Asset allocation with higher order moments and factor models

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Abstract

We provide a comprehensive framework for integrating higher order comoments into the asset allocation decision. In order to reduce the sensitivity of the optimized weights to estimation error in the input parameters, we derive explicit formulas for the higher order comoments under the assumption that stock returns are generated by a multifactor model. We further investigate the potential benefits of (i) switching from a minimum variance objective to a maximum expected utility objective that depends on the covariance, coskewness and cokurtosis, (ii) replacing an equal variance contribution constraint by an equal expected shortfall contribution constraint, and (iii) the use of multifactor models for estimating the higher order comoments instead of the sample estimator or single factor model. In the application we find that accounting for the higher order moments in the portfolio objective and risk diversification constraint, increases out-of-sample returns, decreases the portfolio volatility and leads to an important reduction in the portfolio drawdown. These gains tend to be higher when using the multifactor approach to comoment estimation instead of the sample and single factor model based estimates.

Keywords: investment analysis, factor models, higher order comoments, portfolio selection
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1. Introduction

The world of asset returns is non-normal. Its distribution tends to be asymmetric and extremes occur too often to be compatible with the tails of a normal distribution. An

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important outstanding question is whether and how to integrate this non-normality in the asset allocation decision[1]. If there is no estimation error, most investors would be willing to sacrifice expected return and/or accept a higher volatility in exchange for a higher skewness and lower kurtosis leading to a lower downside risk (e.g. Ang *et al.* (2006), Harvey and Siddique (2000), Scott and Horvath (1980)). This trade-off between positive preferences for odd moments (mean, skewness) and negative preferences for even moments (variance, kurtosis) can be conveniently summarized into a single objective function using a Taylor expansion of the expected utility function as objective (Jondeau and Rockinger (2006), Martellini and Ziemann (2010)) or a portfolio downside risk objective based on the Cornish-Fisher expansion (e.g. Peterson and Boudt (2008), Boudt *et al.* (2013)).

The important caveat is that portfolio moments need to be estimated, and that the estimation error greatly amplifies when the dimension of the investment universe increases. This curse of dimensionality makes the unrestricted estimators of the first four (co)moments almost infeasible for moderately large dimensions. Suppose e.g. that we have a universe of 20 assets, then the number of unique elements in the covariance, coskewness and cokurtosis is 210, 1540 and 8555, respectively. This is clearly an excessive number of parameters compared to the number of observations that are available in realistic applications.

The consequences of estimation error in portfolio optimization are well known. For example, optimized portfolios are often not well-diversified (Green and Hollifield (1992)) and behave like "error maximizers" (Michaud (1998)). As noted e.g. in Kolm *et al.* (2014), this does not imply that the theory of portfolio optimization is flawed, but that the "classical optimization framework has to be modified when used in practice in order to achieve reliability, stability, and robustness with respect to model and estimation errors."

In this article, we follow the approach of adapting the classical framework based on sample comoment estimators by using comoment estimators, which integrate intelligence in relation to the model that has generated the data. Martellini and Ziemann (2010) recommend to use a single factor model. We show that under this approach the total number of unique elements in the covariance, coskewness and cokurtosis matrix of 20 assets is reduced to 83. This however induces possible specification error. Especially as far as asset allocation is concerned, it is unlikely that only one factor would be able to explain the

[1]We will consider distributions that not only have fat tails, but that are also asymmetric. Because of the asymmetry, the asset return distribution is also often not elliptically symmetric. If the distribution is elliptically symmetric, then expected utility is a function of mean and variance Chamberlain (1983) and, in the absence of estimation error, all rational investors will select a portfolio which lies on the Markowitz mean-variance efficient frontier (e.g.Adcock (2014)).

cross-section of asset returns.

In this paper, we are among the first to estimate the coskewness and cokurtois matrix under a general multifactor model. The total number of unique elements in the covariance, coskewness and cokurtosis matrix of 20 assets is 112 and 151 for 2 and 3 factors, respectively. The estimation of such a number of parameters is still feasible and reduces substantially the specification bias of the single factor model.

We illustrate the usefulness of the multifactor approach to higher order comoments in an international portfolio context where the investor allocates with the purpose to maximize his expected utility. The universe consists of four equity benchmarks, nine bond indices and five commodity indices. We find that accounting for the higher order moments using a multifactor approach increases out-of-sample returns, decreases portfolio standard deviations and leads to an important reduction in the portfolio downside risk.

In what follows, we first describe in Section 2 the general higher order moment optimization framework that we propose. This includes our key contribution, which is the derivation of the explicit formula for the higher order comoments when the asset returns are generated by a multifactor model. In Section 3 we then study the out-of-sample performance of portfolios that use these higher order comoments. The major findings are summarized in the conclusions.

2. A general framework for asset allocation with higher order moments

Crama and Schyns (2003) note that a formal framework to handle portfolio selection problems must address at least three types of questions:

- Data modeling;
- The optimization model: Nature of the objective function and the constraints imposed;
- The choice of the optimization technique.

