Assignment 1

Q1a

convolving f with g: $(I*F)[i,j] = \sum_{k,l} I[i-k,j-l]F[k,l]$

$k,l \in [-1,1]$

$$(I*F)[0,0] = \sum_{k,l} I[0-k,0-l]F[k,l]$$

$$= I[1,1]F[-1,-1] + I[1,0]F[-1,0] + I[1,-1]F[-1,1]$$
$$+ I[0,1]F[0,-1] + I[0,0]F[0,0] + I[0,-1]F[0,1]$$
$$+ I[-1,1]F[1,-1] + I[-1,0]F[1,0] + I[-1,-1]F[1,1]$$
$$= (-2)(-1) + (1)(-1) + 0$$
$$+ 0 + (-1)(1) + 0$$
$$+ 0 + 0 + 0 = +2 - 1 - 1 = 0$$

$$(I*F)[0,1] = I[1,2]F[-1,-1] + I[1,1]F[-1,0] + I[1,0]F[-1,1]$$
$$+ I[0,2]F[0,-1] + I[0,1]F[0,0] + I[0,0]F[0,1]$$
$$+ I[-1,2]F[1,-1] + I[-1,1]F[1,0] + I[-1,0]F[1,1]$$
$$= (1)(-1) + (-2)(-1) + (1)(-1)$$
$$+ (2)(1) + 0 + (-1)(1)$$
$$+ 0 + 0 + 0 = \cancel{-1} + \cancel{2}\cancel{-1} + 2 - 1 = 1$$

$$(I*F)[0,2] = I[1,3]F[-1,-1] + I[1,2]F[-1,0] + I[1,1]F[-1,1]$$
$$+ I[0,3]F[0,-1] + I[0,2]F[0,0] + I[0,1]F[0,1]$$
$$+ I[-1,3]F[1,-1] + I[-1,2]F[1,0] + I[-1,1]F[1,1]$$
$$= 0 + (1)(-1) + (-2)(-1)$$
$$+ 0 + (2)(1) + 0$$
$$+ 0 + 0 + 0 = -1 + 2 + 2 = 3$$

$$I = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -1 & 0 & 2 \\ 1 & -2 & 1 \end{bmatrix} \end{array} \qquad F = \begin{array}{c c} k \backslash l & \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{array}{c} -1 \\ 0 \\ 1 \end{array} & \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$(I*F)[1,0] = \sum_{k,l} I[0-k,0-l]F[k,l]$$
$$= I[2,1]F[-1,-1] + I[2,0]F[-1,0] + I[2,-1]F[-1,1]$$
$$+ I[1,1]F[0,-1] + I[1,0]F[0,0] + I[1,-1]F[0,1]$$
$$+ I[0,1]F[1,-1] + I[0,0]F[1,0] + I[0,-1]F[1,1]$$
$$= 0 + 0 + 0$$
$$+ (-2)(1) + (1)(1) + 0$$
$$+ 0 + (-1)(0) + 0 = -2+1 = -1$$

$$(I*F)[1,1] = \sum_{k,l} I[0-k,0-l]F[k,l]$$
$$= I[2,2]F[-1,-1] + I[2,1]F[-1,0] + I[2,0]F[-1,1]$$
$$+ I[1,2]F[0,-1] + I[1,1]F[0,0] + I[1,0]F[0,1]$$
$$+ I[0,2]F[1,-1] + I[0,1]F[1,0] + I[0,0]F[1,1] \Rightarrow 0 \text{ for this row}$$
$$= 0 + 0 + 0$$
$$+ (1)(1) + (-2)(1) + (1)(1) = 1-2+1 = 0$$
$$+ 0 + 0 + 0$$

$$F = \begin{array}{c|ccc} k \backslash l & -1 & 0 & 1 \\ \hline -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{array}$$

$$I = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & -1 & 0 & 2 \\ 1 & 1 & -2 & 1 \end{array}$$

$$(I*F)[1,2] = \sum_{k,l} I[0-k,0-l]F[k,l]$$
$$= I[2,3]F[-1,-1] + I[2,2]F[-1,0] + I[2,1]F[-1,1]$$
$$+ I[1,3]F[0,-1] + I[1,2]F[0,0] + I[1,1]F[0,1]$$
$$+ I[0,3]F[1,-1] + I[0,2]F[1,0] + I[0,1]F[1,1]$$
$$= 0 + 0 + 0$$
$$+ 0 + (1)(1) + (-2)(1)$$
$$+ 0 + 0 + 0 = 1-2 = -1$$

$$(I*F) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 3 \\ 1 & -1 & 0 & -1 \end{array}$$

