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Dimension Reduction in Discrete Time Portfolio Optimization with Partial Information*

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Abstract. This paper considers the problem of portfolio optimization in a market with partial information and discretely observed price processes. Partial information refers to the setting where assets have unobserved factors in the rate of return and the level of volatility. Standard filtering techniques are used to compute the posterior distribution of the hidden variables, but there is difficulty in finding the optimal portfolio because the dynamic programming problem is non-Markovian. However, fast time scale asymptotics can be exploited to obtain an approximate dynamic program (ADP) that *is* Markovian and is therefore much easier to compute. Of consideration is a model where the latent variables (also referred to as hidden states) have fast mean reversion to an invariant distribution that is parameterized by a Markov chain θ_t, where θ_t represents the regime-state of the market and reverts to its own invariant distribution over a much longer time scale. Data and numerical examples are also presented, and there appears to be evidence that unobserved drift results in an information premium.

Key words. filtering, fast mean reversion, partial information, portfolio optimization, approximate dynamic programming, dimension reduction

AMS subject classifications. 93E11, 93E20, 93C70

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1. Introduction.

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $t \geq 0$ denote time. Consider a market with a single risky asset whose price at time t is denoted by S_t, and a risk-free bank account with rate $r \geq 0$. Define another process θ_t to be a Markov chain, and let it model the *regime-state of the market*. In this paper, the market is said to have *undergone a change in regime* if θ_t changes state. In particular, θ_t is taken to be positive recurrent taking values in the set $\{c_1, c_2, \dots, c_m\}$, where each element c_i represents a different regime-state. The evolution of θ_t is determined by an intensity matrix $Q = (q_{ij}) \in \mathbb{R}^{m\times m}$ so that transitions occur at random times $\tau_0 = 0 < \tau_1 < \tau_2 < \dots$ where $\tau_{\ell+1} - \tau_\ell$, $\ell = 0, 1, 2, \dots$, are exponential random variables with parameter $-q_{ii} = \sum_{j\neq i} q_{ij} > 0$ conditional on $\theta_{\tau_\ell} = c_i$, and at time $\tau_{\ell+1}$ the change in state from c_i to c_j occurs with probability $-q_{ij}/q_{ii}$ given $\{\theta_{\tau_\ell} = c_i\}$ and for $i \neq j$. The evolution of θ's distribution is given by a forward Kolmogorov equation,

$$\frac{d}{dt}\mathbb{P}(\theta_t = c_i) = \sum_j q_{ji}\mathbb{P}(\theta_t = c_j). \tag{1.1}$$

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Based on the regime, the log-returns on the risky asset are given by the following stochastic model:

$$\begin{aligned} d\log(S_t) &= \left(\alpha(X_t) - \frac{1}{2}\beta^2(X_t)\right) dt + \beta(X_t)(\rho dB_t + \sqrt{1-\rho^2}dW_t), \\ dX_t &= \frac{1}{\varepsilon}(\theta_t - X_t)dt + \frac{1}{\sqrt{\varepsilon}}dB_t, \end{aligned} \tag{1.2}$$

where α and β are known bounded functions (with β bounded away from zero) and W_t and B_t are Wiener processes independent of each other and independent of θ_t. The parameter $\rho \in (-1, 1)$ is a correlation coefficient, which describes the volatility leverage effect. For small $\varepsilon > 0$ in (1.2), the process X_t will cause rapid movements in volatility and drift, while θ_t represents long-term states in the market and will remain unaffected by the parameter ε. The reciprocal of ε is the rate at which X reverts to a local invariant distribution that is centered around θ_t, and so for small ε the distribution of X_t is approximately Gaussian with mean θ_t and variance 1/2. Throughout this paper, a family of measures $(\mathbb{P}^\varepsilon)_{\varepsilon>0}$ will be used, with each $\mathbb{P}^\varepsilon$ being the probability measure induced by (1.2) for a given $\varepsilon > 0$. Also, throughout the paper, the expectation operator associated with $\mathbb{P}^\varepsilon$ will be denoted with $\mathbb{E}^\varepsilon$.

This paper considers the setting where (X, θ) is latent and observable only through S, and with S observed only in discrete time. This is a realistic setting to consider for financial applications, because prices are quoted only when trades occur. Let n be an index for discrete time with a time-step $\Delta t > 0$ so that the system is observed at times

$$t_n \doteq n\Delta t \qquad \text{for } n = 0, 1, 2, \ldots,$$

and let

$$\begin{aligned} S_n &\doteq S_{t_n}, \\ X_n &\doteq X_{t_n}, \\ \theta_n &\doteq \theta_{t_n}. \end{aligned}$$

At time n, all the information available to the observer is contained in the filtration $\mathcal{F}_n$, which is the σ-algebra generated by the history $S_{0:n} \doteq \{S_\ell : \ell \le n\}$,

$$\mathcal{F}_n \doteq \sigma\{S_{0:n}\}.$$

In other words, X_n and θ_n are not $\mathcal{F}_n$-measurable and must be filtered from the observations $S_{0:n}$. The problem's description so far is one of filtering and parameter estimation: a diffusion model is observed, and filtering methods are to be employed for inference on the hidden states and for estimation of the parameters.

Finance is introduced to the problem by considering an investor who must decide how much or how little of his/her wealth to allocate in the risky asset S_n. The investor's wealth is the value V_n of a portfolio with a self-financing evolution,

$$\begin{aligned} V_{n+1} &= V_n + r\Delta t\,(V_n - a_n) + a_n\frac{\Delta S_{n+1}}{S_n} \\ &= V_n(1 + r\Delta t) + a_n\left(\frac{\Delta S_{n+1}}{S_n} - r\Delta t\right), \end{aligned} \tag{1.3}$$

where $\Delta S_{n+1} = S_{n+1} - S_n$ and a_n is an $\mathcal{F}_n$-adapted process that the investor chooses to be their allocation in S_n. For a time horizon N, the problem is to find an optimal sequence $a \doteq (a_n)_{n\leq N}$ that maximizes expected terminal utility, for which the value function is given by

$$J_n^\varepsilon(v, S_{0:n}) = \operatorname*{ess\,sup}_{a\in\mathcal{A}_n} \mathbb{E}^\varepsilon\{U(V_N)|\mathcal{F}_n \vee \{V_n = v\}\} \qquad \text{for } 0 \leq n \leq N, \tag{1.4}$$

for a given $\varepsilon > 0$. In (1.4), U is a utility function assumed to be increasing and strictly concave (i.e., $U''(v) < 0$ for all v in its domain), and $\mathcal{A}_n$ is the set of admissible strategies adapted to $\mathcal{F}_n$ and is assumed to *not* depend on ε.

The market has only *partial information* because the investor must make trades and choose allocations without observing (X_n, θ_n). In contrast, *full information* is the case when (X_n, θ_n) is an $\mathcal{F}_n$-measurable process, which simplifies things because fully observing (S_n, X_n, θ_n) means the problem is a Markov decision process that can be solved with a straightforward Bellman equation. Partial information makes things harder because the problem is non-Markovian; that is, the optimal allocation depends on the entire history $S_{0:n}$. General non-Markovian dynamic programming is considerably more difficult than the subclass of Markov decision processes for which there are usually a system of Bellman equations. The difficulty lies in the fact that non-Markov programs suffer from a *curse of dimensionality.* Nonetheless, existence of solutions to the problem in (1.4) is shown in [RS05], and given existence, it then becomes appropriate to explore approximate dynamic programming (ADP) algorithms that are suitable to the specific details of the problem. In this paper, an ADP is proposed that exploits small-ε asymptotics and leads to a dimension reduction from the high-dimensionality of the non-Markovian problem.

This paper's proposed ADP is based on the *limiting* or *unperturbed* version of the problem posed in (1.4), namely,

$$\bar{J}_n(v, S_{0:n}) = \operatorname*{ess\,sup}_{a\in\mathcal{A}_n} \bar{\mathbb{E}}\{U(V_N)|\mathcal{F}_n \vee \{V_n = v\}\} \qquad \text{for } 0 \leq n \leq N, \tag{1.5}$$

where $\bar{\mathbb{E}}$ is the expectation associated with the limiting probability measure of $\mathbb{P}^\varepsilon$ as $\varepsilon \to 0$ (this limiting distribution is described in section 2.1). The idea is to approximate the optimal strategy from (1.4) with the optimal from (1.5). More specifically, letting a_n^ε denote the optimal time-n allocation for (1.4) and letting $\bar{a}_n$ be the optimal for (1.5), it will be shown in what follows that

$$a_n^\varepsilon \to \bar{a}_n \qquad \text{as } \varepsilon \to 0, \tag{1.6}$$

pointwise for a given observation vector $s_{0:n}$ in $(\mathbb{R}^+)^{n+1}$ and a given amount of wealth $v \in \mathbb{R}$. In fact, it will be shown that there is an order-$\sqrt{\varepsilon}$ correction (call it a_n') such that

$$a_n^\varepsilon = \bar{a}_n + \sqrt{\varepsilon}a_n' + o(\sqrt{\varepsilon}), \qquad \text{pointwise for any } s_{0:n} \in (\mathbb{R}^+)^{n+1} \text{ and } v \in \mathbb{R}, \tag{1.7}$$

with the term a_n' being derived solely from the objective function of (1.5) and not from the objective function of (1.4). Theorem 4.1 uses the implicit function theorem to prove (1.6) and

(1.7) in the case with no constraints on a_n and the constant absolute relative risk aversion (CARA) utility function; this theorem clearly justifies the approximation

$$a_n^\varepsilon \approx \bar{a}_n + \sqrt{\varepsilon} a_n' \qquad \text{for small } \varepsilon,$$

and hence, an ADP has been proposed that is accurate and has a slight correction for small-ε behavior. It will also be made clear that this is a dimension reduction because the optimization of (1.5) can be solved with a straightforward dynamic program because the unperturbed problem of (1.5) can be reduced to a Markov decision process. Finally, it should be pointed out that the key to success in using this methodology is for the observations on S to have a sampling rate Δt that allows for sufficient *coarse-graining*: for an appropriately chosen $\Delta t > 0$, the time between observations will allow for relaxation of the system's dependence on X yet is frequent enough that filtering can effectively track the hidden variables. If trading in continuous time is a priority, then the method in this paper is not applicable; the ADP proposed here is useful for markets where buy-and-hold strategies can be made more effective with better information.

1.1. Partial information in finance. Commodities markets are a possible application of the results in this paper, because futures contracts in some deliverables have a convenience yield that is not directly observed in the market. Markets with significant convenience yield include oil and gas, where there may be an advantage in holding the physical good until a later date when prices will be higher (e.g., seasonal demand for heating oil as well as weather insurance contracts are mentioned in [Car09a]). Traders of commodities have some sense of the convenience yield, but they cannot directly observe it (see section 5). Another application is stochastic volatility, where exact parameter values are not known and there might also be a latent regime-state that modulates the volatility processes (see [ASK07, AS02]). There is also potential interest in high frequency trading, where multiscale models and latent states are used to quantify the gain/loss in expected utility that might occur when comparing noise traders to informed traders (see [BFL08]).

1.2. Results from the literature. In continuous time with full information, there are well-known results such as [Bjö00, Hen02, MZ10, SZ04], where solutions to a Hamilton–Jacobi–Bellman equation lead to explicit expressions for the optimal allocation with no constraints and for the CARA, power, and logarithmic utility functions. For models with small ε, asymptotic analysis for fast mean-reverting stochastic volatility was explored in [JS02]. For models with hidden Markov chains taking only finitely many values, the continuous time case with partial information is analyzed in [SH04] and also in [CELS07, ES09], and partial information for the linear Gaussian filter is considered in [Bre06, Car09b], where the application is to commodities markets and the loss in utility due to partial information is quantified. Parameter estimation for discretely observed SDEs is addressed in [AS02].

In discrete time, existence of solutions to (1.4) is shown in [RS05] for the case where the set of strategies is unconstrained (i.e., Lagrange multipliers are not used). The case of discrete time with partial information is addressed in [BUV12, BR11, TZ07], wherein it is shown how to reduce (1.5) to a Markov decision process and also that there is a system of Bellman equations. Portfolio optimization with filtering for stochastic volatility is addressed in [BMV06, DLV03] and with transaction costs in [KG12].

More general results on filtering and control in finance are presented in [Ell92], consistency of the particle filter for continuous time models with discrete time observations is shown in [MJP01], and fast mean-reverting asymptotics of the discrete time filter are addressed in [Pap12]. Singular perturbation models for finance are covered in [FPSS11] along with the expansion techniques for solutions to backward equations (these expansion are widely used in this paper). In the optimization literature, stability of perturbed optimization problems was analyzed in [BS96, Rob80] using the implicit function theorem, and ADPs for dealing with the curse of dimensionality are covered in [Pow11, Ber12].

1.3. Results in this paper. This paper's main result is Theorem 4.1, which applies to the unconstrained portfolio problem with the CARA utility function, but the constrained case with hyperbolic absolute risk aversion (HARA) and logarithmic utility are discussed in Appendix D. The proof is technical because application of the implicit function theorem of [BS96, Rob80] in the stochastic setting of the optimal portfolio problem uses conditions such as strict concavity and $\sqrt{\varepsilon}$-regularity of the objective function. One of the main tools in verifying these conditions is the $\sqrt{\varepsilon}$-expansions of solutions to backward equations associated with (1.2). Furthermore, in the process of showing $\sqrt{\varepsilon}$-regularity, an order-$\sqrt{\varepsilon}$ expansion of the nonlinear filter is computed (see section 3.2), which is a new result and an extension of the result in [Pap12] (but for a simplified model, as dependence of Q on X_t is not included). In the sections with numerical experiments, a simple example is used to show how an information premium can occur due to an unobservable drift S_n.

The rest of the paper is organized as follows: section 2 presents the model, describes its limiting behavior, and computes the zero-order and order-$\sqrt{\varepsilon}$ terms in the expansion of (S_n,θ_n)'s marginal transition density conditioned on $(S_{n-1},X_{n-1},\theta_{n-1})$; section 3 presents filtering and some asymptotic results, including an order-$\sqrt{\varepsilon}$ correction to the marginal filter. Section 4 presents the portfolio problem, describes the limiting model and its Markovian reformulation, and presents the main result of Theorem 4.1 that was mentioned above; section 5 gives some examples of data in commodities markets; section 6 presents a numerical example to illustrate how the optimal controls can be computed and also shows that there is an information premium; Appendices A, B, and C contain various results leading up to Theorem 4.1; Appendix D contains a discussion of the implicit function theorem for cases with constraints and power utility; and Appendix E shows pseudocode for computing solutions to the limiting problem.

2. Discrete time model with Markov chain regime change.

The market is taken to be such that prices are observable only at the discrete times $\{t_n\}_{n=0,1,\dots}$ and that $\{S_n\}_{n=0,1,2,\dots}$ are given by a discrete time version of (1.2),

$$\begin{aligned}
&\log\left(S_n/S_{n-1}\right) \overset{d}{=} I_n(X,\theta)+\mathcal{Z}_n(X,\theta)+\varsigma_n(X,\theta)\mathcal{W}_n && \text{(observed)},\\
&I_n(X,\theta)=\int_{t_{n-1}}^{t_n}\left(\alpha(X_u)-\frac{1}{2}\beta^2(X_u)\right)du && \text{(unobserved)},\\
&\mathcal{Z}_n(X,\theta)=\rho\sqrt{\varepsilon}\int_{t_{n-1}}^{t_n}\beta(X_u)\left(dX_u-\frac{1}{\varepsilon}(\theta_u-X_u)du\right) && \text{(unobserved)},\\
&\varsigma_n(X,\theta)=\sqrt{(1-\rho^2)\int_{t_{n-1}}^{t_n}\beta^2(X_u)du} && \text{(unobserved)},
\end{aligned} \tag{2.1}$$

where α and β are known bounded functions (with β bounded away from zero). Here $\stackrel{d}{=}$ stands for equality in distribution, and $\{\mathcal{W}_n\}$ is a sequence of standard normal independent random variables. The discrete time process S_n defined by (2.1) is equivalent in distribution to S_{t_n} defined by (1.2), and from this point forward it will be assumed that S_n is defined by (2.1) as the true model. It is acceptable to work with a model that is only weakly equivalent because all results in what follows will be based on expansions of the probability law. The process S_n is not Markovian by itself, but the triple (S_n, X_n, θ_n) *is* Markov. Note also that θ_n is by itself Markovian, as is the pair (X_n, θ_n).

The transition density function of (S_n, X_n, θ_n) is expressed through a parameterized family of linear operators Γ^ε, each of which is defined such that the action of Γ^ε on any probability measure $\mu : \mathbb{R} \times \Theta \to \mathbb{R}^+$ is the following:

(2.2)
$$\begin{aligned}
&\Gamma^\varepsilon(s_n, s_{n-1})\mu(x', c_j)\\
&\doteq \sum_j \int \frac{\partial^2}{\partial x' \partial s_n} \mathbb{P}^\varepsilon \cdot (S_n \le s_n, X_n \le x', \theta_n = c_j | S_{n-1} = s_{n-1}, X_{n-1} = x, \theta_{n-1} = c_i)\mu(dx, c_i).
\end{aligned}$$

In particular, when $(\Gamma^\varepsilon)_{\varepsilon>0}$ operates on the point-mass $\delta_x \mathbb{1}_{[c_i]}$, the output $\Gamma^\varepsilon \delta_x \mathbb{1}_{[c_i]}$ is the operator's kernel, or, equivalently, it is the transition density of the continuous time process (S_t, X_t, θ_t) from time t_{n-1} to t_n:

$$\begin{aligned}
\text{(2.3)} \qquad &\Gamma^\varepsilon(y', y)\delta_x \mathbb{1}_{[c_i]}(x', c_j)\\
&= \frac{\partial^2}{\partial x' \partial y'} \mathbb{P}^\varepsilon (S_n \le y', X_n \le x', \theta_n = c_j | S_{n-1} = y, X_{n-1} = x, \theta_{n-1} = c_i).
\end{aligned}$$

From (2.1) and (2.3), the kernel can be written more explicitly as

(2.4)
$$\begin{aligned}
&\Gamma^\varepsilon(s_n, s_{n-1})\mu(x', c_j)\\
&= \sum_i \int \frac{\partial^2}{\partial x' \partial s_n} \mathbb{P}^\varepsilon (S_n \le s_n, X_n \le x', \theta_n = c_j | S_{n-1} = s_{n-1}, X_{n-1} = x, \theta_{n-1} = c_i)\mu(dx, c_i)\\
&= \sum_i \frac{1}{s_n\sqrt{2\pi}} \int \mathbb{E}^\varepsilon \left\{ \frac{\delta_{x'}(X_n)\mathbb{1}_{[\theta_n = c_j]}}{\varsigma_n(X, \theta)} \exp\left\{ -\frac{1}{2}\left(\frac{\log(s_n/s_{n-1}) - I_n(X, \theta) - \mathcal{Z}_n(X, \theta)}{\varsigma_n(X, \theta)}\right)^2 \right\} \middle| \theta_{n-1} \right.\\
&\qquad\qquad \left. = c_i, X_{n-1} = x \right\} \mu(dx, c_i),
\end{aligned}$$

which will be useful in section 2.1.