The feature that those three questions have in common is the dimension of the investment universe that we fix at N assets. The overall investment philosophy is that the investor needs to decide on the $N \times 1$ vector of portfolio weights w that optimizes a constrained objective function that depends on the first four portfolio moments.

Implementing this requires first an appropriate estimation tool for the first four portfolio moments. Regarding the first question of data modeling, we assume that the asset return distribution is potentially heavy tailed and asymmetric, implying that the mean-variance utility function is not an adequate choice as objective function. In order to avoid the so-called "error maximization" when the number of parameters to estimate is high compared to the number of observations, we will impose a factor model structure on the return generating process. This will be the object of Subsection 2.1. In Subsection 2.2 we will outline in detail a portfolio decision framework in which the portfolio variance, skewness and kurtosis determine the investment decisions. Finally, Subsection 2.3 describes the application of the Differential Evolution algorithm of Price *et al.* (2006) and Storn and Price (1997) to solving the portfolio optimization problem.

2.1. Data modeling: a multifactor specification of higher order comoments

The key input parameters for the portfolio decision are the covariance, coskewness and cokurtosis matrix of the $N-$ dimensional return vector r with mean μ_r, i.e. the comoments corresponding to (i) the products of two returns, i.e. the covariance of assets i and j:

$$\sigma_{i,j} = E[(r_{(i)} - \mu_{r(i)})(r_{(j)} - \mu_{r(j)})], \tag{1}$$

(ii) the products of three returns, i.e. the coskewness of assets i, j and k:

$$\phi_{i,j,k} = E[(r_{(i)} - \mu_{r(i)})(r_{(j)} - \mu_{r(j)})(r_{(k)} - \mu_{r(k)})], \tag{2}$$

and (iii) the products of four returns, i.e. the cokurtosis of assets i, j, k and l:

$$\psi_{i,j,k,l} = E[(r_{(i)} - \mu_{r(i)})(r_{(j)} - \mu_{r(j)})(r_{(k)} - \mu_{r(k)})(r_{(l)} - \mu_{r(l)})]. \tag{3}$$

It will reveal useful to stack all these comoments into a $N \times N$ covariance matrix Σ, $N \times N^2$ coskewness matrix Φ and $N \times N^3$ cokurtosis matrix Ψ of the return vector r, i.e.:

$$\begin{aligned}
\Sigma &= E[(r-\mu_r)(r-\mu_r)'], \\
\Phi &= E[(r-\mu_r)(r-\mu_r)' \otimes (r-\mu_r)'], \\
\Psi &= E[(r-\mu_r)(r-\mu_r)' \otimes (r-\mu_r)' \otimes (r-\mu_r)'],
\end{aligned} \tag{4}$$

where $\otimes$ denotes the Kronecker product.

In fact, as noted e.g. in Chapter 2 of Jondeau *et al.* (2007), we have the useful property that they allow us to easily calculate the k-th portfolio return moment for a portfolio with weights w:

$$
\begin{aligned}
m_2(w) &= E[(w'(R-\mu_R))^2] = w'\Sigma w, \\
m_3(w) &= E[(w'(R-\mu_R))^3] = w'\Phi(w \otimes w), \\
m_4(w) &= E[(w'(R-\mu_R))^4] = w'\Psi(w \otimes w \otimes w).
\end{aligned} \tag{5}
$$

The challenge is to estimate these comoment matrices Σ, Φ and Ψ. As shown in Table A.1, for moderately sized portfolios of size $N = 20$, it requires to estimate 10,605 parameters. To avoid this curse of dimensionality, the solution we provide is to impose more structure on the data through a factor model.

∼ Insert Table A.1 about here ∼

2.1.1. Review of factor models for asset returns

A common approach is to assume that the variation in asset returns is driven by multiple common factors and idiosyncratic factors that are specific to each asset. Accounting for those factors is an important component of the asset allocation decision (e.g. Boudt and Peterson (2013)). In this subsection, we will review the factor models and in the next subsection derive the expressions for the higher comoments under the multifactor model.

Three types of factor models exist. In the macroeconomic factor model, factors are observable macro-financial variables. Under the fundamental factor model, factors are created from observable asset characteristics. Finally, in statistical factor models, factors are unobservable and extracted from the asset returns. Connor (1995) compares the explanatory power of the three types of factor models for security market returns and finds that the statistical and fundamental factor models outperform the macroeconomic model. In the application on asset allocation, we will use statistical factors, as it is not obvious to find a parsimonious set of fundamental factors.

All three types of factor models can be represented in the same general form. To introduce this notation, suppose that K observable factors are identified as being influential for the portfolio variability, and that, at a given frequency, the asset returns $r_t = (r_{1t}, \cdots, r_{Nt})'$ and the factors $f_t = (f_{1t}, \cdots, f_{Kt})'$ are recorded.

The asset returns are assumed to depend linearly on the factors, whereby the variation in the asset returns that is not explained by the factors, is assumed to be independent of each of the factors and also to be independent across assets. In matrix notation, the system is given by:

$$r_t = a + Bf_t + e_t, \tag{6}$$

where $e_t = (e_{1t}, \cdots, e_{Nt})'$ is the $N \times 1$ vector of asset specific factors and B is the $N \times K$ matrix of factor loadings (also called factor beta's or factor exposures) of the N assets on the K factors.

