Homework 3

FINM 33210: Bayesian Statistical Inference and Machine Learning

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Problem 0.1

(a)

$$
\begin{aligned}
\mathbf{E}[\mathbf{h}'\mathbf{r}] &= \mathbf{E}[\sum_{i=1}^{n} h_i r_i] \\
&= \sum_{i=1}^{n} \mathbf{E}[h_i r_i] \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[r_i] \\
&(\because \mathbf{h} = (1/n, 1/n, \cdots, 1/n)) \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[\beta r_M + \epsilon_i] \\
&= \frac{1}{n} \sum_{i=1}^{n} (\mathbf{E}[\beta r_M] + \mathbf{E}[\epsilon_i]) \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[\beta r_M] + \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[\epsilon_i] \\
&= \frac{\beta}{n} \sum_{i=1}^{n} \mathbf{E}[r_M] + \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[\epsilon_i] \\
&= \frac{\beta}{n} n\mathbf{E}[r_M] + \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[\epsilon_i] \\
&= \beta\mathbf{E}[r_M] + \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[\epsilon_i]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V}[\mathbf{h}'\mathbf{r}] &= \mathbf{V}[\sum_{i=1}^{n} h_i r_i] \\
&= \mathbf{V}[\frac{1}{n}\sum_{i=1}^{n} r_i] \\
&(\because \mathbf{h} = (1/n, 1/n, \cdots, 1/n)) \\
&= \frac{1}{n^2}\mathbf{V}[\sum_{i=1}^{n} r_i] \\
&= \frac{1}{n^2}\mathbf{V}[\sum_{i=1}^{n}(\beta r_M + \epsilon_i)] \\
&= \frac{1}{n^2}\mathbf{V}[n\beta r_M + \sum_{i=1}^{n}\epsilon_i] \\
&= \frac{1}{n^2}\left(\mathbf{V}[n\beta r_M] + \sum_{i=1}^{n}\mathbf{V}[\epsilon_i] + \sum_{i=1}^{n} 2Cov(n\beta r_M, \epsilon_i) + \sum_{i\neq j} 2Cov(\epsilon_i, \epsilon_j)\right) \\
&= \frac{1}{n^2}\left((n\beta)^2\mathbf{V}[r_M] + \sum_{i=1}^{n}\mathbf{V}[\epsilon_i] + \sum_{i=1}^{n} 2n\beta Cov(r_M, \epsilon_i) + \sum_{i\neq j} 2Cov(\epsilon_i, \epsilon_j)\right) \\
&= \frac{1}{n^2}\left((n\beta)^2\mathbf{V}[r_M] + \sum_{i=1}^{n}\mathbf{V}[\epsilon_i] + 2\sum_{i\neq j} Cov(\epsilon_i, \epsilon_j)\right) \\
&(\because \epsilon_i \perp\!\!\!\perp r_M) \\
&= \frac{1}{n^2}\left((n\beta)^2\mathbf{V}[r_M] + \sum_{i=1}^{n}\mathbf{V}[\epsilon_i]\right) \\
&(\because \text{ for } i \neq j,\ \epsilon_i \perp\!\!\!\perp \epsilon_j) \\
&= \beta^2\mathbf{V}[r_M] + \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{V}[\epsilon_i] \\
&= \beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2
\end{aligned}
$$

Therefore, the functions $f()$ and $g()$ can be explicitly defined as:

$$
f(\beta, \sigma_M^2) := \beta^2\sigma_M^2
$$

and

$$
g(\sigma_1^2, \cdots, \sigma_n^2) := \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2
$$

(b)

It was given that $\beta = 0.5$, $\sigma_M = 0.2$, and $\sigma_i \approx 0.03$.