Finally, the marginal transition kernel can also be obtained by solving a partial differential equation (PDE). The transition kernel is a function

$$\begin{aligned}
\text{(2.5)} \qquad &u_n^\varepsilon(t, y', c_j | y, x, c_i)\\
&\doteq \frac{\partial}{\partial y'} \mathbb{P}^\varepsilon (S_t \le y', \theta_t = c_j | S_{n-1} = y, X_{n-1} = x, \theta_{n-1} = c_i) \qquad \forall t \in [t_{n-1}, t_n],
\end{aligned}$$

which satisfies the following backward Kolmogorov equation:

$$\left(\frac{1}{\varepsilon}L_0^i + \frac{1}{\sqrt{\varepsilon}}L_1 + L_2\right) u_n^\varepsilon(t, y', c_j | y, x, c_i) = 0 \qquad \text{for } t \in [t_{n-1}, t_n], \tag{2.6}$$

$$u_n^\varepsilon(t, y', c_j | y, x, c_i)\Big|_{t=t_n} = \delta_{y=y'} \mathbb{1}_{[i=j]}, \tag{2.7}$$

where the operators are defined as

$$\begin{aligned} L_0^i &\doteq \frac{1}{2}\frac{\partial^2}{\partial x^2} + (c_i - x)\frac{\partial}{\partial x}, \\ L_1 &\doteq \rho\beta(x)y\frac{\partial^2}{\partial y \partial x}, \\ L_2 &\doteq \frac{\partial}{\partial t} + \frac{\beta^2(x)y^2}{2}\frac{\partial^2}{\partial y^2} + \alpha(x)y\frac{\partial}{\partial y} + Q, \end{aligned}$$

with $Q = (q_{ij}) \in \mathbb{R}^{m \times m}$ being the matrix of θ_n's transition intensities so that

$$Q\Gamma^\varepsilon(y', y)\delta_x \mathbb{1}_{[c_i]}(x', c_j) = \sum_\ell q_{i\ell}\Gamma^\varepsilon(y', y)\delta_x \mathbb{1}_{[c_\ell]}(x', c_j) \qquad \forall i, j \le m.$$

2.1. The limiting model. Define the Gaussian density functions

$$\upsilon(x, c_i) \doteq \frac{1}{\sqrt{\pi}} e^{-(x-c_i)^2} \tag{2.8}$$

for $x \in \mathbb{R}$ and each $c_i \in \Theta$, $i = 1, \ldots, m$. Expression (2.8) is the invariant density of an Ornstein–Uhlenbeck process with a long-term mean of c_i, which is precisely the same type of process as X_t in the case where θ_t is constant and equal to c_i. In general, when θ_t is variable, there is an ergodic theorem as ε approaches zero:

$$\int_0^t f(X_u)du \to \int_0^t \langle f \rangle_{\theta_u} du \qquad \text{in probability as } \varepsilon \to 0$$

for any $t \in (0, \infty)$ and for any integrable function $f : \mathbb{R} \to \mathbb{R}$ with $\langle f \rangle_{c_i}$ defined as

$$\langle f \rangle_{c_i} \doteq \int f(x)\upsilon(x, c_i)dx. \tag{2.9}$$

As $\varepsilon \to 0$, there is a limiting probability measure $\bar{\mathbb{P}}$ with expectation $\bar{\mathbb{E}}$ that is associated with the weak limit of the paths of the discrete time process $(S_n, X_n, \theta_n)_{0 \le n \le N}$ defined in (2.1). Tightness and weak convergence of the family $(\mathbb{P}^\varepsilon)_{\varepsilon > 0}$ were shown in [Pap12] and apply here as well (see also the boundedness/compactness result in Appendix C.1). In particular, there is weak convergence of the measures $(\mathbb{P}^\varepsilon)_{\varepsilon > 0}$ on marginal events of $(S_t, \theta_t)_{t \le T < \infty}$ as $\varepsilon \to 0$. Equivalently, the measures converge to $\bar{\mathbb{P}}$ so that

$$\mathbb{E}^\varepsilon g(S_\cdot, \theta_\cdot) \to \bar{\mathbb{E}} g(S_\cdot, \theta_\cdot)$$

as $\varepsilon \to 0$ for all bounded continuous functions g.

Based on this ergodic theory and convergence of the family $(\mathbb{P}^\varepsilon)_{\varepsilon>0}$, the discretely observed model in (2.1) converges to a limiting model.

Proposition 2.1. *The auxiliary fields in* (2.1) *converge as ε goes to zero,*

$$I_n(X,\theta) \to \bar{I}_n(\theta) = \int_{t_{n-1}}^{t_n} \left(\langle\alpha\rangle_{\theta_u} - \frac{1}{2}\left\langle\beta^2\right\rangle_{\theta_u} \right) du \qquad \textit{in probability,} \tag{2.10}$$

$$\mathcal{Z}_n(X,\theta) \Rightarrow \overline{\mathcal{Z}}_n(\theta) = \mathcal{N}\left(0\ ,\ \rho^2 \int_{t_{n-1}}^{t_n} \left\langle\beta^2\right\rangle_{\theta_u} du\right), \tag{2.11}$$

$$\varsigma_n(X,\theta) \to \bar{\varsigma}_n(\theta) = \sqrt{(1-\rho^2)\int_{t_{n-1}}^{t_n} \left\langle\beta^2\right\rangle_{\theta_u} du} \qquad \textit{in probability,} \tag{2.12}$$

where $\Rightarrow$ denotes weak convergence, $\left\langle\beta^2\right\rangle$ and $\langle\alpha\rangle$ are averages as defined in (2.9)*, and $\mathcal{N}$ denotes the distribution of an independent Gaussian random variable.*

Proof. See Appendix A.1. ■

The limits in (2.10), (2.11), and (2.12) indicate that X does not play a role in the limiting behavior of the discretely observed model in (2.1). This being the case, there should be a limiting transition kernel for (S_n, θ_n) that is constant in X_n. Furthermore, such a kernel should be the pointwise limit of the kernel in (2.4). Indeed, there is such a limit.

Proposition 2.2. *Suppose that (X_{n-1}, θ_{n-1}) is distributed according to some probability measure $\mu : \mathbb{R}\times\Theta \to \mathbb{R}^+$, and take any $c_j \in \Theta$, and $s_n, s_{n-1} \in \mathbb{R}^+$. The marginal probability (up to a normalizing constant) of the event $\{\theta_n = c_j\}$ given $\{S_n = s_n, S_{n-1} = s_{n-1}\}$ has the following limit:*

$$\begin{aligned} &\mathbb{P}^\varepsilon(\theta_n = c_j | S_n = s_n, S_{n-1} = s_{n-1}) \\ &\propto \int \Gamma^\varepsilon(s_n, s_{n-1})\mu(dx', c_j) \\ &\to \sum_i \frac{1}{s_n\sqrt{2\pi}} \bar{\mathbb{E}}\left\{ \frac{\mathbb{1}_{[\theta_n = c_j]}}{\sqrt{\int_{t_{n-1}}^{t_n} \left\langle\beta^2\right\rangle_{\theta_u} du}} \right. \\ &\qquad \left. \cdot \exp\left\{ -\frac{1}{2}\left(\frac{\log(s_n/s_{n-1}) - \bar{I}_n(\theta)}{\sqrt{\int_{t_{n-1}}^{t_n} \left\langle\beta^2\right\rangle_{\theta_u} du}} \right)^2 \right\} \middle| \theta_{n-1} = c_i \right\} \int \mu(dx, c_i) \end{aligned} \tag{2.13}$$

as $\varepsilon \to 0$, where $\bar{\mathbb{E}}$ is the expectation of the probability law associated with the limiting model in (2.10)*,* (2.11)*, and* (2.12)*.*

Proof. See Appendix A.2. ■

Of course, there is a semigroup operator for the model in (2.10), (2.11), and (2.12). Indeed, inspection of the model leads to a matrix $\bar{\Gamma}$ that operates on any probability measure $\mu : \Theta \to$

$\mathbb{R}^+$ such that

(2.14)

$$\overline{\Gamma}(s_{n+1},s_n)\mu(c_j) = \sum_i \overline{\Gamma}_{ij}(s_{n+1},s_n)\mu(c_i)$$

$$= \frac{1}{s_n\sqrt{2\pi}} \sum_i \bar{\mathbb{E}} \left\{ \frac{\mathbb{1}_{[\theta_{n+1}=c_j]}}{\sqrt{\int_{t_{n-1}}^{t_n} \langle\beta^2\rangle_{\theta_u} du}} \exp\left\{ -\frac{1}{2} \left(\frac{\log(s_{n+1}/s_n) - \bar{I}_n(\theta)}{\sqrt{\int_{t_{n-1}}^{t_n} \langle\beta^2\rangle_{\theta_u} du}} \right)^2 \right\} \Big| \theta_n = c_i \right\} \mu(c_i).$$

By taking μ such that $\int \mu(x,c_i)dx = \mathbb{1}_{[\theta_{n-1}=c_i]}$, it follows from Proposition 2.2 that $\bar{\Gamma}$ is indeed the limiting semigroup operator, giving the limit of the marginal transition distribution,

$$\mathbb{P}^\varepsilon(S_n \le s_n, \theta_n = c_j | S_{n-1} = s_{n-1}, \theta_{n-1} = c_i) \xrightarrow{\varepsilon\to 0} \int_0^{s_n} \overline{\Gamma}_{ij}(y, s_{n-1})dy.$$

2.2. Expansion of the transition density. A central theme in this paper is perturbation theory, where solutions to (2.6) are expanded in powers of $\sqrt{\varepsilon}$. Following the methodology of [FPSS11], the marginal transition density of (S,θ) can be expanded around the limiting transition density, with terms of the expansion being ordered in powers of $\sqrt{\varepsilon}$.

The marginal transition kernel from (2.5) is expanded in powers of $\sqrt{\varepsilon}$ as follows:

$$\begin{aligned} u_n^\varepsilon(t,y',c_j|y,x,c_i) &= \bar{u}_n(t,y',c_j|y,x,c_i) + \sqrt{\varepsilon}u_n'(t,y',c_j|y,x,c_i) \\ &\quad + \varepsilon u_n''(t,y',c_j|y,x,c_i) + \varepsilon^{3/2}u_n'''(t,y',c_j|y,x,c_i) + o(\varepsilon^{3/2}) \end{aligned} \tag{2.15}$$

for $t < t_n$. The function u_n^ε satisfies the PDE of (2.6) and (2.7), and applying the PDE to (2.15) yields

$$\begin{aligned} &\frac{1}{\varepsilon}L_0^j\bar{u}_n + \frac{1}{\sqrt{\varepsilon}}\left(L_1\bar{u}_n + L_0^j u_n'\right) + \left(L_2\bar{u}_n + L_1 u_n' + L_0^j u_n''\right) \\ &\quad + \sqrt{\varepsilon}\left(L_2 u_n' + L_1\bar{u}_n'' + L_0^j u_n'''\right) + \cdots = 0 \end{aligned} \tag{2.16}$$

for $t < t_n$. Sections 2.2.1 and 2.2.2 compute formulas for $\bar{u}_n$ and u_n', respectively. The zero-order term $\bar{u}_n$ agrees with the limiting transition kernel from Proposition 2.2, and the order-$\sqrt{\varepsilon}$ correction u_n' is a solution of a Poisson equation. These first two terms are important because neither depends on x, which means that formulas are of reduced dimension, and hence less time is needed for numerical computation. Furthermore, the correction u_n' is also used to compute a small-$\sqrt{\varepsilon}$ expansion of the filtering distribution in section 3 and the optimal portfolio strategy in section 4.

After computing the expansions, it is necessary to understand the behavior in the expansion's error. If the terminal condition of (2.7) were a bounded and smooth function, then the analysis of [FPSS11] would apply so that rigorous *Big-Oh* estimates of error would be of order ε, that is, $|u_n^\varepsilon - \bar{u}_n - \sqrt{\varepsilon}u_n'| = \mathcal{O}(\varepsilon)$ for $t \le t_n$, if the right-hand side of (2.7) were a bounded smooth function. In this paper, the expansions are made for a transition density that has a δ-function in the terminal condition. In this case, the analysis of [FPSS03] is required to prove accuracy of the expansions, and rigorous error estimates will be only of order $o(\sqrt{\varepsilon})$, that is, $|u_n^\varepsilon - \bar{u}_n - \sqrt{\varepsilon}u_n'| = o(\sqrt{\varepsilon})$ for $t < t_n$, i.e., for t in the semiopen interval $[t_{n-1}, t_n)$. With regard to Big-Oh error estimates, unbounded and nonsmooth terminal conditions will make the error estimates larger than order-ε. For instance, in [FPSS03] it was shown that the error estimate for a derivative price with nonsmooth payoff (e.g., a call option) is $|u_n^\varepsilon - \bar{u}_n - \sqrt{\varepsilon}u_n'| = \mathcal{O}(\varepsilon \log \varepsilon)$.

2.2.1. The zero-order term $\bar{u}_n$. The following proposition gives the first term in the expansion of (2.15).

Proposition 2.3. *The zero-order term $\bar{u}_n$ does not depend on x and satisfies*

$$\begin{aligned} \langle L_2 \rangle \, \bar{u}_n(t, y', c_j | y, c_i) &= 0 \qquad \text{for } t \in [t_{n-1}, t_n), \\ \bar{u}_n \Big|_{t=t_n} &= \delta_{y=y'} \mathbb{1}_{[i=j]}, \end{aligned}$$

where

$$\langle L_2 \rangle = \frac{\partial}{\partial t} + \frac{\left\langle \beta^2 \right\rangle_{c_i} y^2}{2} \frac{\partial^2}{\partial y^2} + \langle \alpha \rangle_{c_i} y \frac{\partial}{\partial y} + Q,$$

with $\left\langle \beta^2 \right\rangle_{c_i}$ and $\langle \alpha \rangle_{c_i}$ being the invariant averages, respectively, as defined in (2.9). In fact, $\bar{u}_n|_{t=t_{n-1}}$ is the limiting operator defined in (2.14),

$$\bar{u}_n(t_{n-1}, s_n, c_j | s_{n-1}, c_i) = \overline{\Gamma}(s_n, s_{n-1}) \mathbb{1}_{[c_i]}(c_j).$$

Proof. The proof follows the steps of section 4.2 on pages 125–135 of [FPSS11]. To ensure that the solution does not blow up, it must be that

$$L_0^i \bar{u}_n(t, y', c_j | y, x, c_i) = 0 \qquad \forall i, j \text{ and } \forall y, y', x;$$

therefore, $\bar{u}_n$ must be in the null-space of the operator $L_0^i = \frac{1}{2}\frac{\partial^2}{\partial x^2} + (c_i - x)\frac{\partial}{\partial x}$. But this space is well known to be spanned by the constant functions of x, and therefore $\bar{u}_n$ must be constant in x. In a similar fashion, by comparing terms of order $\varepsilon^{-1/2}$ in (2.16), it is seen that

$$\underbrace{L_1 \bar{u}_n}_{=0} + L_0^i u_n' = 0 \qquad \forall i,$$

and so it must be that u_n' is also in the null-space of the operator L_0^i and therefore must also be constant in x. Now, comparing the order-1 terms in (2.16), it must be that the following Poisson equation holds:

$$L_2 \bar{u}_n + \underbrace{L_1 u_n'}_{=0} + L_0^i u_n'' = 0 \qquad \forall i, \tag{2.17}$$

for which the Fredholm alternative guarantees a solution for u_n' iff

$$\langle L_2 \bar{u}_n \rangle_i \doteq \int_{\mathbb{R}} L_2 \bar{u}_n(t, y', x', c_j | y, c_i) \upsilon(x, c_i) dx = 0 \tag{2.18}$$

(because υ is the single element in the null-space of the adjoint to the operator L_0^i). The solution $\bar{u}_n$ must satisfy the PDE

$$\langle L_2\rangle_i\,\bar{u}_n = 0 \qquad \text{for } t < t_n,$$

where $\langle L_2\rangle_i \doteq (\frac{\partial}{\partial t} + \frac{\langle\beta^2\rangle_{c_i} y^2}{2}\frac{\partial^2}{\partial y^2} + \langle\alpha\rangle_{c_i} y\frac{\partial}{\partial y} + Q)$. Obviously, $\langle L_2\rangle_i$ is the generator of (Y_t,θ_t) under the limiting probability law, and so it is clear that $\bar{u}_n|_{t=t_{n-1}} = \overline{\Gamma}$. Finally, the terminal condition needs to be defined. Since $\bar{u}_n$ is the transition density based on the limiting model of (2.10)–(2.12), it follows that

$$\lim_{t\nearrow t_n} u_n(t,y',c_j|y,x,c_i) = \lim_{t\nearrow t_n}\frac{\partial}{\partial s_n}\mathbb{P}^0(S_n\leq s_n,\theta_n=c_j|S_t=y,\theta_t=c_i) = \delta_{y=y'}\mathbb{1}_{[i=j]}.$$

Hence, $\overline{u}_n$ solves the system

$$\begin{aligned}&\langle L_2\rangle\,\bar{u}_n = 0,\\ &\bar{u}_n\Big|_{t=t_n} = \delta_{y=y'}\mathbb{1}_{[i=j]}.\end{aligned}\qquad\blacksquare$$

2.2.2. The order-$\sqrt{\varepsilon}$ correction u_n'. To compute the order-$\sqrt{\varepsilon}$ correction in (2.15), it is simply a matter of going back to the proof of Proposition 2.3 and solving a Poisson equation. The computation is essentially the same as equation (4.31) on page 129 of [FPSS11].

Proposition 2.4. *Given $\bar{u}_n$, the order-$\sqrt{\varepsilon}$ term in the expansion of u_n^ε is constant in x and is the solution to a (centered) Poisson equation*

(2.19)

$$\begin{aligned}&\langle L_2\rangle\,u_n'(t,y',c_j|y,c_i)\\ &= \left\langle L_1\phi^{(1)}\right\rangle y^2\frac{\partial^2}{\partial y^2}\bar{u}_n(t,y',c_j|y,c_i) + \left\langle L_1\phi^{(2)}\right\rangle y\frac{\partial}{\partial y}\bar{u}_n(t,y',c_j|y,c_i)\qquad \textit{for } t\in[t_{n-1},t_n],\end{aligned}$$

where $\phi^{(\ell)} = \phi^{(\ell)}(y,x,c_i)$ for $\ell = 1,2$ are each a solution to another Poisson equation,

$$L_0^i\phi^{(1)}(x,c_i) = \frac{1}{2}(\beta^2(x) - \langle\beta^2\rangle_{c_i}), \tag{2.20}$$

$$L_0^i\phi^{(2)}(x,c_i) = \alpha(x) - \langle\alpha\rangle_{c_i}. \tag{2.21}$$

Proof. The proof follows the steps of section 4.2 on pages 125–135 of [FPSS11]. From the terms of order $\sqrt{\varepsilon}$ in (2.16), consider another Poisson equation,

$$L_2u_n' + L_1u_n'' + L_0^iu_n''' = 0,$$

which according to the Fredholm alternative has a solution iff

$$\left\langle L_2u_n' + L_1u_n''\right\rangle = 0. \tag{2.22}$$

From the proof of Proposition 2.3 it is known that u_n' is constant in x so that $\langle L_2 u_n'\rangle = \langle L_2\rangle u_n'$, and from the Poisson equation (2.17) it follows that

$$u_n'' = -L_0^{-1} L_2 \bar{u}_n = -L_0^{-1}(L_2 - \langle L_2\rangle)\bar{u}_n \tag{2.23}$$

(recall that $\bar{u}_n$ satisfies $\langle L_2\rangle \bar{u}_n = 0$), where $L_0 = \mathrm{diag}(L_0^1, L_0^2, \ldots, L_0^m)$. Using (2.22) and (2.23), a centered equation emerges,

$$\langle L_2\rangle u_n' = -\left\langle L_1 u_n''\right\rangle = \left\langle L_1 L_0^{-1}(L_2 - \langle L_2\rangle)\bar{u}_n\right\rangle = \left\langle L_1 L_0^{-1}(L_2 - \langle L_2\rangle)\right\rangle \bar{u}_n,$$

where $\bar{u}_n$ comes outside the bracket because it is constant in x, and plugging in (2.20) and (2.21) yields

$$L_2 - \langle L_2\rangle = \frac{1}{2}(\beta^2(x) - \left\langle\beta^2\right\rangle_{c_i})y^2\frac{\partial^2}{\partial y^2} + \left(\alpha(x) - \langle\alpha\rangle_{c_i}\right) y\frac{\partial}{\partial y} = L_0\left(\phi^{(1)} y^2\frac{\partial^2}{\partial y^2} + \phi^{(2)} y\frac{\partial}{\partial y}\right),$$

and hence the formula in (2.19) holds. ∎

In fact, there is commutativity of differential operators $\langle L_2\rangle$ and $y^k\frac{d^k}{dy^k}$ for any $k \in \mathbb{N}$, and so the solution to (2.19) is given explicitly as

$$u_n' = (t_n - t)\left(\left\langle L_1\phi^{(1)}\right\rangle y^2\frac{\partial^2}{\partial y^2}\bar{u}_n + \left\langle L_1\phi^{(2)}\right\rangle y\frac{\partial}{\partial y}\bar{u}_n\right) \qquad \text{for any } t < t_n. \tag{2.24}$$

At time $t = t_{n-1}$, it follows from the proof of Proposition 2.3 that

$$\begin{aligned}
\bar{u}_n(t_{n-1}, s_n, c_j | s_{n-1}, c_i) &= \frac{\partial^2}{\partial s_n}\mathbb{P}^0(S_n \le s_n, \theta_n = c_j | S_{n-1} = s_{n-1}, \theta_{n-1} = c_i)\\
&= \frac{\partial^2}{\partial s_n}\mathbb{P}^0(S_n \le s_n, \theta_n = c_j | S_{n-1} = s_{n-1}, \theta_{n-1} = c_i)\\
&= \bar{\Gamma}(s_n, s_{n-1})\mathbb{1}_{[c_i]}(c_j),
\end{aligned}$$

where $\bar{\Gamma}$ is the limiting operator defined in (2.14).