2.1.2. Comoments under the linear factor model

Let S be the $K \times K$ covariance matrix of the K factors, G the $K \times K^2$ coskewness matrix of the K factors and P their $K \times K^3$ cokurtosis matrix:

$$\begin{aligned}
\mu_f &= E[f_t], \\
S &= E[(f_t - \mu_f)(f_t - \mu_f)'], \\
G &= E[(f_t - \mu_f)(f_t - \mu_f)' \otimes (f_t - \mu_f)'], \\
P &= E[(f_t - \mu_f)(f_t - \mu_f)' \otimes (f_t - \mu_f)' \otimes (f_t - \mu_f)'].
\end{aligned} \tag{7}$$

We rewrite the comoment matrices Σ, Φ and Ψ as the sum of the comoment of the return explained by the factor (i.e. Bf_t) and a residual matrix denoted by Δ, Ω and Υ:

$$\begin{aligned}
\Sigma &= BSB' + \Delta, \\
\Phi &= BG(B \otimes B') + \Omega, \\
\Psi &= BP(B' \otimes B' \otimes B') + \Upsilon.
\end{aligned} \tag{8}$$

Because of the assumption that the unexplained asset return variation e_t is independent of the factors, Δ is a diagonal matrix with i-th diagonal element equal to the variance of the $i-$th error term and Ω is a $N \times N^2$ matrix of zeros except for the i, j elements where $j = (i-1)N + l$, which is corresponding to the expected third moment of the idiosyncratic factors.

The definition of Υ is slightly more complex. Like the other residual matrices, it consists mostly of zeros, except for the cokurtosis elements corresponding to the decomposition of: the kurtosis of one asset, the cokurtosis between two assets and the cokurtosis between 3 assets.

For the kurtosis of one asset (i.e. $i = j = k = l$), the corresponding element in Υ should be:

$$6b_i' S b_l E[e_{it}^2] + E[e_{it}^4]. \tag{9}$$

For the cokurtosis between two assets when $(i = j = k)$ and $l \neq i$, the corresponding element in Υ should be:

$$3b_i' S b_i E[e_{it}^2], \tag{10}$$

and analogously for $(i = j = l)$ and $k \neq l$, or $(j = k = l)$ and $i \neq k$. When $(i = j) \neq (k = l)$, the corresponding element in Υ should be:

$$b_i' S b_i E[e_{kt}^2] + b_k' S b_k E[e_{it}^2] + E[e_{it}^2] E[e_{kt}^2], \tag{11}$$

and similarly for $(i = l) \neq (j = k)$ and $(i = k) \neq (j = l)$. Finally, for the cokurtosis between 3 assets, i.e. $i = j$ and $k \neq i$, $k \neq l$, $i \neq l$, the corresponding element in Υ should be:

$$b_k' S b_l E[e_{it}^2], \tag{12}$$

and analogously for similar combinations. See Appendix A for the proof.

2.1.3. Estimation

There is considerable evidence of autocorrelation in the variance, skewness and kurtosis (see e.g. Jondeau and Rockinger (2003) and Leon *et al.* (2005)). Several approaches exist to take this time-variation in the higher order moments into account. In the application, we will take the rolling estimation sample approach and avoid making assumptions on the functional form of the dynamic linkage between future and past returns. Alternative parametric approaches are based on garch-type of modelling of the dynamic comoments (e.g. Boudt *et al.* (2013)) or accounting for regime switches in the return distribution (e.g. Bae *et al.* (2014)). The factor exposures are estimated by the ordinary least squares estimator and population moments are estimated by the corresponding sample averages.

2.2. The optimization model: Inclusion of higher order comoments in the objective function and the constraints imposed

The investment universe we analyze consists out of N assets and the investor needs to decide on the $N \times 1$ vector of portfolio weights w that optimizes a constrained objective function that depends on the first four portfolio moments. Like in Martellini and Ziemann (2010) we will assume that the investor maximizes the expected value of the fourth order expansion of the Constant Relative Risk Aversion (CRRA) utility function with risk aversion parameter γ and we neutralize the effect of the mean by assuming it to be zero. This then leads to the following expected utility function:

$$EU_\gamma(w) = -\frac{\gamma}{2} m_{(2)}(w) + \frac{\gamma(\gamma+1)}{6} m_{(3)}(w) - \frac{\gamma(\gamma+1)(\gamma+2)}{24} m_{(4)}(w), \tag{13}$$

where $m_{(2)}(w)$, $m_{(3}(w)$ and $m_{(4)}(w)$ is the second, third and fourth portfolio moment as defined in (5).

Let us take a closer look at the objective function in (13). First, note that the expected utility, ceteris paribus, decreases with the portfolio variance and kurtosis, and increases with the portfolio skewness. This reflects the general preference of investors for portfolios with low downside risk and allows to investor to rank several candidate portfolios with different portfolio return distributions.

