In the lecture notes, we have:

$$ {}_{(12)3}\langle p_{12}' \; p_3' \; \alpha' \; M_L \,|\, p_{12} \; p_3 \; \alpha \; M_L)_{(23)1} $$

by rotational inv.

$$ = \frac{1}{(2L+1)} \sum_{M_L} {}_{(12)3}\langle p_{12}' \; p_3' \; \alpha' \; M_L \,|\, p_{12} \; p_3 \; \alpha \; M_L)_{(23)1} \qquad \forall L = L' $$

$$ = \frac{1}{(2L+1)} \sum_{M_L} {}_{(12)3}\langle p_{12}' \, p_3' \, \alpha' \, M_L | \cdot \int d^3\tilde{p}_{12}' \, d^3\tilde{p}_3' \; |\vec{\tilde{p}}_{12}{}' \, \vec{\tilde{p}}_3{}' \rangle \langle \vec{\tilde{p}}_{12}{}' \, \vec{\tilde{p}}_3{}' | $$

$$ \int d^3\tilde{p}_{12} \; d^3\tilde{p}_3 \quad |\vec{\tilde{p}}_{12} \; \vec{\tilde{p}}_3 \rangle \langle \vec{\tilde{p}}_{12} \; \vec{\tilde{p}}_3 | \cdot \quad |p_{12} \; p_3 \; \alpha M_L)_{(23)1} $$

$$ = \frac{1}{(2L+1)} \sum_{M_L} \int_{d^3\tilde{p}_{12} \, d^3\tilde{p}_3} d^3\tilde{p}_{12}' \, d^3\tilde{p}_3' \; \langle p_{12}' p_3' \alpha' M_L | \vec{\tilde{p}}_{12}{}' \, \vec{\tilde{p}}_3{}' \rangle \; \langle \vec{\tilde{p}}_{12}{}' \, \vec{\tilde{p}}_3{}' \,|\, \vec{\tilde{p}}_{12} \; \vec{\tilde{p}}_3 \rangle \; \langle \vec{\tilde{p}}_{12} \; \vec{\tilde{p}}_3 | \; p_{12} p_3 \alpha M_L) $$

$$ \langle \vec{\tilde{p}}_{12}{}' \, \vec{\tilde{p}}_3{}' \,|\, \vec{\tilde{p}}_{12} \; \vec{\tilde{p}}_3 \rangle = \delta^{(3)}\left(\vec{p}_{23} - \tfrac{1}{2} \vec{\tilde{p}}_{12}{}' + \tfrac{3}{4} \vec{\tilde{p}}_3{}' \right) \delta^{(3)}\left(\vec{p}_1 - \vec{\tilde{p}}_{12}{}' + \tfrac{1}{2} \vec{\tilde{p}}_3{}' \right) $$

$$ * \quad |p_{12} \; p_3 \; \alpha M_L) = \sum_{m_{12} m_3} \underbrace{(l_{12} \; l_3 \; L, \; m_{12} m_3 \; M_L)}_{=: \, C_L} \; |p_{12} \; l_{12} \; m_{12}) \, |p_3 l_3 \; m_3) $$

with $\alpha = (l_{12} \; l_3) L$

and:

$$ \langle \vec{\tilde{p}}_{12} \; \vec{\tilde{p}}_3 \,|\, p_{12} p_3 \; \alpha M_L) $$

$$ = \sum_{m_{12} m_3} C_L \cdot \langle \vec{\tilde{p}}_{12} | p_{12} \; l_{12} \; m_{12}) \; \langle \vec{\tilde{p}}_3 | p_3 \; l_3 \; m_3) $$

$$ = \frac{\delta(\tilde{p}_{12} - p_{12})}{\tilde{p}_{12} \, p_{12}} \cdot \frac{\delta(\tilde{p}_3 - p_3)}{\tilde{p}_3 \; p_3} \sum_{m_{12} m_3} C_L \cdot Y_{l_{12} m_{12}}(\hat{p}_{12}) \; Y_{l_3, m_3}(\hat{p}_3) $$

$$ =: \frac{\delta(\tilde{p}_{12} - p_{12})}{\tilde{p}_{12} \, p_{12}} \cdot \frac{\delta(\tilde{p}_3 - p_3)}{\tilde{p}_3 \; p_3} \cdot \mathcal{Y}_{l_{12} \; l_3}^{L \, M_L} \left(\hat{p}_{12}, \; \hat{p}_3 \right) $$

where we get the second line in lecture 9, page 6.

