Applications

Implications for portfolio optimization

- Standard unconstrained portfolio optimization, without transaction costs, maximizes a mean-variance objective

$$\max_{\boldsymbol{h}\in\mathbb{R}^n} \left(\boldsymbol{h}'\mathbb{E}[\boldsymbol{r}] - \frac{1}{2}\kappa\boldsymbol{h}'\Sigma\boldsymbol{h}\right) \tag{4.1}$$

$$\Sigma := \mathbb{V}[\boldsymbol{r}] \in S^n_{++} \tag{4.2}$$

where $\kappa > 0$.

- In the absence of constraints and transaction costs, this problem can be easily solved in closed form by solving the first-order condition.
- The solution is

$$\boldsymbol{h}^* = (\kappa\Sigma)^{-1}\mathbb{E}[\boldsymbol{r}] \tag{4.3}$$

- The problem (4.1) is mathematically equivalent to various constrained versions (e.g. minimize variance subject to a target return equality constraint) by Lagrange duality.

- For US equities $n \approx 1500$ to 3000 depending on whether our strategy includes small caps.
- With T historical periods, one has:

$$
\begin{aligned}
nT &\text{ data points} \\
n(n+1)/2 &\text{ free parameters in cov matrix} \\
2T/(n+1) &\text{ data points per parameter}
\end{aligned}
$$

- If $n = 2500$ we need 10 years' history to get 2 data points per parameter!

- Let B be a $T \times n$ matrix having the stock return time series as columns.

$$B = \begin{pmatrix} r_{1,t-1} & r_{2,t-1} & \cdots & r_{n,t-1} \\ r_{1,t-2} & r_{2,t-2} & \cdots & r_{n,t-2} \\ \vdots & \vdots & & \vdots \\ r_{1,t-T} & r_{2,t-T} & \cdots & r_{n,t-T} \end{pmatrix}$$

- Now transform B by subtracting the mean from each column.
- The maximum likelihood estimator of the covariance matrix Σ is then

$$\hat{\Sigma} = \frac{1}{T-1} B'B. \tag{4.4}$$

- Now $B'B$ is an $n \times n$ matrix with

$$\mathrm{rank}(B'B) = \mathrm{rank}(B) \leq \min(n, T)$$

  so if $T < n$ it is impossible that (4.4) is invertible, and if T is slightly more than n, the matrix (4.4) may be invertible, but remains extremely ill-conditioned.
- In any case (4.4) is not suitable for use in mean-variance optimization (4.3).
- It seems we made a mistake somewhere, but where?

- The issue is not with the original problem formulation (4.1), which can be derived from sound economic principles (ie. expected utility theory due to Arrow (1963) and Pratt (1964)) as long as the distribution of asset returns $p(\boldsymbol{r})$ is somewhat well-behaved.
- Rather, the main issue is that (4.4) generally produces bad variance forecasts for large classes of portfolios.
- For example, if Σ has a null vector v,

$$\Sigma v = 0,$$

  then construct a long-short portfolio with holdings proportional to v.
- This portfolio is forecasted to have zero volatility by Σ.
- We shall see by the end of the lecture how to produce better covariance forecasts that give believable results for all portfolios.

Asset pricing

- For the purposes of the present discussion, a reasonable convention is to assume that t is the market close time on a given day, and $p_{i,t}$ is the official close price of the i-th asset on the given day.
- If time is measured in days, then $t + 1$ denotes the time of the market closing on the next trading day after t.

- The *return* of the i-th asset over the interval $[t, t+1]$ will be denoted

$$
\begin{aligned}
r_{i,t+1} &= p_{i,t+1}/p_{i,t} - 1 && (4.5)\\
&\approx \log(1 + r_{i,t+1}) && (4.6)\\
&= \log p_{i,t+1} - \log p_{i,t} && (4.7)
\end{aligned}
$$

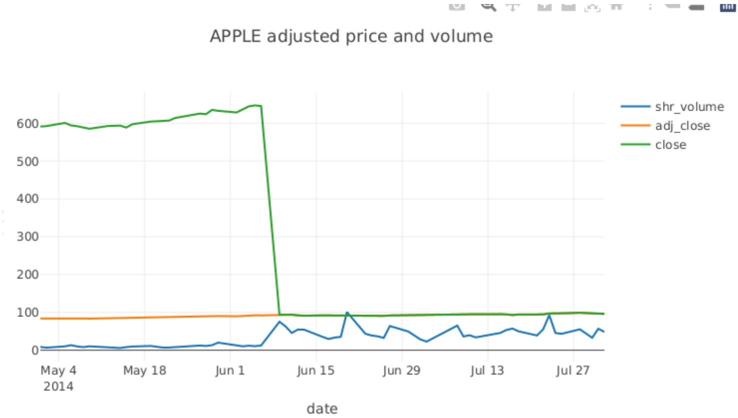
  where the approximation (4.6) is the first-order Taylor approximant, and hence is valid for small returns, i.e. a few percent.
- The times t and $t+1$ may be in the past or in the future.
- Data pertaining to future intervals should be regarded as a random variable.
- For example, if we specify that t is the present time, then

$$p_{i,t+1} \quad \text{and} \quad r_{i,t+1}$$

  for all $i = 1, \ldots, n$ are automatically regarded as random variables.

- The prices $p_{i,t}$ sometimes need to be adjusted in order that the returns (4.5) represent the total return from holding the asset.
- For example, in the equity markets, stock splits and dividends are common occurrences.
- If there are stock splits or dividends between t and $t+1$, the effect of those should be included in the return variable, for example by adjusting the price before the split to be in the same units as price after the split before calculating (4.5).
- Once this adjustment has been made, the quantity (4.5) is called *total return*.

The following example data illustrates a 7-to-1 split for Apple Inc (AAPL), sourced from the CRSP daily stock file.

