Homework 3

FINM 33210: Bayesian Statistical Inference and Machine Learning

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Due: 23:59 (CT) May 19th 2023

Problem 0.1

(a)

$$
\begin{aligned}
\mathbf{E}[\mathbf{h}'\mathbf{r}] &= \mathbf{E}[\sum_{i=1}^{n} h_i r_i] \\
&= \sum_{i=1}^{n} \mathbf{E}[h_i r_i] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[r_i] \\
&(\because \mathbf{h} = (1/n, 1/n, \cdots, 1/n)) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[\beta r_M + \epsilon_i] \\
&= \frac{1}{n}\sum_{i=1}^{n} (\mathbf{E}[\beta r_M] + \mathbf{E}[\epsilon_i]) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[\beta r_M] + \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[\epsilon_i] \\
&= \frac{\beta}{n}\sum_{i=1}^{n} \mathbf{E}[r_M] + \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[\epsilon_i] \\
&= \frac{\beta}{n} n\mathbf{E}[r_M] + \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[\epsilon_i] \\
&= \beta\mathbf{E}[r_M] + \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[\epsilon_i]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V}[\mathbf{h}'\mathbf{r}] &= \mathbf{V}[\sum_{i=1}^{n} h_i r_i] \\
&= \mathbf{V}[\frac{1}{n}\sum_{i=1}^{n} r_i] \\
&(\because \mathbf{h} = (1/n, 1/n, \cdots, 1/n)) \\
&= \frac{1}{n^2}\mathbf{V}[\sum_{i=1}^{n} r_i] \\
&= \frac{1}{n^2}\mathbf{V}[\sum_{i=1}^{n} (\beta r_M + \epsilon_i)] \\
&= \frac{1}{n^2}\mathbf{V}[n\beta r_M + \sum_{i=1}^{n} \epsilon_i] \\
&= \frac{1}{n^2}\left(\mathbf{V}[n\beta r_M] + \sum_{i=1}^{n}\mathbf{V}[\epsilon_i] + \sum_{i=1}^{n} 2Cov(n\beta r_M, \epsilon_i) + \sum_{i\neq j} 2Cov(\epsilon_i, \epsilon_j)\right) \\
&= \frac{1}{n^2}\left((n\beta)^2\mathbf{V}[r_M] + \sum_{i=1}^{n}\mathbf{V}[\epsilon_i] + \sum_{i=1}^{n} 2n\beta Cov(r_M, \epsilon_i) + \sum_{i\neq j} 2Cov(\epsilon_i, \epsilon_j)\right) \\
&= \frac{1}{n^2}\left((n\beta)^2\mathbf{V}[r_M] + \sum_{i=1}^{n}\mathbf{V}[\epsilon_i] + 2\sum_{i\neq j} Cov(\epsilon_i, \epsilon_j)\right) \\
&(\because \epsilon_i \perp\!\!\!\perp r_M) \\
&= \frac{1}{n^2}\left((n\beta)^2\mathbf{V}[r_M] + \sum_{i=1}^{n}\mathbf{V}[\epsilon_i]\right) \\
&(\because \text{ for } i \neq j,\ \epsilon_i \perp\!\!\!\perp \epsilon_j) \\
&= \beta^2\mathbf{V}[r_M] + \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{V}[\epsilon_i] \\
&= \beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2
\end{aligned}
$$

Therefore, the functions $f()$ and $g()$ can be explicitly defined as:

$$f(\beta, \sigma_M^2) := \beta^2\sigma_M^2$$

and

$$g(\sigma_1^2, \cdots, \sigma_n^2) := \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$$

(b)

It was given that $\beta = 0.5$, $\sigma_M = 0.2$, and $\sigma_i \approx 0.03$.

