FAQ's in Option Pricing Theory

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Abstract

We consider several Frequently Asked Questions (FAQ's) in option pricing theory.

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I Introduction

The recent award of the Nobel prize in economics to Professors Merton and Scholes has attracted many newcomers to the field of derivative securities pricing. These newcomers often pose certain frequently asked questions (FAQ's) which the author has struggled with for many years. Sometimes the questions are posed to newcomers by veterans in the field, either in the classroom or during an interview, in the hopes that the next Einstein can be identified.

The purpose of this paper is to present the current state of this continuing confrontation. The responses given in this paper to these FAQ's hardly qualify as answers, since they are practically devoid of rigor and generally raise more questions than they answer. However, it is hoped that these responses will spur others to give their answers in the hopes that these questions can be finally laid to rest.

Here then are the list of questions addressed in the remainder of the paper:

1. Why isn't an option's value just its discounted average payout?
2. Why don't the statistical probabilities matter in the binomial model?
3. Why doesn't the expected rate of return appear in the Black Scholes formula?
4. Is the hedging argument given in the Black Scholes paper correct? Why doesn't one differentiate the number of shares/options held in the hedge portfolio?
5. Can one hedge options in a trinomial model?
6. Can jumps be hedged in a continuous time model?
7. Why does the "market price of risk" appear in many stochastic interest rate or volatility models? Why doesn't it appear in the HJM model?
8. Which volatility should one hedge at - historical or implied?

II Why Isn't an Option's Value Just Its Discounted Average Payout?

Suppose a stock is presently priced at a dollar and suppose that the economy has equal probability of either being good or bad over the next year. If the economy is good, the stock price will double to \$2, while if the economy is bad, the stock price will halve to fifty cents. For simplicity, we assume no dividends are paid on the stock over the next year. If interest rates are zero, it seems reasonable to value an at-the-money call at its expected payout of fifty cents. One problem with this answer is that the same reasoning implies that the stock should be priced at \$1.25: If the economy is in a good state, one share is worth \$2 and if the economy is in a bad state, one share is worth fifty cents, so the the stock is worth \$1.25. However, by the same token, in a good state, one dollar is worth half a share and in a bad state, one dollar is worth two shares, so a dollar is worth 1.25 shares. Equivalently, the stock is apparently now worth 80 cents. If we are not sure about the stock's value, are we really so sure that the call is worth 50 cents?

Suppose we go ahead and open a 2 way market in the at-the-money call at our estimated value of fifty cents. We also offer a 2 way market in the stock at the market price of a dollar. With unusual urgency, a hedge fund sells two calls to us and buys one share from us. Since we get two commissions, our option pricing strategy seems to be generating revenues and accomplishing the objective of capturing market share. After all, we are now long two calls and short one share. And if the stock price doubles to \$2, then the total call payoff of \$2 offsets our cost of closing the short position. However, if the stock price halves, then both calls expire worthless and we pay 50 cents to cover our short stock position. Since the hedge fund makes what we lose, the hedge fund has arbed us into giving them a free at-the-money put.

Clearly, valuing options at their average payout is not such a good idea. To value an option correctly, recall that when interest rates are positive, time value of money implies that a dollar received in one year has a lower value than a dollar received today. It is wise to take into account time value of money when valuing a bond. Whether or not interest rates are positive, there is a state value of money as well: a dollar received in a good economy has a lower value than a dollar received in a bad economy, even if the two states are equally likely. When valuing an option, one should take into account both the time and the state value of money. Since under zero interest rates, there is no time value, we only need state values. To learn them, note that buying two thirds of a share and borrowing one third of a dollar creates a portfolio worth a dollar in the good state and zero otherwise. Thus, the value of a good state dollar is one third of a dollar. Since the value of a dollar in both states is just one dollar, the value of a bad state dollar must be two thirds of a dollar. To replicate the at-the-money call with payoffs one and zero in the good and bad states respectively, we buy one good state dollar and zero bad state dollars. Since the net cost is a third of a dollar, this is also the arbitrage-free value of the at-the-money call. By put-call parity, the value of an at-the-money put must also be one third of a dollar.

Time value of money argues that under positive interest rates, a dollar paid immediately is worth more than a dollar paid in a year because by putting a dollar in the bank, one can have more than a dollar in one year. Similarly, state value of money says that a bad state dollar is worth more than a good state dollar because by doing a few trades, one can get more than one good state dollar. To see how, recall that the at-the-money call and put were each priced at a third of a dollar. Thus if one is scheduled to receive a bad state dollar, then one can sell two puts and use the proceeds to buy two calls. Thus, by foregoing a dollar in the bad state, one gets two dollars in the good state.