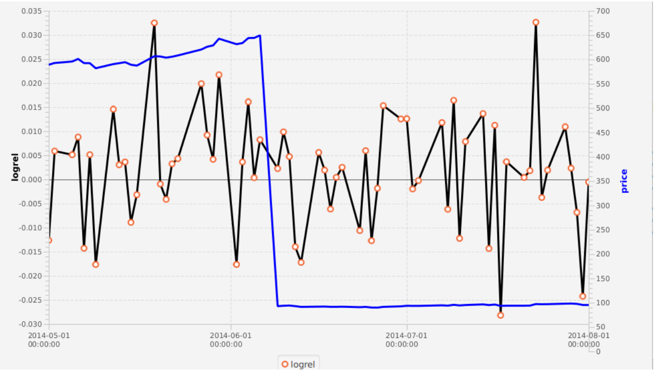


(image credit: Gordon Ritter with CRSP data)

The exact prices around the split, and the total return calculation, are shown in the following table.

t	p[t]	t+1	p[t+1]	r[t+1]
2014-06-04	644.82	2014-06-05	647.35	0.003924
2014-06-05	647.35	2014-06-06	645.57	-0.00275
2014-06-06	645.57	2014-06-09	93.7	0.016001
2014-06-09	93.7	2014-06-10	94.25	0.00587
2014-06-10	94.25	2014-06-11	93.86	-0.004138

We can also plot the price and the log-return through the split date, using right and left y-axes to control the vastly different ranges of the two time series.



(image credit: Gordon Ritter with CRSP data)

Definition 4.1

Let

$$\boldsymbol{r}_{t+1} = \begin{pmatrix} r_{1,t+1} \\ r_{2,t+1} \\ \vdots \\ r_{n,t+1} \end{pmatrix} \tag{4.8}$$

denote the $n \times 1$ *column vector of asset returns over the time interval* $[t, t+1]$*. The vector* (4.8) *is called a cross section, for added clarity to distinguish it from a time series.*

The Capital Asset Pricing Model

Throughout this and the coming sections, we will keep the above notation for prices and returns, except for two small modifications which will make our lives easier.

1. When the time interval has otherwise been specified, we may drop the explicit time index, so $r_{i,t+1}$ becomes r_i when the interval $[t, t+1]$ is known, and similarly $\boldsymbol{r}_{t+1}$ from (4.8) becomes simply $\boldsymbol{r}$.
2. Returns henceforth will be *excess returns* above the risk-free rate, so e.g. from now on

$$r_{i,t+1} = \frac{p_{i,t+1}}{p_{i,t}} - 1 - r_f$$

where r_f is the one-day return on domestically held short-dated government bonds.

- We now recall the main result of the capital asset pricing model (CAPM).
- Suppose that the current time is t, and $t+1$ is in the future.
- Hence $r_i = r_{i,t+1}$ is a random variable.
- In the CAPM, all investors share a common joint probability distribution for asset returns

$$p(r_1, r_2, \ldots, r_n) = p(\boldsymbol{r}) \tag{4.9}$$

- Similarly, r_M is the expected return of the single efficient fund, and σ_M is the ex ante volatility of r_M.
- Since all investors know (4.9), they also share knowledge of σ_M and $\mathbb{E}[r_M]$.

- Since there's only one risky portfolio in this model, all investors must hold some combination of this portfolio and the risk-free asset.
- Any portfolio with weights adding to 1 must be of the form $1-h$ units of the risk-free asset, and h units of M where $h \in \mathbb{R}$.
- The expected return and risk of this portfolio are:

$$\bar{r} = (1-h)r_f + h\bar{r}_M, \quad \sigma = h\sigma_M$$

where, for notational convenience, we will denote $\mathbb{E}[\boldsymbol{r}]$ as $\bar{r}$, and continue this notation throughout the lecture.

- We can then eliminate $h = \sigma/\sigma_M$ to find

$$\bar{r} = (1 - \frac{\sigma}{\sigma_M})r_f + \frac{\sigma}{\sigma_M}\bar{r}_M = r_f + \frac{\sigma}{\sigma_M}(\bar{r}_M - r_f) \tag{4.10}$$

- Eq. (4.10) is called *the capital market line*.
- It tells us that the expected excess return of any *efficient* portfolio is a constant times its risk, where the constant is the so-called *price of risk*

$$\frac{\bar{r}_M - r_f}{\sigma_M} \tag{4.11}$$

- Note that eq. (4.10) is for an efficient portfolio; it does not hold for an inefficient portfolio (such as a single stock).
- The analogous single-stock relation is as follows.

Theorem 4.2

Under the assumptions of the capital asset pricing model, for any asset i we have

$$\mathbb{E}[r_i] = \beta_i \mathbb{E}[r_M] \tag{4.12}$$

where

$$\beta_i = \frac{\operatorname{cov}(r_i, r_M)}{\sigma_M^2}. \tag{4.13}$$

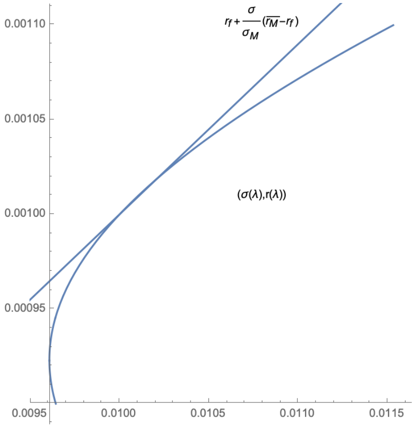
Eq. (4.12) *is called the security market line.*

- Under the strict assumptions of the CAPM, (4.12) is not a regression; rather it is an ex ante constraint on investor expectations, which holds exactly.
- The variances and covariances appearing in (4.13) are the *ex ante* values agreed upon by all investors, under the assumptions of the model.

- *Proof of Theorem 4.2.* Form a portfolio with holding λ in asset i and $1-\lambda$ in the market M.
- Trivially, this portfolio has expected return and variance

$$\begin{aligned} \overline{r}(\lambda) &= \lambda(\overline{r}_i - \overline{r}_M) + \overline{r}_M \qquad (4.14) \\ \sigma^2(\lambda) &= \lambda^2(\sigma_i^2 + \sigma_M^2 - 2\,\mathrm{cov}(r_i, r_M)) \\ &\quad + 2\lambda(\mathrm{cov}(r_i, r_M) - \sigma_M^2) + \sigma_M^2 \end{aligned}$$

- Think of (4.14) as parametrizing a curve in (σ, r) space, just as (4.10) is a line in (σ, r) space.



$r_f + \frac{\sigma}{\sigma_M}(\overline{r_M} - r_f)$
$(\sigma(\lambda), r(\lambda))$
0.00110
0.00105
0.00100
0.00095
0.0095
0.0100
0.0105
0.0110
0.0115

- The line (4.10) is tangent to the curve (4.14) at the point $\lambda = 0$.
- This follows by evaluating (4.14) at $\lambda = 0$ to show that it is the market point $(\overline{r}_M, \sigma_M)$, and other points on the curve aren't efficient portfolios, so they lie below the line (4.10).
- The slope of this tangent line is, of course, (4.11), so we can equate this to the slope calculated by taking the derivative:

$$\frac{\overline{r}_M - r_f}{\sigma_M} = \left.\frac{d\overline{r}(\lambda)}{d\sigma(\lambda)}\right|_{\lambda=0} = \frac{\overline{r}_i - \overline{r}_M}{(\mathrm{cov}(r_i, r_M) - \sigma_M^2)/\sigma_M}$$

- Solving this equation for the unknown variable $\overline{r}_i$ leads directly to (4.12), completing the proof. $\square$

CAPM time series regressions

- We now review how, in practice, one could use this model to derive *ex ante* risk or return expectations.

