Bread Crust Week Problem

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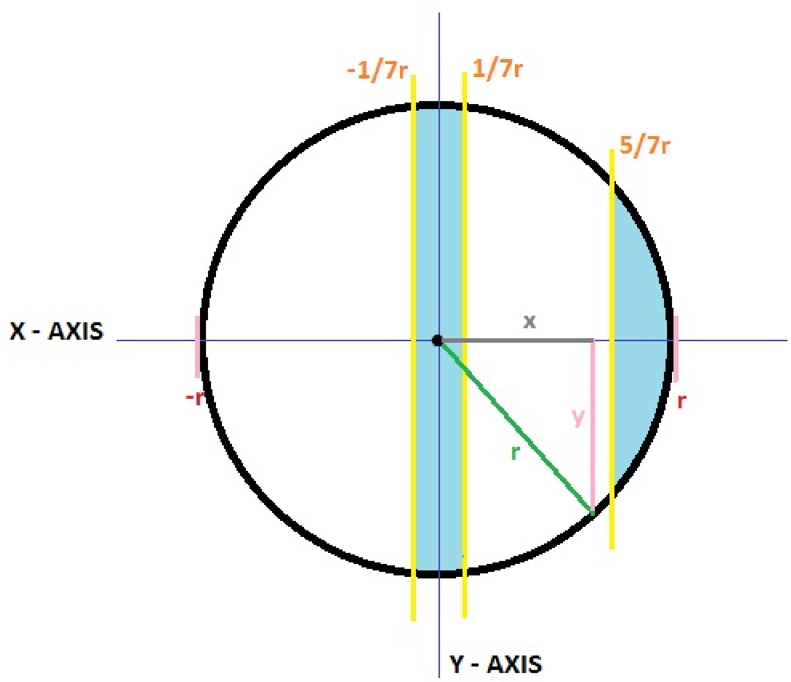
1 Given Information and Question

Spherical loaf of bread, cut into 7 pieces (equally thick).
How much crust does the outside piece have compared to the middle piece? (surface area of outside piece:surface area of middle piece; assume that crust is very thin)

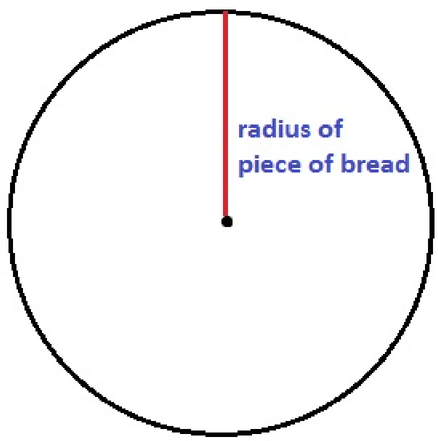
2 Setting Up



The figure above is a representation of the sphere and how it is cut. The blue in the middle is the middle piece of bread, the blue at the end on the right is the outside piece.
From above, r can be calculated in terms of x and y, using the Pythagorean's Theorem:

$$x^2 + y^2 = r^2 \tag{1}$$

It can also be stated that x is equal to y, $x = y$, because the bread is a sphere. As for the radius of the piece(s) of bread, the figure below has been drawn for the help of understanding and clarity. Imagine the piece of bread that has been cut is taken out and viewed from the side (where the crust can be seen as a ring around the bread meat).