The expected value calculation doesn't work because it mixes apples and oranges. For example, the stock is not priced at its statistical expected value because adding one half of two good state dollars to a half of a half of a bad state dollar does not compute. However, the so-called risk-neutral probabilities can be recovered from the stock price and the price of a discount bond. When these probabilities of one third for the up state and two thirds for the down state are used to compute expected option payoffs, then arbitrage-free values are given by this so-called risk-neutral expected value.

III Why Don't Statistical Probabilities Matter in the Binomial Model?

Again consider valuing a call on a stock priced at a dollar which can double or halve over a single period. The graph below shows the 2 possible payouts: If we draw a line through the two points, then the slope is

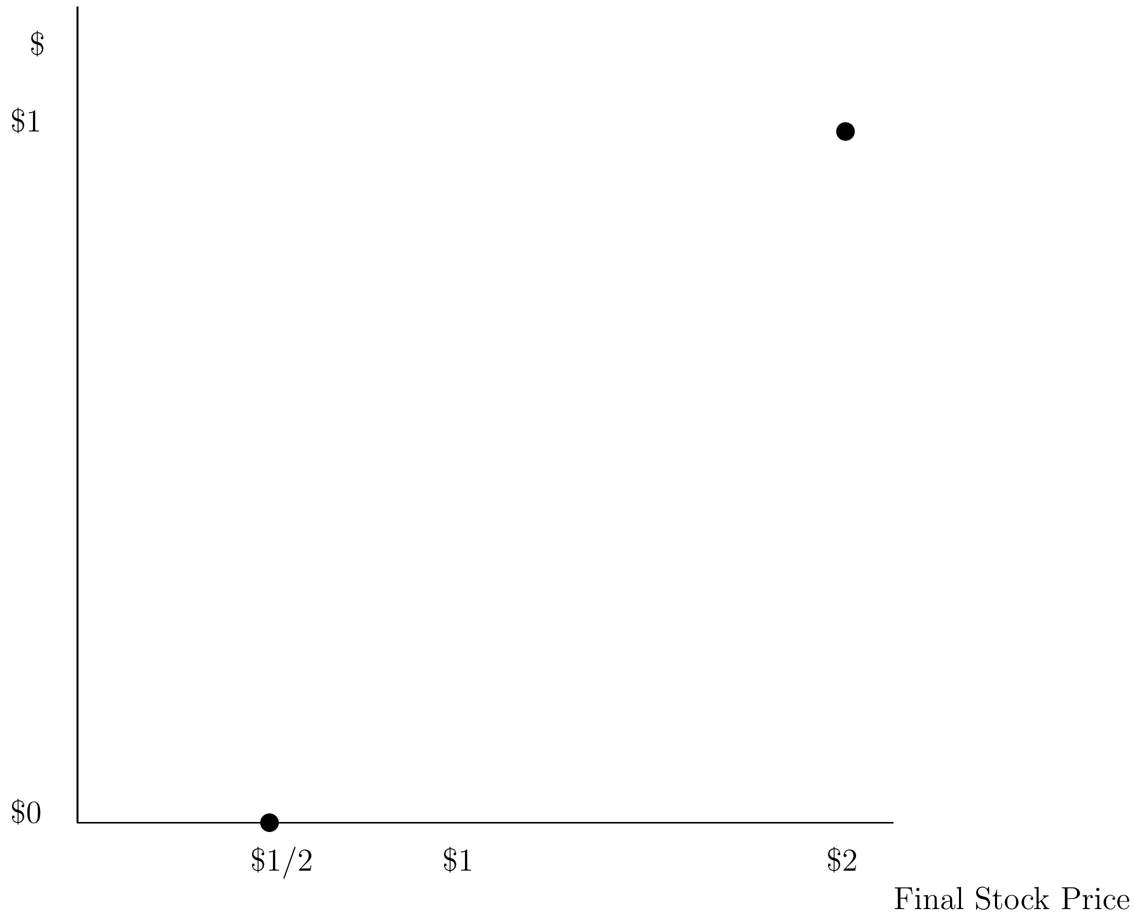
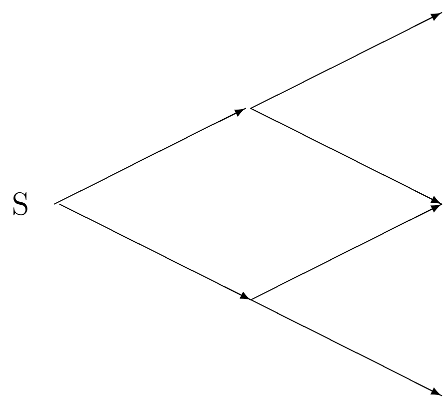


Figure 1: Payoffs in Binomial Model.

