of A by B is the differential form $\iota_A B$ of degree $q - p$ defined by

$$(\iota_A B)_{i_1 \ldots i_{q-p}} = A^{k_1 \ldots k_p} B_{k_1 \ldots k_p i_1 \ldots i_{q-p}}$$

Note: `A.interior_product(B)` yields the same result as `A.contract(0, ..., p-1, B, 0, ..., p-1)` (cf. *contract()*), but `interior_product` is more efficient, the alternating character of A being not used to reduce the computation in *contract()*

INPUT:

- `form` – differential form B (instance of *DiffForm*); the degree of B must be at least equal to the degree of `self`

OUTPUT:

- scalar field (case $p = q$) or *DiffForm* (case $p < q$) representing the interior product $\iota_A B$, where A is `self`

See also:

interior_product() for the interior product of a differential form with a multivector field

EXAMPLES:

Interior product of a vector field ($p = 1$) with a 2-form ($q = 2$) on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() # stereographic coord. North
sage: c_uv.<u,v> = V.chart() # stereographic coord. South
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.vector_field({e_xy: [-y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
```

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```
sage: b = M.diff_form(2, name='b')
sage: b[e_xy,1,2] = 4/(x^2+y^2+1)^2   # the standard area 2-form
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: b.display(e_xy)
b = 4/(x^2 + y^2 + 1)^2 dx/\dy
sage: b.display(e_uv)
b = -4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du/\dv
sage: s = a.interior_product(b); s
1-form i_a b on the 2-dimensional differentiable manifold S^2
sage: s.display(e_xy)
i_a b = -4*x/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx
 - 4*y/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dy
sage: s.display(e_uv)
 i_a b = 4*u/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du
  + 4*v/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv
sage: s == a.contract(b)
True
```

Example with $p = 2$ and $q = 2$:

```
sage: a = M.multivector_field(2, name='a')
sage: a[e_xy,1,2] = x*y
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: a.display(e_xy)
a = x*y d/dx/\d/dy
sage: a.display(e_uv)
a = -u*v d/du/\d/dv
sage: s = a.interior_product(b); s
Scalar field i_a b on the 2-dimensional differentiable manifold S^2
sage: s.display()
i_a b: S^2 --> R
on U: (x, y) |--> 8*x*y/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) |--> 8*u*v/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
```

Some checks:

```
sage: s == a.contract(0, 1, b, 0, 1)
True
sage: s.restrict(U) == 2 * a[[e_xy,1,2]] * b[[e_xy,1,2]]
True
sage: s.restrict(V) == 2 * a[[e_uv,1,2]] * b[[e_uv,1,2]]
True
```

wedge (*other*)

Exterior product with another multivector field.

INPUT:

- `other` – another multivector field (on the same manifold)

OUTPUT:

- instance of *MultivectorField* representing the exterior product `self/\other`

EXAMPLES:

Exterior product of two vector fields on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1) # the sphere S^2
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() # stereographic coord. North
sage: c_uv.<u,v> = V.chart() # stereographic coord. South
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: e_xy = c_xy.frame() ; e_uv = c_uv.frame()
sage: a = M.vector_field({e_xy: [y, x]}, name='a')
sage: a.add_comp_by_continuation(e_uv, W, c_uv)
sage: b = M.vector_field({e_xy: [x^2 + y^2, y]}, name='b')
sage: b.add_comp_by_continuation(e_uv, W, c_uv)
sage: c = a.wedge(b); c
2-vector field a/\b on the 2-dimensional differentiable
 manifold S^2
sage: c.display(e_xy)
a/\b = (-x^3 - (x - 1)*y^2) d/dx/\d/dy
sage: c.display(e_uv)
a/\b = (-v^2 + u) d/du/\d/dv
```

class `sage.manifolds.differentiable.multivectorfield.`**`MultivectorFieldParal`**(*vector_field_module*, *degree*, *name=None*, *latex_name=None*)

Bases: `sage.tensor.modules.alternating_contr_tensor.AlternatingContrTensor`, *`sage.manifolds.differentiable.tensorfield_paral.TensorFieldParal`*

Multivector field with values on a parallelizable manifold.

Given a differentiable manifold U, a differentiable map $\Phi : U \to M$ to a parallelizable manifold M and a positive integer p, a *multivector field of degree* p (or p*-vector field*) *along* U *with values on* $M \supset \Phi(U)$ is a differentiable map

$$a:\ U \longrightarrow T^{(p,0)}M$$

($T^{(p,0)}M$ being the tensor bundle of type $(p,0)$ over M) such that

$$\forall x \in U, \quad a(x) \in \Lambda^p(T_{\Phi(x)}M),$$

where $T_{\Phi(x)}M$ is the vector space tangent to M at $\Phi(x)$ and Λ^p stands for the exterior power of degree p (cf. `ExtPowerFreeModule`). In other words, $a(x)$ is an alternating contravariant tensor of degree p of the tangent vector space $T_{\Phi(x)}M$.

The standard case of a multivector field *on* a manifold M corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

Note: If M is not parallelizable, the class *`MultivectorField`* must be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U,\Phi)$ of vector fields along U with values on M via the map Φ

- `degree` – the degree of the multivector field (i.e. its tensor rank)
- `name` – (default: `None`) name given to the multivector field
- `latex_name` – (default: `None`) LaTeX symbol to denote the multivector field; if none is provided, the LaTeX symbol is set to `name`

EXAMPLES:

A 2-vector field on a 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: c_txyz.<t,x,y,z> = M.chart()
sage: a = M.multivector_field(2, name='a') ; a
2-vector field a on the 4-dimensional differentiable manifold M
sage: a.parent()
Free module A^2(M) of 2-vector fields on the 4-dimensional
 differentiable manifold M
```

A multivector field is a tensor field of purely contravariant type:

```
sage: a.tensor_type()
(2, 0)
```

It is antisymmetric, its components being `CompFullyAntiSym`:

```
sage: a.symmetries()
no symmetry;  antisymmetry: (0, 1)
sage: a[0,1] = 2*x
sage: a[1,0]
-2*x
sage: a.comp()
Fully antisymmetric 2-indices components w.r.t. Coordinate frame
 (M, (d/dt,d/dx,d/dy,d/dz))
sage: type(a.comp())
<class 'sage.tensor.modules.comp.CompFullyAntiSym'>
```

Setting a component with repeated indices to a non-zero value results in an error:

```
sage: a[1,1] = 3
Traceback (most recent call last):
...
ValueError: by antisymmetry, the component cannot have a nonzero value
 for the indices (1, 1)
sage: a[1,1] = 0  # OK, albeit useless
sage: a[1,2] = 3  # OK
```

The expansion of a multivector field with respect to a given frame is displayed via the method `display()`:

```
sage: a.display() # expansion w.r.t. the default frame
a = 2*x d/dt/\d/dx + 3 d/dx/\d/dy
sage: latex(a.display()) # output for the notebook
a = 2 \, x \frac{\partial}{\partial t }\wedge \frac{\partial}{\partial x }
 + 3 \frac{\partial}{\partial x }\wedge \frac{\partial}{\partial y }
```

Multivector fields can be added or subtracted:

```
sage: b = M.multivector_field(2)
sage: b[0,1], b[0,2], b[0,3] = y, 2, x+z
```

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```
sage: s = a + b ; s
2-vector field on the 4-dimensional differentiable manifold M
sage: s.display()
(2*x + y) d/dt/\d/dx + 2 d/dt/\d/dy + (x + z) d/dt/\d/dz + 3 d/dx/\d/dy
sage: s = a - b ; s
2-vector field on the 4-dimensional differentiable manifold M
sage: s.display()
(2*x - y) d/dt/\d/dx - 2 d/dt/\d/dy + (-x - z) d/dt/\d/dz + 3 d/dx/\d/dy
```

An example of 3-vector field in $\mathbf{R}^3$ with Cartesian coordinates:

```
sage: M = Manifold(3, 'R3', latex_name=r'\RR^3', start_index=1)
sage: c_cart.<x,y,z> = M.chart()
sage: a = M.multivector_field(3, name='a')
sage: a[1,2,3] = x^2+y^2+z^2  # the only independent component
sage: a[:] # all the components are set from the previous line:
[[[0, 0, 0], [0, 0, x^2 + y^2 + z^2], [0, -x^2 - y^2 - z^2, 0]],
 [[0, 0, -x^2 - y^2 - z^2], [0, 0, 0], [x^2 + y^2 + z^2, 0, 0]],
 [[0, x^2 + y^2 + z^2, 0], [-x^2 - y^2 - z^2, 0, 0], [0, 0, 0]]]
sage: a.display()
a = (x^2 + y^2 + z^2) d/dx/\d/dy/\d/dz
```

Spherical components from the tensorial change-of-frame formula:

```
sage: c_spher.<r,th,ph> = M.chart(r'r:[0,+oo) th:[0,pi]:\theta ph:[0,2*pi):\phi')
sage: spher_to_cart = c_spher.transition_map(c_cart,
....:                 [r*sin(th)*cos(ph), r*sin(th)*sin(ph), r*cos(th)])
sage: cart_to_spher = spher_to_cart.set_inverse(sqrt(x^2+y^2+z^2),
....:                                atan2(sqrt(x^2+y^2),z), atan2(y, x))
sage: a.comp(c_spher.frame()) # computation of components w.r.t. spherical frame
Fully antisymmetric 3-indices components w.r.t. Coordinate frame
 (R3, (d/dr,d/dth,d/dph))
sage: a.comp(c_spher.frame())[1,2,3, c_spher]
1/sin(th)
sage: a.display(c_spher.frame())
a = sqrt(x^2 + y^2 + z^2)/sqrt(x^2 + y^2) d/dr/\d/dth/\d/dph
sage: a.display(c_spher.frame(), c_spher)
a = 1/sin(th) d/dr/\d/dth/\d/dph
```

As a shortcut of the above command, on can pass just the chart `c_spher` to `display`, the vector frame being then assumed to be the coordinate frame associated with the chart:

```
sage: a.display(c_spher)
a = 1/sin(th) d/dr/\d/dth/\d/dph
```

The exterior product of two multivector fields is performed via the method `wedge()`:

```
sage: a = M.vector_field([x*y, -z*x, y], name='A')
sage: b = M.vector_field([y, z+y, x^2-z^2], name='B')
sage: ab = a.wedge(b) ; ab
2-vector field A/\B on the 3-dimensional differentiable manifold R3
sage: ab.display()
A/\B = (x*y^2 + 2*x*y*z) d/dx/\d/dy + (x^3*y - x*y*z^2 - y^2) d/dx/\d/dz
 + (x*z^3 - y^2 - (x^3 + y)*z) d/dy/\d/dz
```

Let us check the formula relating the exterior product to the tensor product for vector fields:

```
sage: a.wedge(b) == a*b - b*a
True
```

The tensor product of a vector field and a 2-vector field is not a 3-vector field but a tensor field of type $(3, 0)$ with less symmetries:

```
sage: c = a*ab ; c
Tensor field A*(A/\B) of type (3,0) on the 3-dimensional differentiable
 manifold R3
sage: c.symmetries()  # the antisymmetry is only w.r.t. the last 2 arguments:
no symmetry;  antisymmetry: (1, 2)
```

The Lie derivative of a 2-vector field is a 2-vector field:

```
sage: ab.lie_der(a)
2-vector field on the 3-dimensional differentiable manifold R3
```

bracket (*other*)

Return the Schouten-Nijenhuis bracket of `self` with another multivector field.

The Schouten-Nijenhuis bracket extends the Lie bracket of vector fields (cf. `bracket()`) to multivector fields.

Denoting by $A^p(M)$ the $C^k(M)$-module of p-vector fields on the C^k-differentiable manifold M over the field K (cf. `MultivectorModule`), the *Schouten-Nijenhuis bracket* is a K-bilinear map

$$\begin{array}{ccc} A^p(M) \times A^q(M) & \longrightarrow & A^{p+q-1}(M) \\ (a, b) & \longmapsto & [a, b] \end{array}$$

which obeys the following properties:

- if $p = 0$ and $q = 0$, (i.e. a and b are two scalar fields), $[a, b] = 0$
- if $p = 0$ (i.e. a is a scalar field) and $q \geq 1$, $[a, b] = -\iota_{\mathrm{d}a} b$ (minus the interior product of the differential of a by b)
- if $p = 1$ (i.e. a is a vector field), $[a, b] = \mathcal{L}_a b$ (the Lie derivative of b along a)
- $[a, b] = -(-1)^{(p-1)(q-1)}[b, a]$
- for any multivector field c and $(a, b) \in A^p(M) \times A^q(M)$, $[a, .]$ obeys the *graded Leibniz rule*

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(p-1)q} b \wedge [a, c]$$

- for $(a, b, c) \in A^p(M) \times A^q(M) \times A^r(M)$, the *graded Jacobi identity* holds:

$$(-1)^{(p-1)(r-1)}[a, [b, c]] + (-1)^{(q-1)(p-1)}[b, [c, a]] + (-1)^{(r-1)(q-1)}[c, [a, b]] = 0$$

Note: There are two definitions of the Schouten-Nijenhuis bracket in the literature, which differ from each other when p is even by an overall sign. The definition adopted here is that of [**?**], [**?**] and Wikipedia article Schouten-Nijenhuis_bracket. The other definition, adopted e.g. by [**?**], [**?**] and [**?**], is $[a, b]' = (-1)^{p+1}[a, b]$.

INPUT:

- `other` – a multivector field

OUTPUT:

- instance of `MultivectorFieldParal` (or of `DiffScalarField` if $p = 1$ and $q = 0$) representing the Schouten-Nijenhuis bracket $[a, b]$, where a is `self` and b is `other`

EXAMPLES:

Let us consider two vector fields on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: a = M.vector_field([x*y+z, x+y-z, z-2*x+y], name='a')
sage: b = M.vector_field([y+2*z-x, x^2-y+z, z-x], name='b')
```

and form their Schouten-Nijenhuis bracket:

```
sage: s = a.bracket(b); s
Vector field [a,b] on the 3-dimensional differentiable manifold M
sage: s.display()
[a,b] = (-x^3 + (x + 3)*y - y^2 - (x + 2*y + 1)*z - 2*x) d/dx
 + (2*x^2*y - x^2 + 2*x*z - 3*x) d/dy
 + (-x^2 - (x - 4)*y - 3*x + 2*z) d/dz
```

Check that $[a, b]$ is actually the Lie bracket:

```
sage: f = M.scalar_field({X: x+y*z}, name='f')
sage: s(f) == a(b(f)) - b(a(f))
True
```

Check that $[a, b]$ coincides with the Lie derivative of b along a:

```
sage: s == b.lie_derivative(a)
True
```

Schouten-Nijenhuis bracket for $p = 0$ and $q = 1$:

```
sage: s = f.bracket(a); s
Scalar field -i_df a on the 3-dimensional differentiable manifold M
sage: s.display()
-i_df a: M --> R
   (x, y, z) |--> x*y - y^2 - (x + 2*y + 1)*z + z^2
```

Check that $[f, a] = -\iota_{df} a = -df(a)$:

```
sage: s == - f.differential()(a)
True
```

Schouten-Nijenhuis bracket for $p = 0$ and $q = 2$:

```
sage: c = M.multivector_field(2, name='c')
sage: c[0,1], c[0,2], c[1,2] = x+z+1, x*y+z, x-y
sage: s = f.bracket(c); s
Vector field -i_df c on the 3-dimensional differentiable manifold M
sage: s.display()
-i_df c = (x*y^2 + (x + y + 1)*z + z^2) d/dx
 + (x*y - y^2 - x - z - 1) d/dy + (-x*y - (x - y + 1)*z) d/dz
```

Check that $[f, c] = -\iota_{df} c$:

```
sage: s == - f.differential().interior_product(c)
True
```

Schouten-Nijenhuis bracket for $p = 1$ and $q = 2$:

```
sage: s = a.bracket(c); s
2-vector field [a,c] on the 3-dimensional differentiable manifold M
sage: s.display()
[a,c] = ((x - 1)*y - (y - 2)*z - 2*x - 1) d/dx/\d/dy
 + ((x + 1)*y - (x + 1)*z - 3*x - 1) d/dx/\d/dz
 + (-5*x + y - z - 2) d/dy/\d/dz
```

Again, since a is a vector field, the Schouten-Nijenhuis bracket coincides with the Lie derivative:

```
sage: s == c.lie_derivative(a)
True
```

Schouten-Nijenhuis bracket for $p = 2$ and $q = 2$:

```
sage: d = M.multivector_field(2, name='d')
sage: d[0,1], d[0,2], d[1,2] = x-y^2, x+z, z-x-1
sage: s = c.bracket(d); s
3-vector field [c,d] on the 3-dimensional differentiable manifold M
sage: s.display()
[c,d] = (-y^3 + (3*x + 1)*y - y^2 - x - z + 2) d/dx/\d/dy/\d/dz
```

Let us check the component formula (with respect to the manifold's default coordinate chart, i.e. `X`) for $p = q = 2$, taking into account the tensor antisymmetries:

```
sage: s[0,1,2] == - sum(c[i,0]*d[1,2].diff(i)
....:                   + c[i,1]*d[2,0].diff(i) + c[i,2]*d[0,1].diff(i)
....:                   + d[i,0]*c[1,2].diff(i) + d[i,1]*c[2,0].diff(i)
....:                   + d[i,2]*c[0,1].diff(i) for i in M.irange())
True
```

Schouten-Nijenhuis bracket for $p = 1$ and $q = 3$:

```
sage: e = M.multivector_field(3, name='e')
sage: e[0,1,2] = x+y*z+1
sage: s = a.bracket(e); s
3-vector field [a,e] on the 3-dimensional differentiable manifold M
sage: s.display()
[a,e] = (-(2*x + 1)*y + y^2 - (y^2 - x - 1)*z - z^2
 - 2*x - 2) d/dx/\d/dy/\d/dz
```

Again, since $p = 1$, the bracket coincides with the Lie derivative:

```
sage: s == e.lie_derivative(a)
True
```

Schouten-Nijenhuis bracket for $p = 2$ and $q = 3$:

```
sage: s = c.bracket(e); s
4-vector field [c,e] on the 3-dimensional differentiable manifold M
```

Since on a 3-dimensional manifold, any 4-vector field is zero, we have:

```
sage: s.display()
[c,e] = 0
```

Let us check the graded commutation law $[a, b] = -(-1)^{(p-1)(q-1)}[b, a]$ for various values of p and q:

```
sage: f.bracket(a) == - a.bracket(f)   # p=0 and q=1
True
sage: f.bracket(c) == c.bracket(f)     # p=0 and q=2
True
sage: a.bracket(b) == - b.bracket(a)   # p=1 and q=1
True
sage: a.bracket(c) == - c.bracket(a)   # p=1 and q=2
True
sage: c.bracket(d) == d.bracket(c)     # p=2 and q=2
True
```

Let us check the graded Leibniz rule for $p = 1$ and $q = 1$:

```
sage: a.bracket(b.wedge(c)) == a.bracket(b).wedge(c) + b.wedge(a.bracket(c))
True
```

as well as for $p = 2$ and $q = 1$:

```
sage: c.bracket(a.wedge(b)) == c.bracket(a).wedge(b) - a.wedge(c.bracket(b))
True
```

Finally let us check the graded Jacobi identity for $p = 1$, $q = 1$ and $r = 2$:

```
sage: a.bracket(b.bracket(c)) + b.bracket(c.bracket(a)) \
....: + c.bracket(a.bracket(b)) == 0
True
```

as well as for $p = 1$, $q = 2$ and $r = 2$:

```
sage: a.bracket(c.bracket(d)) + c.bracket(d.bracket(a)) \
....: - d.bracket(a.bracket(c)) == 0
True
```

interior_product (*form*)

Interior product with a differential form.

If `self` is a multivector field A of degree p and B is a differential form of degree $q \geq p$ on the same manifold as A, the interior product of A by B is the differential form $\iota_A B$ of degree $q - p$ defined by

$$(\iota_A B)_{i_1 \ldots i_{q-p}} = A^{k_1 \ldots k_p} B_{k_1 \ldots k_p i_1 \ldots i_{q-p}}$$

Note: `A.interior_product(B)` yields the same result as `A.contract(0, ..., p-1, B, 0, ..., p-1)` (cf. *contract()*), but `interior_product` is more efficient, the alternating character of A being not used to reduce the computation in *contract()*

INPUT:

- `form` – differential form B (instance of *DiffFormParal*); the degree of B must be at least equal to the degree of `self`

OUTPUT:

- scalar field (case $p = q$) or *DiffFormParal* (case $p < q$) representing the interior product $\iota_A B$, where A is `self`

See also:

interior_product() for the interior product of a differential form with a multivector field

EXAMPLES:

Interior product with $p = 1$ and $q = 1$ on 4-dimensional manifold:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: a = M.vector_field([x, 1+t^2, x*z, y-3], name='a')
sage: b = M.one_form([-z^2, 2, x, x-y], name='b')
sage: s = a.interior_product(b); s
Scalar field i_a b on the 4-dimensional differentiable manifold M
sage: s.display()
i_a b: M --> R
   (t, x, y, z) |--> x^2*z - x*z^2 + 2*t^2 + (x + 3)*y - y^2
    - 3*x + 2
```

In this case, we have $\iota_a b = a^i b_i = a(b) = b(a)$:

```
sage: all([s == a.contract(b), s == a(b), s == b(a)])
True
```

Case $p = 1$ and $q = 3$:

```
sage: c = M.diff_form(3, name='c')
sage: c[0,1,2], c[0,1,3] = x*y - z, -3*t
sage: c[0,2,3], c[1,2,3] = t+x, y
sage: s = a.interior_product(c); s
2-form i_a c on the 4-dimensional differentiable manifold M
sage: s.display()
i_a c = (x^2*y*z - x*z^2 - 3*t*y + 9*t) dt/\dx
 + (-(t^2*x - t)*y + (t^2 + 1)*z - 3*t - 3*x) dt/\dy
 + (3*t^3 - (t*x + x^2)*z + 3*t) dt/\dz
 + ((x^2 - 3)*y + y^2 - x*z) dx/\dy
 + (-x*y*z - 3*t*x) dx/\dz + (t*x + x^2 + (t^2 + 1)*y) dy/\dz
sage: s == a.contract(c)
True
```

Case $p = 2$ and $q = 3$:

```
sage: d = M.multivector_field(2, name='d')
sage: d[0,1], d[0,2], d[0,3] = t-x, 2*z, y-1
sage: d[1,2], d[1,3], d[2,3] = z, y+t, 4
sage: s = d.interior_product(c); s
1-form i_d c on the 4-dimensional differentiable manifold M
sage: s.display()
i_d c = (2*x*y*z - 6*t^2 - 6*t*y - 2*z^2 + 8*t + 8*x) dt
 + (-4*x*y*z + 2*(3*t + 4)*y + 4*z^2 - 6*t) dx
 + (2*((t - 1)*x - x^2 - 2*t)*y - 2*y^2 - 2*(t - x)*z + 2*t
 + 2*x) dy + (-6*t^2 + 6*t*x + 2*(2*t + 2*x + y)*z) dz
sage: s == d.contract(0, 1, c, 0, 1)
True
```

wedge (*other*)

Exterior product of `self` with another multivector field.

INPUT:

- `other` – another multivector field

OUTPUT:

- instance of *MultivectorFieldParal* representing the exterior product `self/\other`

EXAMPLES:

Exterior product of a vector field and a 2-vector field on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: a = M.vector_field([2, 1+x, y*z], name='a')
sage: b = M.multivector_field(2, name='b')
sage: b[1,2], b[1,3], b[2,3] = y^2, z+x, z^2
sage: a.display()
a = 2 d/dx + (x + 1) d/dy + y*z d/dz
sage: b.display()
b = y^2 d/dx/\d/dy + (x + z) d/dx/\d/dz + z^2 d/dy/\d/dz
sage: s = a.wedge(b); s
3-vector field a/\b on the 3-dimensional differentiable
 manifold M
sage: s.display()
a/\b = (-x^2 + (y^3 - x - 1)*z + 2*z^2 - x) d/dx/\d/dy/\d/dz
```

Check:

```
sage: s[1,2,3] == a[1]*b[2,3] + a[2]*b[3,1] + a[3]*b[1,2]
True
```

2.12 Affine Connections

The class *AffineConnection* implements affine connections on smooth manifolds.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Marco Mancini (2015) : parallelization of some computations
- Florentin Jaffredo (2018) : series expansion with respect to a given parameter

REFERENCES:

- [?]
- [?]
- [?]

class `sage.manifolds.differentiable.affine_connection.`**`AffineConnection`**(*domain*, *name*, *latex_name=None*)

Bases: `sage.structure.sage_object.SageObject`

Affine connection on a smooth manifold.

Let M be a differentiable manifold of class C^∞ (smooth manifold) over a non-discrete topological field K (in most applications $K = \mathbf{R}$ or $K = \mathbf{C}$), let $C^\infty(M)$ be the algebra of smooth functions $M \to K$ (cf. *DiffScalarFieldAlgebra*) and let $\mathfrak{X}(M)$ be the $C^\infty(M)$-module of vector fields on M (cf.

VectorFieldModule). An *affine connection* on M is an operator

$$\begin{array}{cccc}\nabla : & \mathfrak{X}(M)\times\mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ & (u,v) & \longmapsto & \nabla_u v\end{array}$$

that

- is K-bilinear, i.e. is bilinear when considering $\mathfrak{X}(M)$ as a vector space over K
- is $C^\infty(M)$-linear w.r.t. the first argument: $\forall f\in C^\infty(M),\ \nabla_{fu}v = f\nabla_u v$
- obeys Leibniz rule w.r.t. the second argument: $\forall f\in C^\infty(M),\ \nabla_u(fv) = \mathrm{d}f(u)\,v + f\nabla_u v$

The affine connection ∇ gives birth to the *covariant derivative operator* acting on tensor fields, denoted by the same symbol:

$$\begin{array}{cccc}\nabla : & T^{(k,l)}(M) & \longrightarrow & T^{(k,l+1)}(M) \\ & t & \longmapsto & \nabla t\end{array}$$

where $T^{(k,l)}(M)$ stands for the $C^\infty(M)$-module of tensor fields of type (k,l) on M (cf. TensorFieldModule), with the convention $T^{(0,0)}(M) := C^\infty(M)$. For a vector field v, the covariant derivative ∇v is a type-(1,1) tensor field such that

$$\forall u\in\mathfrak{X}(M),\ \nabla_u v = \nabla v(.,u)$$

More generally for any tensor field $t\in T^{(k,l)}(M)$, we have

$$\forall u\in\mathfrak{X}(M),\ \nabla_u t = \nabla t(\ldots,u)$$

Note: The above convention means that, in terms of index notation, the "derivation index" in ∇t is the *last* one:

$$\nabla_c t^{a_1\ldots a_k}{}_{b_1\ldots b_l} = (\nabla t)^{a_1\ldots a_k}{}_{b_1\ldots b_l c}$$

INPUT:

- `domain` – the manifold on which the connection is defined (must be an instance of class DifferentiableManifold)
- `name` – name given to the affine connection
- `latex_name` – (default: `None`) LaTeX symbol to denote the affine connection; if `None`, it is set to `name`.

EXAMPLES:

Affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the 3-dimensional differentiable manifold M
```

A just-created connection has no connection coefficients:

```
sage: nab._coefficients
{}
```

The connection coefficients relative to the manifold's default frame [here $(\partial/\partial x,\partial/\partial y,\partial/\partial z)$], are created by providing the relevant indices inside square brackets:

```
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} = yz
sage: nab._coefficients
{Coordinate frame (M, (d/dx,d/dy,d/dz)): 3-indices components w.r.t.
 Coordinate frame (M, (d/dx,d/dy,d/dz))}
```

If not the default one, the vector frame w.r.t. which the connection coefficients are defined can be specified as the first argument inside the square brackets; hence the above definition is equivalent to:

```
sage: nab[c_xyz.frame(), 1,1,2], nab[c_xyz.frame(),3,2,3] = x^2, y*z
sage: nab._coefficients
{Coordinate frame (M, (d/dx,d/dy,d/dz)): 3-indices components w.r.t.
 Coordinate frame (M, (d/dx,d/dy,d/dz))}
```

Unset components are initialized to zero:

```
sage: nab[:] # list of coefficients relative to the manifold's default vector␣
↪frame
[[[0, x^2, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, y*z], [0, 0, 0]]]
```

The treatment of connection coefficients in a given vector frame is similar to that of tensor components; see therefore the class `TensorField` for the documentation. In particular, the square brackets return the connection coefficients as instances of `ChartFunction`, while the double square brackets return a scalar field:

```
sage: nab[1,1,2]
x^2
sage: nab[1,1,2].display()
(x, y, z) |--> x^2
sage: type(nab[1,1,2])
<class 'sage.manifolds.chart_func.ChartFunctionRing_with_category.element_class'>
sage: nab[[1,1,2]]
Scalar field on the 3-dimensional differentiable manifold M
sage: nab[[1,1,2]].display()
M --> R
(x, y, z) |--> x^2
sage: nab[[1,1,2]].coord_function() is nab[1,1,2]
True
```

Action on a scalar field:

```
sage: f = M.scalar_field(x^2 - y^2, name='f')
sage: Df = nab(f) ; Df
1-form df on the 3-dimensional differentiable manifold M
sage: Df[:]
[2*x, -2*y, 0]
```

The action of an affine connection on a scalar field must coincide with the differential:

```
sage: Df == f.differential()
True
```

A generic affine connection has some torsion:

```
sage: DDf = nab(Df) ; DDf
Tensor field nabla(df) of type (0,2) on the 3-dimensional
 differentiable manifold M
```

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```
sage: DDf.antisymmetrize()[:] # nabla does not commute on scalar fields:
[    0 -x^3    0]
[ x^3    0    0]
[    0    0    0]
```

Let us check the standard formula

$$\nabla_j \nabla_i f - \nabla_i \nabla_j f = T^k_{\ \ ij} \nabla_k f,$$

where the $T^k_{\ \ ij}$'s are the components of the connection's torsion tensor:

```
sage: 2*DDf.antisymmetrize() == nab.torsion().contract(0,Df)
True
```

The connection acting on a vector field:

```
sage: v = M.vector_field(y*z, x*z, x*y, name='v')
sage: Dv = nab(v) ; Dv
Tensor field nabla(v) of type (1,1) on the 3-dimensional differentiable
 manifold M
sage: Dv[:]
[             0 (x^2*y + 1)*z             y]
[             z             0             x]
[             y             x       x*y*z^2]
```

Another example: connection on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
sage: eUW = c_xyW.frame() ; eVW = c_uvW.frame()
sage: nab = M.affine_connection('nabla', r'\nabla')
```

The connection is first defined on the open subset U by means of its coefficients w.r.t. the frame eU (the manifold's default frame):

```
sage: nab[0,0,0], nab[1,0,1] = x, x*y
```

The coefficients w.r.t the frame eV are deduced by continuation of the coefficients w.r.t. the frame eVW on the open subset $W = U \cap V$:

```
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
....:             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
....:
```

At this stage, the connection is fully defined on all the manifold:

```
sage: nab.coef(eU)[:]
[[[x, 0], [0, 0]], [[0, x*y], [0, 0]]]
sage: nab.coef(eV)[:]
[[[1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v, -1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v],
  [1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v, -1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v]],
 [[-1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v, 1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v],
  [-1/16*u^2 + 1/16*v^2 + 1/8*u + 1/8*v, 1/16*u^2 - 1/16*v^2 + 1/8*u + 1/8*v]]]
```

We may let it act on a vector field defined globally on M:

```
sage: a = M.vector_field({eU: [-y,x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y d/dx + x d/dy
sage: a.display(eV)
a = v d/du - u d/dv
sage: da = nab(a) ; da
Tensor field nabla(a) of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: da.display(eU)
nabla(a) = -x*y d/dx*dx - d/dx*dy + d/dy*dx - x*y^2 d/dy*dy
sage: da.display(eV)
nabla(a) = (-1/16*u^3 + 1/16*u^2*v + 1/16*(u + 2)*v^2 - 1/16*v^3 - 1/8*u^2) d/
↪du*du
 + (1/16*u^3 - 1/16*u^2*v - 1/16*(u - 2)*v^2 + 1/16*v^3 - 1/8*u^2 + 1) d/du*dv
 + (1/16*u^3 - 1/16*u^2*v - 1/16*(u - 2)*v^2 + 1/16*v^3 - 1/8*u^2 - 1) d/dv*du
 + (-1/16*u^3 + 1/16*u^2*v + 1/16*(u + 2)*v^2 - 1/16*v^3 - 1/8*u^2) d/dv*dv
```

A few tests:

```
sage: nab(a.restrict(V)) == da.restrict(V)
True
sage: nab.restrict(V)(a) == da.restrict(V)
True
sage: nab.restrict(V)(a.restrict(U)) == da.restrict(W)
True
sage: nab.restrict(U)(a.restrict(V)) == da.restrict(W)
True
```

Same examples with SymPy as the engine for symbolic calculus:

```
sage: M.set_calculus_method('sympy')
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[0,0,0], nab[1,0,1] = x, x*y
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
....:             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
....:
```

At this stage, the connection is fully defined on all the manifold:

```
sage: nab.coef(eU)[:]
[[[x, 0], [0, 0]], [[0, x*y], [0, 0]]]
sage: nab.coef(eV)[:]
[[[u**2/16 + u/8 - v**2/16 + v/8, -u**2/16 + u/8 + v**2/16 + v/8],
 [u**2/16 + u/8 - v**2/16 + v/8, -u**2/16 + u/8 + v**2/16 + v/8]],
```

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```
[[-u**2/16 + u/8 + v**2/16 + v/8, u**2/16 + u/8 - v**2/16 + v/8],
 [-u**2/16 + u/8 + v**2/16 + v/8, u**2/16 + u/8 - v**2/16 + v/8]]]
```

We may let it act on a vector field defined globally on M:

```
sage: a = M.vector_field({eU: [-y,x]}, name='a')
sage: a.add_comp_by_continuation(eV, W, c_uv)
sage: a.display(eU)
a = -y d/dx + x d/dy
sage: a.display(eV)
a = v d/du - u d/dv
sage: da = nab(a) ; da
Tensor field nabla(a) of type (1,1) on the 2-dimensional differentiable
 manifold M
sage: da.display(eU)
nabla(a) = -x*y d/dx*dx - d/dx*dy + d/dy*dx - x*y**2 d/dy*dy
sage: da.display(eV)
nabla(a) = (-u**3/16 + u**2*v/16 - u**2/8 + u*v**2/16 - v**3/16 + v**2/8) d/du*du
 + (u**3/16 - u**2*v/16 - u**2/8 - u*v**2/16 + v**3/16 + v**2/8 + 1) d/du*dv
 + (u**3/16 - u**2*v/16 - u**2/8 - u*v**2/16 + v**3/16 + v**2/8 - 1) d/dv*du
 + (-u**3/16 + u**2*v/16 - u**2/8 + u*v**2/16 - v**3/16 + v**2/8) d/dv*dv
```

add_coef (*frame=None*)

Return the connection coefficients in a given frame for assignment, keeping the coefficients in other frames.

See method *coef()* for details about the definition of the connection coefficients.

To delete the connection coefficients in other frames, use the method *set_coef()* instead.

INPUT:

- frame – (default: None) vector frame in which the connection coefficients are defined; if None, the default frame of the connection's domain is assumed.

> **Warning:** If the connection has already coefficients in other frames, it is the user's responsibility to make sure that the coefficients to be added are consistent with them.