In summary, the expansion in (2.15) can be interpreted in terms of Γ^ε:

$$\begin{aligned}
&\int \Gamma^\varepsilon(s_n, s_{n-1})[\delta_x \mathbb{1}_{[c_i]}](x', c_j)dx'\\
&\qquad = \bar{\Gamma}(s_n, s_{n-1})\mathbb{1}_{[c_i]}(c_j) + \sqrt{\varepsilon}u_n'(t_{n-1}, s_n, c_j | s_{n-1}, c_i) + o(\sqrt{\varepsilon}).
\end{aligned} \tag{2.25}$$

From (2.25) it is clear that the order-$\sqrt{\varepsilon}$ correction term is the $\sqrt{\varepsilon}$ derivative (taken from the right-hand side as $\varepsilon \searrow 0$):

$$u_n'(t_{n-1}, s_n, c_j | s_{n-1}, c_i) = \lim_{\varepsilon\searrow 0}\frac{\int \Gamma^\varepsilon(s_n, s_{n-1})\delta_{x_{n-1}}\mathbb{1}_{[c_i]}(x', c_j)dx' - \bar{\Gamma}(s_n, s_{n-1})\mathbb{1}_{[c_i]}(c_j)}{\sqrt{\varepsilon}}. \tag{2.26}$$

This expansion in (2.25) and the derivative in (2.26) will be instrumental in what follows; derivatives/corrections of order $\sqrt{\varepsilon}$ will be needed for the filtering distribution and for the optimal portfolio problem.

2.3. Effects of correlation parameter ρ. From the limiting model of section 2.1 and the expansions in section 2.2, it should be somewhat clear that the order-$\sqrt{\varepsilon}$ correction term is a *correction for correlation.* Indeed, the limiting kernel in (2.13) does not have the ρ parameter. Furthermore, $L_1 = 0$ if $\rho = 0$, and so the correction term will be zero in the uncorrelated case. For instance, in the continuous time option pricing theory of [FPSS11], if $\rho \approx -.6$, then option prices based on this model and these expansions would have a negative skew in their implied volatilities, but $\rho = 0$ would mean that the correction is zero and the expansion's implied volatility curve would be flat.

In discrete time, infinitesimal correlations between S_n and X_n are not seen so easily. Indeed, X_n is not present in the limiting model. However, the correlation parameter still appears in the limiting model of (2.10), (2.11), and (2.12). The interpretation is that correlation effects happen on the faster time scales, and their presence over longer time periods is felt mostly through the statistical parameterization of the model.

3. Filtering. Filtering is the tool used to track (X_n, θ_n) given only the observations on S. The term "filter" refers to the posterior distribution of the hidden variables. As mentioned in section 1, the filtration $\mathcal{F}_n$ is defined as the σ-algebra generated by the observations up to (and including) time n,

$$\mathcal{F}_n \doteq \sigma\{S_{0:n}\}, \quad n \geq 0,$$

where $S_{0:n} \doteq (S_0, S_1, \dots, S_n)$ is the vector of the observed prices. The filter has a density π_n^ε on $\mathbb{R} \times \Theta$ conditional on $\mathcal{F}_n$, such that

$$\mathbb{E}^\varepsilon\{g(X_n, \theta_n)|\mathcal{F}_n\} = \sum_i \int g(x, c_i)\pi_n^\varepsilon(x, c_i; S_{0:n})dx,$$

where $g : \mathbb{R} \times \Theta \to \mathbb{R}$ is an integrable function, and the filter is written as $\pi_n^\varepsilon(x, c_i) = \pi_n^\varepsilon(x, c_i; S_{0:n})$ omitting the conditioning. Verification of the fact that the filter has a density follows from the Bayesian recursive formula for π_n^ε below. To derive the filter recursion, the operators Γ^ε defined by (2.2) are used, and the following recursive formula yields the updated density:

$$\pi_{n+1}^\varepsilon = \mathcal{E}^\varepsilon(S_{n+1}, S_n, \pi_n^\varepsilon) \doteq \frac{\Gamma^\varepsilon(S_{n+1}, S_n)\pi_n^\varepsilon}{\sum_i \int \Gamma^\varepsilon(S_{n+1}, S_n)\pi_n^\varepsilon(x, c_i)dx}, \quad n \geq 0, \tag{3.1}$$

with the initial density π_0^ε assumed to be known and independent of ε. After recognizing the Bayesian structure in the filter as shown in (3.1), the fact that π_n^ε is a density is verified because Γ^ε is the kernel to the strongly parabolic PDE in (2.6), which admits smooth solutions, and therefore its kernel is a density. So, given the observations $S_{0:n+1}$, $\Gamma^\varepsilon(S_{n+1}, S_n)\pi_n^\varepsilon$ is an unnormalized density, and the left-hand side of (3.1) is a probability density.

3.1. Limit of filtering distribution. In the limit as $\varepsilon \to 0$, there is a filter of reduced dimension. Denote this reduced filter with $\bar{\pi}$, and define it to be a probability distribution on the finite set Θ of the regime switching process θ_n. In particular, the reduced filter is a vector $\bar{\pi}_n = (\bar{\pi}_n(c_1), \bar{\pi}_n(c_2), \dots, \bar{\pi}_n(c_m))$ with $\bar{\pi}_n(c_i) \geq 0$, $\sum_{j=1}^m \bar{\pi}_n(c_j) = 1$, that is, $\bar{\pi}_n$ is an element of the m-dimensional simplex, and it is obtained recursively by

$$\bar{\pi}_{n+1}(c_i) = \bar{\mathcal{E}}(S_{n+1}, S_n, \bar{\pi}_n) \doteq \frac{\overline{\Gamma}(S_{n+1}, S_n)\bar{\pi}_n(c_i)}{\sum_j \overline{\Gamma}(S_{n+1}, S_n)\bar{\pi}_n(c_j)}. \tag{3.2}$$

Notice for the filter in (3.2) that all dependence on X has gone away as a consequence of the ergodic limit as $\varepsilon \to 0$, and hence it is a filter of reduced dimension. In terms of the limiting probability measure, the limiting filter is equivalent to

$$\bar{\mathbb{E}}\{g(\theta_n)|S_{0:n} = s_{0:n}\} = \sum_i g(c_i)\bar{\pi}_n(c_i),$$

and from (2.13) there is the following result.

Theorem 3.1. *Fix a vector $s_{0:n} \in \mathbb{R}^{n+1}$ with nonnegative entries. Suppose that π_0 has no dependence on the parameter ε and that $\bar{\pi}_0(c_i) = \int \pi_0(dx, c_i)$. Then for any bounded function $g(\theta)$, the filter has the following limit:*

$$\mathbb{E}^\varepsilon\{g(\theta_n)|S_{0:n} = s_{0:n}\} \to \bar{\mathbb{E}}\{g(\theta_n)|S_{0:n} = s_{0:n}\} \qquad \text{as } \varepsilon \to 0,$$

where $\bar{\pi}_n$ is given by (3.2).

Proof. The filter can be written as

$$\mathbb{E}^\varepsilon\{g(\theta_n)|S_{0:n} = s_{0:n}\} = \frac{\sum_i g(c_i) \int \left[\prod_{\ell=1}^n \Gamma^\varepsilon(s_\ell, s_{\ell-1})\right] \pi_0(x, c_i)dx}{\sum_i \int \left[\prod_{\ell=1}^n \Gamma^\varepsilon(s_\ell, s_{\ell-1})\right] \pi_0(x, c_i)dx},$$

and the fact that the limiting filter is given by $\bar{\pi}_n$ follows inductively for $n = 1, 2, 3, \dots$ by using the result of Proposition 2.2, (2.13), and the definition $\overline{\Gamma}$ given in (2.14). ■

Remark 1. Theorem 3.1 is almost identical to the limit shown in [Pap12], except there it was shown for a system where Q depended on X_t.

Remark 2. Theorem 3.1 could be proven for the case where π_0 depends on ε, but this generalization is excluded for the sake of simplicity. If one were to consider such cases, then there would need to be some mild assumptions on π_0's dependence on ε. For instance, if π_0 could have an expansion in powers of $\sqrt{\varepsilon}$, then the theorem could easily be generalized.

3.2. The $\sqrt{\varepsilon}$-correction to the filtering distribution. This section computes an order-$\sqrt{\varepsilon}$ correction for the marginal filtering distribution on θ_n. The result relies on an inductive argument as a means to prove that the expansion is in fact valid. Ultimately, the $\sqrt{\varepsilon}$-correction is simply an $\sqrt{\varepsilon}$-derivative of the filter evaluated at $\varepsilon = 0$, and the proof is nothing more than a differentiation in $\sqrt{\varepsilon}$. However, the expansion of section 2.2 will be important for the inductive step.

Consider an expansion for the marginal filter on θ_n with the base term given by (3.2):

$$\mathbb{E}^\varepsilon\left\{g(\theta_n)|S_{0:n} = s_{0:n}\right\} = \sum_i g(c_i)\bar{\pi}_n(c_i) + \sqrt{\varepsilon}\sum_i g(c_i)\bar{\pi}_n'(c_i) + o(\sqrt{\varepsilon}) \tag{3.3}$$

for any bounded function g. It follows from the expansion in (2.25) that the correction $\bar{\pi}_n'$ is an $\sqrt{\varepsilon}$-derivative and is given by a recursive formula.

Proposition 3.2. *For $n > 0$ the correction term $\bar{\pi}_n'$ is the $\sqrt{\varepsilon}$-derivative of the marginal filter taken from the right as $\varepsilon \searrow 0$,*

$$\bar{\pi}_n'(c_i) = \frac{\partial}{\partial\sqrt{\varepsilon}} \int \pi_n^\varepsilon(dx, c_i)\Big|_{\varepsilon=0} = \lim_{\varepsilon\searrow 0} \frac{1}{\sqrt{\varepsilon}}\left(\int \pi_n^\varepsilon(x, c_i)dx - \bar{\pi}_n(c_i)\right), \tag{3.4}$$

and is given recursively by

$$\bar\pi_n'(c_i) = \frac{\bar\Gamma(s_n,s_{n-1})\bar\pi_{n-1}'(c_i) + \sum_j u_n'(t_{n-1},s_n,c_i|s_{n-1},c_j)\bar\pi_{n-1}(c_j)}{\sum_\ell \bar\Gamma(s_n,s_{n-1})\bar\pi_{n-1}(c_\ell)} - \bar\pi_n(c_i)\frac{\sum_\ell\left(\bar\Gamma(s_n,s_{n-1})\bar\pi_{n-1}'(c_\ell) + \sum_j u_n'(t_{n-1},s_n,c_\ell|s_{n-1},c_j)\bar\pi_{n-1}(c_j)\right)}{\sum_\ell \bar\Gamma(s_n,s_{n-1})\bar\pi_{n-1}(c_\ell)}, \tag{3.5}$$

where $\bar\pi_n$ is the average filter given by (3.2) *and u_n' is the correction term given by Proposition* 2.4*.*

Remark 3. Notice from (3.5) that $\sum_i \bar\pi_n'(c_i) = 0$. This means the approximation $\int \pi_n^\varepsilon(dx, c_i) \approx \bar\pi_n(c_i) + \sqrt{\varepsilon}\bar\pi_n'(c_i)$ will sum over the index i to unity.

Proof. See Appendix A.3. ■

4. The portfolio problem in discrete time with partial information. The expansions of the marginal transition kernel in section 2.2 and the expansion of the marginal filtering distribution in section 3 are instrumental to the proofs of the results in this section, wherein the partial information portfolio optimization problem is presented and solved with an ADP. It will be important to continue thinking of the order-$\sqrt{\varepsilon}$ corrections as the right-hand derivatives as $\varepsilon \searrow 0$, as was done in the proof of Proposition 3.2. When appropriate, the order-$\sqrt{\varepsilon}$ expansion term will be denoted as $\frac{\partial}{\partial\sqrt{\varepsilon}}$.

The task in the portfolio problem is to find a portfolio strategy that is adapted to the filtration of observations. The filtration $\mathcal{F}_n$ represents all of the information that is available to the investor at time n. Neither X_n nor θ_n is $\mathcal{F}_n$-measurable (observable), and so the optimal investment problem with partial information becomes one that maximizes the expectation of terminal wealth conditional on $\mathcal{F}_n$.

Recall the self-financing condition of (1.3). For a given terminal time $N > 0$, the objective is to maximize the expectation of the terminal utility over the set of $\mathcal{F}_n$-adapted admissible strategies,

$$(\mathcal{P}^\varepsilon) \qquad \begin{aligned} &J_n^\varepsilon(v, S_{0:n}) \doteq \operatorname*{ess\,sup}_{a\in\mathcal{A}_n} \mathbb{E}^\varepsilon\left\{U\left(V_N\right)\middle|\mathcal{F}_n \vee \{V_n = v\}\right\} \qquad \text{for } n < N,\\ &J_N^\varepsilon(v, S_N) = U(v), \end{aligned}$$

where $U(\cdot)$ is a utility function reflecting the investor's risk preferences, and $\mathcal{A}_n$ is the set of admissible strategies that may place constraints on borrowing and short selling (see Appendix D.1) but is contained within a more general set of strategies (see [BUV12, MZ10]):

$$\mathcal{A}_n \subset \left\{a : \Omega \to \mathbb{R}^{N-n}\middle| \text{ where } \textstyle\sum_{\ell=n}^{N-1} |a_\ell| < \infty \text{ a.s. and each } a_\ell \text{ is } \mathcal{F}_\ell\text{-adapted}\right\}.$$

Also, it is assumed that there are no transaction costs or other effects caused by an investor's buying and selling of assets.

The optimal portfolio problem ($\mathcal{P}^\varepsilon$) can be written formally in recursive form

$$\begin{aligned} &J_n^\varepsilon(v, S_{0:n})\\ &\quad= \sup_{a_n\in\mathbf{a}_n} \mathbb{E}^\varepsilon\left\{J_{n+1}^\varepsilon\left(v(1+r\Delta t)+a_n\left(\frac{\Delta S_{n+1}}{S_n}-r\Delta t\right), S_{0:n+1}\right)|\mathcal{F}_n \vee \{V_n = v\}\right\}, \quad n<N, \end{aligned} \tag{4.1}$$

where $\mathbf{a}_n \subset \mathbb{R}$ is the set of admissible $\mathcal{F}_n$-adapted allocations. At time $n < N$, the $\mathcal{F}_n$-adapted solution to ($\mathcal{P}^\varepsilon$) is denoted $a_n^\varepsilon = a_n^\varepsilon(v; S_{0:n})$, so that

$$
\begin{aligned}
(4.2)\quad & J_n^\varepsilon(v, S_{0:n}) \\
&= \mathbb{E}^\varepsilon \left\{ J_{n+1}^\varepsilon \left(v(1 + r\Delta t) + a_n^\varepsilon \left(\frac{\Delta S_{n+1}}{S_n} - r\Delta t \right), S_{0:n+1} \right) \Big| \mathcal{F}_n \vee \{V_n = v\} \right\}, \quad n < N.
\end{aligned}
$$

A dynamic programming algorithm could be implemented (at least in theory) based on this recursive formula, but this would come at a tremendous computational cost. The obvious reason for the computational overhead is that the program in (4.1) is non-Markovian. However, there is a Markovian formulation of this problem in the limit as $\varepsilon \to 0$, and hence there is a Bellman equation to solve the limiting problem. Furthermore, section 4.3 will use the small ε asymptotics to show that ($\mathcal{P}^\varepsilon$) can be approximated by its limiting behavior and that there is an order-$\sqrt{\varepsilon}$ correction to the optimal strategy.

4.1. The limit problem. For the limiting model in (2.10), (2.11), and (2.12), the optimization of expected terminal utility is referred to as the *unperturbed* or *limit problem*,

$$
(\overline{\mathcal{P}}) \qquad
\begin{aligned}
&\bar{J}_n(v; S_{0:n}) \doteq \operatorname*{ess\,sup}_{a \in \mathcal{A}_n} \bar{\mathbb{E}} \left\{ U(V_N) \Big| \mathcal{F}_n \vee \{V_n = v\} \right\} \quad \text{for } n < N, \\
&\bar{J}_N(v, S_N) = U(v),
\end{aligned}
$$

which is similar to ($\mathcal{P}^\varepsilon$) but with the limiting probability law. For $0 \leq n < N$, the optimal strategy for ($\overline{\mathcal{P}}$) is denoted as $\bar{a}_n = \bar{a}_n(v, s_{0:n})$, and the optimal value function is given recursively as

$$
\begin{aligned}
(4.3)\quad & \bar{J}_n(v, S_{0:n}) \\
&= \bar{\mathbb{E}} \left\{ \bar{J}_{n+1} \left(v(1 + r\Delta t) + \bar{a}_n \left(\frac{\Delta S_{n+1}}{S_n} - r\Delta t \right), S_{0:n+1} \right) \Big| \mathcal{F}_n \vee \{V_n = v\} \right\}, \quad n < N.
\end{aligned}
$$

Now, suppose that $a_\ell^\varepsilon \to \bar{a}_\ell$ in probability as $\varepsilon \to 0$ for all $\ell \in \{n+1, n+2, \ldots, N\}$. Then for any $a \in \mathbf{a_n}$ and any $v \in \mathbb{R}$, and for almost every $S_{0:n} = s_{0:n}$, the objective function from the right-hand side of (4.1) will converge:

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \mathbb{E}^\varepsilon \left\{ J_{n+1}^\varepsilon \left(v(1 + r\Delta t) + a \left(\frac{\Delta S_{n+1}}{S_n} - r\Delta t \right), S_{0:n+1} \right) \Big| \mathcal{F}_n \vee \{V_n = v\} \right\} \\
(4.4)\quad &= \bar{\mathbb{E}} \left\{ \bar{J}_{n+1} \left(v(1 + r\Delta t) + a \left(\frac{\Delta S_{n+1}}{S_n} - r\Delta t \right), S_{0:n+1} \right) \Big| \mathcal{F}_n \vee \{V_n = v\} \right\}
\end{aligned}
$$

(this limit is proven in Appendix C). Using (among other things) (4.4), section 4.3 will show that $J^\varepsilon(v; s_{0:n}) \to \bar{J}(v; s_{0:n})$ as $\varepsilon \to 0$ pointwise in $s_{0:n}$, and that the optimal solution to ($\mathcal{P}^\varepsilon$) is close to the optimal solution to ($\overline{\mathcal{P}}$) with an order-$\sqrt{\varepsilon}$ correction.

4.2. Markovian formulation of the limit problem. As formulated in ($\overline{\mathcal{P}}$), the value function $\bar{J}_n(v; S_{0:n})$ cannot be obtained via a Bellman equation because the joint discrete time price and wealth process (S_n, V_n) is not Markovian. However, the maximization problem can be written as a Markov decision process by defining it as a function of (S_n, V_n), the posterior distribution of θ_n, and as a function of the nonlinear operator $\overline{\mathcal{E}}$ defined in (3.2). This allows for the optimal expected utility to be given by a recursive Bellman equation.

