HOW TO USE THE HOLES IN BLACK-SCHOLES

*by Fischer Black, Goldman, Sachs & Co.**

The Black-Scholes formula is still around, even though it depends on at least 10 unrealistic assumptions. Making the assumptions more realistic hasn't produced a formula that works better across a wide range of circumstances.

In special cases, though, we can improve the formula. If you think investors are making an unrealistic assumption like one of those we used in deriving the formula, there is a strategy you may want to follow that focuses on that assumption.

The same unrealistic assumptions that led to the Black-Scholes formula are behind some versions of "portfolio insurance." As people have shifted to more realistic assumptions, they have changed the way they use portfolio insurance. Some people have dropped it entirely, or have switched to the opposite strategy.

People using incorrect assumptions about market conditions may even have caused the rise and sudden fall in stocks during 1987. One theory of the crash relies on incorrect beliefs, held before the crash, about the extent to which investors were using portfolio insurance, and about how changes in stock prices cause changes in expected returns.

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THE FORMULA

The Black-Scholes formula looks like this:

$$w(x,t) = xN(d_1) - ce^{-r(t^* - t)}N(d_2)$$

where

$$d_1 = \frac{\ln(x/c) + (r + 1/2v^2)(t^* - t)}{v\sqrt{t^* - t}}$$

and

$$d_2 = \frac{\ln(x/c) + (r - 1/2v^2)(t^* - t)}{v\sqrt{t^* - t}}$$

In this expression, **w** is the value of a call option or warrant on the stock, **t** is today's date, **x** is the stock price, **c** is the strike price, r is the interest rate, **t*** is the maturity date, **V** is the standard deviation of the stock's return, and **N** is something called the "cumulative normal density function." (You can approximate **N** using a simple algebraic expression.)

The value of the option increases with increases in the stock's price, the interest rate, the time remaining until the option expires, and the stock's volatility. Except for volatility, which can be estimated several ways, we can observe all of the factors the Black-Scholes formula requires for valuing options.

Note that the stock's expected return doesn't appear in the formula. If you are bullish on the stock, you may buy shares or call options, but you won't change your estimate of the option's value. A higher expected return on the stock means a higher expected return on the option, but it doesn't affect the option's value for a given stock price.

This feature of the formula is very general. I don't know of any variation of the formula where the stock's expected return affects the option's value for a given stock price.

HOW TO IMPROVE THE ASSUMPTIONS

In our original derivation of the formula, Myron Scholes and I made the following unrealistic assumptions:

- The stock's volatility is known, and doesn't change over the life of the option.
- The stock price changes smoothly: it never jumps up or down a large amount in a short time.
- The short-term interest rate never changes.
- Anyone can borrow or lend as much as he wants at a single rate.
- An investor who sells the stock or the option short will have the use of all the proceeds of the sale and receive any returns from investing these proceeds.
- There are no trading costs for either the stock or the option.
- An investor's trades do not affect the taxes he pays.
- The stock pays no dividends.
- An investor can exercise the option only at expiration.
- There are no takeovers or other events that can end the option's life early.

Since these assumptions are mostly false, we know the formula must be wrong. But we may not be able to find any other formula that gives better results in a wide range of circumstances. Here we look at each of these 10 assumptions and describe how we might change them to improve the formula. We also look at strategies that make sense if investors continue to make unrealistic assumptions.

Volatility Changes

The volatility of a stock is not constant. Changes in the volatility of a stock may have a major impact on the values of certain options, especially far-out-of-the-money options. For example, if we use a volatility estimate of 0.20 for the annual standard deviation of the stock, and if we take the interest rate to be zero, we get a value of $0.00884 for a six-month call option with a $40 strike price written on a $28 stock. Keeping everything else the same, but doubling the volatility to 0.40, we get a value of $0.465.

For this out-of-the-money option, doubling the volatility estimate multiplies the value by a factor of 53.

Since the volatility can change, we should really include the ways it can change in the formula. The option value will depend on the entire future path that we expect the volatility to take, and on the uncertainty about what the volatility will be at each point in the future. One measure of that uncertainty is the "volatility of the volatility."

A formula that takes account of changes in volatility will include both current and expected future levels of volatility. Though the expected return on the stock will not affect option values, expected changes in volatility will affect them. And the volatility of volatility will affect them too.