OUTPUT:

- connection coefficients in the given frame, as an instance of the class Components; if such connection coefficients did not exist previously, they are created. See method *coef()* for the storage convention of the connection coefficients.

EXAMPLES:

Setting the coefficients of an affine connection w.r.t. some coordinate frame:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: eX = X.frame(); eX
Coordinate frame (M, (d/dx,d/dy))
sage: nab.add_coef(eX)
3-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy))
sage: nab.add_coef(eX)[1,2,1] = x*y
sage: nab.display(eX)
Gam^x_yx = x*y
```

Since `eX` is the manifold's default vector frame, its mention may be omitted:

```
sage: nab.add_coef()[1,2,1] = x*y
sage: nab.add_coef()
3-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy))
sage: nab.add_coef()[1,2,1] = x*y
sage: nab.display()
Gam^x_yx = x*y
```

Adding connection coefficients w.r.t. to another vector frame:

```
sage: e = M.vector_frame('e')
sage: nab.add_coef(e)
3-indices components w.r.t. Vector frame (M, (e_1,e_2))
sage: nab.add_coef(e)[2,1,1] = x+y
sage: nab.add_coef(e)[2,1,2] = x-y
sage: nab.display(e)
Gam^2_11 = x + y
Gam^2_12 = x - y
```

The coefficients w.r.t. the frame `eX` have been kept:

```
sage: nab.display(eX)
Gam^x_yx = x*y
```

To delete them, use the method *set_coef()* instead.

coef (*frame=None*)

Return the connection coefficients relative to the given frame.

n being the manifold's dimension, the connection coefficients relative to the vector frame (e_i) are the n^3 scalar fields $\Gamma^k{}_{ij}$ defined by

$$\nabla_{e_j} e_i = \Gamma^k{}_{ij} e_k$$

If the connection coefficients are not known already, they are computed from the above formula.

INPUT:

- `frame` – (default: `None`) vector frame relative to which the connection coefficients are required; if none is provided, the domain's default frame is assumed

OUTPUT:

- connection coefficients relative to the frame `frame`, as an instance of the class `Components` with 3 indices ordered as (k, i, j)

EXAMPLES:

Connection coefficient of an affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} =
↪yz
sage: nab.coef()
3-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy,d/dz))
sage: type(nab.coef())
<class 'sage.tensor.modules.comp.Components'>
```

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```
sage: M.default_frame()
Coordinate frame (M, (d/dx,d/dy,d/dz))
sage: nab.coef() is nab.coef(c_xyz.frame())
True
sage: nab.coef()[:]  # full list of coefficients:
[[[0, x^2, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, y*z], [0, 0, 0]]]
```

connection_form(*i*, *j*, *frame=None*)

Return the connection 1-form corresponding to the given index and vector frame.

The *connection 1-forms* with respect to the frame (e_i) are the n^2 1-forms $\omega^i{}_j$ defined by

$$\nabla_v e_j = \langle \omega^i{}_j, v \rangle \, e_i$$

for any vector v.

The components of $\omega^i{}_j$ in the coframe (e^i) dual to (e_i) are nothing but the connection coefficients $\Gamma^i{}_{jk}$ relative to the frame (e_i):

$$\omega^i{}_j = \Gamma^i{}_{jk} e^k$$

INPUT:

- `i`, `j` – indices identifying the 1-form $\omega^i{}_j$
- `frame` – (default: `None`) vector frame relative to which the connection 1-forms are defined; if `None`, the default frame of the connection's domain is assumed.

OUTPUT:

- the 1-form $\omega^i{}_j$, as an instance of `DiffForm`

EXAMPLES:

Connection 1-forms on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,1], nab[1,1,2], nab[1,1,3] = x*y*z, x^2, -y*z
sage: nab[1,2,3], nab[1,3,1], nab[1,3,2] = -x^3, y^2*z, y^2-x^2
sage: nab[2,1,1], nab[2,1,2], nab[2,2,1] = z^2, x*y*z^2, -x^2
sage: nab[2,3,1], nab[2,3,3], nab[3,1,2] = x^2+y^2+z^2, y^2-z^2, x*y+z^2
sage: nab[3,2,1], nab[3,2,2], nab[3,3,3] = x*y+z, z^3 -y^2, x*z^2 - z*y^2
sage: nab.connection_form(1,1)  # connection 1-form (i,j)=(1,1) w.r.t. M's
↪default frame
1-form nabla connection 1-form (1,1) on the 3-dimensional
 differentiable manifold M
sage: nab.connection_form(1,1)[:]
[x*y*z, x^2, -y*z]
```

The result is cached (until the connection is modified via `set_coef()` or `add_coef()`):

```
sage: nab.connection_form(1,1) is nab.connection_form(1,1)
True
```

Connection 1-forms w.r.t. a non-holonomic frame:

```
sage: ch_basis = M.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2], ch_basis[3,3] = y, z, x
sage: e = M.default_frame().new_frame(ch_basis, 'e')
sage: e[1][:], e[2][:], e[3][:]
([y, 0, 0], [0, z, 0], [0, 0, x])
sage: nab.connection_form(1,1,e)
1-form nabla connection 1-form (1,1) on the 3-dimensional
 differentiable manifold M
sage: nab.connection_form(1,1,e).comp(e)[:]
[x*y^2*z, (x^2*y + 1)*z/y, -x*y*z]
```

Check of the formula $\omega^i{}_j = \Gamma^i{}_{jk} e^k$:

```
sage: #... on the manifold's default frame (d/dx, d/dy, d:dz)
sage: dx = M.default_frame().coframe() ; dx
Coordinate coframe (M, (dx,dy,dz))
sage: check = []
sage: for i in M.irange():
....:     for j in M.irange():
....:         check.append( nab.connection_form(i,j) == \
....:               sum( nab[[i,j,k]]*dx[k] for k in M.irange() ) )
....:
sage: check
[True, True, True, True, True, True, True, True, True]
sage: #... on the frame e
sage: ef = e.coframe() ; ef
Coframe (M, (e^1,e^2,e^3))
sage: check = []
sage: for i in M.irange():
....:     for j in M.irange():
....:         s = nab.connection_form(i,j,e).comp(c_xyz.frame(), from_basis=e)
....:         check.append( nab.connection_form(i,j,e) == sum( nab.coef(e)[[i,
↪j,k]]*ef[k] for k in M.irange() ) )
....:
sage: check
[True, True, True, True, True, True, True, True, True]
```

Check of the formula $\nabla_v e_j = \langle \omega^i{}_j, v \rangle e_i$:

```
sage: v = M.vector_field()
sage: v[:] = (x*y, z^2-3*x, z+2*y)
sage: b = M.default_frame()
sage: for j in M.irange():  # check on M's default frame
....:     nab(b[j]).contract(v) == \
....:      sum( nab.connection_form(i,j)(v)*b[i] for i in M.irange())
True
True
True
sage: for j in M.irange():  # check on frame e
....:     nab(e[j]).contract(v) == \
....:      sum( nab.connection_form(i,j,e)(v)*e[i] for i in M.irange())
True
True
True
```

curvature_form(*i*, *j*, *frame=None*)

Return the curvature 2-form corresponding to the given index and vector frame.

The *curvature 2-forms* with respect to the frame (e_i) are the n^2 2-forms $\Omega^i{}_j$ defined by

$$\Omega^i{}_j(u, v) = R(e^i, e_j, u, v)$$

where R is the connection's Riemann curvature tensor (cf. `riemann()`), (e^i) is the coframe dual to (e_i) and (u, v) is a generic pair of vectors.

INPUT:

- `i, j` – indices identifying the 2-form $\Omega^i{}_j$
- `frame` – (default: `None`) vector frame relative to which the curvature 2-forms are defined; if `None`, the default frame of the connection's domain is assumed.

OUTPUT:

- the 2-form $\Omega^i{}_j$, as an instance of `DiffForm`

EXAMPLES:

Curvature 2-forms on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,1], nab[1,1,2], nab[1,1,3] = x*y*z, x^2, -y*z
sage: nab[1,2,3], nab[1,3,1], nab[1,3,2] = -x^3, y^2*z, y^2-x^2
sage: nab[2,1,1], nab[2,1,2], nab[2,2,1] = z^2, x*y*z^2, -x^2
sage: nab[2,3,1], nab[2,3,3], nab[3,1,2] = x^2+y^2+z^2, y^2-z^2, x*y+z^2
sage: nab[3,2,1], nab[3,2,2], nab[3,3,3] = x*y+z, z^3 -y^2, x*z^2 - z*y^2
sage: nab.curvature_form(1,1)  # long time
2-form curvature (1,1) of connection nabla w.r.t. Coordinate frame
 (M, (d/dx,d/dy,d/dz)) on the 3-dimensional differentiable manifold M
sage: nab.curvature_form(1,1).display()  # long time (if above is skipped)
curvature (1,1) of connection nabla w.r.t. Coordinate frame
 (M, (d/dx,d/dy,d/dz)) = (y^2*z^3 + (x*y^3 - x)*z + 2*x) dx/\dy
  + (x^3*z^2 - x*y) dx/\dz + (x^4*y*z^2 - z) dy/\dz
```

Curvature 2-forms w.r.t. a non-holonomic frame:

```
sage: ch_basis = M.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2], ch_basis[3,3] = y, z, x
sage: e = M.default_frame().new_frame(ch_basis, 'e')
sage: e[1].display(), e[2].display(), e[3].display()
(e_1 = y d/dx, e_2 = z d/dy, e_3 = x d/dz)
sage: ef = e.coframe()
sage: ef[1].display(), ef[2].display(), ef[3].display()
(e^1 = 1/y dx, e^2 = 1/z dy, e^3 = 1/x dz)
sage: nab.curvature_form(1,1,e)  # long time
2-form curvature (1,1) of connection nabla w.r.t. Vector frame
 (M, (e_1,e_2,e_3)) on the 3-dimensional differentiable manifold M
sage: nab.curvature_form(1,1,e).display(e)  # long time (if above is skipped)
 curvature (1,1) of connection nabla w.r.t. Vector frame
 (M, (e_1,e_2,e_3)) =
  (y^3*z^4 + 2*x*y*z + (x*y^4 - x*y)*z^2) e^1/\e^2
  + (x^4*y*z^2 - x^2*y^2) e^1/\e^3 + (x^5*y*z^3 - x*z^2) e^2/\e^3
```

Cartan's second structure equation is

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

where the $\omega^i{}_j$'s are the connection 1-forms (cf. `connection_form()`). Let us check it on the frame e:

```
sage: omega = nab.connection_form
sage: check = []
sage: for i in M.irange():  # long time
....:     for j in M.irange():
....:         check.append( nab.curvature_form(i,j,e) == \
....:                       omega(i,j,e).exterior_derivative() + \
....:         sum( omega(i,k,e).wedge(omega(k,j,e)) for k in M.irange()) )
....:
sage: check  # long time
[True, True, True, True, True, True, True, True, True]
```

del_other_coef (*frame=None*)

Delete all the coefficients but those corresponding to `frame`.

INPUT:

- `frame` – (default: `None`) vector frame, the connection coefficients w.r.t. which are to be kept; if `None`, the default frame of the connection's domain is assumed.

EXAMPLES:

We first create two sets of connection coefficients:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: eX = X.frame()
sage: nab.set_coef(eX)[1,2,1] = x*y
sage: e = M.vector_frame('e')
sage: nab.add_coef(e)[2,1,1] = x+y
sage: nab.display(eX)
Gam^x_yx = x*y
sage: nab.display(e)
Gam^2_11 = x + y
```

Let us delete the connection coefficients w.r.t. all frames except for frame `eX`:

```
sage: nab.del_other_coef(eX)
sage: nab.display(eX)
Gam^x_yx = x*y
```

The connection coefficients w.r.t. frame `e` have indeed been deleted:

```
sage: nab.display(e)
Traceback (most recent call last):
...
ValueError: no common frame found for the computation
```

display (*frame=None, chart=None, symbol=None, latex_symbol=None, index_labels=None, index_latex_labels=None, coordinate_labels=True, only_nonzero=True, only_nonredundant=False*)

Display all the connection coefficients w.r.t. to a given frame, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `frame` – (default: `None`) vector frame relative to which the connection coefficients are defined; if `None`, the default frame of the connection's domain is used

- `chart` – (default: `None`) chart specifying the coordinate expression of the connection coefficients; if `None`, the default chart of the connection's domain is used
- `symbol` – (default: `None`) string specifying the symbol of the connection coefficients; if `None`, 'Gam' is used
- `latex_symbol` – (default: `None`) string specifying the LaTeX symbol for the components; if `None`, '\Gamma' is used
- `index_labels` – (default: `None`) list of strings representing the labels of each index; if `None`, integer labels are used, except if `frame` is a coordinate frame and `coordinate_symbols` is set to `True`, in which case the coordinate symbols are used
- `index_latex_labels` – (default: `None`) list of strings representing the LaTeX labels of each index; if `None`, integer labels are used, except if `frame` is a coordinate frame and `coordinate_symbols` is set to `True`, in which case the coordinate LaTeX symbols are used
- `coordinate_labels` – (default: `True`) boolean; if `True`, coordinate symbols are used by default (instead of integers) as index labels whenever `frame` is a coordinate frame
- `only_nonzero` – (default: `True`) boolean; if `True`, only nonzero connection coefficients are displayed
- `only_nonredundant` – (default: `False`) boolean; if `True`, only nonredundant connection coefficients are displayed in case of symmetries

EXAMPLES:

Coefficients of a connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z
```

By default, only the nonzero connection coefficients are displayed:

```
sage: nab.display()
Gam^x_xy = x^2
Gam^z_yz = y*z
sage: latex(nab.display())
\begin{array}{lcl} \Gamma_{ \phantom{\, x} \, x \, y }^{ \, x \phantom{\, x}
↪\phantom{\, y} }
& = & x^{2} \\
\Gamma_{ \phantom{\, z} \, y \, z }^{ \, z \phantom{\, y} \phantom{\, z} }
& = & y z \end{array}
```

By default, the displayed connection coefficients are those w.r.t. to the default frame of the connection's domain, so the above is equivalent to:

```
sage: nab.display(frame=M.default_frame())
Gam^x_xy = x^2
Gam^z_yz = y*z
```

Since the default frame is a coordinate frame, coordinate symbols are used to label the indices, but one may ask for integers instead:

```
sage: M.default_frame() is c_xyz.frame()
True
sage: nab.display(coordinate_labels=False)
```

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```
Gam^1_12 = x^2
Gam^3_23 = y*z
```

The index labels can also be customized:

```
sage: nab.display(index_labels=['(1)', '(2)', '(3)'])
Gam^(1)_(1),(2) = x^2
Gam^(3)_(2),(3) = y*z
```

The symbol 'Gam' can be changed:

```
sage: nab.display(symbol='C', latex_symbol='C')
C^x_xy = x^2
C^z_yz = y*z
sage: latex(nab.display(symbol='C', latex_symbol='C'))
\begin{array}{lcl} C_{ \phantom{\, x} \, x \, y }^{ \, x \phantom{\, x}␣
↪\phantom{\, y} }
& = & x^{2} \\
C_{ \phantom{\, z} \, y \, z }^{ \, z \phantom{\, y} \phantom{\, z} }
& = & y z \end{array}
```

Display of Christoffel symbols, skipping the redundancy associated with the symmetry of the last two indices:

```
sage: M = Manifold(3, 'R^3', start_index=1)
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):\phi')
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2 , (r*sin(th))^2
sage: g.display()
g = dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: g.connection().display(only_nonredundant=True)
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

By default, the parameter `only_nonredundant` is set to `False`:

```
sage: g.connection().display()
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
```

domain()

Return the manifold subset on which the affine connection is defined.

OUTPUT:

- instance of class `DifferentiableManifold` representing the manifold on which `self` is defined.

EXAMPLES:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab.domain()
3-dimensional differentiable manifold M
sage: U = M.open_subset('U', coord_def={c_xyz: x>0})
sage: nabU = U.affine_connection('D')
sage: nabU.domain()
Open subset U of the 3-dimensional differentiable manifold M
```

restrict (*subdomain*)

Return the restriction of the connection to some subdomain.

If such restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset U of the connection's domain (must be an instance of `DifferentiableManifold`)

OUTPUT:

- instance of `AffineConnection` representing the restriction.

EXAMPLES:

Restriction of a connection on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,2], nab[2,1,1] = x^2, x+y
sage: nab[:]
[[[0, x^2], [0, 0]], [[x + y, 0], [0, 0]]]
sage: U = M.open_subset('U', coord_def={c_xy: x>0})
sage: nabU = nab.restrict(U) ; nabU
Affine connection nabla on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: nabU.domain()
Open subset U of the 2-dimensional differentiable manifold M
sage: nabU[:]
[[[0, x^2], [0, 0]], [[x + y, 0], [0, 0]]]
```

The result is cached:

```
sage: nab.restrict(U) is nabU
True
```

until the connection is modified:

```
sage: nab[1,2,2] = -y
sage: nab.restrict(U) is nabU
False
sage: nab.restrict(U)[:]
[[[0, x^2], [0, -y]], [[x + y, 0], [0, 0]]]
```

ricci()

Return the connection's Ricci tensor.

The *Ricci tensor* is the tensor field Ric of type (0,2) defined from the Riemann curvature tensor R by

$$Ric(u, v) = R(e^i, u, e_i, v)$$

for any vector fields u and v, (e_i) being any vector frame and (e^i) the dual coframe.

OUTPUT:

- the Ricci tensor Ric, as an instance of `TensorField`

EXAMPLES:

Ricci tensor of an affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the 3-dimensional differentiable
 manifold M
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} =␣
↪yz
sage: r = nab.ricci() ; r
Tensor field of type (0,2) on the 3-dimensional differentiable
 manifold M
sage: r[:]
[  0 2*x   0]
[  0  -z   0]
[  0   0   0]
```

The result is cached (until the connection is modified via `set_coef()` or `add_coef()`):

```
sage: nab.ricci() is r
True
```

riemann()

Return the connection's Riemann curvature tensor.

The *Riemann curvature tensor* is the tensor field R of type (1,3) defined by

$$R(\omega, w, u, v) = \left\langle \omega, \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \right\rangle$$

for any 1-form ω and any vector fields u, v and w.

OUTPUT:

- the Riemann curvature tensor R, as an instance of `TensorField`

EXAMPLES:

Curvature of an affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla') ; nab
Affine connection nabla on the 3-dimensional differentiable
 manifold M
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} =␣
↪yz
```

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```
sage: r = nab.riemann() ; r
Tensor field of type (1,3) on the 3-dimensional differentiable
 manifold M
sage: r.parent()
Free module T^(1,3)(M) of type-(1,3) tensors fields on the
 3-dimensional differentiable manifold M
```

By construction, the Riemann tensor is antisymmetric with respect to its last two arguments (denoted u and v in the definition above), which are at positions 2 and 3 (the first argument being at position 0):

```
sage: r.symmetries()
no symmetry;  antisymmetry: (2, 3)
```

The components:

```
sage: r[:]
[[[[0, 2*x, 0], [-2*x, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]]],
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]]],
[[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, z], [0, -z, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]]]]
```

The result is cached (until the connection is modified via `set_coef()` or `add_coef()`):

```
sage: nab.riemann() is r
True
```

Another example: Riemann curvature tensor of some connection on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
sage: eUW = c_xyW.frame() ; eVW = c_uvW.frame()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[0,0,0], nab[0,1,0], nab[1,0,1] = x, x-y, x*y
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
....:             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
....:
sage: r = nab.riemann() ; r
Tensor field of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: r.parent()
```

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```
Module T^(1,3)(M) of type-(1,3) tensors fields on the 2-dimensional
 differentiable manifold M
sage: r.display(eU)
(x^2*y - x*y^2) d/dx*dx*dx*dy + (-x^2*y + x*y^2) d/dx*dx*dy*dx + d/dx*dy*dx*dy
 - d/dx*dy*dy*dx - (x^2 - 1)*y d/dy*dx*dx*dy + (x^2 - 1)*y d/dy*dx*dy*dx
 + (-x^2*y + x*y^2) d/dy*dy*dx*dy + (x^2*y - x*y^2) d/dy*dy*dy*dx
sage: r.display(eV)
(1/32*u^3 - 1/32*u*v^2 - 1/32*v^3 + 1/32*(u^2 + 4)*v - 1/8*u - 1/4) d/
↪du*du*du*dv
 + (-1/32*u^3 + 1/32*u*v^2 + 1/32*v^3 - 1/32*(u^2 + 4)*v + 1/8*u + 1/4) d/
↪du*du*dv*du
 + (1/32*u^3 - 1/32*u*v^2 + 3/32*v^3 - 1/32*(3*u^2 - 4)*v - 1/8*u + 1/4) d/
↪du*dv*du*dv
 + (-1/32*u^3 + 1/32*u*v^2 - 3/32*v^3 + 1/32*(3*u^2 - 4)*v + 1/8*u - 1/4) d/
↪du*dv*dv*du
 + (-1/32*u^3 + 1/32*u*v^2 + 5/32*v^3 - 1/32*(5*u^2 + 4)*v + 1/8*u - 1/4) d/
↪dv*du*du*dv
 + (1/32*u^3 - 1/32*u*v^2 - 5/32*v^3 + 1/32*(5*u^2 + 4)*v - 1/8*u + 1/4) d/
↪dv*du*dv*du
 + (-1/32*u^3 + 1/32*u*v^2 + 1/32*v^3 - 1/32*(u^2 + 4)*v + 1/8*u + 1/4) d/
↪dv*dv*du*dv
 + (1/32*u^3 - 1/32*u*v^2 - 1/32*v^3 + 1/32*(u^2 + 4)*v - 1/8*u - 1/4) d/
↪dv*dv*dv*du
```

The same computation parallelized on 2 cores:

```
sage: Parallelism().set(nproc=2)
sage: r_backup = r
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[0,0,0], nab[0,1,0], nab[1,0,1] = x, x-y, x*y
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
....:             nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
....:
sage: r = nab.riemann() ; r
Tensor field of type (1,3) on the 2-dimensional differentiable
 manifold M
sage: r.parent()
Module T^(1,3)(M) of type-(1,3) tensors fields on the 2-dimensional
 differentiable manifold M
sage: r == r_backup
True
sage: Parallelism().set(nproc=1)  # switch off parallelization
```

set_calc_order(*symbol*, *order*, *truncate=False*)

Trigger a series expansion with respect to a small parameter in computations involving `self`.

This property is propagated by usual operations. The internal representation must be `SR` for this to take effect.

INPUT:

- `symbol` – symbolic variable (the "small parameter" ϵ) with respect to which the connection coefficients are expanded in power series
- `order` – integer; the order n of the expansion, defined as the degree of the polynomial representing the truncated power series in `symbol`

- truncate – (default: False) determines whether the connection coefficients are replaced by their expansions to the given order

EXAMPLES:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
sage: g = M.metric()
sage: h = M.tensor_field(0, 2, sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h[0, 1] = x
sage: g.set(g + e*h)
sage: g[:]
[ -1 e*x   0   0]
[e*x   1   0   0]
[  0   0   1   0]
[  0   0   0   1]
sage: nab = g.connection()
sage: nab[0, 1, 1]
-e/(e^2*x^2 + 1)
sage: nab.set_calc_order(e, 1, truncate=True)
sage: nab[0, 1, 1]
-e
```

set_coef (*frame=None*)

Return the connection coefficients in a given frame for assignment.

See method *coef()* for details about the definition of the connection coefficients.

The connection coefficients with respect to other frames are deleted, in order to avoid any inconsistency. To keep them, use the method *add_coef()* instead.

INPUT:

- frame – (default: None) vector frame in which the connection coefficients are defined; if None, the default frame of the connection's domain is assumed.

OUTPUT:

- connection coefficients in the given frame, as an instance of the class Components; if such connection coefficients did not exist previously, they are created. See method *coef()* for the storage convention of the connection coefficients.

EXAMPLES:

Setting the coefficients of an affine connection w.r.t. some coordinate frame:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: nab = M.affine_connection('nabla', latex_name=r'\nabla')
sage: eX = X.frame(); eX
Coordinate frame (M, (d/dx,d/dy))
sage: nab.set_coef(eX)
3-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy))
sage: nab.set_coef(eX)[1,2,1] = x*y
sage: nab.display(eX)
Gam^x_yx = x*y
```

Since eX is the manifold's default vector frame, its mention may be omitted:

```
sage: nab.set_coef()[1,2,1] = x*y
sage: nab.set_coef()
3-indices components w.r.t. Coordinate frame (M, (d/dx,d/dy))
sage: nab.set_coef()[1,2,1] = x*y
sage: nab.display()
Gam^x_yx = x*y
```

To set the coefficients in the default frame, one can even bypass the method `set_coef()` and call directly the operator `[]` on the connection object:

```
sage: nab[1,2,1] = x*y
sage: nab.display()
    Gam^x_yx = x*y
```

Setting the connection coefficients w.r.t. to another vector frame:

```
sage: e = M.vector_frame('e')
sage: nab.set_coef(e)
3-indices components w.r.t. Vector frame (M, (e_1,e_2))
sage: nab.set_coef(e)[2,1,1] = x+y
sage: nab.set_coef(e)[2,1,2] = x-y
sage: nab.display(e)
Gam^2_11 = x + y
Gam^2_12 = x - y
```

The coefficients w.r.t. the frame `eX` have been deleted:

```
sage: nab.display(eX)
Traceback (most recent call last):
...
ValueError: no common frame found for the computation
```

To keep them, use the method *add_coef()* instead.

torsion()

Return the connection's torsion tensor.

The torsion tensor is the tensor field T of type (1,2) defined by

$$T(\omega, u, v) = \langle \omega, \nabla_u v - \nabla_v u - [u, v] \rangle$$

for any 1-form ω and any vector fields u and v.

OUTPUT:

- the torsion tensor T, as an instance of *TensorField*

EXAMPLES:

Torsion of an affine connection on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,2], nab[3,2,3] = x^2, y*z  # Gamma^1_{12} = x^2, Gamma^3_{23} =␣
↪yz
sage: t = nab.torsion() ; t
Tensor field of type (1,2) on the 3-dimensional differentiable
 manifold M
```

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```
sage: t.symmetries()
no symmetry;  antisymmetry: (1, 2)
sage: t[:]
[[[0, -x^2, 0], [x^2, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, 0], [0, 0, 0]],
[[0, 0, 0], [0, 0, -y*z], [0, y*z, 0]]]
```

The torsion expresses the lack of commutativity of two successive derivatives of a scalar field:

```
sage: f = M.scalar_field(x*z^2 + y^2 - z^2, name='f')
sage: DDf = nab(nab(f)) ; DDf
Tensor field nabla(df) of type (0,2) on the 3-dimensional
 differentiable manifold M
sage: DDf.antisymmetrize()[:]  # two successive derivatives do not commute:
[             0   -1/2*x^2*z^2              0]
[   1/2*x^2*z^2              0 -(x - 1)*y*z^2]
[             0  (x - 1)*y*z^2              0]
sage: 2*DDf.antisymmetrize() == nab.torsion().contract(0,nab(f))
True
```

The above identity is the standard formula

$$\nabla_j \nabla_i f - \nabla_i \nabla_j f = T^k_{\ \, ij} \nabla_k f,$$

where the $T^k_{\ \, ij}$'s are the components of the torsion tensor.

The result is cached:

```
sage: nab.torsion() is t
True
```

as long as the connection remains unchanged:

```
sage: nab[2,1,3] = 1+x     # changing the connection
sage: nab.torsion() is t  # a new computation of the torsion has been made
False
sage: (nab.torsion() - t).display()
(-x - 1) d/dy*dx*dz + (x + 1) d/dy*dz*dx
```

Another example: torsion of some connection on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)    # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: transf = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                              restrictions1= x>0, restrictions2= u+v>0)
sage: inv = transf.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
sage: eUW = c_xyW.frame() ; eVW = c_uvW.frame()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[0,0,0], nab[0,1,0], nab[1,0,1] = x, x-y, x*y
sage: for i in M.irange():
....:     for j in M.irange():
....:         for k in M.irange():
```

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```
....:                 nab.add_coef(eV)[i,j,k] = nab.coef(eVW)[i,j,k,c_uvW].expr()
....:
sage: t = nab.torsion() ; t
Tensor field of type (1,2) on the 2-dimensional differentiable
 manifold M
sage: t.parent()
Module T^(1,2)(M) of type-(1,2) tensors fields on the 2-dimensional
 differentiable manifold M
sage: t[eU,:]
[[[0, x - y], [-x + y, 0]], [[0, -x*y], [x*y, 0]]]
sage: t[eV,:]
[[[0, 1/8*u^2 - 1/8*v^2 - 1/2*v], [-1/8*u^2 + 1/8*v^2 + 1/2*v, 0]],
 [[0, -1/8*u^2 + 1/8*v^2 - 1/2*v], [1/8*u^2 - 1/8*v^2 + 1/2*v, 0]]]
```

Check of the torsion formula:

```
sage: f = M.scalar_field({c_xy: (x+y)^2, c_uv: u^2}, name='f')
sage: DDf = nab(nab(f)) ; DDf
Tensor field nabla(df) of type (0,2) on the 2-dimensional
 differentiable manifold M
sage: DDf.antisymmetrize().display(eU)
(-x^2*y - (x + 1)*y^2 + x^2) dx/\dy
sage: DDf.antisymmetrize().display(eV)
(1/8*u^3 - 1/8*u*v^2 - 1/2*u*v) du/\dv
sage: 2*DDf.antisymmetrize() == nab(f).contract(nab.torsion())
True
```

torsion_form(*i*, *frame=None*)

Return the torsion 2-form corresponding to the given index and vector frame.

The *torsion 2-forms* with respect to the frame (e_i) are the n 2-forms θ^i defined by

$$\theta^i(u, v) = T(e^i, u, v)$$

where T is the connection's torsion tensor (cf. `torsion()`), (e^i) is the coframe dual to (e_i) and (u, v) is a generic pair of vectors.

INPUT:

- `i` – index identifying the 2-form θ^i
- `frame` – (default: `None`) vector frame relative to which the torsion 2-forms are defined; if `None`, the default frame of the connection's domain is assumed.

OUTPUT:

- the 2-form θ^i, as an instance of `DiffForm`

EXAMPLES:

Torsion 2-forms on a 3-dimensional manifold:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: c_xyz.<x,y,z> = M.chart()
sage: nab = M.affine_connection('nabla', r'\nabla')
sage: nab[1,1,1], nab[1,1,2], nab[1,1,3] = x*y*z, x^2, -y*z
sage: nab[1,2,3], nab[1,3,1], nab[1,3,2] = -x^3, y^2*z, y^2-x^2
sage: nab[2,1,1], nab[2,1,2], nab[2,2,1] = z^2, x*y*z^2, -x^2
sage: nab[2,3,1], nab[2,3,3], nab[3,1,2] = x^2+y^2+z^2, y^2-z^2, x*y+z^2
```

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```
sage: nab[3,2,1], nab[3,2,2], nab[3,3,3] = x*y+z, z^3 -y^2, x*z^2 - z*y^2
sage: nab.torsion_form(1)
2-form torsion (1) of connection nabla w.r.t. Coordinate frame
 (M, (d/dx,d/dy,d/dz)) on the 3-dimensional differentiable manifold M
sage: nab.torsion_form(1)[:]
[               0            -x^2      (y^2 + y)*z]
[             x^2               0  x^3 - x^2 + y^2]
[    -(y^2 + y)*z -x^3 + x^2 - y^2                0]
```

Torsion 2-forms w.r.t. a non-holonomic frame:

```
sage: ch_basis = M.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2], ch_basis[3,3] = y, z, x
sage: e = M.default_frame().new_frame(ch_basis, 'e')
sage: e[1][:], e[2][:], e[3][:]
([y, 0, 0], [0, z, 0], [0, 0, x])
sage: ef = e.coframe()
sage: ef[1][:], ef[2][:], ef[3][:]
([1/y, 0, 0], [0, 1/z, 0], [0, 0, 1/x])
sage: nab.torsion_form(1, e)
2-form torsion (1) of connection nabla w.r.t. Vector frame
 (M, (e_1,e_2,e_3)) on the 3-dimensional differentiable manifold M
sage: nab.torsion_form(1, e).comp(e)[:]
[                       0                   -x^2*z          (x*y^2 + x*y)*z]
[                   x^2*z                        0  (x^4 - x^3 + x*y^2)*z/y]
[        -(x*y^2 + x*y)*z -(x^4 - x^3 + x*y^2)*z/y                        0]
```

Cartan's first structure equation is

$$\theta^i = \mathrm{d}e^i + \omega^i{}_j \wedge e^j$$

where the $\omega^i{}_j$'s are the connection 1-forms (cf. `connection_form()`). Let us check it on the frame e:

```
sage: for i in M.irange():  # long time
....:     nab.torsion_form(i, e) == ef[i].exterior_derivative() + \
....:      sum(nab.connection_form(i,j,e).wedge(ef[j]) for j in M.irange())
....:
True
True
True
```

2.13 Submanifolds of differentiable manifolds

Given two differentiable manifolds N and M, an *immersion* ϕ is a differentiable map $N \to M$ whose differential is everywhere injective. One then says that N is an *immersed submanifold* of M, via ϕ.

If in addition, ϕ is a differentiable embedding (i.e. ϕ is an immersion that is a homeomorphism onto its image), then N is called an *embedded submanifold* of M (or simply a *submanifold*).

ϕ can also depend on one or multiple parameters. As long as the differential of ϕ remains injective in these parameters, it represents a *foliation*. The *dimension* of the foliation is defined as the number of parameters.

AUTHORS:

- Florentin Jaffredo (2018): initial version

REFERENCES:

- [**?**]

class sage.manifolds.differentiable.differentiable_submanifold.**DifferentiableSubmanifold**(*n*, *na* *fie* *str* *tu* *am* *bi* *em* *ba* *di* *la* *te* *st* *ca* *e-* *go* *un*

Bases: *sage.manifolds.differentiable.manifold.DifferentiableManifold*, *sage.manifolds.topological_submanifold.TopologicalSubmanifold*

Submanifold of a differentiable manifold.

Given two differentiable manifolds N and M, an *immersion* ϕ is a differentiable map $N \to M$ whose differential is everywhere injective. One then says that N is an *immersed submanifold* of M, via ϕ.

If in addition, ϕ is a differentiable embedding (i.e. ϕ is an immersion that is a homeomorphism onto its image), then N is called an *embedded submanifold* of M (or simply a *submanifold*).

ϕ can also depend on one or multiple parameters. As long as the differential of ϕ remains injective in these parameters, it represents a *foliation*. The *dimension* of the foliation is defined as the number of parameters.

