Motivating Examples I

Learning Outcomes

- describe the proof of several elementary facts
- apply similar proof methods to similar elementary facts

Before we delve into the finer details of logic and proofs, it will be beneficial to see some examples of proofs to help motivate the ideas we are learning. While one might think that proofs all have to do with high-level abstract math, there are actually many elementary facts (i.e. things from elementary/high school) that have interesting and inspired proofs.

Note: At times we will take some facts that we use for granted.
It will be noted when this happens.

The first proof that we will encounter is (probably) the most iconic basic proof in math (you may have even encountered it before):

Theorem

The number $\sqrt{2}$ is irrational. $\rightsquigarrow$ what does this statement mean mathematically?

Before we can prove it, we first need to recall the definition of an irrational number.

Def'n

A number α is rational if $\alpha = \frac{p}{q}$, where p and q are intergers (i.e. $0, \pm 1, \pm 2, \ldots$), and $q \neq 0$.

A number α is irrational if it is $\underbrace{\text{not rational}}_{\text{i.e. } \alpha \neq \frac{p}{q}}$.

In other words, the theorem states that $\sqrt{2}$ cannot be written as a fraction.

<u>Theorem proof</u>:

Suppose that $\sqrt{2}$ is rational. This means we can write

(i) $$\sqrt{2} = \frac{p}{q}$$

for some integers p and q, $q \neq 0$.

Let's assume that p/q is written in lowest terms, i.e. p and q share no common factors. (e.g. $\frac{21}{6}$ vs. $\frac{7}{2}$).

⌜Note: the fact that we can express a fraction in "lowest terms" is a fact that we will take for granted ⌟

Squaring both sides of the expression (i), we get

$$2 = \frac{p^2}{q^2}.$$

We can rearrange this to get

(ii) $$p^2 = 2q^2,$$

i.e. p^2 is an even number.

If p^2 is even, then p is also even. [note: another fact that we will (temporarily) take for granted] (spoilers)

This means we can write

(iii) $$p = 2k$$

where k is some integer.

If we plug this expression for p (iii), into (ii), we get

$$2q^2 = p^2 = (2k)^2 = 4k^2.$$

Rearranging this we get

$$q^2 = 2k^2.$$

This means that q^2 is even, and so q is also even.

Both p and q are even, which contradicts our assumption that p/q is in lowest terms, meaning that $\sqrt{2}$ cannot be rational. $\blacksquare$

end of proof

This proof is an example of "proof by contradiction", where we assume the conclusion (i.e. $\sqrt{2}$ is irrational) is not true, and aim to find an impossible consequence (i.e. p/q is both in lowest terms and not in lowest terms).

Once a result is proved, mathematicians often look to generalize that result by find other specific cases (e.g. is $\sqrt{3}$ irrational? is $\sqrt[4]{2}$ (spoilers) irrational?) or by unifying many cases in a single grand theorem (e.g. what are the conditions on n and k to make $\sqrt[k]{n}$ irrational?).

It is not always the case that the proof of a generalizations follows a similar proof method (but for our example some do).

Theorem

The number $\sqrt{3}$ is irrational.

Proof:

Suppose $\sqrt{3}$ is rational with $\sqrt{3} = p/q$ in lowest terms.

Squaring and rearranging we have

$$p^2 = 3q^2,$$

so p^2 is a multiple of 3.

This implies that p must be a multiple of 3 or else it would not arise in the prime factorization of p. (ie. how can $p^2 = 3\alpha$ but $p \neq 3\beta$)

[Note: prime factorization is a major fact that we are taking for granted]

This means that $p = 3k$, for some integer k.

Plugging this into p^2, we get

$$3q^2 = p^2 = (3k)^2 = 9k^2,$$

meaning that $q^2 = 3k^2$, and thus q is also a multiple of 3.

This means both p and q are multiples of 3, which contradicts the assumption that p/q is in lowest terms.

This means that $\sqrt{3}$ must be irrational. $\blacksquare$

This proof employed a very similar method to the first, however we needed to use an extra fact (a notably non-trivial fact compared to the notions of even and odd).

Once a fact has been proved, it is a tool that can be used for other proofs (similar to the facts that we have been using thus far).

Corollary

theorem-like statement whose proof is the direct result of a previous theorem

The number $\sqrt{18}$ is irrational.

Proof.

Suppose $\sqrt{18}$ is rational, and $\sqrt{18} = \frac{p}{q}$ is in lowest terms.

Note that

$$\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}.$$

We can then rearrange the fraction to get

$$\sqrt{2} = \frac{p}{3q}.$$

We know that $\sqrt{2}$ is not rational, so this is a contradiction.

This means $\sqrt{18}$ is irrational. ■

Notably, if we did not already know that $\sqrt{2}$ is irrational, we could have proved this corollary directly, but when we have a tool, we should use it to make our work easier.