Another measure of the uncertainty about the future volatility is the relation between the future stock price and its volatility. A decline in the stock price implies a substantial increase in volatility, while an increase in the stock price implies a substantial decrease in volatility. The effect is so strong that it is even possible that a stock with a price of $20 and a typical daily move

To buy pure volatility, buy both puts and calls in a ratio that gives you no added exposure to the stock; to sell pure volatility, sell both puts and calls in the same ratio.

of $0.50 will start having a typical daily move of only $0.375 if the stock price doubles to $40.

John Cox and Stephen Ross have come up with two formulas that take account of the relation between the future stock price and its volatility.[1] To see the effects of using one of their formulas on the pattern of option values for at-the-money and out-of-the money options, let's look at the values using both Black-Scholes and Cox-Ross formulas for a six-month call option on a $40 stock, taking the interest rate as zero and the volatility as 0.20 per year. For three exercise prices, the value are as follows:

Exercise Price	Black Scholes	Cox-Ross
40.00	2.2600	2.2600
50.00	0.1550	0.0880
57.10	0.0126	0.0020

The Cox-Ross formula implies lower values for out-of-the-money call options than the Black-Scholes formula. But putting in uncertainty about the future volatility will often imply higher values for these same options. We can't tell how the option values will change when we put in both effects.

What should you do if you think a stock's volatility will change in ways that other people do not yet understand? Also suppose that you feel the market values options correctly in all other respects.

You should "buy volatility" if you think volatility will rise, and "sell volatility" if you think it will fall. To buy volatility, buy options; to sell volatility, sell options. Instead of buying stock, you can buy calls or buy stock and sell calls. Or you can take the strongest position on volatility by adding a long or short position in straddles to your existing position. To buy pure volatility, buy both puts and calls in a ratio that gives you no added exposure to the stock; to sell pure volatility, sell both puts and calls in the same ratio.

Jumps

In addition to showing changes in volatility in general and changes in volatility related to changes in stock price, a stock may have jumps. A major news development may cause a sudden large change in the stock price, often accompanied by a temporary suspension of trading in the stock.

When the big news is just as likely to be good as bad, a jump will look a lot like a temporary large increase in volatility. When the big news, if it comes, is sure to be good, or is sure to be bad, the resulting jump is not like a change in volatility. Up jumps and down jumps have different effects on option values than symmetric jumps, where there is an equal chance of an up jump or a down jump.

Robert Merton has a formula that reflects possible symmetric jumps.[2] Compared to the Black-Scholes formula, his formula gives higher values for both in-the-money and out-of-the-money options and lower values for at-the-money options. The differences are especially large for short-term options.

Short-term options also show strikingly different effects for up jumps and down jumps. An increase in the probability of an up jump will cause out-of-the-money calls to go way up in value relative to out-of-the-money puts. An increase in the probability of a down jump will do the reverse. After the crash, people were afraid of another down jump, and out-of-the-money puts were priced very high relative to their Black-Scholes values, while out-of-the-money calls were priced very low.

More than a year after the crash, this fear continues to affect option values.

What should you do if you think jumps are more likely to occur than the market thinks? If you expect a symmetric jump, buy short-term out-of-the-money options. Instead of stock, you can hold call options or more stock plus put options. Or you can sell at-the-money options. Instead of stock, you can hold more stock and sell call options. For a pure play on symmetric jumps, buy out-of-the-money calls and puts, and sell at-the-money calls and puts.

For up jumps, use similar strategies that involve buying short-term out-of-the-money calls, or selling short-term out-of-the-money puts, or both. For down jumps, do the opposite.

Interest Rate Changes

The Black-Scholes formula assumes a constant interest rate, but the yields on bonds with different maturities tell us that the market expects the rate to change. If future changes in the interest rate are known, we can just replace the short-term rate with the yield on a zero-coupon bond that matures when the option expires.

1. See John Cox and Stephen Ross, *Journal of Financial Economics* (January/March 1976).

2. See John Cox, Robert Merton, and Stephen Ross, *Journal of Financial Economics* (January/March 1976).

Suppose you want to short a stock but you face penalties if you sell the stock short directly...You can short it indirectly by holding put options, or by taking a naked short position in call options.

But, of course, future changes in the interest rate are uncertain. When the stock's volatility is known Robert Merton has shown that the zero-coupon bond yield will still work, even when both short-term and long-term interest rates are shifting.[3] At a given point in time, we can find the option value by using the zero-coupon bond yield at that moment for the short-term rate. When both the volatility and the interest rate are shifting, we will need a more complex adjustment.

In general, the effects of interest rate changes on option values do not seem nearly as great as the effects of volatility changes. If you have an opinion on which way interest rates are going, you maybe better off with direct positions in fixed-income securities rather than in options.