Let $\mathcal{M}$ be the space of probability measures on Θ, i.e., the m-dimensional simplex on Θ. For $\varepsilon = 0$, solutions to $(\overline{\mathcal{P}})$ are given by an unperturbed Bellman equation that relies on the lower-dimensional averaged filter of (3.2). In particular, this Bellman equation yields an optimal value function $H_n : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ such that

(4.5)
$$H_n(s_n, v, \mu) = \sup_{a \in \mathbf{a}_n} \sum_i \int H_{n+1}\left(y, v(1 + r\Delta t) + a(y/s_n - (1 + r\Delta t)), \bar{\mathcal{E}}_{n+1}(y, s_n, \mu)\right) \times \bar{\Gamma}(y, s_n)\mu(c_i)dy, \quad n < N,$$

(4.6)
$$H_N(y, v, \mu) = U(v).$$

Solutions to this limiting equation are numerically tractable because H_n is the value function from a Markov decision process. The equivalence of (4.5) and (4.6) to the unperturbed problem $(\overline{\mathcal{P}})$ is shown in Chapters 5 and 6 of [BR11], and it follows that the set of solutions $\bar{a}_n$ solving the unperturbed problem equals the set of solutions to (4.5) and (4.6) when $\mu = \bar{\pi}_n$, and that the value functions are equal:

$$\bar{J}_n(v; S_{0:n}) = H_n(S_n, v, \bar{\pi}_n).$$

The main difference between (4.5) and $(\overline{\mathcal{P}})$ is that the Markovian formulation has a structure that allows for the application of dynamic programming via recursive Bellman equations [Pow11, Ber12]. These algorithms are well suited for computations because they can be iterated without having to simulate and store entire path histories of the observed stochastic processes. The program for computing H_n is given in pseudocode in Appendix E along with a similar algorithm for computing the value function under full information.

Remark 4. In the case of no regime switching (i.e., $Q = 0$), the marginal filter will degenerate to a point-mass as n grows. This means that for large enough N, the no-regime-switching case will behave similarly to the full information case but with lesser utility due to the cost of learning the correct state at the earlier times during the investment period. This loss in utility is the *information premium* that will be discussed later in section 6.

4.3. Small-ε perturbation theory of $(\mathcal{P}^\varepsilon)$ for CARA utility. This section will combine the equations and propositions from sections 2 and 3 and early results from section 4. The main result comes in Theorem 4.1, where the implicit function theorem is used to show the existence of an optimal allocation for $(\mathcal{P}^\varepsilon)$, and which can be approximated by the optimal allocation for $(\overline{\mathcal{P}})$ plus an order-$\sqrt{\varepsilon}$ correction. Here, for simplicity, the theory is presented for the unconstrained case where the controls $\{a_n,\ n \le N\}$ are nonanticipating and finite but otherwise unrestricted and with the CARA utility which requires no constraints because U is defined for negative wealth. Potential results for power or logarithmic utility with linear constraints are presented in Appendix D. Appendix B.1 addresses some technical conditions that are required to apply the implicit function theorem in the stochastic setting of ($\mathcal{P}^\varepsilon$) and ($\overline{\mathcal{P}}$), and these conditions are analogous to the stability condition for the implicit function theorem in [BS96, Rob80].

Recall the iterative structure of problem ($\mathcal{P}^\varepsilon$) shown in (4.2). The CARA utility is the exponential

$$U(v) = -\frac{1}{\gamma}e^{-\gamma v},$$

where $\gamma > 0$. Assuming there are no constraints on borrowing or short selling, the CARA utility leads to a considerable simplification in solving the limit Bellman equation in (4.5) because the investor's level of risk aversion does not depend on their wealth. To see how, simply notice that $e^{-\gamma v}$ and all dependence on V_n can be removed from the supremum because there are no constraints on a:

$$\begin{aligned} H_n(s,v,\mu) &= \operatorname*{ess\,sup}_{a\in\mathcal{A}_n} \bar{\mathbb{E}}\left\{-\frac{1}{\gamma}\exp\left\{-\gamma\left(v+\sum_{\ell=n}^{N-1} a_\ell \frac{\Delta S_{\ell+1}}{S_\ell}\right)\right\}\Big| S_n = s, V_n = v, \bar{\pi}_n = \mu\right\} \\ &= e^{-\gamma v}\operatorname*{ess\,sup}_{a\in\mathcal{A}_n} \bar{\mathbb{E}}\left\{-\frac{1}{\gamma}\exp\left\{-\gamma\sum_{\ell=n}^{N-1} a_\ell \frac{\Delta S_{\ell+1}}{S_\ell}\right\}\Big| S_n = s, V_n = v, \bar{\pi}_n = \mu\right\} \\ &= e^{-\gamma v} H_n(s,0,\mu). \end{aligned}$$

Since dependence on v drops out of the optimization, the objective function is written without v:

$$J_n^\varepsilon(S_{0:n}) = \sup_{a\in\mathbf{a}_n} \mathbb{E}^\varepsilon\left\{e^{-\gamma a\left(\frac{\Delta S_{n+1}}{S_n} - r\Delta t\right)} J_{n+1}^\varepsilon\left(S_{0:n+1}\right)\Big|\mathcal{F}_n\right\}.$$

For the unconstrained case where $\mathbf{a}_n$ is taken to be any $\mathcal{F}_n$-adapted allocation in $\mathbb{R}$, strict concavity of problem ($\mathcal{P}^\varepsilon$) (see Proposition B.1) means that the first-order condition yields the optimal strategy. The first-order condition is denoted as

$$F_n^\varepsilon(a) = \frac{\partial}{\partial a}\mathbb{E}^\varepsilon\left\{e^{-\gamma a\left(\frac{\Delta S_{n+1}}{S_n} - r\Delta t\right)} J_{n+1}^\varepsilon\left(S_{0:n+1}\right)\Big|\mathcal{F}_n\right\} \qquad \text{for } n < N, \tag{4.7}$$

$$F_N^\varepsilon(a) = 0 \tag{4.8}$$

so that the optimal allocation $a_n^\varepsilon \in \mathbf{a}_n$ achieves a root $F_n^\varepsilon(a_n^\varepsilon) = 0$.

Letting $\bar{F}_n$ denote the first-order conditions for the limiting problem,

$$\begin{aligned} \bar{F}_n(a) &= \frac{\partial}{\partial a}\bar{\mathbb{E}}\left\{e^{-\gamma a\left(\frac{\Delta S_{n+1}}{S_n} - r\Delta t\right)} \bar{J}_{n+1}\left(S_{0:n+1}\right)\Big|\mathcal{F}_n\right\} \qquad \text{for } n < N, \\ \bar{F}_N(a) &= 0, \end{aligned}$$

and it follows from the limit in (4.4) that $F_n^\varepsilon(a) \to \bar{F}_n(a)$ pointwise in $a \in \mathbf{a_n}$ and pointwise in $s_{0:n}$. Furthermore, Proposition B.2 shows that the optimal portfolio strategy is unaffected by the overall level of the asset's price, and so H_n is constant in the wealth and asset price, so that $H_n(v, s_n, \mu) = H_n(\mu)$. Hence, the filter $\bar{\pi}_n$ is a sufficient statistic in the limiting problem and $\bar{a}_n$ is a function of it:

$$\bar{a}_n(v, s_{0:n}) = \bar{a}_n(\bar{\pi}_n).$$

This simplification of the portfolio problem is useful because it allows for a (relatively) simple set of proofs to show strict concavity and $\sqrt{\varepsilon}$-regularity of F_n^ε (see Appendix B). It should also be pointed out that the $\sqrt{\varepsilon}$-derivative of the marginal filter is instrumental in showing $\sqrt{\varepsilon}$-regularity of F_n^ε. Using these facts, the following theorem shows how the optimal strategy $a_n^\varepsilon(s_{0:n})$ can be expanded around $\bar{a}_n(\bar{\pi}_n)$.

Theorem 4.1 (the unconstrained case with CARA utility). *For $n \leq N < \infty$, consider any positive vector $s_{0:n}$ to be the observed path of prices, and denote the filter as a function $\bar{\pi}_n$ that is a mapping of $s_{0:n} \mapsto \mathcal{M}$. Suppose that ($\overline{\mathcal{P}}$) has an optimal strategy $\bar{a}_n(\bar{\pi}_n)$. Then there exist unique optimal admissible strategies for ($\mathcal{P}^\varepsilon$) such that*

$$a_n^\varepsilon(s_{0:n}) = \bar{a}_n(\bar{\pi}_n) + \sqrt{\varepsilon}\ a_n'(\bar{\pi}_n) + o(\sqrt{\varepsilon}) \qquad \text{for } \varepsilon < \varepsilon_0, \tag{4.9}$$

pointwise in $s_{0:n}$, where $a_n'(\bar{\pi}_n)$ is a correction term given by

$$a_n'(\bar{\pi}_n) = -\frac{\frac{\partial}{\partial\sqrt{\varepsilon}} F_n^\varepsilon\left(\bar{a}_n(\bar{\pi}_n)\right)\Big|_{\varepsilon=0}}{\frac{\partial}{\partial a}\bar{F}_n\left(\bar{a}_n(\bar{\pi}_n)\right)} \tag{4.10}$$

and is square-integrable: $\bar{\mathbb{E}}\{(a_n'(\bar{\pi}_n))^2\} < \infty$. Furthermore, the optimized value functions converge,

$$J_n^\varepsilon(s_{0:n}) \to \bar{J}_n(s_{0:n}) \qquad \text{as } \varepsilon \to 0,$$

pointwise for positive $s_{0:n}$ and for each $0 \leq n \leq N$.

Proof. With the results of Appendix B.1, the proof is based on a routine application of the implicit function theorem. Let $\bar{a}_n$ denote the solution such that $\bar{F}_n(\bar{a}_n) = 0$. Of concern are all the pairs

$$\left\{(\varepsilon, a_n^\varepsilon)\Big| F_n^\varepsilon(a_n^\varepsilon) = 0\right\}.$$

From Lemma B.3, there is a constant $\mathcal{K} < 0$ such that

$$\frac{\partial}{\partial a}\bar{F}_n(a) \leq \mathcal{K} < 0$$

locally for $a \in nbh(\bar{a}_n(\bar{\pi}_n))$, and so by the implicit function theorem there exist $\varepsilon_0 > 0$, a ball $\mathbb{B}_{\mathbb{R}}(0, R)$, and a function $\tilde{a}_n(\varepsilon)$ that is continuous for $\varepsilon < \varepsilon_0$, such that

$$\left\{(\varepsilon, \tilde{a}_n(\varepsilon))\Big|\varepsilon < \varepsilon_0\right\} = \left\{(\varepsilon, w) \in \mathbb{B}(0, \varepsilon_0) \times \mathbb{B}_{\mathbb{R}}(0, R)\Big| F_n^\varepsilon(\bar{a}_n + w) = 0\right\}.$$

Furthermore, from the $\sqrt{\varepsilon}$-expansion of Proposition B.4 it follows that the regularity of the objective function carries over to the implied function, which allows for an application of the chain rule for the right-hand derivative in $\sqrt{\varepsilon}$:

$$0 = \frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(a_n^\varepsilon)\Big|_{\varepsilon=0} = \frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(\bar{a}_n + \tilde{a}_n(\varepsilon))\Big|_{\varepsilon=0} = \frac{\partial}{\partial a}\bar{F}_n(\bar{a}_n)\frac{\partial}{\partial\sqrt{\varepsilon}}\tilde{a}_n(\varepsilon)\Big|_{\varepsilon=0} + \frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(\bar{a}_n)\Big|_{\varepsilon=0},$$

where the fact that $\tilde{a}_n(0) = 0$ was used. Then solving yields

$$a_n' = \frac{\partial}{\partial\sqrt{\varepsilon}}\tilde{a}_n(0) = -\frac{\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(\bar{a}_n)\Big|_{\varepsilon=0}}{\frac{\partial}{\partial a}\bar{F}_n(\bar{a}_n)},$$

which leaves the expansion of the optimal strategy,

$$a_n^\varepsilon = \bar{a}_n + \tilde{a}_n(\varepsilon) = \bar{a}_n + \sqrt{\varepsilon}\frac{\partial}{\partial\sqrt{\varepsilon}}\tilde{a}_n(0) + o(\sqrt{\varepsilon}).$$

In the correction to the strategy, we have $0 \leq -\frac{1}{\frac{\partial}{\partial a}\bar{F}_n(\bar{a}_n)} \leq -\frac{1}{\mathcal{K}} < \infty$ where existence of constant $\mathcal{K} < 0$ was shown in Lemma B.3, and from the details of Proposition B.4 it can be verified that $\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(\bar{a}_n)|_{\varepsilon=0}$ is square-integrable. Hence, the correction is square-integrable. The proof of convergence of the value function invokes the dominated convergence theorem and relies on convergence of a_n^ε to $\bar{a}_n$; it is very similar to the proof of (4.4). ■

Remark 5. Integrability of the correction term in Theorem 4.1 is important for $\sqrt{\varepsilon}$-regularity in F_n^ε. It is the condition allowing the implicit function theorem to be applied at each backward iteration of the dynamic program. In fact, Theorem 4.1 could apply to just about any discrete time model, but technical conditions such as uniform concavity and $\sqrt{\varepsilon}$-regularity are required, and hence the particularities of the model in (1.2) and (2.1) are important for verifying these facts. For more details, see Appendix B.1.2.

Remark 6. Full information is not covered by Theorem 4.1, unless it is the full information case wherein $\int \pi_0(dx, c_i) = \mathbb{1}_{[\theta_0=c_i]}$ and $Q = 0$. In general, full information is a Markov decision process that can be solved with a standard discrete time Bellman equation (see [Pow11, Ber12]).

It should be clear from Theorem 4.1 that dimension reduction has taken place since all quantities involved in filtering and control are of reduced dimension. Namely, $\bar{\pi}_n$ does not depend on x, the functions $\bar{F}_n$ and $\frac{\partial}{\partial a}\bar{F}_n$ are obtained through the Markovian formulation in (4.5) and do not depend on x, and $\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon|_{\varepsilon=0}$ is given by the formula of reduced dimension in Proposition B.4 and also does not depend on x. Indeed, the ADP that uses the approximation given by (4.9) and (4.10) will be a much faster program than the non-Markovian program proposed in (4.1).

There should also be some clarification that it is an actual improvement to use the approximation $a_n^\varepsilon \approx \bar{a}_n + \sqrt{\varepsilon}a_n'$. In particular, it is possible that the $\sqrt{\varepsilon}$-correction results in lesser utility, but this would be an unlikely event, particularly for ε very small. To understand why, one should interpret the order-$\sqrt{\varepsilon}$ correction as the optimal step in a *single-step gradient ascent algorithm*. A single-step gradient ascent algorithm is the following:

$$a_n^\varepsilon = \bar{a}_n + \gamma_n^\varepsilon F_n^\varepsilon(\bar{a}_n),$$

where the step-size γ_n^ε needs to be optimally determined. Rearranging the terms and dividing by $\sqrt{\varepsilon}$ yields

$$a_n' \simeq \frac{a_n^\varepsilon - \bar{a}_n}{\sqrt{\varepsilon}} = \gamma_n^\varepsilon \frac{1}{\sqrt{\varepsilon}}F_n^\varepsilon(\bar{a}_n) \simeq \gamma_n^\varepsilon \frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(\bar{a}_n)\Big|_{\varepsilon=0},$$

and from (4.10) it can be deduced that (up to terms of $O(\sqrt{\varepsilon})$) the optimal step-size is approximately the denominator in (4.10),

$$\gamma_n^\varepsilon \simeq -\frac{1}{\frac{\partial}{\partial a}\bar{F}_n(\bar{a}_n)},$$

which shows that (4.9) is actually gradient ascent with γ^ε chosen to suit an algorithm that takes only one step away from the initial guess, with the initial guess being $\bar{a}_n$.

5. Example application: Convenience yield in commodities markets. In commodities markets, the convenience yield is the portion of the future contract's yield that can be attributed to the benefit of direct access to the physical good. For instance, crude oil is a market with a convenience yield because sometimes it may be beneficial to buy and store crude, which results in a positive convenience yield. Sometimes it is costly to store oil, and there is incentive to sell at a lower price, which results in a negative convenience yield.

Consider a very simple commodities market, where there is trading in a single future contract for delivery of the good at time T. For times $t \in [0, T]$, let p_t denote the "fair" spot price, and let it satisfy the following SDE:

$$\frac{dp_t}{p_t} = \alpha(X_t)dt + \beta dW_t,$$

where α is the expected rate of return and $\beta > 0$ is the volatility factor. Now let $F_{t,T}$ be the future contract given by

$$F_{t,T} = \mathbb{E}^Q\{p_T | (p_u)_{u \leq t}\} = p_t \mathbb{E}^Q\{e^{\int_t^T \alpha^*(X_u)du} | \mathcal{F}_t\},$$

where X_t is the fast ergodic process and α^* is the risk-neutral drift of the spot price, with

$$\begin{aligned} &\mathbb{E}^Q \text{ being a risk-neutral measure,} \\ &\alpha^*(X_t) = \text{ risk-free rate} - \text{convenience yield (adjusted for the cost-of-carry).} \end{aligned}$$

The unobserved process θ_t is in the model to capture the state of the forward curve. In other words, is the market in backwardation or contango? For instance, in a market with relatively stable interest rates,

- the market is in contango if $F_{t,T} > \mathbb{E}\{p_T | \mathcal{F}_t\}$
  - buyers of crude pay a premium to lock-in prices early,
  - not much benefit for direct access,
  - low (perhaps negative) convenience yield,
- and the market is in backwardation if $F_{t,T} < \mathbb{E}\{p_T | \mathcal{F}_t\}$
  - oil producers offer incentive to lock in prices early,
  - there is some benefit to holding the physical good,
  - there is a positive convenience yield.

Such models would be useful in determining contango and backwardation, and such inference is relevant to how commodities portfolios are managed. In practice, the monthly rollover from expiring futures contracts into new contracts depends on the futures curve, and premia such as a convenience yield are key in determining a portfolio's rollover yield and how much it is *losing to contango*. Figures 1 and 2 show a time series of estimated convenience yields for Brent Crude oil. The figures suggest the possibility of time scales in the market, where reversion to the mean in the fast process X happens intradaily and switches in θ occur every few weeks. That is, daily sampling with $\Delta t = 1/252$ has sufficient coarse-graining to allow for relaxation of the fast state to its invariant distribution (i.e., $\varepsilon \ll \Delta t \ll \min_i(-1/q_{ii})$). Also in the figures, notice how the level of convenience yield undergoes regime change, which is evidence in support of the modeling choice of θ being a finite-state Markov chain.

Partial information comes into play because the fair spot price may not be observed (or even known to exist) prior to maturity of the future, but the future contract is highly liquid and is certainly observable. One explanation is that commodities are mined or produced before their delivery to meet demand. For instance, crude oil may need to be drilled and transported over an ocean in order to meet demand in a specific location. Therefore, futures can be considered as noisy observation on the hidden convenience yield(s), the filtration $\mathcal{F}_n$ will be generated by these observed futures, and then a filtering distribution can be calculated. In this case, the traded asset S_n could be taken to be a commodities portfolio, which generates the same filtration as the commodities futures from which it is composed as a linear combination.

