

Evaluating the Spin-Dependence of the Three-Body Fermionic Wavefunction

Keito Watanabe (s6kewata)* and Haveesh Singirikonda (s6gusing)[†]

(Dated: March 4, 2022)

Three-body particle systems are quite interesting to study because of the various physical phenomena which can affect the interactions between the particles. These systems have been given a lot of focus in literature, especially to study the three-body forces involved and the different kinds of interaction potentials possible between the particles. In this work, we studied the bound state of a system of three identical fermions and how their properties differ from a bosonic system of the same kind. This was performed by utilizing the Faddeev equation to evaluate an eigenvalue problem, yielding bound state energy of $E = -5.221$ MeV. The resulting wavefunction showed a strong peak at low momenta values and yielded a larger amplitude overall compared to the bosonic wavefunction. The kinetic energy $\langle H_0 \rangle$ obtained for $J = 0, \frac{1}{2}$ were 57.46 MeV and 15.02 MeV respectively, disagreeing with physical interpretations. When comparing with spin-dependent interactions, we observed that such considerations realize a more realistic form of the fermionic wavefunction.

I. INTRODUCTION

Ever since the advent of quantum mechanics to study elementary physics, we have been interested in solving many kinds of particle systems. Years of work has been put into devising techniques to study these properties, especially due to the unsolvable nature of the Schrödinger equation for most systems that are more complicated than a simple Hydrogen atom. Numerous ways, both analytical and numerical, have been put forward and thus led to the solidification of the Standard Model in particle physics.

Of the many systems, the three-body particle system is of quite an interest, which pertains to many physical situations, especially for interactions involved in nuclear structure. The nuances involved here are quite interesting as the three-body forces and even the two-body forces between the particles depend a lot on the type of particles under consideration. In a system of bosons, we have a simple situation with only orbital angular momentum coupling. Even for this, different possibilities exist for the multitude of potentials available to study these interactions. The coupling of spin and total angular momenta would be important for a system with fermions. The additional degree of freedom from spin and the anti-symmetry of fermions gives rise to distinct differences of the properties in a fermionic system, which also depends on the spin even when interactions are spin-independent [1]. Along with this, for nucleons, the isospin due to the up and down quarks also becomes important, and would also result in changes to the system. All of the above-described situations are for systems with only two-body interactions (commonly referred to as NN interactions in the literature for nucleons), i.e, neglecting the three-body interaction potential (commonly referred to as the 3N,

for nucleons). Studying this is also very interesting, considering the many possibilities and how it changes the system. There have also been works studying the three-body force [3, 5]. Most current work in literature is concerned about studying the two body and three-body (usually for nucleons with isospins) interactions. The universality of such interactions is also of great interest [1] .

In our lectures, we have considered the interaction of three bosons, while neglecting the three-body force term. In this work, we do a similar analysis of a system of three fermions again neglecting the three-body term. We will see that even with a spin-independent potential, the results are quite different due to the significant coupling of the orbital and spin parts of the angular momentum. The potential used here is the same as the boson case, the One Boson Exchange interaction potential (OBE). Spin-dependent interactions lead to even more changes, and in this work, we try to see how the system changes with such a spin-dependent interaction. The spin-dependent interaction used is the One Pion Exchange interaction potential (OPE).

These three body state calculations of nucleons (commonly used systems for a set of three fermions) are usually carried out by solving the Faddeev equations in partial wave basis [2]. More techniques to how these calculations are done can be found in the references within this paper.

In this work, we study the properties of a three-fermion bound system by drawing similarities to the three-body boson problem studied in the lectures. We apply the developed techniques to apply to this system and see what changes need to be accounted for by the differences that arise by considering Fermions.

This report is structured as follows. Section II described the theory involved, firstly using the concepts which are similar to the case of the three boson system and then bringing up the differences and the complexities which arise for a three-fermion system. In Section

* s6kewata@uni-bonn.de

† haveesh.s@uni-bonn.de

III we go into depth about how the analytical form of the Permutation operator is computed for such a system. Again, we build upon the known form of the Permutation operator for the 3 boson system. After this, in Section **IV**, we describe how these analytical integral equations can be discretized for a numerical treatment. In Section **V**, we describe how these equations are implemented numerically on Python. Here, we again build upon the implementation of the 3 boson system done in the Lecture. In Section **VI** we present the resulting three-body fermionic wavefunction and the off-shell t -matrix utilized in our simulation, for both spin-independent and spin-dependent interaction potentials. We also present the differences between the results obtained from the three-fermion and three-boson system.

II. BASIC THEORY

A. Schrödinger Equations and Interactions

Put together, the entire process of solving the three-body system is just about solving the non-relativistic Schrödinger equation for this bound system of particles. Referring to the Lecture slides (Lecture08) the Schrödinger equation for this system would be

$$\hat{H} |\Psi\rangle = (\hat{H}_0 + \hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31} + \hat{V}_{123}) |\Psi\rangle$$

where, \hat{H}_0 is the internal Kinetic Energy and \hat{V} are the interaction potentials between the indicated particles. But in this situation, we will be neglecting the three body interaction \hat{V}_{123} . From this we get,

$$\hat{H} |\Psi\rangle = (\hat{H}_0 + \hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31}) |\Psi\rangle \quad (1)$$

Similar to the three-boson case, we define a potential based on the exchange of one boson, the so-called one-boson exchange (OBE) potential. This potential includes both long-ranged and short-ranged interactions, corresponding to Coulomb and contact interactions respectively. Such a potential can be defined as follows:

$$V_{12}(\vec{p}, \vec{p}') = \frac{A}{2\pi} \frac{1}{(\vec{p} - \vec{p}')^2 + m_B^2} \exp\left(-\frac{(\vec{p} - \vec{p}')^2 + m_B^2}{\Lambda^2}\right) + \frac{C_0}{4\pi} \exp\left(-\frac{p^2 + p'^2}{\Lambda^2}\right) \quad (2)$$

where m_B is the mass of the boson mediating the exchange and C_0 and A defines the strength of the short and long-ranged interaction respectively. The parameter Λ defines the cutoff-value of the potential, which is required to regularize the expression.

In addition, we also consider a spin-dependent potential energy for the interaction between the fermions. In this work, we use a One Pion Exchange (OPE) potential between the nucleons and incorporate it with our calculations. The analytical form of this OPE potential is out of the scope of this report.

For convenience, we will ignore the hats in the notations from hereon, i.e, $V = \hat{V}$ and $H = \hat{H}$. We can write the interaction terms in terms of the other interactions using the permutation operators P_{ij} ,

$$V_{23} = P_{12}P_{23}V_{12}P_{23}P_{12}$$

$$V_{31} = P_{13}P_{23}V_{12}P_{23}P_{13}$$

Inserting these relations into Equation 1, we get:

$$(E - H_0) |\Psi\rangle = V_{12} |\Psi\rangle + P_{12}P_{23}V_{12}P_{23}P_{12} |\Psi\rangle + P_{13}P_{23}V_{12}P_{23}P_{13} |\Psi\rangle$$

As we have a three Fermion bound state, $P_{23}P_{12} |\Psi\rangle = |\Psi\rangle$ and $P_{23}P_{13} |\Psi\rangle = |\Psi\rangle$ (As $P_{ij} |\Psi\rangle = -|\Psi\rangle$). This property of the Permutation operator with Fermions simplifies the expression,

$$(E - H_0) |\Psi\rangle = (\mathbb{1} + P_{12}P_{23} + P_{13}P_{23})V_{12} |\Psi\rangle$$

$$|\Psi\rangle = \frac{1}{E - H_0} (\mathbb{1} + P)V_{12} |\Psi\rangle \quad (3)$$

where P is defined as $P = P_{12}P_{23} + P_{13}P_{23}$.