But your opinion may affect your decisions to buy or sell options. Higher interest rates mean higher call values and lower put values. If you think interest rates will rise more than the market thinks, you should be more inclined to buy calls, and more inclined to buy more stocks and sell puts, as a substitute for a straight stock position. If you think interest rates will fall more than the market thinks, these preferences should be reversed.

Borrowing Penalties

The rate at which an individual can borrow, even with securities as collateral, is higher than the rate at which he can lend. Sometimes his borrowing rate is substantially higher than his lending rate. Also, margin requirements or restrictions put on by lenders may limit the amount he can borrow.

High rates and limits on borrowing may cause a general increase in call option values, since calls provide leverage that can substitute for borrowing. The interest rates implied by option values may be higher than lending rates. If this happens and you have borrowing limits but no limits on option investments, you may still want to buy calls. But if you can borrow freely at a rate close to the lending rate, you may want to get leverage by borrowing rather than by buying calls.

When implied interest rates are high, conservative investors might buy puts or sell calls to protect a portfolio instead of selling stock. Fixed-income investors might even choose to buy stocks and puts, and sell calls, to create a synthetic fixed-income position with a yield higher than market yields.

Short-Selling Penalties

Short-selling penalties are generally even worse than borrowing penalties. On U.S. exchanges, an investor can sell a stock short only on or after an uptick. He must go to the expense of borrowing stock if he wants to sell it short. Part of his expense involves putting up cash collateral with the person who lends the stock; he generally gets no interest, or interest well below market rates, on this collateral. Also, he may have to put up margin with his broker in cash, and may not receive interest on cash balances with his broker.

For options, the penalties tend to be much less severe. An investor need not borrow an option to sell it short. There is no uptick rule for options. And an investor loses much less interest income in selling an option short than in selling a stock short.

Penalties on short selling that apply to all investors will affect option values. When even professional investors have trouble selling a stock short, we will want to include an element in the option formula to reflect the strength of these penalties. Sometimes we approximate this by assuming an extra dividend yield on the stock, in an amount up to the cost of maintaining a short position as part of a hedge.

Suppose you want to short a stock but you face penalties if you sell the stock short directly. Perhaps you're not even allowed to short the stock directly. You can short it indirectly by holding put options, or by taking a naked short position in call options. (Though most investors who can't short stock directly also can't take naked short positions.)

When you face penalties in selling short, you often face rewards for lending stock to those who want to short it. In this situation, strategies that involve holding the stock and lending it out may dominate other strategies. For example, you might create a position with a limited downside by holding a stock and a put on the stock, and by lending the stock to those who want to short it.

Trading Costs

Trading costs can make it hard for an investor to create an option-like payoff by trading in the underlying stock. They can also make it hard to create a stock-like payoff by trading in the option.

3. Robert Merton, *Bell Journal of Economics and Management Science* (1977).

If you pay taxes on gains and cannot deduct losses, you may want to limit the volatility of your positions and have the freedom to control the timing of gains and losses.

Sometimes they can increase an option's value, and sometimes they can decrease it.

We can't tell how trading costs will affect an option's value, so we can think of them as creating a "band" of possible values. Within this band, it will be impractical for most investors to take advantage of mispricing by selling the option and buying the stock, or by selling the stock and buying the option.

The bigger the stock's trading costs are, the more important it is for you to choose a strategy that creates the payoffs you want with little trading. Trading costs can make options especially useful if you want to shift exposure to the stock after it goes up or down.

If you want to shift your exposure to the market as a whole, rather than to a stock, you will find options even more useful. It is often more costly to trade in a basket of stocks than in a single stock. But you can use index options to reduce your trading in the underlying stocks or futures.

Taxes

Some investors pay no taxes; some are taxed as individuals, paying taxes on dividends, interest, and capital gains; and some are taxed as corporations, also paying taxes on dividends, interest, and capital gains, but at different rates.

The very existence of taxes will affect option values. A hedged position that should give the same return as lending may have a tax that differs from the tax on interest. So if all investors faced the same tax rate, we would use a modified interest rate in the option formula.

The fact that investor tax rates differ will affect values too. Without rules to restrict tax arbitrage, investors could use large hedged positions involving options to cut their taxes sharply or to alter them indefinitely. Thus tax authorities adopt a variety of rules to restrict tax arbitrage. There maybe rules to limit interest deductions or capital loses deductions, or rules to tax gains and losses before a position is closed out. For example, most U.S. index option positions are now taxed each year—partly as short-term capital gains and partly as long-term capital gains—whether or not the taxpayer has closed out his positions.