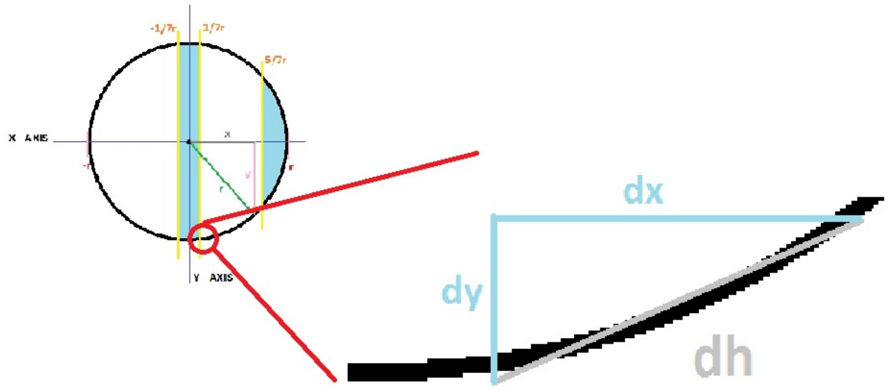


The radius of the piece of bread is the same as length y, which is the same as length x. So from equation 1, the equation can easily be reorganized to find the radius length, which from now on will be called x to avoid confusion (r is radius of spherical bread):

$$x = \sqrt{r^2 - x^2} \tag{2}$$

Due to the spherical shape of the bread, the crust from the pieces in the middle cannot be simply found by finding the surface area of a cylinder. This is because there is a slanted edge on the crust, because of the curvature of the sphere. The curved edge will need to be taken into account when finding the surface area of the crust.

In this part, it is assumed that the slant in the edge is a straight line, instead of a curve. The supposed line is called dh and it can be put into terms of dx



and dy, also by the Pythagorean's Theorem:

$$dy^2 + dx^2 = dh^2 \tag{3}$$

$$dh = \sqrt{dy^2 + dx^2} \tag{4}$$

With the equations above, the surface area of the crust of a piece of bread cut from point a to point b of the spherical loaf can be modelled.

3 Modeling Equation

From the second picture, it was seen that the crust is a circular ring. To model the surface area A_S of the spherical bread is to find how much crust is altogether (how many rings stack up and what is the total area). The surface area of the crust of one infinitely thin piece of bread is the circumference of that bread, which is the circumference of a circle which is $2\pi r$.
In this case, r is the radius of the piece of bread, which is said to be x; therefore, the circumference is $2\pi x$.
To model the surface area of the crust of a certain piece of bread with a width from a to b, the difference in the height or dh has to be multiplied to the circumference. This is explained by this equation:

$$A_S = \int_a^b 2\pi x dh \tag{5}$$

Since dh is known (equation 4), it can be plugged into the equation above and simplified:

$$A_S = 2\pi \int_a^b x\sqrt{dy^2 + dx^2}$$

$$= 2\pi \int_a^b x\sqrt{dy^2 + dx^2} * (dx\sqrt{\frac{1}{dx^2}})$$

$$= 2\pi \int_a^b x\sqrt{\frac{dy^2}{dx^2} + \frac{dx^2}{dx^2}}dx$$

$$= 2\pi \int_a^b x\sqrt{\frac{dy^2}{dx^2} + 1}dx$$

$$A_S = 2\pi \int_a^b x\sqrt{(\frac{dy}{dx})^2 + 1}dx \tag{6}$$

Knowing that x and y are equal, we can find $\frac{dy}{dx}$ by deriving the second equation, defined for x:

$$y = x = \sqrt{r^2 - x^2} = (r^2 - x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{\frac{-1}{2}} * (-2x)$$

$$= \frac{1}{2}(\frac{1}{\sqrt{r^2 - x^2}})(-2x)$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}} \tag{7}$$

Having found $\frac{dy}{dx}$, it can be plugged into equation 6 and the modeling equation can be continued to be simplified:

$$A_S = 2\pi \int_a^b xdx\sqrt{(\frac{dy}{dx})^2 + 1}$$

$$= 2\pi \int_a^b x\sqrt{(\frac{-x}{\sqrt{r^2 - x^2}})^2 + 1}dx$$

$$= 2\pi \int_a^b x\sqrt{\frac{x^2}{r^2 - x^2} + 1}dx$$

$$= 2\pi \int_a^b x\sqrt{\frac{x^2}{r^2 - x^2} + \frac{r^2 - x^2}{r^2 - x^2}}dx$$

