Proof of 1.3

Aximaris

In order to prove $Fib(n)$ is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$, we need to first prove $Fib(n) = \frac{(\varphi^n - \psi^n)}{\sqrt{5}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Here is the definition of $Fib(n)$:

$$Fib(n) = \begin{cases} 0 & \text{if n = 0,} \\ 1 & \text{if n = 1,} \\ Fib(n-1) + Fib(n-2) & \text{otherwise.} \end{cases} \tag{1}$$

Consider there is a matrix Q that satisfies $Q \begin{pmatrix} Fib(n-1) \\ Fib(n) \end{pmatrix} = \begin{pmatrix} Fib(n) \\ Fib(n+1) \end{pmatrix}$. Use the definition above we can easily find such a Q, which is to solve the equation:

$$Q \begin{pmatrix} Fib(n-1) \\ Fib(n) \end{pmatrix} = \begin{pmatrix} Fib(n) \\ Fib(n) + Fib(n-1) \end{pmatrix}$$

And thus $Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Therefore, for any given interger n, the formula $Q^n \begin{pmatrix} Fib(0) \\ Fib(1) \end{pmatrix} = \begin{pmatrix} Fib(n) \\ Fib(n+1) \end{pmatrix}$ will tell us $Fib(n)$. Here $\begin{pmatrix} Fib(0) \\ Fib(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Our idea is to find two eigenvectors of matrix Q, and substitute $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with the combination of two eigenvectors. Assume they are v_1 and v_2.

Solve the equation:

$$\begin{aligned} Qv &= \lambda v \\ (Q - \lambda E)v &= 0 \end{aligned} \tag{2}$$

$$det(Q - \lambda E) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \tag{3}$$

$$\begin{aligned} \lambda(\lambda - 1) - 1 &= 0 \\ \lambda^2 - \lambda - 1 &= 0 \end{aligned}$$

So we get $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$, which is φ and ψ. And the corresponding eigenvectors are $v_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} \end{pmatrix}$ and $v_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} \end{pmatrix}$.

Once the eigenvectors have been set, the initial vector of $Fib(n)$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ now can be expressed by the combination of v_1 and v_2. But consider a better vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = Q \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and here we get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} (v_1 - v_2)$$

And thus, we can tell for any given n, the $Fib(n)$ will be:

$$\begin{aligned}\begin{pmatrix} Fib(n) \\ Fib(n+1) \end{pmatrix} &= Q^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= Q^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \left(\varphi^{n-1} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{3+\sqrt{5}}{2} \end{pmatrix} - \psi^{n-1} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} \end{pmatrix} \right)\end{aligned}$$

Immediately we get $Fib(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$. Notice that $Fib(n)$ will always be an integer. Therefore, if $\frac{1}{\sqrt{5}}\varphi^n < Fib(n) + \frac{1}{2}$, then the $Fib(n)$ is the closest integer to φ^n.

Notice that $\frac{1}{\sqrt{5}}\varphi^n = Fib(n) + \frac{1}{\sqrt{5}}\psi^n$, so all we have to do is to prove:

$$\frac{1}{\sqrt{5}}\psi^n < \frac{1}{2}$$

Clearly,

$$\frac{1}{\sqrt{5}} < \frac{1}{2} \text{ , and } \psi < 1 \text{ , which means } \psi^n < 1 \text{ too}$$

Therefore, $\frac{1}{\sqrt{5}}\psi^n < \frac{1}{2}$ is true.

So our final conclusion is that $Fib(n)$ is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$.