General Definitions:

Wigner 9j-Symbols (Def. I.)

$$|(j_1 j_4) j_7, (j_2 j_5) j_8; \overset{j_9}{J} M\rangle := \sum_{j_3, j_6} |(j_1 j_2) j_3, (j_4 j_5) j_6; \overset{j_9}{J} M\rangle \sqrt{\hat{j}_3 \hat{j}_6 \hat{j}_7 \hat{j}_8} \underbrace{\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}}_{\hat{=}\ 9j\text{-symbol}} \quad (1)$$

J

with
$$\begin{cases} \hat{j}_3 := 2j_3 + 1 \\ \hat{j}_6 := 2j_6 + 1 \\ \hat{j}_7 := 2j_7 + 1 \\ \hat{j}_8 := 2j_8 + 1 \end{cases}$$

Wigner 6j-symbols (Def. II.)

$$|j_1, j_{23}; JM\rangle := \sum_{j_{12}} |j_{12}, j_3; JM\rangle (-1)^{j_1 + j_2 + j_3 + J} \sqrt{\hat{j}_{12} \hat{j}_{23}} \underbrace{\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix}}_{\hat{=}\ 6j\text{-symbol}} \quad (2)$$

with
$$\begin{cases} \hat{j}_{12} := 2j_{12} + 1 \\ \hat{j}_{23} := 2j_{23} + 1 \end{cases}$$

In order to account for the spin-contribution we have to derive the permutation operator for arbitrary spins.

So, we start writing down the permutation operator using the partial-wave eigenbasis

$$\boxed{|P_{12} P_3 ((\ell_{12} S_{12}) J_{12}, (\ell_3 S_3) I_3) J\rangle \equiv |P_{12} P_3 \overbrace{\alpha_{12} \alpha_3}^{\equiv \alpha}\rangle} \quad (3)$$

Basis states in the three fermion case

with
$$\begin{cases} \alpha_{12} := (\ell_{12} S_{12}) J_{12}\ J \\ \alpha_3 := (\ell_3 S_3) I_3\ J \end{cases}$$

see "3-fermion-1.png" in your repository. From the lecture notes "Lecture09.pdf", p.6, we know that

~~We can write the permutation operator in partial wave representation as~~

$$\cancel{\frac{1}{2L+1}\sum_{M} \langle p_{12}' p_3' \alpha' M | p_{12} p_3 \alpha M \rangle = 8\pi^2 \int_{-1}^{1} dx \, \frac{\delta(p_{12} - \pi_{12}(p_3' p_3 x))}{p_{12}^2} \, \frac{\delta(p_{12}' - \pi_{12}'(p_3' p_3 x))}{p_{12}'^2} \, \frac{1}{2L+1}\sum_{M} \mathcal{Y}^{*L'M'}_{l_{12}' l_3'}(\dots)\, \mathcal{Y}^{LM}_{l_{12} l_3}(\dots) =: G_{\alpha'\alpha}(p_3' p_3 x)}$$

We can write the <u>permutation operator</u> (<u>without</u> spin contribution) in partial wave representation as

$$\frac{1}{2L+1}\sum_{M} \langle p_{12}' p_3' \alpha_L' M | p_{12} p_3 \alpha_L M \rangle = \int d\hat{p}_3 \, d\hat{p}_3' \, \frac{\delta(p_{12} - |-\vec{p}_3{}' - \frac{1}{2}\vec{p}_3|)}{p_{12}^2} \, \frac{\delta(p_{12}' - |\vec{p}_3 + \frac{1}{2}\vec{p}_3{}'|)}{p_{12}'^2}$$

$$\times \frac{1}{2L+1}\sum_{M} \mathcal{Y}^{*L'M'}_{l_{12}' l_3'}(\dots)\, \mathcal{Y}^{LM}_{l_{12} l_3}(\dots)$$

$$\equiv 8\pi^2 \int_{-1}^{1} dx \, \frac{\delta(p_{12} - \pi_{12}(p_3' p_3 x))}{p_{12}^2} \, \frac{\delta(p_{12}' - \pi_{12}'(p_3' p_3 x))}{p_{12}'^2}$$

$$\times \frac{1}{2L+1}\sum_{M} \mathcal{Y}^{*L'M'}_{l_{12}' l_3'}(\dots)\, \mathcal{Y}^{LM}_{l_{12} l_3}(\dots) \qquad (4)$$

<u>with</u> $\int d\hat{p}_3 \, d\hat{p}_3' \longrightarrow 8\pi^2 \int_{-1}^{1} dx$,

see "Lecture09.pdf", p. 6

<u>Note</u> that in Eq. (4) we have

$$\boxed{\alpha_L := (l_{12} l_3) L}$$

Next, we set up the <u>permutation operator</u> <u>with</u> spin contribution and then we use Eq. (1), Eq. (2) and Eq. (4) to <u>decouple</u> the <u>orbital part</u> and the <u>spin part</u>.