So in fermionic case, we consider the spin.

$\Rightarrow$ our state looks like:

$$|p_{12}\ p_3 \underbrace{\left(\overset{\nearrow \alpha_{12}}{(l_{12}\ s_{12})\ j_{12}}\ \overset{\nearrow \alpha_3}{(l_3\ s_3)\ j_3} \right) J}_{=:\tilde{\alpha}}\ M_L \rangle$$

$$= |p_{12}\ p_3\ \alpha_{12}\ \alpha_3\ M_L\rangle =: |p_{12}\ p_3\ \tilde{\alpha}\ M_L\rangle$$

We want:

$\equiv$ permutation operator

$${}_{(12)3}\langle p_{12}'\ p_3'\ \alpha'\ M\ |\ p_{12}\ p_3\ \alpha\ m\rangle_{(23)1}$$

where $\alpha = \left((l_{12}\ s_{12})\ j_{12}\ (l_3\ s_3)\ j_3 \right) J$, $\quad \alpha' = \left((l_{12}'\ s_{12}')\ j_{12}'\ (l_3'\ s_3')\ j_3 \right) J' \right)$

$$= \langle p_{12}'\ p_3' \left((l_{12}'\ s_{12}')\ j_{12}'\ (l_3'\ s_3')\ j_3' \right) J\ M\ |\ \sum_{L'S'} |p_{12}'\ p_3' \left((l_{12}'\ l_3')\ L'\ (s_{12}'\ s_3')\ S' \right) J\ M\rangle$$

completeness, given in notes.

$$\langle p_{12}'\ p_3' \left((l_{12}'\ l_3')\ L'\ (s_{12}'\ s_3')\ S' \right) J\ M\ |\ \sum_{LS} |p_{12}\ p_3 \left((l_{12}\ l_3)\ L\ (s_{12}\ s_3)\ S \right) J\ M\rangle$$

$$\langle p_{12}\ p_3 \left((l_{12}\ l_3)\ L\ (s_{12}\ s_3)\ S \right) J\ M\ |\ p_{12}\ p_3 \left((l_{12}\ l_3)\ j_{12}\ (s_{12}\ s_3)\ j_3 \right) J\ M\rangle$$

$$= \sum_{LS} \sum_{L'S'} \sqrt{\hat{J}_{12}'\ \hat{J}_3'\ \hat{L}'\ \hat{S}'} \begin{Bmatrix} l_{12}' & s_{12}' & j_{12}' \\ l_3' & s_3' & j_3' \\ L' & S' & J \end{Bmatrix} \cdot \sqrt{\hat{J}_{12}\ \hat{J}_3\ \hat{L}\ \hat{S}} \begin{Bmatrix} l_{12} & s_{12} & j_{12} \\ l_3 & s_3 & j_3 \\ L & S & J \end{Bmatrix}$$

$$\langle p_{12}'\ p_3' \left((l_{12}'\ l_3')\ L',\ (s_{12}'\ s_3')\ S' \right) J\ M\ |\ p_{12}\ p_3 \left((l_{12}\ l_3)\ L,\ (s_{12}\ s_3)\ S \right) J\ M\rangle$$

Now the system is rotationally invariant wrt Spin & ang. mom.

rot. invariance

$$= \frac{1}{(2S+1)(2L+1)} \sum_{M_S} \sum_{M_L} \sum_{LS} \sqrt{\hat{J}_{12}\ \hat{J}_{12}'\ \hat{J}_3\ \hat{J}_3'\ \hat{L}^2\ \hat{S}^2} \begin{Bmatrix} l_{12}' & s_{12}' & j_{12}' \\ l_3' & s_3' & j_3' \\ L & & J \end{Bmatrix} \begin{Bmatrix} l_{12} & s_{12} & j_{12} \\ l_3 & s_3 & j_3 \\ L & S & J \end{Bmatrix}$$

$$\langle p_{12}'\ p_3' \left((l_{12}'\ l_3')\ L,\ (s_{12}'\ s_3')\ S \right) J\ M\ |\ p_{12}\ p_3 \left((l_{12}\ l_3)\ L,\ (s_{12}\ s_3)\ S \right) J\ M\rangle$$

where this holds only when $L = L'$, $S = S'$

Now we can decouple the state into angular & spin states separately.

$$|p_{12}\, p_3\, ((l_{12}\, l_3)\, L, (s_{12}\, s_3)\, S\,)\, J\, M\rangle = |p_{12}\, p_3\, (l_{12} l_3) L\, M_L\rangle |(s_{12}\, s_3)\, S\, M_s\rangle$$

completeness with m_l, m_s states.