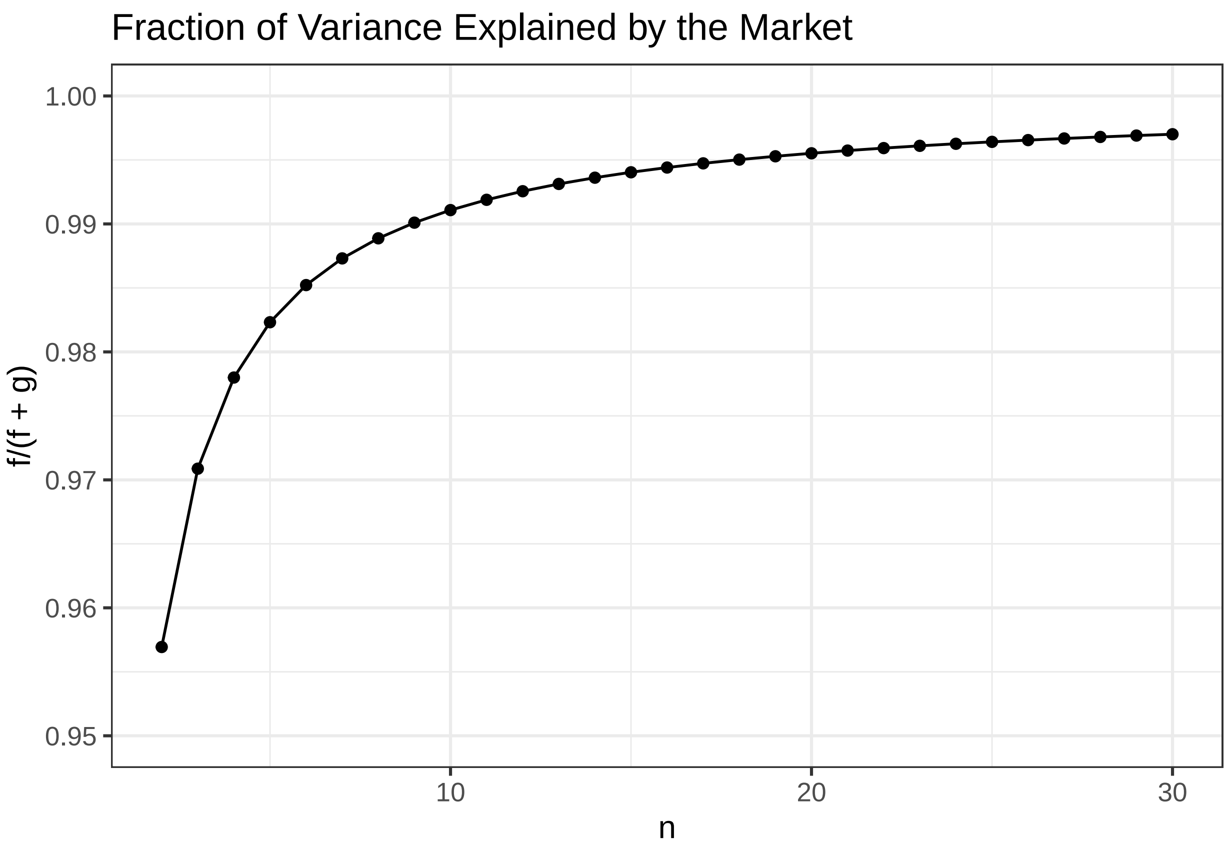
Using the definition in (a) for $f()$ and $g()$,

$$\frac{f}{f+g} = \frac{\beta^2 \sigma_M^2}{\beta^2 \sigma_M^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2}$$

Now substituting the values $\beta = 0.5$, $\sigma_M = 0.2$, and $\sigma_i \approx 0.03$.

$$\frac{f}{f+g}(n) \approx \frac{0.5^2 \cdot 0.2^2}{0.5^2 \cdot 0.2^2 + \frac{1}{n^2} n \cdot 0.03^2} = \frac{0.01}{0.01 + \frac{0.0009}{n}} = \frac{1}{1 + \frac{0.09}{n}}$$

If we numerically compute and plot $f/(f+g)$ as a function of n for $n = 2, \ldots, 30$:



(c)

We know from (a) that

$$\mathbf{E}[\mathbf{h}'\mathbf{r}] = \beta\mathbf{E}[r_M] + \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[\epsilon_i]$$