The permutation operator in partial wave representation with respect to spin can be constructed from the partial-wave eigenbasis, cf. Eq. (3):

$$\langle p_{12}' p_3' \overset{\equiv \alpha_{12}'\alpha_3'}{\alpha'} M' | p_{12}\, p_3 \overset{\equiv \alpha_{12}\alpha_3}{\alpha} M \rangle$$

was neglected in the notation above

$$= \langle p_{12}' p_3' \alpha_{12}' \alpha_3' M' | p_{12} p_3 \alpha_{12} \alpha_3 M \rangle$$

$$= \langle p_{12}' p_3' ((l_{12}' s_{12}') J_{12}', (l_3' s_3') I_3') J M | p_{12} p_3 ((l_{12} s_{12}) J_{12}, (l_3 s_3) I_3) J M \rangle$$

$$\boxed{\text{Now, we insert appropriate full sets of states in order to do the decoupling procedure (note that we drop sum signs here for reasons of clarity)}}$$

$$\overset{!}{=} \langle p_{12}' p_3' ((l_{12}' s_{12}') J_{12}', (l_3' s_3') I_3') J M |$$

$$\underbrace{| p_{12}' p_3' ((l_{12}' l_3') L', (s_{12}' s_3') S') J M \rangle \langle p_{12}' p_3' ((l_{12}' l_3') L', (s_{12}' s_3') S') J M |}_{= \mathbb{1}}$$

$$\underbrace{| p_{12} p_3 ((l_{12} l_3) L, (s_{12} s_3) S) J M \rangle \langle p_{12} p_3 ((l_{12} l_3) L, (s_{12} s_3) S) J M |}_{= \mathbb{1}}$$

$$| p_{12} p_3 ((l_{12} s_{12}) J_{12}, (l_3 s_3) I_3) J M \rangle$$

$$= \underbrace{\langle p_{12}' p_3' ((l_{12}' s_{12}') J_{12}', (l_3' s_3') I_3') J M | p_{12}' p_3' ((l_{12}' l_3') L', (s_{12}' s_3') S') J M \rangle}$$

Use Def. I. (Eq. (1)) to rewrite this in terms of 9j-Symbols

$$\times \underbrace{\langle p_{12}' p_3' ((l_{12}' l_3') L', (s_{12}' s_3') S') J M | p_{12} p_3 ((l_{12} l_3) L, (s_{12} s_3) S) J M \rangle}$$

$$= \underbrace{\langle p_{12}' p_3' (l_{12}' l_3') L' | p_{12} p_3 (l_{12} l_3) L \rangle}_{\text{see Eq. (4)}} \delta_{LL'} \underbrace{\langle (s_{12}' s_3') S' | (s_{12} s_3) S \rangle}_{\text{use Def II (Eq. (2)) to rewrite this in terms of 6j-Symbols}} \delta_{SS'}$$

$$\times \underbrace{\langle p_{12} p_3 ((l_{12} l_3) L, (s_{12} s_3) S) J M | p_{12} p_3 ((l_{12} s_{12}) J_{12}, (l_3 s_3) I_3) J M \rangle}$$

Use Def. I. (Eq. (1)) to rewrite this in terms of 9j-Symbols

$$= \sum_{L'S'} \sqrt{\hat{L}' \hat{S}' \hat{J}'_{12} \hat{I}'_3} \begin{Bmatrix} l'_{12} & s'_{12} & J'_{12} \\ l'_3 & s'_3 & I'_3 \\ L' & S' & J \end{Bmatrix}$$

$$\times \underbrace{\langle p'_{12} p'_3 (l'_{12} l'_3) L' | p_{12} p_3 (l_{12} l_3) L \rangle}_{\text{see Eq. (4)}} \delta_{LL'} \underbrace{\langle (s'_{12} s'_3) S' | (s_{12} s_3) S \rangle}_{\text{Def. II.}} \delta_{SS'}$$

$$\times \sum_{LS} \sqrt{\hat{L}\hat{S}\hat{J}_{12}\hat{I}_3} \begin{Bmatrix} l_{12} & s_{12} & J_{12} \\ l_3 & s_3 & I_3 \\ L & S & J \end{Bmatrix}$$

we have $s_3 = s'_3 = \frac{1}{2}$

using this and Def. II. we obtain

$$\langle (s'_{12} s'_3) S' | (s_{12} s_3) S \rangle = (-1)^{s_{12}} \sqrt{\hat{s}'_{12} \hat{s}_{12}} \begin{Bmatrix} \overset{s_3}{\tfrac{1}{2}} & \tfrac{1}{2} & s'_{12} \\ \underset{s'_3}{\tfrac{1}{2}} & S & s_{12} \end{Bmatrix}$$

the phase is changed because of re-ordering of the coupling

$$(s_{12} s_3) S \longrightarrow (s_3 s_{12}) S$$

This is needed to use

$$|s_3 s_{12} S\rangle = \sum_{s_{12}} |s'_{12} s'_3 S\rangle \underbrace{\langle s'_{12} s'_3 S | s_3 s_{12} S \rangle}_{\sqrt{\hat{s}'_{12} \hat{s}_{12}}\, (-1)^{\cdots} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s'_{12} \\ \frac{1}{2} & S' & s_{12} \end{Bmatrix}}$$