Second, we observe that the trade-off between variance, skewness and kurtosis depends on the risk aversion parameter γ. Like in Martellini and Ziemann (2010) we will consider $\gamma = 10$ as our base case but will also try the more common choice of $\gamma = 5$. The higher the risk aversion γ , the more weight is put on the higher moments. Finally, note that under the assumption of normality, maximizing expected utility is equivalent to the minimum variance portfolio.

We will apply this to an asset allocation portfolio. In order to avoid that this portfolio's risk exposure is concentrated in the least risky assets, we additionally impose the equal risk contribution (ERC) diversification constraint that all asset classes contribute equally to portfolio risk. Assuming normality, the equal risk contribution constraint, implies that all assets contribute equally to the portfolio variance. The properties of imposing the ERC constraint in variance based portfolios have been studied in Maillard *et al.* (2010) and Lee (2011), among others.

For asymmetric distributions, the variance is no longer acceptable as a risk measure and a downside risk measure needs to be used. We will use the modified Expected Shortfall estimator proposed by (Boudt *et al.* (2008), Boudt *et al.* (2013)) in which the third order Cornish-Fisher expansion is used to account for the skewness and kurtosis when estimating the portfolio risk.

More precisely, the estimator for the expected shortfall at the level α (e.g. 5 per cent) is given by:

$$ES_\alpha(w) = -w'\mu + \sqrt{m_{(2)}(w)} \times \frac{1}{\alpha}[a_\alpha + b_\alpha k(w) + c_\alpha s(w) + d_\alpha s^2(w)]. \tag{14}$$

with $a_\alpha, b_\alpha, c_\alpha$, and d_α numbers that depend on the choice of the loss level α (see Boudt *et* *al.* (2008)) and $s(w)$ and $k(w)$ are the portfolio skewness and excess kurtosis respectively:

$$\begin{aligned} s(w) &= m_{(3)}(w)/m_{(2)}^{3/2}(w), \\ k(w) &= m_{(4)}(w)/m_{(2)}^{2}(w) - 3. \end{aligned} \tag{15}$$

2.3. The choice of the optimization technique: the Differential Evolution heuristic

The optimizer of the asset allocation problem needs to search for the portfolio weight vector w that minimizes an objective function which (besides w) depends non-linearly on the higher order comoment matrices Σ, Φ and Ψ under a non-linear risk diversification constraint that also depend on Σ, Φ and Ψ, as well as a full investment constraint and box constraints on the portfolio weights. We will solve this problem using a heuristic optimization method called Differential Evolution in which the box and full investment constraints are directly imposed through a mapping of the generated populations into the feasible space, while the equal risk contribution constraint is imposed as a penalty in the objective function.

Differential Evolution belongs to the class of genetic algorithms which use biology-inspired operations of crossover, mutation, and selection on a population in order to minimize an objective function over the course of successive generations. It was developed by Storn and Price (1997) and is distinct from most genetic algorithms because of the use of arithmetic operations instead of logical operation in mutation. We generated the first generation by drawing a population of 500 $N-$dimensional members u_i from the independent uniform distribution between 0 and 1 and standardized each vector to sum up to 1. Denote the corresponding standardized vector x_i, with $x_{i(j)} = u_i(j)/\sum_{j=1}^{N} u_{i(j)}(i = 1, \cdots, 500)$. Further generations are created by choosing three members of the population (x_1, x_2 and x_3), at random, and for each solution in the parent generation, create a mutant vector v as

$$v = x_1 + F(x_2 - x_3), \tag{16}$$

where the scale factor $F = 0.8$. The cross-over probability determining the relative fraction of the parent and mutant that are copied into the candidate child is set to 0.5. After the cross-over, the solutions that do not satisfy the $[0, 1]$ bound constraint are replaced by a draw from the uniform distribution on the allowed $[0, 1]$ interval and all solutions are standardized to sum up to unity such that the solutions satisfy by construction the full investment constraint. If a trial vector improves the objective function, it replaces the previous vector in the population, otherwise the previous vector remains. It follows that, the closer the population gets to the global optimum, the more the distribution of solutions in the current generation will shrink and therefore reinforce the generation of smaller difference vectors.

3. Empirical illustration

The proposed methodology has important applications in the design of optimal asset allocation portfolios. We illustrate this for a realistic universe of four equity benchmarks (Europe, North America, Pacific and Emerging Markets), eight bond indices (corporate developed high yield index in EUR and USD, corporate developed investment grade index in EUR, corporate emerging investment grade in USD, sovereign developed investment grade in USD, EUR, JPY and sovereign developed and emerging in USD) and five commodity indices (agriculture, energy, industrial metals, livestock and precious metals).

The cumulative return evolution of each of the asset classes over the period 1999-2012 is shown in Figure A.1. Besides the differences in volatility and return over the period, the graph clearly shows the diversification potential across the different investment universes. The shaded area in Figure A.1 corresponds to the out-of-sample evaluation period used to compare the different portfolio allocation methodologies.

∼ Insert Figure A.1 about here ∼

Table A.2 summarizes the performance of the different assets over this period (expressed in EUR).