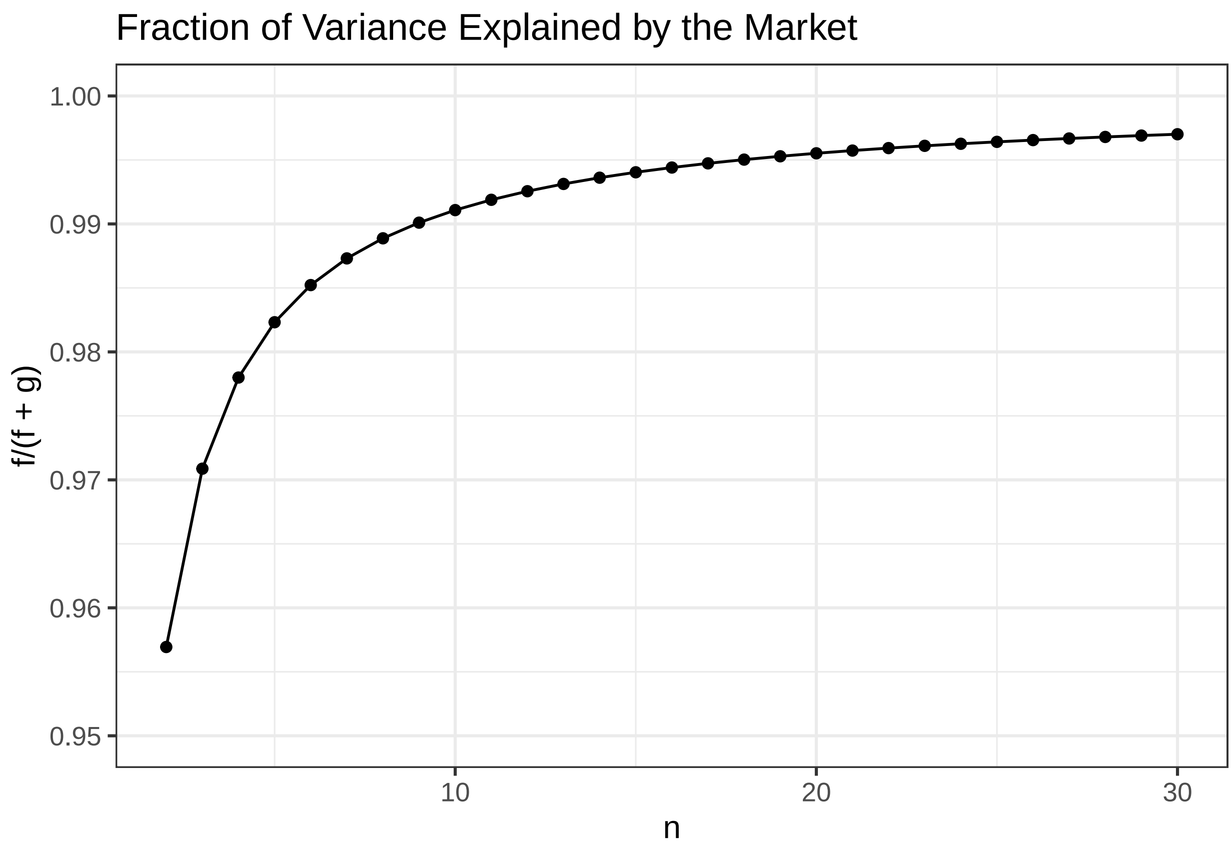
Using the definition in (a) for $f()$ and $g()$,

$$\frac{f}{f+g} = \frac{\beta^2 \sigma_M^2}{\beta^2 \sigma_M^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2}$$

Now substituting the values $\beta = 0.5$, $\sigma_M = 0.2$, and $\sigma_i \approx 0.03$.

$$\frac{f}{f+g}(n) \approx \frac{0.5^2 \cdot 0.2^2}{0.5^2 \cdot 0.2^2 + \frac{1}{n^2} n \cdot 0.03^2} = \frac{0.01}{0.01 + \frac{0.0009}{n}} = \frac{1}{1 + \frac{0.09}{n}}$$

If we numerically compute and plot $f/(f+g)$ as a function of n for $n = 2, \ldots, 30$:



(c)