$\hat{L}^2$ $\hat{S}^2$ due to $\delta_{LL'}, \delta_{SS'}$

$$= \sum_{L'S'} \sum_{LS} \sqrt{\hat{L}'\hat{L}\hat{S}'\hat{S}\hat{J}'_{12}\hat{J}_{12}\hat{I}'_3\hat{I}_3} \begin{Bmatrix} l'_{12} & s'_{12} & J'_{12} \\ l'_3 & s'_3 & I'_3 \\ L' & S' & J \end{Bmatrix} \begin{Bmatrix} l_{12} & s_{12} & J_{12} \\ l_3 & s_3 & I_3 \\ L & S & J \end{Bmatrix}$$

$$\times (-1)^{s_{12}} \sqrt{\hat{s}'_{12} \hat{s}_{12}} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s'_{12} \\ \frac{1}{2} & S' & s_{12} \end{Bmatrix} \delta_{LL'} \delta_{SS'}$$

$$\times \underbrace{\frac{8\pi^2}{2L+1}}_{=\hat{L}} \int_{-1}^{1} dx \, \frac{\delta(p_{12} - \pi_{12}(p'_3 p_3 x))}{p_{12}^2} \, \frac{\delta(p'_{12} - \pi'_{12}(p'_3 p_3 x))}{p'^2_{12}}$$

$$\times \sum_M Y^{*L'M*}_{l'_{12} l'_3} \, Y^{LM}_{l_{12} l_3}$$

Eq. (4)

$$= \int_{-1}^{1} dx \, \frac{\delta(p_{12} - \pi_{12}(p'_3 p_3 x))}{p_{12}^2} \, \frac{\delta(p'_{12} - \pi'_{12}(p'_3 p_3 x))}{p'^2_{12}} \, G_{\alpha'\alpha}(p'_3 p_3 x)$$

Where we defined the numerical representation of the permutation $G_{\alpha'\alpha}(p_3' p_3 x)$ as

$$G_{\alpha'\alpha}(p_3' p_3 x) := \sum_L \sum_S \hat{S} \sqrt{\hat{j}_{12} \hat{j}_{12}' \hat{I}_3 \hat{I}_3'}$$

$$\times \begin{Bmatrix} l_{12}' & s_{12}' & j_{12}' \\ l_3' & \frac{1}{2} & I_3' \\ L & S' & J \end{Bmatrix} \begin{Bmatrix} l_{12} & s_{12} & j_{12} \\ l_3 & \frac{1}{2} & I_3 \\ L & S' & J \end{Bmatrix}$$

$$\times (-1)^{s_{12}} \sqrt{\hat{s}_{12} \hat{s}_{12}'} \begin{Bmatrix} 1/2 & 1/2 & s_{12}' \\ 1/2 & S' & s_{12} \end{Bmatrix}$$

$$\times 8\pi^2 \sum_M \mathcal{Y}^{*LM}_{l_{12}' l_3'} \mathcal{Y}^{LM}_{l_{12} l_3}$$

Including Isospin (t_{12}, t_{12}', T) leads to the result

$$G_{\alpha'\alpha}(p_3' p_3 x) := \sum_L \sum_S \hat{S} \sqrt{\hat{j}_{12} \hat{j}_{12}' \hat{I}_3 \hat{I}_3'}$$

$$\times \begin{Bmatrix} l_{12}' & s_{12}' & j_{12}' \\ l_3' & \frac{1}{2} & I_3' \\ L & S' & J \end{Bmatrix} \begin{Bmatrix} l_{12} & s_{12} & j_{12} \\ l_3 & \frac{1}{2} & I_3 \\ L & S' & J \end{Bmatrix}$$

$$\times (-1)^{s_{12}} \sqrt{\hat{s}_{12} \hat{s}_{12}'} \begin{Bmatrix} 1/2 & 1/2 & s_{12}' \\ 1/2 & S' & s_{12} \end{Bmatrix}$$

$$\times (-1)^{t_{12}} \sqrt{\hat{t}_{12} \hat{t}_{12}'} \begin{Bmatrix} 1/2 & 1/2 & t_{12}' \\ 1/2 & T & t_{12} \end{Bmatrix}$$

$$\times 8\pi^2 \sum_M \mathcal{Y}^{*LM}_{l_{12}' l_3'} \mathcal{Y}^{LM}_{l_{12} l_3}$$