> *The value of β may be given an interpretation similar to that found in regression analysis utilizing historic data, although in the context of the CAPM it is to be interpreted strictly as an ex ante value based on probabilistic beliefs about future outcomes.*
>
> — Sharpe (1990)

- Of course, a historically estimated regression coefficient may be a good *ex ante* forecast; this is likely to be true if (and only if) the capital structure of the underlying company remains stationary.
- If the company has just undergone a major spinoff or restructuring, or a major debt issue, then the historical beta estimated via regression is likely less accurate as a forecast of ex ante beta.

- In those cases where the capital structure has not changed dramatically, we can construct a time-series regression for the i-th stock,

$$r_{i,t} = \alpha_i + \beta_i r_{M,t} + \epsilon_{i,t} \tag{4.15}$$

where $r_{i,t}$ is the realized historical excess return on the i-th stock on day t, and $r_{M,t}$ is the historical excess return on some market-capitalization-weighted basket.

- Once the data needed by (4.15) is collected for t in some window,

$$t = 1, 2, \ldots, T$$

and the regression is run, we will have coefficient estimates

$$\hat{\alpha}_i, \ \hat{\beta}_i$$

and fitted residuals $\hat{\epsilon}_{i,t}$.
- The regression also gives the standard errors of the coefficient estimates, and the t-statistic which tests the null hypothesis that the coefficients are zero under the assumption of Gaussian errors.
- Furthermore, the regression gives an estimate of the standard deviation of the residuals,

$$\hat{\sigma}_\epsilon^2 = \frac{1}{T-1} \hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{\epsilon}}$$

- If one is willing to take these historical estimates and promote them to *ex ante* forecasts, as would be appropriate in a stationary model, then one can predict the variance for the i-th asset as

$$\mathbb{V}[r_i] = \hat{\beta}_i^2 \sigma_M^2 + \hat{\sigma}_\epsilon^2$$

- Estimates of σ_M^2 can be obtained from time-series of market returns, or they can be implied by the options market; for example, the VIX gives one such estimate.

The cross-sectional CAPM

- Interestingly, one can invert the estimation problem, turning it into a cross-sectional estimation.
- Understanding the "cross-sectional CAPM" is an excellent introduction to multi-factor models.

- Suppose that the cross-sectional vector containing the beta of each stock
$$\boldsymbol{\beta} := \{\beta_i : i = 1, \ldots, n\}$$
is exogenously determined, fixed, and provided to you.
- Also assume given the cross-section of asset returns
$$\{r_i : 1 = 1, \ldots, n\}$$
which occurred on *one particular day* in the past.
- Assume for now that these are all the data you have; in particular, you do not have access to the market capitalizations, or the market's return r_M.
- Your task is to infer the most likely value for the market's return on the given day.

- In this setup, the one missing parameter is r_M and we can attempt to infer it by the maximum likelihood estimate, by minimizing the sum of squared residuals

$$\operatorname*{argmin}_{r_M} ||r_M \boldsymbol{\beta} - \boldsymbol{r}||^2$$

where as usual, double vertical bar denotes the length of a vector.

- The solution is

$$\hat{r}_M = \frac{\boldsymbol{\beta} \cdot \boldsymbol{r}}{||\boldsymbol{\beta}||^2}. \tag{4.16}$$

- I call this the *cross-sectional CAPM*.

The oil factor

- Now suppose that there is a commodity called oil, whose price is volatile and which fluctuates in the market due to supply shocks which are only weakly correlated to the market portfolio's returns.
- Some companies, such as Exxon Mobil, are net producers of oil, and so their economic outlook should be positively impacted when oil prices rise, while other companies, such as transportation and shipping companies, use oil in the course of providing services to their customers, and hence their costs go up when oil price rises.

- Let $p_{t,\text{oil}}$ denote the spot price, and let

$$r_{t,\text{oil}} = \frac{p_{t,\text{oil}}}{p_{t-1,\text{oil}}} - 1 \tag{4.17}$$

  denote the simple price return on day t to an investor holding a fixed quantity of the commodity.
- This ignores certain real-world frictions such as storage costs.

- The analogous model to the CAPM (4.15) is then

$$r_{i,t} = \alpha_i + \beta_i r_{M,t} + x_{i,\text{oil}} r_{t,\text{oil}} + \epsilon_{i,t} \tag{4.18}$$

- As before we can proceed in two ways; we can estimate (4.18) on time series data, which is now a multivariate regression, obtaining new estimates

$$\hat{\alpha}_i,\ \hat{\beta}_i, \hat{x}_{i,\text{oil}}$$

and the residual estimates would be slightly different as well.

- Alternatively, suppose that some benevolent power had given us values for CAPM betas,

$$\beta_1, \ldots, \beta_n$$

for all assets for one particular day, as before in the cross-sectional CAPM.

- As before we would like to use these exogenously given values to impute the missing value for r_M, but in a world where oil has an influence on stock prices, the univariate regression we used before in (4.16) becomes mis-specified and must be replaced with a multivariate one.

- We could attempt to guess a reasonable value of

$$x_{i,\text{oil}}$$

for each company for all $i = 1, \ldots, n$.
- For example, we could start by defining $x_{i,\text{oil}} = 1$ whenever i corresponds to an oil-producer stock such as Exxon.
- Some companies might have engaged in borrowing in order to finance oil exploration and production projects (and related hardware, such as offshore oil rigs).
- Such companies are actually more like call options on the price of oil, because they only start to become profitable once the oil price crosses a threshold where they make back their costs, and so we assign them higher than 1 exposure, after we analyze their capital structure to figure out just how levered they really are.
- Companies for whom oil is a major cost (e.g. shipping companies, airlines) should get negative exposure, but this can be tempered based on whether they have internally hedged (e.g. buying forward contracts).

- These sorts of structural analyses of companies' business models, even if done well, won't give a perfectly exact exposure to large fluctuations in the oil spot price, but *neither will a time series regression*, as all such regressions are subject to estimation error.
- It's entirely plausible that the structural exposures, if they were prepared by competent security analysts, could give less noisy and more stable representations of each company's true exposure to the underlying risk source.