Therefore,

$$\begin{aligned}
\mathbf{E}[\mathbf{h}'\mathbf{r} - 0.01] &= \mathbf{E}[\mathbf{h}'\mathbf{r}] - \mathbf{E}[0.01] \\
&= \beta\mathbf{E}[r_M] + \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[\epsilon_i] - 0.01 \\
&= \beta \cdot 0.07 + \frac{1}{n}\sum_{i=1}^{n} 1.5 \cdot \sigma_i - 0.01 \\
&= 0.07\beta + \frac{1.5}{n}\sum_{i=1}^{n}\sigma_i - 0.01
\end{aligned}$$

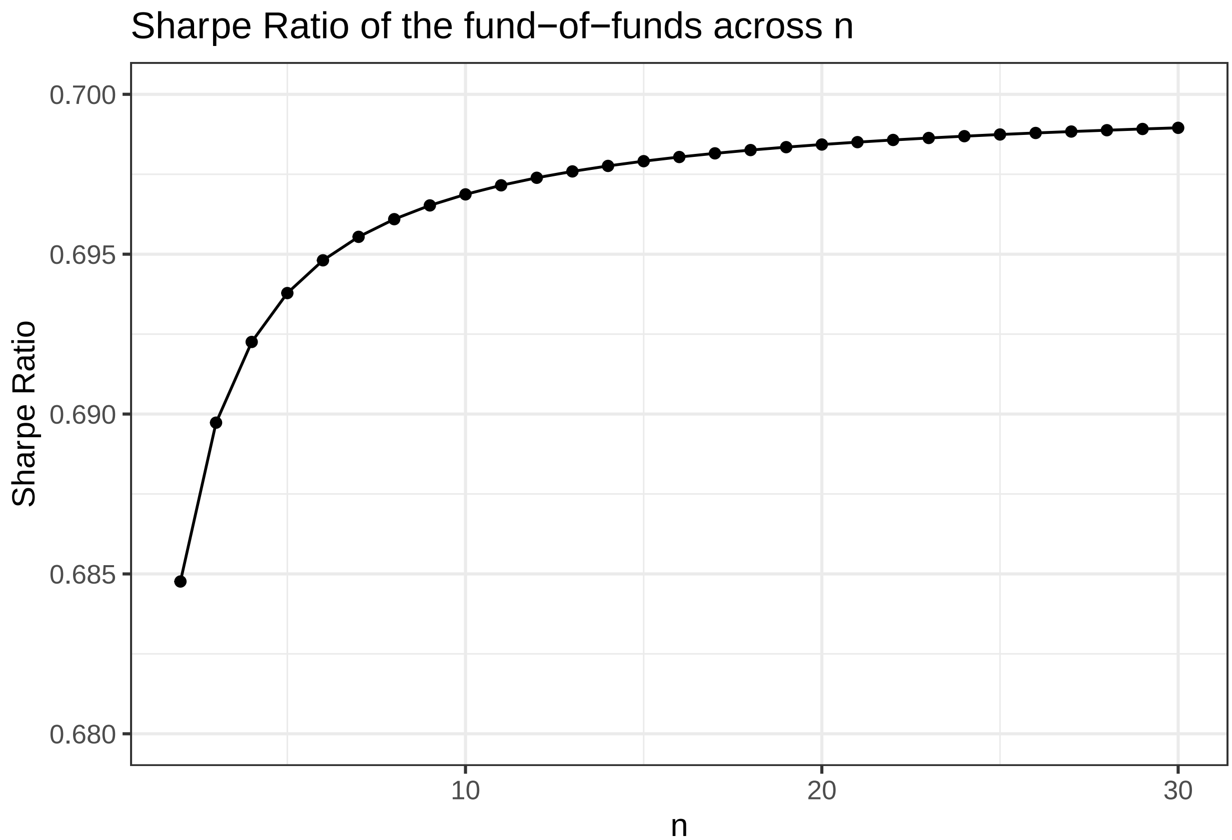
Moreover, we also know from (a) that

$$\mathbf{V}[\mathbf{h}'\mathbf{r}] = \beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$$