2/3 so the vertical intercept is -1/3. Since only two points on the call's hockey stick payoff are considered possible, one is indifferent between the linear payoff and the nonlinear payoff. However, the linear payoff is easily created by buying 2/3 of a share and borrowing 1/3 of a unit bond. It follows that the value of the option is the cost of this replicating portfolio, which is independent of the probabilities of up and down. To be clear, if market probability assessments change, then the stock and bond prices may change, and if they do, then option prices will almost surely change. However, the functional relationship between the option price and the stock and bond prices will remain independent of the probabilities. In essence, these prices are sufficient statistics for the statistical probabilities.

Now consider a two period recombining binomial tree as shown below:



Although an at-the-money call maturing at the end of the second period has a nonlinear payoff, at each intermediate node, the payoff is locally linear. Thus terminal values can be propagated back one period to generate intermediate values. If these intermediate values are regarded as the payoffs of a single period claim, this payoff is again locally linear when viewed from the root node and hence susceptible to valuation by replication. The next section explores the implications of the linearity principle in the continuous time setup.

IV Why Doesn't the Expected Rate of Return Appear in the Black Scholes Formula?

There are now many derivations of the Black Scholes formula. While mathematically rigorous, these derivations generally do not focus on explaining why the expected rate of return on the stock does not appear in the Black Scholes formula. This section attempts to explain why the expected rate of return over the option's life $[0, T]$ is irrelevant for valuing an option at some fixed future time $t \in [0, T]$, *given* the stock price at t. We note that if the option price at t is related to the level of the standard Brownian motion at t rather than the stock price, then the expected rate of return over $[0, t]$ does appear.

The remainder of this section presents a derivation of the Black Scholes formula, which eschews rigor in pursuit of the intuition behind this fundamental irrelevance result. It relies heavily on the following continuous time version of the Linearity Principle. Consider a derivative security whose payoff at T is linear in the price of a stock:

$$D_T = N_0 + N_1 S_T, \tag{1}$$

where N_0 and N_1 are known at some prior time $t < T$. For simplicity, suppose a constant interest rate r and a constant dividend yield of δ. Then the absence of arbitrage requires that the time t price of the derivative must be:

$$D_t = N_0 e^{-r(T-t)} + N_1 e^{-\delta(T-t)} S_t, t < T. \tag{2}$$

To prove this linearity principle, note that if the derivative were priced at an amount which is greater than the LHS, then the arbitrage involves selling the derivative, lending $N_0 e^{-r(T-t)}$ dollars, and buying $N_1 e^{-\delta(T-t)}$ shares. By leaving the interest in the bank and reinvesting the dividends back into the stock, these holdings will grow to N_0 dollars and N_1 shares, whose value covers the payout on the short derivative. Conversely, if the derivative were priced below the LHS, the arbitrage is to buy the derivative and sell the replicating portfolio. Thus, the LHS of (2) defines a function relating the value V_t to only N_0, $e^{-r(T-t)}$, N_1, and $S_t e^{-\delta(T-t)}$. Given the values of these 4 entities, one does not need to know *anything* about the parameters governing changes in the price. If the price follows a diffusion, this means that one does not need to know the expected rate of return *or the volatility*, given the values of N_0, $e^{-r(T-t)}$, N_1, and $S_t e^{-\delta(T-t)}$.

One of the triumphs of the Black Scholes Merton (BSM) analysis is that the triumvirate were able to apply the above argument to a situation where D_T is apparently non-linear in S_T. They further showed that given time t information such as S_t, N_0 and N_1 depend on the volatility of the process, but not on the expected rate of return. They accomplished this task by assuming continuity in time of both the price process and trading opportunities. While both assumptions can be challenged on empirical grounds, the positivist view is that the model should be judged by its conclusions rather than its assumptions. On this basis, the model is an unparalleled success.

Borrowing an insight from Newton, Liebnitz, and Itô, BSM realized that when viewed very closely, nonlinear functions can be treated as "locally linear". More formally, suppose that the price process of the stock is:

$$dS_t = [\mu(S_t, t) - \delta]S_t dt + \sigma(S_t, t)S_t dW_t, \qquad t \in [0, T], \tag{3}$$

where the functions μ and σ are termed the expected rate of return and the volatility respectively.