- For want of a better term, let's call the vector

$$x_{i,\text{oil}}, \;\; i = 1, \ldots, n$$

the *structural exposures* to oil, to distinguish it from regressed exposures, because it is derived from the structure of the company and their business model, rather than being derived purely from regression on past data.

- We can then form a cross-sectional regression analogous to (4.16),

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} \beta_1 & x_{1,\text{oil}} \\ \beta_2 & x_{2,\text{oil}} \\ \vdots & \\ \beta_n & x_{n,\text{oil}} \end{pmatrix} \begin{pmatrix} r_M \\ r_{\text{oil}} \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} \tag{4.19}$$

in which the only unknown parameters are (r_M, r_{oil}).

- Running the regression gives parameter estimates

$$\hat{r}_M, \ \hat{r}_{\text{oil}}$$

which pertain to the same date, or same time interval, over which we measured $r_1, \ldots, r_n$.

- Note that the estimate $\hat{r}_{\text{oil}}$ obtained this way will not equal the simple price return measured on the oil spot price as in (4.17).
- Rather, it is an indirect measure of how oil is doing, based on the returns on stocks which depend on oil.

- What is it about oil that makes (4.19) a reasonable model, while doing the same thing with, say the time series of temperatures on Neptune would not give a reasonable model?
- The difference is that with oil, one can see that structurally, companies depend on it and they use it and transact in it, as part of their normal business operations.
- Small fluctuations in oil prices, such as normal market microstructure noise, are unlikely to be impactful for stock prices, but if the move is large enough, at some point investors will be forced to re-calibrate predictions of the company's earnings and cash flows to reflect the changing energy costs.
- In that sense, the large fluctuations in oil can be seen as like the proverbial tide that rises and floats all boats, or falls and sinks all boats.

- There are other examples of random variables which we could identify as catalysts generating random shocks, to which all stock prices respond, but to varying degrees.
- In many cases it is easier and/or less error prone to identify the structural exposures, like we did for oil, than to compute regressed exposures.
- In the case of oil, there is a time series of spot prices that is amenable to time-series regression, and so one does not strictly *need* to use structural exposures.
- But for other common sources of risk, perhaps only the structural exposure is available and there is no observable time series to regress upon.

- The logical path this takes us on leads to an expanded version of the cross-sectional model (4.19) having more than two columns.

Multi-factor models

- Arbitrage pricing theory relaxes several of the assumptions made in the CAPM.
- In particular, we relax the assumption that there is a single efficient frontier allocation of risky assets.
- This allows the possibility that a CAPM-like relation may hold, but with multiple underlying sources of risk.

- Specifically, let

$$r_i, \ i = 1, \ldots, n$$

denote the cross-section of asset returns over a given time period $[t, t+1]$, in excess of the risk-free rate, and as usual $\boldsymbol{r}$ denotes the n-dimensional random vector with r_i as components.
- In a fully-general model, the multivariate distribution $p(\boldsymbol{r})$ could have arbitrary covariance and higher-moment structures, but remember that for n large there is typically never enough data to estimate such over-parameterized models.

- Instead of allowing very complex forms for the multivariate distribution $p(\boldsymbol{r})$, APT models constrain the distribution's complexity by assuming a structural model of the following form:

$$r_i = x_{i,1} f_1 + x_{i,2} f_2 + \cdots + x_{i,p} f_p + \epsilon_i, \tag{4.20}$$
$$\epsilon_i \sim N(0, \sigma_i^2) \tag{4.21}$$

  for all $i = 1, \ldots, n$.
- Think of this as an expanded version of the cross-sectional model (4.19) where the number of factors, p, can be arbitrary.
- For uniformity, we re-label the first column's exposures from β_i to $x_{i,1}$.
- Likewise for uniformity of notation, we use f_1, f_2 instead of r_M and r_{oil}.

- If $p = 1$ the structural equation (4.20) reduces to

$$r_i = x_{i,1} f_1 + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma_i^2)$$

- Taking expectations of both sides and dropping the needless "1" subscript allow us to compare this to the CAPM, where we proved a comparable relation (4.12), with f corresponding to r_M, the return on the single risk source.
- APT differs from the CAPM in that we do not generally need to assume that all investors have the same expected returns.
- If we are willing to make that assumption, though, then stronger conclusions can be reached.

- In the CAPM, we were able to identify the single efficient fund by arguing that its weights must equal the market-capitalization weights.
- Hence we were given "for free" a very nice proxy for the single efficient fund: a capitalization-weighted basket such as the Russell 3000.
- Time series of that fund's return are readily available.
- We could estimate β_i from a time series regression.
- If $p > 1$ then the underlying assumptions of that argument break down: there is no longer such a direct way to identify f_j nor $x_{i,j}$ $(j = 1, \ldots, p)$.
- We shall return to the estimation problem in due course.

- The APT structural equation (4.20) is more conveniently expressed in matrix form:

$$\underset{n\times 1}{\boldsymbol{r}} = \underset{n\times p}{X} \cdot \underset{p\times 1}{\boldsymbol{f}} + \underset{n\times 1}{\boldsymbol{\epsilon}}, \quad \mathbb{E}[\boldsymbol{\epsilon}] = 0, \quad \mathbb{V}[\boldsymbol{\epsilon}] = D \tag{4.22}$$

- By our conventions, $\boldsymbol{r}$ and $\boldsymbol{f}$ are returns from the close on day t to the close on day $t+1$, and hence are random variables as viewed from time t or any time before.
- Structural factor exposures X should be calculated entirely from data known before time t.
- Conditional on the information set available at time t, the matrix X is not random.

Definition 4.3

A strict factor model is one in which the variance-covariance matrix of the residuals is diagonal:

$$D := \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2) \quad \textit{with all} \quad \sigma_i^2 > 0. \tag{4.23}$$

- We will henceforth restrict attention to strict factor models.
- Eq. (4.23) entails that all significant sources of correlation are already captured by factors, represented as columns of X.
- Note, for later use, that in strict factor models D^{-1} exists and can be computed in $O(n)$ time because it is diagonal.