B. The Faddeev Component

Three body bound states are usually computed using the Faddeev Equation, so for this it is convenient to define the Faddeev component as well. This is defined as

$$|\psi_{12}\rangle = \frac{1}{E - H_0} V_{12} |\Psi\rangle$$

So using this, Equation 3 can be written as

$$|\Psi\rangle = (\mathbb{1} + P) |\psi_{12}\rangle$$

From this, we can obtain an equation for the Faddeev component.

$$|\psi_{12}\rangle = \frac{1}{E - H_0} V_{12} |\Psi\rangle = \frac{1}{E - H_0} V_{12} (\mathbb{1} + P) |\psi_{12}\rangle \quad (4)$$

C. Faddeev Equation

To solve Equation 4 analytically, we make use of the Lippmann-Schwinger equation used in the Lecture to

solve two-body scattering. The Lippmann-Schwinger equation is (from the Lecture):

$$t_{12} = V_{12} + V_{12} \frac{1}{E - H_0} t_{12}$$

Rearranging the terms in Equation 4, we get:

$$\left(\mathbb{1} - \frac{1}{E - H_0} V_{12} \right) |\psi_{12}\rangle = \frac{1}{E - H_0} V_{12} P |\psi_{12}\rangle$$

Multiplying on sides by a factor involving the t-matrix,

$$\left(\mathbb{1} + \frac{1}{E - H_0} t_{12} \right) \left(\mathbb{1} - \frac{1}{E - H_0} V_{12} \right) = \mathbb{1}$$

The right hand side is obtained on the application of the Lippmann-Schwinger equation. From this the Faddeev equations can be derived (from the Lecture),

$$\begin{aligned} |\psi_{12}\rangle &= \left(\mathbb{1} + \frac{1}{E - H_0} t_{12} \right) \frac{1}{E - H_0} V_{12} P |\psi_{12}\rangle \\ |\psi_{12}\rangle &= \frac{1}{E - H_0} t_{12} P |\psi_{12}\rangle \end{aligned} \quad (5)$$

Where the Lippmann-Schwinger equation is used to obtain the expression below. These are the Faddeev Equations.

D. Jacobi Momenta

It is very convenient to use Jacobi momenta for our problem as we have condensed the whole system of equations to be dependent on the interaction term between particles 1 and 2, through V_{12} . Information of the other interactions is simply given by the action of the Permutation operator. Due to this reason, we can say that particles 1 and 2 are interacting and the 3rd particle is a spectator particle.

Using these conventions, we can define a set of Jacobi coordinates for these set of particles, and taking \vec{k}_1 , \vec{k}_2 and \vec{k}_3 as the single particle momenta,

$$\begin{aligned} \vec{p}_{12} &= \frac{1}{2}(\vec{k}_1 - \vec{k}_2) \\ \vec{p}_3 &= \frac{2}{3}\vec{k}_3 - \frac{1}{3}(\vec{k}_1 + \vec{k}_2) \\ \vec{P} &= \vec{k}_1 + \vec{k}_2 + \vec{k}_3 \end{aligned}$$

Other permutations can similarly be written as

$$\begin{aligned} \vec{p}_{23} &= \frac{1}{2}(\vec{k}_2 - \vec{k}_3) \\ \vec{p}_1 &= \frac{2}{3}\vec{k}_1 - \frac{1}{3}(\vec{k}_2 + \vec{k}_3) \\ \vec{p}_{31} &= \frac{1}{2}(\vec{k}_3 - \vec{k}_1) \\ \vec{p}_2 &= \frac{2}{3}\vec{k}_2 - \frac{1}{3}(\vec{k}_1 + \vec{k}_3) \end{aligned}$$

These momenta would satisfy,

$$\begin{aligned} \vec{p}_{23} &= -\frac{1}{2}\vec{p}_{12} - \frac{3}{4}\vec{p}_3 \\ \vec{p}_1 &= \vec{p}_{12} - \frac{1}{2}\vec{p}_3 \\ \vec{p}_{31} &= -\frac{1}{2}\vec{p}_{12} + \frac{3}{4}\vec{p}_3 \\ \vec{p}_2 &= -\vec{p}_{12} - \frac{1}{2}\vec{p}_3 \end{aligned}$$

Using these properties, we can derive the matrix elements of coordinate transformations.

$$\langle \vec{p}'_{12} \vec{p}'_3 | \vec{p}_{23} \vec{p}_1 \rangle = \delta^{(3)}(\vec{p}_{23} + \frac{1}{2}\vec{p}'_{12} + \frac{3}{4}\vec{p}'_3) \delta^{(3)}(\vec{p}_1 - \vec{p}'_{12} + \frac{1}{2}\vec{p}'_3) \quad (6)$$

III. THE PERMUTATION OPERATOR

The permutation operator being a discrete operator we need to find ways to incorporate this into the numerical analysis. And this is the most important challenge we have in this problem. The major change when compared to a three-body Boson system also would be seen here.

Before this, we introduce a convention (taken from the Lecture) that would make it easier for us to denote which particles are paired up and which one is the spectator particle (essentially the type of Jacobi coordinate used). In Equation 6, on the left hand side, we wrote

$$(12)3 \langle \vec{p}'_{12} \vec{p}'_3 | \vec{p}_{23} \vec{p}_1 \rangle_{(23)1}$$

Here, (23)1 represents the Jacobi coordinate used. From now on, we will use this to indicate the Jacobi momenta used.

A. Permutation operator using Jacobi coordinates

The permutation operator can be represented in the momentum basis as

$$(12)3 \langle \vec{p}'_{12} \vec{p}'_3 | P_{12} P_{23} | \vec{p}_{12} \vec{p}_3 \rangle_{(12)3}$$

On the application of the permutation operator, we would get:

$${}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 | P_{12} P_{23} | \vec{p}'_{12} \vec{p}'_3 \rangle {}_{(12)3} = {}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 | \vec{p}'_{12} \vec{p}'_3 \rangle {}_{(23)1}$$

So we use the braket on the right-hand side of the above equation to denote the Permutation operators ($P_{12} P_{23}$) on this basis. The braket notation for $P_{13} P_{23}$ can be written using the properties that:

$$\begin{aligned} \langle \vec{p}'_{12} \vec{p}'_3 | P_{12} P_{23} + P_{13} P_{23} | \vec{p}'_{12} \vec{p}'_3 \rangle &= \\ \langle \vec{p}'_{12} \vec{p}'_3 | P_{12} P_{23} + P_{12} P_{13} P_{23} P_{12} | \vec{p}'_{12} \vec{p}'_3 \rangle &= \\ \langle \vec{p}'_{12} \vec{p}'_3 | P_{12} P_{23} + P_{12} P_{23} P_{12} P_{12} | \vec{p}'_{12} \vec{p}'_3 \rangle &= \\ 2 \langle \vec{p}'_{12} \vec{p}'_3 | P_{12} P_{23} | \vec{p}'_{12} \vec{p}'_3 \rangle & \end{aligned}$$

So the Permutation operator can be represented in the momentum basis as:

$${}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 | P | \vec{p}'_{12} \vec{p}'_3 \rangle {}_{(12)3} = 2 {}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 | \vec{p}'_{12} \vec{p}'_3 \rangle {}_{(23)1}$$

So we need to compute the braket ${}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 | \vec{p}'_{12} \vec{p}'_3 \rangle {}_{(23)1}$. This will be addressed in the next section.

B. Partial Wave Representation of the Permutation Operator

The analytical form of the permutation operator is obtained by using the partial wave representation of the states. In this, the quantity we want to compute which represents the Permutation operator would be in terms of the quantum numbers l , s , and j . In the case of three-body Bosons, only the orbital angular momentum was considered. Here, as we have a three-body system of Fermions we will have to take in the consideration of the spins and isospins (for systems with quarks), and these, in turn, give us the total angular momentum J .