The broad picture of Table A.2 is that, over the period, the bond asset class clearly outperformed the equity and commodities asset classes, but there is substantial variation in return and risk of the components of the same asset class. In equities, the North American benchmark yielded the highest annualized return (4.35%), the lowest annualized volatility (16.11%) and the most positive skewness (-0.428), while European markets showed the lowest annualized returns (0.11%) and the most negative skewness (-0.736). In the bond market, the impact of the issuer is as intuitively expected. Corporate HY bonds exhibit a higher return and risk than corporate IG and sovereign developed. Currency also has a substantial impact that seems to interact with the effect of the issuer. Finally, except for precious metals, commodities had negative returns over the period.

Table A.3 compares the portfolio methodologies, which either aim to minimize variance, or to maximize the CRRA expected utility with risk aversion parameter γ (5 and 10), under the risk diversification constraint that all three major asset classes contribute equally to portfolio variance or to the 95% portfolio expected shortfall. Portfolios are fully invested and short sales positions are not allowed. This leads to six constrained portfolio objectives. Each objective depends on the estimated comoments, for which we evaluate three options: the sample estimator, the single factor model based estimate and a 3-factor approach. In total we have thus 18 portfolio allocation schemes. The portfolios are optimized using the Differential Evolution global optimization method, as described in Subsection 2.3. The comoments are estimated using weekly returns in local currency and a rolling sample of six years.

∼ Insert Table A.2 about here ∼

The standard approach is in the first row of Table A.3. It consists in minimizing the portfolio variance under the equal variance contribution constraint. Comparing this method with the other 17 portfolio allocation methodologies, the overall picture appears to be that the following three steps tend to increase the out-of-sample return and reduce the portfolio risk: (i) first, the estimation of comoments under a multifactor specification; (ii) second, the inclusion of higher order comoments in the objective function by using a maximum CRRA expected utility objective instead of a minimum variance objective; and (iii) third, controlling diversification through an equal expected shortfall contribution constraint instead of an equal variance contribution constraint.

These effects are expected and in line with prior research that emphasized the risk of noise optimization when using the simple sample average-based estimates of the comoments in portfolio optimization (e.g. Jagannathan and Ma (2003), Ledoit and Wolf (2003), Martellini and Ziemann (2010)). Jondeau and Rockinger (2006) argued that, while the mean-variance criterion provides a good approximation of the expected utility maximization under moderate non-normality, it may be ineffective under large departure from normality, as the one in our sample, as shown by the skewness and excess kurtosis of the asset returns in Table A.2. Because of this non-normality, risk diversification constraints based on expected shortfall are more appropriate than risk constraints based on the portfolio variance, as they take the non-linear dependence across asset returns into account.

In our out-of-sample analysis we thus confirm the result of Martellini and Ziemann (2010) and Jondeau and Rockinger (2012) that accounting for higher order moments can substantially improve portfolio performance. They showed this for equity portfolios, while our application is on asset allocation. Like in Martellini and Ziemann (2010), the size of the universe is relatively large (17 assets), explaining the need of structured estimates in order to increase the efficiency of the estimates and the resulting portfolio weights.

For our sample, the best risk-adjusted portfolio performance is observed for the maximum expected utility objective with risk aversion parameter γ=10 and diversification imposed through an equal contribution to the expected shortfall constraint, whereby the comoments are estimated under the multifactor approach. This scenario shows an annualized return of 9.47%, an annualized volatility of 9.35%, positive skewness and a 95% portfolio VaR of 2.25%. Its maximum drawdown (8.06%) is less than the maximum drawdown on the least risky asset of the universe (Sov Developed IG, EUR: 8.46% maximum drawdown) at a substantially higher return (9.47% instead of 5.28%), and, importantly, because of the diversification constraint, it has a much smaller idiosyncratic risk.

∼ Insert Table A.3 about here ∼

Finally, we report in Figure A.2 the monthly levels of the benchmark minimum variance portfolio using the sample estimator, and compare with the higher order comoment based maximum expected utility ($\gamma = 10$) portfolios using the three types of estimator. Clearly, the objective function matters and over the period, the maximum expected utility objective with expected shortfall as risk measure leads to significantly better performance than the standard approach based on the variance as risk measure. As mentioned above, in the sample, the choice of the estimator seems to matter, as the portfolio using the comoments estimated under the three-factor model has a substantially higher performance over the period.

∼ Insert Figure A.2 about here ∼

4. Conclusion

We propose a novel methodology to account for non-normality and the risk of "error maximization" when optimized portfolios take the classical higher order comoment estimates as input parameter. The building block of the proposed methodology is the assumption of a multifactor model generating the asset returns. This assumption allows for parsimoniously estimating the higher order comoments. Parsimony is achieved by striking a balance between generality (multiple factors) and the number of parameters to estimate. It extends the single factor approach in Martellini and Ziemann (2010). This is the key contribution of this article.