INPUT:

- n – positive integer; dimension of the manifold
- name – string; name (symbol) given to the manifold
- field – field K on which the manifold is defined; allowed values are
  - **'real' or an object of type RealField (e.g., RR) for** a manifold over $\mathbf{R}$
  - **'complex' or an object of type ComplexField (e.g., CC)** for a manifold over $\mathbf{C}$
  - an object in the category of topological fields (see Fields and TopologicalSpaces) for other types of manifolds
- structure – manifold structure (see *TopologicalStructure* or *RealTopologicalStructure*)
- ambient – (default: None) manifold of destination of the immersion. If None, set to self
- base_manifold – (default: None) if not None, must be a topological manifold; the created object is then an open subset of base_manifold
- latex_name – (default: None) string; LaTeX symbol to denote the manifold; if none are provided, it is set to name
- start_index – (default: 0) integer; lower value of the range of indices used for “indexed objects” on the manifold, e.g., coordinates in a chart

- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(field)` is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from *`ManifoldSubset`* would return the previously constructed object corresponding to these arguments)

EXAMPLES:

Let N be a 2-dimensional submanifold of a 3-dimensional manifold M:

```
sage: M = Manifold(3, 'M')
sage: N = Manifold(2, 'N', ambient=M)
sage: N
2-dimensional differentiable submanifold N embedded in 3-dimensional
 differentiable manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
```

Let us define a 1-dimension foliation indexed by t. The inverse map is needed in order to compute the adapted chart in the ambient manifold:

```
sage: t = var('t')
sage: phi = N.diff_map(M,{(CN, CM):[u, v, t+u**2+v**2]}); phi
Differentiable map from the 2-dimensional differentiable submanifold N
 embedded in 3-dimensional differentiable manifold M to the
 3-dimensional differentiable manifold M
sage: phi_inv = M.diff_map(N, {(CM, CN):[x, y]})
sage: phi_inv_t = M.scalar_field({CM: z-x**2-y**2})
```

ϕ can then be declared as an embedding $N \to M$:

```
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                 t_inverse={t: phi_inv_t})
```

The foliation can also be used to find new charts on the ambient manifold that are adapted to the foliation, ie in which the expression of the immersion is trivial. At the same time, the appropriate coordinate changes are computed:

```
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
sage: len(M.coord_changes())
2
```

See also:

`manifold` and *`topological_submanifold`*

CHAPTER

THREE

PSEUDO-RIEMANNIAN MANIFOLDS

3.1 Pseudo-Riemannian Manifolds

A *pseudo-Riemannian manifold* is a pair (M, g) where M is a real differentiable manifold M (see `DifferentiableManifold`) and g is a field of non-degenerate symmetric bilinear forms on M, which is called the *metric tensor*, or simply the *metric* (see `PseudoRiemannianMetric`).

Two important subcases are

- *Riemannian manifold*: the metric g is positive definite, i.e. its signature is $n = \dim M$;
- *Lorentzian manifold*: the metric g has signature $n - 2$ (positive convention) or $2 - n$ (negative convention).

On a pseudo-Riemannian manifold, one may use various standard `operators` acting on scalar and tensor fields, like `grad()` or `div()`.

All pseudo-Riemannian manifolds, whatever the metric signature, are implemented via the class `PseudoRiemannianManifold`.

Example 1: the sphere as a Riemannian manifold of dimension 2

We start by declaring S^2 as a 2-dimensional Riemannian manifold:

```
sage: M = Manifold(2, 'S^2', structure='Riemannian')
sage: M
2-dimensional Riemannian manifold S^2
```

We then cover S^2 by two stereographic charts, from the North pole and from the South pole respectively:

```
sage: U = M.open_subset('U')
sage: stereoN.<x,y> = U.chart()
sage: V = M.open_subset('V')
sage: stereoS.<u,v> = V.chart()
sage: M.declare_union(U,V)
sage: stereoN_to_S = stereoN.transition_map(stereoS,
....:                 [x/(x^2+y^2), y/(x^2+y^2)], intersection_name='W',
....:                 restrictions1= x^2+y^2!=0, restrictions2= u^2+v^2!=0)
sage: W = U.intersection(V)
sage: stereoN_to_S
Change of coordinates from Chart (W, (x, y)) to Chart (W, (u, v))
sage: stereoN_to_S.display()
u = x/(x^2 + y^2)
v = y/(x^2 + y^2)
sage: stereoN_to_S.inverse().display()
```

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```
x = u/(u^2 + v^2)
y = v/(u^2 + v^2)
```

We get the metric defining the Riemannian structure by:

```
sage: g = M.metric()
sage: g
Riemannian metric g on the 2-dimensional Riemannian manifold S^2
```

At this stage, the metric g is defined as a Python object but there remains to initialize it by setting its components with respect to the vector frames associated with the stereographic coordinates. Let us begin with the frame of chart `stereoN`:

```
sage: eU = stereoN.frame()
sage: g[eU, 0, 0] = 4/(1 + x^2 + y^2)^2
sage: g[eU, 1, 1] = 4/(1 + x^2 + y^2)^2
```

The metric components in the frame of chart `stereoS` are obtained by continuation of the expressions found in $W = U \cap V$ from the known change-of-coordinate formulas:

```
sage: eV = stereoS.frame()
sage: g.add_comp_by_continuation(eV, W)
```

At this stage, the metric g is well defined in all S^2:

```
sage: g.display(eU)
g = 4/(x^2 + y^2 + 1)^2 dx*dx + 4/(x^2 + y^2 + 1)^2 dy*dy
sage: g.display(eV)
g = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du*du
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv*dv
```

The expression in frame `eV` can be given a shape similar to that in frame `eU`, by factorizing the components:

```
sage: g[eV, 0, 0].factor()
4/(u^2 + v^2 + 1)^2
sage: g[eV, 1, 1].factor()
4/(u^2 + v^2 + 1)^2
sage: g.display(eV)
g = 4/(u^2 + v^2 + 1)^2 du*du + 4/(u^2 + v^2 + 1)^2 dv*dv
```

Let us consider a scalar field f on S^2:

```
sage: f = M.scalar_field({stereoN: 1/(1+x^2+y^2)}, name='f')
sage: f.add_expr_by_continuation(stereoS, W)
sage: f.display()
f: S^2 --> R
on U: (x, y) |--> 1/(x^2 + y^2 + 1)
on V: (u, v) |--> (u^2 + v^2)/(u^2 + v^2 + 1)
```

The gradient of f (with respect to the metric g) is:

```
sage: gradf = f.gradient()
sage: gradf
Vector field grad(f) on the 2-dimensional Riemannian manifold S^2
sage: gradf.display(eU)
grad(f) = -1/2*x d/dx - 1/2*y d/dy
```

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```
sage: gradf.display(eV)
grad(f) = 1/2*u d/du + 1/2*v d/dv
```

It is possible to write `grad(f)` instead of `f.gradient()`, by importing the standard differential operators of vector calculus:

```
sage: from sage.manifolds.operators import *
sage: grad(f) == gradf
True
```

The Laplacian of f (with respect to the metric g) is obtained either as `f.laplacian()` or, thanks to the above import, as `laplacian(f)`:

```
sage: Df = laplacian(f)
sage: Df
Scalar field Delta(f) on the 2-dimensional Riemannian manifold S^2
sage: Df.display()
Delta(f): S^2 --> R
on U: (x, y) |--> (x^2 + y^2 - 1)/(x^2 + y^2 + 1)
on V: (u, v) |--> -(u^2 + v^2 - 1)/(u^2 + v^2 + 1)
```

Let us check the standard formula $\Delta f = \mathrm{div}(\mathrm{grad}\, f)$:

```
sage: Df == div(gradf)
True
```

Since each open subset of S^2 inherits the structure of a Riemannian manifold, we can get the metric on it via the method `metric()`:

```
sage: gU = U.metric()
sage: gU
Riemannian metric g on the Open subset U of the 2-dimensional Riemannian
 manifold S^2
sage: gU.display()
g = 4/(x^2 + y^2 + 1)^2 dx*dx + 4/(x^2 + y^2 + 1)^2 dy*dy
```

Of course, `gU` is nothing but the restriction of g to U:

```
sage: gU is g.restrict(U)
True
```

Example 2: Minkowski spacetime as a Lorentzian manifold of dimension 4

We start by declaring a 4-dimensional Lorentzian manifold M:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: M
4-dimensional Lorentzian manifold M
```

We define Minkowskian coordinates on M:

```
sage: X.<t,x,y,z> = M.chart()
```

We construct the metric tensor by:

```
sage: g = M.metric()
sage: g
Lorentzian metric g on the 4-dimensional Lorentzian manifold M
```

and initialize it to the Minkowskian value:

```
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: g.display()
g = -dt*dt + dx*dx + dy*dy + dz*dz
sage: g[:]
[-1  0  0  0]
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]
```

We may check that the metric is flat, i.e. has a vanishing Riemann curvature tensor:

```
sage: g.riemann().display()
Riem(g) = 0
```

A vector field on M:

```
sage: u = M.vector_field(name='u')
sage: u[0] = cosh(t)
sage: u[1] = sinh(t)
sage: u.display()
u = cosh(t) d/dt + sinh(t) d/dx
```

The scalar square of u is:

```
sage: s = u.dot(u); s
Scalar field u.u on the 4-dimensional Lorentzian manifold M
```

Scalar products are taken with respect to the metric tensor:

```
sage: u.dot(u) == g(u,u)
True
```

u is a unit timelike vector, i.e. its scalar square is identically -1:

```
sage: s.display()
u.u: M --> R
   (t, x, y, z) |--> -1
sage: s.expr()
-1
```

Let us consider a unit spacelike vector:

```
sage: v = M.vector_field(name='v')
sage: v[0] = sinh(t)
sage: v[1] = cosh(t)
sage: v.display()
v = sinh(t) d/dt + cosh(t) d/dx
sage: v.dot(v).display()
v.v: M --> R
   (t, x, y, z) |--> 1
sage: v.dot(v).expr()
1
```

u and v are orthogonal vectors with respect to Minkowski metric:

```
sage: u.dot(v).display()
    u.v: M --> R
   (t, x, y, z) |--> 0
sage: u.dot(v).expr()
0
```

The divergence of u is:

```
sage: s = u.div(); s
Scalar field div(u) on the 4-dimensional Lorentzian manifold M
sage: s.display()
div(u): M --> R
   (t, x, y, z) |--> sinh(t)
```

while its d'Alembertian is:

```
sage: Du = u.dalembertian(); Du
Vector field Box(u) on the 4-dimensional Lorentzian manifold M
sage: Du.display()
Box(u) = -cosh(t) d/dt - sinh(t) d/dx
```

AUTHORS:

- Eric Gourgoulhon (2018): initial version

REFERENCES:

- B. O'Neill : *Semi-Riemannian Geometry* [**?**]
- J. M. Lee : *Riemannian Manifolds* [**?**]

class sage.manifolds.differentiable.pseudo_riemannian.**PseudoRiemannianManifold**(*n, name, metric_name='g', signature=None, base_manifold=N, diff_degree=+Inf, latex_name=None, metric_latex_name=, start_index=0, category=None, unique_tag=Non*

Bases: `sage.manifolds.differentiable.manifold.DifferentiableManifold`

PseudoRiemannian manifold.

A *pseudo-Riemannian manifold* is a pair (M, g) where M is a real differentiable manifold M (see `DifferentiableManifold`) and g is a field of non-degenerate symmetric bilinear forms on M, which is called the *metric tensor*, or simply the *metric* (see `PseudoRiemannianMetric`).

Two important subcases are

- *Riemannian manifold*: the metric g is positive definite, i.e. its signature is $n = \dim M$;
- *Lorentzian manifold*: the metric g has signature $n - 2$ (positive convention) or $2 - n$ (negative convention).

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `metric_name` – (default: `'g'`) string; name (symbol) given to the metric
- `signature` – (default: `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)
- `ambient` – (default: `None`) if not `None`, must be a differentiable manifold; the created object is then an open subset of `ambient`
- `diff_degree` – (default: `infinity`) degree k of differentiability
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none is provided, it is set to `name`
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the metric; if none is provided, it is set to `metric_name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g. coordinates in a chart
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(RR).Differentiable()` (or `Manifolds(RR).Smooth()` if `diff_degree = infinity`) is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from *`ManifoldSubset`*, via *`DifferentiableManifold`* and *`TopologicalManifold`*, would return the previously constructed object corresponding to these arguments).

EXAMPLES:

Pseudo-Riemannian manifolds are constructed via the generic function *`Manifold()`*, using the keyword `structure`:

```
sage: M = Manifold(4, 'M', structure='pseudo-Riemannian', signature=0)
sage: M
4-dimensional pseudo-Riemannian manifold M
sage: M.category()
Category of smooth manifolds over Real Field with 53 bits of precision
```

The metric associated with `M` is:

```
sage: M.metric()
Pseudo-Riemannian metric g on the 4-dimensional pseudo-Riemannian
 manifold M
sage: M.metric().signature()
0
sage: M.metric().tensor_type()
(0, 2)
```

Its value has to be initialized either by setting its components in various vector frames (see the above examples regarding the 2-sphere and Minkowski spacetime) or by making it equal to a given field of symmetric bilinear forms (see the method *set()* of the metric class). Both methods are also covered in the documentation of method *metric()* below.

The metric object belongs to the class *PseudoRiemannianMetric*:

```
sage: isinstance(M.metric(), sage.manifolds.differentiable.metric.
....:                        PseudoRiemannianMetric)
True
```

See the documentation of this class for all operations available on metrics.

The default name of the metric is `g`; it can be customized:

```
sage: M = Manifold(4, 'M', structure='pseudo-Riemannian',
....:              metric_name='gam', metric_latex_name=r'\gamma')
sage: M.metric()
Riemannian metric gam on the 4-dimensional Riemannian manifold M
sage: latex(M.metric())
\gamma
```

A Riemannian manifold is constructed by the proper setting of the keyword `structure`:

```
sage: M = Manifold(4, 'M', structure='Riemannian'); M
4-dimensional Riemannian manifold M
sage: M.metric()
Riemannian metric g on the 4-dimensional Riemannian manifold M
sage: M.metric().signature()
4
```

Similarly, a Lorentzian manifold is obtained by:

```
sage: M = Manifold(4, 'M', structure='Lorentzian'); M
4-dimensional Lorentzian manifold M
sage: M.metric()
Lorentzian metric g on the 4-dimensional Lorentzian manifold M
```

The default Lorentzian signature is taken to be positive:

```
sage: M.metric().signature()
2
```

but one can opt for the negative convention via the keyword `signature`:

```
sage: M = Manifold(4, 'M', structure='Lorentzian', signature='negative')
sage: M.metric()
Lorentzian metric g on the 4-dimensional Lorentzian manifold M
sage: M.metric().signature()
-2
```

metric(*name=None*, *signature=None*, *latex_name=None*, *dest_map=None*)

Return the metric giving the pseudo-Riemannian structure to the manifold, or define a new metric tensor on the manifold.

INPUT:

- `name` – (default: `None`) name given to the metric; if `name` is `None` or matches the name of the metric defining the pseudo-Riemannian structure of `self`, the latter metric is returned
- `signature` – (default: `None`; ignored if `name` is `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of

negative terms) in any diagonal writing of the metric components; if `signature` is not provided, S is set to the manifold's dimension (Riemannian signature)

- `latex_name` – (default: `None`; ignored if `name` is `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`
- `dest_map` – (default: `None`; ignored if `name` is `None`) instance of class *DiffMap* representing the destination map $\Phi : U \to M$, where U is the current manifold; if `None`, the identity map is assumed (case of a metric tensor field *on* U)

OUTPUT:

- instance of *PseudoRiemannianMetric*

EXAMPLES:

Metric of a 3-dimensional Riemannian manifold:

```
sage: M = Manifold(3, 'M', structure='Riemannian', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric(); g
Riemannian metric g on the 3-dimensional Riemannian manifold M
```

The metric remains to be initialized, for instance by setting its components in the coordinate frame associated to the chart `X`:

```
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: g.display()
g = dx*dx + dy*dy + dz*dz
```

Alternatively, the metric can be initialized from a given field of nondegenerate symmetric bilinear forms; we may create the former object by:

```
sage: X.coframe()
Coordinate coframe (M, (dx,dy,dz))
sage: dx, dy, dz = X.coframe()[1], X.coframe()[2], X.coframe()[3]
sage: b = dx*dx + dy*dy + dz*dz
sage: b
Field of symmetric bilinear forms dx*dx+dy*dy+dz*dz on the
 3-dimensional Riemannian manifold M
```

We then use the metric method *set()* to make `g` being equal to `b` as a symmetric tensor field of type `(0,2)`:

```
sage: g.set(b)
sage: g.display()
g = dx*dx + dy*dy + dz*dz
```

Another metric can be defined on `M` by specifying a metric name distinct from that chosen at the creation of the manifold (which is `g` by default, but can be changed thanks to the keyword `metric_name` in *Manifold()*):

```
sage: h = M.metric('h'); h
Riemannian metric h on the 3-dimensional Riemannian manifold M
sage: h[1,1], h[2,2], h[3,3] = 1+y^2, 1+z^2, 1+x^2
sage: h.display()
h = (y^2 + 1) dx*dx + (z^2 + 1) dy*dy + (x^2 + 1) dz*dz
```

The metric tensor `h` is distinct from the metric entering in the definition of the Riemannian manifold `M`:

```
sage: h is M.metric()
False
```

while we have of course:

```
sage: g is M.metric()
True
```

Providing the same name as the manifold's default metric returns the latter:

```
sage: M.metric('g') is M.metric()
True
```

In the present case (M is diffeomorphic to $\mathbf{R}^3$), we can even create a Lorentzian metric on M:

```
sage: h = M.metric('h', signature=1); h
Lorentzian metric h on the 3-dimensional Riemannian manifold M
```

open_subset (*name*, *latex_name=None*, *coord_def={}*)

Create an open subset of self.

An open subset is a set that is (i) included in the manifold and (ii) open with respect to the manifold's topology. It is a differentiable manifold by itself. Moreover, equipped with the restriction of the manifold metric to itself, it is a pseudo-Riemannian manifold. Hence the returned object is an instance of *PseudoRiemannianManifold*.

INPUT:

- name – name given to the open subset
- latex_name – (default: None) LaTeX symbol to denote the subset; if none is provided, it is set to name
- coord_def – (default: {}) definition of the subset in terms of coordinates; coord_def must a be dictionary with keys charts in the manifold's atlas and values the symbolic expressions formed by the coordinates to define the subset.

OUTPUT:

- instance of *PseudoRiemannianManifold* representing the created open subset

EXAMPLES:

Open subset of a 2-dimensional Riemannian manifold:

```
sage: M = Manifold(2, 'M', structure='Riemannian')
sage: X.<x,y> = M.chart()
sage: U = M.open_subset('U', coord_def={X: x>0}); U
Open subset U of the 2-dimensional Riemannian manifold M
sage: type(U)
<class 'sage.manifolds.differentiable.pseudo_riemannian.
↪PseudoRiemannianManifold_with_category'>
```

We initialize the metric of M:

```
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
```

Then the metric on U is determined as the restriction of g to U:

```
sage: gU = U.metric(); gU
Riemannian metric g on the Open subset U of the 2-dimensional Riemannian␣
↪manifold M
sage: gU.display()
g = dx*dx + dy*dy
sage: gU is g.restrict(U)
True
```

volume_form(*contra=0*)

Volume form (Levi-Civita tensor) ϵ associated with `self`.

This assumes that `self` is an orientable manifold.

The volume form ϵ is a n-form (n being the manifold's dimension) such that for any vector basis (e_i) that is orthonormal with respect to the metric of the pseudo-Riemannian manifold `self`,

$$\epsilon(e_1, \ldots, e_n) = \pm 1$$

There are only two such n-forms, which are opposite of each other. The volume form ϵ is selected such that the default frame of `self` is right-handed with respect to it.

INPUT:

- `contra` – (default: 0) number of contravariant indices of the returned tensor

OUTPUT:

- if `contra = 0` (default value): the volume n-form ϵ, as an instance of `DiffForm`
- if `contra = k`, with $1 \leq k \leq n$, the tensor field of type (k,n-k) formed from ϵ by raising the first k indices with the metric (see method `up()`); the output is then an instance of `TensorField`, with the appropriate antisymmetries, or of the subclass `MultivectorField` if $k = n$

EXAMPLES:

Volume form of the Euclidean 3-space:

```
sage: M = Manifold(3, 'M', structure='Riemannian', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric()
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: eps = M.volume_form(); eps
3-form eps_g on the 3-dimensional Riemannian manifold M
sage: eps.display()
eps_g = dx/\dy/\dz
```

Raising the first index:

```
sage: eps1 = M.volume_form(1); eps1
Tensor field of type (1,2) on the 3-dimensional Riemannian
 manifold M
sage: eps1.display()
d/dx*dy*dz - d/dx*dz*dy - d/dy*dx*dz + d/dy*dz*dx + d/dz*dx*dy
 - d/dz*dy*dx
sage: eps1.symmetries()
no symmetry; antisymmetry: (1, 2)
```

Raising the first and second indices:

```
sage: eps2 = M.volume_form(2); eps2
Tensor field of type (2,1) on the 3-dimensional Riemannian
 manifold M
sage: eps2.display()
d/dx*d/dy*dz - d/dx*d/dz*dy - d/dy*d/dx*dz + d/dy*d/dz*dx
 + d/dz*d/dx*dy - d/dz*d/dy*dx
sage: eps2.symmetries()
no symmetry; antisymmetry: (0, 1)
```

Fully contravariant version:

```
sage: eps3 = M.volume_form(3); eps3
3-vector field on the 3-dimensional Riemannian manifold M
sage: eps3.display()
d/dx/\d/dy/\d/dz
```

3.2 Euclidean Spaces and Vector Calculus

3.2.1 Euclidean Spaces

An *Euclidean space of dimension* n is an affine space E, whose associated vector space is a n-dimensional vector space over $\mathbf{R}$ and is equipped with a positive definite symmetric bilinear form, called the *scalar product* or *dot product* [**?**]. An Euclidean space of dimension n can also be viewed as a Riemannian manifold that is diffeomorphic to $\mathbf{R}^n$ and that has a flat metric g. The Euclidean scalar product is then that defined by the Riemannian metric g.

The current implementation of Euclidean spaces is based on the second point of view. This allows for the introduction of various coordinate systems in addition to the usual the Cartesian systems. Standard curvilinear systems (planar, spherical and cylindrical coordinates) are predefined for 2-dimensional and 3-dimensional Euclidean spaces, along with the corresponding transition maps between them. Another benefit of such an implementation is the direct use of methods for vector calculus already implemented at the level of Riemannian manifolds (see, e.g., the methods *`cross_product()`* and *`curl()`*, as well as the module *`operators`*).

Euclidean spaces are implemented via the following classes:

- *`EuclideanSpace`* for generic values n,
- *`EuclideanPlane`* for $n = 2$,
- *`Euclidean3dimSpace`* for $n = 3$.

The user interface is provided by *`EuclideanSpace`*.

Example 1: the Euclidean plane

We start by declaring the Euclidean plane `E`, with `(x, y)` as Cartesian coordinates:

```
sage: E.<x,y> = EuclideanSpace()
sage: E
Euclidean plane E^2
sage: dim(E)
2
```

`E` is automatically endowed with the chart of Cartesian coordinates:

```
sage: E.atlas()
[Chart (E^2, (x, y))]
sage: cartesian = E.default_chart(); cartesian
Chart (E^2, (x, y))
```

Thanks to the use of `<x,y>` when declaring `E`, the coordinates (x, y) have been injected in the global namespace, i.e. the Python variables `x` and `y` have been created and are available to form symbolic expressions:

```
sage: y
y
sage: type(y)
<type 'sage.symbolic.expression.Expression'>
sage: assumptions()
[x is real, y is real]
```

The metric tensor of `E` is predefined:

```
sage: g = E.metric(); g
Riemannian metric g on the Euclidean plane E^2
sage: g.display()
g = dx*dx + dy*dy
sage: g[:]
[1 0]
[0 1]
```

It is a *flat* metric, i.e. it has a vanishing Riemann tensor:

```
sage: g.riemann()
Tensor field Riem(g) of type (1,3) on the Euclidean plane E^2
sage: g.riemann().display()
Riem(g) = 0
```

Polar coordinates (r, ϕ) are introduced by:

```
sage: polar.<r,ph> = E.polar_coordinates()
sage: polar
Chart (E^2, (r, ph))
```

`E` is now endowed with two coordinate charts:

```
sage: E.atlas()
[Chart (E^2, (x, y)), Chart (E^2, (r, ph))]
```

The ranges of the coordinates introduced so far are:

```
sage: cartesian.coord_range()
x: (-oo, +oo); y: (-oo, +oo)
sage: polar.coord_range()
r: (0, +oo); ph: [0, 2*pi] (periodic)
```

The transition map from polar coordinates to Cartesian ones is:

```
sage: E.coord_change(polar, cartesian).display()
x = r*cos(ph)
y = r*sin(ph)
```

while the reverse one is:

```
sage: E.coord_change(cartesian, polar).display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
```

A point of E is constructed from its coordinates (by default in the Cartesian chart):

```
sage: p = E((-1,1), name='p'); p
Point p on the Euclidean plane E^2
sage: p.parent()
Euclidean plane E^2
```

The coordinates of a point are obtained by letting the corresponding chart act on it:

```
sage: cartesian(p)
(-1, 1)
sage: polar(p)
(sqrt(2), 3/4*pi)
```

At this stage, E is endowed with three vector frames:

```
sage: E.frames()
[Coordinate frame (E^2, (e_x,e_y)),
 Coordinate frame (E^2, (d/dr,d/dph)),
 Vector frame (E^2, (e_r,e_ph))]
```

The third one is the standard orthonormal frame associated with polar coordinates, as we can check from the metric components in it:

```
sage: polar_frame = E.polar_frame(); polar_frame
Vector frame (E^2, (e_r,e_ph))
sage: g[polar_frame,:]
[1 0]
[0 1]
```

The expression of the metric tensor in terms of polar coordinates is:

```
sage: g.display(polar)
g = dr*dr + r^2 dph*dph
```

A vector field on E:

```
sage: v = E.vector_field(-y, x, name='v'); v
Vector field v on the Euclidean plane E^2
sage: v.display()
v = -y e_x + x e_y
sage: v[:]
[-y, x]
```

By default, the components of v, as returned by display or the bracket operator, refer to the Cartesian frame on E; to get the components with respect to the orthonormal polar frame, one has to specify it explicitly, generally along with the polar chart for the coordinate expression of the components:

```
sage: v.display(polar_frame, polar)
v = r e_ph
sage: v[polar_frame,:,polar]
[0, r]
```

Note that the default frame for the display of vector fields can be changed thanks to the method *set_default_frame()*; in the same vein, the default coordinates can be changed via the method *set_default_chart()*:

```
sage: E.set_default_frame(polar_frame)
sage: E.set_default_chart(polar)
sage: v.display()
v = r e_ph
sage: v[:]
[0, r]
sage: E.set_default_frame(E.cartesian_frame())  # revert to Cartesian frame
sage: E.set_default_chart(cartesian)            # and chart
```

When defining a vector field from components relative to a vector frame different from the default one, the vector frame has to be specified explicitly:

```
sage: v = E.vector_field(1, 0, frame=polar_frame)
sage: v.display(polar_frame)
e_r
sage: v.display()
x/sqrt(x^2 + y^2) e_x + y/sqrt(x^2 + y^2) e_y
```

The argument `chart` must be used to specify in which coordinate chart the components are expressed:

```
sage: v = E.vector_field(0, r, frame=polar_frame, chart=polar)
sage: v.display(polar_frame, polar)
r e_ph
sage: v.display()
-y e_x + x e_y
```

It is also possible to pass the components as a dictionary, with a pair (vector frame, chart) as a key:

```
sage: v = E.vector_field({(polar_frame, polar): (0, r)})
sage: v.display(polar_frame, polar)
r e_ph
```

The key can be reduced to the vector frame if the chart is the default one:

```
sage: v = E.vector_field({polar_frame: (0, 1)})
sage: v.display(polar_frame)
e_ph
```

Finally, it is possible to construct the vector field without initializing any component:

```
sage: v = E.vector_field(); v
Vector field on the Euclidean plane E^2
```

The components can then by set in a second stage, via the square bracket operator, the unset components being assumed to be zero:

```
sage: v[1] = -y
sage: v.display()  # v[2] is zero
-y e_x
sage: v[2] = x
sage: v.display()
-y e_x + x e_y
```

The above is equivalent to:

```
sage: v[:] = -y, x
sage: v.display()
-y e_x + x e_y
```

The square bracket operator can also be used to set components in a vector frame that is not the default one:

```
sage: v = E.vector_field(name='v')
sage: v[polar_frame, 2, polar] = r
sage: v.display(polar_frame, polar)
v = r e_ph
sage: v.display()
v = -y e_x + x e_y
```

The value of the vector field v at point p:

```
sage: vp = v.at(p); vp
Vector v at Point p on the Euclidean plane E^2
sage: vp.display()
v = -e_x - e_y
sage: vp.display(polar_frame.at(p))
v = sqrt(2) e_ph
```

A scalar field on E:

```
sage: f = E.scalar_field(x*y, name='f'); f
Scalar field f on the Euclidean plane E^2
sage: f.display()
f: E^2 --> R
   (x, y) |--> x*y
   (r, ph) |--> r^2*cos(ph)*sin(ph)
```

The value of f at point p:

```
sage: f(p)
-1
```

The gradient of f:

```
sage: from sage.manifolds.operators import * # to get grad, div, etc.
sage: w = grad(f); w
Vector field grad(f) on the Euclidean plane E^2
sage: w.display()
grad(f) = y e_x + x e_y
sage: w.display(polar_frame, polar)
grad(f) = 2*r*cos(ph)*sin(ph) e_r + (2*cos(ph)^2 - 1)*r e_ph
```

The dot product of two vector fields:

```
sage: s = v.dot(w); s
Scalar field v.grad(f) on the Euclidean plane E^2
sage: s.display()
v.grad(f): E^2 --> R
   (x, y) |--> x^2 - y^2
   (r, ph) |--> (2*cos(ph)^2 - 1)*r^2
sage: s.expr()
x^2 - y^2
```

The norm is related to the dot product by the standard formula:

```
sage: norm(v)^2 == v.dot(v)
True
```

The divergence of the vector field v:

```
sage: s = div(v); s
Scalar field div(v) on the Euclidean plane E^2
sage: s.display()
div(v): E^2 --> R
   (x, y) |--> 0
   (r, ph) |--> 0
```

Example 2: Vector calculus in the Euclidean 3-space

We start by declaring the 3-dimensional Euclidean space E, with (x,y,z) as Cartesian coordinates:

```
sage: E.<x,y,z> = EuclideanSpace()
sage: E
Euclidean space E^3
```

A simple vector field on E:

```
sage: v = E.vector_field(-y, x, 0, name='v')
sage: v.display()
v = -y e_x + x e_y
sage: v[:]
[-y, x, 0]
```

The Euclidean norm of v:

```
sage: s = norm(v); s
Scalar field |v| on the Euclidean space E^3
sage: s.display()
|v|: E^3 --> R
   (x, y, z) |--> sqrt(x^2 + y^2)
sage: s.expr()
sqrt(x^2 + y^2)
```

The divergence of v is zero:

```
sage: from sage.manifolds.operators import *
sage: div(v)
Scalar field div(v) on the Euclidean space E^3
sage: div(v).display()
div(v): E^3 --> R
   (x, y, z) |--> 0
```

while its curl is a constant vector field along e_z:

```
sage: w = curl(v); w
Vector field curl(v) on the Euclidean space E^3
sage: w.display()
curl(v) = 2 e_z
```

The gradient of a scalar field:

```
sage: f = E.scalar_field(sin(x*y*z), name='f')
sage: u = grad(f); u
Vector field grad(f) on the Euclidean space E^3
sage: u.display()
grad(f) = y*z*cos(x*y*z) e_x + x*z*cos(x*y*z) e_y + x*y*cos(x*y*z) e_z
```

The curl of a gradient is zero:

```
sage: curl(u).display()
curl(grad(f)) = 0
```

The dot product of two vector fields:

```
sage: s = u.dot(v); s
Scalar field grad(f).v on the Euclidean space E^3
sage: s.expr()
(x^2 - y^2)*z*cos(x*y*z)
```

The cross product of two vector fields:

```
sage: a = u.cross(v); a
Vector field grad(f) x v on the Euclidean space E^3
sage: a.display()
grad(f) x v = -x^2*y*cos(x*y*z) e_x - x*y^2*cos(x*y*z) e_y
 + 2*x*y*z*cos(x*y*z) e_z
```

The scalar triple product of three vector fields:

```
sage: triple_product = E.scalar_triple_product()
sage: s = triple_product(u, v, w); s
Scalar field epsilon(grad(f),v,curl(v)) on the Euclidean space E^3
sage: s.expr()
4*x*y*z*cos(x*y*z)
```

Let us check that the scalar triple product of u, v and w is $u \cdot (v \times w)$:

```
sage: s == u.dot(v.cross(w))
True
```

AUTHORS:

- Eric Gourgoulhon (2018): initial version

REFERENCES:

- M. Berger: *Geometry I* [**?**]

class sage.manifolds.differentiable.euclidean.**Euclidean3dimSpace**(*name=None, latex_name=None, coordinates='Cartesian', symbols=None, metric_name='g', metric_latex_name=None, start_index=1, base_manifold=None, category=None, unique_tag=None*)

Bases: *sage.manifolds.differentiable.euclidean.EuclideanSpace*

3-dimensional Euclidean space.

A *3-dimensional Euclidean space* is an affine space E, whose associated vector space is a 3-dimensional vector space over $\mathbf{R}$ and is equipped with a positive definite symmetric bilinear form, called the *scalar product* or *dot product*.

The class *Euclidean3dimSpace* inherits from *PseudoRiemannianManifold* (via *EuclideanSpace*) since a 3-dimensional Euclidean space can be viewed as a Riemannian manifold that is diffeomorphic to $\mathbf{R}^3$ and that has a flat metric g. The Euclidean scalar product is the one defined by the Riemannian metric g.

INPUT:

- `name` – (default: `None`) string; name (symbol) given to the Euclidean 3-space; if `None`, the name will be set to `'E^3'`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the Euclidean 3-space; if `None`, it is set to `'\mathbb{E}^{3}'` if `name` is `None` and to `name` otherwise
- `coordinates` – (default: `'Cartesian'`) string describing the type of coordinates to be initialized at the Euclidean 3-space creation; allowed values are `'Cartesian'` (see *cartesian_coordinates()*), `'spherical'` (see *spherical_coordinates()*) and `'cylindrical'` (see *cylindrical_coordinates()*)
- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*, namely `symbols` is a string of coordinate fields separated by a blank space, where each field contains the coordinate's text symbol and possibly the coordinate's LaTeX symbol (when the latter is different from the text symbol), both symbols being separated by a colon (`:`); if `None`, the symbols will be automatically generated according to the value of `coordinates`
- `metric_name` – (default: `'g'`) string; name (symbol) given to the Euclidean metric tensor
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the Euclidean metric tensor; if none is provided, it is set to `metric_name`
- `start_index` – (default: 1) integer; lower value of the range of indices used for "indexed objects" in the Euclidean 3-space, e.g. coordinates of a chart
- `base_manifold` – (default: `None`) if not `None`, must be an Euclidean 3-space; the created object is then an open subset of `base_manifold`

- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(RR).Differentiable()` (or `Manifolds(RR).Smooth()` if `diff_degree = infinity`) is assumed (see the category `Manifolds`)
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)
- `init_coord_methods` – (default: `None`) dictionary of methods to initialize the various type of coordinates, with each key being a string describing the type of coordinates; to be used by derived classes only
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from *`PseudoRiemannianManifold`* would return the previously constructed object corresponding to these arguments)

EXAMPLES:

A 3-dimensional Euclidean space:

```
sage: E = EuclideanSpace(3); E
Euclidean space E^3
sage: latex(E)
\mathbb{E}^{3}
```

`E` belongs to the class *`Euclidean3dimSpace`* (actually to a dynamically generated subclass of it via SageMath's category framework):

```
sage: type(E)
<class 'sage.manifolds.differentiable.euclidean.Euclidean3dimSpace_with_category'>
```

`E` is a real smooth manifold of dimension 3:

```
sage: E.category()
Category of smooth manifolds over Real Field with 53 bits of precision
sage: dim(E)
3
```

It is endowed with a default coordinate chart, which is that of Cartesian coordinates (x, y, z):

```
sage: E.atlas()
[Chart (E^3, (x, y, z))]
sage: E.default_chart()
Chart (E^3, (x, y, z))
sage: cartesian = E.cartesian_coordinates()
sage: cartesian is E.default_chart()
True
```

A point of `E`:

```
sage: p = E((3,-2,1)); p
Point on the Euclidean space E^3
sage: cartesian(p)
(3, -2, 1)
sage: p in E
True
sage: p.parent() is E
True
```

`E` is endowed with a default metric tensor, which defines the Euclidean scalar product:

```
sage: g = E.metric(); g
Riemannian metric g on the Euclidean space E^3
sage: g.display()
g = dx*dx + dy*dy + dz*dz
```

Curvilinear coordinates can be introduced on `E`: see *spherical_coordinates()* and *cylindrical_coordinates()*.

See also:

Example 2: Vector calculus in the Euclidean 3-space

cartesian_coordinates(*symbols=None*, *names=None*)

Return the chart of Cartesian coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the Cartesian chart has not been already defined; if `None` the symbols are generated as (x, y, z).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of Cartesian coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.cartesian_coordinates()
Chart (E^3, (x, y, z))
sage: E.cartesian_coordinates().coord_range()
x: (-oo, +oo); y: (-oo, +oo); z: (-oo, +oo)
```

An example where the Cartesian coordinates have not been previously created:

```
sage: E = EuclideanSpace(3, coordinates='spherical')
sage: E.atlas()  # only spherical coordinates have been initialized
[Chart (E^3, (r, th, ph))]
sage: E.cartesian_coordinates(symbols='X Y Z')
Chart (E^3, (X, Y, Z))
sage: E.atlas()  # the Cartesian chart has been added to the atlas
[Chart (E^3, (r, th, ph)), Chart (E^3, (X, Y, Z))]
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.cartesian_coordinates()[1]
X
sage: E.cartesian_coordinates()[3]
Z
sage: E.cartesian_coordinates()[:]
(X, Y, Z)
```

It is also possible to use the operator `<,>` to set symbolic variable containing the coordinates:

```
sage: E = EuclideanSpace(3, coordinates='spherical')
sage: cartesian.<u,v,w> = E.cartesian_coordinates()
sage: cartesian
```

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```
Chart (E^3, (u, v, w))
sage: u, v, w
(u, v, w)
```

The command `cartesian.<u,v,w> = E.cartesian_coordinates()` is actually a shortcut for:

```
sage: cartesian = E.cartesian_coordinates(symbols='u v w')
sage: u, v, w = cartesian[:]
```

cylindrical_coordinates(*symbols=None*, *names=None*)
Return the chart of cylindrical coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the cylindrical chart has not been already defined; if `None` the symbols are generated as (ρ, ϕ, z).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of cylindrical coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.cylindrical_coordinates()
Chart (E^3, (rh, ph, z))
sage: latex(_)
\left(\mathbb{E}^{3},({\rho}, {\phi}, z)\right)
sage: E.cylindrical_coordinates().coord_range()
rh: (0, +oo); ph: [0, 2*pi] (periodic); z: (-oo, +oo)
```

The relation to Cartesian coordinates is:

```
sage: E.coord_change(E.cylindrical_coordinates(),
....:                E.cartesian_coordinates()).display()
x = rh*cos(ph)
y = rh*sin(ph)
z = z
sage: E.coord_change(E.cartesian_coordinates(),
....:                E.cylindrical_coordinates()).display()
rh = sqrt(x^2 + y^2)
ph = arctan2(y, x)
z = z
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.cylindrical_coordinates()[1]
rh
sage: E.cylindrical_coordinates()[3]
z
sage: E.cylindrical_coordinates()[:]
(rh, ph, z)
```

They can also be obtained via the operator `<,>`:

```
sage: cylindrical.<rh,ph,z> = E.cylindrical_coordinates()
sage: cylindrical
Chart (E^3, (rh, ph, z))
sage: rh, ph, z
(rh, ph, z)
```

Actually, `cylindrical.<rh,ph,z> = E.cylindrical_coordinates()` is a shortcut for:

```
sage: cylindrical = E.cylindrical_coordinates()
sage: rh, ph, z = cylindrical[:]
```

The coordinate symbols can be customized:

```
sage: E = EuclideanSpace(3)
sage: E.cylindrical_coordinates(symbols=r"R Phi:\Phi Z")
Chart (E^3, (R, Phi, Z))
sage: latex(E.cylindrical_coordinates())
\left(\mathbb{E}^{3},(R, {\Phi}, Z)\right)
```

Note that if the cylindrical coordinates have been already initialized, the argument `symbols` has no effect:

```
sage: E.cylindrical_coordinates(symbols=r"rh:\rho ph:\phi z")
Chart (E^3, (R, Phi, Z))
```

cylindrical_frame()

Return the orthonormal vector frame associated with cylindrical coordinates.