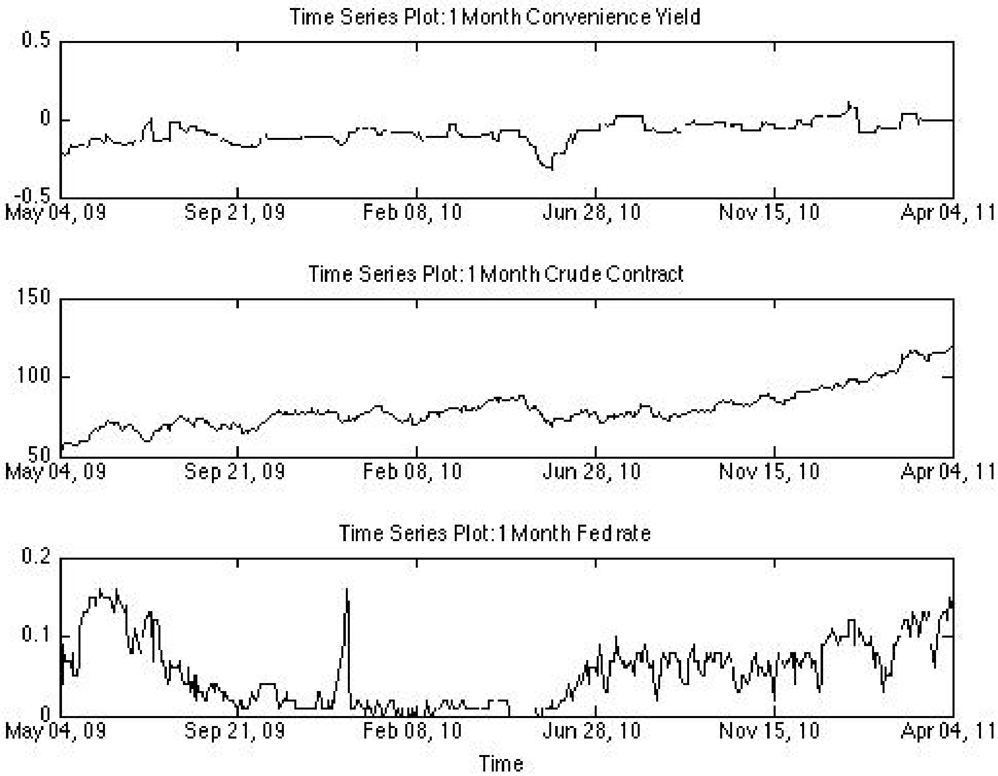

Figure 1. *Top: One month convenience yield;* $Y_{conv} = Y_{t\text{-}bill} - \frac{1}{T-t}\log(F_{t,T}/S_t)$. *Middle: One month future on Brent Crude. Bottom: One month constant maturity T-Bill.*

6. Numerical simulations of $(\overline{\mathcal{P}})$ with CARA utility.

In the beginning of section 4.3, it was shown how the portfolio value V_n does not affect the investor's risk aversion when there is CARA utility. Hence, the Bellman equation need only be solved for $v = 0$, and (4.5) simplifies to

$$H_n(s_n, 0, \mu) = \sup_{a_n \in \mathbf{a}_n} \sum_i \int e^{-\gamma a_n (y - s_n)} H_{n+1}\left(y,\ 0,\ \bar{\mathcal{E}}(y, s_n, \mu)\right) \bar{\Gamma}(dy, s_n)\mu(c_i). \tag{6.1}$$

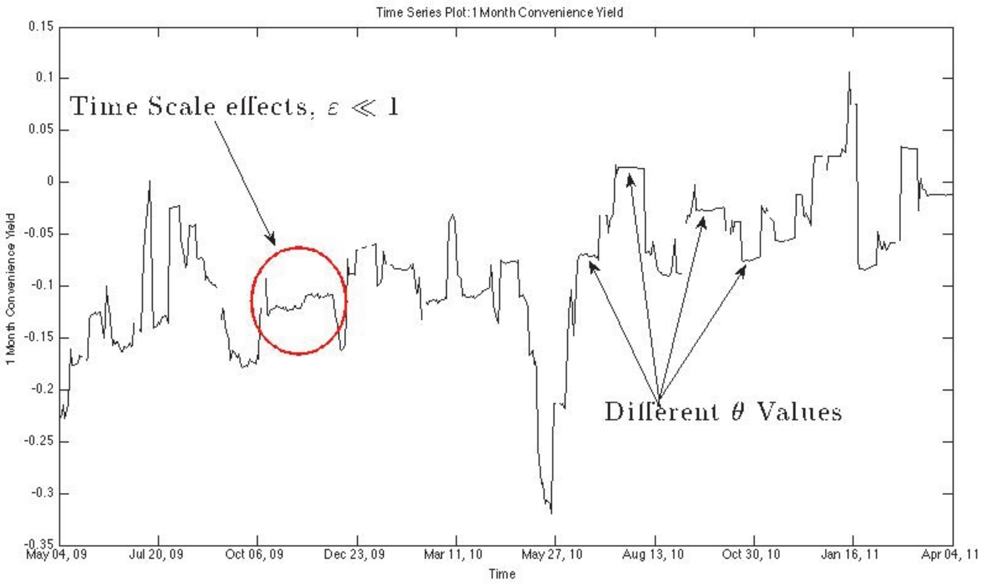

Figure 2. *A closer look at the convenience yield in the top plot of Figure 1. The red circle highlights small scale fluctuations in convenience yield, possibly correlated with short term volatility movements.*

The Bellman equation in (6.1) is computable at very little computational cost and will be at the center of the upcoming numerical experiments in this section. For algorithms to compute H_n, see Appendix E.

6.1. The example. The following example explores solutions to $(\overline{\mathcal{P}})$ with a two-state Markov chain $\theta_t \in \{c_1, c_2\}$ such that the model in (2.1) is specified as

$$d\log(S_t) = \left(\alpha(X_t) - \frac{1}{2}\beta^2\right) dt + \beta dB_t, \qquad S_0 = 50, \tag{6.2}$$

where $\beta = .2$ and the drift function averages to

$$\langle\alpha\rangle_{c_1} = \int_{\mathbb{R}} \alpha(x)\upsilon(dx, c_1) = .01, \tag{6.3}$$

$$\langle\alpha\rangle_{c_2} = \int_{\mathbb{R}} \alpha(x)\upsilon(dx, c_2) = .05, \tag{6.4}$$

and where υ is the density from (2.8). The transition intensity matrix of θ_t will be

$$Q = 10^{\xi} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{6.5}$$

where 10^{ξ} is a multiplier that can be adjusted to illustrate the effects of higher/lower transition rates. The chosen time horizon for investment will be three months. The assumptions regarding the time scales are that ε is small enough so that X_n models intraday effects that are averaged away on a daily basis, and so there is enough coarse-graining if observations are sampled daily, i.e., $\Delta t = 1/252 = 1$ day.

6.2. Using the continuous time results. Under full information and for small Δt, optimal strategies for the model of (6.2), (6.3), (6.4), and (6.5) can be well approximated with the analytic solutions from the continuous time theory [Hen02]. Basically, discrete time with full information amounts to a probabilistically based numerical scheme for solving the continuous time problem with full information (see [BW11]). In [Hen02] it is shown that the optimal strategy for $(\mathcal{P}^\varepsilon)$ is

$$a_t^{cont} \doteq \frac{\alpha(X_t) - r}{\gamma\beta^2}.$$

Under partial information, this strategy is unattainable because it depends on the unobserved X_t and would also be (at least slightly) suboptimal because $(\mathcal{P}^\varepsilon)$ is in discrete time. However, it might be a good starting point for a gradient ascent algorithm (see Appendix E). Appendix E has the numerical methods with a gradient ascent algorithm implemented at each backward step in the computation of a Bellman equation. The continuous time strategy can be used to get an initial approximation,

$$\bar{a}_n \simeq \frac{\langle\alpha\rangle_{\widehat{\theta}_n^{map}} - r}{\gamma\beta^2},$$

where $\widehat{\theta}_n^{map} \doteq \arg\max_{c_i} \bar{\pi}_n^0(c_i)$ is the maximum a posteriori estimator.

6.3. Indifference price of a European call option. Let $k \geq 0$ be the strike-price of a European call option. The holder of the option receives the amount $(S_N - k)^+$ at time N, and the writer/hedger of the option has an indifference price $\mathcal{C}_n \in \mathcal{F}_n$ such that

$$\begin{aligned}\bar{J}_n(0, S_{0:n}) = \operatorname*{ess\,sup}_{a\in\mathcal{A}_n} \bar{\mathbb{E}}\Bigg\{U\Bigg(\mathcal{C}_n(1+r\Delta t)^{N-n} + \sum_{\ell=n}^{N-1} a_\ell\left(\frac{\Delta S_{\ell+1}}{S_\ell} - r\Delta t\right) \\ - (S_N - k)^+\Bigg)\Bigg|\mathcal{F}_n \vee \{V_n = \mathcal{C}_n\}\Bigg\}\end{aligned}$$

for all $n \leq N$ (see [Car09a, Car09b, HH09, SZ04, MZ09]).[1]

In the example set forth in (6.2), (6.3), (6.4), and (6.5), the volatility function is constant in X so that $\beta(X_n) = \beta$ (i.e., known and constant volatility), and so simulated experiments will show that $\mathcal{C}_n$ can be very close to the Black–Scholes price, and the optimal strategy can be very close to the Black–Scholes delta hedge. Indeed, we let $a_n^{0,call}$ denote the optimal strategy

[1]It should be pointed out that the proof of Theorem 4.1 does not apply to indifference pricing, even for CARA utility. The reason is that the optimal portfolio allocation depends on S_n if the terminal wealth is short a call option. In order for the perturbation theory to be applicable, there needs to be a revised proof of Lemma B.3, the details of which are not discussed in this paper because Theorem 4.1 is not applied to indifference pricing; this paper looks only at indifference pricing in the unperturbed problem $(\overline{\mathcal{P}})$.

for the unperturbed problem of optimizing a portfolio that is short a call option. A good guess of the strategy for small Δt is

$$a_n^{0,call} \simeq S_n \Delta_{BS}(S_n, k),$$

where Δ_{BS} is the Black–Scholes delta and can also be used to initialize gradient ascent. For $\varepsilon \ll \Delta t \ll 1$ the complete market results from continuous time theory provide a good approximation, although the level of risk aversion will play a role in its accuracy.

The indifference prices for the European call option are listed in Tables 1–6. Notice in Tables 1, 2, and 3 that indifference prices for $\gamma = 1$ are somewhat similar for different ξ and different $\bar{\pi}_0(c_1)$. On the other hand, Tables 4, 5, and 6 show that for $\gamma = .01$ there is a premium for $\bar{\pi}_0(c_1)$ closer to zero, which is an indication that a hedger with low risk aversion likes to know when the return on the asset is certain to remain in the higher state, and he/she will charge more for the option in this case. This is an information premium.

6.4. The information premium.

Let $\mathcal{G}_n$ denote the set of admissible strategies under full information, and let $\bar{\mathcal{J}}_n(v, s, c_i)$ denote the maximum expected utility under full information,

$$\bar{\mathcal{J}}_n(v, s, c_i) = \operatorname*{ess\,sup}_{a \in \mathcal{G}_n} \bar{\mathbb{E}}\{U(V_N) | S_n = s, \theta_n = c_i, V_n = v\}.$$

The Bellman equations for the full-information problem have the form

$$\bar{\mathcal{J}}_n(v, s, \theta) = \operatorname*{ess\,sup}_{a_n \in \mathbf{g}_n} \bar{\mathbb{E}} \left\{ \bar{\mathcal{J}}_{n+1} \left(v(1 + r\Delta t) + a_n \left(\frac{\Delta S_{n+1}}{S_n} - r\Delta t \right), \theta_{n+1} \right) \Big| S_n = s, \theta_n = \theta \right\} \tag{6.6}$$

for $0 \leq n < N$, with $\bar{\mathcal{J}}_N(v, s, \theta) = U(v)$, and with $\mathbf{g}_n \subset \mathbb{R}$ being the set of admissible full-information portfolio allocations.

An important thing to notice is that the investor under partial information can expect the fully informed investor to have higher utility:

$$\begin{aligned}
\bar{J}_n(v, S_{0:n}) &= \operatorname*{ess\,sup}_{a \in \mathcal{A}_n} \bar{\mathbb{E}} \left\{ U \left(v(1 + r\Delta t)^{N-n} + \sum_{\ell=n}^{N-1} a_\ell \left(\frac{\Delta S_{\ell+1}}{S_\ell} - r\Delta t \right) \right) \Big| \mathcal{F}_n \vee \{V_n = v\} \right\} \\
&\leq \bar{\mathbb{E}} \left\{ \operatorname*{ess\,sup}_{a \in \mathcal{G}_n} \bar{\mathbb{E}} \left\{ U \left(v(1 + r\Delta t)^{N-n} + \sum_{\ell=n}^{N-1} a_\ell \left(\frac{\Delta S_{\ell+1}}{S_\ell} - r\Delta t \right) \right) \Big| S_n, \theta_n \right\} \Big| \mathcal{F}_n \vee \{V_n = v\} \right\} \\
&= \bar{\mathbb{E}} \left\{ \bar{\mathcal{J}}_n(v, S_n, \theta_n) | \mathcal{F}_n \vee \{V_n = v\} \right\} \\
&= \sum_{j=1}^{m} \bar{\mathcal{J}}_n(v, S_n, c_j) \bar{\pi}_n(c_j).
\end{aligned}$$

Thus, strategies under partial information result in less expected utility and on average will not perform as well as those under full information. This expected loss in utility due to partial information was quantified in [Bre06], and while it cannot easily be quantified for $(\overline{\mathcal{P}})$ with a general nonlinear framework, the numerical experiments suggest that there is a premium.

Table 1
The indifference price for a call option $\gamma = 1$, $\xi = 1$.

		$\bar{\pi}_0(c_1)$					
Strike	Black–Scholes	0	.2	.4	.6	.8	1
45	5.3562	5.3562	5.3560	5.3560	5.3560	5.3560	5.3560
50	1.9939	2.0024	2.0023	2.0022	2.0022	2.0022	2.0022
55	.4770	0.4824	0.4822	0.4822	0.4821	0.4821	0.4821

Table 2
The indifference price for a call option $\gamma = 1$, $\xi = -20$.

		$\bar{\pi}_0(c_1)$					
Strike	Black–Scholes	0	.2	.4	.6	.8	1
45	5.3562	5.3566	5.3560	5.3560	5.3560	5.3560	5.3560
50	1.9939	2.0030	2.0023	2.0022	2.0022	2.0022	2.0022
55	.4770	.4830	.4822	.4821	.4821	.4821	.4821

Table 3
The indifference price for a call option $\gamma = 1$, $\xi = -100$.

		$\bar{\pi}_0(c_1)$					
Strike	Black–Scholes	0	.2	.4	.6	.8	1
45	5.3562	5.3573	5.3560	5.3560	5.3560	5.3560	5.3560
50	1.9939	2.0038	2.0023	2.0022	2.0022	2.0022	2.0022
55	.4770	0.4840	0.4822	0.4821	0.4821	0.4821	0.4821

Table 4
The indifference price for a call option $\gamma = .01$, $\xi = 1$.

		$\bar{\pi}_0(c_1)$					
Strike	Black–Scholes	0	.2	.4	.6	.8	1
45	5.3562	5.3983	5.3841	5.3796	5.3778	5.3770	5.3763
50	1.9939	2.0311	2.0169	2.0124	2.0106	2.0099	2.0091
55	.4770	0.5191	0.5049	0.5004	0.4985	0.4978	0.4971

Table 5
The indifference price for a call option $\gamma = .01$, $\xi = -20$.

		$\bar{\pi}_0(c_1)$					
Strike	Black–Scholes	0	.2	.4	.6	.8	1
45	5.3562	5.4496	5.3843	5.3796	5.3778	5.3770	5.3761
50	1.9939	2.0824	2.0172	2.0124	2.0106	2.0099	2.0089
55	.4770	0.5704	0.5051	0.5004	0.4985	0.4978	0.4968

Table 6
The indifference price for a call option $\gamma = .01$, $\xi = -100$.

		$\bar{\pi}_0(c_1)$					
Strike	Black–Scholes	0	.2	.4	.6	.8	1
45	5.3562	5.5277	5.3843	5.3796	5.3778	5.3770	5.3761
50	1.9939	2.1605	2.0172	2.0124	2.0106	2.0099	2.0089
55	.4770	0.6484	0.5051	0.5004	0.4985	0.4978	0.4968

The presence of the information premium can be deduced by looking at the certainty equivalents for various filter values. The certainty equivalent is the amount of cash CE such that

$$CE(\bar{\pi}_0) = U^{-1}(H_0(S_0, 0, \bar{\pi}_0)) = -\frac{1}{\gamma}\log(-\gamma H_0(S_0, 0, \bar{\pi}_0)),$$

and so the price of information on the possibility of the higher growth state can be quantified by looking at the ratio of the CE's,

$$IP(\bar{\pi}_0) \doteq \frac{CE(\bar{\pi}_0)}{CE((1,0))} = \frac{\log(-\gamma H_0(S_0, 0, \bar{\pi}_0))}{\log(-\gamma H_0(S_0, 0, (1,0)))},$$

where the quantity IP represents the information premium. The denominator of IP is the logarithm of optimal expected utility given the full information that θ starts in the low-growth state (i.e., $\theta_0 = c_1$ or $\bar{\pi}_0 = (\bar{\pi}_0(c_1), \bar{\pi}_0(c_2)) = (1, 0)$), which means that IP is a denomination of wealth over the low-growth state. In other words, for every dollar that an investor would pay for the portfolio when he/she knows that $\theta_0 = c_1$, he/she would pay $IP(\bar{\pi}_0)$ dollars for the portfolio with another $\bar{\pi}_0$.

Figure 3 is a plot of IP for various values of $\bar{\pi}_0(c_1)$ and ξ, and it appears that investors will pay a premium for $\xi \ll 0$ (i.e., for $Q \approx 0$) and $\bar{\pi}_n^0(c_1) \approx 0$. In particular, partial information can have a certainty equivalent that is as much as .04 dollars higher than that of being fully informed that the market is in the low-growth state. Furthermore, there appears to be a jump of about .02 dollars as $\bar{\pi}_0$ tends toward $(0, 1)$, indicating that investors have a higher certainty equivalent if they know that the asset is in its high-growth state.

Section 6.3 pointed out an information premium in the indifference prices of Tables 1–6, which indicated that less risk-averse hedgers put a premium on information when the intensity matrix of the Markov chain was $Q \approx 0$. Comparing the premium in those tables to the premium seen in Figure 3, there appears to be an agreement in how the investors view the optimal portfolio for various $\bar{\pi}_0$ and Q. It might appear that there was a greater premium in the tables of indifference prices, but that is only because indifference prices were shown with very low risk aversion levels. Comparing only Figure 3 with only the indifference prices that took $\gamma = 1$, it can be seen that the premium is more pronounced for the portfolio problem than it was for the indifference prices, and that is because the growth function $\alpha(x)$ is more significant to the former; the unobserved drift function $\alpha(X_t)$ becomes insignificant for the hedging of European options as $\Delta t \to 0$.

In summary, Tables 1–6 and Figure 3 illustrate the premium that will be paid for knowing the initial state of θ. The investor is willing to pay a premium for a low probability of the low-growth state occurring during the period of investment.

7. Conclusions. This paper has proposed an approximate dynamic program (ADP) for a portfolio optimization problem in a market with partial information and discretely observed prices. The problem is difficult to solve due to a curse of dimensionality associated with its non-Markovian structure, but there is a good way to approximate the problem for the model in (1.2). In particular, there is an ergodic theory for the model that can be used to show that the original optimization problem can be well approximated by a limiting problem that is of reduced dimension. In Theorem 4.1, the implicit function theorem is used to show that the

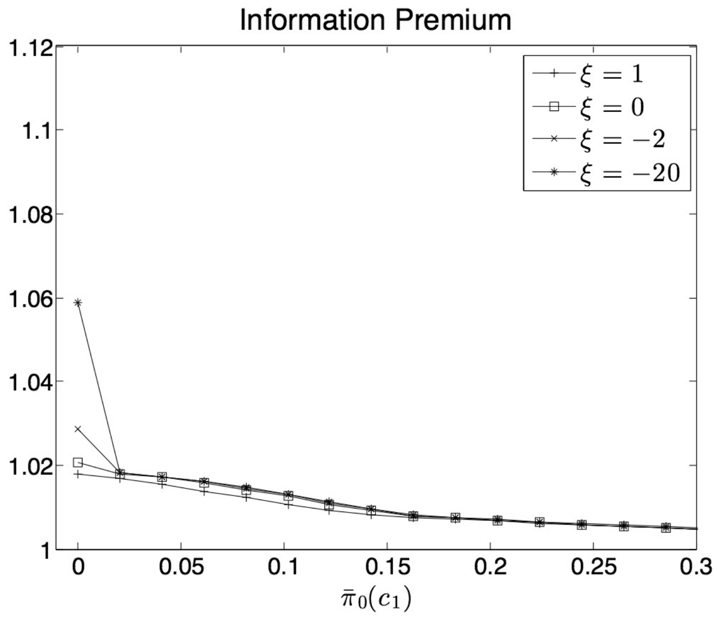

Figure 3. *The information premium which is defined as the ratio of expected returns on the portfolio maximization problems* $IP(\bar{\pi}_0) = \log(-\gamma H_0(S_0, 0, \bar{\pi}_0)) / \log(-\gamma H_0(S_0, 0, (1, 0)))$. *The plot shows this premium for the example described in* (6.2), (6.3), (6.4), *and* (6.5), *with the parameters* $\gamma = 1$, $\Delta t = 1/252$, *and* $N = 63$.

limiting problem has an optimal strategy that is "close" to that of the original problem, and that there is also a small ε correction term to the strategy. Approximation by the limiting problem with a correction term is the ADP, and it is of reduced dimension because the limiting problem is considerably easier to solve, namely because the hidden state is a finite dimensional Markov chain and a Bellman equation can be written for such problems. In numerical studies it came out that information can come at a premium in financial markets.