Additionally, we conduct an empirical analysis in order to assess the potential gains of including higher order comoments in the asset allocation decision. For this, we apply comoment estimates in an out-of-sample portfolio analysis for a heterogeneous universe of 17 equity, bonds and commodity indices over the period January 2007-July 2013.

Our analysis seems to suggest that that the following three actions may increase the out-of-sample return and reduce the portfolio risk: (i) imposing more structure on the estimates (moving from the sample estimator to the factor model based estimates); (ii) switching from an equal variance contribution constraint to an equal expected shortfall constraint; (iii) switching from a minimum variance objective to a CRRA expected utility objective.

We believe that our findings are promising but require more research in terms of sensitivity to the sample studied, turnover analysis, risk budgets, the treatment of currency effects, the handling of outliers and, importantly, the choice of factors.

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Appendix A. Proofs of decomposition higher order comoments under factor model specification

For notational convenience and without loss of generality, we assume here the returns and factors to be de-meaned such that $r_{it} = b_i' f_t + e_{it}$. Because we assume independence between the factors and residual terms, we have that, for $j \neq k$, and powers p and q:

$$E[(b_j' f_t)^p e_{jt}^q] = E[(b_j' f_t)^p] E[e_{jt}^q].$$

The independence across residual terms implies further that, for $j \neq k$:

$$E[e_{jt}^p e_{kt}^q] = E[e_{jt}^p]E[e_{kt}^q].$$

These independence assumptions allow us to simplify greatly the higher order comoments, as we show next.

We first consider the elements in the coskewness matrix $\Phi = BG(B' \otimes B') + \Omega$. We have that for the skewness of asset i:

$$\begin{aligned}
\Phi_{i,i,i} &= E[(b_i' f_t + e_{it})^3], \\
&= E[(b_i' f_t)^3 + 3(b_i' f_t)^2 e_{it} + 3(b_i' f_t) e_{it}^2 + e_{it}^3], \\
&= b_i' G(b_i \otimes b_i) + E[e_{it}^3].
\end{aligned}$$

Hence the residual element in Ω is $E[e_{it}^3]$. All other elements in Ω are 0, since the coskewness only comes from the common factor exposures:

$$\begin{aligned}
\Phi_{i,i,j} &= E[(b_i' f_t + e_{it})^2 (b_j' f_t + e_{jt})], \\
&= E[(b_i' f_t)^2 (b_j' f_t) + (b_i' f_t)^2 e_{jt} + 2(b_i' f_t) e_{it} (b_j' f_t) + 2(b_i' f_t) e_{it} e_{jt} + e_{it}^2 (b_j' f_t) + e_{it}^2 e_{jt}], \\
&= b_i' G(b_i \otimes b_j),
\end{aligned}$$

for $i \neq j$, and, for $i \neq j$, $j \neq k$, and $i \neq k$:

$$\begin{aligned}
\Phi_{i,j,k} &= E[(b_i' f_t + e_{it})(b_j' f_t + e_{jt})(b_k' f_t + e_{kt})], \\
&= b_i' G(b_j \otimes b_k).
\end{aligned}$$

We now consider the cokurtosis matrix $\Psi = BP(B' \otimes B' \otimes B') + \Upsilon$. For the kurtosis of asset i:

$$\begin{aligned}
\Psi_{i,i,i,i} &= E[(b_i' f_t + e_{it})^4], \\
&= E[(b_i' f_t)^4 + 4(b_i' f_t)^3 e_{it} + 6(b_i' f_t)^2 e_{it}^2 + 4(b_i' f_t) e_{it}^3 + e_{it}^4], \\
&= b_i' P(b_i \otimes b_i \otimes b_i) + 6 b_i' S b_i E[e_{it}^2] + E[e_{it}^4].
\end{aligned}$$

The result in (9) then follows. Then for the cokurtosis element, where $(i = j = k)$ and $l \neq i$:

$$\Psi_{i,i,i,l} = E[(b_i' f_t + e_{it})^3 (b_l' f_t + e_{lt})],$$

$$
\begin{aligned}
&= E[(b_i'f_t)^3 + 3(b_i'f_t)^2 e_{it} + 3(b_i'f_t)^2 e_{it}^2 + e_{it}^3)(b_l'f_t + e_{lt})], \\
&= b_i'P(b_i \otimes b_i \otimes b_j) + 3E[(b_i'f_t)(b_l'f_t)e_{it}^2], \\
&= b_i'P(b_i \otimes b_i \otimes b_j) + 3b_i'Sb_l E[e_{it}^2].
\end{aligned}
$$

This proves the result in (10). We obtain (11) from:

$$
\begin{aligned}
\Psi_{i,i,k,k} &= E[(b_i'f_t + e_{it})^2(b_k'f_t + e_{kt})^2], \\
&= E[((b_i'f_t)^2 + 2(b_i'f_t)^2 e_{it} + e_{it}^2)((b_k'f_t)^2 + 2(b_k'f_t)^2 e_{kt} + e_{kt}^2)], \\
&= b_i'P(b_i \otimes b_k \otimes b_k) + b_i'Sb_i E[e_{kt}^2] + b_k'Sb_k E[e_{it}^2] + E[e_{it}^2]E[e_{kt}^2].
\end{aligned}
$$

