

Assignment 3

Please upload a **clear** and **legible** submission onto Gradescope by 6pm.

Question 1

Sets are essential in mathematics for expressing quantified statements and constructing rigorous proofs. They provide a structured framework for organizing mathematical objects, facilitating the formulation of universally applicable truths about specific classes of elements. The different set notations (sentence, roster, and set-builder), offer versatility in expressing mathematical sets. We need to be able to transition between the different set notations, specifically going to or from sentence notation.

For each set S given below, describe the set using the other two notations: sentence notation, roster notation, or set-builder notation. (Hint: To find a suitable sentence for b. and c., think about what the set is describing. The set-builder form for a. and b. is not unique.)(Note: When describing the set in roster notation, you need to provide enough elements to make the set clear.)

[6 marks]

- a. The set S of integers divisible by 3 and 5.
- b. $S = \{1, 2, 3, 4, 6, 12\}$
- c. $S = \{4^k : k \in \mathbb{Z}\}$

Question 2

If you revisit our proof that $\sqrt{2}$ is irrational, you will see that negation played an important role. There are in fact two methods of proof for which negation is the necessary first step: proof by contradiction and proof by contraposition (we will explore these both later in the course). It is then important that we understand how to find the negation of a statement.

Answer each of the following.

[12 marks]

- a. Identify if each of the following sentences are statements. For each statement, write the negation.
 - i. There exists a pair of irrational numbers a and b such that $a \cdot b$ is a rational number.
 - ii. The set $\{3n : n \in \mathbb{N}\}$.
 - iii. The sum of interior angles of a triangle is 180° .

- iv. The limit of a sequence approaches infinity as n goes to infinity.
- b. Compose two mathematical statements (one quantified and one non-quantified), and two mathematical non-statements. For each statement, write the negation.

Question 3

We currently live in a time of rapidly developing technology. One particular topic of interest is generative AI, specifically ChatGPT. It has shown itself to be a useful (but also controversial) tool in a diverse range of situations. It is increasingly important to understand the role that such technology can play in various echelons and domains of science.

Below is a proof of the statement *If the product of one pair of positive integers is larger than the product of another pair, then the sum is also larger*, given by ChatGPT from the prompt “please prove that if the product of one pair of positive integers is larger than the product of another pair, then the sum is also larger”.

Proof. Let's consider two pairs of positive integers: (a, b) and (c, d) , where a, b, c, d are positive integers. We are given that the product of the first pair is larger than the product of the second pair, i.e., $ab > cd$.

Now, let's analyze the sum of each pair:

Sum of the first pair: $a + b$

Sum of the second pair: $c + d$

We want to prove that if $ab > cd$, then $a + b > c + d$. Let's assume, for the sake of contradiction, that $a + b \leq c + d$. Now, we can rearrange this inequality to get:

$$a \leq c + d - b$$

Since a and b are positive integers, we can conclude that:

$$a < c + d$$

Now, multiply both sides by b :

$$ab < b(c + d)$$

Since $ab > cd$ (as given in the problem), we have a contradiction because ab cannot be both greater than and less than $b(c + d)$. Therefore, our assumption that $a + b \leq c + d$ must be false. This implies that $a + b > c + d$. So, we have successfully proven that if the product of one pair of positive integers is larger than the product of another pair, then the sum is also larger. \square

Answer each of the following.

[12 marks]

- a. As we saw in Assignment 2 Question 1f., this statement is in fact false. Critique the “proof” given by ChatGPT. (Hint: What statements in the proof are correct? What statements in the proof are false? What statements in the proof are out of place? What statements in the proof make no sense?)
- b. Prompt ChatGPT to prove that $\sqrt{2}$ is irrational. How does the proof compare to our proof during class? (Hint: If you are unsure how to prompt it, you can use “please prove that root(2) is irrational”.)
- c. Why do you think ChatGPT is able to provide a correct proof of that $\sqrt{2}$ is irrational, when it failed in the above example?
- d. In what ways do you think ChatGPT can be a useful tool regarding proofs in mathematics?