$$= 2\pi \int_a^b x\sqrt{\frac{r^2}{r^2 - x^2}}dx$$

$$= 2\pi \int_a^b rx\frac{1}{\sqrt{r^2 - x^2}}dx$$

Because r is a constant, it can be taken outside of the integral, giving us a simplified modeling equation:

$$A_S = 2\pi r \int_a^b \frac{x}{\sqrt{r^2 - x^2}}dx \tag{8}$$

4 Middle Piece

The modeling equation (equation 8) can be used to find the amount of crust for the middle piece. Since the spherical bread is divided into seven pieces, and if the center of the sphere is considered to be 0, then the boundary of the integral a to b should be $-\frac{1}{7}r$ and $\frac{1}{7}r$. (See first picture at top of paper.)

$$\begin{aligned} A_S &= 2\pi r \int_{-\frac{1}{7}r}^{\frac{1}{7}r} \frac{x}{\sqrt{r^2 - x^2}} dx \\ &= 2 * 2\pi r \int_{0}^{\frac{1}{7}r} \frac{x}{\sqrt{r^2 - x^2}} dx \\ &= 4\pi r \int_{0}^{\frac{1}{7}r} \frac{x}{\sqrt{r^2 - x^2}} dx \end{aligned}$$

To easily integrate this, it can be noticed that the portion inside the integral is the same expression as the derivative of y, $\frac{dy}{dx}$, except on the other end of the spectrum (the derivative was negative, the integral portion is positive). This means that the integral of the above equation is equal to $-y$, which is also $-x$ (equation 2).

$$A_S = 4\pi r(-\sqrt{r^2 - x^2} \mid_a^b) \tag{9}$$

for $a = 0$ and $b = \frac{1}{7}r$

$$\begin{aligned} A_S &= 4\pi r[(-(\sqrt{r^2 - (\frac{1}{7}r)^2})) - (-\sqrt{r^2 - 0})] \\ &= 4\pi r(-(\sqrt{r^2 - (\frac{1}{49}r^2)}) + \sqrt{r^2}) \\ &= 4\pi r\sqrt{-r^2 + \frac{1}{49}r^2 + r^2} \\ &= 4\pi r\sqrt{\frac{1}{49}r^2} \\ &= 4\pi r * \frac{1}{7}r \end{aligned}$$

$$A_S = \frac{4}{7}\pi r^2 \tag{10}$$

The above value is the amount of crust on the middle piece.

5 Outside Piece

For the outside piece, it is also the same method as the middle piece (deriving answer from modeling equation, equation 8), except the boundaries are now between $\frac{5}{7}r$ and r. (See first picture at top of paper.)

$$A_S = 2\pi r \int_{\frac{5}{7}r}^{r} \frac{x}{\sqrt{r^2 - x^2}} dx$$

Again,the integral is the negative value of y or x (equation 2).

$$A_S = 2\pi r(-\sqrt{r^2 - x^2} \mid_a^b) \tag{11}$$

for $a = \frac{5}{7}r$ and $b = r$

$$A_S = 2\pi r[(-(\sqrt{r^2 - (r)^2})) - (-\sqrt{r^2 - (\frac{5}{7}r)^2})]$$

$$= 2\pi r(\sqrt{r^2 - (\frac{25}{49}r^2)})$$

$$= 2\pi r\sqrt{r^2 - (\frac{25}{49}r^2)}$$

$$= 2\pi r\sqrt{\frac{24}{49}r^2}$$

$$= 2\pi r * \frac{2\sqrt{6}}{7}r$$

$$A_S = \frac{4\sqrt{6}}{7}\pi r^2 \tag{12}$$

Above is the value for the amount of crust on the outside piece.

6 Conclusion

Comparing the two values, the outside piece has $\sqrt{6}$ times more crust than the middle piece.