OUTPUT:

- *VectorFrame*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.cylindrical_frame()
Vector frame (E^3, (e_rh,e_ph,e_z))
sage: E.cylindrical_frame()[1]
Vector field e_rh on the Euclidean space E^3
sage: E.cylindrical_frame()[:]
(Vector field e_rh on the Euclidean space E^3,
 Vector field e_ph on the Euclidean space E^3,
 Vector field e_z on the Euclidean space E^3)
```

The cylindrical frame expressed in terms of the Cartesian one:

```
sage: for e in E.cylindrical_frame():
....:     e.display(E.cartesian_frame(), E.cylindrical_coordinates())
e_rh = cos(ph) e_x + sin(ph) e_y
e_ph = -sin(ph) e_x + cos(ph) e_y
e_z = e_z
```

The orthonormal frame (e_r, e_ϕ, e_z) expressed in terms of the coordinate frame $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial z}\right)$:

```
sage: for e in E.cylindrical_frame():
....:     e.display(E.cylindrical_coordinates())
e_rh = d/drh
e_ph = 1/rh d/dph
e_z = d/dz
```

scalar_triple_product (*name=None*, *latex_name=None*)

Return the scalar triple product operator, as a 3-form.

The *scalar triple product* (also called *mixed product*) of three vector fields u, v and w defined on an Euclidean space E is the scalar field

$$\epsilon(u, v, w) = u \cdot (v \times w).$$

The scalar triple product operator ϵ is a *3-form*, i.e. a field of fully antisymmetric trilinear forms; it is also called the *volume form* of E or the *Levi-Civita tensor* of E.

INPUT:

- `name` – (default: `None`) string; name given to the scalar triple product operator; if `None`, `'epsilon'` is used
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the scalar triple product; if `None`, it is set to `r'\epsilon'` if `name` is `None` and to `name` otherwise.

OUTPUT:

- the scalar triple product operator ϵ, as an instance of `DiffFormParal`

EXAMPLES:

```
sage: E.<x,y,z> = EuclideanSpace()
sage: triple_product = E.scalar_triple_product()
sage: triple_product
3-form epsilon on the Euclidean space E^3
sage: latex(triple_product)
\epsilon
sage: u = E.vector_field(x, y, z, name='u')
sage: v = E.vector_field(-y, x, 0, name='v')
sage: w = E.vector_field(y*z, x*z, x*y, name='w')
sage: s = triple_product(u, v, w); s
Scalar field epsilon(u,v,w) on the Euclidean space E^3
sage: s.display()
epsilon(u,v,w): E^3 --> R
   (x, y, z) |--> x^3*y + x*y^3 - 2*x*y*z^2
sage: s.expr()
x^3*y + x*y^3 - 2*x*y*z^2
sage: latex(s)
\epsilon\left(u,v,w\right)
sage: s == - triple_product(w, v, u)
True
```

Check of the identity $\epsilon(u, v, w) = u \cdot (v \times w)$:

```
sage: s == u.dot(v.cross(w))
True
```

Customizing the name:

```
sage: E.scalar_triple_product(name='S')
3-form S on the Euclidean space E^3
sage: latex(_)
S
sage: E.scalar_triple_product(name='Omega', latex_name=r'\Omega')
3-form Omega on the Euclidean space E^3
sage: latex(_)
\Omega
```

spherical_coordinates(*symbols=None*, *names=None*)
Return the chart of spherical coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the spherical chart has not been already defined; if `None` the symbols are generated as (r, θ, ϕ).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of spherical coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.spherical_coordinates()
Chart (E^3, (r, th, ph))
sage: latex(_)
\left(\mathbb{E}^{3},(r, {\theta}, {\phi})\right)
sage: E.spherical_coordinates().coord_range()
r: (0, +oo); th: (0, pi); ph: [0, 2*pi] (periodic)
```

The relation to Cartesian coordinates is:

```
sage: E.coord_change(E.spherical_coordinates(),
....:                E.cartesian_coordinates()).display()
x = r*cos(ph)*sin(th)
y = r*sin(ph)*sin(th)
z = r*cos(th)
sage: E.coord_change(E.cartesian_coordinates(),
....:                E.spherical_coordinates()).display()
r = sqrt(x^2 + y^2 + z^2)
th = arctan2(sqrt(x^2 + y^2), z)
ph = arctan2(y, x)
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.spherical_coordinates()[1]
r
sage: E.spherical_coordinates()[3]
ph
sage: E.spherical_coordinates()[:]
(r, th, ph)
```

They can also be obtained via the operator `<,>`:

```
sage: spherical.<r,th,ph> = E.spherical_coordinates()
sage: spherical
Chart (E^3, (r, th, ph))
sage: r, th, ph
(r, th, ph)
```

Actually, `spherical.<r,th,ph> = E.spherical_coordinates()` is a shortcut for:

```
sage: spherical = E.spherical_coordinates()
sage: r, th, ph = spherical[:]
```

The coordinate symbols can be customized:

```
sage: E = EuclideanSpace(3)
sage: E.spherical_coordinates(symbols=r"R T:\Theta F:\Phi")
Chart (E^3, (R, T, F))
sage: latex(E.spherical_coordinates())
\left(\mathbb{E}^{3},(R, {\Theta}, {\Phi})\right)
```

Note that if the spherical coordinates have been already initialized, the argument `symbols` has no effect:

```
sage: E.spherical_coordinates(symbols=r"r th:\theta ph:\phi")
Chart (E^3, (R, T, F))
```

spherical_frame()

Return the orthonormal vector frame associated with spherical coordinates.

OUTPUT:

- *VectorFrame*

EXAMPLES:

```
sage: E = EuclideanSpace(3)
sage: E.spherical_frame()
Vector frame (E^3, (e_r,e_th,e_ph))
sage: E.spherical_frame()[1]
Vector field e_r on the Euclidean space E^3
sage: E.spherical_frame()[:]
(Vector field e_r on the Euclidean space E^3,
 Vector field e_th on the Euclidean space E^3,
 Vector field e_ph on the Euclidean space E^3)
```

The spherical frame expressed in terms of the Cartesian one:

```
sage: for e in E.spherical_frame():
....:     e.display(E.cartesian_frame(), E.spherical_coordinates())
e_r = cos(ph)*sin(th) e_x + sin(ph)*sin(th) e_y + cos(th) e_z
e_th = cos(ph)*cos(th) e_x + cos(th)*sin(ph) e_y - sin(th) e_z
e_ph = -sin(ph) e_x + cos(ph) e_y
```

The orthonormal frame (e_r, e_θ, e_ϕ) expressed in terms of the coordinate frame $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)$:

```
sage: for e in E.spherical_frame():
....:     e.display(E.spherical_coordinates())
e_r = d/dr
e_th = 1/r d/dth
e_ph = 1/(r*sin(th)) d/dph
```

class sage.manifolds.differentiable.euclidean.**EuclideanPlane**(*name=None*, *latex_name=None*, *coordinates='Cartesian'*, *symbols=None*, *metric_name='g'*, *metric_latex_name=None*, *start_index=1*, *base_manifold=None*, *category=None*, *unique_tag=None*)

Bases: *sage.manifolds.differentiable.euclidean.EuclideanSpace*

Euclidean plane.

An *Euclidean plane* is an affine space E, whose associated vector space is a 2-dimensional vector space over $\mathbf{R}$ and is equipped with a positive definite symmetric bilinear form, called the *scalar product* or *dot product*.

The class *EuclideanPlane* inherits from *PseudoRiemannianManifold* (via *EuclideanSpace*) since an Euclidean plane can be viewed as a Riemannian manifold that is diffeomorphic to $\mathbf{R}^2$ and that has a flat metric g. The Euclidean scalar product is the one defined by the Riemannian metric g.

INPUT:

- `name` – (default: `None`) string; name (symbol) given to the Euclidean plane; if `None`, the name will be set to `'E^2'`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the Euclidean plane; if `None`, it is set to `'\mathbb{E}^{2}'` if `name` is `None` and to `name` otherwise
- `coordinates` – (default: `'Cartesian'`) string describing the type of coordinates to be initialized at the Euclidean plane creation; allowed values are `'Cartesian'` (see *cartesian_coordinates()*) and `'polar'` (see *polar_coordinates()*)
- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*, namely `symbols` is a string of coordinate fields separated by a blank space, where each field contains the coordinate's text symbol and possibly the coordinate's LaTeX symbol (when the latter is different from the text symbol), both symbols being separated by a colon (`:`); if `None`, the symbols will be automatically generated according to the value of `coordinates`
- `metric_name` – (default: `'g'`) string; name (symbol) given to the Euclidean metric tensor
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the Euclidean metric tensor; if none is provided, it is set to `metric_name`
- `start_index` – (default: 1) integer; lower value of the range of indices used for "indexed objects" in the Euclidean plane, e.g. coordinates of a chart
- `base_manifold` – (default: `None`) if not `None`, must be an Euclidean plane; the created object is then an open subset of `base_manifold`
- `category` – (default: `None`) to specify the category; if `None`, `Manifolds(RR).Differentiable()` (or `Manifolds(RR).Smooth()` if `diff_degree = infinity`) is assumed (see the category `Manifolds`)
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

- `init_coord_methods` – (default: `None`) dictionary of methods to initialize the various type of coordinates, with each key being a string describing the type of coordinates; to be used by derived classes only
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from *`PseudoRiemannianManifold`* would return the previously constructed object corresponding to these arguments)

EXAMPLES:

One creates an Euclidean plane `E` with:

```
sage: E.<x,y> = EuclideanSpace(); E
Euclidean plane E^2
```

`E` is a real smooth manifold of dimension 2:

```
sage: E.category()
Category of smooth manifolds over Real Field with 53 bits of precision
sage: dim(E)
2
```

It is endowed with a default coordinate chart, which is that of Cartesian coordinates (x, y):

```
sage: E.atlas()
[Chart (E^2, (x, y))]
sage: E.default_chart()
Chart (E^2, (x, y))
sage: cartesian = E.cartesian_coordinates()
sage: cartesian is E.default_chart()
True
```

A point of `E`:

```
sage: p = E((3,-2)); p
Point on the Euclidean plane E^2
sage: cartesian(p)
(3, -2)
sage: p in E
True
sage: p.parent() is E
True
```

`E` is endowed with a default metric tensor, which defines the Euclidean scalar product:

```
sage: g = E.metric(); g
Riemannian metric g on the Euclidean plane E^2
sage: g.display()
g = dx*dx + dy*dy
```

Curvilinear coordinates can be introduced on `E`: see *`polar_coordinates()`*.

See also:

Example 1: the Euclidean plane

`cartesian_coordinates`(*symbols=None*, *names=None*)

Return the chart of Cartesian coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the Cartesian chart has not been already defined; if `None` the symbols are generated as (x, y).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of Cartesian coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(2)
sage: E.cartesian_coordinates()
Chart (E^2, (x, y))
sage: E.cartesian_coordinates().coord_range()
x: (-oo, +oo); y: (-oo, +oo)
```

An example where the Cartesian coordinates have not been previously created:

```
sage: E = EuclideanSpace(2, coordinates='polar')
sage: E.atlas()  # only polar coordinates have been initialized
[Chart (E^2, (r, ph))]
sage: E.cartesian_coordinates(symbols='X Y')
Chart (E^2, (X, Y))
sage: E.atlas()  # the Cartesian chart has been added to the atlas
[Chart (E^2, (r, ph)), Chart (E^2, (X, Y))]
```

Note that if the Cartesian coordinates have been already initialized, the argument `symbols` has no effect:

```
sage: E.cartesian_coordinates(symbols='x y')
Chart (E^2, (X, Y))
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.cartesian_coordinates()[1]
X
sage: E.cartesian_coordinates()[2]
Y
sage: E.cartesian_coordinates()[:]
(X, Y)
```

It is also possible to use the operator `<,>` to set symbolic variable containing the coordinates:

```
sage: E = EuclideanSpace(2, coordinates='polar')
sage: cartesian.<u,v> = E.cartesian_coordinates()
sage: cartesian
Chart (E^2, (u, v))
sage: u,v
(u, v)
```

The command `cartesian.<u,v> = E.cartesian_coordinates()` is actually a shortcut for:

```
sage: cartesian = E.cartesian_coordinates(symbols='u v')
sage: u, v = cartesian[:]
```

polar_coordinates(*symbols=None, names=None*)

Return the chart of polar coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the polar chart has not been already defined; if `None` the symbols are generated as (r, ϕ).
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of polar coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(2)
sage: E.polar_coordinates()
Chart (E^2, (r, ph))
sage: latex(_)
\left(\mathbb{E}^{2},(r, {\phi})\right)
sage: E.polar_coordinates().coord_range()
r: (0, +oo); ph: [0, 2*pi] (periodic)
```

The relation to Cartesian coordinates is:

```
sage: E.coord_change(E.polar_coordinates(),
....:                E.cartesian_coordinates()).display()
x = r*cos(ph)
y = r*sin(ph)
sage: E.coord_change(E.cartesian_coordinates(),
....:                E.polar_coordinates()).display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
```

The coordinate variables are returned by the square bracket operator:

```
sage: E.polar_coordinates()[1]
r
sage: E.polar_coordinates()[2]
ph
sage: E.polar_coordinates()[:]
(r, ph)
```

They can also be obtained via the operator `<,>`:

```
sage: polar.<r,ph> = E.polar_coordinates(); polar
Chart (E^2, (r, ph))
sage: r, ph
(r, ph)
```

Actually, `polar.<r,ph> = E.polar_coordinates()` is a shortcut for:

```
sage: polar = E.polar_coordinates()
sage: r, ph = polar[:]
```

The coordinate symbols can be customized:

```
sage: E = EuclideanSpace(2)
sage: E.polar_coordinates(symbols=r"r th:\theta")
```

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```
Chart (E^2, (r, th))
sage: latex(E.polar_coordinates())
\left(\mathbb{E}^{2},(r, {\theta})\right)
```

Note that if the polar coordinates have been already initialized, the argument `symbols` has no effect:

```
sage: E.polar_coordinates(symbols=r"R Th:\Theta")
Chart (E^2, (r, th))
```

polar_frame()

Return the orthonormal vector frame associated with polar coordinates.

OUTPUT:

- instance of *VectorFrame*

EXAMPLES:

```
sage: E = EuclideanSpace(2)
sage: E.polar_frame()
Vector frame (E^2, (e_r,e_ph))
sage: E.polar_frame()[1]
Vector field e_r on the Euclidean plane E^2
sage: E.polar_frame()[:]
(Vector field e_r on the Euclidean plane E^2,
 Vector field e_ph on the Euclidean plane E^2)
```

The orthonormal polar frame expressed in terms of the Cartesian one:

```
sage: for e in E.polar_frame():
....:     e.display(E.cartesian_frame(), E.polar_coordinates())
e_r = cos(ph) e_x + sin(ph) e_y
e_ph = -sin(ph) e_x + cos(ph) e_y
```

The orthonormal frame (e_r, e_ϕ) expressed in terms of the coordinate frame $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}\right)$:

```
sage: for e in E.polar_frame():
....:     e.display(E.polar_coordinates())
e_r = d/dr
e_ph = 1/r d/dph
```

class sage.manifolds.differentiable.euclidean.**EuclideanSpace**(*n*, *name=None*, *latex_name=None*, *coordinates='Cartesian'*, *symbols=None*, *metric_name='g'*, *metric_latex_name=None*, *start_index=1*, *base_manifold=None*, *category=None*, *init_coord_methods=None*, *unique_tag=None*)

Bases: *sage.manifolds.differentiable.pseudo_riemannian.PseudoRiemannianManifold*

Euclidean space.

An *Euclidean space of dimension* n is an affine space E, whose associated vector space is a n-dimensional vector space over $\mathbf{R}$ and is equipped with a positive definite symmetric bilinear form, called the *scalar product* or *dot product*.

Euclidean space of dimension n can be viewed as a Riemannian manifold that is diffeomorphic to $\mathbf{R}^n$ and that has a flat metric g. The Euclidean scalar product is the one defined by the Riemannian metric g.

INPUT:

- `n` – positive integer; dimension of the space over the real field
- `name` – (default: `None`) string; name (symbol) given to the Euclidean space; if `None`, the name will be set to `'E^n'`
- `latex_name` – (default: `None`) string; LaTeX symbol to denote the space; if `None`, it is set to `'\mathbb{E}^{n}'` if `name` is `None` and to `name` otherwise
- `coordinates` – (default: `'Cartesian'`) string describing the type of coordinates to be initialized at the Euclidean space creation; allowed values are
  - `'Cartesian'` (canonical coordinates on $\mathbf{R}^n$)
  - `'polar'` for `n=2` only (see `polar_coordinates()`)
  - `'spherical'` for `n=3` only (see `spherical_coordinates()`)
  - `'cylindrical'` for `n=3` only (see `cylindrical_coordinates()`)
- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in `RealDiffChart`, namely `symbols` is a string of coordinate fields separated by a blank space, where each field contains the coordinate's text symbol and possibly the coordinate's LaTeX symbol (when the latter is different from the text symbol), both symbols being separated by a colon (`:`); if `None`, the symbols will be automatically generated according to the value of `coordinates`
- `metric_name` – (default: `'g'`) string; name (symbol) given to the Euclidean metric tensor
- `metric_latex_name` – (default: `None`) string; LaTeX symbol to denote the Euclidean metric tensor; if none is provided, it is set to `metric_name`
- `start_index` – (default: 1) integer; lower value of the range of indices used for "indexed objects" in the Euclidean space, e.g. coordinates of a chart
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must then be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

If `names` is specified, then `n` does not have to be specified.

EXAMPLES:

Constructing a 2-dimensional Euclidean space:

```
sage: E = EuclideanSpace(2); E
Euclidean plane E^2
```

Each call to `EuclideanSpace` creates a different object:

```
sage: E1 = EuclideanSpace(2)
sage: E1 is E
False
sage: E1 == E
False
```

The LaTeX symbol of the Euclidean space is by default $\mathbb{E}^n$, where n is the dimension:

```
sage: latex(E)
\mathbb{E}^{2}
```

But both the name and LaTeX names of the Euclidean space can be customized:

```
sage: F = EuclideanSpace(2, name='F', latex_name=r'\mathcal{F}'); F
Euclidean plane F
sage: latex(F)
\mathcal{F}
```

By default, an Euclidean space is created with a single coordinate chart: that of Cartesian coordinates:

```
sage: E.atlas()
[Chart (E^2, (x, y))]
sage: E.cartesian_coordinates()
Chart (E^2, (x, y))
sage: E.default_chart() is E.cartesian_coordinates()
True
```

The coordinate variables can be initialized, as the Python variables `x` and `y`, by:

```
sage: x, y = E.cartesian_coordinates()[:]
```

However, it is possible to both construct the Euclidean space and initialize the coordinate variables in a single stage, thanks to SageMath operator `<,>`:

```
sage: E.<x,y> = EuclideanSpace()
```

Note that providing the dimension as an argument of `EuclideanSpace` is not necessary in that case, since it can be deduced from the number of coordinates within `<,>`. Besides, the coordinate symbols can be customized:

```
sage: E.<X,Y> = EuclideanSpace()
sage: E.cartesian_coordinates()
Chart (E^2, (X, Y))
```

By default, the LaTeX symbols of the coordinates coincide with the text ones:

```
sage: latex(X+Y)
X + Y
```

However, it is possible to customize them, via the argument `symbols`, which must be a string, usually prefixed by `r` (for *raw* string, in order to allow for the backslash character of LaTeX expressions). This string contains the coordinate fields separated by a blank space; each field contains the coordinate's text symbol and possibly the coordinate's LaTeX symbol (when the latter is different from the text symbol), both symbols being separated by a colon (`:`):

```
sage: E.<xi,ze> = EuclideanSpace(symbols=r"xi:\xi ze:\zeta")
sage: E.cartesian_coordinates()
Chart (E^2, (xi, ze))
sage: latex(xi+ze)
{\xi} + {\zeta}
```

Thanks to the argument `coordinates`, an Euclidean space can be constructed with curvilinear coordinates initialized instead of the Cartesian ones:

```
sage: E.<r,ph> = EuclideanSpace(coordinates='polar')
sage: E.atlas()   # no Cartesian coordinates have been constructed
[Chart (E^2, (r, ph))]
sage: polar = E.polar_coordinates(); polar
Chart (E^2, (r, ph))
sage: E.default_chart() is polar
True
sage: latex(r+ph)
{\phi} + r
```

The Cartesian coordinates, along with the transition maps to and from the curvilinear coordinates, can be constructed at any time by:

```
sage: cartesian.<x,y> = E.cartesian_coordinates()
sage: E.atlas()  # both polar and Cartesian coordinates now exist
[Chart (E^2, (r, ph)), Chart (E^2, (x, y))]
```

The transition maps have been initialized by the command `E.cartesian_coordinates()`:

```
sage: E.coord_change(polar, cartesian).display()
x = r*cos(ph)
y = r*sin(ph)
sage: E.coord_change(cartesian, polar).display()
r = sqrt(x^2 + y^2)
ph = arctan2(y, x)
```

The default name of the Euclidean metric tensor is g:

```
sage: E.metric()
Riemannian metric g on the Euclidean plane E^2
sage: latex(_)
g
```

But this can be customized:

```
sage: E = EuclideanSpace(2, metric_name='h')
sage: E.metric()
Riemannian metric h on the Euclidean plane E^2
sage: latex(_)
h
sage: E = EuclideanSpace(2, metric_latex_name=r'\mathbf{g}')
sage: E.metric()
Riemannian metric g on the Euclidean plane E^2
sage: latex(_)
\mathbf{g}
```

A 4-dimensional Euclidean space:

```
sage: E = EuclideanSpace(4); E
4-dimensional Euclidean space E^4
sage: latex(E)
\mathbb{E}^{4}
```

`E` is a real smooth manifold of dimension 4:

```
sage: E.category()
Category of smooth manifolds over Real Field with 53 bits of precision
```

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```
sage: dim(E)
4
```

It is endowed with a default coordinate chart, which is that of Cartesian coordinates (x_1, x_2, x_3, x_4):

```
sage: E.atlas()
[Chart (E^4, (x1, x2, x3, x4))]
sage: E.default_chart()
Chart (E^4, (x1, x2, x3, x4))
sage: E.default_chart() is E.cartesian_coordinates()
True
```

`E` is also endowed with a default metric tensor, which defines the Euclidean scalar product:

```
sage: g = E.metric(); g
Riemannian metric g on the 4-dimensional Euclidean space E^4
sage: g.display()
g = dx1*dx1 + dx2*dx2 + dx3*dx3 + dx4*dx4
```

cartesian_coordinates(*symbols=None*, *names=None*)

Return the chart of Cartesian coordinates, possibly creating it if it does not already exist.

INPUT:

- `symbols` – (default: `None`) string defining the coordinate text symbols and LaTeX symbols, with the same conventions as the argument `coordinates` in *RealDiffChart*; this is used only if the Cartesian chart has not been already defined; if `None` the symbols are generated as $(x_1, \ldots, x_n)$.
- `names` – (default: `None`) unused argument, except if `symbols` is not provided; it must be a tuple containing the coordinate symbols (this is guaranteed if the shortcut operator `<,>` is used)

OUTPUT:

- the chart of Cartesian coordinates, as an instance of *RealDiffChart*

EXAMPLES:

```
sage: E = EuclideanSpace(4)
sage: X = E.cartesian_coordinates(); X
Chart (E^4, (x1, x2, x3, x4))
sage: X.coord_range()
x1: (-oo, +oo); x2: (-oo, +oo); x3: (-oo, +oo); x4: (-oo, +oo)
sage: X[2]
x2
sage: X[:]
(x1, x2, x3, x4)
sage: latex(X[:])
\left({x_{1}}, {x_{2}}, {x_{3}}, {x_{4}}\right)
```

cartesian_frame()

Return the orthonormal vector frame associated with Cartesian coordinates.

OUTPUT:

- *CoordFrame*

EXAMPLES:

```
sage: E = EuclideanSpace(2)
sage: E.cartesian_frame()
Coordinate frame (E^2, (e_x,e_y))
sage: E.cartesian_frame()[1]
Vector field e_x on the Euclidean plane E^2
sage: E.cartesian_frame()[:]
(Vector field e_x on the Euclidean plane E^2,
 Vector field e_y on the Euclidean plane E^2)
```

For Cartesian coordinates, the orthonormal frame coincides with the coordinate frame:

```
sage: E.cartesian_frame() is E.cartesian_coordinates().frame()
True
```

3.2.2 Operators for vector calculus

This module defines the following operators for scalar, vector and tensor fields on any pseudo-Riemannian manifold (see `pseudo_riemannian`), and in particular on Euclidean spaces (see `euclidean`) :

- `grad()`: gradient of a scalar field
- `div()`: divergence of a vector field, and more generally of a tensor field
- `curl()`: curl of a vector field (3-dimensional case only)
- `laplacian()`: Laplace-Beltrami operator acting on a scalar field, a vector field, or more generally a tensor field
- `dalembertian()`: d'Alembert operator acting on a scalar field, a vector field, or more generally a tensor field, on a Lorentzian manifold

All these operators are implemented as functions that call the appropriate method on their argument. The purpose is to allow one to use standard mathematical notations, e.g. to write `curl(v)` instead of `v.curl()`.

Note that the `norm()` operator is defined in the module `functional`.

See also:

Examples 1 and 2 in `euclidean` for examples involving these operators in the Euclidean plane and in the Euclidean 3-space.

AUTHORS:

- Eric Gourgoulhon (2018): initial version

`sage.manifolds.operators.`**`curl`**`(`*vector*`)`

Curl operator.

The *curl* of a vector field v on an orientable pseudo-Riemannian manifold (M, g) of dimension 3 is the vector field defined by

$$\mathrm{curl}\, v = (*(\mathrm{d}v^\flat))^\sharp$$

where $v^\flat$ is the 1-form associated to v by the metric g (see `down()`), $*(\mathrm{d}v^\flat)$ is the Hodge dual with respect to g of the 2-form $\mathrm{d}v^\flat$ (exterior derivative of $v^\flat$) (see `hodge_dual()`) and $(*(\mathrm{d}v^\flat))^\sharp$ is corresponding vector field by g-duality (see `up()`).

An alternative expression of the curl is

$$(\mathrm{curl}\, v)^i = \epsilon^{ijk} \nabla_j v_k$$

where ∇ is the Levi-Civita connection of g (cf. `LeviCivitaConnection`) and ϵ the volume 3-form (Levi-Civita tensor) of g (cf. `volume_form()`)

INPUT:

- `vector` – vector field on an orientable 3-dimensional pseudo-Riemannian manifold, as an instance of `VectorField`

OUTPUT:

- instance of `VectorField` representing the curl of `vector`

EXAMPLES:

Curl of a vector field in the Euclidean 3-space:

```
sage: E.<x,y,z> = EuclideanSpace()
sage: v = E.vector_field(sin(y), sin(x), 0, name='v')
sage: v.display()
v = sin(y) e_x + sin(x) e_y
sage: from sage.manifolds.operators import curl
sage: s = curl(v); s
Vector field curl(v) on the Euclidean space E^3
sage: s.display()
curl(v) = (cos(x) - cos(y)) e_z
sage: s[:]
[0, 0, cos(x) - cos(y)]
```

See the method `curl()` of `VectorField` for more details and examples.

`sage.manifolds.operators.`**`dalembertian`**(*field*)

d'Alembert operator.

