Question 2.5

The problem

Q2.5 For any integer x and y, if $x + y$ is odd, then x and y have opposite parity.

Define: Even Number: A number is even if it is divisible by 2, or if it is equal to $2k$ for some integer k.

Define: Odd Number: A number is odd if it is not even, or if it is equal to $2k + 1$ for some integer k.

Logical Variables: $S(x) = x + y = 2k + 1, p = x, q = y$

Expressed as a Universal Statement: $\forall x, y \in \mathbb{Z}, S(x) \rightarrow p \wedge q$

Thoughts on the problem: We can prove this problem by contradiction. We can assume that $x + y$ is odd and x and y have the same parity. Then we can show that this assumption leads to a contradiction.

Contradiction: $p \wedge \neg q$

Proof by Contradiction

Proof. Using the definition of an odd number assume that $x + y$ is odd and x and y have the same parity. Then there exists an integer k such that $x + y = 2k + 1$ and there exists an integer l such that $x = 2l$ and $y = 2l$.

Defining Set: $x, y, k, l, i \in \mathbb{Z}$

$$\begin{aligned}
x + y &= 2k + 1 \\
x &= 2l \\
y &= 2l \\
2l + 2l &= 2k + 1 \\
4l &= 2k + 1 \\
2(2l) &= 2k + 1 \\
i &= 2l \\
2i &= 2k + 1
\end{aligned}$$

This is a contradiction because $2i$ is even and $2k + 1$ is odd.

Therefore, the assumption that $x + y$ is odd and x and y have the same parity is false. Our original statement is true.

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