Proof of 1.3

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In order to prove $Fib(n)$ is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$, we need to first prove $Fib(n) = \frac{(\varphi^n - \psi^n)}{\sqrt{5}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Here is the definition of $Fib(n)$:

$$Fib(n) = \begin{cases} 0 & \text{if n = 0,} \\ 1 & \text{if n = 1,} \\ Fib(n-1) + Fib(n-2) & \text{otherwise.} \end{cases} \tag{1}$$

Consider there is a matrix Q that satisfies $Q \begin{pmatrix} Fib(n-1) \\ Fib(n) \end{pmatrix} = \begin{pmatrix} Fib(n) \\ Fib(n+1) \end{pmatrix}$. Use the definition above we can easily find such a Q, which is to solve the equation:

$$Q \begin{pmatrix} Fib(n-1) \\ Fib(n) \end{pmatrix} = \begin{pmatrix} Fib(n) \\ Fib(n) + Fib(n-1) \end{pmatrix}$$

And thus $Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Therefore, for any given interger n, the formula $Q^n \begin{pmatrix} Fib(0) \\ Fib(1) \end{pmatrix} = \begin{pmatrix} Fib(n) \\ Fib(n+1) \end{pmatrix}$ will tell us $Fib(n)$. Here $\begin{pmatrix} Fib(0) \\ Fib(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Our idea is to find two eigen vectors of matrix Q, and substitute $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with the combination of two eigen vectors. Assume they are v_1 and v_2.

Solve the equation:

$$\begin{aligned} Qv &= \lambda v \\ (Q - \lambda E)v &= 0 \end{aligned} \tag{2}$$

$$det(Q - \lambda E) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \tag{3}$$

$$\begin{aligned} \lambda(\lambda - 1) - 1 &= 0 \\ \lambda^2 - \lambda - 1 &= 0 \end{aligned}$$

So we get $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$, which is φ and ψ. And the corresponding eigen vectors are $v_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} \end{pmatrix}$ and $v_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} \end{pmatrix}$