So in this partial wave eigenbasis, this would be written as

$${}_{(12)3} \langle p'_{12} p'_3 \alpha' M' | p_{12} p_3 \alpha M \rangle {}_{(23)1}$$

where, $\alpha M = \alpha_{12} \alpha_3 M$, where

$$\begin{aligned} \alpha_{12} &= (l_{12} s_{12}) j_{12} \\ \alpha_3 &= (l_3 s_3) j_3 \\ M &= JM \end{aligned}$$

And due to rotational invariance, we have $M' = M$. This state would then be (From now on, we will be ignoring the subscripts (12)3 and (23)1 for ease, and it is assumed that the brakets are calculated for these permutations and Jacobi momenta.):

$$\begin{aligned} \langle p'_{12} p'_3 \alpha' M' | p_{12} p_3 \alpha M \rangle &= \\ \langle p'_{12} p'_3 \alpha'_{12} \alpha'_3 M' | p_{12} p_3 \alpha_{12} \alpha_3 M \rangle & \end{aligned} \quad (7)$$

This coupling of the different components can be treated by incorporating the Wigner 9j and 6j symbols. The Wigner 9j symbol is defined as

$$\begin{aligned} \langle a(bc), (de)f; im | (ad)g, (be)h; im \rangle &= \\ \sqrt{(2c+1)(2f+1)(2g+1)(2h+1)} \left\{ \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \right\} & \end{aligned}$$

And the Wigner 6j symbol is defined as

$$\begin{aligned} \langle a, (bc)d; JM | (ab)e, c; JM \rangle &= \\ (-1)^{a+b+c+J} \sqrt{(2d+1)(2e+1)} \left\{ \begin{array}{ccc} a & b & e \\ c & J & d \end{array} \right\} & \end{aligned}$$

Equation 7 must be modified to be able to use the definitions of the Wigner 9j symbols. This can be done using the completeness relations:

$$\begin{aligned} \langle p'_{12} p'_3 ((l'_{12} s'_{12}) j'_{12}, (l'_3 s'_3) j'_3) JM | p_{12} p_3 ((l_{12} s_{12}) j_{12}, (l_3 s_3) j_3) JM \rangle &= \\ \langle p'_{12} p'_3 ((l'_{12} s'_{12}) j'_{12}, (l'_3 s'_3) j'_3) JM | p'_{12} p'_3 ((l'_{12} l'_3) L', (s'_{12} s'_3) S') JM \rangle & \\ \langle p'_{12} p'_3 ((l'_{12} l'_3) L', (s'_{12} s'_3) S') JM | p_{12} p_3 ((l_{12} l_3) L, (s_{12} s_3) S) JM \rangle & \\ \langle p_{12} p_3 ((l_{12} l_3) L, (s_{12} s_3) S) JM | p_{12} p_3 ((l_{12} s_{12}) j_{12}, (l_3 s_3) j_3) JM \rangle & \end{aligned}$$

Here,

$$\int dp'_{12} dp'_3 p'_{12}^2 p'_3^2 | p'_{12} p'_3 ((l'_{12} l'_3) L', (s'_{12} s'_3) S') JM \rangle \\ \langle p'_{12} p'_3 ((l'_{12} l'_3) L', (s'_{12} s'_3) S') JM |$$

and

$$\begin{aligned} \int dp_{12} dp_3 p_{12}^2 p_3^2 | p_{12} p_3 ((l_{12} l_3) L, (s_{12} s_3) S) JM \rangle \\ \langle p_{12} p_3 ((l_{12} l_3) L, (s_{12} s_3) S) JM | \end{aligned}$$

represent the identity operators $\mathbb{1}$. In here, we have not written the integrals on the top to avoid clutter. And it is to be assumed hereon, that the equation has integrals wrt to $p'_{12}, p_{12}, p'_3, p_3$.

From the definition of the Wigner symbols, the above equation reduces to:

$$\begin{aligned} \langle p'_{12} p'_3 ((l'_{12} s'_{12}) j'_{12}, (l'_3 s'_3) j'_3) JM | p_{12} p_3 ((l_{12} s_{12}) j_{12}, (l_3 s_3) j_3) JM \rangle &= \\ \sum_{LS} \sum_{L'S'} \sqrt{\tilde{j}_{12} \tilde{j}'_{12} \tilde{j}_3 \tilde{j}'_3 \tilde{L} \tilde{L}' \tilde{S} \tilde{S}'} \left\{ \begin{array}{ccc} l_{12} & s_{12} & j_{12} \\ l_3 & s_3 & j_3 \\ L & S & J \end{array} \right\} \left\{ \begin{array}{ccc} l'_{12} & s'_{12} & j'_{12} \\ l'_3 & s'_3 & j'_3 \\ L' & S' & J \end{array} \right\} & \\ \langle p'_{12} p'_3 ((l'_{12} l'_3) L', (s'_{12} s'_3) S') JM | p_{12} p_3 ((l_{12} l_3) L, (s_{12} s_3) S) JM \rangle & \end{aligned}$$

where, $\tilde{j}_{12} = 2j_{12} + 1$, and the other tilde terms are similarly defined. The last bracket term needs to be evaluated. In this term, the orbital and spin parts can be separated out as they are independent of each other. Doing this,

$$\begin{aligned} & \langle p'_{12}p'_3((l'_{12}l'_3)L', (s'_{12}s'_3)S')JM | p_{12}p_3((l_{12}l_3)L, (s_{12}s_3)S)JM \rangle \\ & \langle p'_{12}p'_3(l'_{12}l'_3)L' | p_{12}p_3(l_{12}l_3)L \rangle \langle (s'_{12}s'_3)S' | (s_{12}s_3)S \rangle \end{aligned} \quad (8)$$

The first part (orbital) can be computed using the partial wave representation of the state, similar to the case of three-body Bosons. The second part (spin) is evaluated using the Wigner 6j symbols.

1. Orbital Part

We use the fact that:

This can be obtained in terms of the Spherical Harmonics (Y_{lm}).

$$\begin{aligned} \langle \vec{p}_{12} | p'_{12}l'_{12}m'_{12} \rangle &= \frac{\delta(p_{12} - p'_{12})}{p_{12}p'_{12}} Y_{l'_{12}m'_{12}}(\hat{p}_{12}) \\ \langle \vec{p}_3 | p'_3l'_3m'_3 \rangle &= \frac{\delta(p_3 - p'_3)}{p_3p'_3} Y_{l'_3m'_3}(\hat{p}_3) \end{aligned}$$

We need to compute:

$$\langle p'_{12}p'_3(l'_{12}l'_3)L' | p_{12}p_3(l_{12}l_3)L \rangle$$

This part was computed for the bosonic case, where only orbital angular momentum coupling takes place. From the Lecture:

$$\begin{aligned} & \frac{1}{2L+1} \sum_M \langle p'_{12}p'_3(l'_{12}l'_3)L' | p_{12}p_3(l_{12}l_3)L \rangle \\ &= \int d\hat{\vec{p}}_3 \hat{\vec{p}}'_3 \frac{\delta(p_{12} - |\vec{p}'_3 - \frac{\vec{p}_3}{2}|)}{p_{12}^2} \frac{\delta(p'_{12} - |\vec{p}'_3 + \frac{\vec{p}_3}{2}|)}{p'_{12}^2} \\ & \times \frac{1}{2L+1} \sum_M \mathcal{Y}^{L'M'}_{l'_{12}l'_3}(\vec{p}_3 + \frac{1}{2}\vec{p}'_3) \mathcal{Y}^{LM}_{l_{12}l_3}(-\vec{p}'_3 - \frac{1}{2}\vec{p}_3) \\ &= 8\pi^2 \int_{-1}^1 dx \frac{\delta(p_{12} - \pi_{12}(p'_3 p_3 x))}{p_{12}^2} \frac{\delta(p'_{12} - \pi'_{12}(p'_3 p_3 x))}{p'_{12}^2} \\ & \times \frac{1}{2L+1} \mathcal{Y}^{L'M'}_{l'_{12}l'_3} \mathcal{Y}^{LM}_{l_{12}l_3} \end{aligned} \quad (9)$$