We know from (a) that

$$\mathbf{E}[\mathbf{h}'\mathbf{r}] = \beta\mathbf{E}[r_M] + \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[\epsilon_i]$$

Therefore,

$$\begin{aligned}
\mathbf{E}[\mathbf{h}'\mathbf{r} - 0.01] &= \mathbf{E}[\mathbf{h}'\mathbf{r}] - \mathbf{E}[0.01] \\
&= \beta\mathbf{E}[r_M] + \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[\epsilon_i] - 0.01 \\
&= \beta \cdot 0.07 + \frac{1}{n}\sum_{i=1}^{n} 1.5 \cdot \sigma_i - 0.01 \\
&= 0.07\beta + \frac{1.5}{n}\sum_{i=1}^{n}\sigma_i - 0.01
\end{aligned}$$

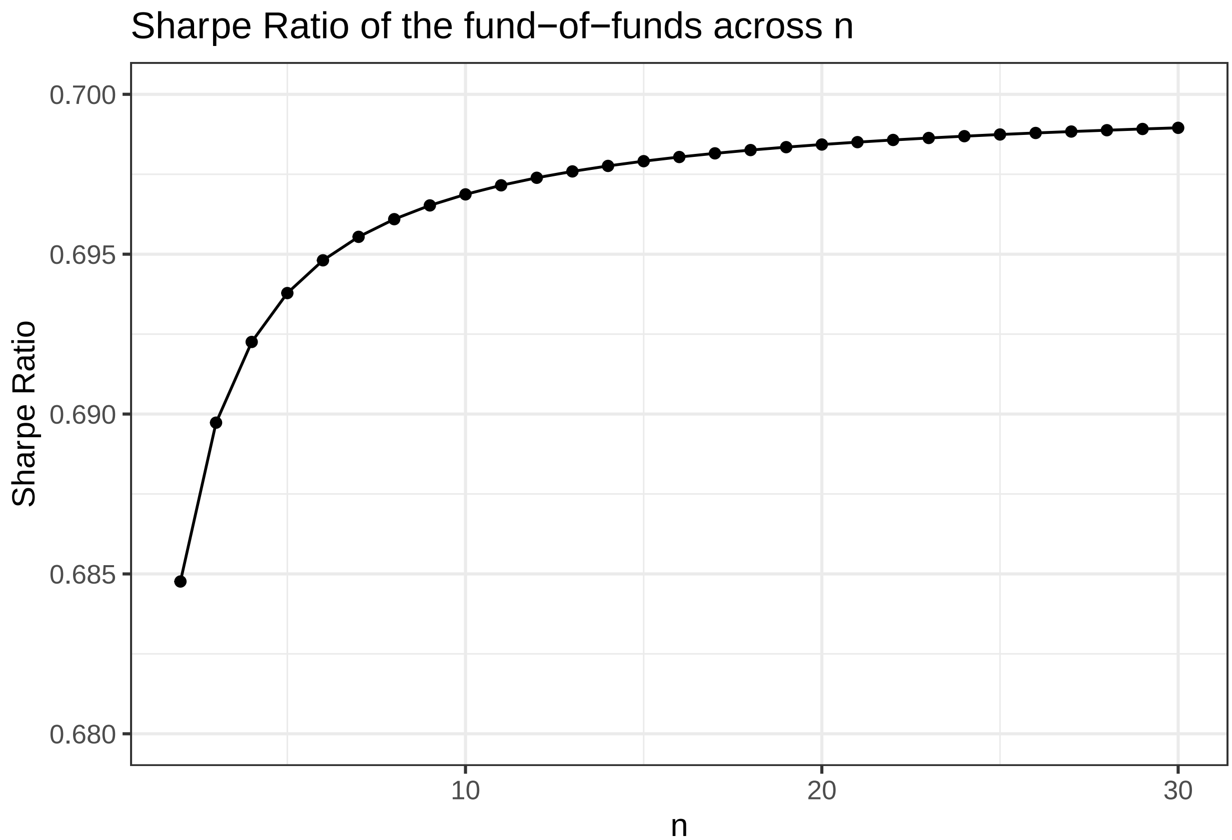
Moreover, we also know from (a) that

$$\mathbf{V}[\mathbf{h}'\mathbf{r}] = \beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$$

