An explicit, generic parameterization for the Shallue–van de Woestijne map

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1 Introduction

In this note, we derive an explicit mapping based on the work of Shallue and van de Woestijne [SvdW06] that can be applied to essentially any elliptic curve over a base field $\mathbb{F}$ satisfying $\#\mathbb{F} > 5$. Our derivation is similar to the one described by Fouque and Tibouchi [FT10], but applies more generally. In particular, because the work of Fouque and Tibouchi focuses only on pairing-friendly curves in the Barreto-Naehrig family [BN06], it is restricted to curves $\mathrm{E}(\mathbb{F}_p) : y^2 = x^3 + b$ satisfying $p \equiv 1 \bmod 3$. In contrast, the mapping given in this note does not restrict p, and applies to any non-singular curve $\mathrm{E}(\mathbb{F}_p) : y^2 = x^3 + ax + b$. In other words, we require only that $4a^3 + 27b^2 \neq 0 \in \mathbb{F}_p$.

This note owes a textual debt to [WB19], particularly in the description of the background and notation. For a brief survey of related work, see Section 1.1 of that paper.

2 Background

Notation. We write $\mathrm{E}(\mathbb{F})$ for the group (in multiplicative notation) of rational points on elliptic curve E over field $\mathbb{F}$; this group's order is $\#\mathrm{E}(\mathbb{F})$.

$\mathrm{Sgn}_0(\beta)$ is a function that returns the "sign" of β. For $\beta \in \mathbb{F}_p$, let $\mathrm{Sgn}_0(\beta) = -1$ if $\beta > (p-1)/2$, and 1 otherwise. For extensions of $\mathbb{F}_p$, Sgn_0 generalizes in a natural way.

We regard the square root in $\mathbb{F}$ as a function, so we fix a canonical representation, namely, $\beta \triangleq \sqrt{\alpha} \in \mathbb{F}$ such that $\mathrm{Sgn}_0(\beta) = 1$.

2.1 The Shallue–van de Woestijne map

For any elliptic curve $\mathrm{E}(\mathbb{F}) : y^2 = f(x) = x^3 + ax + b$, $\#\mathbb{F} > 5$, Shallue and van de Woestijne give a map from $L \subseteq \mathbb{F}$ to the curve $\mathrm{E}(\mathbb{F})$ [SvdW06]. They observe, generalizing and simplifying the result of Skałba [Ska05], that for any rational point on the threefold

$$V(\mathbb{F}) : f(x_1)f(x_2)f(x_3) = x_4^2$$

such that $x_4 \neq 0$, at least one of $f(x_j), j \in \{1, 2, 3\}$ must be a square. This implies that one of the x_j is the x-coordinate of a rational point on $\mathrm{E}(\mathbb{F})$.

To construct a rational point on $V(\mathbb{F})$, the authors define the surface $S(\mathbb{F})$ and the rational map $\phi_1 : S(\mathbb{F}) \mapsto V(\mathbb{F})$, which is invertible on its image [SvdW06, Lemma 6]:

$$S(\mathbb{F}) : y^2 \left(u^2 + uv + v^2 + a\right) = -f(u)$$

$$\phi_1 : (u, v, y) \mapsto \left(v, -u - v, u + y^2, f(u + y^2) \cdot \frac{y^2 + uv + v^2 + a}{y}\right).$$

Next, the authors observe [SvdW06, Lemma 7] that fixing $u = u_0$ satisfying $f(u_0) \neq 0$ and $3u_0^2 + 4a \neq 0$ specializes $S(\mathbb{F})$ to a curve that is birational to a conic with a rational parameterization. This gives a rational map $\phi_2 : \mathbb{A}^1 \mapsto S(\mathbb{F})$ that is invertible on its image.

Putting it all together, define $L = \{t \in \mathbb{F} : \phi_1(\phi_2(t)) \text{ is defined}\}$. Then, to map $t \in L$ to E($\mathbb{F}$), first compute $\phi_1(\phi_2(t))$, which is a rational point (x_1, x_2, x_3, x_4) on $V(\mathbb{F})$, so at least one $f(x_j), j \in \{1, 2, 3\}$ is square. Choose the smallest j where this is the case, compute the corresponding y-coordinate, and return (x_j, y).

3 A generic parameterization

We now give a generic Shallue–van de Woestijne mapping (§2.1) for the elliptic curve E($\mathbb{F}$) : $y^2 = f(x) = x^3 + ax + b$. To begin, we work with $S(\mathbb{F})$ generically in terms of $u = u_0$; we discuss how to choose u_0 below. Rewriting as in [SvdW06, Lemma 7]:

$$y^2\left(\frac{3}{4}u_0^2 + \left(v + \frac{u_0}{2}\right)^2\right) = -f(u_0) - ay^2$$

$$\frac{z^2}{f(u_0)} + w^2 = -\frac{3u_0^2 + 4a}{4f(u_0)} \qquad \text{where } z = v + \frac{u_0}{2}, \quad w = \frac{1}{y}$$