2. Spin Part

For evaluating the spin part, we will use the Wigner 6j symbols. Also,

$$s_3 = s'_3 = \frac{1}{2}$$

Using this along with the definition of the Wigner 6j symbols, we get

$$\begin{aligned} & \langle (s'_{12}s'_3)S' | (s_{12}s_3)S \rangle \\ &= (-1)^{s_{12}} \sqrt{(2s'_{12} + 1)(2s_{12} + 1)} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & s'_{12} \\ \frac{1}{2} & S & s_{12} \end{array} \right\} \end{aligned} \quad (10)$$

Putting everything together, we now have the representation of the permutation operator. Using the results from Equations 9 and 10 into Equation 8,

$$\begin{aligned} & {}_{(12)3} \langle p'_{12}p'_3\alpha'M' | p_{12}p_3\alpha M \rangle_{(23)1} \\ &= \int_{-1}^1 dx \frac{\delta(p_{12} - \pi_{12}(p'_3 p_3 x))}{p_{12}^2} \frac{\delta(p'_{12} - \pi'_{12}(p'_3 p_3 x))}{p'_{12}^2} \\ & \times G_{\alpha'\alpha}(p'_3 p_3 x) \end{aligned}$$

where, we have defined $G_{\alpha'\alpha}$ as (and using rotational invariance, so $L' = L$ and $S' = S$):

$$\begin{aligned} G_{\alpha'\alpha} &= \sum_L \sum_S (2S+1) \sqrt{\tilde{j}_{12}\tilde{j}'_{12}\tilde{j}_3\tilde{j}'_3} \\ & \times \left\{ \begin{array}{ccc} l_{12} & s_{12} & j_{12} \\ l_3 & \frac{1}{2} & j_3 \\ L & S & J \end{array} \right\} \left\{ \begin{array}{ccc} l'_{12} & s'_{12} & j'_{12} \\ l'_3 & \frac{1}{2} & j'_3 \\ L & S & J \end{array} \right\} \\ & \times (-1)^{s_{12}} \sqrt{(2s'_{12} + 1)(2s_{12} + 1)} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & s'_{12} \\ \frac{1}{2} & S & s_{12} \end{array} \right\} \\ & \times 8\pi^2 \sum_M \mathcal{Y}^{LM}_{l'_{12}l'_3} \mathcal{Y}^{LM}_{l_{12}l_3} \end{aligned} \quad (11)$$

If we include isospin in the previous theory, we would get:

$$\begin{aligned} G_{\alpha'\alpha} &= \sum_L \sum_S (2S+1) \sqrt{\tilde{j}_{12}\tilde{j}'_{12}\tilde{j}_3\tilde{j}'_3} \\ & \times \left\{ \begin{array}{ccc} l_{12} & s_{12} & j_{12} \\ l_3 & \frac{1}{2} & j_3 \\ L & S & J \end{array} \right\} \left\{ \begin{array}{ccc} l'_{12} & s'_{12} & j'_{12} \\ l'_3 & \frac{1}{2} & j'_3 \\ L & S & J \end{array} \right\} \\ & \times (-1)^{s_{12}} \sqrt{(2s'_{12} + 1)(2s_{12} + 1)} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & s'_{12} \\ \frac{1}{2} & S & s_{12} \end{array} \right\} \\ & \times (-1)^{\tau_{12}} \sqrt{(2\tau'_{12} + 1)(2\tau_{12} + 1)} \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \tau'_{12} \\ \frac{1}{2} & T & \tau_{12} \end{array} \right\} \\ & \times 8\pi^2 \sum_M \mathcal{Y}^{LM}_{l'_{12}l'_3} \mathcal{Y}^{LM}_{l_{12}l_3} \end{aligned} \quad (12)$$

where τ_{12} is the coupled isospin of both the particles and T is the total isospin.

IV. DISCRETIZATION

After obtaining the form for the permutation operator, we now need to put everything together in the Lippmann-Schwinger and Faddeev equations. Again, we use Partial

wave representations. Using this the Lippman-Schwinger equation (from the Lecture) would look like:

$$t_{12}(p'_{12}, p_{12}; p_3) = V_{12}(p'_{12}, p_{12}) + \int dp''_{12} p''_{12}^2 V_{l_{12}} \frac{1}{E - \frac{p''_{12}^2}{m}} t_{12}(p''_{12}, p_{12}; p_3) \quad (13)$$

and the Faddeev equation would be (again from the Lecture)

$$\begin{aligned} \psi_{12}^{\alpha'}(p'_{12} p'_3) &= \frac{1}{E - \frac{p'_{12}^2}{m} - \frac{3p'_3^2}{4m}} \sum_{\tilde{\alpha}_{12}\tilde{\alpha}_3} \sum_{\alpha_{12}\alpha_3} \delta_{\tilde{\alpha}_3\alpha'_3} \\ &\times \int_{-1}^1 dx \int_0^\infty dp_3 p_3^2 t_{12}(p'_{12} \alpha'_{12}; \pi'_{12}(p'_3 p_3 x); p'_3) \\ &\times G_{\tilde{\alpha}\alpha}(p'_3 p_3 x) \psi_{12}^{\alpha}(\pi_{12}(p'_3 p_3 x); p_3) \end{aligned}$$

Now, using these expressions we will be able to discretize the equations, which can then be used for a numerical implementation. From the Lecture,

$$\psi_{12}^{\alpha} \rightarrow \psi(p_i q_i \alpha) \equiv \psi(i + j N_p + \alpha N_p N_q)$$

And also interpolation is necessary as the points will not be continuous. This is done using spline interpolation (here we use a cubic spline, as done for the Two-body system). This interpolation is done as a sum of function values.

$$f(\pi) = \sum_{i=0}^{N_p-1} S_i(\pi) f(p_i)$$

All of this, leads to a matrix equation (over three indices) :

$$\psi(i'j'\alpha') = \sum_{ij\alpha} K(i'j'\alpha'; ij\alpha) \psi(ij\alpha) \quad (14)$$

This is the discretized Faddeev equation. The matrix K is (essentially collecting all the terms together):

$$\begin{aligned} K(i'j'\alpha'; ij\alpha) &= \frac{1}{E - \frac{p'_{12}^2}{m} - \frac{3p'_3^2}{4m}} \omega_{j'}^q q_{j'}^2 \sum_{\tilde{\alpha}} \delta_{\tilde{\alpha}_3\alpha'_3} \\ &\times \sum_k \omega_k^x \sum_m S_m(\pi'_{12}(q_{j'} q_j x_k)) \times t_{12}(p_i, \alpha'_{12}; p_m \tilde{\alpha}_m; q_{j'}) \\ &\times G_{\tilde{\alpha}\alpha}(q_{j'} q_j x_k) \times S_i(\pi_{12}(q_{j'} q_j x_k)) \end{aligned} \quad (15)$$

Here, the integrals are converted to summations using weights from standard quadrature techniques. ω_i^p , ω_j^q and ω_k^x are the corresponding integration weights for integrals over p_i , q_j and x_k respectively.