Now using the assumptions of (b)

$$\begin{aligned}
\frac{\mathbf{E}[\mathbf{h}'\mathbf{r} - 0.01]}{\sqrt{\mathbf{V}[\mathbf{h}'\mathbf{r}]}} &= \frac{0.07\beta + \frac{1.5}{n}\sum_{i=1}^{n}\sigma_i - 0.01}{\sqrt{\beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2}} \\
&\approx \frac{0.07 \cdot 0.5 + \frac{1.5}{n}\sum_{i=1}^{n} 0.03 - 0.01}{\sqrt{0.5^2 \cdot 0.2^2 + \frac{1}{n^2}\sum_{i=1}^{n} 0.03^2}} \\
&= \frac{0.07 \cdot 0.5 + 1.5 \cdot 0.03 - 0.01}{\sqrt{0.5^2 \cdot 0.2^2 + \frac{0.03^2}{n}}} \\
&= \frac{0.35 + 0.45 - 0.1}{\sqrt{1 + \frac{0.09}{n}}} \\
&= \frac{0.7}{\sqrt{1 + \frac{0.09}{n}}}
\end{aligned}$$

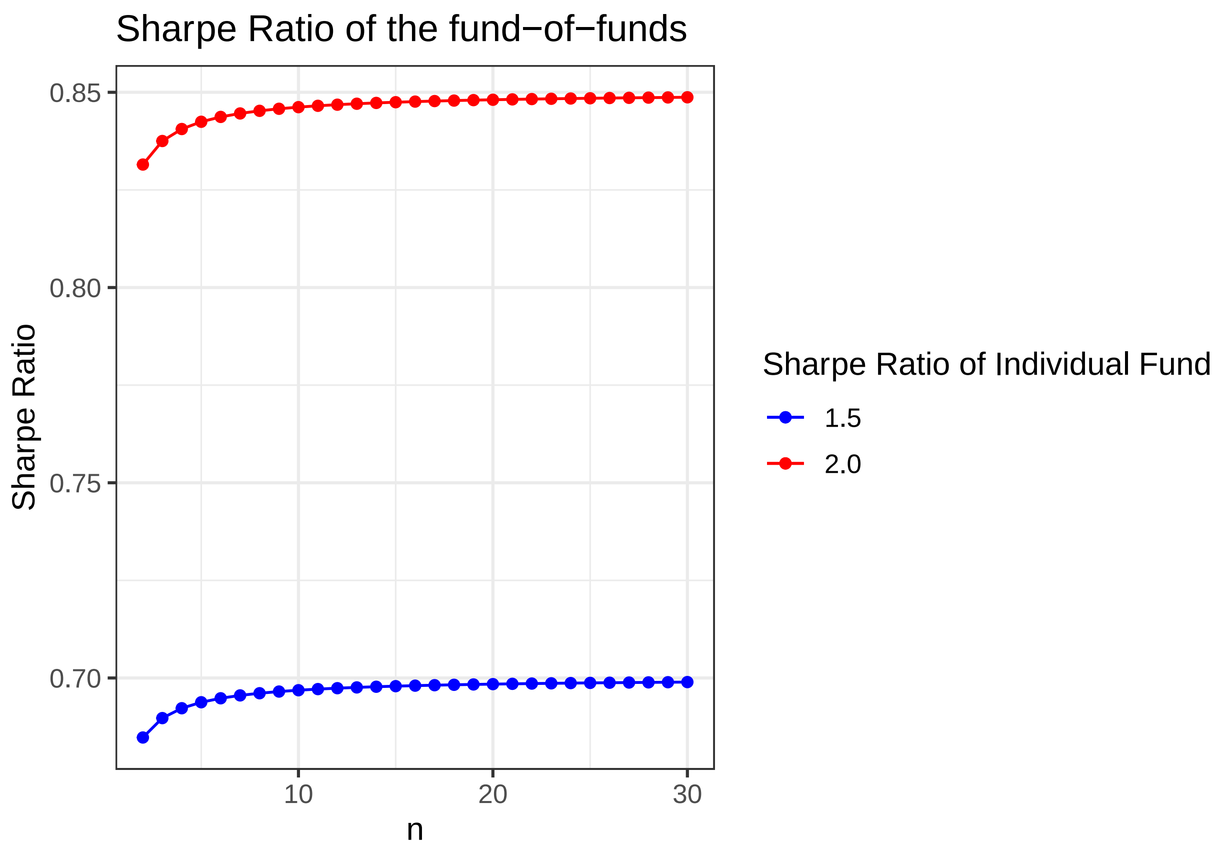
If we numerically compute and plot the Sharpe ratio as a function of n for $n = 2, \ldots, 30$:



If the Sharpe ratio of ϵ_i is 2.0,

$$\begin{aligned}
\frac{\mathbf{E}[\mathbf{h}'\mathbf{r} - 0.01]}{\sqrt{\mathbf{V}[\mathbf{h}'\mathbf{r}]}} &= \frac{0.07\beta + \frac{2.0}{n}\sum_{i=1}^{n}\sigma_i - 0.01}{\sqrt{\beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2}} \\
&\approx \frac{0.07 \cdot 0.5 + \frac{2.0}{n}\sum_{i=1}^{n} 0.03 - 0.01}{\sqrt{0.5^2 \cdot 0.2^2 + \frac{1}{n^2}\sum_{i=1}^{n} 0.03^2}} \\
&= \frac{0.07 \cdot 0.5 + 2.0 \cdot 0.03 - 0.01}{\sqrt{0.5^2 \cdot 0.2^2 + \frac{0.03^2}{n}}} \\
&= \frac{0.35 + 0.6 - 0.1}{\sqrt{1 + \frac{0.09}{n}}} \\
&= \frac{0.85}{\sqrt{1 + \frac{0.09}{n}}}
\end{aligned}$$

Moreover, if we plot the two different ϵ_i-Sharpe-ratio together:



The general trend in how the Sharpe Ratio changes as n changes is similar in both cases. However, all the Sharpe Ratios for each n is larger for 2.0 ϵ_i-Sharpe-ratio case, when compared to 1.5 ϵ_i-Sharpe-ratio case. To be precise, the former's Sharpe Ratio is scaled to $\frac{0.85}{0.7}$ of the latter.

(d)

First of all, the expected return of a single fund in this setting is:

$$\mathbf{E}[r_i] = \beta\mathbf{E}[r_M] + \mathbf{E}[\epsilon_i]$$

Therefore, using the properties $\beta = 0$, $\sigma_i = 0.1$ and $\mathbf{E}[\epsilon_i] = 1.5 \cdot \sigma_i$,

$$\mathbf{E}[r_i - 0.01] = 0 \cdot \mathbf{E}[r_M] + 1.5 \cdot 0.1 - 0.01 = 0.14$$

Moreover, the variance of a single fund is:

$$\begin{aligned}
\mathbf{V}[r_i] &= \mathbf{V}[\beta r_M + \epsilon_i] \\
&= \mathbf{V}[\beta r_M] + \mathbf{V}[\epsilon_i] + 2Cov(\beta r_M, \epsilon_i) \\
&= \beta^2\mathbf{V}[r_M] + \sigma_i^2 + 2\beta Cov(r_M, \epsilon_i) \\
&= \beta^2\mathbf{V}[r_M] + \sigma_i^2 \\
&(\because \epsilon_i \perp\!\!\!\perp r_M)
\end{aligned}$$