If you can use capital losses to offset gains, you may act roughly the same way whether your tax rate is high or low. If your tax rate stays the same from year to year, you may act about the same whether you are forced to realize gains and losses or are able to choose the year you realize them.

But if you pay taxes on gains and cannot deduct losses, you may want to limit the volatility of your positions and have the freedom to control the timing of gains and losses. This will affect how you use options, and may affect option values as well. I find it hard to predict, though, whether it will increase or decrease option values.

Investors who buy a put option will have a capital gain or loses at the end of the year, or when the option expires. Investors who simulate the put option by trading in the underlying stock will sell after a decline, and buy after a rise. By choosing which lots of stock to buy and which lots to sell, they will be able to generate a series of realized capital losses and unrealized gains. The tax advantages of this strategy may reduce put values for many taxable investors. By a similar argument, the tax advantages of a simulated call option may reduce call values for most taxable investors.

Dividends and Early Exercise

The original Black-Scholes formula does not take account of dividends. But dividends reduce call option values and increase put option values, at least when there are no offsetting adjustments in the terms of the options. Dividends make early exercise of a call option more likely, and early exercise of a put option less likely.

We now have several ways to change the formula to account for dividends. One way assumes that the dividend yield is constant for all possible stock price levels and at all future times. Another assumes that the issuer has money set aside to pay the dollar dividends due before the option expires. Yet another assumes that the dividend depends in a known way on the stock price at each ex-dividend date.

John Cox, Stephen Ross, and Mark Rubinstein have shown how to figure option values using a "tree" of possible future stock prices.[4] The tree gives the same values as the formula when we use the same assumptions. But the tree is more flexible, and lets us relax some of the assumptions. For example, we can put on the tree the dividend that the firm will pay for each possible future stock price at each future time. We can also test, at each node of the tree,

4. John Cox, Mark Rubinstein, and Stephen Ross, "Option Pricing: A Simplified Approach," *Journal of Financial Economics* Vol.7 (1979), 229-263.

The true cost of portfolio insurance, in my view, is a factor that doesn't even affect option values. It is the mean reversion in the market: the rate at which the expected return on the market falls as the market rises.

whether an investor will exercise the option early for that stock price at that time.

Option values reflect the market's belief about the stock's future dividends and the likelihood of early exercise. When you think that dividends will be higher than the market thinks, you will want to buy puts or sell calls, other things equal. When you think that option holders will exercise too early or too late, you will want to sell options to take advantage of the opportunities the holders create.

Takeovers

The original formula assumes the underlying stock will continue trading for the life of the option. Takeovers can make this assumption false.

If firm A takes over firm B through an exchange of stock, options on firm B's stock will normally become options on firm A's stock. We will use A's volatility rather than B's in valuing the option.

If firm A takes over firm B through a cash tender offer, there are two effects. First, outstanding options on B will expire early. This will tend to reduce values for both puts and calls. Second, B's stock price will rise through the tender offer premium. This will increase call values and decrease put values.

But when the market knows of a possible tender offer from firm A, B's stock price will be higher than it might otherwise be. It will be between its normal level and its normal level increased by the tender offer. Then if A fails to make an offer, the price will fall, or will show a smaller-than-normal rise.

All these factors work together to influence option values. The chance of a takeover will make an option's value sometimes higher and sometimes lower. For a short-term out-of-the-money call option, the chance of a takeover will generally increase the option value. For a short-term out-of-the-money put option, the chance of a takeover will generally reduce the option value.

The effects of takeover probability on values can be dramatic for these short-term out-of-the-money options. If you think your opinion of the chance of a takeover is more accurate than the market's, you can express your views clearly with options like these.

The October 19 crash is the opposite of a takeover as far as option values go. Option values then, and since then, have reflected the fear of another crash. Out-of-the-money puts have been selling for high values, and out-of-the-money calls have been selling for low values. If you think another crash is unlikely, you may want to buy out-of-the-money calls, or sell out-of-the-money puts, or do both.

Now that we've looked at the 10 assumptions in the Black-Scholes formula, let's see what role, if any, they play in portfolio insurance strategies.

PORTFOLIO INSURANCE

In the months before the crash, people in the U.S. and elsewhere became more and more interested in portfolio insurance. As I define it, portfolio insurance is any strategy where you reduce your stock positions when prices fall, and increase them when prices rise.