The complete wavefunction $|\Psi\rangle$ defined in Eq. II B can be then discretized following the method used above, yielding:

$$\begin{aligned} \Psi(p'_i, q'_j, \alpha'_{12}) &= \sum_{p_i, q_j, \alpha_{12}} \sum_k \omega_k^x (\mathbb{1} + \alpha G_{\tilde{\alpha}\alpha}(p_i, q_j, x_k)) \\ &\times S_i(\pi_{12}(p_i, q_j, x_k)) S_j(\pi_3(p_i, q_j, x_k)) \psi(p_i, q_j, \alpha_{12}) \end{aligned} \quad (16)$$

The expectation values of the problem (e.g. kinetic energy, potential energy) can then be evaluated by taking the (discretized) inner product between the complete wavefunctions $|\Psi\rangle$. Note that here, we utilize the fact that the normalization of the wavefunction can be defined by the Faddeev component:

$$\langle \Psi | \Psi \rangle = \langle \Psi | (\mathbb{1} + P) |\psi\rangle = 3 \langle \Psi | |\psi\rangle \quad (17)$$

which is satisfied due to the symmetry of the wavefunction.

The Lippmann-Schwinger equation is discretized in a procedure similar to the lectures (namely Lecture 8) into a matrix equation as such:

$$\begin{aligned} \sum_{\alpha''_{12}, m} A(p_i, \alpha'_{12}, p_m, \alpha''_{12}) \cdot t_{12}(p_m, \alpha'_{12}; p_m \tilde{\alpha}_m; q_k) \\ = V_{12}(p_i, \alpha'_{12}, p_j, \alpha_{12}) \end{aligned} \quad (18)$$

where the matrix A is defined as such:

$$\begin{aligned} A(p_i, \alpha'_{12}, p_m, \alpha''_{12}) &= \delta_{p_i p_m} \delta_{\alpha'_{12} \alpha''_{12}} \\ &- \omega_{p_m} p_m^2 V_{12}(p_i, \alpha'_{12}, p_m, \alpha''_{12}) \frac{1}{E - \frac{p_m^2}{m} - \frac{3p_3^2}{4m}} \end{aligned} \quad (19)$$

V. IMPLEMENTATION

The problem was implemented much like the bosonic case; only minor modifications were required to account for the spin and isospin states of the particles, along with the orbital angular states. As of such, our problem also reduced to solving the Faddeev equation (given in Eq. 5) as an eigenvalue equation with $\eta_{true} = 1$. The energies that yield eigenvalues closest to unity were then used as the bound state energy for that particular state. The complete wavefunction was then evaluated following the discretization defined in Eq. 16 above¹.

To observe the different partial wave channels, i.e. different configurations for the quantum numbers, we define a maximal value for each quantum number utilized in this simulation:

The maximal values were chosen based on the constraints due to how these quantum numbers are defined. For example, the spin of each particle $s_i = \frac{1}{2}$ so $s_{12,max}$

¹ <https://github.com/kwat0308/CompPhysWS2122>

$l_{12,max}$	$l_{3,max}$	$s_{12,max}$	$j_{12,max}$	$j_{3,max}$	$\tau_{12,max}$	$\tau_{3,max}$
1	0	1	2	$1/2$	1	$1/2$

TABLE I: Maximal values of quantum numbers utilized in the simulation.

cannot be larger than 1. Otherwise, such values were chosen to yield eigenvalues that converge to unity. The corresponding values for J are then chosen from such a choice of maximal quantum numbers.

The corresponding grid sizes for the momentum values p, q, x are chosen as $N_x = 8$, $N_p = 32$ and $N_q = 32$, and further the coefficients C_0, Λ that are defined for the OBE potential in Eq. 2 were chosen to have the following values: $C_0 = 0.020167$, $\Lambda = 700$ MeV. The same values of Λ are used for the spin-dependent potential as well. These grid and potential parameters were chosen such that the eigenvalue equation yield eigenvalues closest to unity.

With the given choice of parameters, we obtained an energy value of $E = -5.221$ MeV that converges with an eigenvalue of $|\eta_{true} - \eta| = 2.07 \times 10^{-13}$. This energy value is then used to determine the Faddeev component and thus the complete wave function.

VI. RESULTS

A. Off-Shell t -matrix

In Fig. 1 we show the dependence of the quantum numbers $\alpha_{12} = (l_{12}, s_{12}, j_{12}, \tau_{12})$ of the off-shell t -matrix evaluated from the Lippmann-Schwinger equation defined from Eq. 13. Four configurations were taken at $q = 0.00$ fm $^{-1}$.

From Fig. 1, we observe that the off-shell t -matrix has a clear dependence on the quantum number used to evaluate the matrix. From Figs. 1a and 1b, we see that varying the spin quantum number does not yield in any significant differences between the t -matrix element. However, if we compare Fig. 1a with those at higher l_{12} states (Figs. 1c, 1d), we observe that the number of oscillatory modes increase. This indicates that higher momentum values contribute for higher angular momentum states. Such contributions, however, are small as the amplitudes of the t -matrix decrease for larger angular momentum states.

Furthermore, we observe that the oscillations are linear with respect to p and p' , indicating that the diagonal elements, i.e. where $p = p'$, contribute most to the solution.

B. Three-Body Fermionic Wavefunction

In Fig. 2, we present our results for the three-body fermion wavefunction $|\Psi|$ for different quantum states, namely with $J = \frac{1}{2}$ and $J = \frac{3}{2}$. We also show the projection of the wavefunction onto the spectator momentum q in Fig. 3 for the same quantum states.

From Fig. 2, we see that the wavefunction peaks close to zero with a maximal amplitude of $|\Psi_{max}| \approx 25$ fm 3 . Furthermore, the wavefunction decays strongly as the momentum of both the two-body state and the spectator (p, q respectively) as the wavefunction increases. The structure of the wavefunction between the two quantum states remains the same, however, we observe that the peaks are more well-defined for $J = \frac{3}{2}$.

Furthermore, from Fig. 3 we observe that the behavior of the wavefunction strongly depends on the spectator momentum q , as the wavefunction is strongly peaked for lower momenta and spreads out with a lower amplitude for higher momenta. This is consistent with the results obtained from the off-shell t -matrix, as we observed that lower angular states yield oscillations only for small momenta. Here, we observe that for lower momenta, the $J = \frac{1}{2}$ solution yields a smoother structure as compared to the higher angular state. This may be caused by the higher oscillatory modes that are contained in the off-shell t -matrix for higher angular states.

However, the wavefunction that we have obtained is not smooth, indicating that the choice of our parameters was not sufficient enough to yield a stable solution. In the future, we should perform searches for maximal quantum numbers, grid parameters, and parameters related to the OBE potential that yield the most stable solution.

C. Comparison to Bosonic Wavefunction

In Fig. 4 we compare the obtained three-body fermionic wavefunction to the bosonic case as discussed in the lectures. We compared the same total orbital angular states $L = 0$, which corresponds to $J = 0$ and $J = \frac{1}{2}$ for the bosonic and fermionic case respectively. We also show the contour and surface plots for the bosonic wavefunction in the Appendix section.

We also evaluated the expectation value of the kinetic energy of the system for different values of the total angular momentum J . Those with integer and half-integer values of J are bosonic and fermionic wavefunctions respectively. The results are shown in Table II.

J	$\langle H_0 \rangle / \text{MeV}$
0	57.46
$1/2$	15.02
$3/2$	3.46

TABLE II: Expectation value of the kinetic energy H_0 of the three-body system with different total angular momenta.

