3 Fermion Project

1)

A 3 fermion system can be described using partial-wave momentum representation:

$$\boxed{|p_{12}\, p_3\, ((l_{12}\, s_{12}) J_{12}\, (l_3 s_3) I_3) J\rangle \equiv |p_{12}\, p_3\, \alpha_{12}\, \alpha_3\rangle}$$

PARTIAL WAVE EIGENBASIS

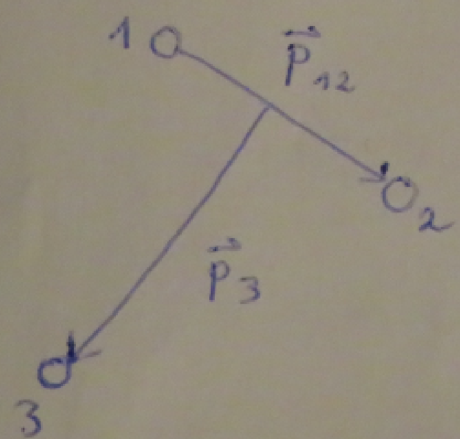
1

$J \hat{=}$ total angular momentum (conserved)

$\alpha_{12} := (l_{12}\, s_{12}) J_{12}\, J$

$\alpha_3 := (l_3\, s_3) I_3\, J$

Jacobi-momenta for 3-particle system:



The total spin s_{12} of fermion 1 and 2:

$$s_{12} = (s_1, s_2), \quad s_1 = s_2 = \frac{1}{2}$$

And the spin s_3 of fermion 3:

$$s_3 = \frac{1}{2}$$

2) Projecting the Fadeev - equation

Starting from the Schrödinger equation for a bound state wave function, we can define a Fadeev component and relate it to the wave fct. (→ lecture 08.pdf, p. 14)

Using the Lippmann - Schwinger equation we can rewrite the expression for the Faddeev component $|\Psi_{12}\rangle$ and find the Faddeev equations

$$\boxed{|\Psi_{12}\rangle = \frac{1}{E - H_0} t_{12} P |\Psi_{12}\rangle}$$

(→ lecture 08.pdf, p. 15)

Using Eq ①: $\langle p_{12}' \, p_3' \underbrace{\alpha_{12}' \, \alpha_3'}_{\equiv \alpha'} |\Psi_{12}\rangle = \Psi_{12}(p_{12}', p_3')$ ← In general $\Psi_\alpha(p_{12} \, p_3 \, \alpha) = \langle p_{12} \, p_3 \, \alpha | \Psi \rangle$ is the application of the permutation operator to any state $|\Psi\rangle$

$$\Rightarrow \underbrace{\langle p_{12}' \, p_3' \, \alpha' | \Psi_{12}\rangle}_{\equiv \Psi_{12}^{\alpha'}(p_{12}' p_3' \cancel{\alpha'})} = \langle p_{12}' \, p_3' \, \alpha' | \frac{1}{E - H_0} t_{12} P |\Psi_{12}\rangle$$

$$= \langle p_{12}' \, p_3' \, \alpha' | \frac{1}{E - H_0} t_{12} | \tilde{p}_{12} \, \tilde{p}_3 \, \tilde{\alpha}\rangle \langle \tilde{p}_{12} \, \tilde{p}_3 \, \tilde{\alpha} | P | p_{12} \, p_3 \, \alpha\rangle \times \underbrace{\langle p_{12} \, p_3 \, \alpha | \Psi_{12}\rangle}_{\equiv \Psi_{12}^{\alpha}(p_{12} \, p_3 \cancel{\alpha})}$$

In the previous step we used the completeness-of-states relation, e.g.

$$\boxed{\int d\tilde{p}_{12} \, \tilde{p}_{12}^2 \underbrace{\sum_{\tilde{\alpha}}}_{\equiv \sum_{\tilde{\alpha}_{12} \tilde{\alpha}_3}} |\tilde{p}_{12} \, \tilde{p}_3 \, \tilde{\alpha}\rangle \langle \tilde{p}_{12} \, \tilde{p}_3 \, \tilde{\alpha}| = \mathbb{1}}$$

$\int d\tilde{p}_3 \, \tilde{p}_3^2$

omitting $\int$ and Σ for reasons of clarity.

Then, we can take the kinetic energy term out.