Finally, (12) is obtained using:

$$
\begin{aligned}
\Psi_{i,i,k,l} &= E[(b_i'f_t + e_{it})^2(b_k'f_t + e_{kt})(b_l'f_t + e_{lt})], \\
&= E[((b_i'f_t)^2 + 2(b_i'f_t)^2 e_{it} + e_{it}^2)((b_k'f_t) + e_{kt})((b_l'f_t) + e_{lt})], \\
&= b_i'P(b_i \otimes b_k \otimes b_l) + b_k'Sb_l E[e_{it}^2],
\end{aligned}
$$

where $i \neq k$, $k \neq l$, and $i \neq l$. For all other elements that have no common index:

$$
\begin{aligned}
\Psi_{i,j,k,l} &= E[(b_i'f_t + e_{it})(b_j'f_t + e_{jt})(b_k'f_t + e_{kt})(b_l'f_t + e_{lt})], \\
&= b_i'P(b_i \otimes b_k \otimes b_l).
\end{aligned}
$$

Table A.1: Number of elements to estimate under the unrestricted and multifactor model approach.

	Unrestricted	Factor model with K factors		
Σ	$N(N+1)/2$	$N(K+1)+K(K+1)/2$		
Φ	$N(N+1)(N+2)/6$	$N(K+1)+K(K+1)(K+2)/6$		
Ψ	$N(N+1)(N+2)(N+3)/24$	$N(K+2)$ $+K(K+1)/2$ $+K(K+1)(K+2)(K+3)/24$		
Total	$N(N+1)/2$ $+N(N+1)(N+2)/6$ $+N(N+1)(N+2)(N+3)/24$	$N(K+3)$ $+[(1+(K+2)/3$ $+(K+2)(K+3)/12]K(K+1)/2$		
	Total	$K=1$	$K=2$	$K=3$
$N=5$	120	23	37	61
$N=20$	10,605	83	112	151
$N=100$	4,598,025	403	512	631

Table A.2: Monthly returns analysis of single-asset benchmarks over the period January 2007-July 2013 (in EUR).

	Annualized return	Annualized standard deviation	Skewness	Excess kurtosis	95% Historical VaR	Max Drawdown
Equity Europe	0.11%	18.05%	-0.736	1.162	-9.28%	-53.92%
Equity North America	4.35%	16.11%	-0.428	0.173	-7.67%	-46.46%
Equity Emerging Markets	3.00%	20.46%	-0.639	1.275	-10.01%	-55.83%
Equity Pacific	1.07%	15.10%	-0.470	-0.069	-7.51%	-45.99%
Corp Developed HY, EUR	7.34%	14.32%	-1.128	7.808	-6.65%	-36.06%
Corp Developed HY, USD	8.22%	13.18%	1.955	8.827	-2.44%	-26.78%
Corp Developed IG, EUR	5.23%	11.51%	0.458	-0.226	-4.52%	-15.00%
Corp Emerging IG, USD	6.79%	11.32%	0.740	3.178	-3.81%	-21.39%
Sov Developed IG, EUR	5.28%	5.39%	0.103	2.657	-1.96%	-8.49%
Sov Developed IG, JPY	4.95%	16.48%	1.808	7.219	-3.86%	-25.47%
Sov Developed IG, USD	6.24%	13.35%	0.932	1.700	-4.51%	-15.02%
Sov Emerging IG,USD	7.47%	10.22%	0.560	0.803	-3.65%	-12.70%
Precious metals	9.68%	19.75%	0.089	0.209	-8.22%	-33.86%
Energy	-2.72%	28.70%	-1.387	4.194	-15.64%	-69.36%
Industry Metals	-5.48%	24.23%	-0.479	1.353	-12.33%	-62.46%
Agriculture	-0.29%	23.72%	0.274	0.081	-10.44%	-45.44%
Livestock	-8.45%	17.58%	0.133	0.541	-8.65%	-52.83%

Table A.3: Monthly returns analysis of minimum variance/maximum expected utility strategies under equal variance/ES constraint over the period January 2007-July 2013 (in EUR).