If u_0 is chosen such that the RHS is square, a solution to the above equation is given by $(z_0, w_0) \triangleq (0, \sqrt{-(3u_0^2 + 4a)/(4f(u_0))})$. Setting $w = w_0 + tz$ and substituting yields

$$2tw_0 f(u_0) + (1 + t^2 f(u_0))z = 0 \qquad z \neq 0$$

$$z = -\frac{2tw_0 f(u_0)}{1 + t^2 f(u_0)}$$

$$w = w_0 + tz = w_0 \frac{1 - t^2 f(u_0)}{1 + t^2 f(u_0)}$$

Solving for y and v,

$$y = \frac{1}{w} = \frac{1}{w_0} \cdot \frac{1 + t^2 f(u_0)}{1 - t^2 f(u_0)}$$

$$v = z - \frac{u_0}{2} = -\frac{u_0}{2} - \frac{2tw_0 f(u_0)}{1 + t^2 f(u_0)}$$

Finally, from the map ϕ_1 (§2.1), we have

$$x_1 = v = -\frac{u_0}{2} - \frac{2tw_0 f(u_0)}{1 + t^2 f(u_0)}$$

$$x_2 = -u_0 - v = -\frac{u_0}{2} + \frac{2tw_0 f(u_0)}{1 + t^2 f(u_0)}$$

$$x_3 = u_0 + y^2 = u_0 + \left(\frac{1}{w_0} \cdot \frac{1 + t^2 f(u_0)}{1 - t^2 f(u_0)}\right)^2$$

This map is undefined when $t^2 f(u_0) = \pm 1$. To handle this case, we start by applying Montgomery's trick [Mon87], i.e., evaluating the above map in one inversion by computing the inverse of $\left(1 + t^2 f(u_0)\right)\left(1 - t^2 f(u_0)\right) = 1 - t^4 f(u_0)^2$. If one's inversion algorithm returns zero on input zero (as is true, e.g., for inversion by Fermat's little theorem), then in the exceptional cases we will have $x_1 = x_2 = -u_0/2$ and $x_3 = u_0$. Thus, if u_0 is chosen such that $f(u_0)$ or $f(-u_0/2)$ is square in $\mathbb{F}$, the map will be exception-free.

Putting it all together. Fix u_0 such that $f(u_0) \neq 0 \in \mathbb{F}$, $-(3u_0^2 + 4a)/(4f(u_0))$ is a nonzero square in $\mathbb{F}$, and $f(u_0)$ or $f(-u_0/2)$ is square in $\mathbb{F}$. On input $t \in \mathbb{F}$, evaluate the x_j with Montgomery's trick, using an inversion function that returns 0 on input 0. Finally, compute the result as follows:

$$(x, y) = \begin{cases} \left(x_1, \sqrt{f(x_1)}\right) & \text{if } f(x_1) \text{ is square} \in \mathbb{F}\,\text{; else} \\ \left(x_2, \sqrt{f(x_2)}\right) & \text{if } f(x_2) \text{ is square} \in \mathbb{F}\,\text{; else} \\ \left(x_3, \sqrt{f(x_3)} \cdot \mathrm{Sgn}_0(t)\right) & \text{otherwise} \end{cases}$$

The reason that only the third case involves multiplication by $\mathrm{Sgn}_0(t)$ is that x_1 and x_2 depend on the sign of t, while x_3 does not. Thus, if $j \in \{1, 2\}$, t and $-t$ yield distinct points. We ensure the same when $j = 3$ by fixing $\mathrm{Sgn}_0(y) = \mathrm{Sgn}_0(t)$.

Does a suitable u_0 exist? We show that, under a seemingly mild assumption, a suitable u_0 can be found with overwhelming probability for any curve of cryptographic interest.

Consider that case that $a \neq 0$. Then u_0 meets all requirements if $f(u_0)$ and $-3u_0^2 - 4a$ are nonzero squares. Consider the sets $U \triangleq \{u_0 : f(u_0) \text{ is a nonzero square} \in \mathbb{F}\}$ and $V \triangleq \{-3u_0^2 - 4a : u_0 \in U\}$. Because $f(\cdot)$ has degree 3, $|U|$ is a constant fraction of $\#\mathbb{F}$; similarly, by the definition of V, $|V|$ is a constant fraction of $|U|$, and thus of $\#\mathbb{F}$. If any element of V is a nonzero square in $\mathbb{F}$, the corresponding value of u_0 is suitable by construction. Under the assumption that $\Pr\left[-3u_0^2 - 4a \text{ is square} \in \mathbb{F}\right]$ is independent of $\Pr\left[f(u_0) \text{ is square} \in \mathbb{F}\right]$, with overwhelming probability a suitable u_0 exists: half of the elements of $\mathbb{F}$ are square, and thus the probability that V does not contain a square is $2^{-|V|}$, which is at worst polynomially smaller than $2^{-\#\mathbb{F}}$.

Next, consider the case that $a = 0$ and $\mathbb{F} \triangleq \mathbb{F}_p, p \equiv 2 \bmod 3$. The condition on p guarantees that -3 is nonsquare in $\mathbb{F}_p$, so u_0 must be chosen such that $f(u_0)$ is nonzero and nonsquare in $\mathbb{F}_p$ in order to satisfy the condition that $-3u_0^2/4f(u_0)$ is square in $\mathbb{F}_p$. Consider the sets $S \triangleq \{u_0 : f(u_0) \text{ is a nonzero nonsquare} \in \mathbb{F}_p\}$ and $T \triangleq \{f(-u_0/2) : u_0 \in S\}$. By an argument similar to the above, $|T|$ is a constant fraction of $\#\mathbb{F}_p$. If any element of T is a nonzero square in $\mathbb{F}_p$, the corresponding value of u_0 is suitable by construction. Under the assumption that $\Pr\left[f(-u_0/2) \text{ is square} \in \mathbb{F}_p\right]$ is independent of $\Pr\left[f(u_0) \text{ is nonsquare} \in \mathbb{F}_p\right]$, with overwhelming probability a suitable u_0 exists.

Finally, consider the case that $a = 0$ and $\mathbb{F} \triangleq \mathbb{F}_p, p \equiv 1 \bmod 3$. The condition on p guarantees that -3 is square in $\mathbb{F}_p$, so any u_0 such that $f(u_0)$ is square in $\mathbb{F}_p$ is suitable.

We note that the assumptions stated above appear to be true in practice, but to our knowledge they are not easily proved.

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