Kinetic energy: $\boxed{H_0 = \frac{p_{12}^2}{m} + \frac{3 p_3^2}{4m}}$ (→ lecture 09.pdf, p. 8)

Using this as well as the partial wave representation of the permutation operator

$$\boxed{\langle \tilde{p}_{12} \, \tilde{p}_3 \, \tilde{\alpha} | P | p_{12} \, p_3 \, \alpha\rangle}$$ (→ lecture 09.pdf)

~~for $\Psi_{12}^{\alpha'}(p_{12}' p_3')$~~

and the t-matrix embedded in 3-body space (→ lecture 09.pdf, p. 8)

$$\boxed{\langle p_{12}' \, p_3' \, \alpha' | t_{12} | p_{12} \, p_3 \, \alpha\rangle = \delta_{\alpha_3 \alpha_3'} \frac{\delta(p_3 \, p_3')}{p_3 \, p_3'} t_{12}(p_{12}' \alpha_{12}' ; p_{12} \, \alpha_{12} ; p_3)}$$

we obtain for $\Psi_{12}^{\alpha'}(p_{12}' p_3')$:

$$\psi_{12}^{\alpha'}(p'_{12}\, p'_3) = \frac{1}{E - \frac{p_{12}'^2}{m} - \frac{3p_3'^2}{4m}} \int d\tilde{p}_{12}\, \tilde{p}_{12}^2 \int d\tilde{p}_3\, \tilde{p}_3^2 \sum_{\tilde{\alpha}_{12}\tilde{\alpha}_3} \delta_{\alpha'_3\tilde{\alpha}_3} \frac{\delta(p'_3 - \tilde{p}_3)}{p'_3\, \tilde{p}_3} \times t_{12}(p'_{12}\alpha'_{12};\, \tilde{p}_{12}\tilde{\alpha}_{12};\, p'_3)$$

t-matrix in 3-body space

$$\times \sum_{\alpha_{12}\alpha_3} \int dp_{12}\, p_{12}^2 \int dp_3\, p_3^2 \int_{-1}^{1} dx\, \frac{\delta(p_{12} - \pi_{12}(\tilde{p}_3 p_3 x))}{p_{12}^2} \frac{\delta(\tilde{p}_{12} - \pi'_{12}(\tilde{p}_3 p_3 x))}{\tilde{p}_{12}^2} \times G_{\tilde{\alpha}\alpha}(\tilde{p}_3 p_3 x)$$

permutation operator;
$G_{\tilde{\alpha}\alpha}(\tilde{p}_3 p_3 x) \hat{=}$ numerical representation of permutation

$$\times \psi_{12}^{\alpha}(p_{12}\, p_3)$$

Performing the momentum integrals except $\int dp_3\, p_3^2$ allows to simplify this equation to

$$\psi_{12}^{\alpha'}(p'_{12}\, p'_3) = \frac{1}{E - \frac{p_{12}'^2}{m} - \frac{3p_3'^2}{4m}} \sum_{\tilde{\alpha}_{12}\tilde{\alpha}_3} \sum_{\alpha_{12}\alpha_3} \delta_{\tilde{\alpha}_3\alpha'_3} \int_{-1}^{1} dx \int dp_3\, p_3^2$$

$$\times t_{12}(p'_{12}\alpha'_{12};\, \pi'_{12}(p'_3 p_3 x)\, \tilde{\alpha}_{12};\, p'_3) \times G_{\tilde{\alpha}\alpha}(p'_3 p_3 x)$$

$$\times \psi_{12}^{\alpha}(\pi_{12}(p'_3 p_3 x);\, p_3) \quad (2)$$

$\longrightarrow$ check differences betw. this and corresponding expr. in lecture

3) Discretization of $\psi_{12}^{\alpha'}(p'_{12}\, p'_3)$

Similar to lecture09.pdf, p. 9:
We discretize the momenta by using N_p and N_q momenta:

$$\psi_{12}^{\alpha'}(p'_{12}\, p'_3) \equiv \psi_{12}(p'_{12}\, p'_3\alpha') \longrightarrow \psi(p'_i\, q'_j\, \alpha') = \psi(i' + j' N_p + \alpha' N_p N_q)$$