Now using the assumptions of (b)

$$\begin{aligned}
\frac{\mathbf{E}[\mathbf{h}'\mathbf{r} - 0.01]}{\sqrt{\mathbf{V}[\mathbf{h}'\mathbf{r}]}} &= \frac{0.07\beta + \frac{1.5}{n}\sum_{i=1}^{n}\sigma_i - 0.01}{\sqrt{\beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2}} \\
&\approx \frac{0.07 \cdot 0.5 + \frac{1.5}{n}\sum_{i=1}^{n} 0.03 - 0.01}{\sqrt{0.5^2 \cdot 0.2^2 + \frac{1}{n^2}\sum_{i=1}^{n} 0.03^2}} \\
&= \frac{0.07 \cdot 0.5 + 1.5 \cdot 0.03 - 0.01}{\sqrt{0.5^2 \cdot 0.2^2 + \frac{0.03^2}{n}}} \\
&= \frac{0.35 + 0.45 - 0.1}{\sqrt{1 + \frac{0.09}{n}}} \\
&= \frac{0.7}{\sqrt{1 + \frac{0.09}{n}}}
\end{aligned}$$

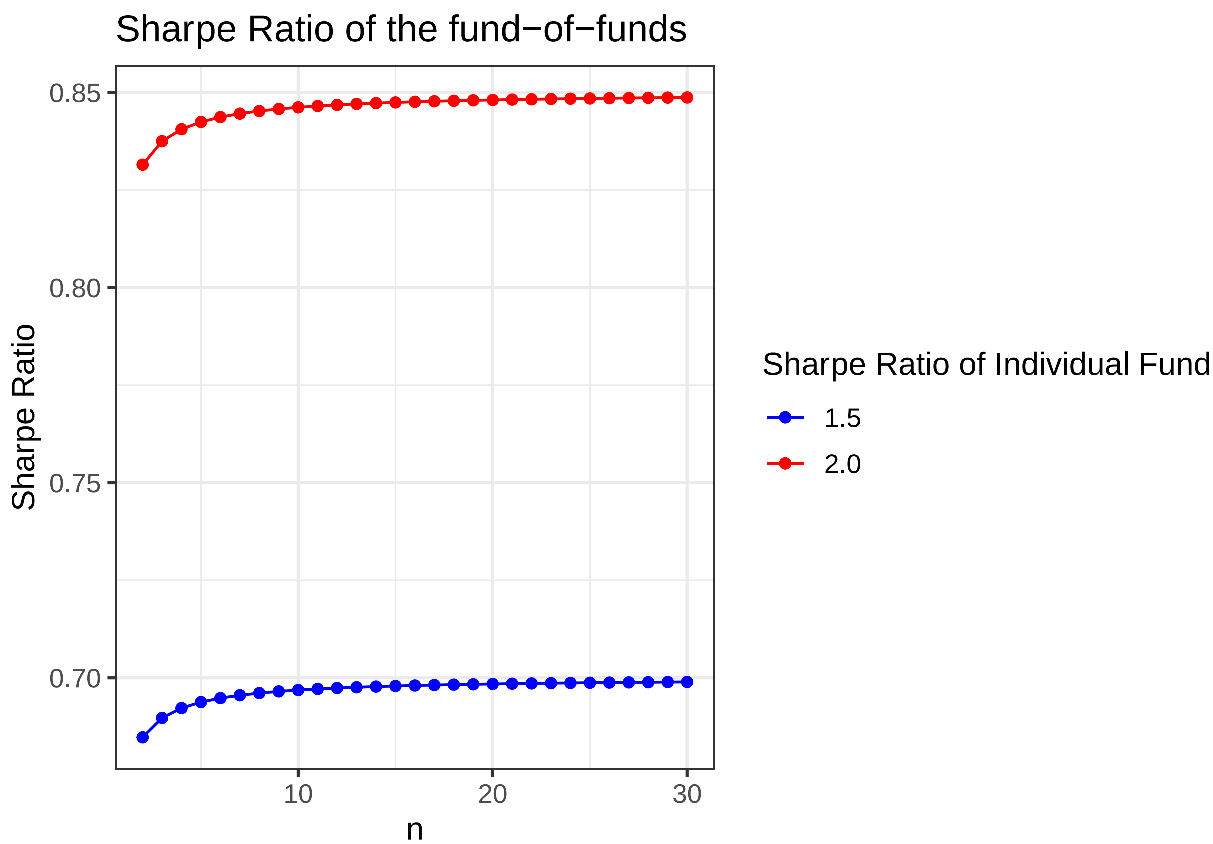
If we numerically compute and plot the Sharpe ratio as a function of n for $n = 2, \ldots, 30$:



If the Sharpe ratio of ϵ_i is 2.0,

$$\begin{aligned}
\frac{\mathbf{E}[\mathbf{h}'\mathbf{r} - 0.01]}{\sqrt{\mathbf{V}[\mathbf{h}'\mathbf{r}]}} &= \frac{0.07\beta + \frac{2.0}{n}\sum_{i=1}^{n}\sigma_i - 0.01}{\sqrt{\beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2}} \\
&\approx \frac{0.07 \cdot 0.5 + \frac{2.0}{n}\sum_{i=1}^{n} 0.03 - 0.01}{\sqrt{0.5^2 \cdot 0.2^2 + \frac{1}{n^2}\sum_{i=1}^{n} 0.03^2}} \\
&= \frac{0.07 \cdot 0.5 + 2.0 \cdot 0.03 - 0.01}{\sqrt{0.5^2 \cdot 0.2^2 + \frac{0.03^2}{n}}} \\
&= \frac{0.35 + 0.6 - 0.1}{\sqrt{1 + \frac{0.09}{n}}} \\
&= \frac{0.85}{\sqrt{1 + \frac{0.09}{n}}}
\end{aligned}$$

Moreover, if we plot the two different ϵ_i-Sharpe-ratio together:



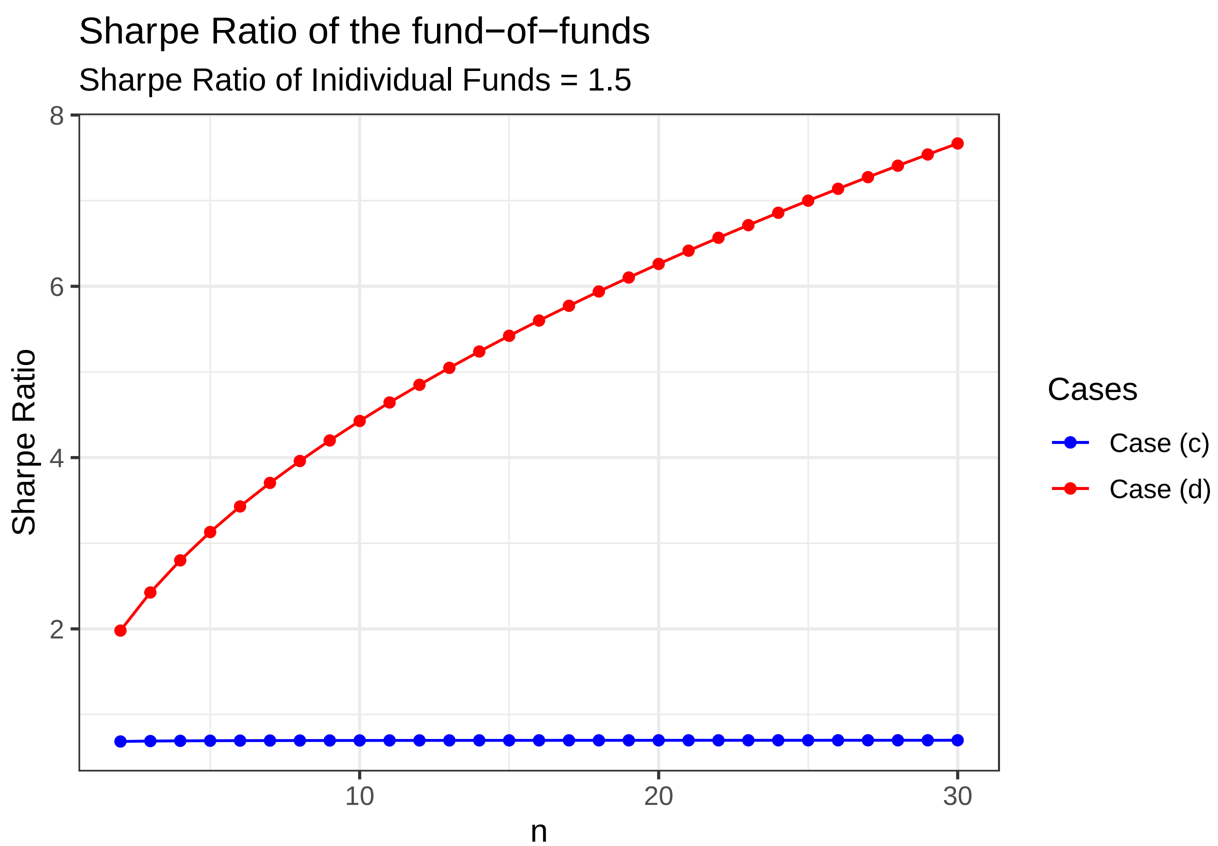
The general trend in how the Sharpe Ratio changes as n changes is similar in both cases. However, all the Sharpe Ratios for each n is larger for 2.0 ϵ_i-Sharpe-ratio case, when compared to 1.5 ϵ_i-Sharpe-ratio case. To be precise, the former's Sharpe Ratio is scaled to $\frac{0.85}{0.7}$ of the latter.

