Question 1

We can write:

$$t_\ell(p,p') = V_\ell(p,p') + \int_0^\infty dp'' \frac{f(p'')-f(q)}{q^2-p''^2} + f(q)\int_0^\infty dp'' \frac{1}{q^2-p''^2} - i\pi\frac{f(q)}{2q}$$

$$\Rightarrow V_\ell(p,p') = t_\ell(p,p') - \int_0^\infty dp'' \frac{f(p'')-f(q)}{q^2-p''^2} - f(q)\int_0^\infty dp'' \frac{1}{q^2-p''^2} + i\pi\frac{f(q)}{2q} \quad (A)$$

Now by discretizing the integrals in this equation, we can write the 2-D momentum array $(p,p') \to (p_i, p_j)$ with $i,j = 0,\dots,N$ (with $p_N = q$).

Now looking at the second term:

$$\int_0^\infty dp'' \frac{f(p'')-f(q)}{q^2-p''^2} = \int_0^\infty dp'' \frac{f(p'')}{q^2-p''^2} - f(q)\int_0^\infty \frac{dp''}{q^2-p''^2}$$

$$= 2\mu\left[\int_0^\infty dp'' \frac{p''^2 V_\ell(p,p'')\, t_\ell(p'',p')}{q^2-p''^2} - V_\ell(p,q)\cdot t_\ell(q,p')\cdot\int_0^\infty \frac{dp''}{q^2-p''^2}\right]$$

We discretize the $\int_0^\infty dp''$ integrals from $i = 0,\dots,N-1$ with associated weights W_i:

$$= 2\mu\left[\sum_{k=0}^{N-1} \frac{p_k^2\cdot V_{ik}\cdot t_{kj}}{q^2-p_k^2} W_k - V_{iN}\cdot t_{Nj}\, q^2 \cdot \sum_{m=0}^{N-1}\frac{W_m}{q^2-p''^2_m}\right] \quad ①$$

Now look at the third term:

$$f(q)\int_0^{\infty} dp'' \frac{1}{q^2-p''^2} \simeq f(q)\int_0^{p_{max}} dp'' \frac{1}{q^2-p''^2}$$

$$\underset{\text{notes}}{=} f(q)\,\frac{1}{2q}\ln\left(\frac{p_{max}+q}{p_{max}-q}\right)$$

$$= 2\mu\, q^2\, V_\ell(p,q)\, t_\ell(q,p')\cdot\frac{1}{2q}\ln\left(\frac{p_{max}+q}{p_{max}-q}\right)$$

$$= \mu q\, V_{iN}\cdot t_{Nj}\,\ln\left(\frac{p_{max}+q}{p_{max}-q}\right) \qquad ②$$

And finally the last term:

$$i\pi\,\frac{f(q)}{2q} = i\pi\cdot\frac{1}{2q}\cdot 2\mu q^2\, V_\ell(p,q)\, t_\ell(q,p')$$

$$= i\pi\,\mu q\, V_{iN}\, t_{Nj} \qquad ③$$

Now combining ①, ②, ③ to (*), we see that:

$$V_\ell(p,p') \rightarrow V_{ij} = \sum_{k=0}^{N} \delta_{ik}\cdot t_{kj} + \sum_{k=0}^{N-1}\frac{2\mu V_{ik}\cdot p_k^2}{q^2-p_k^2} w_k\, t_{kj}$$

$$+ \sum_{m=0}^{N-1}\frac{2\mu V_{iN}\cdot q^2}{q^2-p_m^2} w_m\, t_{Nj} - \mu q\, V_{iN}\ln\left(\frac{p_{max}+q}{p_{max}-q}\right) t_{Nj}$$

$$+ i\pi\,\mu q\, V_{iN}\, t_{Nj}$$

So with $A_{ik}\cdot t_{kj} = V_{ij}$, we obtain the expression of A_{ik} as desired:

$$\boxed{A_{ik} = \begin{cases} \delta_{ik} - \dfrac{2\mu V_{ik} p_k^2}{q^2-p_k^2} w_k & : \quad k \neq N \\ \delta_{ik} + \left(\displaystyle\sum_{m=0}^{N-1}\frac{2\mu V_{iN} q^2}{q^2-p_m^2} w_m\right) - \mu q\, V_{iN}\ln\left(\dfrac{p_{max}+q}{p_{max}-q}\right) + i\pi\,\mu q\, V_{iN} & : \quad k = N \end{cases}}$$

Question 5

Using the relation given in the sheet, we have:

$$|t(\vec{q}_f, \vec{q}_i)|^2 = \left[\sum_{l,m} Y_{lm}(\hat{q}_i) \cdot Y^*_{lm}(\hat{q}_f)\, t_l(q_f, q_i)\right] \cdot \left[\sum_{l',m'} Y_{l'm'}(\hat{q}_i)\, Y^*_{l'm'}(\hat{q}_f)\, t_{l'}(q_f, q_i)\right]^*$$

$$= \sum_{\substack{l,m,\\ l',m'}} Y_{lm}(\hat{q}_i)\, Y^*_{lm}(\hat{q}_f)\, Y^*_{l'm'}(\hat{q}_i)\, Y_{l'm'}(\hat{q}_f) \cdot t_l(q_f, q_i) \cdot t^*_{l'}(q_f, q_i)$$

$$= \sum_{l,l'} t_l(q_f, q_i) \cdot t^*_{l'}(q_f, q_i) \cdot \left(\frac{2l+1}{4\pi}\right)\left[\frac{4\pi}{2l+1} \sum_m Y_{lm}(\hat{q}_i)\, Y^*_{lm}(\hat{q}_f)\right] \cdot \left(\frac{2l'+1}{4\pi}\right) \cdot \left[\frac{4\pi}{2l'+1} \sum_{m'} Y_{l'm'}(\hat{q}_i)\, Y^*_{l'm'}(\hat{q}_f)\right]^*$$

$$= \sum_{l,l'} \frac{1}{16\pi^2} \cdot (2l+1)(2l'+1)\, t_l(q_f, q_i) \cdot t^*_{l'}(q_f, q_i) \cdot P_l(\hat{q}_f \cdot \hat{q}_i) \cdot P_{l'}(\hat{q}_f \cdot \hat{q}_i)$$

$$= \sum_{l,l'} \frac{1}{16\pi^2} \cdot (2l+1)(2l'+1)\, t_l(q_f, q_i) \cdot t^*_{l'}(q_f, q_i) \cdot P_l(\cos\theta) \cdot P_{l'}(\cos\theta)$$

$\therefore$ We can write the differential cross section as such:

$$\frac{d\sigma}{d\hat{q}_f} = (2\pi)^4 \mu^2 \sum_{l,l'} \frac{1}{16\pi^2} \cdot (2l+1)(2l'+1)\, t_l(q_f, q_i) \cdot t^*_{l'}(q_f, q_i) \cdot P_l(\cos\theta) \cdot P_{l'}(\cos\theta)$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\hat{q}_f} = \pi^2 \mu^2 \sum_{l,l'} (2l+1)(2l'+1)\, t_l(q_f, q_i)\, t^*_{l'}(q_f, q_i)\, P_l(\cos\theta)\, P_{l'}(\cos\theta)}$$