The leverage factor

- Typically, there can be several mathematical methods for capturing what is essentially one economic idea.
- For example, consider the notion of financial leverage.
- All measures of leverage are dimensionless ratios.
- One could consider a leverage factor implied by the current market capitalization:

$$\mathrm{MLEV} = \frac{\mathrm{ME}_t + \mathrm{PE}_t + \mathrm{LD}_t}{\mathrm{ME}_t} \tag{4.24}$$

  where ME_t is the market value of common equity, PE_t is the book value of preferred equity, and LD_t is the book value of long-term debt.
- The value of preferred equity and long-term debt are as of the end of the most recent fiscal year.
- The market value of equity is computed using a recent closing price of the stock.

- Note that (4.24) is, for better or worse, strongly dependent on ME_t which fluctuates with the stock price.
- Another measure of leverage, called *book leverage*, is obtained from (4.24) by replacing the market value of common equity with the book value of common equity, leaving the other terms in the numerator unchanged:

$$\mathrm{BLEV} = \frac{\mathrm{CEQ}_t + \mathrm{PE}_t + \mathrm{LD}_t}{\mathrm{CEQ}_t}$$

  where CEQ_t is the book value of common equity.
- Note that CEQ_t is a differential between an asset calculation and a liabilities calculation.
- It is an accounting function, and as such it does not fluctuate with the market price.
- Rather, it is a function of other accounting metrics mentioned on the company's public financial statements (e.g. Form 10-Qs and related).

- Another accounting ratio relevant to leverage is the debt/assets ratio:

$$\mathrm{DA} = \frac{\mathrm{LD}_t + \mathrm{DCL}_t}{\mathrm{TA}_t}$$

where DCL_t is the value of debt in current liabilities, and TA_t is the book value of total assets.

- Finally, one can construct a multi-level indicator variable based on the debt rating of a company.
- The rationale for such an indicator would be that the ratings agencies are doing calculations that are more sophisticated than the above, to estimate the company's distance to a default or a related credit event.
- If the reason leverage is a risk in the first place, is that it can increase default risk, then it makes sense as part of the leverage factor.
- These calculations are all different proxies for a presumed risk source that is real, but whose returns are not observed directly – we cannot simply look up on Bloomberg the current return of "the" leverage factor in the same way one can look up the spot price of oil.

The liquidity factor

- Generally speaking, liquidity risk is real.
- If investors are entering a position that they may not be able to exit, this is a real risk and there should be a risk premium associated to that.
- In other words, we would expect investors to price that risk when deciding what they would pay for the stock.
- As we saw before, there can be several mathematical methods for capturing what is essentially one economic idea.

- STOA is the annualized share turnover rate using data from the last 12 months.
- It is equal to
$$\text{STOA} = \frac{V_{ann}}{\overline{N}_{out}}$$
where V_{ann} is the total trading volume (in number of shares) over the last 12 months and $\overline{N}_{out}$ is the average number of shares outstanding over the same period.
- As a second formulation, one can compute the same ratio, but using the most recent quarter and multiplying by 4 to annualize.

The volatility factor

- Perhaps there is a single risk source related to how the market prices volatility.
- Yet different single-name equities will respond differently to such shifts in the market's price of risk.
- To first order, the exposure should be strongly related to the stock's volatility itself, but even that is a concept with multiple definitions.
- For example, one has historical volatility over various backward-looking windows.
- One has implied volatility at one or more future horizons, represented by different option expiration dates.
- If there is a strong serial dependence in the stock's residuals, then its true response to a volatility shift is higher than a time series model assuming iid residuals would predict.

- As before, we can construct a composite factor by defining different descriptors, and then combining them statistically.
- Here are a few of the commonly used descriptors for the volatility factor.

$$\mathrm{DSTD} = \sqrt{\sum_{t=1}^{T} w_t r_t^2}$$

where r_t is the return over day t, w_t is the weight for day t, and T is the number of trading days used to compute this descriptor (eg. 65 days).

$$\mathrm{HL} = \log(P_H/P_L)$$

where P_H and P_L are the maximum price and minimum price attained over the last one month, split adjusted.

- One can also look at cumulative range in log space.
- Define

$$Z_t = \sum_{s=1}^{t} \log(1 + r_{i,s}) - \sum_{s=1}^{t} \log(1 + r_{f,s})$$

where $r_{i,s}$ is the return on stock i in month s, and $r_{f,s}$ is the risk-free rate for month s.

- In other words, Z_t is the cumulative return of the stock over the risk-free rate at the end of month t.
- Define Z_{max} and Z_{min} as the maximum and minimum values of Z_t over the last 12 months.
- Define

$$\text{CMRA} = \log \frac{1 + Z_{max}}{1 + Z_{min}}$$

- As mentioned above, there may be serial dependence in residuals from the market model regressions.
- One can construct a dimensionless ratio capturing this up to lag 2 as follows:

$$\text{SDP} = \frac{\sum_{t=3}^{T}(e_t + e_{t-1} + e_{t-2})^2}{\sum_{t=3}^{T}(e_t^2 + e_{t-1}^2 + e_{t-2}^2)}$$

  where e_t is the residual from the market model regression in month t, and T is the number of months over which this regression is run (typically, $T = 60$ months).

- Finally, some measure of option implied vol can be included.
- General guidelines are to use liquid options (usually, out of the money options are more liquid).
- One should also decide on a forward-looking time horizon over which to measure; the VIX calculation uses 30 days, but there may not be options expiring precisely at 30 calendar days from now, so one can interpolate in time to obtain a constant-horizon calculation.

The momentum factor

- The first descriptor is simple relative strength (relative log-return above the risk-free rate),

$$\mathrm{RSTR} = \sum_{t=1}^{T} \log(1 + r_{i,t}) - \sum_{t=1}^{T} \log(1 + r_{f,t})$$

  where $r_{i,t}$ is the arithmetic return of stock i in month t, and $r_{f,t}$ is the arithmetic risk-free rate for month t.
- This measure is usually computed over the last one year ie. $T = 12$.

- The above relative strength measure will be positive for most stocks in a year when the whole market had strong returns.
- This effect could be removed, eg. by standardizing in the cross section, but this would not remove the effect of market beta.
- Stocks with higher beta would go up more than the market and simply de-meaning in the cross section wouldn't differentiate between high-beta and low-beta stocks.

- A less beta-sensitive descriptor is equal to the alpha term (i.e., the intercept term) from a 60-month regression of the stock's excess returns on the S&P 500 excess returns (or the relevant benchmark for our universe, e.g. the TOPIX in Japan).

