Wigner 9_j-Symbols

9_j-symbols are used to couple 4 different angular momenta j_1, j_2, j_4, j_5 leading to a total angular momentum J with 3rd component M.
This coupling causes the formation of 2 different basis systems for the space of the total angular momentum $|JM\rangle$ which is a subspace of the Hilbert space. These 2 basis states read

$$\left.\begin{array}{l} |(j_1 j_2) j_3, (j_4 j_5) j_6 ; JM\rangle \\ \text{and } |(j_1 j_4) j_7, (j_2 j_5) j_8 ; JM\rangle \end{array}\right\} \text{with } J \equiv j_9$$

The Eigenfunctions of the basis depend on each other. Concretely, they are related by the linear transformation

$$|(j_1 j_4) j_7, (j_2 j_5) j_8 ; JM\rangle = \sum_{j_3, j_6} |(j_1 j_2) j_3, (j_4 j_5) j_6 ; JM\rangle$$

$$\times \underbrace{\langle (j_1 j_2) j_3, (j_4 j_5) j_6 ; JM | (j_1 j_4) j_7, (j_2 j_5) j_8 ; JM\rangle}_{=: C_{9j} \,\hat{=}\, 9_j\text{-coefficient}}$$

Thus, the transformation coefficient C_{9j} changes the coupling. This C_{9j} defines the 9_j-symbol according to

$$\langle (j_1 j_2) j_3, (j_4 j_5) j_6 ; JM | (j_1 j_4) j_7, (j_2 j_5) j_8 , JM\rangle$$

$$= \sqrt{(2j_3+1)(2j_6+1)(2j_7+1)(2j_8+1)} \; \underbrace{\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}}_{=: S_{9j} \,\hat{=}\, 9j\text{-symbol}} \leftarrow J$$

So, if the 4 momenta j_1, j_2, j_3, j_4 fulfill the relations

$$j_1 + j_2 = j_{12}, \qquad j_3 + j_4 = j_{34} \longrightarrow j_{12} + j_{34} = J$$

$$j_1 + j_3 = j_{13}, \qquad j_2 + j_4 = j_{24} \longrightarrow j_{13} + j_{24} = J$$

The corresponding basis states are related via

$$|j_{13}, j_{24}; JM\rangle = \sum_{j_{12}, j_{34}} |j_{12}, j_{34}; JM\rangle \underbrace{\langle j_{12}, j_{34}; JM | j_{13}, j_{24}; JM\rangle}_{=: C_{9j}}$$

with

$$\langle j_{12}, j_{34}; JM | j_{13}, j_{24}; JM\rangle = \sqrt{(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)}$$

$$\times \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{Bmatrix}$$

Wigner 6j-Symbols

6j-symbols are used to couple 3 different angular momenta j_1, j_2, j_3 to a total angular momentum $|J\,M\rangle$. This coupling causes the formation of 2 different basis systems in the space of $|J\,M\rangle$ (which is a sub-space of the Hilbert space):

$$|j_{12}, j_3;\ J M\rangle$$
$$\text{and} \quad |j_1, j_{23},\ J M\rangle$$

$$\text{with} \quad \begin{cases} j_1 + j_2 =: j_{12}, & j_{12} + j_3 = J \\ j_2 + j_3 =: j_{23}, & j_{23} + j_1 = J \end{cases}$$

These 2 basis states are related by the linear transformation

$$|j_1, j_{23};\ JM\rangle = \sum_{j_{12}} |j_{12}, j_3;\ JM\rangle \underbrace{\langle j_{12}, j_3;\ JM\,|\,j_1, j_{23};\ JM\rangle}_{=:\, c_{6j}\ \hat{=}\ \text{6j-coefficient}}$$

This coefficient C_{6j} defines the 6j-symbol according to

$$\langle j_{12}, j_3;\ JM\,|\,j_1, j_{23};\ JM\rangle = (-1)^{j_1+j_2+j_3+J}\sqrt{(2j_{12}+1)(2j_{23}+1)}\ \underbrace{\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix}}_{\substack{=:\, S_{6j} \\ \hat{=}\ \text{6j-symbol}}}$$

Now, we want to derive the permutation operator for arbitrary spins.

In our 3-fermion case the basis states read

$$|p'q'\,[(l_{12}\,s_{12})j_{12},(l_3\,s_3)\,I_3\,]\,JM\rangle =: |p'q'\alpha'\rangle$$
$$|pq\,[(l_{23}\,s_{23})j_{23},(l_1\,s_1)\,I_1\,]\,JM\rangle = |pq\alpha\rangle$$

isospin is neglected here. Call it T, t, then how do you have to modify these basis states?

<u>1^{st} step:</u> Decouple orbital part, spin part (and isospin part)

$$\langle p'q'\alpha' M' | pq\alpha M\rangle$$

$$= \sum_{LS}\sum_{L'S'} \sqrt{\hat{j}_{12}\hat{j}_{23}\hat{I}_3\hat{I}_1\hat{L}\hat{L}'\hat{S}\hat{S}'} \begin{Bmatrix} l_{12} & s_{12} & j_{12} \\ l_3 & s_3 & I_3 \\ L' & S' & J \end{Bmatrix} \begin{Bmatrix} l_{23} & s_{23} & j_{23} \\ l_1 & s_1 & I_1 \\ L & S & J \end{Bmatrix} \times$$

$$\times \langle p'q'[(l_{12}l_3)L',(s_{12}s_3)S'];JM\,|\,pq\,[(l_{23}l_1)L,(s_{23}s_1)S];JM\rangle$$

Note: $\langle p'q'\alpha'M | pq\alpha M\rangle$

$$= \langle p'q'[(l_{12}s_{12})j_{12},(l_3s_3)I_3], JM | p'q'[(l_{12}l_3)L',(s_{12}s_3)S']JM\rangle \times$$

$$\times \langle p'q'[(l_{12}l_3)L',(s_{12}s_3)S'], JM | pq[(l_{23}l_1)L,(s_{23}s_1)S']JM\rangle \times$$

$$\times \langle pq[(l_{23}l_1)L,(s_{23}s_1)S']JM | pq[(l_{23}s_{23})j_{23},(l_1s_1)I_1]JM\rangle$$

and in Eq above: $\hat{J}_{12} := (2j_{12}+1)$

$\hat{J}_{23} := (2j_{23}+1)$

$\hat{I}_3 := (2I_3+1)$ and so on.

Try to continue this calculation!

Try to continue this calculation!

<u>Hint</u>: - orbital part $\rightsquigarrow$ see 3 bosons case
- spin part: Use definition of 6j-coefficients
- isospin part: similar to spin part

If this is done:

- project the Lippmann - Schwinger eq. for t_{12} to the partial wave basis ⤷ very similar to lecture notes
- projecting the Fadeev-equation
- write down the wave fct. $|\Psi\rangle$
- discretization of t_{12}

↳ check what are the differences compared to the lecture notes!