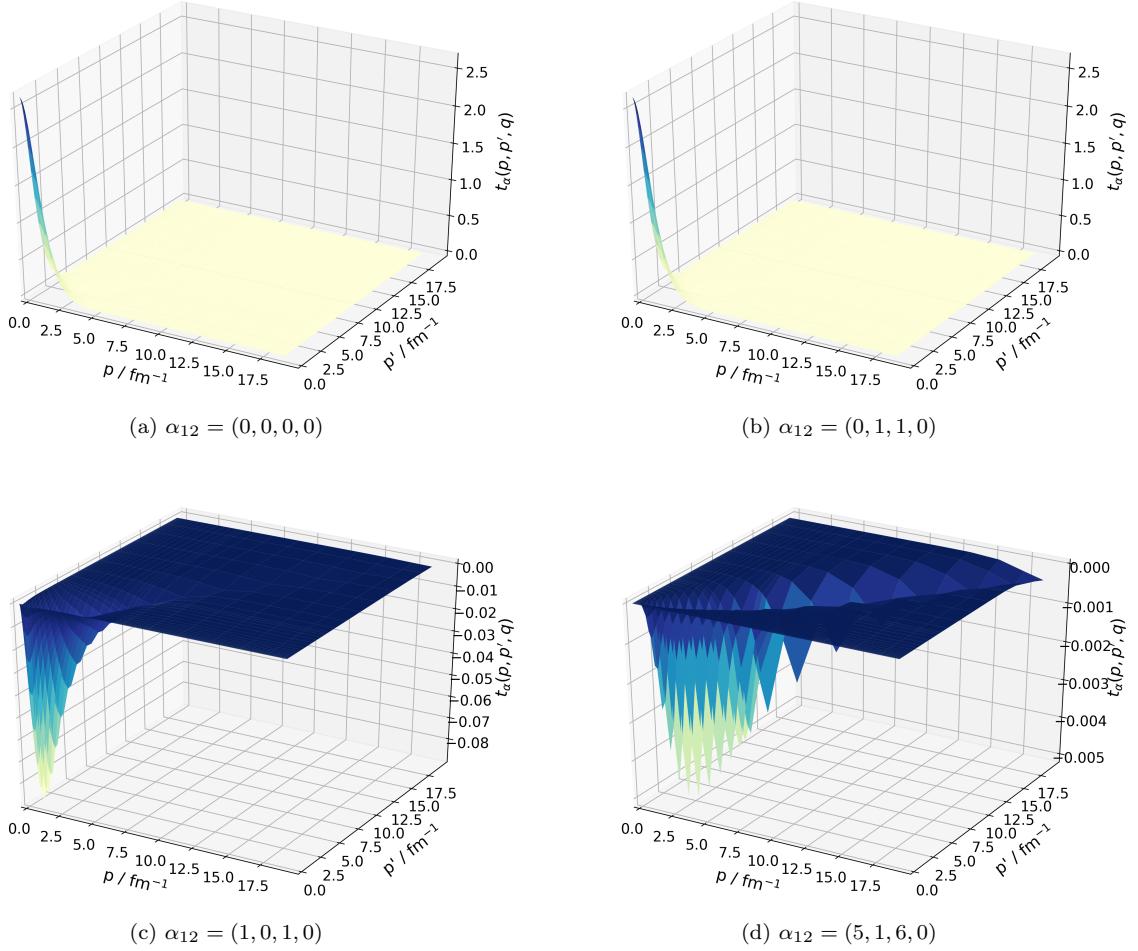


FIG. 1: The off-shell t-matrix element t_{12} for varying α_{12} values at $q = 0.00 \text{ fm}^{-1}$. Here $\alpha_{12} = (l_{12}, s_{12}, j_{12}, \tau_{12})$.

From Fig. 4, we see that the fermionic and bosonic wavefunctions behave similarly for varying spectator momenta. We observe that wavefunctions behave more smoothly at lower momenta. For the lowest momenta with $q = 4.31 \times 10^{-3} \text{ fm}^{-1}$, we observe that peak amplitude of the three-fermion wavefunction yields $|\Psi_{f,max}| \approx 13.35 \text{ fm}^3$, while the three-boson wavefunction yields a lower peak amplitude of $|\Psi_{b,max}| \approx 3.37 \text{ fm}^3$. The high probability amplitude of the fermion wavefunction implies that it is more realized in nature, and as such is a better description to describe the interaction between protons and neutrons.

By comparing the expectation values of the energies, we observe that as the angular states increase, the kinetic energy $\langle H_0 \rangle$ decreases. This is not consistent with the behavior that we would expect as higher states should increase the number of phase space and states available to contribute to the expectation value. This may be due to the fact that we have not considered the parity of the states when performing the simulation. In the future, this should be accounted for as well.

D. Spin-Dependent Potential

In Fig. 5, we compare the behaviour of the three-body fermion wavefunction under spin-independent and spin-dependent interactions and further compare with the bosonic wavefunction as well. We compare the wavefunctions for varying values of the spectator momenta. In the Appendix, we show the contour and surface plots for the spin-dependent fermion wavefunction as well.

We observe that by considering a spin-dependent potential, the wavefunction behaves similarly to the case of the bosonic wavefunction in Fig. 4. This is due to the fact that when considering fermionic interactions, the coupling due to the additional quantum numbers must be considered. In the spin-independent case, we have neglected this, and as such its structure deviates away from the bosonic case. This implies that by accounting for the spin dependence of the potential, we were able to construct the form of our wave function that is more realistic in nature.

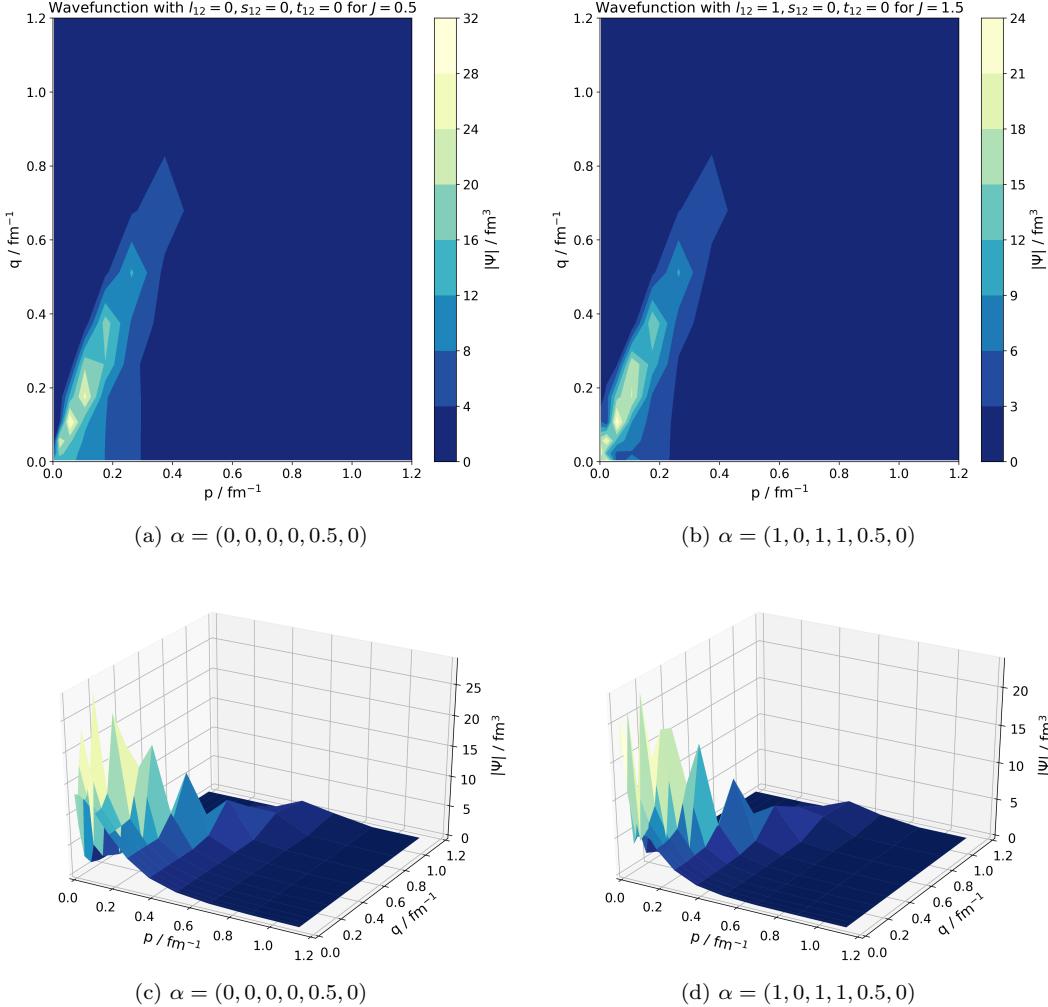


FIG. 2: The three-body fermionic wavefunction for $J = \frac{1}{2}$ (left) and $J = \frac{3}{2}$ (right). Here $\alpha = (l_{12}, l_3, s_{12}, j_{12}, j_3, \tau_{12})$. (a), (b) and (c), (d) show the contour and surface plots of the wavefunction respectively.