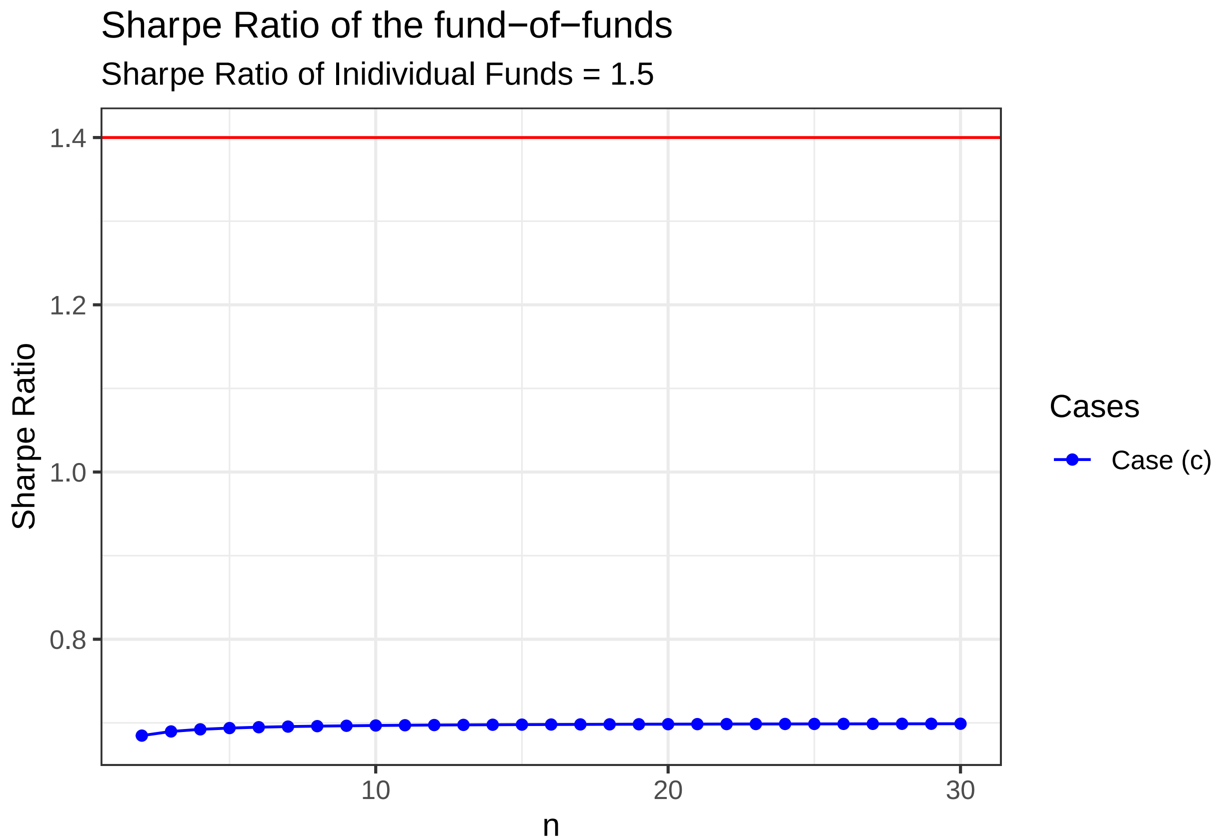
Similarly, using the given properties:

$$\mathbf{V}[r_i] = 0^2 \cdot \mathbf{V}[r_M] + 0.1^2 = 0.1^2$$

Therefore, the Sharpe ratio of the fund becomes

$$\frac{\mathbf{E}[r_i - 0.01]}{\sqrt{\mathbf{V}[r_i]}} = \frac{0.14}{\sqrt{0.1^2}} = 1.4$$

If we plot the (d)-case and (c)-case on the same graph:



The single fund seems to be much better in terms of Sharpe ratio than the earlier scenarios. This would mostly be attributed to the fact that the single fund contains no market risk yet has an already high Sharpe ratio of 1.5 by itself.

The comparison for when the Sharpe ratio of ϵ_i is 2.0 would yield a similar result with the Sharpe ratio of the fund being 1.9