The *d'Alembert operator* or *d'Alembertian* on a Lorentzian manifold (M, g) is nothing but the Laplace-Beltrami operator:

$$\Box = \nabla_i \nabla^i = g^{ij} \nabla_i \nabla_j$$

where ∇ is the Levi-Civita connection of the metric g (cf. `LeviCivitaConnection`) and $\nabla^i := g^{ij} \nabla_j$

INPUT:

- `field` – a scalar field f (instance of `DiffScalarField`) or a tensor field f (instance of `TensorField`) on a pseudo-Riemannian manifold

OUTPUT:

- $\Box f$, as an instance of `DiffScalarField` or of `TensorField`

EXAMPLES:

d'Alembertian of a scalar field in the 2-dimensional Minkowski spacetime:

```
sage: M = Manifold(2, 'M', structure='Lorentzian')
sage: X.<t,x> = M.chart()
sage: g = M.metric()
sage: g[0,0], g[1,1] = -1, 1
sage: f = M.scalar_field((x-t)^3 + (x+t)^2, name='f')
sage: from sage.manifolds.operators import dalembertian
sage: Df = dalembertian(f); Df
Scalar field Box(f) on the 2-dimensional Lorentzian manifold M
sage: Df.display()
```

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```
Box(f): M --> R
   (t, x) |--> 0
```

See the method `dalembertian()` of `DiffScalarField` and the method `dalembertian()` of `TensorField` for more details and examples.

`sage.manifolds.operators.div(tensor)`

Divergence operator.

Let t be a tensor field of type $(k, 0)$ with $k \geq 1$ on a pseudo-Riemannian manifold (M, g). The *divergence* of t is the tensor field of type $(k-1, 0)$ defined by

$$(\operatorname{div} t)^{a_1 \ldots a_{k-1}} = \nabla_i t^{a_1 \ldots a_{k-1} i} = (\nabla t)^{a_1 \ldots a_{k-1} i}{}_i$$

where ∇ is the Levi-Civita connection of g (cf. `LeviCivitaConnection`).

Note that the divergence is taken on the *last* index of the tensor field. This definition is extended to tensor fields of type (k, l) with $k \geq 0$ and $l \geq 1$, by raising the last index with the metric g: $\operatorname{div} t$ is then the tensor field of type $(k, l-1)$ defined by

$$(\operatorname{div} t)^{a_1 \ldots a_k}{}_{b_1 \ldots b_{l-1}} = \nabla_i (g^{ij} t^{a_1 \ldots a_k}{}_{b_1 \ldots b_{l-1} j}) = (\nabla t^\sharp)^{a_1 \ldots a_k i}{}_{b_1 \ldots b_{l-1} i}$$

where $t^\sharp$ is the tensor field deduced from t by raising the last index with the metric g (see `up()`).

INPUT:

- `tensor` – tensor field t on a pseudo-Riemannian manifold (M, g), as an instance of `TensorField` (possibly via one of its derived classes, like `VectorField`)

OUTPUT:

- the divergence of `tensor` as an instance of either `DiffScalarField` if $(k, l) = (1, 0)$ (`tensor` is a vector field) or $(k, l) = (0, 1)$ (`tensor` is a 1-form) or of `TensorField` if $k + l \geq 2$

EXAMPLES:

Divergence of a vector field in the Euclidean plane:

```
sage: E.<x,y> = EuclideanSpace()
sage: v = E.vector_field(cos(x*y), sin(x*y), name='v')
sage: v.display()
v = cos(x*y) e_x + sin(x*y) e_y
sage: from sage.manifolds.operators import div
sage: s = div(v); s
Scalar field div(v) on the Euclidean plane E^2
sage: s.display()
div(v): E^2 --> R
   (x, y) |--> x*cos(x*y) - y*sin(x*y)
sage: s.expr()
x*cos(x*y) - y*sin(x*y)
```

See the method `divergence()` of `TensorField` for more details and examples.

`sage.manifolds.operators.grad(scalar)`

Gradient operator.

The *gradient* of a scalar field f on a pseudo-Riemannian manifold (M, g) is the vector field $\operatorname{grad} f$ whose components in any coordinate frame are

$$(\operatorname{grad} f)^i = g^{ij} \frac{\partial F}{\partial x^j}$$

where the x^j's are the coordinates with respect to which the frame is defined and F is the chart function representing f in these coordinates: $f(p) = F(x^1(p), \ldots, x^n(p))$ for any point p in the chart domain. In other words, the gradient of f is the vector field that is the g-dual of the differential of f.

INPUT:

- `scalar` – scalar field f, as an instance of `DiffScalarField`

OUTPUT:

- instance of `VectorField` representing grad f

EXAMPLES:

Gradient of a scalar field in the Euclidean plane:

```
sage: E.<x,y> = EuclideanSpace()
sage: f = E.scalar_field(sin(x*y), name='f')
sage: from sage.manifolds.operators import grad
sage: grad(f)
Vector field grad(f) on the Euclidean plane E^2
sage: grad(f).display()
grad(f) = y*cos(x*y) e_x + x*cos(x*y) e_y
sage: grad(f)[:]
[y*cos(x*y), x*cos(x*y)]
```

See the method `gradient()` of `DiffScalarField` for more details and examples.

`sage.manifolds.operators.laplacian(field)`

Laplace-Beltrami operator.

The *Laplace-Beltrami operator* on a pseudo-Riemannian manifold (M, g) is the operator

$$\Delta = \nabla_i \nabla^i = g^{ij} \nabla_i \nabla_j$$

where ∇ is the Levi-Civita connection of the metric g (cf. `LeviCivitaConnection`) and $\nabla^i := g^{ij} \nabla_j$

INPUT:

- `field` – a scalar field f (instance of `DiffScalarField`) or a tensor field f (instance of `TensorField`) on a pseudo-Riemannian manifold

OUTPUT:

- Δf, as an instance of `DiffScalarField` or of `TensorField`

EXAMPLES:

Laplacian of a scalar field on the Euclidean plane:

```
sage: E.<x,y> = EuclideanSpace()
sage: f = E.scalar_field(sin(x*y), name='f')
sage: from sage.manifolds.operators import laplacian
sage: Df = laplacian(f); Df
Scalar field Delta(f) on the Euclidean plane E^2
sage: Df.display()
Delta(f): E^2 --> R
   (x, y) |--> -(x^2 + y^2)*sin(x*y)
sage: Df.expr()
-(x^2 + y^2)*sin(x*y)
```

The Laplacian of a scalar field is the divergence of its gradient:

```
sage: from sage.manifolds.operators import div, grad
sage: Df == div(grad(f))
True
```

See the method `laplacian()` of `DiffScalarField` and the method `laplacian()` of `TensorField` for more details and examples.

3.3 Pseudo-Riemannian Metrics

The class `PseudoRiemannianMetric` implements pseudo-Riemannian metrics on differentiable manifolds over $\mathbf{R}$. The derived class `PseudoRiemannianMetricParal` is devoted to metrics with values on a parallelizable manifold.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Pablo Angulo (2016) : Schouten, Cotton and Cotton-York tensors
- Florentin Jaffredo (2018) : series expansion for the inverse metric

REFERENCES:

- [?]
- [?]
- [?]

class `sage.manifolds.differentiable.metric.`**`PseudoRiemannianMetric`**(*vector_field_module*, *name*, *signature=None*, *latex_name=None*)

Bases: `sage.manifolds.differentiable.tensorfield.TensorField`

Pseudo-Riemannian metric with values on an open subset of a differentiable manifold.

An instance of this class is a field of nondegenerate symmetric bilinear forms (metric field) along a differentiable manifold U with values on a differentiable manifold M over $\mathbf{R}$, via a differentiable mapping $\Phi : U \rightarrow M$. The standard case of a metric field *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

A *metric* g is a field on U, such that at each point $p \in U$, $g(p)$ is a bilinear map of the type:

$$g(p) : \ T_q M \times T_q M \longrightarrow \mathbf{R}$$

where $T_q M$ stands for the tangent space to the manifold M at the point $q = \Phi(p)$, such that $g(p)$ is symmetric: $\forall (u, v) \in T_q M \times T_q M, \ g(p)(v, u) = g(p)(u, v)$ and nondegenerate: $(\forall v \in T_q M, \ \ g(p)(u, v) = 0) \Longrightarrow u = 0$.

Note: If M is parallelizable, the class `PseudoRiemannianMetricParal` should be used instead.

INPUT:

- `vector_field_module` – module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $\Phi(U) \subset M$
- `name` – name given to the metric

- `signature` – (default: `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is `None`, S is set to the dimension of manifold M (Riemannian signature)
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

EXAMPLES:

Standard metric on the sphere S^2:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: # The two open domains covered by stereographic coordinates (North and␣
↪South):
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic coord
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                  intersection_name='W', restrictions1= x^2+y^2!=0,
....:                  restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V) # The complement of the two poles
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: c_xyW = c_xy.restrict(W) ; c_uvW = c_uv.restrict(W)
sage: eUW = c_xyW.frame() ; eVW = c_uvW.frame()
sage: g = M.metric('g') ; g
Riemannian metric g on the 2-dimensional differentiable manifold S^2
```

The metric is considered as a tensor field of type (0,2) on S^2:

```
sage: g.parent()
Module T^(0,2)(S^2) of type-(0,2) tensors fields on the 2-dimensional
 differentiable manifold S^2
```

We define g by its components on domain U (factorizing them to have a nicer view):

```
sage: g[eU,1,1], g[eU,2,2] = 4/(1+x^2+y^2)^2, 4/(1+x^2+y^2)^2
sage: g.display(eU)
g = 4/(x^2 + y^2 + 1)^2 dx*dx + 4/(x^2 + y^2 + 1)^2 dy*dy
```

A matrix view of the components:

```
sage: g[eU,:]
[4/(x^2 + y^2 + 1)^2                   0]
[                  0 4/(x^2 + y^2 + 1)^2]
```

The components of g on domain V expressed in terms of (u,v) coordinates are similar to those on domain U expressed in (x,y) coordinates, as we can check explicitly by asking for the component transformation on the common subdomain W:

```
sage: g.display(eVW, c_uvW)
g = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du*du
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv*dv
```

Therefore, we set:

```
sage: g[eV,1,1], g[eV,2,2] = 4/(1+u^2+v^2)^2, 4/(1+u^2+v^2)^2
sage: g[eV,1,1].factor() ; g[eV,2,2].factor()
```

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```
4/(u^2 + v^2 + 1)^2
4/(u^2 + v^2 + 1)^2
sage: g.display(eV)
g = 4/(u^2 + v^2 + 1)^2 du*du + 4/(u^2 + v^2 + 1)^2 dv*dv
```

At this stage, the metric is fully defined on the whole sphere. Its restriction to some subdomain is itself a metric (by default, it bears the same symbol):

```
sage: g.restrict(U)
Riemannian metric g on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: g.restrict(U).parent()
Free module T^(0,2)(U) of type-(0,2) tensors fields on the Open subset
 U of the 2-dimensional differentiable manifold S^2
```

The parent of $g|_U$ is a free module because is U is a parallelizable domain, contrary to S^2. Actually, g and $g|_U$ have different Python type:

```
sage: type(g)
<class 'sage.manifolds.differentiable.metric.PseudoRiemannianMetric'>
sage: type(g.restrict(U))
<class 'sage.manifolds.differentiable.metric.PseudoRiemannianMetricParal'>
```

As a field of bilinear forms, the metric acts on pairs of tensor fields, yielding a scalar field:

```
sage: a = M.vector_field({eU: [x, 2+y]}, name='a')
sage: a.add_comp_by_continuation(eV, W, chart=c_uv)
sage: b = M.vector_field({eU: [-y, x]}, name='b')
sage: b.add_comp_by_continuation(eV, W, chart=c_uv)
sage: s = g(a,b) ; s
Scalar field g(a,b) on the 2-dimensional differentiable manifold S^2
sage: s.display()
g(a,b): S^2 --> R
on U: (x, y) |--> 8*x/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1)
on V: (u, v) |--> 8*(u^3 + u*v^2)/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1)
```

The inverse metric is:

```
sage: ginv = g.inverse() ; ginv
Tensor field inv_g of type (2,0) on the 2-dimensional differentiable
 manifold S^2
sage: ginv.parent()
Module T^(2,0)(S^2) of type-(2,0) tensors fields on the 2-dimensional
 differentiable manifold S^2
sage: latex(ginv)
g^{-1}
sage: ginv.display(eU) # again the components are expanded
inv_g = (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) d/dx*d/dx
 + (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) d/dy*d/dy
sage: ginv.display(eV)
inv_g = (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) d/du*d/du
 + (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) d/dv*d/dv
```

We have:

```
sage: ginv.restrict(U) is g.restrict(U).inverse()
True
sage: ginv.restrict(V) is g.restrict(V).inverse()
True
sage: ginv.restrict(W) is g.restrict(W).inverse()
True
```

The volume form (Levi-Civita tensor) associated with g:

```
sage: eps = g.volume_form() ; eps
2-form eps_g on the 2-dimensional differentiable manifold S^2
sage: eps.display(eU)
eps_g = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx/\dy
sage: eps.display(eV)
eps_g = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du/\dv
```

The unique non-trivial component of the volume form is nothing but the square root of the determinant of g in the corresponding frame:

```
sage: eps[[eU,1,2]] == g.sqrt_abs_det(eU)
True
sage: eps[[eV,1,2]] == g.sqrt_abs_det(eV)
True
```

The Levi-Civita connection associated with the metric g:

```
sage: nabla = g.connection() ; nabla
Levi-Civita connection nabla_g associated with the Riemannian metric g
 on the 2-dimensional differentiable manifold S^2
sage: latex(nabla)
\nabla_{g}
```

The Christoffel symbols ${\Gamma^i}_{jk}$ associated with some coordinates:

```
sage: g.christoffel_symbols(c_xy)
3-indices components w.r.t. Coordinate frame (U, (d/dx,d/dy)), with
 symmetry on the index positions (1, 2)
sage: g.christoffel_symbols(c_xy)[:]
[[[-2*x/(x^2 + y^2 + 1), -2*y/(x^2 + y^2 + 1)],
  [-2*y/(x^2 + y^2 + 1), 2*x/(x^2 + y^2 + 1)]],
 [[2*y/(x^2 + y^2 + 1), -2*x/(x^2 + y^2 + 1)],
  [-2*x/(x^2 + y^2 + 1), -2*y/(x^2 + y^2 + 1)]]]
sage: g.christoffel_symbols(c_uv)[:]
[[[-2*u/(u^2 + v^2 + 1), -2*v/(u^2 + v^2 + 1)],
  [-2*v/(u^2 + v^2 + 1), 2*u/(u^2 + v^2 + 1)]],
 [[2*v/(u^2 + v^2 + 1), -2*u/(u^2 + v^2 + 1)],
  [-2*u/(u^2 + v^2 + 1), -2*v/(u^2 + v^2 + 1)]]]
```

The Christoffel symbols are nothing but the connection coefficients w.r.t. the coordinate frame:

```
sage: g.christoffel_symbols(c_xy) is nabla.coef(c_xy.frame())
True
sage: g.christoffel_symbols(c_uv) is nabla.coef(c_uv.frame())
True
```

Test that ∇ is the connection compatible with g:

```
sage: t = nabla(g) ; t
Tensor field nabla_g(g) of type (0,3) on the 2-dimensional
 differentiable manifold S^2
sage: t.display(eU)
nabla_g(g) = 0
sage: t.display(eV)
nabla_g(g) = 0
sage: t == 0
True
```

The Riemann curvature tensor of g:

```
sage: riem = g.riemann() ; riem
Tensor field Riem(g) of type (1,3) on the 2-dimensional differentiable
 manifold S^2
sage: riem.display(eU)
Riem(g) = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) d/dx*dy*dx*dy
 - 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) d/dx*dy*dy*dx
 - 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) d/dy*dx*dx*dy
 + 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) d/dy*dx*dy*dx
sage: riem.display(eV)
Riem(g) = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) d/du*dv*du*dv
 - 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) d/du*dv*dv*du
 - 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) d/dv*du*du*dv
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) d/dv*du*dv*du
```

The Ricci tensor of g:

```
sage: ric = g.ricci() ; ric
Field of symmetric bilinear forms Ric(g) on the 2-dimensional
 differentiable manifold S^2
sage: ric.display(eU)
Ric(g) = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx*dx
 + 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dy*dy
sage: ric.display(eV)
Ric(g) = 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) du*du
 + 4/(u^4 + v^4 + 2*(u^2 + 1)*v^2 + 2*u^2 + 1) dv*dv
sage: ric == g
True
```

The Ricci scalar of g:

```
sage: r = g.ricci_scalar() ; r
Scalar field r(g) on the 2-dimensional differentiable manifold S^2
sage: r.display()
r(g): S^2 --> R
on U: (x, y) |--> 2
on V: (u, v) |--> 2
```

In dimension 2, the Riemann tensor can be expressed entirely in terms of the Ricci scalar r:

$$R^i{}_{jlk} = \frac{r}{2} \left(\delta^i{}_k g_{jl} - \delta^i{}_l g_{jk} \right)$$

This formula can be checked here, with the r.h.s. rewritten as $-r g_{j[k} \delta^i{}_{l]}$:

```
sage: delta = M.tangent_identity_field()
sage: riem == - r*(g*delta).antisymmetrize(2,3)
True
```

christoffel_symbols(*chart=None*)

Christoffel symbols of `self` with respect to a chart.

INPUT:

- `chart` – (default: `None`) chart with respect to which the Christoffel symbols are required; if none is provided, the default chart of the metric's domain is assumed.

OUTPUT:

- the set of Christoffel symbols in the given chart, as an instance of `CompWithSym`

EXAMPLES:

Christoffel symbols of the flat metric on $\mathbf{R}^3$ with respect to spherical coordinates:

```
sage: M = Manifold(3, 'R3', r'\RR^3', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: X.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2, r^2*sin(th)^2
sage: g.display()  # the standard flat metric expressed in spherical␣
↪coordinates
g = dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: Gam = g.christoffel_symbols() ; Gam
3-indices components w.r.t. Coordinate frame (U, (d/dr,d/dth,d/dph)),
 with symmetry on the index positions (1, 2)
sage: type(Gam)
<class 'sage.tensor.modules.comp.CompWithSym'>
sage: Gam[:]
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(th)^2]],
[[0, 1/r, 0], [1/r, 0, 0], [0, 0, -cos(th)*sin(th)]],
[[0, 0, 1/r], [0, 0, cos(th)/sin(th)], [1/r, cos(th)/sin(th), 0]]]
sage: Gam[1,2,2]
-r
sage: Gam[2,1,2]
1/r
sage: Gam[3,1,3]
1/r
sage: Gam[3,2,3]
cos(th)/sin(th)
sage: Gam[2,3,3]
-cos(th)*sin(th)
```

Note that a better display of the Christoffel symbols is provided by the method *christoffel_symbols_display()*:

```
sage: g.christoffel_symbols_display()
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

christoffel_symbols_display(*chart=None, symbol=None, latex_symbol=None, index_labels=None, index_latex_labels=None, coordinate_labels=True, only_nonzero=True, only_nonredundant=True*)

Display the Christoffel symbols w.r.t. to a given chart, one per line.

The output is either text-formatted (console mode) or LaTeX-formatted (notebook mode).

INPUT:

- `chart` – (default: `None`) chart with respect to which the Christoffel symbols are defined; if none is provided, the default chart of the metric's domain is assumed.
- `symbol` – (default: `None`) string specifying the symbol of the connection coefficients; if `None`, 'Gam' is used
- `latex_symbol` – (default: `None`) string specifying the LaTeX symbol for the components; if `None`, '\Gamma' is used
- `index_labels` – (default: `None`) list of strings representing the labels of each index; if `None`, coordinate symbols are used except if `coordinate_symbols` is set to `False`, in which case integer labels are used
- `index_latex_labels` – (default: `None`) list of strings representing the LaTeX labels of each index; if `None`, coordinate LaTeX symbols are used, except if `coordinate_symbols` is set to `False`, in which case integer labels are used
- `coordinate_labels` – (default: `True`) boolean; if `True`, coordinate symbols are used by default (instead of integers)
- `only_nonzero` – (default: `True`) boolean; if `True`, only nonzero connection coefficients are displayed
- `only_nonredundant` – (default: `True`) boolean; if `True`, only nonredundant (w.r.t. the symmetry of the last two indices) connection coefficients are displayed

EXAMPLES:

Christoffel symbols of the flat metric on $\mathbf{R}^3$ with respect to spherical coordinates:

```
sage: M = Manifold(3, 'R3', r'\RR^3', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: X.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2, r^2*sin(th)^2
sage: g.display()  # the standard flat metric expressed in spherical␣
↪coordinates
g = dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: g.christoffel_symbols_display()
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

To list all nonzero Christoffel symbols, including those that can be deduced by symmetry, use `only_nonredundant=False`:

```
sage: g.christoffel_symbols_display(only_nonredundant=False)
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

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```
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
```

Listing all Christoffel symbols (except those that can be deduced by symmetry), including the vanishing one:

```
sage: g.christoffel_symbols_display(only_nonzero=False)
Gam^r_r,r = 0
Gam^r_r,th = 0
Gam^r_r,ph = 0
Gam^r_th,th = -r
Gam^r_th,ph = 0
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,r = 0
Gam^th_r,th = 1/r
Gam^th_r,ph = 0
Gam^th_th,th = 0
Gam^th_th,ph = 0
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,r = 0
Gam^ph_r,th = 0
Gam^ph_r,ph = 1/r
Gam^ph_th,th = 0
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,ph = 0
```

Using integer labels:

```
sage: g.christoffel_symbols_display(coordinate_labels=False)
Gam^1_22 = -r
Gam^1_33 = -r*sin(th)^2
Gam^2_12 = 1/r
Gam^2_33 = -cos(th)*sin(th)
Gam^3_13 = 1/r
Gam^3_23 = cos(th)/sin(th)
```

connection(*name=None*, *latex_name=None*, *init_coef=True*)

Return the unique torsion-free affine connection compatible with `self`.

This is the so-called Levi-Civita connection.

INPUT:

- `name` – (default: `None`) name given to the Levi-Civita connection; if `None`, it is formed from the metric name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Levi-Civita connection; if `None`, it is set to `name`, or if the latter is None as well, it formed from the symbol ∇ and the metric symbol
- `init_coef` – (default: `True`) determines whether the connection coefficients are initialized, as Christoffel symbols in the top charts of the domain of `self` (i.e. disregarding the subcharts)

OUTPUT:

- the Levi-Civita connection, as an instance of `LeviCivitaConnection`

EXAMPLES:

Levi-Civita connection associated with the Euclidean metric on $\mathbf{R}^3$:

```
sage: M = Manifold(3, 'R^3', start_index=1)
sage: # Let us use spherical coordinates on R^3:
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):\phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2 , (r*sin(th))^2  # the Euclidean metric
sage: g.connection()
Levi-Civita connection nabla_g associated with the Riemannian
 metric g on the Open subset U of the 3-dimensional differentiable
 manifold R^3
sage: g.connection().display()  # Nonzero connection coefficients
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
```

Test of compatibility with the metric:

```
sage: Dg = g.connection()(g) ; Dg
Tensor field nabla_g(g) of type (0,3) on the Open subset U of the
 3-dimensional differentiable manifold R^3
sage: Dg == 0
True
sage: Dig = g.connection()(g.inverse()) ; Dig
Tensor field nabla_g(inv_g) of type (2,1) on the Open subset U of
 the 3-dimensional differentiable manifold R^3
sage: Dig == 0
True
```

cotton(*name=None*, *latex_name=None*)

Return the Cotton conformal tensor associated with the metric. The tensor has type (0,3) and is defined in terms of the Schouten tensor S (see `schouten()`):

$$C_{ijk} = (n-2)\left(\nabla_k S_{ij} - \nabla_j S_{ik}\right)$$

INPUT:

- `name` – (default: `None`) name given to the Cotton conformal tensor; if `None`, it is set to "Cot(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Cotton conformal tensor; if `None`, it is set to "\mathrm{Cot}(g)", where "g" is the metric's name

OUTPUT:

- the Cotton conformal tensor Cot, as an instance of `TensorField`

EXAMPLES:

Checking that the Cotton tensor identically vanishes on a conformally flat 3-dimensional manifold, for instance the hyperbolic space H^3:

```
sage: M = Manifold(3, 'H^3', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: X.<rh,th,ph> = U.chart(r'rh:(0,+oo):\rho th:(0,pi):\theta  ph:(0,
↪2*pi):\phi')
sage: g = U.metric('g')
sage: b = var('b')
sage: g[1,1], g[2,2], g[3,3] = b^2, (b*sinh(rh))^2, (b*sinh(rh)*sin(th))^2
sage: g.display()  # standard metric on H^3:
g = b^2 drh*drh + b^2*sinh(rh)^2 dth*dth
 + b^2*sin(th)^2*sinh(rh)^2 dph*dph
sage: Cot = g.cotton() ; Cot # long time
Tensor field Cot(g) of type (0,3) on the Open subset U of the
 3-dimensional differentiable manifold H^3
sage: Cot == 0 # long time
True
```

cotton_york (*name=None*, *latex_name=None*)

Return the Cotton-York conformal tensor associated with the metric. The tensor has type (0,2) and is only defined for manifolds of dimension 3. It is defined in terms of the Cotton tensor C (see `cotton()`) or the Schouten tensor S (see `schouten()`):

$$CY_{ij} = \frac{1}{2}\epsilon^{kl}{}_{i}C_{jlk} = \epsilon^{kl}{}_{i}\nabla_k S_{lj}$$

INPUT:

- `name` – (default: `None`) name given to the Cotton-York tensor; if `None`, it is set to "CY(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Cotton-York tensor; if `None`, it is set to "\mathrm{CY}(g)", where "g" is the metric's name

OUTPUT:

- the Cotton-York conformal tensor CY, as an instance of `TensorField`

EXAMPLES:

Compute the determinant of the Cotton-York tensor for the Heisenberg group with the left invariant metric:

```
sage: M = Manifold(3, 'Nil', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.riemannian_metric('g')
sage: g[1,1], g[2,2], g[2,3], g[3,3] = 1, 1+x^2, -x, 1
sage: g.display()
g = dx*dx + (x^2 + 1) dy*dy - x dy*dz - x dz*dy + dz*dz
sage: CY = g.cotton_york() ; CY # long time
Tensor field CY(g) of type (0,2) on the 3-dimensional
 differentiable manifold Nil
sage: CY.display()  # long time
CY(g) = 1/2 dx*dx + (-x^2 + 1/2) dy*dy + x dy*dz + x dz*dy - dz*dz
sage: det(CY[:]) # long time
-1/4
```

det (*frame=None*)

Determinant of the metric components in the specified frame.

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the components g_{ij} of the metric are defined; if `None`, the default frame of the metric's domain is used. If a chart is provided instead of a frame, the associated coordinate frame is used

OUTPUT:

- the determinant $\det(g_{ij})$, as an instance of `DiffScalarField`

EXAMPLES:

Metric determinant on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: g[:]
[ x + 1     x*y]
[   x*y -y + 1]
sage: s = g.determinant()  # determinant in M's default frame
sage: s.expr()
-x^2*y^2 - (x + 1)*y + x + 1
```

A shortcut is `det()`:

```
sage: g.det() == g.determinant()
True
```

The notation `det(g)` can be used:

```
sage: det(g) == g.determinant()
True
```

Determinant in a frame different from the default's one:

```
sage: Y.<u,v> = M.chart()
sage: ch_X_Y = X.transition_map(Y, [x+y, x-y])
sage: ch_X_Y.inverse()
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: g.comp(Y.frame())[:, Y]
[ 1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2                            1/4*u]
[                           1/4*u -1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2]
sage: g.determinant(Y.frame()).expr()
-1/4*x^2*y^2 - 1/4*(x + 1)*y + 1/4*x + 1/4
sage: g.determinant(Y.frame()).expr(Y)
-1/64*u^4 - 1/64*v^4 + 1/32*(u^2 + 2)*v^2 - 1/16*u^2 + 1/4*v + 1/4
```

A chart can be passed instead of a frame:

```
sage: g.determinant(X) is g.determinant(X.frame())
True
sage: g.determinant(Y) is g.determinant(Y.frame())
True
```

The metric determinant depends on the frame:

```
sage: g.determinant(X.frame()) == g.determinant(Y.frame())
False
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g = M.metric('g')
```

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```
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: s = g.determinant()  # determinant in M's default frame
sage: s.expr()
-x**2*y**2 + x - y*(x + 1) + 1
```

determinant (*frame=None*)

Determinant of the metric components in the specified frame.

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the components g_{ij} of the metric are defined; if `None`, the default frame of the metric's domain is used. If a chart is provided instead of a frame, the associated coordinate frame is used

OUTPUT:

- the determinant $\det(g_{ij})$, as an instance of `DiffScalarField`

EXAMPLES:

Metric determinant on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: g[:]
[ x + 1    x*y]
[   x*y -y + 1]
sage: s = g.determinant()  # determinant in M's default frame
sage: s.expr()
-x^2*y^2 - (x + 1)*y + x + 1
```

A shortcut is `det()`:

```
sage: g.det() == g.determinant()
True
```

The notation `det(g)` can be used:

```
sage: det(g) == g.determinant()
True
```

Determinant in a frame different from the default's one:

```
sage: Y.<u,v> = M.chart()
sage: ch_X_Y = X.transition_map(Y, [x+y, x-y])
sage: ch_X_Y.inverse()
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: g.comp(Y.frame())[:, Y]
[ 1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2                            1/4*u]
[                           1/4*u -1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2]
sage: g.determinant(Y.frame()).expr()
-1/4*x^2*y^2 - 1/4*(x + 1)*y + 1/4*x + 1/4
sage: g.determinant(Y.frame()).expr(Y)
-1/64*u^4 - 1/64*v^4 + 1/32*(u^2 + 2)*v^2 - 1/16*u^2 + 1/4*v + 1/4
```

A chart can be passed instead of a frame:

```
sage: g.determinant(X) is g.determinant(X.frame())
True
sage: g.determinant(Y) is g.determinant(Y.frame())
True
```

The metric determinant depends on the frame:

```
sage: g.determinant(X.frame()) == g.determinant(Y.frame())
False
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: s = g.determinant()  # determinant in M's default frame
sage: s.expr()
-x**2*y**2 + x - y*(x + 1) + 1
```

hodge_star(*pform*)

Compute the Hodge dual of a differential form with respect to the metric.

If the differential form is a p-form A, its *Hodge dual* with respect to the metric g is the $(n-p)$-form $*A$ defined by

$$*A_{i_1\ldots i_{n-p}} = \frac{1}{p!} A_{k_1\ldots k_p} \epsilon^{k_1\ldots k_p}{}_{i_1\ldots i_{n-p}}$$

where n is the manifold's dimension, ϵ is the volume n-form associated with g (see `volume_form()`) and the indices $k_1,\ldots,k_p$ are raised with g.

INPUT:

- `pform`: a p-form A; must be an instance of `DiffScalarField` for $p = 0$ and of `DiffForm` or `DiffFormParal` for $p \geq 1$.

OUTPUT:

- the $(n-p)$-form $*A$

EXAMPLES:

Hodge dual of a 1-form in the Euclidean space R^3:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, 1, 1
sage: var('Ax Ay Az')
(Ax, Ay, Az)
sage: a = M.one_form(Ax, Ay, Az, name='A')
sage: sa = g.hodge_star(a) ; sa
2-form *A on the 3-dimensional differentiable manifold M
sage: sa.display()
*A = Az dx/\dy - Ay dx/\dz + Ax dy/\dz
sage: ssa = g.hodge_star(sa) ; ssa
1-form **A on the 3-dimensional differentiable manifold M
sage: ssa.display()
**A = Ax dx + Ay dy + Az dz
sage: ssa == a  # must hold for a Riemannian metric in dimension 3
True
```

Hodge dual of a 0-form (scalar field) in R^3:

```
sage: f = M.scalar_field(function('F')(x,y,z), name='f')
sage: sf = g.hodge_star(f) ; sf
3-form *f on the 3-dimensional differentiable manifold M
sage: sf.display()
*f = F(x, y, z) dx/\dy/\dz
sage: ssf = g.hodge_star(sf) ; ssf
Scalar field **f on the 3-dimensional differentiable manifold M
sage: ssf.display()
**f: M --> R
   (x, y, z) |--> F(x, y, z)
sage: ssf == f # must hold for a Riemannian metric
True
```

Hodge dual of a 0-form in Minkowski spacetime:

```
sage: M = Manifold(4, 'M')
sage: X.<t,x,y,z> = M.chart()
sage: g = M.lorentzian_metric('g')
sage: g[0,0], g[1,1], g[2,2], g[3,3] = -1, 1, 1, 1
sage: g.display()  # Minkowski metric
g = -dt*dt + dx*dx + dy*dy + dz*dz
sage: var('f0')
f0
sage: f = M.scalar_field(f0, name='f')
sage: sf = g.hodge_star(f) ; sf
4-form *f on the 4-dimensional differentiable manifold M
sage: sf.display()
*f = f0 dt/\dx/\dy/\dz
sage: ssf = g.hodge_star(sf) ; ssf
Scalar field **f on the 4-dimensional differentiable manifold M
sage: ssf.display()
**f: M --> R
   (t, x, y, z) |--> -f0
sage: ssf == -f  # must hold for a Lorentzian metric
True
```

Hodge dual of a 1-form in Minkowski spacetime:

```
sage: var('At Ax Ay Az')
(At, Ax, Ay, Az)
sage: a = M.one_form(At, Ax, Ay, Az, name='A')
sage: a.display()
A = At dt + Ax dx + Ay dy + Az dz
sage: sa = g.hodge_star(a) ; sa
3-form *A on the 4-dimensional differentiable manifold M
sage: sa.display()
*A = -Az dt/\dx/\dy + Ay dt/\dx/\dz - Ax dt/\dy/\dz - At dx/\dy/\dz
sage: ssa = g.hodge_star(sa) ; ssa
1-form **A on the 4-dimensional differentiable manifold M
sage: ssa.display()
**A = At dt + Ax dx + Ay dy + Az dz
sage: ssa == a  # must hold for a Lorentzian metric in dimension 4
True
```

Hodge dual of a 2-form in Minkowski spacetime:

```
sage: F = M.diff_form(2, name='F')
sage: var('Ex Ey Ez Bx By Bz')
(Ex, Ey, Ez, Bx, By, Bz)
sage: F[0,1], F[0,2], F[0,3] = -Ex, -Ey, -Ez
sage: F[1,2], F[1,3], F[2,3] = Bz, -By, Bx
sage: F[:]
[  0 -Ex -Ey -Ez]
[ Ex   0  Bz -By]
[ Ey -Bz   0  Bx]
[ Ez  By -Bx   0]
sage: sF = g.hodge_star(F) ; sF
2-form *F on the 4-dimensional differentiable manifold M
sage: sF[:]
[  0  Bx  By  Bz]
[-Bx   0  Ez -Ey]
[-By -Ez   0  Ex]
[-Bz  Ey -Ex   0]
sage: ssF = g.hodge_star(sF) ; ssF
2-form **F on the 4-dimensional differentiable manifold M
sage: ssF[:]
[  0  Ex  Ey  Ez]
[-Ex   0 -Bz  By]
[-Ey  Bz   0 -Bx]
[-Ez -By  Bx   0]
sage: ssF.display()
**F = Ex dt/\dx + Ey dt/\dy + Ez dt/\dz - Bz dx/\dy + By dx/\dz
 - Bx dy/\dz
sage: F.display()
F = -Ex dt/\dx - Ey dt/\dy - Ez dt/\dz + Bz dx/\dy - By dx/\dz
 + Bx dy/\dz
sage: ssF == -F  # must hold for a Lorentzian metric in dimension 4
True
```

Test of the standard identity

$$*(A \wedge B) = \epsilon(A^\sharp, B^\sharp, ., .)$$

where A and B are any 1-forms and $A^\sharp$ and $B^\sharp$ the vectors associated to them by the metric g (index raising):

```
sage: var('Bt Bx By Bz')
(Bt, Bx, By, Bz)
sage: b = M.one_form(Bt, Bx, By, Bz, name='B')
sage: b.display()
B = Bt dt + Bx dx + By dy + Bz dz
sage: epsilon = g.volume_form()
sage: g.hodge_star(a.wedge(b)) == epsilon.contract(0,a.up(g)).contract(0,b.
↪up(g))
True
```

inverse(*expansion_symbol=None*, *order=1*)

Return the inverse metric.

INPUT:

- `expansion_symbol` – (default: `None`) symbolic variable; if specified, the inverse will be expanded in power series with respect to this variable (around its zero value)
- `order` – integer (default: 1); the order of the expansion if `expansion_symbol` is not `None`;

the *order* is defined as the degree of the polynomial representing the truncated power series in `expansion_symbol`; currently only first order inverse is supported

If `expansion_symbol` is set, then the zeroth order metric must be invertible. Moreover, subsequent calls to this method will return a cached value, even when called with the default value (to enable computation of derived quantities). To reset, use `_del_derived()`.

OUTPUT:

- instance of *TensorField* with `tensor_type` = (2,0) representing the inverse metric

EXAMPLES:

Inverse of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)  # S^2 is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart() # stereographic coord.
sage: xy_to_uv = c_xy.transition_map(c_uv, (x/(x^2+y^2), y/(x^2+y^2)),
....:                 intersection_name='W', restrictions1= x^2+y^2!=0,
....:                 restrictions2= u^2+v^2!=0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)  # the complement of the two poles
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: g = M.metric('g')
sage: g[eU,1,1], g[eU,2,2] = 4/(1+x^2+y^2)^2, 4/(1+x^2+y^2)^2
sage: g.add_comp_by_continuation(eV, W, c_uv)
sage: ginv = g.inverse(); ginv
Tensor field inv_g of type (2,0) on the 2-dimensional differentiable manifold␣
↪S^2
sage: ginv.display(eU)
inv_g = (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) d/dx*d/dx
 + (1/4*x^4 + 1/4*y^4 + 1/2*(x^2 + 1)*y^2 + 1/2*x^2 + 1/4) d/dy*d/dy
sage: ginv.display(eV)
inv_g = (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) d/du*d/du
 + (1/4*u^4 + 1/4*v^4 + 1/2*(u^2 + 1)*v^2 + 1/2*u^2 + 1/4) d/dv*d/dv
```

Let us check that `ginv` is indeed the inverse of `g`:

```
sage: s = g.contract(ginv); s  # contraction of last index of g with first␣
↪index of ginv
Tensor field of type (1,1) on the 2-dimensional differentiable manifold S^2
sage: s == M.tangent_identity_field()
True
```

restrict (*subdomain*, *dest_map=None*)

Return the restriction of the metric to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset U of the metric's domain (must be an instance of *DifferentiableManifold*)
- `dest_map` – (default: `None`) destination map $\Phi : U \to V$, where V is a subdomain of `self._codomain` (type: *DiffMap*) If None, the restriction of `self._vmodule._dest_map` to U is used.