Objective/ Risk Contribution	Estimator	Ann. return	Annualized standard deviation	Skewness	Excess kurtosis	95% Historical VaR	Max Drawdown
Min Variance/	Sample	1.02%	10.69%	-1.60	9.02	-5.60%	-25.52%
Variance	Single factor	3.02%	7.54%	0.49	1.67	-2.89%	-8.99%
	Multifactor	3.48%	7.84%	0.71	1.73	-2.83%	-18.56%
Min Variance/	Sample	5.08%	8.46%	0.17	0.90	-3.38%	-16.45%
E.Shortfall	Single factor	5.54%	7.90%	0.40	0.55	-2.96%	-7.86%
	Multifactor	4.70%	8.35%	0.33	0.52	-3.27%	-12.40%
Max E.Utility/	Sample	3.36%	8.32%	0.06	2.98	-3.27%	-9.89%
Variance	Single factor	5.85%	8.57%	0.64	0.70	-3.04%	-10.26%
($\gamma = 5$)	Multifactor	4.23%	8.73%	1.06	3.41	-3.44%	-13.27%
Max E.Utility/	Sample	5.47%	7.07%	0.27	0.77	-2.68%	-9.17%
E.Shortfall	Single factor	4.43%	9.59%	-0.88	6.75	-4.40%	-18.24%
($\gamma = 5$)	Multifactor	5.21%	9.67%	-0.98	6.88	-4.43%	-13.31%
Max E.Utility/	Sample	3.64%	8.03%	0.38	0.29	-3.19%	-11.65%
Variance	Single factor	1.80%	7.97%	0.01	0.57	-3.55%	-11.67%
($\gamma = 10$)	Multifactor	4.53%	8.90%	0.27	-0.18	-3.59%	-11.87%
Max E.Utility/	Sample	4.82%	9.05%	-0.05	0.38	-3.86%	-11.99%
E.Shortfall	Single factor	4.49%	8.25%	0.26	0.91	-3.28%	-10.27%
($\gamma = 10$)	**Multifactor**	**9.47%**	**9.35%**	**1.43**	**3.32**	**-2.25%**	**-8.06%**

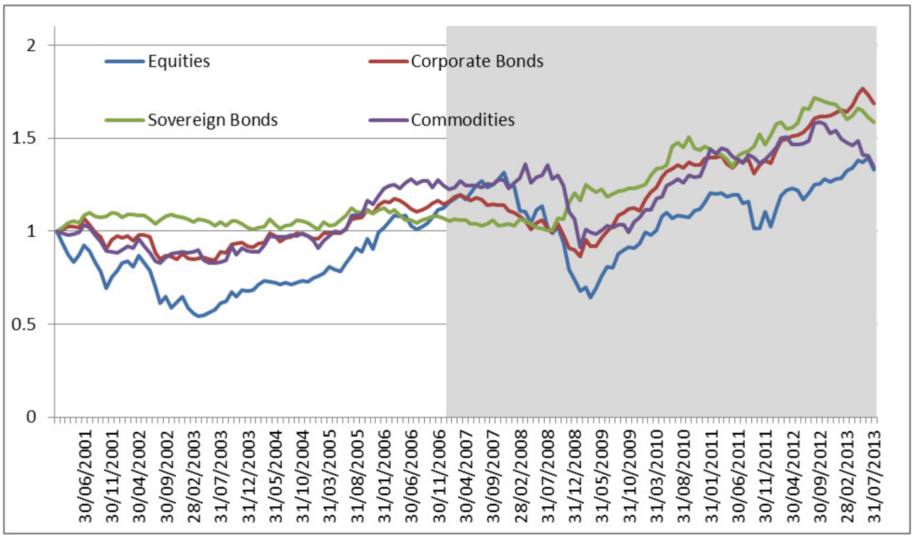

Figure A.1: Monthly cumulative level of equal-weighted equity, equal-weighted sovereign bonds, equal-weighted corporate bonds and equal-weighted commodities over the period February 2001-July 2013 (in EUR). The grey area indicates the out-of-sample evaluation period used.

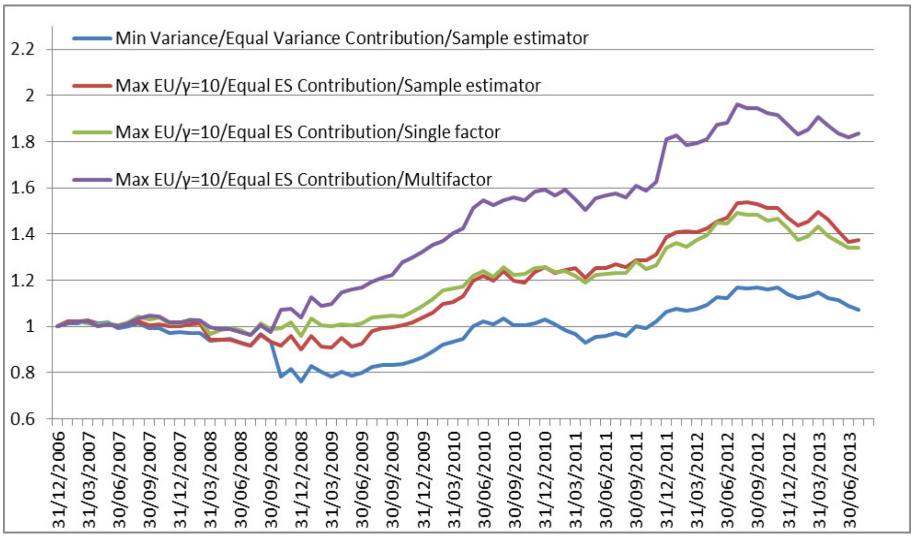

Figure A.2: Monthly cumulative level in EUR of the weekly rebalanced international asset allocation portfolio that either minimized variance or maximizes the expected utility with γ=10, under the diversification constraint of equal variance or equal ES contribution of each asset class. The comoments are estimated using the sample approach, single factor model or three-factor model.