Possible directions in which to continue this work include more analysis of the crude oil data and convenience yields, generalizations of Theorem 4.1 for other utility functions, and further numerical studies of the information premia in other incomplete markets. There is also growing interest in volatility uncertainty, and so it could be worthwhile to extend the results of this paper to see how pricing models such as Heston or Stein–Stein change with filtering.

Appendix A. Expansion proofs.

A.1. Proof of Proposition 2.1.

Proof. The limits in (2.10) and (2.12) follow from the ergodic theory of the Ornstein–Uhlenbeck process (see [FPSS11]). To show (2.11), start by defining the following:

$$Z_t \doteq \rho \int_{t_{n-1}}^{t} \beta(X_u) dB_u \qquad \forall t \geq t_{n-1},$$

from which it is clear that $\mathcal{Z}_n(X,\theta) = Z_{t_n}$. The limit in (2.11) will follow via weak convergence arguments of [EK86], namely, through tightness of measures on $Z.$ and convergence of a martingale problem. The proof will also rely on the expansion techniques of [FPSS11].

Tightness of the measures on $Z.$ follows because it is a stochastic integral with bounded integrand; hence the Kolmogorov continuity criterion applies. Then, for any bounded continuous function f there is the following backward equation:

$$\left(\frac{1}{\varepsilon}L_0^i + \frac{\rho\beta(x)}{\sqrt{\varepsilon}}\frac{\partial^2}{\partial z\partial x} + \frac{\partial}{\partial t} + \frac{\rho^2\beta^2(x)}{2}\frac{\partial^2}{\partial z^2} + Q\right)\mathbb{E}^\varepsilon\{f(Z_T)|Z_t = z, X_t = x, \theta_t = c_i\} = 0$$

for all $t \in [t_{n-1}, T)$. Following the methodology of [FPSS11], solutions to this backward equation are expanded in powers of $\sqrt{\varepsilon}$, and then formulas for each term in the expansion are sequentially identified. Denote the base term in this expansion as $\bar{\mathbb{E}}\{f(Z_T)|Z_t = z, X_t = x, \theta_t = c_i\}$ and the first two corrections as $\Phi^{(1)}$ and $\Phi^{(2)}$, respectively, and then write the expansion as

$$\begin{aligned}&\mathbb{E}^\varepsilon\{f(Z_T)|Z_t = z, X_t = x, \theta_t = c_i\}\\ &= \bar{\mathbb{E}}\{f(Z_T)|Z_t = z, X_t = x, \theta_t = c_i\} + \sqrt{\varepsilon}\Phi^{(1)}(t,z,x,c_i) + \varepsilon\Phi^{(2)}(t,z,x,c_i) + o(\varepsilon),\end{aligned}$$

from which it follows that

$$\begin{aligned}&L_0^i\bar{\mathbb{E}}\{f(Z_T)|Z_t = z, X_t = x, \theta_t = c_i\} = 0,\\ &L_0^i\Phi^{(1)}(t,z,x,c_i) = 0.\end{aligned}$$

Regularity of the expansion requires that the first two terms of the expansion be constant in x so that

$$\begin{aligned}\bar{\mathbb{E}}\{f(Z_T)|Z_t = z, X_t = x, \theta_t = c_i\} &= \bar{\mathbb{E}}\{f(Z_T)|Z_t = z, \theta_t = c_i\},\\ \Phi^{(1)}(t,z,x,c_i) &= \Phi^{(1)}(t,z,c_i).\end{aligned}$$

Then, using the order-ε term it follows that

$$\left(\frac{\partial}{\partial t} + \frac{\rho^2\beta^2(x)}{2}\frac{\partial^2}{\partial z^2} + Q\right)\bar{\mathbb{E}}\{f(Z_T)|Z_t = z, \theta_t = c_i\} = -L_0^i\Phi^{(2)}(t,z,x,c_i),$$

and from the Fredholm alternative (see [FPSS11]) there is a solution iff the base term satisfies

$$\left(\frac{\partial}{\partial t} + \frac{\rho^2\left\langle\beta^2\right\rangle_{c_i}}{2}\frac{\partial^2}{\partial z^2} + Q\right)\bar{\mathbb{E}}\{f(Z_T)|Z_t = z, \theta_t = c_i\} = 0.$$

Furthermore, this base term is the limit

$$\mathbb{E}^\varepsilon\{f(Z_T)|Z_t = z, X_t = x, \theta_t = c_i\} \to \bar{\mathbb{E}}\{f(Z_T)|Z_t = z, \theta_t = c_i\} \qquad \text{as } \varepsilon \to 0,$$

and this implies that $Z. \Rightarrow \rho\int_{t_{n-1}}^{\cdot}\sqrt{\langle\beta^2\rangle_{\theta_u}}\,dB_u$ as $\varepsilon \to 0$, which proves the limit in (2.11). ■

A.2. Proof of Proposition 2.2.

Proof. The proof is a straightforward computation:

$$
\begin{aligned}
&\int \Gamma^\varepsilon(s_n, s_{n-1})\mu(x', c_j)dx' \\
&= \sum_i \int\int \frac{\partial^2}{\partial x' \partial s_n}\mathbb{P}^\varepsilon(S_n \le s_n, X_n \le x', \theta_n = c_j | S_{n-1} = s_{n-1}, X_{n-1} = x, \theta_{n-1} = c_i)\mu(x, c_i)dxdx' \\
&= \sum_i \int \frac{1}{s_n\sqrt{2\pi}}\mathbb{E}^\varepsilon\left\{\frac{\mathbb{1}_{[\theta_n = c_j]}}{\varsigma_n(X,\theta)}\exp\left\{-\frac{1}{2}\left(\frac{\log(s_n/s_{n-1}) - I_n(X,\theta) - \mathcal{Z}_n(X,\theta)}{\varsigma_n(X,\theta)}\right)^2\right\}\Big|\theta_{n-1} = c_i,\right. \\
&\qquad\qquad \left. X_{n-1} = x\right\}\mu(dx, c_i) \\
&\overset{\varepsilon\to 0}{\longrightarrow} \sum_i \frac{1}{s_n\sqrt{2\pi}}\bar{\mathbb{E}}\left\{\frac{\mathbb{1}_{[\theta_n = c_j]}}{\bar\varsigma_n(\theta)}\exp\left\{-\frac{1}{2}\left(\frac{\log(s_n/s_{n-1}) - \bar I_n(\theta) - \overline{\mathcal{Z}}_n(\theta)}{\bar\varsigma_n(\theta)}\right)^2\right\}\Big|\theta_{n-1} = c_i\right\}\int \mu(dx, c_i),
\end{aligned}
$$

where uniform boundedness of α and β (and from below for β) allows for the limit to be obtained through weak convergence. Now, the expression in this limit is the convolution of two independent Gaussian densities, and so this limit can be rewritten as (2.13). ∎

A.3. Proof of Proposition 3.2.

Proof. Without loss of generality, we can take $g(c_i) = \mathbb{1}_{[c_i]}$ in (3.3). Suppose the expansion in (3.3) is valid at time $n-1$, so that

$$
\int \pi^\varepsilon_{n-1}(dx, c_i) = \bar\pi_{n-1}(c_i) + \sqrt{\varepsilon}\bar\pi'_{n-1}(c_i) + o(\sqrt{\varepsilon}).
$$

Using the expansion of (2.25), the proof is an application of the chain rule,

$$
\begin{aligned}
&\bar\pi'_n(c_i) \\
&= \lim_{\varepsilon\searrow 0}\frac{1}{\sqrt{\varepsilon}}\left(\int \pi^\varepsilon_n(x, c_i)dx - \bar\pi_n(c_i)\right) \\
&= \lim_{\varepsilon\searrow 0}\frac{1}{\sqrt{\varepsilon}}\left(\frac{\int \Gamma^\varepsilon(s_n, s_{n-1})\pi^\varepsilon_{n-1}(dx, c_i)}{\sum_\ell \int \Gamma^\varepsilon(s_n, s_{n-1})\pi^\varepsilon_{n-1}(dx', c_\ell)} - \frac{\overline{\Gamma}(s_n, s_{n-1})\bar\pi_{n-1}(c_i)}{\sum_\ell \overline{\Gamma}(s_n, s_{n-1}n)\bar\pi_{n-1}(c_\ell)}\right) \\
&= \lim_{\varepsilon\searrow 0}\frac{1}{\sqrt{\varepsilon}}\left(\frac{\bar\Gamma(s_n, s_{n-1})\int \pi^\varepsilon_{n-1}(dx, c_i) + \sqrt{\varepsilon}\sum_j u'_n(t_{n-1}, s_n, c_i|s_{n-1}, c_j)\int \pi^\varepsilon_{n-1}(dx, c_j) + o(\sqrt{\varepsilon})}{\sum_\ell\left(\bar\Gamma(s_n, s_{n-1})\int \pi^\varepsilon_{n-1}(dx, c_\ell) + \sqrt{\varepsilon}\sum_j u'_n(t_{n-1}, s_n, c_\ell|s_{n-1}, c_j)\int \pi^\varepsilon_{n-1}(dx, c_j) + o(\sqrt{\varepsilon})\right)}\right. \\
&\qquad\qquad \left. - \frac{\overline{\Gamma}(s_n, s_{n-1})\bar\pi_{n-1}(c_i)}{\sum_\ell \overline{\Gamma}(s_n, s_{n-1}n)\bar\pi_{n-1}(c_\ell)}\right) \\
&= \frac{\bar\Gamma(s_n, s_{n-1})\bar\pi'_{n-1}(c_i) + \sum_j u'_n(t_{n-1}, s_n, c_i|s_{n-1}, c_j)\bar\pi_{n-1}(c_j)}{\sum_\ell \bar\Gamma(s_n, s_{n-1})\bar\pi_{n-1}(c_\ell)} \\
&\quad - \bar\pi_n(c_i)\frac{\sum_\ell\left(\bar\Gamma(s_n, s_{n-1})\bar\pi'_{n-1}(c_\ell) + \sum_j u'_n(t_{n-1}, s_n, c_\ell|s_{n-1}, c_j)\bar\pi_{n-1}(c_j)\right)}{\sum_\ell \bar\Gamma(s_n, s_{n-1})\bar\pi_{n-1}(c_\ell)}. \qquad \blacksquare
\end{aligned}
$$

Appendix B. Preliminaries for the proof of Theorem 4.1.

B.1. Properties of the objective function. This appendix contains some technical results which are used to invoke the implicit function theorem. The first result is strict concavity of both ($\mathcal{P}^\varepsilon$) and ($\overline{\mathcal{P}}$), which is sufficient to conclude that the optimal portfolio can be obtained through first-order conditions; it should also be mentioned that concavity of the problem holds due to concavity of U and not due to the choice of CARA utility. The second result shows that ($\overline{\mathcal{P}}$) is in fact uniformly strictly concave locally for all a_n in a neighborhood of the optimum $\bar{a}_n$, which is shown here in the case of CARA utility. Uniform strict concavity near the optimum also holds for power and logarithmic utility, but showing it involves a more complicated proof (Appendix D alludes to these further technicalities). Finally, this subsection contains a regularity result that is necessary to obtain the order-$\sqrt{\varepsilon}$ correction to the expansion of a_n^ε.

The following proposition ensures that the first-order condition in (4.7) and (4.8) yields the unique solution to the problem ($\mathcal{P}^\varepsilon$) or ($\overline{\mathcal{P}}$).

Proposition B.1. *For all $\varepsilon \geq 0$, the first derivative of F_n^ε and the second derivative of the objective function are strictly negative,*

$$\frac{\partial}{\partial a} F_n^\varepsilon(a) < 0 \quad \textit{for } 0 \leq n < N. \tag{B.1}$$

Proof. Without loss of generality and for ease in notation, the proposition is proven for $r = 0$. Then notice that

$$\begin{aligned}
\frac{\partial}{\partial a} F_n^\varepsilon(a) &= \frac{\partial^2}{\partial a^2} \mathbb{E}^\varepsilon \left\{ e^{-\gamma a \frac{\Delta S_{n+1}}{S_n}} J_{n+1}^\varepsilon \left(S_{0:n+1}\right) \middle| \mathcal{F}_n \right\} \\
&= \frac{\partial^2}{\partial a^2} \mathbb{E}^\varepsilon \left\{ U \left(a \frac{\Delta S_{n+1}}{S_n} + \sum_{\ell=n+1}^{N} a_\ell^\varepsilon \frac{\Delta S_{\ell+1}}{S_\ell} \right) \middle| \mathcal{F}_n \right\} \\
&= \mathbb{E}^\varepsilon \left\{ U'' \left(a \frac{\Delta S_{n+1}}{S_n} + \sum_{\ell=n+1}^{N} a_\ell^\varepsilon \frac{\Delta S_{\ell+1}}{S_\ell} \right) \left(\frac{\Delta S_{n+1}}{S_n} \right)^2 \middle| \mathcal{F}_n \right\} \\
&< 0
\end{aligned}$$

for all $n \leq N$ because the utility $U(\cdot)$ is assumed to be strictly concave with $U'' < 0$. ∎

B.1.1. Uniform bound. Proposition B.1 would be sufficient for solving ($\overline{\mathcal{P}}$) at a single time t_n, but the backward iterations for dynamic programming will require the objective function to take expectations over future solutions. This could pose a problem because upper bounds on $\frac{\partial}{\partial a} \bar{F}_n(a)$ at future times are random variables, which means there can be instabilities. However, a uniform upper bound on the second derivative will eliminate any such instability.

Recall the Markovian formulation's value function,

$$H_n(s_n, 0, \bar{\pi}_n) = \bar{J}_n(s_{0:n}).$$

The first step in showing uniformity is to prove that H_n is constant in s_n.

Proposition B.2. *The optimal amount of wealth invested in the risky asset does not depend on the price but only on the parameters of the returns. Hence, $H_n(s_n, 0, \bar{\pi})$ is constant in s_n.*

Proof. Without loss of generality and for ease in notation, the proposition is proven for $r = 0$. Obviously, the claim of the proposition is true for $n = N$ because $H_N(s_n, 0, \mu) = U(0)$. Now, suppose that H_{n+1} is constant in s_{n+1} so that $H_{n+1}(s_{n+1}, 0, \mu) = H_{n+1}(0, \mu)$. It follows from (2.14) that $\bar{\Gamma}(y, s_n) = \frac{1}{s_n}\bar{\Gamma}(\frac{y}{s_n}, 1)$ for all $y, s_n \in \mathbb{R}^+$, and from (3.2) that $\bar{\mathcal{E}}(y, s_n, \mu) = \bar{\mathcal{E}}(\frac{y}{s_n}, 1, \mu)$ for all $\mu \in \mathcal{M}$. From here the result follows inductively:

$$\begin{aligned}
H_n(s_n, 0, \mu) &= \sup_{a_n \in \mathbf{a_n}} \sum_i \int e^{-\gamma a_n \frac{y - s_n}{s_n}} H_{n+1}\left(0,\ \bar{\mathcal{E}}(y, s_n, \mu)\right) \bar{\Gamma}(y, s_n)\mu(c_i)dy \\
&= \sup_{a_n \in \mathbf{a_n}} \sum_i \int e^{-\gamma a_n \left(\frac{y}{s_n} - 1\right)} H_{n+1}\left(0,\ \bar{\mathcal{E}}\left(\frac{y}{s_n}, 1, \mu\right)\right) \frac{1}{s_n}\bar{\Gamma}\left(\frac{y}{s_n}, 1\right)\mu(c_i)dy \\
&= \sup_{a_n \in \mathbf{a_n}} \sum_i \int e^{-\gamma a_n (y' - 1)} H_{n+1}\left(0,\ \bar{\mathcal{E}}\left(y', 1, \mu\right)\right) \bar{\Gamma}\left(y', 1\right)\mu(c_i)dy',
\end{aligned}$$

which is constant in s_n, and hence $H_n(s_n, 0, \mu) = H_n(0, \mu)$. ■

The proof of Proposition B.2 differs significantly for power utility, but the author highly suspects it to be true for this case as well. Indeed, Proposition B.2 is sufficient and instrumental in proving the theorem of section 4.3, and it is worth noting that Proposition B.2 is an instance where the Markovian formulation plays a role in the theory. In fact, the Markovian formulation is instrumental in proving the following lemma for uniformly strict concavity.

Lemma B.3. *There is a constant $\mathcal{K} < 0$ such that*

$$\frac{\partial}{\partial a}\bar{F}_n(a) \leq \mathcal{K} < 0$$

for all $n \leq N$, locally for all $a \in nbh(\bar{a}_n)$ (a in a neighborhood of $\bar{a}_n$), and that

$$\frac{\partial}{\partial a}\bar{F}_n(\bar{a}_n) \leq \mathcal{K} < 0$$

for almost every observation sequence $s_{0:n}$ with positive entries.

Proof. Without loss of generality and for ease in notation, the proposition is proven for $r = 0$. It follows from Proposition B.1 that

$$\frac{\partial}{\partial a}\bar{F}_n(a) = \frac{\partial^2}{\partial a^2}\sum_i \int e^{-\gamma a(y'-1)} H_{n+1}\left(0,\ \bar{\mathcal{E}}\left(y', 1, \mu\right)\right) \bar{\Gamma}\left(y', 1\right)\mu(c_i)dy' < 0$$

for all $\mu \in \mathcal{M}$. It should be clear from this equation that μ is a sufficient statistic for the distribution (also mentioned prior to Theorem 4.1), and so the optimal strategy can be written as a function of it, $\bar{a}_n = \bar{a}_n(\mu)$. But $\mathcal{M}$ is the lattice of probability distributions on the finite set Θ and is therefore a compact set. Using the deterministic perturbation theory of [BS96, Rob80] it can be verified that $\frac{\partial}{\partial a}\bar{F}_n(a)|_{a=\bar{a}_n(\mu)}$ is continuous in the vector μ and $\frac{\partial}{\partial a}\bar{F}_n(a)$ is continuous in the scalar a, and so $\frac{\partial}{\partial a}\bar{F}_n(a)$ achieves its maximum for some $\mu' \in \mathcal{M}$ at a point $a' \in nbh(\bar{a}_n(\mu'))$. Call this maximum $\mathcal{K}_n$. It then follows that

$$\max_{a \in nbh(\bar{a}_n(\bar{\pi}_n))} \frac{\partial}{\partial a}\bar{F}_n(a) \leq \frac{\partial}{\partial a}\bar{F}_n(a') \leq \max_{n' \leq N}\mathcal{K}_{n'} \doteq \mathcal{K} < 0 \qquad \forall n \leq N,$$

for almost every $s_{0:n}$ with positive entries, which proves the lemma. ■

B.1.2. The $\sqrt{\varepsilon}$-regularity of F_n^ε. The $\sqrt{\varepsilon}$-correction to the optimal strategy requires there to be regularity of $F_n^\varepsilon(a)$ in ε. In other words, there needs to be verification of existence for

$$\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(a)\Big|_{\varepsilon=0} \doteq \lim_{\varepsilon\searrow 0}\frac{F_n^\varepsilon(a)-\bar{F}_n(a)}{\sqrt{\varepsilon}} \qquad \forall a\in\mathbf{a_n},$$

where it should be pointed out that $\mathbf{a_n}$ has no ε-dependence. This $\sqrt{\varepsilon}$-derivative can also be thought of as the correction in the $\sqrt{\varepsilon}$-expansion

$$F_n^\varepsilon(a) = \bar{F}_n(a) + \sqrt{\varepsilon}\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(a)\Big|_{\varepsilon=0} + o(\sqrt{\varepsilon}) \qquad \forall a\in\mathbf{a_n}.$$

Theorem 4.1 uses $\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(a)|_{\varepsilon=0}$ to get regularity of the optimal strategy a_n^ε, that is,

$$\frac{\partial}{\partial\sqrt{\varepsilon}}a_n^\varepsilon\Big|_{\varepsilon=0} = \lim_{\varepsilon\searrow 0}\frac{a_n^\varepsilon-\bar{a}_n}{\sqrt{\varepsilon}},$$

where existence of the derivative $\frac{\partial}{\partial\sqrt{\varepsilon}}a_n^\varepsilon|_{\varepsilon=0}$ depends on $\sqrt{\varepsilon}$-regularity of F_n^ε. The following proposition contains the necessary analysis of $\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(a)|_{\varepsilon=0}$ at time n and uses an inductive argument based on the assumption that $(\frac{\partial}{\partial\sqrt{\varepsilon}}a_\ell^\varepsilon|_{\varepsilon=0})_{\ell\geq n+1}$ exist and are square-integrable.