OUTPUT:

- instance of `PseudoRiemannianMetric` representing the restriction.

EXAMPLES:

```
sage: M = Manifold(5, 'M')
sage: g = M.metric('g', signature=3)
sage: U = M.open_subset('U')
sage: g.restrict(U)
Lorentzian metric g on the Open subset U of the
 5-dimensional differentiable manifold M
sage: g.restrict(U).signature()
3
```

See the top documentation of `PseudoRiemannianMetric` for more examples.

ricci (*name=None*, *latex_name=None*)

Return the Ricci tensor associated with the metric.

This method is actually a shortcut for `self.connection().ricci()`

The Ricci tensor is the tensor field Ric of type (0,2) defined from the Riemann curvature tensor R by

$$Ric(u, v) = R(e^i, u, e_i, v)$$

for any vector fields u and v, (e_i) being any vector frame and (e^i) the dual coframe.

INPUT:

- `name` – (default: `None`) name given to the Ricci tensor; if none, it is set to "Ric(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Ricci tensor; if none, it is set to "\mathrm{Ric}(g)", where "g" is the metric's name

OUTPUT:

- the Ricci tensor Ric, as an instance of `TensorField` of tensor type (0,2) and symmetric

EXAMPLES:

Ricci tensor of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: a = var('a') # the sphere radius
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a^2, a^2*sin(th)^2
sage: g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth*dth + a^2*sin(th)^2 dph*dph
sage: g.ricci()
Field of symmetric bilinear forms Ric(g) on the Open subset U of
 the 2-dimensional differentiable manifold S^2
sage: g.ricci()[:]
[        1         0]
[        0 sin(th)^2]
sage: g.ricci() == a^(-2) * g
True
```

ricci_scalar (*name=None*, *latex_name=None*)

Return the Ricci scalar associated with the metric.

The Ricci scalar is the scalar field r defined from the Ricci tensor Ric and the metric tensor g by

$$r = g^{ij} Ric_{ij}$$

INPUT:

- name – (default: None) name given to the Ricci scalar; if none, it is set to "r(g)", where "g" is the metric's name
- latex_name – (default: None) LaTeX symbol to denote the Ricci scalar; if none, it is set to "\mathrm{r}(g)", where "g" is the metric's name

OUTPUT:

- the Ricci scalar r, as an instance of DiffScalarField

EXAMPLES:

Ricci scalar of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: a = var('a') # the sphere radius
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a^2, a^2*sin(th)^2
sage: g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth*dth + a^2*sin(th)^2 dph*dph
sage: g.ricci_scalar()
Scalar field r(g) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: g.ricci_scalar().display() # The Ricci scalar is constant:
r(g): U --> R
   (th, ph) |--> 2/a^2
```

riemann(*name=None*, *latex_name=None*)

Return the Riemann curvature tensor associated with the metric.

This method is actually a shortcut for `self.connection().riemann()`

The Riemann curvature tensor is the tensor field R of type (1,3) defined by

$$R(\omega, u, v, w) = \left\langle \omega, \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \right\rangle$$

for any 1-form ω and any vector fields u, v and w.

INPUT:

- name – (default: None) name given to the Riemann tensor; if none, it is set to "Riem(g)", where "g" is the metric's name
- latex_name – (default: None) LaTeX symbol to denote the Riemann tensor; if none, it is set to "\mathrm{Riem}(g)", where "g" is the metric's name

OUTPUT:

- the Riemann curvature tensor R, as an instance of TensorField

EXAMPLES:

Riemann tensor of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: a = var('a') # the sphere radius
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a^2, a^2*sin(th)^2
sage: g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth*dth + a^2*sin(th)^2 dph*dph
sage: g.riemann()
Tensor field Riem(g) of type (1,3) on the Open subset U of the
 2-dimensional differentiable manifold S^2
sage: g.riemann()[:]
[[[[0, 0], [0, 0]], [[0, sin(th)^2], [-sin(th)^2, 0]]],
 [[[0, -1], [1, 0]], [[0, 0], [0, 0]]]]
```

In dimension 2, the Riemann tensor can be expressed entirely in terms of the Ricci scalar r:

$$R^i_{\ jlk} = \frac{r}{2}\left(\delta^i_{\ k} g_{jl} - \delta^i_{\ l} g_{jk}\right)$$

This formula can be checked here, with the r.h.s. rewritten as $-r g_{j[k} \delta^i_{\ l]}$:

```
sage: g.riemann() == \
....:  -g.ricci_scalar()*(g*U.tangent_identity_field()).antisymmetrize(2,3)
True
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a**2, a**2*sin(th)**2
sage: g.riemann()[:]
[[[[0, 0], [0, 0]],
  [[0, sin(2*th)/(2*tan(th)) - cos(2*th)],
   [-sin(2*th)/(2*tan(th)) + cos(2*th), 0]]],
 [[[0, -1], [1, 0]], [[0, 0], [0, 0]]]]
```

schouten(*name=None*, *latex_name=None*)

Return the Schouten tensor associated with the metric.

The Schouten tensor is the tensor field Sc of type (0,2) defined from the Ricci curvature tensor Ric (see *ricci()*) and the scalar curvature r (see *ricci_scalar()*) and the metric g by

$$Sc(u, v) = \frac{1}{n-2}\left(Ric(u, v) + \frac{r}{2(n-1)} g(u, v)\right)$$

for any vector fields u and v.

INPUT:

- `name` – (default: `None`) name given to the Schouten tensor; if none, it is set to "Schouten(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Schouten tensor; if none, it is set to "\mathrm{Schouten}(g)", where "g" is the metric's name

OUTPUT:

- the Schouten tensor Sc, as an instance of *TensorField* of tensor type (0,2) and symmetric

EXAMPLES:

Schouten tensor of the left invariant metric of Heisenberg's Nil group:

```
sage: M = Manifold(3, 'Nil', start_index=1)
sage: X.<x,y,z> = M.chart()
sage: g = M.riemannian_metric('g')
sage: g[1,1], g[2,2], g[2,3], g[3,3] = 1, 1+x^2, -x, 1
sage: g.display()
g = dx*dx + (x^2 + 1) dy*dy - x dy*dz - x dz*dy + dz*dz
sage: g.schouten()
Field of symmetric bilinear forms Schouten(g) on the 3-dimensional
 differentiable manifold Nil
sage: g.schouten().display()
Schouten(g) = -3/8 dx*dx + (5/8*x^2 - 3/8) dy*dy - 5/8*x dy*dz
 - 5/8*x dz*dy + 5/8 dz*dz
```

set (*symbiform*)

Defines the metric from a field of symmetric bilinear forms

INPUT:

- `symbiform` – instance of *TensorField* representing a field of symmetric bilinear forms

EXAMPLES:

Metric defined from a field of symmetric bilinear forms on a non-parallelizable 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: U = M.open_subset('U') ; V = M.open_subset('V')
sage: M.declare_union(U,V)   # M is the union of U and V
sage: c_xy.<x,y> = U.chart() ; c_uv.<u,v> = V.chart()
sage: xy_to_uv = c_xy.transition_map(c_uv, (x+y, x-y), intersection_name='W',
....:                               restrictions1= x>0, restrictions2= u+v>0)
sage: uv_to_xy = xy_to_uv.inverse()
sage: W = U.intersection(V)
sage: eU = c_xy.frame() ; eV = c_uv.frame()
sage: h = M.sym_bilin_form_field(name='h')
sage: h[eU,0,0], h[eU,0,1], h[eU,1,1] = 1+x, x*y, 1-y
sage: h.add_comp_by_continuation(eV, W, c_uv)
sage: h.display(eU)
h = (x + 1) dx*dx + x*y dx*dy + x*y dy*dx + (-y + 1) dy*dy
sage: h.display(eV)
h = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du*du + 1/4*u du*dv
 + 1/4*u dv*du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv*dv
sage: g = M.metric('g')
sage: g.set(h)
sage: g.display(eU)
g = (x + 1) dx*dx + x*y dx*dy + x*y dy*dx + (-y + 1) dy*dy
sage: g.display(eV)
g = (1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2) du*du + 1/4*u du*dv
 + 1/4*u dv*du + (-1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2) dv*dv
```

signature ()

Signature of the metric.

OUTPUT:

- signature S of the metric, defined as the integer $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components

EXAMPLES:

Signatures on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M')
sage: g = M.metric('g') # if not specified, the signature is Riemannian
sage: g.signature()
2
sage: h = M.metric('h', signature=0)
sage: h.signature()
0
```

sqrt_abs_det (*frame=None*)

Square root of the absolute value of the determinant of the metric components in the specified frame.

INPUT:

- `frame` – (default: `None`) vector frame with respect to which the components g_{ij} of `self` are defined; if `None`, the domain's default frame is used. If a chart is provided, the associated coordinate frame is used

OUTPUT:

- $\sqrt{|\det(g_{ij})|}$, as an instance of *DiffScalarField*

EXAMPLES:

Standard metric in the Euclidean space $\mathbf{R}^3$ with spherical coordinates:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):\phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2, (r*sin(th))^2
sage: g.display()
g = dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: g.sqrt_abs_det().expr()
r^2*sin(th)
```

Metric determinant on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: g[:]
[ x + 1    x*y]
[   x*y -y + 1]
sage: s = g.sqrt_abs_det() ; s
Scalar field on the 2-dimensional differentiable manifold M
sage: s.expr()
sqrt(-x^2*y^2 - (x + 1)*y + x + 1)
```

Determinant in a frame different from the default's one:

```
sage: Y.<u,v> = M.chart()
sage: ch_X_Y = X.transition_map(Y, [x+y, x-y])
sage: ch_X_Y.inverse()
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
```

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```
sage: g[Y.frame(),:,Y]
[ 1/8*u^2 - 1/8*v^2 + 1/4*v + 1/2                             1/4*u]
[                           1/4*u -1/8*u^2 + 1/8*v^2 + 1/4*v + 1/2]
sage: g.sqrt_abs_det(Y.frame()).expr()
1/2*sqrt(-x^2*y^2 - (x + 1)*y + x + 1)
sage: g.sqrt_abs_det(Y.frame()).expr(Y)
1/8*sqrt(-u^4 - v^4 + 2*(u^2 + 2)*v^2 - 4*u^2 + 16*v + 16)
```

A chart can be passed instead of a frame:

```
sage: g.sqrt_abs_det(Y) is g.sqrt_abs_det(Y.frame())
True
```

The metric determinant depends on the frame:

```
sage: g.sqrt_abs_det(X.frame()) == g.sqrt_abs_det(Y.frame())
False
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g = M.metric('g')
sage: g[1,1], g[1, 2], g[2, 2] = 1+x, x*y , 1-y
sage: g.sqrt_abs_det().expr()
sqrt(-x**2*y**2 - x*y + x - y + 1)
sage: g.sqrt_abs_det(Y.frame()).expr()
sqrt(-x**2*y**2 - x*y + x - y + 1)/2
sage: g.sqrt_abs_det(Y.frame()).expr(Y)
sqrt(-u**4 + 2*u**2*v**2 - 4*u**2 - v**4 + 4*v**2 + 16*v + 16)/8
```

volume_form(*contra=0*)

Volume form (Levi-Civita tensor) ϵ associated with the metric.

This assumes that the manifold is orientable.

The volume form ϵ is a n-form (n being the manifold's dimension) such that for any vector basis (e_i) that is orthonormal with respect to the metric,

$$\epsilon(e_1, \ldots, e_n) = \pm 1$$

There are only two such n-forms, which are opposite of each other. The volume form ϵ is selected such that the domain's default frame is right-handed with respect to it.

INPUT:

- `contra` – (default: 0) number of contravariant indices of the returned tensor

OUTPUT:

- if `contra = 0` (default value): the volume n-form ϵ, as an instance of `DiffForm`
- if `contra = k`, with $1 \leq k \leq n$, the tensor field of type (k,n-k) formed from ϵ by raising the first k indices with the metric (see method `up()`); the output is then an instance of `TensorField`, with the appropriate antisymmetries, or of the subclass `MultivectorField` if $k = n$

EXAMPLES:

Volume form on $\mathbf{R}^3$ with spherical coordinates:

```
sage: M = Manifold(3, 'M', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: c_spher.<r,th,ph> = U.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):\phi')
sage: g = U.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2, (r*sin(th))^2
sage: g.display()
g = dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: eps = g.volume_form() ; eps
3-form eps_g on the Open subset U of the 3-dimensional
 differentiable manifold M
sage: eps.display()
eps_g = r^2*sin(th) dr/\dth/\dph
sage: eps[[1,2,3]] == g.sqrt_abs_det()
True
sage: latex(eps)
\epsilon_{g}
```

The tensor field of components $\epsilon^i{}_{jk}$ (`contra=1`):

```
sage: eps1 = g.volume_form(1) ; eps1
Tensor field of type (1,2) on the Open subset U of the
 3-dimensional differentiable manifold M
sage: eps1.symmetries()
no symmetry;  antisymmetry: (1, 2)
sage: eps1[:]
[[[0, 0, 0], [0, 0, r^2*sin(th)], [0, -r^2*sin(th), 0]],
 [[0, 0, -sin(th)], [0, 0, 0], [sin(th), 0, 0]],
 [[0, 1/sin(th), 0], [-1/sin(th), 0, 0], [0, 0, 0]]]
```

The tensor field of components $\epsilon^{ij}{}_k$ (`contra=2`):

```
sage: eps2 = g.volume_form(2) ; eps2
Tensor field of type (2,1) on the Open subset U of the
 3-dimensional differentiable manifold M
sage: eps2.symmetries()
no symmetry;  antisymmetry: (0, 1)
sage: eps2[:]
[[[0, 0, 0], [0, 0, sin(th)], [0, -1/sin(th), 0]],
 [[0, 0, -sin(th)], [0, 0, 0], [1/(r^2*sin(th)), 0, 0]],
 [[0, 1/sin(th), 0], [-1/(r^2*sin(th)), 0, 0], [0, 0, 0]]]
```

The tensor field of components ϵ^{ijk} (`contra=3`):

```
sage: eps3 = g.volume_form(3) ; eps3
3-vector field on the Open subset U of the 3-dimensional
 differentiable manifold M
sage: eps3.tensor_type()
(3, 0)
sage: eps3.symmetries()
no symmetry;  antisymmetry: (0, 1, 2)
sage: eps3[:]
[[[0, 0, 0], [0, 0, 1/(r^2*sin(th))], [0, -1/(r^2*sin(th)), 0]],
 [[0, 0, -1/(r^2*sin(th))], [0, 0, 0], [1/(r^2*sin(th)), 0, 0]],
 [[0, 1/(r^2*sin(th)), 0], [-1/(r^2*sin(th)), 0, 0], [0, 0, 0]]]
sage: eps3[1,2,3]
1/(r^2*sin(th))
```

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```
sage: eps3[[1,2,3]] * g.sqrt_abs_det() == 1
True
```

weyl(*name=None*, *latex_name=None*)

Return the Weyl conformal tensor associated with the metric.

The Weyl conformal tensor is the tensor field C of type (1,3) defined as the trace-free part of the Riemann curvature tensor R

INPUT:

- `name` – (default: `None`) name given to the Weyl conformal tensor; if `None`, it is set to "C(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Weyl conformal tensor; if `None`, it is set to "\mathrm{C}(g)", where "g" is the metric's name

OUTPUT:

- the Weyl conformal tensor C, as an instance of `TensorField`

EXAMPLES:

Checking that the Weyl tensor identically vanishes on a 3-dimensional manifold, for instance the hyperbolic space H^3:

```
sage: M = Manifold(3, 'H^3', start_index=1)
sage: U = M.open_subset('U') # the complement of the half-plane (y=0, x>=0)
sage: X.<rh,th,ph> = U.chart(r'rh:(0,+oo):\rho th:(0,pi):\theta  ph:(0,
↪2*pi):\phi')
sage: g = U.metric('g')
sage: b = var('b')
sage: g[1,1], g[2,2], g[3,3] = b^2, (b*sinh(rh))^2, (b*sinh(rh)*sin(th))^2
sage: g.display()  # standard metric on H^3:
g = b^2 drh*drh + b^2*sinh(rh)^2 dth*dth
 + b^2*sin(th)^2*sinh(rh)^2 dph*dph
sage: C = g.weyl() ; C
Tensor field C(g) of type (1,3) on the Open subset U of the
 3-dimensional differentiable manifold H^3
sage: C == 0
True
```

class `sage.manifolds.differentiable.metric.`**`PseudoRiemannianMetricParal`**(*vector_field_module*, *name*, *signature=None*, *latex_name=None*)

Bases: `sage.manifolds.differentiable.metric.PseudoRiemannianMetric`, `sage.manifolds.differentiable.tensorfield_paral.TensorFieldParal`

Pseudo-Riemannian metric with values on a parallelizable manifold.

An instance of this class is a field of nondegenerate symmetric bilinear forms (metric field) along a differentiable manifold U with values in a parallelizable manifold M over $\mathbf{R}$, via a differentiable mapping $\Phi : U \to M$. The standard case of a metric field *on* a manifold corresponds to $U = M$ and $\Phi = \mathrm{Id}_M$. Other common cases are Φ being an immersion and Φ being a curve in M (U is then an open interval of $\mathbf{R}$).

A *metric* g is a field on U, such that at each point $p \in U$, $g(p)$ is a bilinear map of the type:

$$g(p): T_q M \times T_q M \longrightarrow \mathbf{R}$$

where $T_q M$ stands for the tangent space to manifold M at the point $q = \Phi(p)$, such that $g(p)$ is symmetric: $\forall (u, v) \in T_q M \times T_q M, \ g(p)(v, u) = g(p)(u, v)$ and nondegenerate: $(\forall v \in T_q M, \ \ g(p)(u, v) = 0) \Longrightarrow u = 0$.

Note: If M is not parallelizable, the class *PseudoRiemannianMetric* should be used instead.

INPUT:

- `vector_field_module` – free module $\mathfrak{X}(U, \Phi)$ of vector fields along U with values on $\Phi(U) \subset M$
- `name` – name given to the metric
- `signature` – (default: `None`) signature S of the metric as a single integer: $S = n_+ - n_-$, where n_+ (resp. n_-) is the number of positive terms (resp. number of negative terms) in any diagonal writing of the metric components; if `signature` is `None`, S is set to the dimension of manifold M (Riemannian signature)
- `latex_name` – (default: `None`) LaTeX symbol to denote the metric; if `None`, it is formed from `name`

EXAMPLES:

Metric on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: g = M.metric('g') ; g
Riemannian metric g on the 2-dimensional differentiable manifold M
sage: latex(g)
g
```

A metric is a special kind of tensor field and therefore inheritates all the properties from class *TensorField*:

```
sage: g.parent()
Free module T^(0,2)(M) of type-(0,2) tensors fields on the
 2-dimensional differentiable manifold M
sage: g.tensor_type()
(0, 2)
sage: g.symmetries()  # g is symmetric:
symmetry: (0, 1);  no antisymmetry
```

Setting the metric components in the manifold's default frame:

```
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-x
sage: g[:]
[ x + 1    x*y]
[   x*y -x + 1]
sage: g.display()
g = (x + 1) dx*dx + x*y dx*dy + x*y dy*dx + (-x + 1) dy*dy
```

Metric components in a frame different from the manifold's default one:

```
sage: c_uv.<u,v> = M.chart()  # new chart on M
sage: xy_to_uv = c_xy.transition_map(c_uv, [x+y, x-y]) ; xy_to_uv
Change of coordinates from Chart (M, (x, y)) to Chart (M, (u, v))
sage: uv_to_xy = xy_to_uv.inverse() ; uv_to_xy
```

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```
Change of coordinates from Chart (M, (u, v)) to Chart (M, (x, y))
sage: M.atlas()
[Chart (M, (x, y)), Chart (M, (u, v))]
sage: M.frames()
[Coordinate frame (M, (d/dx,d/dy)), Coordinate frame (M, (d/du,d/dv))]
sage: g[c_uv.frame(),:]  # metric components in frame c_uv.frame() expressed in M
↪'s default chart (x,y)
[ 1/2*x*y + 1/2         1/2*x]
[        1/2*x -1/2*x*y + 1/2]
sage: g.display(c_uv.frame())
g = (1/2*x*y + 1/2) du*du + 1/2*x du*dv + 1/2*x dv*du
 + (-1/2*x*y + 1/2) dv*dv
sage: g[c_uv.frame(),:,c_uv]   # metric components in frame c_uv.frame()␣
↪expressed in chart (u,v)
[ 1/8*u^2 - 1/8*v^2 + 1/2            1/4*u + 1/4*v]
[           1/4*u + 1/4*v -1/8*u^2 + 1/8*v^2 + 1/2]
sage: g.display(c_uv.frame(), c_uv)
g = (1/8*u^2 - 1/8*v^2 + 1/2) du*du + (1/4*u + 1/4*v) du*dv
 + (1/4*u + 1/4*v) dv*du + (-1/8*u^2 + 1/8*v^2 + 1/2) dv*dv
```

As a shortcut of the above command, on can pass just the chart `c_uv` to `display`, the vector frame being then assumed to be the coordinate frame associated with the chart:

```
sage: g.display(c_uv)
g = (1/8*u^2 - 1/8*v^2 + 1/2) du*du + (1/4*u + 1/4*v) du*dv
 + (1/4*u + 1/4*v) dv*du + (-1/8*u^2 + 1/8*v^2 + 1/2) dv*dv
```

The inverse metric is obtained via *inverse()*:

```
sage: ig = g.inverse() ; ig
Tensor field inv_g of type (2,0) on the 2-dimensional differentiable
 manifold M
sage: ig[:]
[ (x - 1)/(x^2*y^2 + x^2 - 1)      x*y/(x^2*y^2 + x^2 - 1)]
[     x*y/(x^2*y^2 + x^2 - 1) -(x + 1)/(x^2*y^2 + x^2 - 1)]
sage: ig.display()
inv_g = (x - 1)/(x^2*y^2 + x^2 - 1) d/dx*d/dx
 + x*y/(x^2*y^2 + x^2 - 1) d/dx*d/dy + x*y/(x^2*y^2 + x^2 - 1) d/dy*d/dx
 - (x + 1)/(x^2*y^2 + x^2 - 1) d/dy*d/dy
```

inverse(*expansion_symbol=None*, *order=1*)

Return the inverse metric.

INPUT:

- `expansion_symbol` – (default: `None`) symbolic variable; if specified, the inverse will be expanded in power series with respect to this variable (around its zero value)
- `order` – integer (default: 1); the order of the expansion if `expansion_symbol` is not `None`; the *order* is defined as the degree of the polynomial representing the truncated power series in `expansion_symbol`; currently only first order inverse is supported

If `expansion_symbol` is set, then the zeroth order metric must be invertible. Moreover, subsequent calls to this method will return a cached value, even when called with the default value (to enable computation of derived quantities). To reset, use `_del_derived()`.

OUTPUT:

- instance of *TensorFieldParal* with `tensor_type` = (2,0) representing the inverse metric

EXAMPLES:

Inverse metric on a 2-dimensional manifold:

```
sage: M = Manifold(2, 'M', start_index=1)
sage: c_xy.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-x
sage: g[:]  # components in the manifold's default frame
[ x + 1    x*y]
[   x*y -x + 1]
sage: ig = g.inverse() ; ig
Tensor field inv_g of type (2,0) on the 2-dimensional
  differentiable manifold M
sage: ig[:]
[ (x - 1)/(x^2*y^2 + x^2 - 1)      x*y/(x^2*y^2 + x^2 - 1)]
[     x*y/(x^2*y^2 + x^2 - 1) -(x + 1)/(x^2*y^2 + x^2 - 1)]
```

If the metric is modified, the inverse metric is automatically updated:

```
sage: g[1,2] = 0 ; g[:]
[ x + 1      0]
[     0 -x + 1]
sage: g.inverse()[:]
[ 1/(x + 1)          0]
[         0 -1/(x - 1)]
```

Using SymPy as symbolic engine:

```
sage: M.set_calculus_method('sympy')
sage: g[1,1], g[1,2], g[2,2] = 1+x, x*y, 1-x
sage: g[:]  # components in the manifold's default frame
[x + 1   x*y]
[  x*y 1 - x]
sage: g.inverse()[:]
[ (x - 1)/(x**2*y**2 + x**2 - 1)      x*y/(x**2*y**2 + x**2 - 1)]
[     x*y/(x**2*y**2 + x**2 - 1) -(x + 1)/(x**2*y**2 + x**2 - 1)]
```

Demonstration of the series expansion capabilities:

```
sage: M = Manifold(4, 'M', structure='Lorentzian')
sage: C.<t,x,y,z> = M.chart()
sage: e = var('e')
sage: g = M.metric()
sage: h = M.tensor_field(0, 2, sym=(0,1))
sage: g[0, 0], g[1, 1], g[2, 2], g[3, 3] = -1, 1, 1, 1
sage: h[0, 1], h[1, 2], h[2, 3] = 1, 1, 1
sage: g.set(g + e*h)
```

If `e` is a small parameter, `g` is a tridiagonal approximation of the Minkowski metric:

```
sage: g[:]
[-1  e  0  0]
[ e  1  e  0]
[ 0  e  1  e]
[ 0  0  e  1]
```

The inverse, truncated to first order in `e`, is:

```
sage: g.inverse(expansion_symbol=e)[:]
[-1  e  0  0]
[ e  1 -e  0]
[ 0 -e  1 -e]
[ 0  0 -e  1]
```

If inverse() is called subsequently, the result will be the same. This allows for all computations to be made to first order:

```
sage: g.inverse()[:]
[-1  e  0  0]
[ e  1 -e  0]
[ 0 -e  1 -e]
[ 0  0 -e  1]
```

restrict (*subdomain*, *dest_map=None*)

Return the restriction of the metric to some subdomain.

If the restriction has not been defined yet, it is constructed here.

INPUT:

- subdomain – open subset U of self._domain (must be an instance of *DifferentiableManifold*)
- dest_map – (default: None) destination map $\Phi : U \rightarrow V$, where V is a subdomain of self._codomain (type: *DiffMap*) If None, the restriction of self._vmodule._dest_map to U is used.

OUTPUT:

- instance of *PseudoRiemannianMetricParal* representing the restriction.

EXAMPLES:

Restriction of a Lorentzian metric on $\mathbf{R}^2$ to the upper half plane:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: g = M.lorentzian_metric('g')
sage: g[0,0], g[1,1] = -1, 1
sage: U = M.open_subset('U', coord_def={X: y>0})
sage: gU = g.restrict(U); gU
Lorentzian metric g on the Open subset U of the 2-dimensional
 differentiable manifold M
sage: gU.signature()
0
sage: gU.display()
g = -dx*dx + dy*dy
```

ricci_scalar (*name=None*, *latex_name=None*)

Return the metric's Ricci scalar.

The Ricci scalar is the scalar field r defined from the Ricci tensor Ric and the metric tensor g by

$$r = g^{ij} Ric_{ij}$$

INPUT:

- name – (default: None) name given to the Ricci scalar; if none, it is set to "r(g)", where "g" is the metric's name

- `latex_name` – (default: `None`) LaTeX symbol to denote the Ricci scalar; if none, it is set to "\mathrm{r}(g)", where "g" is the metric's name

OUTPUT:

- the Ricci scalar r, as an instance of `DiffScalarField`

EXAMPLES:

Ricci scalar of the standard metric on the 2-sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: U = M.open_subset('U') # the complement of a meridian (domain of␣
↪spherical coordinates)
sage: c_spher.<th,ph> = U.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: a = var('a') # the sphere radius
sage: g = U.metric('g')
sage: g[1,1], g[2,2] = a^2, a^2*sin(th)^2
sage: g.display() # standard metric on the 2-sphere of radius a:
g = a^2 dth*dth + a^2*sin(th)^2 dph*dph
sage: g.ricci_scalar()
Scalar field r(g) on the Open subset U of the 2-dimensional
 differentiable manifold S^2
sage: g.ricci_scalar().display() # The Ricci scalar is constant:
r(g): U --> R
   (th, ph) |--> 2/a^2
```

set (*symbiform*)

Define the metric from a field of symmetric bilinear forms.

INPUT:

- `symbiform` – instance of `TensorFieldParal` representing a field of symmetric bilinear forms

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: s = M.sym_bilin_form_field(name='s')
sage: s[0,0], s[0,1], s[1,1] = 1+x^2, x*y, 1+y^2
sage: g = M.metric('g')
sage: g.set(s)
sage: g.display()
g = (x^2 + 1) dx*dx + x*y dx*dy + x*y dy*dx + (y^2 + 1) dy*dy
```

3.4 Levi-Civita Connections

The class `LeviCivitaConnection` implements the Levi-Civita connection associated with some pseudo-Riemannian metric on a smooth manifold.

AUTHORS:

- Eric Gourgoulhon, Michal Bejger (2013-2015) : initial version
- Marco Mancini (2015) : parallelization of some computations

REFERENCES:

- [?]
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class sage.manifolds.differentiable.levi_civita_connection.**LeviCivitaConnection**(*metric*, *name*, *latex_name=None*, *init_coef=True*)

Bases: *sage.manifolds.differentiable.affine_connection.AffineConnection*

Levi-Civita connection on a pseudo-Riemannian manifold.

Let M be a differentiable manifold of class C^∞ (smooth manifold) over $\mathbf{R}$ endowed with a pseudo-Riemannian metric g. Let $C^\infty(M)$ be the algebra of smooth functions $M \to \mathbf{R}$ (cf. *DiffScalarFieldAlgebra*) and let $\mathfrak{X}(M)$ be the $C^\infty(M)$-module of vector fields on M (cf. *VectorFieldModule*). The *Levi-Civita connection associated with* g is the unique operator

$$\begin{array}{cccc} \nabla : & \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ & (u, v) & \longmapsto & \nabla_u v \end{array}$$

that

- is $\mathbf{R}$-bilinear, i.e. is bilinear when considering $\mathfrak{X}(M)$ as a vector space over $\mathbf{R}$
- is $C^\infty(M)$-linear w.r.t. the first argument: $\forall f \in C^\infty(M),\ \nabla_{fu} v = f \nabla_u v$
- obeys Leibniz rule w.r.t. the second argument: $\forall f \in C^\infty(M),\ \nabla_u (f v) = \mathrm{d}f(u)\, v + f \nabla_u v$
- is torsion-free
- is compatible with g: $\forall (u, v, w) \in \mathfrak{X}(M)^3,\ u(g(v, w)) = g(\nabla_u v, w) + g(v, \nabla_u w)$

The Levi-Civita connection ∇ gives birth to the *covariant derivative operator* acting on tensor fields, denoted by the same symbol:

$$\begin{array}{cccc} \nabla : & T^{(k,l)}(M) & \longrightarrow & T^{(k,l+1)}(M) \\ & t & \longmapsto & \nabla t \end{array}$$

where $T^{(k,l)}(M)$ stands for the $C^\infty(M)$-module of tensor fields of type (k, l) on M (cf. *TensorFieldModule*), with the convention $T^{(0,0)}(M) := C^\infty(M)$. For a vector field v, the covariant derivative ∇v is a type-(1,1) tensor field such that

$$\forall u \in \mathfrak{X}(M),\ \nabla_u v = \nabla v(., u)$$

More generally for any tensor field $t \in T^{(k,l)}(M)$, we have

$$\forall u \in \mathfrak{X}(M),\ \nabla_u t = \nabla t(\ldots, u)$$

Note: The above convention means that, in terms of index notation, the "derivation index" in ∇t is the *last* one:

$$\nabla_c t^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = (\nabla t)^{a_1 \ldots a_k}{}_{b_1 \ldots b_l c}$$

INPUT:

- `metric` – the metric g defining the Levi-Civita connection, as an instance of class *PseudoRiemannianMetric*
- `name` – name given to the connection
- `latex_name` – (default: `None`) LaTeX symbol to denote the connection

- `init_coef` – (default: `True`) determines whether the Christoffel symbols are initialized (in the top charts on the domain, i.e. disregarding the subcharts)

EXAMPLES:

Levi-Civita connection associated with the Euclidean metric on $\mathbf{R}^3$ expressed in spherical coordinates:

```
sage: forget() # for doctests only
sage: M = Manifold(3, 'R^3', start_index=1)
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2 , (r*sin(th))^2
sage: g.display()
g = dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: nab = g.connection(name='nabla', latex_name=r'\nabla') ; nab
Levi-Civita connection nabla associated with the Riemannian metric g on
 the 3-dimensional differentiable manifold R^3
```

Let us check that the connection is compatible with the metric:

```
sage: Dg = nab(g) ; Dg
Tensor field nabla(g) of type (0,3) on the 3-dimensional
 differentiable manifold R^3
sage: Dg == 0
True
```

and that it is torsionless:

```
sage: nab.torsion() == 0
True
```

As a check, let us enforce the computation of the torsion:

```
sage: sage.manifolds.differentiable.affine_connection.AffineConnection.
↪torsion(nab) == 0
True
```

The connection coefficients in the manifold's default frame are Christoffel symbols, since the default frame is a coordinate frame:

```
sage: M.default_frame()
Coordinate frame (R^3, (d/dr,d/dth,d/dph))
sage: nab.coef()
3-indices components w.r.t. Coordinate frame (R^3, (d/dr,d/dth,d/dph)),
 with symmetry on the index positions (1, 2)
```

We note that the Christoffel symbols are symmetric with respect to their last two indices (positions (1,2)); their expression is:

```
sage: nab.coef()[:]  # display as a array
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(th)^2]],
 [[0, 1/r, 0], [1/r, 0, 0], [0, 0, -cos(th)*sin(th)]],
 [[0, 0, 1/r], [0, 0, cos(th)/sin(th)], [1/r, cos(th)/sin(th), 0]]]
sage: nab.display()  # display only the non-vanishing symbols
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_th,r = 1/r
```

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```
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
Gam^ph_ph,r = 1/r
Gam^ph_ph,th = cos(th)/sin(th)
sage: nab.display(only_nonredundant=True)  # skip redundancy due to symmetry
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

The same display can be obtained via the function `christoffel_symbols_display()` acting on the metric:

```
sage: g.christoffel_symbols_display(chart=c_spher)
Gam^r_th,th = -r
Gam^r_ph,ph = -r*sin(th)^2
Gam^th_r,th = 1/r
Gam^th_ph,ph = -cos(th)*sin(th)
Gam^ph_r,ph = 1/r
Gam^ph_th,ph = cos(th)/sin(th)
```

coef (*frame=None*)

Return the connection coefficients relative to the given frame.

n being the manifold's dimension, the connection coefficients relative to the vector frame (e_i) are the n^3 scalar fields $\Gamma^k_{\ ij}$ defined by

$$\nabla_{e_j} e_i = \Gamma^k_{\ ij} e_k$$

If the connection coefficients are not known already, they are computed

- as Christoffel symbols if the frame (e_i) is a coordinate frame
- from the above formula otherwise

INPUT:

- `frame` – (default: `None`) vector frame relative to which the connection coefficients are required; if none is provided, the domain's default frame is assumed

OUTPUT:

- connection coefficients relative to the frame `frame`, as an instance of the class `Components` with 3 indices ordered as (k, i, j); for Christoffel symbols, an instance of the subclass `CompWithSym` is returned.