For the shifted momenta we need to do interpolation. We use the representation as sum of fct values

$$f(\pi) = \sum_{i=0}^{N_p - 1} S_i(\pi)\, f(p_i)$$

Introducing momentum-grid points $p_{i'}$, $q_{j'}$, x_k and corresponding integration weights $\omega_{i'}^{p}$, $\omega_{j'}^{q}$, ω_k^{x}

we can write the Faddeev-equation as

$$\boxed{\psi(i'j'\alpha') = \sum_{ij\alpha} K(i'j'\alpha'; ij\alpha)\, \psi(ij\alpha)}$$ Discretized Faddeev Eq.

with

$$K(i'j'\alpha'; ij\alpha) = \frac{1}{E - \frac{p_i^2}{m} - \frac{3q_{j'}^2}{4m}} \omega_{j'}^{q} q_{j'}^2 \sum_{\tilde{\alpha}} \delta_{\tilde{\alpha}_3 \alpha_3'} \sum_{k} \omega_k^x$$

$$\times \sum_m S_m(\pi_{12}'(q_{j'} q_j x_k))\, t_{12}(p_i \alpha_{12}'; p_m \tilde{\alpha}_m; q_{j'})$$

$$\times G_{\tilde{\alpha}\alpha}(q_{j'} q_j x_k)\, S_i(\pi_{12}(q_{j'} q_j x_k))$$

↳ Check the difference betw. this result and the expression in the lecture

So, the Faddeev eq. (2) is now discretized. **4)**
Now, we discretize the wave function:

$$\boxed{|\Psi\rangle = (\mathbb{1} + P)|\psi_{12}\rangle}$$ ← Meaning: By means of the permutation operator P the wave fct. $|\Psi\rangle$ can be expressed in terms of the Faddeev-component $|\psi_{12}\rangle$

$$\Rightarrow \langle p_{12}' p_3' \alpha' | \Psi\rangle = \langle p_{12}' p_3' \alpha' | (\mathbb{1} + P) | p_{12} p_3 \alpha\rangle \langle p_{12} p_3 \alpha | \psi_{12}\rangle$$

$$\Rightarrow \Psi^{\alpha'}(p_{12}' p_3') = \sum_{\alpha} \int dp_{12}\, p_{12}^2\, dp_3\, p_3^2 \int_{-1}^{1} dx\, \frac{\delta(p_{12} - \pi_{12}(p_{12}' p_3' x))}{p_{12}^2}$$

$$\times \frac{\delta(p_3 - \pi(p_{12}' p_3' x))}{p_3^2} \left(1 + 2 G_{\alpha'\alpha}(p_{12}' p_3' x)\right) \psi_{12}(p_{12} p_3 \alpha)$$

Next, discretize $\boxed{\Psi^{\alpha'}(p_{12}' p_3') \equiv \Psi(p_{12}' p_3' \alpha') \longrightarrow \Psi(i'j'\alpha')}$ and $\psi_{12}(p_{12} p_3 \alpha) \longrightarrow \psi(p_i q_j \alpha)$

$$\boxed{\Psi(i'j'\alpha') = \sum_{ij\alpha} \sum_{k} \omega_k^x \left(1 + \alpha\, G_{\alpha'\alpha}(p_i' q_j x)\right) \times S_i(\pi_{12}(p_{i'} q_{j'} x))\, S_j(\pi_3(p_i' q_j x))\, \psi(p_i q_j \alpha)}$$

The Faddeev component $\psi(p_i q_j \alpha)$ can be obtained by solving the Faddeev equation. So, we calculate the wave fct. for the given Faddeev-component of the 3-fermion bound state system.

5) Project the Lippmann-Schwinger equation for t_{12} onto the partial-wave basis

+ discretize your resulting equation

↳ See next page

Remember: In Exercise 8 we used the wave fct. to obtain the expectation value of the Hamilton operator / Kinetic energy which then was compared to the binding energy known from the lecture code.

5) Now, project the Lippmann - Schwinger equation onto the partial wave basis (→ lecture09.pdf, p. 8) for t_{12}

$$\boxed{t_{12}(\vec{p}\,'_{12}, \vec{p}_{12}; \vec{p}_3) = V_{12}(\vec{p}\,'_{12}, \vec{p}_{12}; \vec{p}_3) + \int d^3\vec{p}\,'' \, V_{12}(\vec{p}\,'_{12}, \vec{p}\,''_{12}) \frac{1}{E-H_0} t_{12}(\vec{p}\,''_{12}, \vec{p}_{12}; \vec{p}_3)}$$