(d)

Using the same assumptions in (c), for the 1.5 ϵ_i-Sharpe-ratio case, the new Sharpe Ratio of the fund of funds becomes:

$$
\begin{aligned}
\frac{\mathbf{E}[\mathbf{h}'\mathbf{r} - 0.01]}{\sqrt{\mathbf{V}[\mathbf{h}'\mathbf{r}]}} &= \frac{0.07\beta + \frac{1.5}{n}\sum_{i=1}^{n}\sigma_i - 0.01}{\sqrt{\beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2}} \\
&= \frac{0.07 \cdot 0 + \frac{1.5}{n}\sum_{i=1}^{n} 0.1 - 0.01}{\sqrt{0^2 \cdot 0.2^2 + \frac{1}{n^2}\sum_{i=1}^{n} 0.1^2}} \\
&= \frac{1.5 \cdot 0.1 - 0.01}{\sqrt{\frac{0.1^2}{n}}} \\
&= \frac{0.14}{0.1}\sqrt{n} \\
&= 1.4\sqrt{n}
\end{aligned}
$$

If we plot the (d)-case and (c)-case on the same graph:

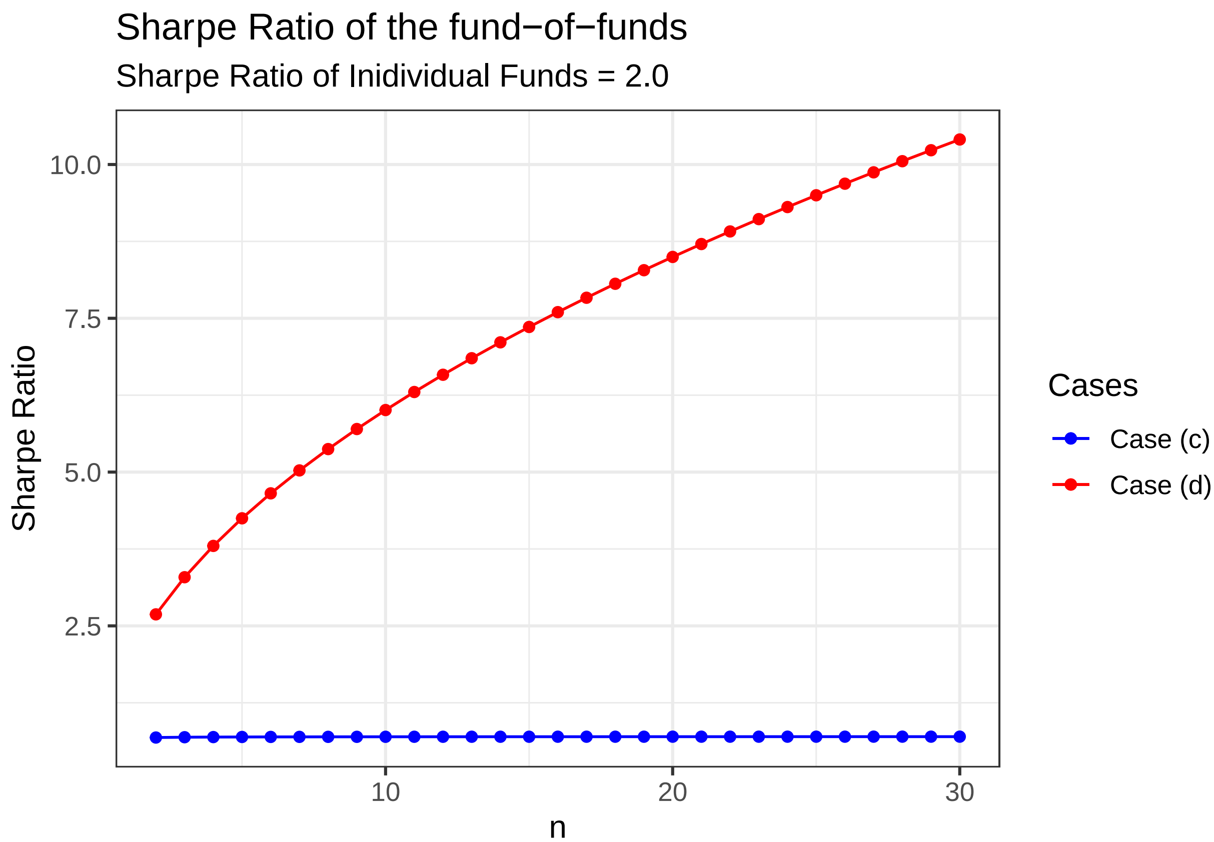


Simply investing in a single fund case of (d) seems to be better in terms of Sharpe ratio when individual fund's Sharpe Ratio is 1.5. The difference in Sharpe ratio between the two cases is greater for larger n values.