EXAMPLES:

Christoffel symbols of the Levi-Civita connection associated to the Euclidean metric on $\mathbf{R}^3$ expressed in spherical coordinates:

```
sage: M = Manifold(3, 'R^3', start_index=1)
sage: c_spher.<r,th,ph> = M.chart(r'r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):\phi')
sage: g = M.metric('g')
sage: g[1,1], g[2,2], g[3,3] = 1, r^2 , (r*sin(th))^2
```

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```
sage: g.display()
g = dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: nab = g.connection()
sage: gam = nab.coef() ; gam
3-indices components w.r.t. Coordinate frame (R^3, (d/dr,d/dth,d/dph)),
 with symmetry on the index positions (1, 2)
sage: gam[:]
[[[0, 0, 0], [0, -r, 0], [0, 0, -r*sin(th)^2]],
[[0, 1/r, 0], [1/r, 0, 0], [0, 0, -cos(th)*sin(th)]],
[[0, 0, 1/r], [0, 0, cos(th)/sin(th)], [1/r, cos(th)/sin(th), 0]]]
sage: # The only non-zero Christoffel symbols:
sage: gam[1,2,2], gam[1,3,3]
(-r, -r*sin(th)^2)
sage: gam[2,1,2], gam[2,3,3]
(1/r, -cos(th)*sin(th))
sage: gam[3,1,3], gam[3,2,3]
(1/r, cos(th)/sin(th))
```

Connection coefficients of the same connection with respect to the orthonormal frame associated to spherical coordinates:

```
sage: ch_basis = M.automorphism_field()
sage: ch_basis[1,1], ch_basis[2,2], ch_basis[3,3] = 1, 1/r, 1/(r*sin(th))
sage: e = c_spher.frame().new_frame(ch_basis, 'e')
sage: gam_e = nab.coef(e) ; gam_e
3-indices components w.r.t. Vector frame (R^3, (e_1,e_2,e_3))
sage: gam_e[:]
[[[0, 0, 0], [0, -1/r, 0], [0, 0, -1/r]],
[[0, 1/r, 0], [0, 0, 0], [0, 0, -cos(th)/(r*sin(th))]],
[[0, 0, 1/r], [0, 0, cos(th)/(r*sin(th))], [0, 0, 0]]]
sage: # The only non-zero connection coefficients:
sage: gam_e[1,2,2], gam_e[2,1,2]
(-1/r, 1/r)
sage: gam_e[1,3,3], gam_e[3,1,3]
(-1/r, 1/r)
sage: gam_e[2,3,3], gam_e[3,2,3]
(-cos(th)/(r*sin(th)), cos(th)/(r*sin(th)))
```

restrict (*subdomain*)

Return the restriction of the connection to some subdomain.

If such restriction has not been defined yet, it is constructed here.

INPUT:

- `subdomain` – open subset U of the connection's domain (must be an instance of *DifferentiableManifold*)

OUTPUT:

- instance of *LeviCivitaConnection* representing the restriction.

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[0,0], g[1,1] = 1+y^2, 1+x^2
```

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```
sage: nab = g.connection()
sage: nab[:]
[[[0, y/(y^2 + 1)], [y/(y^2 + 1), -x/(y^2 + 1)]],
 [[-y/(x^2 + 1), x/(x^2 + 1)], [x/(x^2 + 1), 0]]]
sage: U = M.open_subset('U', coord_def={X: x>0})
sage: nabU = nab.restrict(U); nabU
Levi-Civita connection nabla_g associated with the Riemannian
 metric g on the Open subset U of the 2-dimensional differentiable
 manifold M
sage: nabU[:]
[[[0, y/(y^2 + 1)], [y/(y^2 + 1), -x/(y^2 + 1)]],
 [[-y/(x^2 + 1), x/(x^2 + 1)], [x/(x^2 + 1), 0]]]
```

Let us check that the restriction is the connection compatible with the restriction of the metric:

```
sage: nabU(g.restrict(U)).display()
nabla_g(g) = 0
```

ricci (*name=None*, *latex_name=None*)

Return the connection's Ricci tensor.

This method redefines *sage.manifolds.differentiable.affine_connection.AffineConnection.ricci()* to take into account the symmetry of the Ricci tensor for a Levi-Civita connection.

The Ricci tensor is the tensor field Ric of type (0,2) defined from the Riemann curvature tensor R by

$$Ric(u, v) = R(e^i, u, e_i, v)$$

for any vector fields u and v, (e_i) being any vector frame and (e^i) the dual coframe.

INPUT:

- `name` – (default: `None`) name given to the Ricci tensor; if none, it is set to "Ric(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Ricci tensor; if none, it is set to "\mathrm{Ric}(g)", where "g" is the metric's name

OUTPUT:

- the Ricci tensor Ric, as an instance of *TensorField* of tensor type (0,2) and symmetric

EXAMPLES:

Ricci tensor of the standard connection on the 2-dimensional sphere:

```
sage: M = Manifold(2, 'S^2', start_index=1)
sage: c_spher.<th,ph> = M.chart(r'th:(0,pi):\theta ph:(0,2*pi):\phi')
sage: g = M.metric('g')
sage: g[1,1], g[2,2] = 1, sin(th)^2
sage: g.display() # standard metric on S^2:
g = dth*dth + sin(th)^2 dph*dph
sage: nab = g.connection() ; nab
Levi-Civita connection nabla_g associated with the Riemannian
 metric g on the 2-dimensional differentiable manifold S^2
sage: ric = nab.ricci() ; ric
Field of symmetric bilinear forms Ric(g) on the 2-dimensional
 differentiable manifold S^2
```

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```
sage: ric.display()
Ric(g) = dth*dth + sin(th)^2 dph*dph
```

Checking that the Ricci tensor of the Levi-Civita connection associated to Schwarzschild metric is identically zero (as a solution of the Einstein equation):

```
sage: M = Manifold(4, 'M')
sage: c_BL.<t,r,th,ph> = M.chart(r't r:(0,+oo) th:(0,pi):\theta ph:(0,
↪2*pi):\phi') # Schwarzschild-Droste coordinates
sage: g = M.lorentzian_metric('g')
sage: m = var('m')  # mass in Schwarzschild metric
sage: g[0,0], g[1,1] = -(1-2*m/r), 1/(1-2*m/r)
sage: g[2,2], g[3,3] = r^2, (r*sin(th))^2
sage: g.display()
g = (2*m/r - 1) dt*dt - 1/(2*m/r - 1) dr*dr + r^2 dth*dth
 + r^2*sin(th)^2 dph*dph
sage: nab = g.connection() ; nab
Levi-Civita connection nabla_g associated with the Lorentzian
 metric g on the 4-dimensional differentiable manifold M
sage: ric = nab.ricci() ; ric
Field of symmetric bilinear forms Ric(g) on the 4-dimensional
 differentiable manifold M
sage: ric == 0
True
```

riemann(*name=None*, *latex_name=None*)

Return the Riemann curvature tensor of the connection.

This method redefines *sage.manifolds.differentiable.affine_connection.AffineConnection.riemann()* to set some name and the latex_name to the output.

The Riemann curvature tensor is the tensor field R of type (1,3) defined by

$$R(\omega, w, u, v) = \left\langle \omega, \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \right\rangle$$

for any 1-form ω and any vector fields u, v and w.

INPUT:

- `name` – (default: `None`) name given to the Riemann tensor; if none, it is set to "Riem(g)", where "g" is the metric's name
- `latex_name` – (default: `None`) LaTeX symbol to denote the Riemann tensor; if none, it is set to "\mathrm{Riem}(g)", where "g" is the metric's name

OUTPUT:

- the Riemann curvature tensor R, as an instance of *TensorField*

EXAMPLES:

Riemann tensor of the Levi-Civita connection associated with the metric of the hyperbolic plane (Poincaré disk model):

```
sage: M = Manifold(2, 'M', start_index=1)
sage: X.<x,y> = M.chart('x:(-1,1) y:(-1,1)')  # Cartesian coord. on the␣
↪Poincaré disk
sage: X.add_restrictions(x^2+y^2<1)
sage: g = M.metric('g')
```

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```
sage: g[1,1], g[2,2] = 4/(1-x^2-y^2)^2, 4/(1-x^2-y^2)^2
sage: nab = g.connection()
sage: riem = nab.riemann(); riem
Tensor field Riem(g) of type (1,3) on the 2-dimensional
 differentiable manifold M
sage: riem.display_comp()
Riem(g)^x_yxy = -4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
Riem(g)^x_yyx = 4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
Riem(g)^y_xxy = 4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
Riem(g)^y_xyx = -4/(x^4 + y^4 + 2*(x^2 - 1)*y^2 - 2*x^2 + 1)
```

torsion()

Return the connection's torsion tensor (identically zero for a Levi-Civita connection).

See *sage.manifolds.differentiable.affine_connection.AffineConnection.torsion()* for the general definition of the torsion tensor.

OUTPUT:

- the torsion tensor T, as a vanishing instance of *TensorField*

EXAMPLES:

```
sage: M = Manifold(2, 'M')
sage: X.<x,y> = M.chart()
sage: g = M.metric('g')
sage: g[0,0], g[1,1] = 1+y^2, 1+x^2
sage: nab = g.connection()
sage: t = nab.torsion(); t
Tensor field of type (1,2) on the 2-dimensional differentiable
 manifold M
```

The torsion of a Levi-Civita connection is always zero:

```
sage: t.display()
0
```

3.5 Pseudo-Riemannian submanifolds

An *embedded (resp. immersed) submanifold of a pseudo-Riemannian manifold* (M, g) is an embedded (resp. immersed) submanifold N of M as a differentiable manifold such that pull back of the metric tensor g via the embedding (resp. immersion) endows N with the structure of a pseudo-Riemannian manifold.

A limitation of the current implementation is that a foliation is required to perform nearly all the calculations (except the induced metric). This is because the normal vector is easily computed with a foliation, but otherwise requires some operations which are not yet implemented in Sage (contraction over different domains).

To correctly compute the normal vector, the submanifold must be declared either Riemannian or Lorentzian.

The following example explains how to compute the various quantities associated with the hyperbolic slicing of the 3-dimensional Minkowski space.

The manifolds must first be declared:

```
sage: M = Manifold(3, 'M', structure="Lorentzian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
```

The considered slice being spacelike hypersurfaces, they are Riemannian manifolds.

Let us continue with chart declarations and various free variables:

```
sage: E.<w,x,y> = M.chart()
sage: C.<rh,th> = N.chart(r'rh:(0,+oo):\rho th:(0,2*pi):\theta')
sage: b = var('b',domain='real')
sage: assume(b>0)
sage: t = var('t',domain='real')
```

Here b is the hyperbola semi major axis, and t is the parameter of the foliation.

One must then define the embedding, as well as the inverse embedding and the inverse concerning the foliation parameter:

```
sage: phi = N.diff_map(M,{(C,E): [b*cosh(rh)+t,
....:                             b*sinh(rh)*cos(th),
....:                             b*sinh(rh)*sin(th)]})
sage: phi_inv = M.diff_map(N,{(E,C):[log(sqrt(x^2+y^2+b^2)/b+
....:                                sqrt((x^2+y^2+b^2)/b^2-1)),
....:                                atan2(y,x)]})
sage: phi_inv_t = M.scalar_field({E:w-sqrt(x^2+y^2+b^2)})
```

One can check that the inverse is correct with:

```
sage: (phi*phi_inv).display()
M --> M
   (w, x, y) |--> ((b^2 + x^2 + y^2 + sqrt(b^2 + x^2 + y^2)*(t + sqrt(x^2 +
 y^2)) + sqrt(x^2 + y^2)*t)/(sqrt(b^2 + x^2 + y^2) + sqrt(x^2 + y^2)), x, y)
```

The first parameter cannot be evaluated yet, because the inverse for t is not taken into account. To prove that it is correct, one can temporarily inject it in the result:

```
sage: assume(w-t>0)
sage: (phi*phi_inv).expr()[0].subs({b^2: (w-t)^2-x^2-y^2})\
....:           .simplify().expand().simplify_full()
w
sage: forget(w-t>0)
```

The immersion can then be declared:

```
sage: N.set_immersion(phi, inverse=phi_inv, var=t,
....:                 t_inverse = {t: phi_inv_t})
```

This line doesn't do any calculation yet. It just check the coherence of the arguments, but not the inverse, the user is trusted on this point. The user can also declare that the immersion is in fact an embedding:

```
sage: N.declare_embedding()
```

Finally, we initialize the metric of the Minkowski space:

```
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = -1, 1, 1
```

With this, the declaration the ambient manifold and its foliation is finished, and calculations can be performed.

The first step is always to find a chart adapted to the foliation. This is done by the method "adapted_chart":

```
sage: T = N.adapted_chart(); T
[Chart (M, (rh_M, th_M, t_M))]
```

`T` contains a new chart defined on M. By default, the name of a coordinate will be the name of the coordinate in the submanifold chart indexed by the name of the ambient manifold.

One can check that some coordinates changes have been introduced on M:

```
sage: len(M.coord_changes())
2
```

Let us compute the induced metric (or first fundamental form):

```
sage: N.induced_metric()[:] # long time
[           b^2                0]
[             0 b^2*sinh(rh)^2]
```

the normal vector:

```
sage: N.normal().display()  # long time
n = sqrt(b^2 + x^2 + y^2)/b d/dw + x/b d/dx + y/b d/dy
```

Check that the hypersurface is indeed spacelike:

```
sage: N.ambient_metric()(N.normal(), N.normal()).display()  # long time
g(n,n): M --> R
   (w, x, y) |--> -1
   (rh_M, th_M, t_M) |--> -1
```

The lapse function is:

```
sage: N.lapse().display()  # long time
N: M --> R
   (w, x, y) |--> sqrt(b^2 + x^2 + y^2)/b
   (rh_M, th_M, t_M) |--> cosh(rh_M)
```

while the shift vector is:

```
sage: N.shift().display()  # long time
beta = -(x^2 + y^2)/b^2 d/dw - sqrt(b^2 + x^2 + y^2)*x/b^2 d/dx
 - sqrt(b^2 + x^2 + y^2)*y/b^2 d/dy
```

The extrinsic curvature (or second fundamental form) as a tensor of the ambient manifold:

```
sage: N.ambient_extrinsic_curvature()[:] # long time
[                                          -(x^2 + y^2)/b^3  (b^2*x + x^3 + x*y^2)/(sqrt(b^2 +␣
↪x^2 + y^2)*b^3) (y^3 + (b^2 + x^2)*y)/(sqrt(b^2 + x^2 + y^2)*b^3)]
[                         sqrt(b^2 + x^2 + y^2)*x/b^3                                          -
↪(b^2 + x^2)/b^3                                          -x*y/b^3]
[                         sqrt(b^2 + x^2 + y^2)*y/b^3                                           ␣
↪      -x*y/b^3                                  -(b^2 + y^2)/b^3]
```

The extrinsic curvature (or second fundamental form) as a tensor of the submanifold:

```
sage: N.extrinsic_curvature()[:] # long time
[           -b                0]
[            0 -b*sinh(rh)^2]
```

AUTHORS:

- Florentin Jaffredo (2018): initial version

REFERENCES:

- B. O'Neill : *Semi-Riemannian Geometry* [?]
- J. M. Lee : *Riemannian Manifolds* [?]

class sage.manifolds.differentiable.pseudo_riemannian_submanifold.**PseudoRiemannianSubmanifo**

Bases: *sage.manifolds.differentiable.pseudo_riemannian.PseudoRiemannianManifold*, *sage.manifolds.differentiable.differentiable_submanifold.DifferentiableSubmanifold*

Pseudo-Riemannian submanifold.

An *embedded (resp. immersed) submanifold of a pseudo-Riemannian manifold* (M, g) is an embedded (resp. immersed) submanifold N of M as a differentiable manifold such that pull back of the metric tensor g via the embedding (resp. immersion) endows N with the structure of a pseudo-Riemannian manifold.

INPUT:

- `n` – positive integer; dimension of the manifold
- `name` – string; name (symbol) given to the manifold
- `field` – field K on which the manifold is defined; allowed values are
  - **'real' or an object of type RealField (e.g., RR) for** a manifold over $\mathbf{R}$
  - **'complex' or an object of type ComplexField (e.g., CC)** for a manifold over $\mathbf{C}$
  - an object in the category of topological fields (see `Fields` and `TopologicalSpaces`) for other types of manifolds
- `structure` – manifold structure (see *TopologicalStructure* or *RealTopologicalStructure*)
- `ambient` – (default: `None`) manifold of destination of the immersion. If `None`, set to `self`
- `base_manifold` – (default: `None`) if not `None`, must be a topological manifold; the created object is then an open subset of `base_manifold`

- `latex_name` – (default: `None`) string; LaTeX symbol to denote the manifold; if none are provided, it is set to `name`
- `start_index` – (default: 0) integer; lower value of the range of indices used for "indexed objects" on the manifold, e.g., coordinates in a chart - `category` – (default: `None`) to specify the category; if `None`, `Manifolds(field)` is assumed (see the category `Manifolds`)
- `unique_tag` – (default: `None`) tag used to force the construction of a new object when all the other arguments have been used previously (without `unique_tag`, the `UniqueRepresentation` behavior inherited from *`ManifoldSubset`* would return the previously constructed object corresponding to these arguments)

EXAMPLES:

Let N be a 2-dimensional submanifold of a 3-dimensional manifold M:

```
sage: M = Manifold(3, 'M', structure ="pseudo-Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="pseudo-Riemannian")
sage: N
2-dimensional pseudo-Riemannian submanifold N embedded in 3-dimensional
 differentiable manifold M
sage: CM.<x,y,z> = M.chart()
sage: CN.<u,v> = N.chart()
```

Let us define a 1-dimension foliation indexed by t. The inverse map is needed in order to compute the adapted chart in the ambient manifold:

```
sage: t = var('t')
sage: phi = N.diff_map(M, {(CN,CM):[u, v, t+u^2+v^2]}); phi
Differentiable map from the 2-dimensional pseudo-Riemannian submanifold
 N embedded in 3-dimensional differentiable manifold M to the
 3-dimensional Riemannian manifold M
sage: phi_inv = M.diff_map(N,{(CM, CN): [x,y]})
sage: phi_inv_t = M.scalar_field({CM: z-x^2-y^2})
```

ϕ can then be declared as an embedding $N \to M$:

```
sage: N.set_embedding(phi, inverse=phi_inv, var=t,
....:                  t_inverse={t: phi_inv_t})
```

The foliation can also be used to find new charts on the ambient manifold that are adapted to the foliation, ie in which the expression of the immersion is trivial. At the same time, the appropriate coordinate changes are computed:

```
sage: N.adapted_chart()
[Chart (M, (u_M, v_M, t_M))]
sage: len(M.coord_changes())
2
```

See also:

`manifold` and *`differentiable_submanifold`*

`ambient_extrinsic_curvature`()

Return the second fundamental form of the submanifold as a tensor field on the ambient manifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- (0,2) tensor field on the ambient manifold equal to the second fundamental form once orthogonally projected onto the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                   intersection_name='W',
....:                   restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.ambient_second_fundamental_form()  # long time
Field of symmetric bilinear forms K along the 1-dimensional
 pseudo-Riemannian submanifold N embedded in 2-dimensional
 differentiable manifold M with values on the 2-dimensional
 Riemannian manifold M
sage: N.ambient_second_fundamental_form()[:] # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

An alias is `ambient_extrinsic_curvature`:

```
sage: N.ambient_extrinsic_curvature()[:]  # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

ambient_first_fundamental_form()

Return the first fundamental form of the submanifold as a tensor of the ambient manifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- (0,2) tensor field on the ambient manifold describing the induced metric before projection on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
```

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```
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                        intersection_name='W',
....:                        restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:            (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.ambient_first_fundamental_form()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

An alias is `ambient_induced_metric`:

```
sage: N.ambient_induced_metric()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

ambient_induced_metric()

Return the first fundamental form of the submanifold as a tensor of the ambient manifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- (0,2) tensor field on the ambient manifold describing the induced metric before projection on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                        intersection_name='W',
....:                        restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:            (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.ambient_first_fundamental_form()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

An alias is `ambient_induced_metric`:

```
sage: N.ambient_induced_metric()[:]
[ x^2/(x^2 + 4) -2*x/(x^2 + 4)]
[-2*x/(x^2 + 4)     4/(x^2 + 4)]
```

ambient_metric()

Return the metric of the ambient manifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the metric of the ambient manifold

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                                 r*sin(th)*sin(ph),
....:                                 r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
sage: N.ambient_metric()[:]
[1 0 0]
[0 1 0]
[0 0 1]
sage: N.ambient_metric() is g
True
```

ambient_second_fundamental_form()

Return the second fundamental form of the submanifold as a tensor field on the ambient manifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- (0,2) tensor field on the ambient manifold equal to the second fundamental form once orthogonally projected onto the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
```

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```
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                   intersection_name='W',
....:                   restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:           (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.ambient_second_fundamental_form()  # long time
Field of symmetric bilinear forms K along the 1-dimensional
 pseudo-Riemannian submanifold N embedded in 2-dimensional
 differentiable manifold M with values on the 2-dimensional
 Riemannian manifold M
sage: N.ambient_second_fundamental_form()[:] # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

An alias is `ambient_extrinsic_curvature`:

```
sage: N.ambient_extrinsic_curvature()[:]  # long time
[-x^2/(x^2 + 4)  2*x/(x^2 + 4)]
[ 2*x/(x^2 + 4)   -4/(x^2 + 4)]
```

clear_cache()

Reset all the cached functions and the derived quantities.

Use this function if you modified the immersion (or embedding) of the submanifold. Note that when calling a calculus function after clearing, new Python objects will be created.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
sage: n = N.normal()
sage: n is N.normal()
True
sage: N.clear_cache()
sage: n is N.normal()
```

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```
False
```

difft()

Return the differential of the first scalar field defining the submanifold

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- 1-form field on the ambient manifold.

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
sage: N.difft().display()
dr = x/sqrt(x^2 + y^2 + z^2) dx + y/sqrt(x^2 + y^2 + z^2) dy +
 z/sqrt(x^2 + y^2 + z^2) dz
```

extrinsic_curvature()

Return the second fundamental form of the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the second fundamental form, as a symmetric tensor field of type (0,2) on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                   intersection_name='W',
....:                   restrictions1=x!=0, restrictions2=y!=0)
```

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```
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.second_fundamental_form() # long time
Field of symmetric bilinear forms K on the 1-dimensional
 pseudo-Riemannian submanifold N embedded in 2-dimensional
 differentiable manifold M
sage: N.second_fundamental_form().display() # long time
K = -4/(x^4 + 8*x^2 + 16) dx*dx
```

An alias is `extrinsic_curvature`:

```
sage: N.extrinsic_curvature().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx*dx
```

first_fundamental_form()

Return the first fundamental form of the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the first fundamental form, as an instance of *PseudoRiemannianMetric*

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
sage: N.first_fundamental_form()  # long time
Riemannian metric gamma on the 2-dimensional pseudo-Riemannian
 submanifold N embedded in 3-dimensional differentiable manifold M
sage: N.first_fundamental_form()[:]  # long time
[           r^2              0]
[             0  r^2*sin(th)^2]
```

An alias is `induced_metric`:

```
sage: N.induced_metric()[:]  # long time
[         r^2              0]
[           0 r^2*sin(th)^2]
```

gauss_curvature()

Return the Gauss curvature of the submanifold.

The *Gauss curvature* is the product or the principal curvatures, or equivalently the determinant of the projection operator.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the Gauss curvature as a scalar field on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                 intersection_name='W',
....:                 restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4), x/2/sqrt(1+x^2/4)],
....:           (stereoS, E): [1/sqrt(1+4/y^2), 2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.gauss_curvature().display()  # long time
N --> R
on U: x |--> -1
on V: y |--> -1
```

gradt()

Return the gradient of the first scalar field defining the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- vector field on the ambient manifold.

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
```

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```
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
sage: N.gradt().display()
grad_r = x/sqrt(x^2 + y^2 + z^2) d/dx + y/sqrt(x^2 + y^2 + z^2) d/dy
 + z/sqrt(x^2 + y^2 + z^2) d/dz
```

induced_metric()

Return the first fundamental form of the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the first fundamental form, as an instance of *PseudoRiemannianMetric*

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                              r*sin(th)*sin(ph),
....:                              r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
sage: N.first_fundamental_form()  # long time
Riemannian metric gamma on the 2-dimensional pseudo-Riemannian
 submanifold N embedded in 3-dimensional differentiable manifold M
sage: N.first_fundamental_form()[:]  # long time
[          r^2               0]
[            0 r^2*sin(th)^2]
```

An alias is `induced_metric`:

```
sage: N.induced_metric()[:]  # long time
[          r^2               0]
[            0 r^2*sin(th)^2]
```

lapse()

Return the lapse function of the foliation.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the lapse function, as a scalar field on the ambient manifold

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
sage: N.lapse().display()
N: M --> R
   (x, y, z) |--> 1
   (th_M, ph_M, r_M) |--> 1
```

mean_curvature()

Return the mean curvature of the submanifold.

The *mean curvature* is the arithmetic mean of the principal curvatures, or equivalently the trace of the projection operator.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the mean curvature, as a scalar field on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
```

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```
....:                         intersection_name='W',
....:                         restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:           (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.mean_curvature().display()  # long time
N --> R
on U: x |--> -1
on V: y |--> -1
```

mixed_projection (*tensor*, *indices=0*)

Return de n+1 decomposition of a tensor on the submanifold and the normal vector.

The n+1 decomposition of a tensor of rank k can be obtained by contracting each index either with the normal vector or the projection operator of the submanifold (see `projector()`).

INPUT:

- `tensor` – any tensor field, eventually along the submanifold if no foliation is provided.
- `indices` – (default: `0`) list of integers containing the indices on which the projection is made on the normal vector. By default, all projections are made on the submanifold. If an integer n is provided, the n first contractions are made with the normal vector, all the other ones with the orthogonal projection operator.

OUTPUT:

- tensor field of rank k-`len(indices)`.

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
```

If `indices` is not specified, the mixed projection of the ambient metric coincides with the first fundamental form:

```
sage: gpp = N.mixed_projection(g); gpp  # long time
Tensor field of type (0,2) on the 3-dimensional Riemannian
 manifold M
```

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```
sage: gpp == N.ambient_first_fundamental_form()  # long time
True
```

The other non redundant projections are:

```
sage: gnp =  N.mixed_projection(g, [0]); gnp  # long time
1-form on the 3-dimensional Riemannian manifold M
```

and:

```
sage: gnn = N.mixed_projection(g, [0,1]); gnn
Scalar field on the 3-dimensional Riemannian manifold M
```

which is constant and equal to 1 (the norm of the unit normal vector):

```
sage: gnn.display()
M --> R
(x, y, z) |--> 1
(th_M, ph_M, r_M) |--> 1
```

normal()

Return a normal unit vector to the submanifold.

If a foliation is defined, it is used to compute the gradient of the foliation parameter and then the normal vector. If not, the normal vector is computed using the following formula:

$$n = \vec{*}(\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_{n-1})$$

where the star stands for the Hodge dual operator and the wedge for the exterior product.

This formula does not always define a proper vector field when multiple charts overlap, because of the arbitrariness of the direction of the normal vector. To avoid this problem, this function considers the graph defined by the atlas of the submanifold and the changes of coordinates, and only calculate the normal vector once by connected component. The expression is then propagate by restriction, continuation, or change of coordinates using a breadth-first exploration of the graph.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- vector field on the ambient manifold.

EXAMPLES:

A sphere embedded in Euclidean space foliated on the radius:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
```

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```
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
sage: N.normal().display()  # long time
n = x/sqrt(x^2 + y^2 + z^2) d/dx + y/sqrt(x^2 + y^2 + z^2) d/dy
 + z/sqrt(x^2 + y^2 + z^2) d/dz
```

Or in spherical coordinates:

```
sage: N.normal().display(T[0].frame(),T[0])  # long time
n = d/dr_M
```

The same sphere of constant radius, i.e. not assumed to be part of a foliation, in stereographic coordinates:

```
sage: M = Manifold(3, 'M', structure="Riemannian", start_index=1)
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U, V)
sage: stereoN.<x,y> = U.chart()
sage: stereoS.<xp,yp> = V.chart("xp:x' yp:y'")
sage: stereoN_to_S = stereoN.transition_map(stereoS,
....:                                 (x/(x^2+y^2), y/(x^2+y^2)),
....:                                 intersection_name='W',
....:                                 restrictions1= x^2+y^2!=0,
....:                                 restrictions2= xp^2+yp^2!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: W = U.intersection(V)
sage: stereoN_W = stereoN.restrict(W)
sage: stereoS_W = stereoS.restrict(W)
sage: A = W.open_subset('A', coord_def={stereoN_W: (y!=0, x<0),
....:                                   stereoS_W: (yp!=0, xp<0)})
sage: spher.<the,phi> = A.chart(r'the:(0,pi):\theta phi:(0,2*pi):\phi')
sage: stereoN_A = stereoN_W.restrict(A)
sage: spher_to_stereoN = spher.transition_map(stereoN_A,
....:                              (sin(the)*cos(phi)/(1-cos(the)),
....:                               sin(the)*sin(phi)/(1-cos(the))))
sage: spher_to_stereoN.set_inverse(2*atan(1/sqrt(x^2+y^2)),
....:                                     atan2(-y,-x)+pi)
sage: stereoN_to_S_A = stereoN_to_S.restrict(A)
sage: spher_to_stereoS = stereoN_to_S_A * spher_to_stereoN
sage: stereoS_to_N_A = stereoN_to_S.inverse().restrict(A)
sage: stereoS_to_spher = spher_to_stereoN.inverse() * stereoS_to_N_A
sage: E.<X,Y,Z> = M.chart()
sage: phi = N.diff_map(M, {(stereoN, E): [2*x/(1+x^2+y^2),
....:                                     2*y/(1+x^2+y^2),
....:                                     (x^2+y^2-1)/(1+x^2+y^2)],
....:                      (stereoS, E): [2*xp/(1+xp^2+yp^2),
....:                                    2*yp/(1+xp^2+yp^2),
....:                                   (1-xp^2-yp^2)/(1+xp^2+yp^2)]},
....:                     name='Phi', latex_name=r'\Phi')
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[3,3],g[1,1],g[2,2]=1,1,1
```

The normal vector is computed the same way, but now returns a tensor field along N:

```
sage: n = N.normal()  # long time
sage: n  # long time
Vector field n along the 2-dimensional pseudo-Riemannian submanifold
 N embedded in 3-dimensional differentiable manifold M with values
 on the 3-dimensional Riemannian manifold M
```

Let us check that the choice of orientation is coherent on the two top frames:

```
sage: n.restrict(V).display(format_spec=spher)  # long time
n = -cos(phi)*sin(the) d/dX - sin(phi)*sin(the) d/dY - cos(the) d/dZ
sage: n.restrict(U).display(format_spec=spher)  # long time
n = -cos(phi)*sin(the) d/dX - sin(phi)*sin(the) d/dY - cos(the) d/dZ
```

principal_curvatures(*chart*)

Return the principal curvatures of the submanifold.

The *principal curvatures* are the eigenvalues of the projection operator. The resulting scalar fields are named `k_i` with the index `i` ranging from 0 to the submanifold dimension minus one.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

INPUT:

- `chart` – chart in which the principal curvatures are to be computed

OUTPUT:

- the principal curvatures, as a list of scalar fields on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                 intersection_name='W',
....:                 restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.principal_curvatures(stereoN)[0].display()  # long time
k_0: N --> R
on U: x |--> -1
```

principal_directions(*chart*)

Return the principal directions of the submanifold.

The *principal directions* are the eigenvectors of the projection operator. The result is formatted as a list of couples (eigenvector, eigenvalue).

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

INPUT:

- `chart` – chart in which the principal directions are to be computed

OUTPUT:

- list of couples (vector field, scalar field) representing the principal directions and the associated principal curvatures

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                 intersection_name='W',
....:                 restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.principal_directions(stereoN)[0][0].display()  # long time
e_0 = d/dx
```

project (*tensor*)

Return the orthogonal projection of a tensor field onto the submanifold.

INPUT:

- `tensor` – any tensor field to be projected onto the submanifold. If no foliation is provided, must be a tensor field along the submanifold.

OUTPUT:

- orthogonal projection of `tensor` onto the submanifold, as a tensor field of the *ambient* manifold

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
```

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```
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
```

Let us perform the projection of the ambient metric and check that it is equal to the first fundamental form:

```
sage: pg = N.project(g); pg  # long time
Tensor field of type (0,2) on the 3-dimensional Riemannian manifold M
sage: pg == N.ambient_first_fundamental_form()  # long time
True
```

Note that the result of `project` is not cached.

projector()

Return the orthogonal projector onto the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the orthogonal projector onto the submanifold, as tensor field of type (1,1) on the ambient manifold

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
```

The orthogonal projector onto `N` as a type-(1,1) tensor field on `M`:

```
sage: N.projector()  # long time
Tensor field gamma of type (1,1) on the 3-dimensional Riemannian
 manifold M
```

Check that the orthogonal projector applied to the normal vector is zero:

```
sage: N.projector().contract(N.normal()).display()  # long time
0
```

second_fundamental_form()

Return the second fundamental form of the submanifold.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the second fundamental form, as a symmetric tensor field of type (0,2) on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                   intersection_name='W',
....:                   restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.second_fundamental_form() # long time
Field of symmetric bilinear forms K on the 1-dimensional
 pseudo-Riemannian submanifold N embedded in 2-dimensional
 differentiable manifold M
sage: N.second_fundamental_form().display() # long time
K = -4/(x^4 + 8*x^2 + 16) dx*dx
```

An alias is `extrinsic_curvature`:

```
sage: N.extrinsic_curvature().display()  # long time
K = -4/(x^4 + 8*x^2 + 16) dx*dx
```

shape_operator()

Return the shape operator of the submanifold.

The shape operator is equal to the second fundamental form with one of the indices upped.

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- the shape operator, as a tensor field of type (1,1) on the submanifold

EXAMPLES:

A unit circle embedded in the Euclidean plane:

```
sage: M = Manifold(2, 'M', structure="Riemannian")
sage: N = Manifold(1, 'N', ambient=M, structure="Riemannian")
sage: U = N.open_subset('U')
```

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```
sage: V = N.open_subset('V')
sage: N.declare_union(U,V)
sage: stereoN.<x> = U.chart()
sage: stereoS.<y> = V.chart()
sage: E.<X,Y> = M.chart()
sage: stereoN_to_S = stereoN.transition_map(stereoS, (4/x),
....:                     intersection_name='W',
....:                     restrictions1=x!=0, restrictions2=y!=0)
sage: stereoS_to_N = stereoN_to_S.inverse()
sage: phi = N.diff_map(M, {(stereoN, E): [1/sqrt(1+x^2/4),x/2/sqrt(1+x^2/4)],
....:          (stereoS, E): [1/sqrt(1+4/y^2),2/y/sqrt(1+4/y^2)]})
sage: N.set_embedding(phi)
sage: g = M.metric()
sage: g[0,0], g[1,1] = 1, 1
sage: N.shape_operator()  # long time
Tensor field of type (1,1) on the 1-dimensional pseudo-Riemannian
 submanifold N embedded in 2-dimensional differentiable manifold M
sage: N.shape_operator()[:]  # long time
[-1]
```

shift()

Return the shift function of the foliation

The result is cached, so calling this method multiple times always returns the same result at no additional cost.