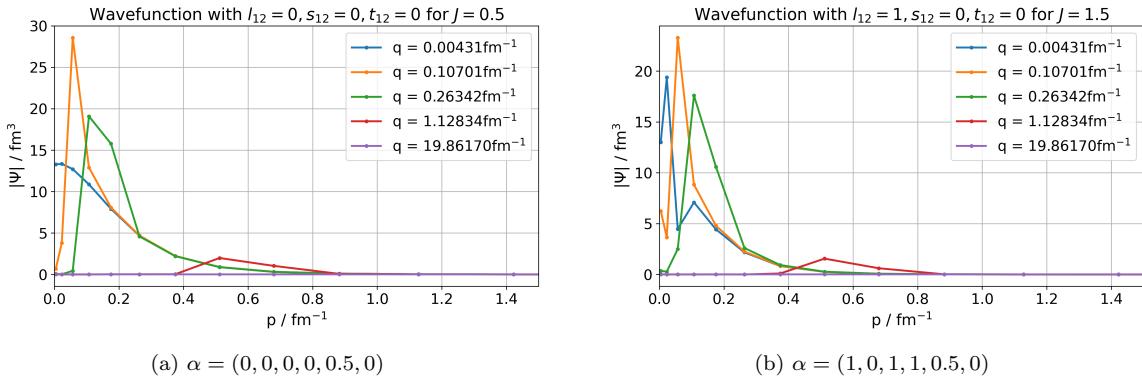


FIG. 3: The fermionic wavefunction at different values of the momentum q for (a) $J = \frac{1}{2}$ and (b) $J = \frac{3}{2}$. The parameter α is defined as above.

VII. CONCLUSION

In this paper, we constructed the three-body fermion wavefunction for different angular states $J = \frac{1}{2}$ and

$J = \frac{3}{2}$. To construct our problem, we modified the implementation constructed for the three-boson wavefunction

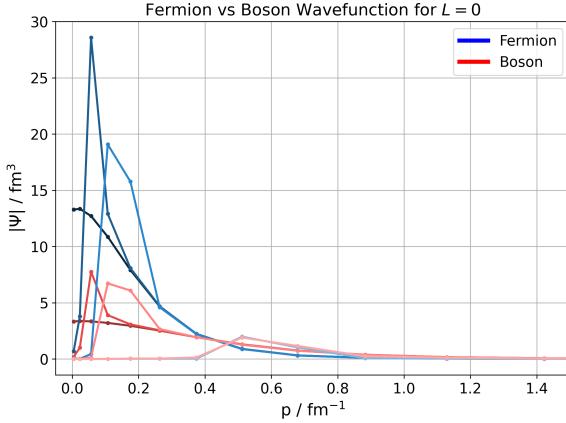


FIG. 4: The three-body fermionic and bosonic wavefunction for varying values of the spectator momentum q with $L = 0$. The following values for q were taken, following from the darker to lighter shades in color: $q = 4.31 \times 10^{-3} \text{ fm}^{-1}$, 0.107 fm^{-1} , 0.263 fm^{-1} , and 1.13 fm^{-1} .

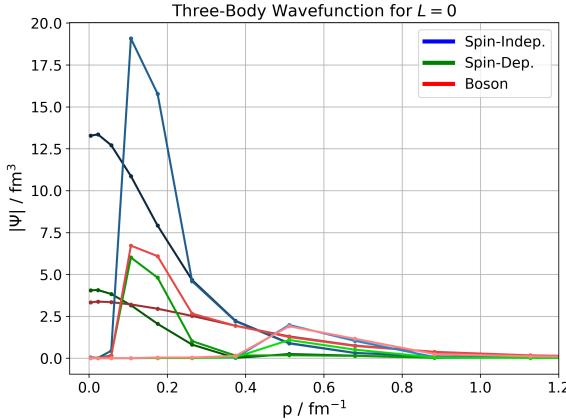


FIG. 5: The three-body fermionic wavefunction considering a spin-dependent and spin-independent interaction, as well as the boson wavefunction (with spin-indep. potential). Here we consider $L = 0$ for all three cases. We take $q = 4.31 \times 10^{-3} \text{ fm}^{-1}$, 0.263 fm^{-1} , and 1.13 fm^{-1} .

by taking into account additional quantum numbers such as the spin s , total angular momentum j , and isospin τ . The problem was then solved by evaluating an eigenvalue equation and using the energy value that yields an eigenvalue closest to unity. In our problem, we obtained an energy value of $E = -5.221 \text{ MeV}$ that yields an eigenvalue of $|1 - \eta| = 2.07 \times 10^{-13}$.

The resulting wavefunctions were shown to be peaked at lower momenta which decayed as the momenta in-

creased. We also observed that the lower angular state yielded a smoother result compared to those at a higher state, caused by the increase in oscillatory modes in the off-shell t -matrix.

Comparing our results to the bosonic case, we observed that the behaviour is similar except for the amplitudes, in which the boson wavefunction had smaller amplitudes. The kinetic energy for the bosonic case, however, was much larger than the fermion case, due to the considerations from additional quantum numbers. Finally, we saw that by considering the spin dependence of the potential, we were able to obtain a more realistic form of the wave function.

To extend this project, one can include the three-body force V_{123} that was neglected in this paper, and determine the resulting three-body fermion wavefunction. Furthermore, one can also increase the degrees of freedom by considering a four-body problem. This requires one to implement the Yakubovsky equations, which utilizes components similar to the Faddeev components used in this paper [4].

VIII. ACKNOWLEDGEMENTS

We would like to thank Nikolas Maximilian Schlage for mentoring us and guiding us through this project. We are extremely grateful for the help he has provided us with. We would also like to thank Dr. Andreas Nogga for providing us with the original implementation for the three Boson system, which was very helpful for us to implement the fermionic three-body system. We would also like to thank them for providing us with the implementation of the spin-dependent potential.

REFERENCES

- [1] E. Braaten and H.-W. Hammer. Universality in few-body systems with large scattering length. *Physics Reports*, 428(5–6):259–390, Jun 2006.
- [2] C. Elster, W. Schadow, A. Nogga, and W. Glöckle. Three-body bound-state calculations without angular-momentum decomposition. *Few-Body Systems*, 27(2):83–105, Nov 1999.
- [3] J. Golak, K. Topolnicki, R. Skibiński, W. Glöckle, H. Kamada, and A. Nogga. A three-dimensional treatment of the three-nucleon bound state. *Few-Body Systems*, 54(12):2427–2446, Jul 2012.
- [4] L. Platter, H.-W. Hammer, and U.-G. Meißner. Four-boson system with short-range interactions. *Physical Review A*, 70(5), Nov 2004.
- [5] A. Stadler, W. Glöckle, and P. U. Sauer. Faddeev equations with three-nucleon force in momentum space. *Phys. Rev. C*, 44:2319–2327, Dec 1991.