Proposition B.4. *Assume that $\frac{\partial}{\partial\sqrt{\varepsilon}}a_\ell^\varepsilon|_{\varepsilon=0}$ exist and are square-integrable for all $\ell>n$:*

$$\bar{\mathbb{E}}\left\{\left(\sum_{\ell=n+1}^{N-1}\frac{\partial}{\partial\sqrt{\varepsilon}}a_\ell^\varepsilon\right)^2\Big|_{\varepsilon=0}\Big|S_{0:n}=s_{0:n},V_n=v\right\}<\infty. \tag{B.2}$$

Then $\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(a)$ exists for all $a\in\mathbb{R}$ and is given by

$$\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(a)\Big|_{\varepsilon=0} = \sum_i\bar{\pi}_n(c_i)\psi_n'(t_n,s_n|s_n,c_i)+\sum_i\psi_n^0(t_n,s_n|s_n,c_i)\pi_n'(c_i), \tag{B.3}$$

with π_n' being the $\sqrt{\varepsilon}$-derivative given by Proposition 3.2 of section 3, with ψ_n^0 satisfying

$$\begin{aligned}&\langle L_2\rangle\,\psi_n^0=0,\\ &\psi_n^0\Big|_{t=t_{n+1}}=\frac{y-\tilde{y}}{\tilde{y}}\bar{\mathbb{E}}\left\{U'\left(v+a\frac{y-\tilde{y}}{\tilde{y}}+\sum_{\ell=n+1}^{N-1}\bar{a}_\ell\frac{\Delta S_{\ell+1}}{S_\ell}\right)\Big|S_{n+1}=y,\theta_{n+1}=c_i,S_{0:n}=s_{0:n}\right\},\end{aligned}$$

where $\langle L_2\rangle$ is the same operator from Proposition 2.3 of section 2.2, and with ψ_n' satisfying a Poisson equation

$$\langle L_2\rangle\,\psi_n' = \left\langle L_1\phi^{(1)}\right\rangle y^2\frac{\partial^2}{\partial y^2}\psi_n^0+\left\langle L_1\phi^{(2)}\right\rangle y\frac{\partial}{\partial y}\psi_n^0, \tag{B.4}$$

where $\phi^{(1)}$ and $\phi^{(2)}$ are those given in (2.20) and (2.21) of section 2.2. In fact, ψ_n' is given explicitly as $\psi_n'=(t_{n+1}-t)(\langle L_1\phi^{(1)}\rangle y^2\frac{\partial^2}{\partial y^2}\psi_n^0+\langle L_1\phi^{(2)}\rangle y\frac{\partial}{\partial y}\psi_n^0)$.

Proof. For simplicity and without loss of generality, the results are proven for $r=0$. The $\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(a)$ should look as follows:

(B.5)

$$\begin{aligned}
&\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(a)\Big|_{\varepsilon=0}\\
&=\frac{\partial}{\partial\sqrt{\varepsilon}}\mathbb{E}^\varepsilon\left\{U'\left(v+a\frac{\Delta S_{n+1}}{S_n}+\sum_{\ell=n+1}^{N-1}\bar{a}_\ell\frac{\Delta S_{\ell+1}}{S_\ell}\right)\frac{\Delta S_{n+1}}{S_n}\Bigg|S_{0:n}=s_{0:n},V_n=v\right\}\Big|_{\varepsilon=0}\\
&+\bar{\mathbb{E}}\left\{U''\left(v+a\frac{\Delta S_{n+1}}{S_n}+\sum_{\ell=n+1}^{N-1}\bar{a}_\ell\frac{\Delta S_{\ell+1}}{S_\ell}\right)\frac{\Delta S_{n+1}}{S_n}\sum_{\ell=n+1}^{N-1}\frac{\Delta S_{\ell+1}}{S_\ell}\frac{\partial}{\partial\sqrt{\varepsilon}}a_\ell^\varepsilon\Big|_{\varepsilon=0}\Bigg|S_{0:n}=s_{0:n},V_n=v\right\}.
\end{aligned}$$

Since it was assumed that $\frac{\partial}{\partial\sqrt{\varepsilon}}a_\ell^\varepsilon|_{\varepsilon=0}$ exist and are square integrable for all $\ell>n$, the validity of (B.5) is confirmed by showing existence of the first term on the right-hand side of (B.5).

To show existence, start by defining the following full-information function:

$$\begin{aligned}
\psi_n^\varepsilon(t,\tilde{y}|y,x,c_i)\doteq\mathbb{E}^\varepsilon\Bigg\{&U'\left(v+a\frac{S_{n+1}-\tilde{y}}{\tilde{y}}+\sum_{\ell=n+1}^{N-1}\bar{a}_\ell\frac{\Delta S_{\ell+1}}{S_\ell}\right)\frac{S_{n+1}-\tilde{y}}{\tilde{y}}\Bigg|S_t=y,X_t=x,\\
&\theta_t=c_i,S_{0:n}=s_{0:n}\Bigg\}
\end{aligned}$$

for $t\in[t_n,t_{n+1}]$, for any $\tilde{y},y\in\mathbb{R}^+$, $x\in\mathbb{R}$, and c_i in θ's state space. The function ψ_n^ε is the solution to a backward PDE-like equation (2.6):

$$\begin{aligned}
&\left(\frac{1}{\varepsilon}L_0+\frac{1}{\sqrt{\varepsilon}}L_1+L_2\right)\psi_n^\varepsilon=0,\\
&\psi_n^\varepsilon\Big|_{t=t_{n+1}}=\frac{y-\tilde{y}}{\tilde{y}}\mathbb{E}^\varepsilon\Bigg\{U'\left(v+a\frac{y-\tilde{y}}{\tilde{y}}+\sum_{\ell=n+1}^{N-1}\bar{a}_\ell\frac{\Delta S_{\ell+1}}{S_\ell}\right)\Bigg|S_{n+1}=y,X_{n+1}=x,\\
&\qquad\theta_{n+1}=c_i,S_{0:n}=s_{0:n}\Bigg\}.
\end{aligned}$$

The solution to this PDE can be expanded in powers of $\sqrt{\varepsilon}$,

$$\begin{aligned}
\psi_n^\varepsilon(t,\tilde{y}|y,x,c_i)=\psi_n^0(t,\tilde{y}|y,x,c_i)&+\sqrt{\varepsilon}\psi_n'(t,\tilde{y}|y,x,c_i)+\varepsilon\psi_n''(t,\tilde{y}|y,x,c_i)\\
&+\varepsilon^{3/2}\psi_n'''(t,\tilde{y}|y,x,c_i)+o(\varepsilon^{3/2}),
\end{aligned}$$

where the order-$\sqrt{\varepsilon}$ term is the first derivative, $\psi_n'=\frac{\partial}{\partial\sqrt{\varepsilon}}\psi_n^\varepsilon|_{\varepsilon=0}$. Repeating the methodology in Chapter 4 of [FPSS11] (and as done in section 2.2), the base term ψ_n^0 is shown to be constant in x and to satisfy

$$\begin{aligned}
&\langle L_2\rangle\,\psi_n^0=0,\\
&\psi_n^0\Big|_{t=t_{n+1}}=\frac{y-\tilde{y}}{\tilde{y}}\bar{\mathbb{E}}\left\{U'\left(v+a\frac{y-\tilde{y}}{\tilde{y}}+\sum_{\ell=n+1}^{N-1}\bar{a}_\ell\frac{\Delta S_{\ell+1}}{S_\ell}\right)\Bigg|S_{n+1}=y,\theta_{n+1}=c_i,S_{0:n}=s_{0:n}\right\},
\end{aligned}$$

where $\langle L_2\rangle$ is the same operator from Proposition 2.3 of section 2.2. Also, the steps taken in Proposition 2.4 can be repeated to find that ψ_n' is constant on x and is the solution of the Poisson equation (B.4). Furthermore, the same commutativity arguments used to obtain (2.24) can be used to obtain the explicit expression $\psi_n' = (t_{n+1}-t)(\langle L_1\phi^{(1)}\rangle y^2\frac{\partial^2}{\partial y^2}\psi_n^0 + \langle L_1\phi^{(2)}\rangle y\frac{\partial}{\partial y}\psi_n^0)$. After computing ψ_n^0 and ψ_n', the chain rule can be applied to get the $\sqrt{\varepsilon}$-derivative as two parts,

$$\begin{aligned}
&\frac{\partial}{\partial\sqrt{\varepsilon}}\mathbb{E}^\varepsilon\left\{U'\left(v + a\frac{\Delta S_{n+1}}{S_n} + \sum_{\ell=n+1}^{N-1}\bar{a}_\ell\frac{\Delta S_{\ell+1}}{S_\ell}\right)\frac{\Delta S_{n+1}}{S_n}\middle| S_{0:n}=s_{0:n}, V_n = v\right\}\Big|_{\varepsilon=0}\\
&= \frac{\partial}{\partial\sqrt{\varepsilon}}\sum_i\int\psi_n^\varepsilon(t_n,s_n|s_n,x,c_i)\pi_n^\varepsilon(dx,c_i)\Big|_{\varepsilon=0}\\
&= \sum_i\bar{\pi}_n(c_i)\psi_n'(t_n,s_n|s_n,c_i) + \sum_i\psi_n^0(t_n,s_n|s_n,c_i)\pi_n'(c_i),
\end{aligned}$$

with π_n' being the $\sqrt{\varepsilon}$-derivative given by Proposition 3.2 of section 3. ∎

Appendix C. Convergence results for objective function.

C.1. The bounds on S_n.

From (1.2) and (2.1), the stochastic recursion for the prices is written as

$$S_n = S_{n-1}e^{I_n(X,\theta)+\mathcal{Z}_n(X,\theta)+\varsigma_n(X,\theta)\mathcal{W}_n},$$

where based on (2.1) the integrated terms are

$$\begin{aligned}
I_n(X,\theta) &= \int_{t_{n-1}}^{t_n}\left(\alpha(X_u) - \frac{1}{2}\beta^2(X_u)\right)du,\\
\mathcal{Z}_n(X,\theta) &= \rho\int_{t_{n-1}}^{t_n}\beta(X_u)dB_u,\\
\varsigma_n(X,\theta) &= \sqrt{(1-\rho^2)\int_{t_{n-1}}^{t_n}\beta^2(X_u)du}
\end{aligned}$$

with $\mathcal{W}_n$ being a standard normal random variable. Recall the assumption that the Brownian motions B_t, $\mathcal{W}_n$ and the Markov chain θ_t are independent processes. By the assumption that α and β are uniformly bounded, it follows that $|I_n|\leq C_I$ for some constant C_I. Thus, letting $\tilde{S}_n = e^{nC_I}S_n$, it follows that

$$\begin{aligned}
\mathbb{E}^\varepsilon\{\tilde{S}_n|\mathcal{F}_{n-1}\} &= \tilde{S}_{n-1}\mathbb{E}^\varepsilon\{e^{C_I+I_n+\mathcal{Z}_n+\varsigma_n\mathcal{W}_n}|\mathcal{F}_{n-1}\}\\
&= \tilde{S}_{n-1}\mathbb{E}^\varepsilon\{e^{C_I+I_n+\mathcal{Z}_n+\frac{1}{2}\varsigma_n^2}|\mathcal{F}_{n-1}\}\\
&\geq \tilde{S}_{n-1}\mathbb{E}^\varepsilon\{e^{C_I+I_n+\mathcal{Z}_n}|\mathcal{F}_{n-1}\}\\
&\geq \tilde{S}_{n-1}\mathbb{E}^\varepsilon\{e^{\mathcal{Z}_n}|\mathcal{F}_{n-1}\}\\
&\geq \tilde{S}_{n-1}e^{\mathbb{E}^\varepsilon\{\mathcal{Z}_n|\mathcal{F}_{n-1}\}} \qquad\qquad \text{(by Jensen's inequality)}\\
&= \tilde{S}_{n-1},
\end{aligned}$$

which means that $\tilde{S}_n$ is an $\mathcal{F}_n$-submartingale. By Doob's inequality, it follows that for any $1 \leq p < \infty$,

$$\mathbb{P}^{\varepsilon}\left\{\max_{0\leq n\leq N} \tilde{S}_n > C\right\} \leq \frac{\mathbb{E}^{\varepsilon}\{(\tilde{S}_N)^p\}}{C^p}.$$

On the other hand, using the upper bound C_I, it follows that for any $\varepsilon > 0$ there is a constant $C_{p,N}$ independent of ε such that

$$\mathbb{E}^{\varepsilon}\{(\tilde{S}_N)^p\} \leq C_{p,N}\mathbb{E}\{(S_0)^p\}, \tag{C.1}$$

where it is assumed that S_0 also has moments of order p independent of ε. It follows that

$$\mathbb{P}^{\varepsilon}\left\{\max_{0\leq n\leq N} S_n > C\right\} \leq \mathbb{P}^{\varepsilon}\left\{\max_{0\leq n\leq N} \tilde{S}_n > C\right\} \leq \frac{\mathbb{E}\{(S_0)^p\}C_{p,N}}{C^p}. \tag{C.2}$$

Given $\delta > 0$, let $C = C_\delta$ be such that $\frac{\mathbb{E}\{(S_0)^p\}C_{p,N}}{C_\delta^p} < \delta$. Then if $K_\delta = \{\sup_{0\leq n\leq N} S_n \leq C_\delta\}$, it follows that $\mathbb{P}^{\varepsilon}\{K_\delta\} \geq 1-\delta$. Furthermore, from the submartingale property of $\tilde{S}_n$ and (C.1), there is a bound for the expectation of the maximum with any $1 < p < \infty$, such that

$$\mathbb{E}^{\varepsilon}\left\{\left(\max_{0\leq n\leq N} S_n\right)^p\right\} \leq \mathbb{E}^{\varepsilon}\left\{\left(\max_{0\leq n\leq N} \tilde{S}_n\right)^p\right\} \leq \frac{p}{p-1}\mathbb{E}^{\varepsilon}\left\{(\tilde{S}_N)^p\right\} \leq \tilde{C}_{p,N}\mathbb{E}\{(S_0)^p\}. \tag{C.3}$$

C.2. Proof of limit in (4.4) for CARA utility.

Proof. Without loss of generality the result is proven for $r = 0$. At time $n-1$ the expected utility is

$$\mathbb{E}^{\varepsilon}\left\{U\left(v + a\frac{\Delta S_n}{S_{n-1}} + \sum_{\ell=n}^{N-1} a_\ell^{\varepsilon}\frac{\Delta S_{\ell+1}}{S_\ell}\right)\middle| S_{0:n-1} = s_{0:n-1}, V_{n-1} = v\right\} \tag{C.4}$$

$$= \sum_i \int\int \mathbb{E}^{\varepsilon}\Bigg\{U\left(v + a\frac{y - s_{n-1}}{s_{n-1}} + \sum_{\ell=n}^{N-1} a_\ell^{\varepsilon}\frac{\Delta S_{\ell+1}}{S_\ell}\right)\Bigg| S_{0:n-1} = s_{0:n-1},$$
$$S_n = y, V_{n-1} = v\Bigg\}\Gamma^{\varepsilon}(y, s_{n-1})\pi_{n-1}^{\varepsilon}(x, c_i)dxdy.$$

By the assumption that $a_\ell^{\varepsilon} \to \bar{a}_\ell$ probably for all $\ell \in \{n+1, n+2, \ldots, N\}$, it follows that the integrand in (C.4) converges,

$$\mathbb{E}^{\varepsilon}\left\{U\left(v + a\frac{y - s_{n-1}}{s_{n-1}} + \sum_{\ell=n}^{N-1} a_\ell^{\varepsilon}\frac{\Delta S_{\ell+1}}{S_\ell}\right)\middle| S_{0:n-1} = s_{0:n-1}, S_n = y, V_{n-1} = v\right\}$$

$$\to \bar{\mathbb{E}}\left\{U\left(v + a\frac{y - s_{n-1}}{s_{n-1}} + \sum_{\ell=n}^{N-1} a_\ell^{\varepsilon}\frac{\Delta S_{\ell+1}}{S_\ell}\right)\middle| S_{0:n-1} = s_{0:n-1}, S_n = y, V_{n-1} = v\right\},$$

pointwise in a, s, and y. Furthermore, the integrand in (C.4) has a finite lower bound,

$$U(v + a(y - s_{n-1}))$$
$$\leq \mathbb{E}^\varepsilon \left\{ U\left(v + a\frac{y - s_{n-1}}{s_{n-1}} + \sum_{\ell=n}^{N-1} a_\ell^\varepsilon \frac{\Delta S_{\ell+1}}{S_\ell} \right) \middle| S_{0:n-1} = s_{0:n-1}, S_n = y, V_{n-1} = v \right\}.$$

The CARA utility is $U(v) = -\frac{1}{\gamma}e^{-\gamma v}$, which obviously has a finite upper bound of zero. Given these bounds, the dominated convergence theorem can be applied to (C.4) to prove the lemma for CARA utility. ■

Appendix D. Stability analysis of ($\mathcal{P}^\varepsilon$) with Lagrangian constraints and cases of power and logarithmic utilities. This appendix is a discussion on a generalized version of Theorem 4.1 for linear constraints and general utility function. The overarching idea is to apply the implicit function theorem of [Rob80] and [BS96] to the stochastic setting of ($\mathcal{P}^\varepsilon$) and ($\overline{\mathcal{P}}$). Without loss of generality the case of $r = 0$ is considered.

D.1. Convex cone of admissible strategies. In discrete time, borrowing and short selling can lead to portfolios with negative value because the risky asset has probability distribution with infinite support. The asset price S_t is log-normal, and so investors must have some tolerance for negative portfolio values if they wish to borrow or short sell. Hence, their risk preferences will restrict them to a set $\mathcal{A}_n$ that will admit borrowing and short selling only if the utility function is defined for negative wealth. Negative wealth can be considered admissible for CARA (exponential) utility, which is useful in settings such as indifference pricing (recall section 6.3). On the other hand, logarithmic and power utilities are not defined for negative wealth, and so admissible strategies will not permit borrowing or short selling (see [BUV12]).

At time n, the set of strategies a_n that are admissible will be defined by a d-dimensional function $G = (G_1, G_2, \dots, G_d)$ which must satisfy d-many equality/inequality constraints:

$$\begin{aligned} G_1(a_n, S_n, V_n) &\leq 0, \\ G_2(a_n, S_n, V_n) &\leq 0, \\ &\vdots \\ &\vdots \\ G_d(a_n, S_n, V_n) &\leq 0. \end{aligned}$$

A common set of constraints is for a_n to lie in a convex cone of $\mathbb{R}^2$. Figure 4 shows admissibility cones for the various utility functions. For utilities defined for negative portfolio values, such as CARA utility, the cone allows for borrowing and short selling. Utilities such as logarithmic and power utility are not defined for negative V_N and so the cone consists of strategies with no borrowing or short selling (i.e., the upper half of the first quadrant).