~~Discretize this equation by introducing an integration weights~~

$$\hookrightarrow t_{12}(p'_{12}\alpha'_{12}; p_{12}\alpha_{12}, p_3) = V_{12}(p'_{12}\alpha'_{12}; p_{12}\alpha_{12}) + \sum_{\alpha''_{12}} \int dp''_{12}\, p''_{12}\; V_{12}(p'_{12}\alpha'_{12}; p''_{12}\alpha''_{12}) \frac{1}{\tilde{E} - \frac{p''^2_{12}}{m}} \times t_{12}(p''_{12}\alpha''_{12}; p_{12}\alpha_{12}; p_3)$$

In the latter equation: $\boxed{\tilde{E} = E - \frac{3p_3^2}{4m}}$

and $\boxed{d^3\vec{p}\,'' = \sum_{\alpha''_{12}\alpha''_3} \int dp''\, p''^2}$

Now, discretize this equation by introducing an integration weight ω_m and discrete grid points p_i:

$$t_{12}(p_i\alpha'_{12}, p_j\alpha_{12}; q_k) = V_{12}(p_i\alpha'_{12}; p_j\alpha_{12}) + \sum_{\alpha''_{12}} \sum_m \omega(p_m)\cdot p_m^2 \cdot V_{12}(p_i\alpha'_{12}, p_m\alpha''_{12}) \cdot \frac{1}{\tilde{E} - \frac{p_m^2}{m}} \times t_{12}(p_m\alpha''_{12}; p_j\alpha_{12}; q_k) \quad (*)$$

Similar to Exercise 7 (HW 8.1) this discretized Lippmann-Schwinger equation can be written in the form

$$\boxed{A_{im} t_{mj} = V_{ij}}$$

To show this, we simply reformulate the previous Eq. (*) as follows:

$$t_{12}(p_i\alpha'_{12}; p_j\alpha_{12}; q_k) - \sum_{\alpha''_{12}} \sum_m \underbrace{\omega_{p_m}}_{\equiv \omega_m} p_m^2 \times V_{12}(p_i\alpha'_{12}, p_m\alpha''_{12}) \frac{1}{\tilde{E} - \frac{p_m^2}{m}} t_{12}(p_m\alpha''_{12}; p_j\alpha_{12}; q_k)$$

$$= V_{12}(p_i\alpha'_{12}, p_j\alpha_{12})$$

$$\Rightarrow \sum_{\alpha''_{12} m} \left[\delta_{p_i p_m} \delta_{\alpha'_{12} \alpha''_{12}} - \omega_{p_m} p_m^2 \cdot V_{12}(p_i \alpha'_{12}, p_m \alpha''_{12}) \frac{1}{\tilde{E} - \frac{p_m^2}{m}} \right] \times$$

$$\times \, t_{12}(p_m \alpha''_{12}, p_j \alpha_{12}; q_K) = V_{12}(p_i \alpha'_{12}, p_j \alpha_{12})$$

This means that we obtained

$$\boxed{A(p_i \alpha'_{12}, p_m \alpha''_{12}) \cdot t_{12}(p_m \alpha''_{12}, p_j \alpha_{12}; q_K) = V_{12}(p_i \alpha'_{12}, p_j \alpha_{12})}$$

with

$$A_{im} \equiv A(p_i \alpha'_{12}, p_m \alpha''_{12}) := \delta_{p_i p_m} \delta_{\alpha'_{12} \alpha''_{12}} - \omega_m \cdot p_m^2 \cdot V_{12}(p_i \alpha'_{12}, p_m \alpha''_{12}) \times$$

$$\times \frac{1}{\tilde{E} - \frac{p_m^2}{m}}$$

This is the desired reformulation of the Lippmann-Schwinger Equation (LSE).

Then, we can proceed as follows, to implement the LSE:

- define grid points and integral weights
- implement the matrix A
- use a <u>linear-equation solver</u> such that t_{KN} can be obtained. In the matrix equations

$(t_{12})_{mj}$ $\boxed{\sum_{m=0}^{N} A_{im} t_{mj} = V_{ij}}$ $(V_{12})_{ij}$ and $(t_{12})_{mN}$ $\boxed{\sum_{m=0}^{N} A_{im} t_{mN} = V_{iN}}$

A_{im} is a matrix while t_{mN} and V_{iN} are column-vectors (V_{iN} is the last column of the matrix V_{ij}). E.g.:
Solving this system of linear equations provides t_{mN} from which t_{NN} can be computed.

- Return t_{NN} (→ see Exercise 7)