Math review: linear combinations of random variables

Theorem 4.4

Let X and Y be random variables defined on the same probability space, and let $a, b \in \mathbb{R}$ be constants. Then

$$\begin{aligned}
\mathbb{E}[aX + bY] &= a\mathbb{E}[X] + b\mathbb{E}[Y] \\
\mathbb{V}[aX + bY] &= a^2\mathbb{V}[X] + b^2\mathbb{V}[Y] + 2ab\,\mathrm{Cov}(X, Y) \\
&= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \mathbb{V}[X] & \mathrm{Cov}(X, Y) \\ \mathrm{Cov}(X, Y) & \mathbb{V}[Y] \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}
\end{aligned}$$

We can easily extend this result, by induction, to arbitrary finite linear combinations of random variables.

Theorem 4.5

Let $X_1, X_2, \ldots, X_n$ be random variables defined on the same probability space, and let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ be constants. Then

$$\mathbb{E}[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i]$$

$$\mathbb{V}[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \operatorname{Cov}(X_i, X_j)$$

- The result of the preceding theorem is more compactly expressed in matrix notation.
- Let $\boldsymbol{X}$ be a random vector valued in $\mathbb{R}^n$, let

$$\boldsymbol{a} \in \mathbb{R}^n$$

be a constant vector, and let $\mathbb{V}[\boldsymbol{X}]$ be the matrix whose i, j entry is $\mathrm{Cov}(X_i, X_j)$.
- Then

$$\mathbb{E}[\boldsymbol{a} \cdot \boldsymbol{X}] = \boldsymbol{a} \cdot \mathbb{E}[\boldsymbol{X}]$$
$$\mathbb{V}[\boldsymbol{a} \cdot \boldsymbol{X}] = \boldsymbol{a} \cdot \mathbb{V}[\boldsymbol{X}]\boldsymbol{a}$$

Implications of multi-factor models for portfolio risk and return

- The variable $\boldsymbol{f}$ in (4.22) denotes a p-dimensional vector-valued stochastic process which cannot be observed directly, but for simplicity we will assume that the $\boldsymbol{f}$ process is stationary with finite first and second moments,

$$\mathbb{E}[\boldsymbol{f}] = \mu_f, \quad \text{and} \quad \mathbb{V}[\boldsymbol{f}] = F. \tag{4.25}$$

- Given a portfolio with holdings vector $\boldsymbol{h}$, priced in dollars at time t, we can calculate the single-period portfolio return,

$$\boldsymbol{h}'\boldsymbol{r} = \boldsymbol{h}'X\boldsymbol{f} + \boldsymbol{h}'\boldsymbol{\epsilon} \tag{4.26}$$

  where prime denotes the transpose.
- Depending on whether the interval $[t, t+1]$ is in the future or the past, (4.26) is either a decomposition of what will happen, or an explanation of what already did.
- In what follows we will almost exclusively use it in the future tense.
- Due to its appearance in (4.26), the $1 \times p$ row vector $\boldsymbol{h}'X$ takes on a special significance.

Definition 4.6

For a portfolio with holdings vector $\boldsymbol{h} \in \mathbb{R}^n$, *the vector*

$$\boldsymbol{h}'X \in \mathbb{R}^p$$

is called the exposure vector of the portfolio. The j*-th element of* $\boldsymbol{h}'X$ *is called the exposure of* $\boldsymbol{h}$ *to the* j*-th factor.*

Statistically-estimated factor returns

- Since the variable $\boldsymbol{f}$ in (4.22) denotes a p-dimensional random vector process which cannot be observed directly, information about the $\boldsymbol{f}$-process must be obtained via statistical inference.
- The primary outputs of statistical parameter inference in the APT model are estimates of the parameters μ_f and F mentioned in (4.25), as well as estimates of the daily realizations $\hat{f}_{t+1}$ for each one-day interval $[t, t+1]$.

- Statistically estimated realizations $\hat{f}$ are called *factor returns* by practitioners (Menchero, Morozov, and Shepard, 2008).
- This nomenclature arises because they can be viewed as returns on certain portfolios, called *factor portfolios*.
- If we take D proportional to the identity matrix for simplicity, and assume that $X'X$ is invertible, then the factor returns are

$$\hat{f} = X^{+}\boldsymbol{r}, \qquad \text{where} \quad X^{+} = (X'X)^{-1}X'. \tag{4.27}$$

- Note that X^{+} has dimensions $p \times n$, like the transpose.
- Since $\boldsymbol{r} \in \mathbb{R}^n$ is a cross-sectional vector of returns, the j-th factor return $\hat{f}_j$ is the return on a long-short portfolio whose holdings are given by the j-th row of X^{+}.

- The j-th row of X^+, when viewed as a portfolio, has rather special properties.
- For example, it has unit exposure to the j-th factor, and 0 exposure to all other factors as is proven by the following identity:
$$X^+X = (X'X)^{-1}X'X = I. \tag{4.28}$$
- The orthogonality implied by (4.28) is actually quite important for building intuition about the factor returns and factor portfolios.
- For example, if one of the columns of X is CAPM beta, then the other factor portfolios are β-neutral.

If X contains an industry classification, then the other factors are industry-neutral, etc.

- Above, we assumed that $X'X$ is invertible, but what if it isn't?
- One can then define

$$X^{+} := \lim_{\delta \to 0^{+}} (X'X + \delta I)^{-1} X'$$

and many standard linear-regression calculations proceed as before; the "least squares" coefficients can still be found as

$$\hat{f} = X^{+} \boldsymbol{r},$$

and the fitted residuals are

$$\hat{\epsilon} = \boldsymbol{r} - X\hat{\boldsymbol{f}} = \boldsymbol{r} - X(X^{+}\boldsymbol{r})$$

where the parentheses indicate the recommended computation order.

Basic portfolio risk calculations

- The model (4.22), (4.23) and (4.25) entails associated reductions of the first and second moments of the asset returns:

$$\mathbb{E}[\boldsymbol{r}] = X\mu_{\boldsymbol{f}}, \quad \text{and} \quad \Sigma := \mathbb{V}[\boldsymbol{r}] = D + XFX' \tag{4.29}$$

where X' denotes the transpose.

- These relations follow immediately from Theorem 4.5 and independence between $\boldsymbol{f}$ and ϵ.

- Eq. (4.29) is quite useful for portfolio construction and for analyzing existing portfolios.
- For example, for a portfolio with dollar holdings

$$\boldsymbol{h} \in \mathbb{R}^n,$$

it says that

$$\boldsymbol{h}'\Sigma\boldsymbol{h} = \boldsymbol{h}'D\boldsymbol{h} + \boldsymbol{h}'XFX'\boldsymbol{h}$$

which expresses the portfolio's variance in terms of the *idiosyncratic variance* $\boldsymbol{h}'D\boldsymbol{h}$ and a second term computable from $\boldsymbol{h}'X$, the exposure vector.