1b

1D conv: $(I * F_n)[i] = \sum_k I[i-k] F_n[k]$

$$F_1 = \begin{array}{c} -1 \\ 0 \\ 1 \end{array}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad I = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & -1 & 0 & 2 \\ 1 & 1 & -2 & 1 \end{array}$$

$$(I*F_1)[0,0] = \sum_k I[0-k,0] F_1[k]$$
$$= I[1,0]F_1[-1] + I[0,0]F_1[0] + I[-1,0]F_1[1]$$
$$= (1)(-1) + (-1)(1) + 0 = -2$$
$$(I*F_1)[1,0] = I[2,0]F_1[-1] + I[1,0]F_1[0] + I[0,0]F_1[1]$$
$$= 0 + (1)(1) + (-1)(0) = 1$$
$$(I*F_1)[0,1] = I[1,1]F_1[-1] + I[0,1]F_1[0] + I[-1,1]F_1[1]$$
$$= (-2)(-1) + 0 + 0 = 2$$
$$(I*F_1)[1,1] = I[2,1]F_1[-1] + I[1,1]F_1[0] + I[0,1]F_1[1]$$
$$= 0 + (-2)(1) + 0 = -2$$
$$(I*F_1)[0,2] = I[1,2]F_1[-1] + I[0,2]F_1[0] + I[-1,2]F_1[1]$$
$$= (1)(-1) + (2)(1) + 0 = -1 + 2 = 1$$
$$(I*F_1)[1,2] = I[2,2]F_1[-1] + I[1,2]F_1[0] + I[0,2]F_1[1]$$
$$= 0 + (1)(1) + (2)(0) = 1$$

$$I * F_1 = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & -2 & 2 & 1 \\ 1 & 1 & -2 & 1 \end{array} = I'$$

$$F_2 = \begin{array}{ccc} -1 & 0 & 1 \\ \hline 1 & 1 & 1 \end{array}$$

1D conv: $(I * F_n)[i] = \sum_k I[i-k] F_n[k]$

$$(I' * F_2)[0,0] = \sum_\ell I'[0, j-\ell] F_2[\ell]$$
$$= I'[0,1]F_2[-1] + I'[0,0]F_2[0] + I'[0,-1]F_2[1]$$
$$= (2) + (-2) = 0$$
$$(I'*F_2)[0,1] = I'[0,2] + I'[0,1] + I'[0,0] = 1 + 2 - 2 = 1$$
$$(I'*F_2)[0,2] = I'[0,3] + I'[0,2] + I'[0,1] = 0 + 1 + 2 = 3$$
$$(I'*F_2)[1,0] = I'[1,1] + I'[1,0] + I'[1,-1] = -2 + 1 = -1$$
$$(I'*F_2)[1,1] = I'[1,2] + I'[1,1] + I'[1,0] = 1 - 2 + 1 = 0$$
$$(I'*F_2)[1,2] = I'[1,3] + I'[1,2] + I'[1,1] = 0 + 1 - 2 = -1$$

$$(I*F_1)*F_2 = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -1 \end{bmatrix}$$

$$(I*F_1)*F_2[0,0] = \sum_\ell F_2[\ell] I'[0, j-\ell]$$

$$(I*F_1)[0,0] = \sum_k I[0-k,0] F_1[k]$$
$$(I*F)[0,1] = \sum_k I[0-k,1] F_1[k]$$

1c

ID conv: $(I * F_n)[i] = \sum_k I[i-k] F_n[k]$

Prove for $F = F_1 F_2$, $I * F = (I * F_1) * F_2$

$\boxed{F = F_1 F_2}$

$\boxed{[I * F_n][i,j] = \sum_k I[i-k, j] F_n[k]}$

$F[k,\ell] = F_1[k] F_2[\ell]$

↳ def of inner product

$$(I * F)[i,j] = \sum_k \sum_\ell I[i-k, j-\ell] F[k,\ell]$$

$$= \sum_k \sum_\ell I[i-k, j-\ell] (F_1[k])(F_2[\ell])$$

$$= \sum_\ell F_2[\ell] \underbrace{\sum_k I[i-k, j-\ell] F_1[k]}_{(I * F_1)[i, \underbrace{j-\ell}_{\text{constant}}]}$$

$$\to \sum_\ell \left(\left(\sum_k I[i-k, j] F_1[k]\right)[j-\ell]\right) F_2[\ell]$$

$$= \sum_\ell F_2[\ell] \left[(I * F_1)[i, j-\ell]\right]$$

$$= (I * F_1) * F_2$$

2a)

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$\frac{d}{dx} g_\sigma(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left(\frac{-2x}{2\sigma^2}\right) = -\frac{x}{\sqrt{2\pi}\,\sigma^3} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$\frac{d^2}{dx^2} g_\sigma(x) = \frac{-1}{\sqrt{2\pi}\,\sigma^3} \exp\left(-\frac{x^2}{2\sigma^2}\right) - \frac{x}{\sqrt{2\pi}\,\sigma^3} \cdot \left(\exp\left(-\frac{x^2}{2\sigma^2}\right) \frac{-\cancel{2}x}{\cancel{2}\sigma^2}\right)$$

$$= \exp\left(\frac{-x^2}{2\sigma^2}\right)\left[\frac{x^2}{\sqrt{2\pi}\,\sigma^5} - \frac{1}{\sqrt{2\pi}\,\sigma^3}\right]$$

$$= \exp\left(-\frac{x^2}{2\sigma^2}\right)\left[\frac{x^2 - \sigma^2}{\sqrt{2\pi}\,\sigma^5}\right]$$