$$= \sum_{m_{12}} \sum_{m_3} \sum_{m_{s_{12}}} \sum_{m_{s_3}} |p_{12}\, l_{12}\, m_{12}\rangle\, |p_3\, l_3\, m_3\rangle\, |s_{12}\, m_{s_{12}}\rangle \cdot |s_3\, m_{s_3}\rangle$$

$$\underbrace{\langle p_{12}\, l_{12}\, m_{12}, p_3 l_3 m_3 | p_{12}\, p_3\, ((l_{12}\, l_3)\, L M_L)\rangle}_{\text{Orbital part, C-G}} =: C_l$$

$$\underbrace{\langle s_{12}\, m_{s_{12}}, s_3\, m_{s_3} | (s_{12}\, s_3)\, S\, M_S\rangle}_{\text{spin part, CG coeff,}} =: C_s$$

$$=: \sum_{m_{12} m_3} \sum_{m_{s_{12}} m_{s_3}} |p_{12}\, l_{12}\, m_{12}\rangle\, |p_3 l_3\, m_3\rangle\, |s_{12}\, m_{s_{12}}\rangle\, |s_3\, m_{s_3}\rangle\, C_l\, C_s$$

Now projecting with an arbitrary 3D state with spin $|\tilde{\vec{p}}_{12}\, \tilde{\vec{p}}_3\, \sigma_{12}\, \sigma_3\rangle$:

$$\Rightarrow \langle \tilde{\vec{p}}_{12}\, \tilde{\vec{p}}_3\, \sigma_{12}\, \sigma_3 | p_{12}\, p_3\, ((l_{12}\, l_3)\, L, (s_{12}\, s_3)\, S\,)\, J\, M\rangle$$

$$= \sum_{m_{12} m_3} C_l \langle \tilde{\vec{p}}_{12} | p_{12} l_{12}\, m_{12}\rangle \langle \tilde{\vec{p}}_3 | p_3 l_3\, m_3\rangle \sum_{m_{s_{12}} m_{s_3}} C_s \langle \sigma_{12} | s_{12}\, m_{s_{12}}\rangle \langle \sigma_3 | s_3\, m_{s_3}\rangle$$

So:

$$\langle p_{12}'\, p_3'\, ((l_{12}'\, l_3')\, L, (s_{12}'\, s_3')\, S\,)\, J M | p_{12}\, p_3\, ((l_{12}\, l_3)\, L, (s_{12}\, s_3)\, S\,)\, J\, M\rangle$$

completeness ↓

$$= \sum_{\substack{\sigma_{12}\sigma_3 \\ \sigma_{12}'\sigma_3'}} \int d^3\tilde{p}_{12}\, d^3\tilde{p}_3\, d^3\tilde{p}_{12}'\, d^3\tilde{p}_3' \langle p_{12}'\, p_3'\, [(l_{12}'\, l_3')L, (s_{12}'\, s_3')S]\, J\, M | \tilde{\vec{p}}_{12}'\, \tilde{\vec{p}}_3'\, \sigma_{12}'\, \sigma_3'\rangle$$

$$\langle \tilde{\vec{p}}_{12}'\, \tilde{\vec{p}}_3' | \tilde{\vec{p}}_{12}\, \tilde{\vec{p}}_3\rangle \langle \sigma_{12}'\, \sigma_3' | \sigma_{12}\, \sigma_3\rangle$$

$$\langle \tilde{\vec{p}}_{12}\, \tilde{\vec{p}}_3\, \sigma_{12}\, \sigma_3 | p_{12}\, p_3\, [(l_{12}\, l_3)L, (s_{12}\, s_3)\, S]\, J\, M\rangle$$

$$= \sum_{\substack{\sigma_{12}\sigma_3 \\ \sigma_{12}'\sigma_3'}} \int d^3\tilde{p}_{12}\, d^3\tilde{p}_3\, d^3\tilde{p}_{12}'\, d^3\tilde{p}_3'\, \delta^{(3)}(\tilde{\vec{p}}_{12}' - \tilde{\vec{p}}_{12})\, \delta^{(3)}(\tilde{\vec{p}}_3' - \tilde{\vec{p}}_3) \cdot \delta_{\sigma_{12}'\sigma_{12}}\, \delta_{\sigma_3'\sigma_3}$$