- Almost anything can now be reduced to a simpler form using the factor structure.
- For example, the CAPM beta of an asset is

$$\beta_i = \frac{\mathrm{cov}(r_i, r_M)}{\mathrm{var}(r_M)} \tag{4.30}$$

- The model (4.22), (4.23) and (4.25) gives us a way of computing both the numerator and denominator in (4.30):

$$\begin{aligned} \mathrm{cov}(r_i, r_M) &= \mathrm{cov}(x_i \cdot f, x_M \cdot f) = x_i' F x_M \\ \mathrm{var}(r_M) &\approx x_M' F x_M \end{aligned}$$

  where x_i is the i-th row of X, which is to say the exposure vector of the i-th security.
- Also x_M is the "exposure vector of the market", which is in practice estimated by choosing a basket, such as the Russell 3000, as a proxy for the market portfolio, and calculating its exposures by definition 4.6.

APT model covariance in portfolio optimization

- Now consider the APT model's point estimate covariance matrix, which is diagonal plus low-rank:

$$\Sigma = XFX' + D.$$

- Recall the Woodbury matrix-inversion lemma,

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1} VA^{-1}, \tag{4.31}$$

  where A, U, C and V all denote matrices of the correct (conformable) sizes.
- The lemma holds in the case when all of the matrix inverses in (4.31) exist.

- Apply (4.31) with

$$A = D, U = X, V = X', C = F$$

to get

$$\Sigma^{-1} = D^{-1} - D^{-1}X\left(F^{-1} + X'D^{-1}X\right)^{-1}X'D^{-1} \qquad (4.32)$$

- The right hand side of (4.32) is computationally efficient; note that D^{-1} involves n scalar reciprocals, since D is diagonal.
- The remaining inverses in (4.32), such as

$$F^{-1} \text{ and } \left(F^{-1} + X'D^{-1}X\right)^{-1}$$

involve $p \times p$ matrices where $p \ll n$.

- Let $\boldsymbol{v}$ be a nontrivial eigenvector of Σ with eigenvalue λ.
- If $\boldsymbol{v}$ is a null vector then $\lambda = 0$.
- Either way, we assume without loss of generality that

$$\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1 \quad \text{and} \quad \Sigma \boldsymbol{v} = \lambda \boldsymbol{v}.$$

- Hence

$$\begin{aligned} \lambda = \langle \boldsymbol{v}, \Sigma \boldsymbol{v} \rangle &= \langle \boldsymbol{v}, (XFX' + D)\boldsymbol{v} \rangle \\ &\geq \langle \boldsymbol{v}, D\boldsymbol{v} \rangle \end{aligned}$$

- Since D is diagonal and has no zero entries, it has no null vector, and so $\lambda > 0$.

- Similarly, one can use (4.32) to put a strong lower bound on eigenvalues of Σ^{-1}.
- Hence, we can see that one of the main issues with covariance matrices and optimization (the existence of portfolios with very low forecasted vol) does not occur when the covariance matrix comes from a reasonably-constructed APT model.

VaR

- Imagine you are an investment bank before the Volcker rule, and management gets nervous if the trading division loses more than $50mm in a single day.
- If π is a random variable representing the profit in a day, then $\ell = -\pi$ is known as the *loss*.
- Suppose the strategists analyze the predictive density $p(\ell)$ and estimate that
$$\int_{5\times 10^7}^{\infty} p(\ell) d\ell \approx 0.01,$$
so the strategy will only make management nervous about once in every 100 days.
- In this situation, the number $50mm equals the 99% VaR.
- By convention, VaR is quoted as an upper quantile of the loss, not a lower quantile of the profit, and is therefore typically, by convention, quoted as a positive number.

- Supposing that

$$F_\ell(x) = \int_{-\infty}^{x} p(\ell)d\ell,$$

  the CDF of the loss distribution, is a one-to-one function, the 99% VaR is $F_\ell^{-1}(0.99)$.
- There's nothing magical about the number 0.99, and if your threshold is different, you can instead consider the $100 \times (1-\alpha)\%$ VaR, defined by $F_\ell^{-1}(1-\alpha)$, where $0 < \alpha < 1$, as long as F_ℓ is invertible.

More generally, F_ℓ may not be invertible, in which case a mathematically proper definition, which reduces to the previous one in the invertible case, is given by:

$$\mathrm{VaR}_\alpha = \inf \{ x \in \mathbb{R} : F_\ell(x) \geq 1 - \alpha \} . \tag{4.33}$$

which again is called the "$100 \times (1 - \alpha)$% VaR" so $\alpha = 0.05$ is 95% VaR etc.

- Suppose the loss follows an $N(0, \sigma^2)$ distribution which is defined by the density

$$p(\ell) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\ell^2}{2\sigma^2}}$$

- One may show that the CDF of the zero-mean normal, and its inverse, are given by:

$$F(x) = \int_{-\infty}^{x} p(\ell) d\ell = \frac{1}{2}\mathrm{erfc}\Big(- \frac{x}{\sigma\sqrt{2}}\Big) \tag{4.34}$$

$$F^{-1}(1-\alpha) = -\sigma\sqrt{2}\,\mathrm{erfc}^{-1}(2(1-\alpha)) \tag{4.35}$$

- It therefore follows by evaluating (4.35) numerically that

$$F^{-1}(0.95) \approx 1.64\,\sigma\,.$$

- The error function $\mathrm{erf}(z)$ is given by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

and the complementary error function $\mathrm{erfc}(z)$ is given simply by

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z).$$

- The problem with VaR is that it has nothing to say about the *size of the loss*, when it does occur.
- A more interesting measure of risk is therefore the *expected tail loss*, defined by

$$\mathsf{ETL}_\alpha = \mathbb{E}[\ell \mid \ell \geq \mathsf{VaR}_\alpha] \tag{4.36}$$

Homogeneous risk metrics

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *homogeneous of degree k* if

$$f(\lambda x) = \lambda^k f(x) \text{ for all } \lambda > 0, x \in \mathbb{R}^n.$$

- Three of the most commonly-used risk metrics are volatility, value-at-risk (4.33), and expected tail loss (4.36); these three are all homogeneous of degree $k = 1$.
- For example, the volatility of a 2:1 levered portfolio is theoretically twice the volatility of the unlevered version.

