Nested Quantifiers

Learning Outcomes

- interpret statements with nested quantifiers
- negate a statement with nested quantifiers

We have probably encountered limits of functions at least once in our academic lives thus far, but have maybe never given real thought to the formal definition of a limit.

Def'n

The limt of $f(x)$ as x approaches a is L, denoted

$$\lim_{x \to a} f(x) = L,$$

if $\underbrace{\forall \varepsilon > 0}_{1}, \underbrace{\exists \delta > 0 \text{ s.t.}}_{2} \underbrace{\forall x \in (a-\delta, a+\delta)}_{3}, |f(x) - L| < \varepsilon.$

As presented, this definition contains 3 quantifiers. It is in fact quite come for statements and definitions in mathematics to contain multiple quantifiers. In this situation, the quantifiers are sometimes referred to as nested quantifiers.

It is important that we understand how to interpret nested quantifiers. Rather than starting with the limit, let's consider two simpler statements:

(i) $\forall s \in \mathbb{R}, \exists t \in \mathbb{R}$ s.t. $t > s$.

(ii) $\exists t \in \mathbb{R}$ s.t. $\forall s \in \mathbb{R}$, $t > s$.

In the process of understanding how to interpret these statements, we also want to determine how or if (i) and (ii) are related (as they look very similar at a first glance).

Q: How do we intepret (i)?

In words,

"for all real numbers s, there is a real number t such that $s > t$".

If we think about this like a game, player S announces a real number s, then player T announces a real number t. The statement is true if player T can pick a number larger than s. Because player S picks first, player T can use this knowledge when choosing t. Statement (i) is then true because player T could always choose $t = s+1$.

Q: How do we interpret (ii)?

In words,

"there exists a real number t such that for all real numbers s, $t > s$". i.e. read from left to right.

Continuing the game analogy, we have a similar situation, but the turn order is reversed, player T announces a real number t before player S announces s. The statement is true if player T can choose a t such that $t > s$ for all real numbers s that player S could choose. Because player T goes first, player S can always choose $s = t$, making the statement false.

Notably, we have seen that the order of the quantifiers matters when we have one universal and one existential quantifier. We can extend and generalize this idea.

Let X, Y be sets, $Q(x,y)$ a property determined by a choice of x in X and y in Y.

pick one in X then pick one in Y

pick one in Y then cycle through all of X

$$\forall x \in X, \exists y \in Y \text{ s.t. } Q(x,y) \not\equiv \exists y \in Y \text{ s.t. } \forall x \in X, Q(x,y)$$

not logically equivalent

$$\forall x \in X, \forall y \in Y, Q(x,y) \equiv \forall y \in Y, \forall x \in X, Q(x,y)$$

logically equivalent

$$\exists x \in X \text{ s.t. } \exists y \in Y \text{ s.t. } Q(x,y) \equiv \exists y \in Y \text{ s.t. } \exists x \in X \text{ s.t. } Q(x,y)$$

Note: When we have two of the same quantifier (in sequence) we can phrase them as "for all x in X and y in X" or "there exists an x in X and y in Y", i.e. combining the quantifiers into one statement.

Q: What happens when we have more than two quatifiers?

We can apply a similar game logic by breaking the statement into layers. Each quantifier is "nested" within the previous layer (hence nested quantifier).

Consider the statement

property depending on x,y,z

$$\exists x \in X \text{ s.t. } \forall y \in Y \ \exists z \in Z \text{ s.t. } R(x,y,z).$$

We can view the layers as

$$\exists x \in X \text{ s.t. } \Big[\forall y \in Y \ \big[\exists z \in Z \text{ s.t. } R(x,y,z) \big] \Big],$$

and rewrite it as

$$\left.\begin{array}{ll} & \exists\, x \in X \text{ s.t. } P(x) \\ \text{where } P(x) \text{ is} & \forall\, y \in Y,\ Q(x,y) \\ \text{where } Q(x,y) \text{ is} & \exists\, z \in Z \text{ s.t. } R(x,y) \end{array}\right\}$$

same resembles nested loops in programming

Ex.

limit: $\underbrace{\forall\, \varepsilon > 0}_{1},\ \underbrace{\exists\, \delta > 0 \text{ s.t.}}_{2}\ \underbrace{\forall\, x \in (a-\delta, a+\delta)}_{3},\ \underbrace{|f(x) - L| < \varepsilon}_{P(\varepsilon,\delta,x)}.$

When is this true? (i.e. $|f(x)-L| < \varepsilon$ is true)

(i) Pick an $\varepsilon > 0$.

(ii) Based on the choice of ε, pick a $\delta > 0$. $\delta = \delta(\varepsilon)$ i.e. it can depend on ε

(iii) Pick any x in $(a-\delta, a+\delta)$. x technically depends on δ and ε because the interval depends on $\delta = \delta(\varepsilon)$.

(iv) Check if $|f(x) - L| < \varepsilon$

(v) Repeat (iii)-(iv) for each x in $(a-\delta, a+\delta)$ ie. cycle through all of X

We next need to figure out how to negate quantified statements with nested quantifiers. Fortunately this is easily done with the tools at our disposal. First, recall that the negation for single quantified statements are given by

$$\neg(\forall x \in X, P(x)) \equiv (\exists x \in X \text{ s.t. } \neg P(x)),$$

$$\neg(\exists x \in X \text{ s.t. } P(x)) \equiv (\forall x \in X, \neg P(x)).$$

By viewing nested quantifiers in layers, we can systematically find the negation of a statement.

Ex.

$$\exists x \in X \text{ s.t. } \forall y \in Y, \exists z \in Z \text{ s.t. } R(x,y,z).$$

$$\neg(\exists x \in X \text{ s.t. } P(x)) \equiv (\forall x \in X, \neg P(x))$$

where $\neg P(x) \equiv \neg(\forall y \in Y, Q(x,y)) \equiv (\exists y \in Y \text{ s.t. } \neg Q(x,y))$

where $\neg Q(x,y) \equiv \neg(\exists z \in Z \text{ s.t. } R(x,y)) \equiv (\forall z \in Z, \neg R(x,y,z))$

Assembling the right side into a single statement, we get

$$\neg(\exists x \in X \text{ s.t. } \forall y \in Y, \exists z \in Z \text{ s.t. } R(x,y,z))$$
$$\equiv (\forall x \in X, \exists y \in Y \text{ s.t. } \forall z \in Z, \neg R(x,y,z))$$

We can make (and generalize) some observations regarding negations based on this example:

(i) variables and domains remain in the same order,

(ii) the quantifiers switch (i.e. $\forall \leftrightarrow \exists$),

(iii) the final property/statement is negated.

Notably, this is not a proof of how a general quantified statement transforms under negation, but any example can be handled systematically as in the example.

Ex.

$\forall s \in \mathbb{R}, \exists t \in \mathbb{R}$ s.t. $t > s$. we saw this true

Following our observations, the negation of this statement is

$\underbrace{\exists s \in \mathbb{R} \text{ s.t.}}_{1} \underbrace{\forall t \in \mathbb{R}}_{2}, t \leq s.$ this must be false because it is the negation.

Q: Is the negation true or false?

1. Pick an $s \in \mathbb{R}$
2. Pick a $t \in \mathbb{R}$
3. Check if $t \leq s$
4. Repeat 2. and 3. for all $t \in \mathbb{R}$.

The negation is clearly false because we could choose $t = s+1$.