The cone constraints described in Figure 4 are quantified with inequalities. For instance, the cone constraints for a CARA utility are

$$\begin{aligned} G_1(a, S_n, V_n) &= -(V_n - \kappa^+ a_n) \leq 0, \\ G_2(a, S_n, V_n) &= \kappa^- a_n - V_n \leq 0. \end{aligned}$$

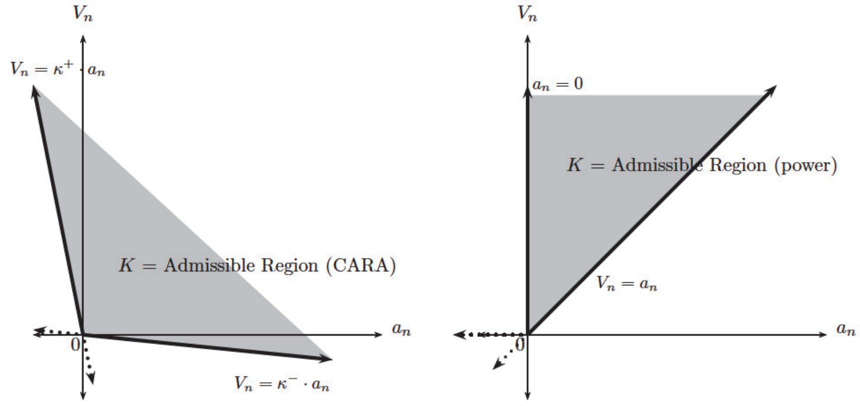

Figure 4. *Left: The admissible cone for CARA (exponential) utility. The darker lines are the constraints on wealth and short-selling. If V_n is positive, then short-selling is allowed up to κ^+ of the portfolio's value, and if V_n is negative, then the portfolio is required to possess at least κ^- of its value in equity. Right: The admissible cone for logarithmic and power utility, neither of which is defined for negative wealth, so all admissible strategies must have $V_n \geq a_n$ and $a_n \geq 0$.*

For logarithmic and power utilities the constraints are

$$G_1(a, S_n, V_n) = a_n - V_n \leq 0,$$
$$G_2(a, S_n, V_n) = -a \leq 0.$$

D.2. Lagrangian formulation. Let $a_n^\varepsilon = a_n^\varepsilon(v, s_{0:n})$ be an $\mathcal{F}_n$-measurable optimal strategy at time n, given the wealth $V_n = v$ and the price history $S_{0:n} = s_{0:n}$. For any possible history of prices $s_{0:n} \in \mathbb{R}^{n+1}$ (with nonnegative entries) and an admissible strategy $a \in \mathbb{R}$, the problem in $(\mathcal{P}^\varepsilon)$ with constraints can be formulated with an objective function that also has Lagrange multipliers:

(D.1)

$$\mathcal{L}_n^\varepsilon(a, \lambda, s_{0:n}, v) \doteq \mathbb{E}^\varepsilon \left\{ U\left(v + a\Delta S_{n+1} + \sum_{\ell=n+1}^{N-1} a_\ell^\varepsilon \Delta S_{\ell+1} \right) \middle| S_{0:n} = s_{0:n}, V_n = v \right\} + \lambda \cdot G(a, s_n, v).$$

Here $\lambda \in \mathbb{R}^d$ is a vector of nonnegative Lagrange multipliers that weight the vector $G(a, s_n, v) \in \mathbb{R}^d$ of inequality constraints. If there is a qualified solution (i.e., the set of Lagrange multipliers is not empty), then the optimal strategy a_n^ε satisfies the first-order condition,

$$0 = \frac{\partial}{\partial a} \mathcal{L}_n^\varepsilon(a, \lambda, s_{0:n}, v)\Big|_{a=a_n^\varepsilon, \lambda=\lambda_n^\varepsilon} \qquad \text{and} \qquad G(a, s_n, v)\Big|_{a=a_n^\varepsilon} = 0, \ 0 \leq n < N,$$

where λ_n^ε is the optimal Lagrange multiplier. Explicitly, this derivative with respect to a is

$$\frac{\partial}{\partial a}\mathcal{L}_n^\varepsilon(a,\lambda,s_{0:n},v)$$
$$= \mathbb{E}^\varepsilon\left\{U'\left(v + a\Delta S_{n+1} + \sum_{\ell=n+1}^{N-1} a_\ell^\varepsilon \Delta S_{\ell+1}\right)\Delta S_{n+1}\middle| S_{0:n} = s_{0:n}, V_n = v\right\} + \lambda\cdot\frac{\partial}{\partial a}G(a,s_n,v).$$

The main hurdle in generalizing Theorem 4.1 to this case is to verify that $\mathcal{L}^\varepsilon$ can fit the criterion for application of the perturbation theory in [BS96, Rob80] and that it applies in the stochastic setting. Dominated convergence can be used to show that the Lagrangian has the limit $\mathcal{L}_{N-1}^\varepsilon(a,\lambda,s_{0:n},v) \to \mathcal{L}_{N-1}^0(a,\lambda,s_{0:n},v)$ pointwise as $\varepsilon \to 0$ for $n = N-1$; the same is done successively for $n < N-1$. However, the compactness arguments used to get uniform strict concavity in (B.3) do not apply because the objective function depends on the level of wealth v, and v can find its way off a compact set. Nonetheless, the probability of v deviating from compact sets is very small, and so it is more than likely that sufficient lower bounds for concavity and $\sqrt{\varepsilon}$-regularity can be obtained, but this has not been done in these appendices.

D.3. Stability of the generalized equations. The analysis of optimal solutions of the Lagrangian in (D.1) is performed using a generalized formulation of the problem. The generalized equation is

$$F_n^\varepsilon(z,s_{0:n},v) \in \mathcal{R}(z), \tag{D.2}$$

where

$$z = (a,\lambda) \in \mathbf{a}_n \times \mathbb{R}^d,$$

and

$$F_n^\varepsilon(z,s_{0:n},v) = \left(\frac{\partial}{\partial a}\mathcal{L}_n^\varepsilon(a,\lambda,s_{0:n},v), G(a,s_n,v)\right),$$

and the set on the right-hand side is defined as $\mathcal{R}(z) = (0,\mathcal{R}_K^{-1}(\lambda))$ with K being the convex cone of constraints on a_n such that

$$\mathcal{R}_K^{-1}(\lambda) = \begin{cases} y \in K : \lambda\cdot y = 0 & \text{if } \lambda \in K^-, \\ \emptyset & \text{otherwise,}\end{cases}$$

with $K^- = \{\lambda \in \mathbb{R}^d : \langle \lambda, y\rangle \le 0\ \forall y \in K\}$, i.e., the polar (negative dual) of $K \subset \mathbb{R}^d$. For the limit or unperturbed problem ($\varepsilon = 0$) with solution z^0, the solution z^ε of the perturbed problem is approximated as

$$z^\varepsilon \simeq z^0 + \tilde{z}^\varepsilon,$$

where $\tilde{z}^\varepsilon$ is a solution to the following generalized linear equation (with unknown $\mathbf{w} \in \mathbb{R}^{d+1}$), representing partial linearization:

$$\bar{F}_n(z^0,s_{0:n},v) + \mathbf{w}\nabla_z \bar{F}_n(z,s_{0:n},v)\Big|_{z=z^0} + \sqrt{\varepsilon}\frac{\partial}{\partial\sqrt{\varepsilon}}F_n^\varepsilon(z^0,s_{0:n},v)\Big|_{\varepsilon=0} \in \mathcal{R}(z^0+\mathbf{w}), \tag{D.3}$$

where $\nabla_z \bar{F}_n$ is the Jacobian matrix of $\bar{F}_n$. Equation (D.3) is a local well-posedness condition for problems with small perturbation. Analysis using the generalized equation was done in [Rob80] and was presented in a more general framework in [BS96]. For the problems they considered, essentially, if the strong stability conditions are met, then through an application of the implicit function theorem the existence, uniqueness, and convergence of the optimal controls follow. The difficulties with problems ($\mathcal{P}^\varepsilon$) and ($\overline{\mathcal{P}}$) is that they involve backward iterations and stochastic features, which make application of the theory not straightforward. However, if the implicit function theorem does apply, then Theorem 4.1 can be generalized as follows.

Theorem D.1. *For $\varepsilon = 0$, suppose that $\{z_n^0\}_{n<N}$ are solutions of the generalized equation (D.2) and (D.3) has a solution pointwise in $(v, s_{0,N})$. Then there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$ the functions z_n^ε and $\tilde{z}_n^\varepsilon$ are well defined in the vicinity of z_n^0 and the origin, respectively, pointwise in $(v, s_{0,N})$. In addition, z_n^ε are Lipschitz continuous in $\sqrt{\varepsilon}$, $\tilde{z}_n^\varepsilon = O(\sqrt{\varepsilon})$, and $z_n^\varepsilon = z_n^0 + \tilde{z}_n^\varepsilon + o(\sqrt{\varepsilon})$.*

Based on Theorem D.1, the correction term for the optimal strategy can be found by solving (D.3). The first component of (D.3) is the Jacobian of the generalized value function, $\nabla_z F_n^\varepsilon$, a quantity which is computable with an application of Fubini's theorem. The other component is the derivative $\frac{\partial}{\partial\sqrt{\varepsilon}} F_n^\varepsilon|_{\varepsilon=0}$, which will be taken as a right-hand limit,

$$\frac{\partial}{\partial\sqrt{\varepsilon}} F_n^\varepsilon(z^0, s_{0:n}, v)\Big|_{\varepsilon=0} = \left(\frac{\partial}{\partial\sqrt{\varepsilon}}\frac{\partial}{\partial a}\mathcal{L}_n^\varepsilon(a, \lambda, s_{0:n}, v), 0\right)\Big|_{\varepsilon=0},$$

where the derivatives of the constraint functions G are zero because we have assumed that the cone of admissible strategies does not depend on ε. Applying the chain rule gives

$$\frac{\partial}{\partial\sqrt{\varepsilon}}\frac{\partial}{\partial a}\mathcal{L}_n^\varepsilon(a, \lambda, s_{0:n}, v)\Big|_{\varepsilon=0}$$

$$= \frac{\partial}{\partial\sqrt{\varepsilon}}\mathbb{E}^\varepsilon\left\{U'\left(v + a\Delta S_{n+1} + \sum_{\ell=n+1}^{N-1}\bar{a}_\ell \Delta S_{\ell+1}\right)\Delta S_{n+1}\middle| S_{0:n} = s_{0:n}, V_n = v\right\}\Big|_{\varepsilon=0} \tag{D.4}$$

$$+\mathbb{E}^\varepsilon\left\{U''\left(v + a\Delta S_{n+1} + \sum_{\ell=n+1}^{N-1} a_\ell^\varepsilon \Delta S_{\ell+1}\right)\Delta S_{n+1}\sum_{\ell=n+1}^{N-1}\Delta S_{\ell+1}\frac{\partial}{\partial\sqrt{\varepsilon}}a_\ell^\varepsilon\middle| S_{0:n} = s_{0:n}, V_n = v\right\}\Bigg|_{\varepsilon=0}.$$

Given the derivative in $\sqrt{\varepsilon}$, the correction term for the ε-optimal strategy can identified in terms of right-hand derivatives at $\varepsilon = 0$,

$$a_n^\varepsilon = \bar{a}_n + \sqrt{\varepsilon}\frac{\partial}{\partial\sqrt{\varepsilon}}a_n^\varepsilon\Big|_{\varepsilon=0} + o(\sqrt{\varepsilon}),$$
$$\lambda_n^\varepsilon = \bar{\lambda}_n + \sqrt{\varepsilon}\frac{\partial}{\partial\sqrt{\varepsilon}}\lambda_n^\varepsilon\Big|_{\varepsilon=0} + o(\sqrt{\varepsilon}),$$

which can be computed by solving (D.3).

Appendix E. Numerical method for CARA utility (no strategy constraints). This appendix contains pseudocode for the reader who is interested in doing numerics themselves. The code is for the case of CARA utility with no constraints but *does* include dependence on the price s_n, which is a superfluous computation in light of Proposition B.2, but having dependence on s_n is included because it is required for indifference pricing. Indeed, the following codes were used to obtain the figures and tables of section 6.

Equation (6.1) is a relatively low-dimensional integration. However, there are a few details that should be explained for the sake of clarity. First, the function $H_n(s, 0, \cdot)$ needs to be computed on a domain $\mathcal{D}_n$ where $\mathbb{P}^0(S_n \notin \mathcal{D}_n) \ll 1$ for each $n = 0, 1, \ldots, N$. The sets $\mathcal{D}_n$ grow with time because the process S_n is not mean-reverting, and so $\mathcal{D}_N$ will have a broad range of values, whereas $\mathcal{D}_0$ will be a singleton at S_0. For a fixed mesh size ΔY, the construction of these domains is described by Algorithm 1.

Algorithm 1. Generate H's domain.

$S_0^{min} = S_0$;

$S_0^{max} = S_0$;

for $n = 0, 1, 2, \ldots, N-1$ **do**

$S_{n+1}^{min} = S_n^{min} \exp\left\{\min_i \left(\left(\alpha(c_i) - \frac{1}{2}\beta^2(c_i)\right)\Delta t - 3\beta(c_i)\sqrt{\Delta t}\right)\right\}$;

$S_{n+1}^{max} = S_n^{max} \exp\left\{\max_i \left(\left(\alpha(c_i) - \frac{1}{2}\beta^2(c_i)\right)\Delta t + 3\beta(c_i)\sqrt{\Delta t}\right)\right\}$;

$\mathrm{M} = \lceil \log(S_{n+1}^{max}/S_{n+1}^{min})/\Delta Y \rceil$;

$\mathcal{D}_{n+1} = \left\{e^{\log(S_{n+1}^{min})}, e^{\log(S_{n+1}^{min})+\Delta Y}, \ldots, e^{\log(S_{n+1}^{min})+\mathrm{M}\Delta Y}\right\}$;

end for

Given the domains of each S_n, the optimization for the Markov decision process described by (6.1) can be computed. The numerical computation of $H_n(s, 0, p)$ is done at all points $s \in \mathcal{D}_n$ and at all points $p \in \Pi$ where Π is a discrete set of probability vectors on Θ. In general,

$$H_n(s, 0, p) \propto \sum_i \sum_{s' \in \mathcal{D}_{n+1}} \sum_{q \in \Pi} e^{-\gamma a(s'-s)} H_{n+1}(s', 0, q)$$
$$\times \mathbb{P}(\bar{\pi}_{n+1} \in dq | S_{n+1} = s', S_n = s, \bar{\pi}_n = p)\mathbb{P}(S_{n+1} \in ds | S_n = s, \bar{\pi}_n = p).$$

However, the structure of the transition probabilities in this equation is degenerate, as

$$\mathbb{P}(\bar{\pi}_{n+1} = dq | S_{n+1} = s', S_n = s, \bar{\pi}_n = p) = \delta_{\bar{\mathcal{E}}(s', s, p)}(q) dq,$$

and so the functions $\bar{\mathcal{E}}(s', s, p) \in [0, 1]$ need to be projected to the discrete subspace Π. Such a projection could be devised any number of ways (e.g., one could round $\bar{\mathcal{E}}(s', s, p)$ to the nearest element of Π). Denote the projection to Π as simply

$$\Pi\left[\bar{\mathcal{E}}(s', s, p)\right].$$

Algorithm 2. The partial information algorithm for H_n with CARA utility.

$H_N(s,0,p) = U(0)$ for all $s \in \mathcal{D}_N$ and $p \in \Pi$;

for $n = N-1, N-2, N-3, \ldots, 0$ **do**

  initialize a_{old} with an approximate solution;

  ctr $= 0$ and err $= \infty$;

  while err>tol && ctr≤ctr$_{\max}$ **do**

    for $s \in \mathcal{D}_n$ **do**

      for $p \in \Pi$ **do**

$$H_n(s,0,p) = \sum_{i,j} \sum_{s' \in \mathcal{D}_{n+1}} e^{-\gamma a_{old}(s,p)(s'-s)} \times H_{n+1}\left(s', 0, \Pi\left[\bar{\mathcal{E}}(s',s,p)\right]\right) \bar{\Gamma}_{ij}(s',s)p(c_i)\Delta Y;$$

$$\frac{\partial}{\partial a} H_n(s,0,p) = -\gamma \sum_{i,j} \sum_{s' \in \mathcal{D}_{n+1}} (s'-s) e^{-\gamma a_{old}(s,p)(s'-s)} \times H_{n+1}\left(s', 0, \Pi\left[\bar{\mathcal{E}}(s',s,p)\right]\right) \bar{\Gamma}_{ij}(s',s)p(c_i)\Delta Y;$$

$$a_{new}(s,p) = a_{old}(s,p) + \eta \frac{\partial}{\partial a} H_n(s,0,p);$$

      end for

    end for

    err $= \|a_{new} - a_{old}\|$;

    $a_{old} = a_{new}$;

    ctr $=$ ctr $+1$;

  end while

end for

The last piece that is necessary for the numerical algorithm is to define the operator $\bar{\Gamma}$ in matrix form. To do so, simply define for any $c_i, c_j \in \Theta$ the matrix

$$\bar{\Gamma}_{ij}(y,s) \doteq \mathbb{E}\left\{ \frac{\mathbb{1}_{[\theta_n = c_j]}}{\bar{\varsigma}_n(\theta)} \exp\left\{ -\frac{1}{2} \left(\frac{\log(y/s) - \bar{I}_n(\theta)}{\bar{\varsigma}_n(\theta)} \right)^2 \right\} \middle| \theta_{n-1} = c_i \right\}$$

for any $y, s \in \mathbb{R}^+$.

Given the domains $\{\mathcal{D}_n\}_{n \le N}$, the matrix operator $\bar{\Gamma}$, the domain Π, and its projection operator, (6.1) can be computed using the method described in Algorithm 2. The optimization is carried out using the gradient ascent with parameter $\eta \in (0,1)$. It should be mentioned that gradient ascent finds a local min/max (not necessarily a global max), and that there should be a cutoff value ctr$_{\max}$ to enforce some stability.

Finally, Algorithm 3 is a description of the method for full information. The full informa-

Algorithm 3. The full information algorithm for H_n with CARA utility.

$H_N(s,0,c_i) = U(0)$ for all $s \in \mathcal{D}_N$ and $i = 1,\dots,m$;

for $n = N-1, N-2, N-3, \dots, 0$ **do**

initialize a_{old} with an approximate solution;

ctr $= 0$ and err $= \infty$;

while err>tol && ctr≤ctr$_{\max}$ **do**

for $s \in \mathcal{D}_n$ **do**

for $i = 1\dots,m$ **do**

$$H_n(s,0,c_i) = \sum_j \sum_{s' \in \mathcal{D}_{n+1}} e^{-\gamma a_{old}(s,p)(s'-s)} \times H_{n+1}(s',0,c_j)\,\bar{\Gamma}_{ij}(s',s)\Delta Y;$$

$$\frac{\partial}{\partial a} H_n(s,0,c_i) = -\gamma \sum_j \sum_{s' \in \mathcal{D}_{n+1}} (s'-s) e^{-\gamma a_{old}(s,p)(s'-s)} \times H_{n+1}(s',0,c_j)\,\bar{\Gamma}_{ij}(s',s)\Delta Y;$$

$$a_{new}(s,c_i) = a_{old}(s,c_i) + \eta \frac{\partial}{\partial a} H_n(s,0,c_i);$$

end for

end for

err $= \|a_{new} - a_{old}\|$;

$a_{old} = a_{new}$;

ctr $=$ ctr $+1$;

end while

end for

tion case does not have degeneracy (and hence does not need the projection operator Π), and the computation is essentially a backward equation with an optimization solved by first-order conditions; it is a standard method from the literature on Markov decision processes (see [Pow11, Ber12]).

Algorithm 4. Gradient ascent algorithm (damping coefficient $\eta \in (0,1)$).

while err_H >tol && err_c >tol && ctr< $\mathrm{ctr}_{\max}$ **do**

$\tilde{a}_n^0 = \bar{a}_n + \eta\bar{\mathbb{E}}\left\{\exp\left\{-\gamma\sum_{\ell=n}^{N-1}\bar{a}_\ell\Delta S_{\ell+1}\right\}\Delta S_{n+1}\middle|\mathcal{F}_n\right\}$;

$\tilde{a}_n^{0,call} = a_n^{0,call} + \eta\bar{\mathbb{E}}\left\{\exp\left\{-\gamma\sum_{\ell=n}^{N-1}a_\ell^{0,call}\Delta S_{\ell+1} + \gamma(S_N-k)^+\right\}\Delta S_{n+1}\middle|\mathcal{F}_n\right\}$;

$err_H = |\tilde{a}_n^0 - \bar{a}_n|$;

$err_c = |\tilde{a}_n^{0,call} - a_n^{0,call}|$;

$\bar{a}_n = \tilde{\tilde{a}}_n$;

$a_n^{0,call} = \tilde{a}_n^{0,call}$;

ctr = ctr+1;

end while

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