- Euler's homogeneous function theorem states that f is homogeneous of degree k if and only if

$$x \cdot \nabla f = \sum_i x_i \frac{\partial f}{\partial x_i} = kf(x)\,. \qquad (4.37)$$

- In finance, this theorem is typically applied with $k = 1$, as follows.

Definition 4.7

We say a scalar-valued random variable r over a future period $[t, t+1]$ has a return decomposition if it obeys a linear relation of the form

$$r = \sum_m x_m g_m \tag{4.38}$$

where x_m are non-random "exposures" known as of time t, (known ex ante, in other words) and g_m are random return sources whose realizations become known ex post, or at $t+1$.

- The simplest example of a return decomposition is the decomposition of a portfolio's return into contributions from individual assets.
- For a portfolio with holdings vector $\boldsymbol{h} \in \mathbb{R}^n$, write

$$r := \boldsymbol{h} \cdot \boldsymbol{r} = \sum_{i=1}^{n} h_i r_i$$

in which the dollar holding h_i plays the role of the ex ante known exposure, and the asset's return r_i is the i-th stochastic return source.

- The next-simplest example of a return decomposition involves an APT model.
- Suppose $\boldsymbol{r} = X\boldsymbol{f} + \boldsymbol{\epsilon}$ and hence

$$\boldsymbol{h} \cdot \boldsymbol{r} = \sum_{m=1}^{p} (\boldsymbol{h} \cdot \boldsymbol{x}_m)\, f_m + \boldsymbol{h} \cdot \boldsymbol{\epsilon} \tag{4.39}$$

  where $\boldsymbol{x}_j$ denotes the j-th column of the exposure matrix X.
- Suppose that the term $\boldsymbol{h} \cdot \epsilon$ is small relative to the total, or we simply want to focus on analyzing the factor contributions.
- Eq (4.39) then gives

$$r = \sum_{m=1}^{p} x_m g_m, \quad x_m = \boldsymbol{h} \cdot \boldsymbol{x}_m, \quad g_m = f_m$$

  and we have yet another return decomposition of the form (4.38).

- We want to be as general as possible, so we will focus first on understanding those relations which apply to any return decomposition, whether the decomposition pertains to assets, APT factors, or other return sources.

- Applying Euler's theorem (4.37) to the volatility of the return, $\sigma(r)$, one can write

$$\sigma(r) = \sum_m x_m \, \mathrm{MCR}_m, \quad \text{where} \tag{4.40}$$

$$\mathrm{MCR}_m := \frac{\partial \sigma(r)}{\partial x_m} \tag{4.41}$$

  where MCR is for *marginal contribution to risk*.
- Eqns (4.40) and (4.41) would immediately generalize to any homogeneous risk measure.

- Applying the definition of covariance one can also derive from (4.38) another variance decomposition,

$$
\begin{aligned}
\sigma^2(r) = \operatorname{cov}(r, r) &= \operatorname{cov}(r, \sum_{m=1}^{p} x_m g_m) \\
&= \sum_m x_m \operatorname{cov}(g_m, r) \\
&= \sum_m x_m \rho(g_m, r)\sigma(g_m)\sigma(r)
\end{aligned}
$$

- Dividing the last equation by $\sigma(r)$ yields the *x-sigma-rho attribution*

$$\sigma(r) = \sum_m x_m \sigma(g_m)\rho(g_m, r) \tag{4.42}$$

where $\rho(g_m, r)$ is the correlation of source m with the portfolio's return.

- By comparing (4.42) with (4.40) we see that

$$\mathrm{MCR}_m = \sigma(g_m)\rho(g_m, r) \tag{4.43}$$

- One can also invert (4.43) to express the correlation in terms of the MCR and the vol,

$$\rho(g_m, r) = \frac{\mathrm{MCR}_m}{\sigma(g_m)} \tag{4.44}$$

- Furthermore, we now see that the right-hand side of (4.44) makes sense mathematically if σ is replaced by any degree-1 homogeneous risk measure, such as VaR, where we could continue to define MCR by (4.41).
- When used in this way, $\rho(g_m, r)$ is called the *generalized correlation*; see Goldberg et al. (2009).

- Define the *information ratio* (IR) as

$$\mathrm{IR} = \mathbb{E}[r]/\sigma(r).$$

- As the next theorem shows, optimal IR portfolio allocations allocate risk and return proportionally.

Theorem 4.8

For an unconstrained maximum information ratio portfolio, the expected source returns are directly proportional to the source marginal contributions to risk,

$$\mathbb{E}[g_m] = \mathrm{IR} \cdot \mathsf{MCR}_m$$

where IR is the portfolio information ratio.

Proof. If we are at optimal $\mathrm{IR} = \mathbb{E}[r]/\sigma(r)$, then

$$0 = \frac{\partial}{\partial x_i}\left[\frac{\mathbb{E}(r)}{\sigma(r)}\right] = \frac{\sigma(r)\mathbb{E}[g_i] - \mathbb{E}[r]\,\mathrm{MCR}_i}{\sigma(r)^2}$$

Hence

$$\sigma(r)\mathbb{E}[g_i] = \mathbb{E}[r]\,\mathrm{MCR}_i,$$

from which we get the desired relation by dividing by $\sigma(r)$ on both sides. $\square$

- Theorem 4.8 provides implied returns that serve as an important reality check on whether the actual portfolio is consistent with the manager's views.
- In any situation where we can calculate MCR's, we can then calculate $\mathrm{IR} \cdot \mathrm{MCR}_m$ for a few reasonable choices of IR.
- Sometimes non-quantitative portfolio managers are surprised by the results.
- The directions and signs line up to make sense: if $\mathrm{MCR}_m < 0$ for some m, it means you could reduce risk by having more exposure to it.

This would happen, for example, if source m were uncorrelated to the other sources, and you were short (negative exposure), which would happen if $\mathbb{E}[g_m] < 0$.

- Arrow, Kenneth J (1963). "Liquidity preference, Lecture VI in "Lecture Notes for Economics 285, The Economics of Uncertainty", pp 33-53". In.
- Goldberg, Lisa R et al. (2009). "Extreme Risk Analysis, July 2009". In: *MSCI Barra Research Paper* 2009-16.
- Menchero, Jose, Andrei Morozov, and Peter Shepard (2008). "The Barra Global Equity Model (GEM2)". In: *MSCI Barra Research Notes*, p. 53.
- Pratt, John W (1964). "Risk aversion in the small and in the large". In: *Econometrica: Journal of the Econometric Society*, pp. 122–136.
- Sharpe, William F (1990). "Capital asset prices with and without negative holdings". In: *Nobel Lecture, December 7, 1990*.