Now when individual fund's Sharpe Ratio is 2.0:

$$
\begin{aligned}
\frac{\mathbf{E}[\mathbf{h}'\mathbf{r} - 0.01]}{\sqrt{\mathbf{V}[\mathbf{h}'\mathbf{r}]}} &= \frac{0.07\beta + \frac{2.0}{n}\sum_{i=1}^{n}\sigma_i - 0.01}{\sqrt{\beta^2\sigma_M^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2}} \\
&= \frac{0.07 \cdot 0 + \frac{2.0}{n}\sum_{i=1}^{n} 0.1 - 0.01}{\sqrt{0^2 \cdot 0.2^2 + \frac{1}{n^2}\sum_{i=1}^{n} 0.1^2}} \\
&= \frac{2.0 \cdot 0.1 - 0.01}{\sqrt{\frac{0.1^2}{n}}} \\
&= \frac{0.19}{0.1}\sqrt{n} \\
&= 1.9\sqrt{n}
\end{aligned}
$$

Moreover,



Similarly, simply investing in a single fund case of (d) seems to be better in terms of Sharpe ratio when individual fund's Sharpe Ratio is 2.0 as well. The difference in Sharpe ratio between the two cases is greater when individual funds have 2.0 Sharpe ratio than when individual funds have 1.5 Sharpe ratio. Within the above graph, the difference in Sharpe ratio between the two cases is greater for larger n values.