Assignment 2

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Question 1

When we learn mathematics growing up, we are not treated to the axioms, definitions, theorems, and proofs that form the foundations. Rather, we are told that various facts are true, and justify these facts by verifying that they are true for many specific cases (but notably not in general). Even when working with basic numbers and operations, there are facts that we always take for granted that can be easily proved, and some facts that we might easily believe are true because we have never seen a specific case where they fail.

Answer each of the following.
[12 marks]

a. Consider the numbers $8, 13, 35, 46$. Compute the sum and difference of each possible pairing (e.g. 8 and 13, 8 and 35, etc.). [Note: You don't need to double up, e.g. you don't need to do $8 + 13$ and $13 + 8$ because we know they are the same] What do you notice about the results (regarding even and odd)?

Let n, m be even numbers, and p, q be odd numbers. (Hint: It will be useful for parts b.-d. to use a general representation of even and odd numbers.)

b. Prove that $n + m$ and $n - m$ are even.

c. Prove that $p + q$ and $p - q$ are even.

d. Prove that $n + p$ and $n - p$ are odd.

For each statement below, find a counterexample.

e. If the sum of one pair of positive integers is larger than the sum of another pair, then the product is also larger.

f. If the product of one pair of positive integers is larger than the product of another pair, then the sum is also larger.

Question 2

When trying to generalize our second main elementary fact (*for any natural number, $n^2 - n$ is even*), we saw that two particular generalizations were equivalent:

- $n^3 - n$ is divisible by 3,
- $(n + 1)n(n - 1)$ is divisible by 3.

It would be interesting to know if these two generalizations continue to be equivalent, such as

- $n^4 - n$ is divisible by 4,
- the product of four consecutive numbers is divisible by 4,

or if 2 and 3 are just special cases, and moreover if these generalizations continue to be true.

Answer each of the following.
[10 marks]

a. Prove that we cannot write $n^4 - n$ as a product of four consecutive numbers (thus showing that the two generalizations are no longer equivalent)? (Hint: Factor $n^4 - n$.)

b. Is $n^4 - n$ divisible by 4? Show your work.

c. Is *the product of four consecutive numbers divisible by* 4? Is *the product of five consecutive numbers divisible by* 5? Explain.

d. Formulate a conjecture about the product of n consecutive numbers. (i.e. write a statement similar to part c.)

e. Formulate four additional generalizations to the our second main elementary fact. (Note: You do not need to prove them.)

Question 3

Proofs that involve pictures (whether we can actually draw them or need to imagine them in our heads) often present very straightforward ways to verify a fact, but can be difficult to generalize. There can be misconceptions in how to picture the generalization, such as creating an incorrect picture that will prove that fact, rather than the actual picture representing the generalization, or misconceptions in how to translate the generalized picture into the language of mathematics, such as incorrectly defining an equation that represents part of the generalized picture.

Answer each of the following.
[8 marks]

a. Critique the following argument.

Statement: *For any natural number, $n^3 - n$ is even.* (Hint: This statement is in fact true, but you should be looking for a logical mistake in the following proof.)

> **Proof:** Consider an $n \times n \times n$ cube grid. There are n^3 small cubes. Remove the n small cubes along the long diagonal through the cube so that there are $n^3 - n$ cubes remaining. Since the cube is separated into two identical and symmetric pieces by the main diagonal, one on each side, it must be that $n^3 - n$ is even.

One generalization of our second main elementary fact that we have (maybe) not yet considered is the following:

Theorem (1)**.** *For any natural number, $n^3 - n^2$ is even.*

b. Prove Theorem (1) using algebra.

c. Prove Theorem (1) using a geometric argument. (Hint: What shapes do n^3 and n^2 look like?)