Appendix A: Supplementary Figures

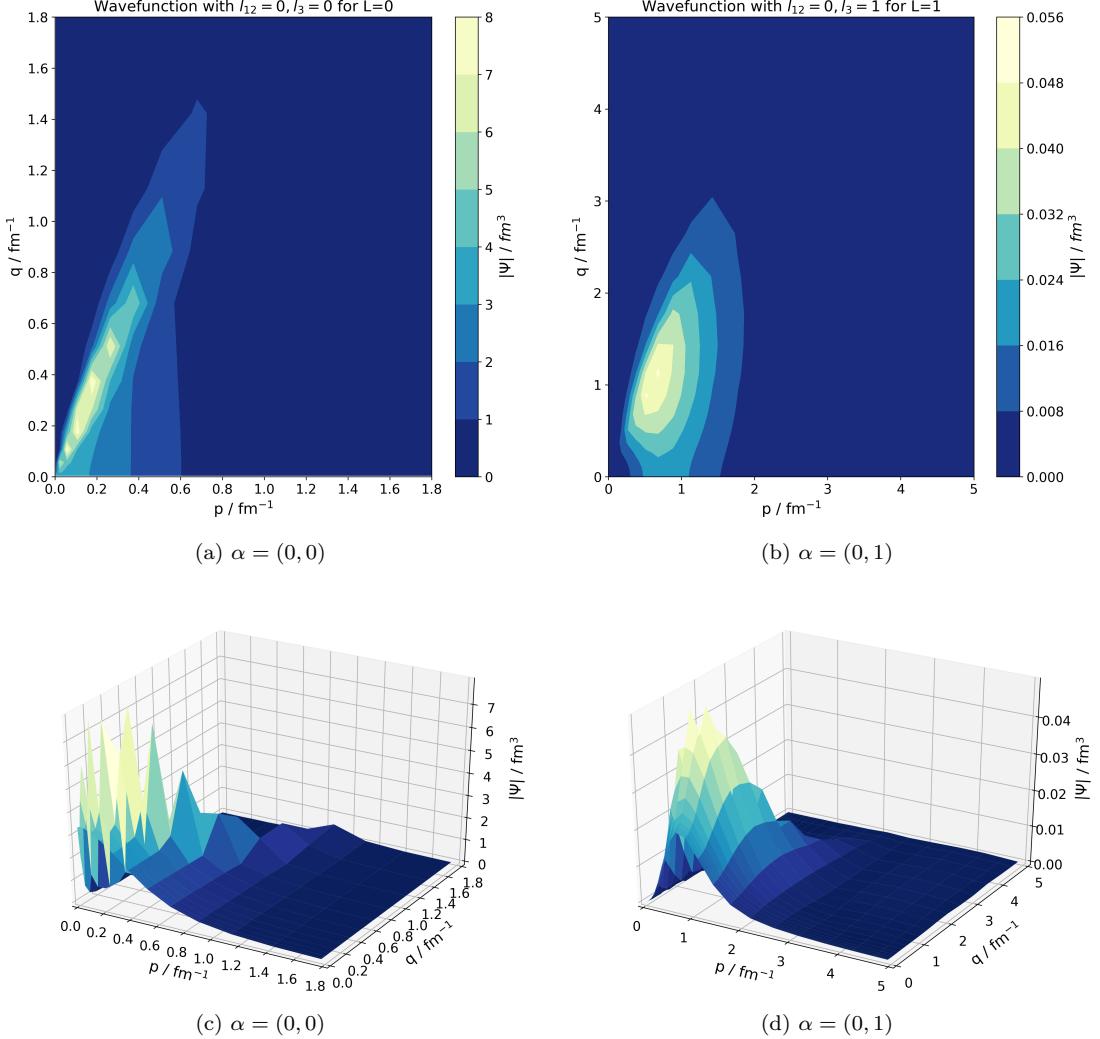


FIG. 6: The three-body boson wavefunction for $L = 0$ (left) and $L = 1$ (right). Here $\alpha = (l_{12}, l_3)$. (a), (b) and (c), (d) show the contour and surface plots of the wavefunction respectively.

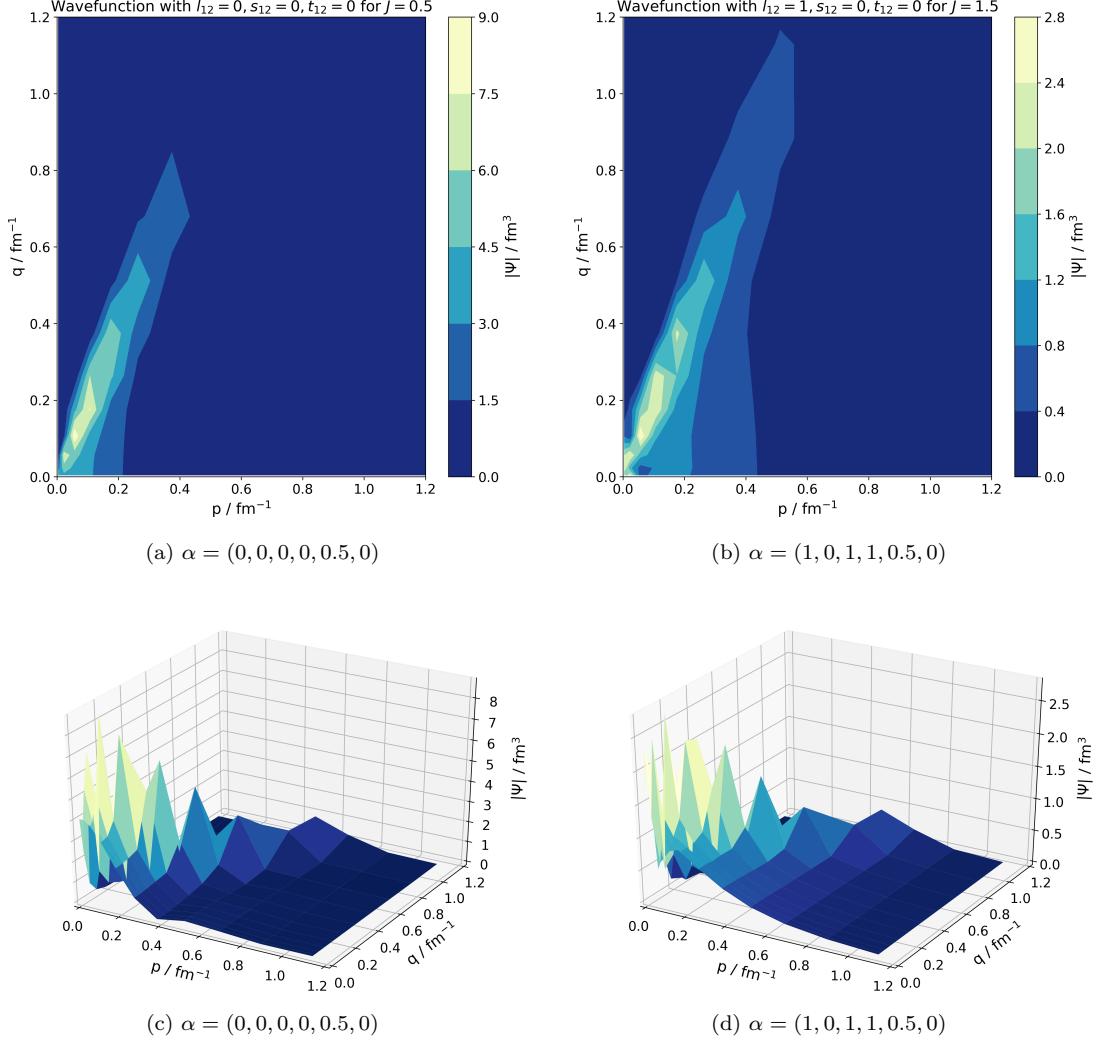


FIG. 7: Same as Fig. 2 but considering the spin-dependent potential.