$$\sum_{m_{12} m_3} C_l \langle \tilde{\vec{p}}_{12} | p_{12} l_{12}\, m_{12}\rangle \langle \tilde{\vec{p}}_3 | p_3 l_3\, m_3\rangle \sum_{m_{s_{12}} m_{s_3}} C_s \langle \sigma_{12} | s_{12}\, m_{s_{12}}\rangle \langle \sigma_3 | s_3\, m_{s_3}\rangle$$

$$\sum_{m_{12}'m_3'} C_\ell \langle p_{12}'\ell_{12}' m_{12}' | \tilde{\vec{p}}_{12}{}' \rangle \langle p_3' \ell_3' m_3' | \tilde{\vec{p}}_3{}' \rangle \sum_{m_{s_{12}}' m_{s_3}'} C_s \langle s_{12}' m_{s_{12}}' | \sigma_{12}' \rangle \langle s_3' m_{s_3}' | \sigma_3' \rangle$$

$$= \Big[\int d^3\tilde{p}_{12}\, d^3\tilde{p}_3\, d^3\tilde{p}_{12}{}'\, d^3\tilde{p}_3{}' \; \delta^{(3)}\!\left(\vec{p}_{23} - \tfrac{1}{2}\vec{p}_{12}{}' + \tfrac{3}{4}\vec{p}_3{}' \right) \delta^{(3)}\!\left(\vec{p}_1 - \vec{p}_{12}{}' + \tfrac{1}{2}\vec{p}_3{}' \right)$$

$$\sum_{m_{12}m_3} C_\ell \langle \tilde{\vec{p}}_{12} | p_{12}\ell_{12} m_{12} \rangle \langle \tilde{\vec{p}}_3 | p_3 \ell_3 m_3 \rangle \sum_{m_{12}'m_3'} C_\ell^* \langle p_{12}'\ell_{12}' m_{12}' | \tilde{\vec{p}}_{12}{}' \rangle \langle p_3' \ell_3' m_3' | \tilde{\vec{p}}_3{}' \rangle \Big]$$

orbital part

$$\Big[\sum_{\sigma_{12}\sigma_3} \sum_{m_{s_{12}} m_{s_3}} C_s \langle \sigma_{12} | s_{12} m_{s_{12}} \rangle \langle \sigma_3 | s_3 m_{s_3} \rangle \sum_{m_{s_{12}}' m_{s_3}'} C_s^{*\prime} \langle s_{12}' m_{s_{12}}' | \sigma_{12} \rangle \langle s_3' m_{s_3}' | \sigma_3 \rangle \Big]$$

spin part

The orbital part is evaluated like in the lecture notes.

For the spin part:

→ there must be some way to represent spin states:
like $\langle \vec{\sigma} | s, m_s \rangle = ?$

→ maybe we can take spin in z-direction without loss of generality?

→ otherwise we can use the completeness relation again:

$$\sum_{m_{s_{12}} m_{s_3}} \sum_{m_{s_{12}}' m_{s_3}'} C_s C_s^{*\prime} \langle s_{12}' m_{s_{12}}' | \sum_{\sigma_{12}} |\sigma_{12}\rangle\langle\sigma_{12}| \, |s_{12}, m_{s_{12}}\rangle$$
$$\cdot \langle s_3', m_{s_3}' | \sum_{\sigma_3} |\sigma_3\rangle\langle\sigma_3| \, |s_3\, m_{s_3}\rangle$$

$$= \sum_{m_{s_{12}} m_{s_3}} \sum_{m_{s_{12}}' m_{s_3}'} C_s C_s^{*\prime} \underbrace{\langle s_{12}' m_{s_{12}}' | s_{12}\, m_{s_{12}} \rangle}_{=\delta_{s_{12}' s_{12}} \delta_{m_{s_{12}} m_{s_{12}}'}} \underbrace{\langle s_3' m_{s_3}' | s_3\, m_{s_3} \rangle}_{=\delta_{s_3' s_3} \delta_{m_{s_3}' m_{s_3}}}$$

$$= \sum_{m_{s_{12}} m_{s_3}} C_s \cdot C_s^{*\prime} \, \delta_{s_{12} s_{12}'} \, \delta_{s_3 s_3'} = \sum_{m_{s_{12}} m_{s_3}} C_s \cdot C_s^* = \sum_{m_{s_{12}} m_{s_3}} |C_s|^2$$

$$= \sum_{m_{s_{12}} m_{s_3}} \left| \langle s_{12}\, m_{s_{12}}, s_3\, m_{s_3} | (s_{12}\, s_3)\, S\, M_S \rangle \right|^2$$