Some investors use option formulas to figure how much to increase or reduce their positions as prices change. They trade in stocks or futures or short-term options to create the effect of having a long-term put against stock, or a long-term call plus T-bills.

You don't need synthetic options or option formulas for portfolio insurance. You can do the same thing with a variety of systems for changing your positions as prices change. However, the assumptions behind the Black-Scholes formula also affect portfolio insurance strategies that don't use the formula.

The higher your trading costs, the less likely you are to create synthetic options or any other adjustment strategy that involves a lot of trading. On October 19, the costs of trading in futures and stocks became much higher than they had been earlier, partly because the futures were priced against the portfolio insurers. The futures were at a discount when portfolio insurers wanted to sell. This made all portfolio insurance strategies less attractive.

Portfolio insurance using synthetic strategies wins when the market makes big jumps, but without much volatility. It loses when market volatility is high, because an investor will sell after a fall, and buy after a rise. He loses money on each cycle.

But the true cost of portfolio insurance, in my view, is a factor that doesn't even affect option values. It is the mean reversion in the market: the rate at which the expected return on the market falls as the market rises.[5]

5. For evidence of mean reversion, see Eugene Fama and Kenneth French, "Permanent and Temporary Components of Stock Prices," *Journal of Political Economy* Vol. 96 No. 2 (April 1988), 246-273; and James Poterba and Lawrence Summers, "Mean Reversion in Stock Prices: Evidence and Implications," *Journal of Financial Economics* Vol. 22 No.1 (October 1988), 27-60.

If your view of mean reversion is higher than the market's, you can buy short-term options and sell long-term options. If you think mean reversion is lower, you can do the reverse.

Mean reversion is what balances supply and demand for portfolio insurance. High mean reversion will discourage portfolio insurers because it will mean they are selling when expected return is higher and buying when expected return is lower. For the same reason, high mean reversion will attract "value investors" or "tactical asset allocators," who buy after a decline and sell after a rise. Value investors use indicators like price-earnings ratios and dividend yields to decide when to buy and sell. They act as sellers of portfolio insurance.

If mean reversion were zero, I think that more investors would want to buy portfolio insurance than to sell it. People have a natural desire to try to limit their losses. But, on balance, there must be as many sellers as buyers of insurance. What makes this happen is a positive normal level of mean reversion.

THE CRASH

During 1987, investors shifted toward wanting more portfolio insurance. This increased the market's mean reversion. But mean reversion is hard to see; it takes years to detect a change in it. So investors did not understand that mean reversion was rising. Since rising mean reversion should restrain an increase in portfolio insurance demand, this misunderstanding caused a further increase in demand.

Because of mean reversion, the market rise during 1987 caused a sharper-than-usual fall in expected return. But investors didn't see this at first. They continued to buy, as their portfolio insurance strategies suggested. Eventually, though, they came to understand the effects of portfolio insurance on mean reversion, partly by observing the large orders that price changes brought into the market.

Around October 19, the full truth of what was happening hit investors. They saw that at existing levels of the market, the expected return *was* much lower than they had assumed. They sold at those levels. The market fell, and expected return rose, until equilibrium was restored.

MEAN REVERSION AND STOCK VOLATILITY

Now that we've explained mean reversion, how can you use your view of it in your investments?

If you have a good estimate of a stock's volatility, the stock's expected return won't affect option values. Since the expected return won't affect values, neither will mean reversion.

But mean reversion may influence your estimate of the stock's volatility. With mean reversion day-to-day volatility will be higher than month-to-month volatility, which will be higher than year-to-year volatility. Your volatility estimates for options with several years of life should be generally lower than your volatility estimates for options with several days or several months of life.

If your view of mean reversion is higher than the market's, you can buy short-term options and sell long-term options. If you think mean reversion is lower, you can do the reverse. If you are a buyer of options, you will favor short-term options when you think mean reversion is high, and long-term options when you think it is low. If you are a seller of options, you will favor long-term options when you think mean reversion is high, and short-term options when you think it's low.

These effects will be most striking in stock index options. But they will also show up in individual stock options, through the effects of market moves on individual stocks and through the influence of "trend followers." Trend followers act like portfolio insurers, but they trade individual stocks rather than portfolios. When the stock rises, they buy; and when it falls, they sell. They act as if the past trend in a stock's price is likely to continue.

In individual stocks, as in portfolios, mean reversion should normally make implied volatilities higher for short-term options than for long-term options. (An option's implied volatility is the volatility that makes its Black-Scholes value equal to its price.) If your views differ from the market's, you may have a chance for a profitable trade.