OUTPUT:

- shift vector field on the ambient manifold.

EXAMPLES:

A sphere embedded in Euclidean space:

```
sage: M = Manifold(3, 'M', structure="Riemannian")
sage: N = Manifold(2, 'N', ambient=M, structure="Riemannian")
sage: C.<th,ph> = N.chart(r'th:(0,pi):\theta ph:(-pi,pi):\phi')
sage: r = var('r')
sage: assume(r>0)
sage: E.<x,y,z> = M.chart()
sage: phi = N.diff_map(M,{(C,E): [r*sin(th)*cos(ph),
....:                             r*sin(th)*sin(ph),
....:                             r*cos(th)]})
sage: phi_inv = M.diff_map(N, {(E,C): [arccos(z/r), atan2(y,x)]})
sage: phi_inv_r = M.scalar_field({E: sqrt(x^2+y^2+z^2)})
sage: N.set_embedding(phi, inverse=phi_inv, var=r,
....:                 t_inverse={r: phi_inv_r})
sage: T = N.adapted_chart()
sage: g = M.metric()
sage: g[0,0], g[1,1], g[2,2] = 1, 1, 1
sage: N.shift().display()  # long time
beta = 0
```

CHAPTER

FOUR

UTILITIES FOR CALCULUS

This module defines helper functions which are used for simplifications and display of symbolic expressions.

AUTHORS:

- Michal Bejger (2015) : class *ExpressionNice*
- Eric Gourgoulhon (2015, 2017) : simplification functions
- Travis Scrimshaw (2016): review tweaks

class `sage.manifolds.utilities.`**`ExpressionNice`**`(ex)`

Bases: `sage.symbolic.expression.Expression`

Subclass of `Expression` for a "human-friendly" display of partial derivatives and the possibility to shorten the display by skipping the arguments of symbolic functions.

INPUT:

- `ex` – symbolic expression

EXAMPLES:

An expression formed with callable symbolic expressions:

```
sage: var('x y z')
(x, y, z)
sage: f = function('f')(x, y)
sage: g = f.diff(y).diff(x)
sage: h = function('h')(y, z)
sage: k = h.diff(z)
sage: fun = x*g + y*(k-z)^2
```

The standard Pynac display of partial derivatives:

```
sage: fun
y*(z - diff(h(y, z), z))^2 + x*diff(f(x, y), x, y)
sage: latex(fun)
y {\left(z - \frac{\partial}{\partial z}h\left(y, z\right)\right)}^{2} + x \frac
↪{\partial^{2}}{\partial x\partial y}f\left(x, y\right)
```

With *ExpressionNice*, the Pynac notation `D[...]` is replaced by textbook-like notation:

```
sage: from sage.manifolds.utilities import ExpressionNice
sage: ExpressionNice(fun)
y*(z - d(h)/dz)^2 + x*d^2(f)/dxdy
sage: latex(ExpressionNice(fun))
y {\left(z - \frac{\partial\,h}{\partial z}\right)}^{2}
 + x \frac{\partial^2\,f}{\partial x\partial y}
```

An example when function variables are themselves functions:

```
sage: f = function('f')(x, y)
sage: g = function('g')(x, f)  # the second variable is the function f
sage: fun = (g.diff(x))*x - x^2*f.diff(x,y)
sage: fun
-x^2*diff(f(x, y), x, y) + (diff(f(x, y), x)*D[1](g)(x, f(x, y)) + D[0](g)(x, f(x,
↪ y)))*x
sage: ExpressionNice(fun)
-x^2*d^2(f)/dxdy + (d(f)/dx*d(g)/d(f(x, y)) + d(g)/dx)*x
sage: latex(ExpressionNice(fun))
-x^{2} \frac{\partial^2\,f}{\partial x\partial y}
 + {\left(\frac{\partial\,f}{\partial x}
   \frac{\partial\,g}{\partial \left( f\left(x, y\right) \right)}
 + \frac{\partial\,g}{\partial x}\right)} x
```

Note that `D[1](g)(x, f(x,y))` is rendered as `d(g)/d(f(x, y))`.

An example with multiple differentiations:

```
sage: fun = f.diff(x,x,y,y,x)*x
sage: fun
x*diff(f(x, y), x, x, x, y, y)
sage: ExpressionNice(fun)
x*d^5(f)/dx^3dy^2
sage: latex(ExpressionNice(fun))
x \frac{\partial^5\,f}{\partial x ^ 3\partial y ^ 2}
```

Parentheses are added around powers of partial derivatives to avoid any confusion:

```
sage: fun = f.diff(y)^2
sage: fun
diff(f(x, y), y)^2
sage: ExpressionNice(fun)
(d(f)/dy)^2
sage: latex(ExpressionNice(fun))
\left(\frac{\partial\,f}{\partial y}\right)^{2}
```

The explicit mention of function arguments can be omitted for the sake of brevity:

```
sage: fun = fun*f
sage: ExpressionNice(fun)
f(x, y)*(d(f)/dy)^2
sage: Manifold.options.omit_function_arguments=True
sage: ExpressionNice(fun)
f*(d(f)/dy)^2
sage: latex(ExpressionNice(fun))
f \left(\frac{\partial\,f}{\partial y}\right)^{2}
sage: Manifold.options._reset()
sage: ExpressionNice(fun)
f(x, y)*(d(f)/dy)^2
sage: latex(ExpressionNice(fun))
f\left(x, y\right) \left(\frac{\partial\,f}{\partial y}\right)^{2}
```

class `sage.manifolds.utilities.`**`SimplifyAbsTrig`**`(ex)`

Bases: `sage.symbolic.expression_conversions.ExpressionTreeWalker`

Class for simplifying absolute values of cosines or sines (in the real domain), by walking the expression tree.

The end user interface is the function *`simplify_abs_trig()`*.

INPUT:

- ex – a symbolic expression

EXAMPLES:

Let us consider the following symbolic expression with some assumption on the range of the variable x:

```
sage: assume(pi/2<x, x<pi)
sage: a = abs(cos(x)) + abs(sin(x))
```

The method simplify_full() is ineffective on such an expression:

```
sage: a.simplify_full()
abs(cos(x)) + abs(sin(x))
```

We construct a *SimplifyAbsTrig* object s from the symbolic expression a:

```
sage: from sage.manifolds.utilities import SimplifyAbsTrig
sage: s = SimplifyAbsTrig(a)
```

We use the __call__ method to walk the expression tree and produce a correctly simplified expression, given that $x \in (\pi/2, \pi)$:

```
sage: s()
-cos(x) + sin(x)
```

Calling the simplifier s with an expression actually simplifies this expression:

```
sage: s(a)  # same as s() since s is built from a
-cos(x) + sin(x)
sage: s(abs(cos(x/2)) + abs(sin(x/2)))  #  pi/4 < x/2 < pi/2
cos(1/2*x) + sin(1/2*x)
sage: s(abs(cos(2*x)) + abs(sin(2*x)))  #  pi < 2 x < 2*pi
abs(cos(2*x)) - sin(2*x)
sage: s(abs(sin(2+abs(cos(x)))))  # nested abs(sin_or_cos(...))
sin(-cos(x) + 2)
```

See also:

simplify_abs_trig() for more examples with *SimplifyAbsTrig* at work.

composition(*ex*, *operator*)

This is the only method of the base class ExpressionTreeWalker that is reimplemented, since it manages the composition of abs with cos or sin.

INPUT:

- ex – a symbolic expression
- operator – an operator

OUTPUT:

- a symbolic expression, equivalent to ex with abs(cos(...)) and abs(sin(...)) simplified, according to the range of their argument.

EXAMPLES:

```
sage: from sage.manifolds.utilities import SimplifyAbsTrig
sage: assume(-pi/2 < x, x<0)
```

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```
sage: a = abs(sin(x))
sage: s = SimplifyAbsTrig(a)
sage: a.operator()
abs
sage: s.composition(a, a.operator())
sin(-x)
```

```
sage: a = exp(function('f')(x))   # no abs(sin_or_cos(...))
sage: a.operator()
exp
sage: s.composition(a, a.operator())
e^f(x)
```

```
sage: forget()   # no longer any assumption on x
sage: a = abs(cos(sin(x)))   # simplifiable since -1 <= sin(x) <= 1
sage: s.composition(a, a.operator())
cos(sin(x))
sage: a = abs(sin(cos(x)))   # not simplifiable
sage: s.composition(a, a.operator())
abs(sin(cos(x)))
```

class sage.manifolds.utilities.**SimplifySqrtReal**(*ex*)

Bases: sage.symbolic.expression_conversions.ExpressionTreeWalker

Class for simplifying square roots in the real domain, by walking the expression tree.

The end user interface is the function *simplify_sqrt_real()*.

INPUT:

- ex – a symbolic expression

EXAMPLES:

Let us consider the square root of an exact square under some assumption:

```
sage: assume(x<1)
sage: a = sqrt(x^2-2*x+1)
```

The method simplify_full() is ineffective on such an expression:

```
sage: a.simplify_full()
sqrt(x^2 - 2*x + 1)
```

and the more agressive method canonicalize_radical() yields a wrong result, given that $x < 1$:

```
sage: a.canonicalize_radical()   # wrong output!
x - 1
```

We construct a *SimplifySqrtReal* object s from the symbolic expression a:

```
sage: from sage.manifolds.utilities import SimplifySqrtReal
sage: s = SimplifySqrtReal(a)
```

We use the __call__ method to walk the expression tree and produce a correctly simplified expression:

```
sage: s()
-x + 1
```

Calling the simplifier s with an expression actually simplifies this expression:

```
sage: s(a)  # same as s() since s is built from a
-x + 1
sage: s(sqrt(x^2))
abs(x)
sage: s(sqrt(1+sqrt(x^2-2*x+1)))  # nested sqrt's
sqrt(-x + 2)
```

Another example where both `simplify_full()` and `canonicalize_radical()` fail:

```
sage: b = sqrt((x-1)/(x-2))*sqrt(1-x)
sage: b.simplify_full()  # does not simplify
sqrt(-x + 1)*sqrt((x - 1)/(x - 2))
sage: b.canonicalize_radical()  # wrong output, given that x<1
(I*x - I)/sqrt(x - 2)
sage: SimplifySqrtReal(b)()  # OK, given that x<1
-(x - 1)/sqrt(-x + 2)
```

See also:

`simplify_sqrt_real()` for more examples with *`SimplifySqrtReal`* at work.

`arithmetic`(*ex*, *operator*)

This is the only method of the base class `ExpressionTreeWalker` that is reimplemented, since square roots are considered as arithmetic operations with `operator` = `pow` and `ex.operands()[1]` = `1/2` or `-1/2`.

INPUT:

- `ex` – a symbolic expression
- `operator` – an arithmetic operator

OUTPUT:

- a symbolic expression, equivalent to `ex` with square roots simplified

EXAMPLES:

```
sage: from sage.manifolds.utilities import SimplifySqrtReal
sage: a = sqrt(x^2+2*x+1)
sage: s = SimplifySqrtReal(a)
sage: a.operator()
<built-in function pow>
sage: s.arithmetic(a, a.operator())
abs(x + 1)
```

```
sage: a = x + 1  # no square root
sage: s.arithmetic(a, a.operator())
x + 1
```

```
sage: a = x + 1 + sqrt(function('f')(x)^2)
sage: s.arithmetic(a, a.operator())
x + abs(f(x)) + 1
```

`sage.manifolds.utilities.`**`exterior_derivative`**(*form*)

Exterior derivative of a differential form.

INPUT:

- `form` – a differential form; this must an instance of either
  - `DiffScalarField` for a 0-form (scalar field)
  - `DiffFormParal` for a p-form ($p \geq 1$) on a parallelizable manifold
  - `DiffForm` for a a p-form ($p \geq 1$) on a non-parallelizable manifold

OUTPUT:

- the $(p+1)$-form that is the exterior derivative of `form`

EXAMPLES:

Exterior derivative of a scalar field (0-form):

```
sage: from sage.manifolds.utilities import exterior_derivative
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: f = M.scalar_field({X: x+y^2+z^3}, name='f')
sage: df = exterior_derivative(f); df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = dx + 2*y dy + 3*z^2 dz
```

An alias is `xder`:

```
sage: from sage.manifolds.utilities import xder
sage: df == xder(f)
True
```

Exterior derivative of a 1-form:

```
sage: a = M.one_form(name='a')
sage: a[:] = [x+y*z, x-y*z, x*y*z]
sage: da = xder(a); da
2-form da on the 3-dimensional differentiable manifold M
sage: da.display()
da = (-z + 1) dx/\dy + (y*z - y) dx/\dz + (x*z + y) dy/\dz
sage: dda = xder(da); dda
3-form dda on the 3-dimensional differentiable manifold M
sage: dda.display()
dda = 0
```

See also:

sage.manifolds.differentiable.diff_form.DiffFormParal.exterior_derivative or *sage.manifolds.differentiable.diff_form.DiffForm.exterior_derivative* for more examples.

`sage.manifolds.utilities.`**`set_axes_labels`**`(`*graph*, *xlabel*, *ylabel*, *zlabel*, ***kwds*`)`

Set axes labels for a 3D graphics object `graph`.

This is a workaround for the lack of axes labels in 3D plots. This sets the labels as `text3d()` objects at locations determined from the bounding box of the graphic object `graph`.

INPUT:

- `graph` – `Graphics3d`; a 3D graphic object
- `xlabel` – string for the x-axis label
- `ylabel` – string for the y-axis label

- `zlabel` – string for the z-axis label
- `**kwds` – options (e.g. color) for text3d

OUTPUT:

- the 3D graphic object with text3d labels added

EXAMPLES:

```
sage: g = sphere()
sage: g.all
[Graphics3d Object]
sage: from sage.manifolds.utilities import set_axes_labels
sage: ga = set_axes_labels(g, 'X', 'Y', 'Z', color='red')
sage: ga.all  # the 3D frame has now axes labels
[Graphics3d Object, Graphics3d Object,
 Graphics3d Object, Graphics3d Object]
```

`sage.manifolds.utilities.`**`simplify_abs_trig`**(*expr*)

Simplify `abs(sin(...))` and `abs(cos(...))` in symbolic expressions.

EXAMPLES:

```
sage: M = Manifold(3, 'M', structure='topological')
sage: X.<x,y,z> = M.chart(r'x y:(0,pi) z:(-pi/3,0)')
sage: X.coord_range()
x: (-oo, +oo); y: (0, pi); z: (-1/3*pi, 0)
```

Since x spans all $\mathbf{R}$, no simplification of `abs(sin(x))` occurs, while `abs(sin(y))` and `abs(sin(3*z))` are correctly simplified, given that $y \in (0, \pi)$ and $z \in (-\pi/3, 0)$:

```
sage: from sage.manifolds.utilities import simplify_abs_trig
sage: simplify_abs_trig( abs(sin(x)) + abs(sin(y)) + abs(sin(3*z)) )
abs(sin(x)) + sin(y) + sin(-3*z)
```

Note that neither `simplify_trig()` nor `simplify_full()` works in this case:

```
sage: s = abs(sin(x)) + abs(sin(y)) + abs(sin(3*z))
sage: s.simplify_trig()
abs(4*cos(-z)^2 - 1)*abs(sin(-z)) + abs(sin(x)) + abs(sin(y))
sage: s.simplify_full()
abs(4*cos(-z)^2 - 1)*abs(sin(-z)) + abs(sin(x)) + abs(sin(y))
```

despite the following assumptions hold:

```
sage: assumptions()
[x is real, y is real, y > 0, y < pi, z is real, z > -1/3*pi, z < 0]
```

Additional checks are:

```
sage: simplify_abs_trig( abs(sin(y/2)) )  # shall simplify
sin(1/2*y)
sage: simplify_abs_trig( abs(sin(2*y)) )  # must not simplify
abs(sin(2*y))
sage: simplify_abs_trig( abs(sin(z/2)) )  # shall simplify
sin(-1/2*z)
sage: simplify_abs_trig( abs(sin(4*z)) )  # must not simplify
abs(sin(-4*z))
```

Simplification of `abs(cos(...))`:

```
sage: forget()
sage: M = Manifold(3, 'M', structure='topological')
sage: X.<x,y,z> = M.chart(r'x y:(0,pi/2) z:(pi/4,3*pi/4)')
sage: X.coord_range()
x: (-oo, +oo); y: (0, 1/2*pi); z: (1/4*pi, 3/4*pi)
sage: simplify_abs_trig( abs(cos(x)) + abs(cos(y)) + abs(cos(2*z)) )
abs(cos(x)) + cos(y) - cos(2*z)
```

Additional tests:

```
sage: simplify_abs_trig(abs(cos(y-pi/2)))  # shall simplify
cos(-1/2*pi + y)
sage: simplify_abs_trig(abs(cos(y+pi/2)))  # shall simplify
-cos(1/2*pi + y)
sage: simplify_abs_trig(abs(cos(y-pi)))  # shall simplify
-cos(-pi + y)
sage: simplify_abs_trig(abs(cos(2*y)))  # must not simplify
abs(cos(2*y))
sage: simplify_abs_trig(abs(cos(y/2)) * abs(sin(z)))  # shall simplify
cos(1/2*y)*sin(z)
```

`sage.manifolds.utilities.`**`simplify_chain_generic`**`(`*expr*`)`

Apply a chain of simplifications to a symbolic expression.

This is the simplification chain used in calculus involving coordinate functions on manifolds over fields different from $\mathbf{R}$, as implemented in *ChartFunction*.

The chain is formed by the following functions, called successively:

1. `simplify_factorial()`
2. `simplify_rectform()`
3. `simplify_trig()`
4. `simplify_rational()`
5. `expand_sum()`

NB: for the time being, this is identical to `simplify_full()`.

EXAMPLES:

We consider variables that are coordinates of a chart on a complex manifold:

```
sage: M = Manifold(2, 'M', structure='topological', field='complex')
sage: X.<x,y> = M.chart()
```

Then neither `x` nor `y` is assumed to be real:

```
sage: assumptions()
[]
```

Accordingly, `simplify_chain_generic` does not simplify `sqrt(x^2)` to `abs(x)`:

```
sage: from sage.manifolds.utilities import simplify_chain_generic
sage: s = sqrt(x^2)
sage: simplify_chain_generic(s)
sqrt(x^2)
```

This contrasts with the behavior of *simplify_chain_real()*.

Other simplifications:

```
sage: s = (x+y)^2 - x^2 -2*x*y - y^2
sage: simplify_chain_generic(s)
0
sage: s = (x^2 - 2*x + 1) / (x^2 -1)
sage: simplify_chain_generic(s)
(x - 1)/(x + 1)
sage: s = cos(2*x) - 2*cos(x)^2 + 1
sage: simplify_chain_generic(s)
0
```

sage.manifolds.utilities.**simplify_chain_generic_sympy**(*expr*)

Apply a chain of simplifications to a sympy expression.

This is the simplification chain used in calculus involving coordinate functions on manifolds over fields different from $\mathbf{R}$, as implemented in *ChartFunction*.

The chain is formed by the following functions, called successively:

1. `combsimp()`
2. `trigsimp()`
3. `expand()`
4. `simplify()`

EXAMPLES:

We consider variables that are coordinates of a chart on a complex manifold:

```
sage: forget()  # for doctest only
sage: M = Manifold(2, 'M', structure='topological', field='complex', calc_method=
↪'sympy')
sage: X.<x,y> = M.chart()
```

Then neither `x` nor `y` is assumed to be real:

```
sage: assumptions()
[]
```

Accordingly, `simplify_chain_generic_sympy` does not simplify `sqrt(x^2)` to `abs(x)`:

```
sage: from sage.manifolds.utilities import simplify_chain_generic_sympy
sage: s = (sqrt(x^2))._sympy_()
sage: simplify_chain_generic_sympy(s)
sqrt(x**2)
```

This contrasts with the behavior of *simplify_chain_real_sympy()*.

Other simplifications:

```
sage: s = ((x+y)^2 - x^2 -2*x*y - y^2)._sympy_()
sage: simplify_chain_generic_sympy(s)
0
sage: s = ((x^2 - 2*x + 1) / (x^2 -1))._sympy_()
sage: simplify_chain_generic_sympy(s)
(x - 1)/(x + 1)
```

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```
sage: s = (cos(2*x) - 2*cos(x)^2 + 1)._sympy_()
sage: simplify_chain_generic_sympy(s)
0
```

sage.manifolds.utilities.**simplify_chain_real**(*expr*)

Apply a chain of simplifications to a symbolic expression, assuming the real domain.

This is the simplification chain used in calculus involving coordinate functions on real manifolds, as implemented in *ChartFunction*.

The chain is formed by the following functions, called successively:

1. simplify_factorial()
2. simplify_trig()
3. simplify_rational()
4. *simplify_sqrt_real()*
5. *simplify_abs_trig()*
6. canonicalize_radical()
7. simplify_log()
8. simplify_rational()
9. simplify_trig()

EXAMPLES:

We consider variables that are coordinates of a chart on a real manifold:

```
sage: M = Manifold(2, 'M', structure='topological')
sage: X.<x,y> = M.chart('x:(0,1) y')
```

The following assumptions then hold:

```
sage: assumptions()
[x is real, x > 0, x < 1, y is real]
```

and we have:

```
sage: from sage.manifolds.utilities import simplify_chain_real
sage: s = sqrt(y^2)
sage: simplify_chain_real(s)
abs(y)
```

The above result is correct since y is real. It is obtained by simplify_real() as well:

```
sage: s.simplify_real()
abs(y)
sage: s.simplify_full()
abs(y)
```

Furthermore, we have:

```
sage: s = sqrt(x^2-2*x+1)
sage: simplify_chain_real(s)
-x + 1
```

which is correct since $x \in (0, 1)$. On this example, neither simplify_real() nor simplify_full(), nor canonicalize_radical() give satisfactory results:

```
sage: s.simplify_real()  # unsimplified output
sqrt(x^2 - 2*x + 1)
sage: s.simplify_full()  # unsimplified output
sqrt(x^2 - 2*x + 1)
sage: s.canonicalize_radical()  # wrong output since x in (0,1)
x - 1
```

Other simplifications:

```
sage: s = abs(sin(pi*x))
sage: simplify_chain_real(s)  # correct output since x in (0,1)
sin(pi*x)
sage: s.simplify_real()  # unsimplified output
abs(sin(pi*x))
sage: s.simplify_full()  # unsimplified output
abs(sin(pi*x))
```

```
sage: s = cos(y)^2 + sin(y)^2
sage: simplify_chain_real(s)
1
sage: s.simplify_real()  # unsimplified output
cos(y)^2 + sin(y)^2
sage: s.simplify_full()  # OK
1
```

sage.manifolds.utilities.**simplify_chain_real_sympy**(*expr*)

Apply a chain of simplifications to a sympy expression, assuming the real domain.

This is the simplification chain used in calculus involving coordinate functions on real manifolds, as implemented in *ChartFunction*.

The chain is formed by the following functions, called successively:

1. combsimp()
2. trigsimp()
3. *simplify_sqrt_real()*
4. *simplify_abs_trig()*
5. expand()
6. simplify()

EXAMPLES:

We consider variables that are coordinates of a chart on a real manifold:

```
sage: forget()  # for doctest only
sage: M = Manifold(2, 'M', structure='topological',calc_method='sympy')
sage: X.<x,y> = M.chart('x:(0,1) y')
```

The following assumptions then hold:

```
sage: assumptions()
[x is real, x > 0, x < 1, y is real]
```

and we have:

```
sage: from sage.manifolds.utilities import simplify_chain_real_sympy
sage: s = (sqrt(y^2))._sympy_()
sage: simplify_chain_real_sympy(s)
Abs(y)
```

Furthermore, we have:

```
sage: s = (sqrt(x^2-2*x+1))._sympy_()
sage: simplify_chain_real_sympy(s)
1 - x
```

Other simplifications:

```
sage: s = (abs(sin(pi*x)))._sympy_()
sage: simplify_chain_real_sympy(s)  # correct output since x in (0,1)
sin(pi*x)
```

```
sage: s = (cos(y)^2 + sin(y)^2)._sympy_()
sage: simplify_chain_real_sympy(s)
1
```

sage.manifolds.utilities.**simplify_sqrt_real**(*expr*)

Simplify `sqrt` in symbolic expressions in the real domain.

EXAMPLES:

Simplifications of basic expressions:

```
sage: from sage.manifolds.utilities import simplify_sqrt_real
sage: simplify_sqrt_real( sqrt(x^2) )
abs(x)
sage: assume(x<0)
sage: simplify_sqrt_real( sqrt(x^2) )
-x
sage: simplify_sqrt_real( sqrt(x^2-2*x+1) )
-x + 1
sage: simplify_sqrt_real( sqrt(x^2) + sqrt(x^2-2*x+1) )
-2*x + 1
```

This improves over `canonicalize_radical()`, which yields incorrect results when `x < 0`:

```
sage: forget()  # removes the assumption x<0
sage: sqrt(x^2).canonicalize_radical()
x
sage: assume(x<0)
sage: sqrt(x^2).canonicalize_radical()
-x
sage: sqrt(x^2-2*x+1).canonicalize_radical() # wrong output
x - 1
sage: ( sqrt(x^2) + sqrt(x^2-2*x+1) ).canonicalize_radical() # wrong output
-1
```

Simplification of nested `sqrt`'s:

```
sage: forget()  # removes the assumption x<0
sage: simplify_sqrt_real( sqrt(1 + sqrt(x^2)) )
sqrt(abs(x) + 1)
```

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```
sage: assume(x<0)
sage: simplify_sqrt_real( sqrt(1 + sqrt(x^2)) )
sqrt(-x + 1)
sage: simplify_sqrt_real( sqrt(x^2 + sqrt(4*x^2) + 1) )
-x + 1
```

Again, `canonicalize_radical()` fails on the last one:

```
sage: (sqrt(x^2 + sqrt(4*x^2) + 1)).canonicalize_radical()
x - 1
```

`sage.manifolds.utilities.`**`xder`**(*form*)

Exterior derivative of a differential form.

INPUT:

- `form` – a differential form; this must an instance of either
  - *`DiffScalarField`* for a 0-form (scalar field)
  - *`DiffFormParal`* for a p-form ($p \geq 1$) on a parallelizable manifold
  - *`DiffForm`* for a a p-form ($p \geq 1$) on a non-parallelizable manifold

OUTPUT:

- the $(p+1)$-form that is the exterior derivative of `form`

EXAMPLES:

Exterior derivative of a scalar field (0-form):

```
sage: from sage.manifolds.utilities import exterior_derivative
sage: M = Manifold(3, 'M')
sage: X.<x,y,z> = M.chart()
sage: f = M.scalar_field({X: x+y^2+z^3}, name='f')
sage: df = exterior_derivative(f); df
1-form df on the 3-dimensional differentiable manifold M
sage: df.display()
df = dx + 2*y dy + 3*z^2 dz
```

An alias is `xder`:

```
sage: from sage.manifolds.utilities import xder
sage: df == xder(f)
True
```

Exterior derivative of a 1-form:

```
sage: a = M.one_form(name='a')
sage: a[:] = [x+y*z, x-y*z, x*y*z]
sage: da = xder(a); da
2-form da on the 3-dimensional differentiable manifold M
sage: da.display()
da = (-z + 1) dx/\dy + (y*z - y) dx/\dz + (x*z + y) dy/\dz
sage: dda = xder(da); dda
3-form dda on the 3-dimensional differentiable manifold M
sage: dda.display()
dda = 0
```

See also:

sage.manifolds.differentiable.diff_form.DiffFormParal.exterior_derivative or *sage.manifolds.differentiable.diff_form.DiffForm.exterior_derivative* for more examples.

CHAPTER FIVE

MANIFOLDS CATALOG

A catalog of manifolds to rapidly create various simple manifolds.

The current entries to the catalog are obtained by typing `manifolds.<tab>`, where `<tab>` indicates pressing the tab key. They are:

- `Sphere()`: sphere embedded in Euclidean space
- `Torus()`: torus embedded in Euclidean space
- `Minkowski()`: 4-dimensional Minkowski space
- `Kerr()`: Kerr spacetime

AUTHORS:

- Florentin Jaffredo (2018) : initial version

`sage.manifolds.catalog.`**`Kerr`**(*m=1*, *a=0*, *coordinates='BL'*, *names=None*)

Generate a Kerr spacetime.

A Kerr spacetime is a 4 dimensional manifold describing a rotating black hole. Two coordinate systems are implemented: Boyer-Lindquist and Kerr (3+1 version).

The shortcut operator `.<,>` can be used to specify the coordinates.

INPUT:

- `m` – (default: `1`) mass of the black hole in natural units ($c = 1$, $G = 1$)
- `a` – (default: `0`) angular momentum in natural units; if set to `0`, the resulting spacetime corresponds to a Schwarzschild black hole
- `coordinates` – (default: `"BL"`) either `"BL"` for Boyer-Lindquist coordinates or `"Kerr"` for Kerr coordinates (3+1 version)
- `names` – (default: `None`) name of the coordinates, automatically set by the shortcut operator

OUTPUT:

- Lorentzian manifold

EXAMPLES:

```
sage: m, a = var('m, a')
sage: K = manifolds.Kerr(m, a)
sage: K
4-dimensional Lorentzian manifold M
sage: K.atlas()
[Chart (M, (t, r, th, ph))]
sage: K.metric().display()
```

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```
g = (2*m*r/(a^2*cos(th)^2 + r^2) - 1) dt*dt
 + 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dt*dph
 + (a^2*cos(th)^2 + r^2)/(a^2 - 2*m*r + r^2) dr*dr
 + (a^2*cos(th)^2 + r^2) dth*dth
 + 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dph*dt
 + (2*a^2*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) + a^2 + r^2)*sin(th)^2 dph*dph

sage: K.<t, r, th, ph> = manifolds.Kerr()
sage: K
4-dimensional Lorentzian manifold M
sage: K.metric().display()
g = (2/r - 1) dt*dt + r^2/(r^2 - 2*r) dr*dr
 + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: K.default_chart().coord_range()
t: (-oo, +oo); r: (0, +oo); th: (0, pi); ph: [-pi, pi] (periodic)

sage: m, a = var('m, a')
sage: K.<t, r, th, ph> = manifolds.Kerr(m, a, coordinates="Kerr")
sage: K
4-dimensional Lorentzian manifold M
sage: K.atlas()
[Chart (M, (t, r, th, ph))]
sage: K.metric().display()
g = (2*m*r/(a^2*cos(th)^2 + r^2) - 1) dt*dt
 + 2*m*r/(a^2*cos(th)^2 + r^2) dt*dr
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dt*dph
 + 2*m*r/(a^2*cos(th)^2 + r^2) dr*dt
 + (2*m*r/(a^2*cos(th)^2 + r^2) + 1) dr*dr
 - a*(2*m*r/(a^2*cos(th)^2 + r^2) + 1)*sin(th)^2 dr*dph
 + (a^2*cos(th)^2 + r^2) dth*dth
 - 2*a*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2) dph*dt
 - a*(2*m*r/(a^2*cos(th)^2 + r^2) + 1)*sin(th)^2 dph*dr
 + (2*a^2*m*r*sin(th)^2/(a^2*cos(th)^2 + r^2)
 + a^2 + r^2)*sin(th)^2 dph*dph
sage: K.default_chart().coord_range()
t: (-oo, +oo); r: (0, +oo); th: (0, pi); ph: [-pi, pi] (periodic)
```

`sage.manifolds.catalog.`**`Minkowski`**(*positive_spacelike=True*, *names=None*)

Generate a Minkowski space of dimension 4.

By default the signature is set to $(- + ++)$, but can be changed to $(+ - --)$ by setting the optionnal argument `positive_spacelike` to `False`. The shortcut operator `.<,>` can be used to specify the coordinates.

INPUT:

- `positive_spacelike` – (default: `True`) if `False`, then the spacelike vectors yield a negative sign (i.e., the signature is $(+ - --)$)
- `names` – (default: `None`) name of the coordinates, automatically set by the shortcut operator

OUTPUT:

- Lorentzian manifold of dimension 4 with (flat) Minkowskian metric

EXAMPLES:

```
sage: M.<t, x, y, z> = manifolds.Minkowski()
sage: M.metric()[:]
[-1  0  0  0]
```

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```
[ 0  1  0  0]
[ 0  0  1  0]
[ 0  0  0  1]

sage: M.<t, x, y, z> = manifolds.Minkowski(False)
sage: M.metric()[:]
[ 1  0  0  0]
[ 0 -1  0  0]
[ 0  0 -1  0]
[ 0  0  0 -1]
```

`sage.manifolds.catalog.Sphere`(*dim=None*, *radius=1*, *names=None*, *stereo2d=False*, *stereo_lim=None*)

Generate a sphere embedded in Euclidean space.

The shortcut operator `.<,>` can be used to specify the coordinates.

INPUT:

- `dim` – (optional) the dimension of the sphere; if not specified, equals to the number of coordinate names
- `radius` – (default: `1`) radius of the sphere
- `names` – (default: `None`) name of the coordinates, automatically set by the shortcut operator
- `stereo2d` – (default: `False`) if `True`, defines only the stereographic charts, only implemented in 2d
- `stereo_lim` – (default: `None`) parameter used to restrict the span of the stereographic charts, so that they don't cover the whole sphere; valid domain will be `x**2 + y**2 < stereo_lim**2`

OUTPUT:

- Riemannian manifold

EXAMPLES:

```
sage: S.<th, ph> = manifolds.Sphere()
sage: S
2-dimensional pseudo-Riemannian submanifold S embedded in
 3-dimensional differentiable manifold E^3
sage: S.atlas()
[Chart (S, (th, ph))]
sage: S.metric().display()
gamma = dth*dth + sin(th)^2 dph*dph

sage: S = manifolds.Sphere(2, stereo2d=True)  # long time
sage: S  # long time
2-dimensional pseudo-Riemannian submanifold S embedded in
 3-dimensional differentiable manifold E^3
sage: S.metric().display()  # long time
gamma = 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dx*dx
 + 4/(x^4 + y^4 + 2*(x^2 + 1)*y^2 + 2*x^2 + 1) dy*dy
```

`sage.manifolds.catalog.Torus`(*R=2*, *r=1*, *names=None*)

Generate a 2-dimensional torus embedded in Euclidean space.

The shortcut operator `.<,>` can be used to specify the coordinates.

INPUT:

- `R` – (default: `2`) distance form the center to the center of the tube

- `r` – (default: `1`) radius of the tube
- `names` – (default: `None`) name of the coordinates, automatically set by the shortcut operator

OUTPUT:

- Riemannian manifold

EXAMPLES:

```
sage: T.<theta, phi> = manifolds.Torus(3, 1)
sage: T
2-dimensional pseudo-Riemannian submanifold M embedded in
 3-dimensional differentiable manifold E^3
sage: T.atlas()
[Chart (M, (theta, phi))]
sage: T.embedding().display()
M --> E^3
   (theta, phi) |--> (X, Y, Z) = ((cos(theta) + 3)*cos(phi),
                                  (cos(theta) + 3)*sin(phi),
                                  sin(theta))
sage: T.metric().display()
gamma = dtheta*dtheta + (cos(theta)^2 + 6*cos(theta) + 9) dphi*dphi
```

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