Assuming that the time t value of the derivative D_t is a sufficiently smooth function V of only S_t and t, a bivariate Taylor series expansion implies:

$$\begin{aligned} \triangle D_t \equiv V(S_t + \triangle S, t + \triangle t) - V(S, t) \; &= \; \frac{\partial V}{\partial t}(S_t, t)\triangle t + O(\triangle t)^2 \\ &+ \; \frac{\partial V}{\partial S}(S_t, t)\triangle S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\triangle S_t)^2 + O(\triangle S)^3 + O((\triangle S)^2 \triangle t), t \in [0, T]. \end{aligned} \tag{4}$$

If one substitutes $\triangle S = \int_t^{t+\triangle t}[\mu(S_u, u) - \delta]S_u du + \sigma(S_u, u)S_u dW_u$, then μ appears in (4) in all four terms in the second row.

However, if we focus on an infinitesimally small time step, then Itô's lemma implies:

$$dD_t = \frac{\partial V}{\partial t}(S_t, t)dt + \frac{\partial V}{\partial S}(S_t, t)dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2, \qquad t \in [0, T]. \tag{5}$$

Since the higher order terms have disappeared, so has the appearance of μ in them. However, substitution of (3) in (5) shows that μ still affects the *change in* the value dD_t. From (5), it appears that the future value of the derivative D_{t+dt} is *quadratic* in the future value of the stock S_{t+dt} since:

$$D_{t+dt} - D_t = \frac{\partial V}{\partial t}(S_t, t)dt + \frac{\partial V}{\partial S}(S_t, t)(S_{t+dt} - S_t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(S_t, t)(S_{t+dt} - S_t)^2, \qquad t \in [0, T]. \tag{6}$$

However, Itô's recognition that $(S_{t+dt} - S_t)^2 = \sigma^2(S_t, t)S_t^2 dt$, which given S_t, is both deterministic and independent of μ, puts (6) in the same linear form as (1):

$$D_{t+dt} = \underbrace{D_t + \left[\frac{\partial V}{\partial t}(S_t, t)dt + \frac{\sigma^2(S_t, t)S_t^2}{2}\frac{\partial^2 V}{\partial S^2}(S_t, t)\right]dt - \frac{\partial V}{\partial S}(S_t, t)S_t}_{N_0} + \underbrace{\frac{\partial V}{\partial S}(S_t, t)}_{N_1} S_{t+dt}, \qquad t \in [0, T]. \tag{7}$$

It is important to realize at this stage that N_0 and N_1 could depend on μ, even given the information at t such as S_t. This is because μ could appear in the function V, which we do not yet know. The failure to recognize this possibility would imply that N_1 does not depend on σ, which will turn out to be false.

If one replaces S_{t+dt} in (7) with $S_t + \mu(S_t,t)S_t dt + \sigma(S_t,t)dW_t$, then μ still appears in (7). However, applying the linearity principle (2) to (7):

$$D_t = \left\{ D_t + \left[\frac{\partial V}{\partial t}(S_t,t) + \frac{\sigma^2(S_t,t)S_t^2}{2}\frac{\partial^2 V}{\partial S^2}(S_t,t) \right] dt - \frac{\partial V}{\partial S}(S_t,t)S_t \right\} e^{-rdt} + \frac{\partial V}{\partial S}(S_t,t)S_t e^{-\delta dt}, \qquad t \in [0,T]. \tag{8}$$

This expression constrains V since replacing e^{-rdt} with $1 - rdt$, $e^{-\delta dt}$ with $1 - \delta dt$, and eliminating terms of order $(dt)^2$ in (8) leaves the Black Scholes p.d.e.:

$$\frac{\partial V}{\partial t}(S_t,t) + \frac{\sigma^2(S_t,t)S_t^2}{2}\frac{\partial^2 V}{\partial S^2}(S_t,t) + (r-\delta)S_t\frac{\partial V}{\partial S}(S_t,t) - rV(S_t,t) = 0. \tag{9}$$

Solving this p.d.e. subject to the terminal condition and the appropriate boundary conditions allows us to determine $V(S,t)$ for all $t \in [0,T]$. Since μ does not appear in the PDE or the boundary conditions, we thus have that D_t is completely independent of μ, given the information at t such as S_t. Furthermore, from (7), the number of shares held from t to $t+dt$ is $N_1(S_t,t) = \frac{\partial V}{\partial S}(S_t,t), t \in [0,T]$. It follows that N_0 and N_1 are also independent of μ, given the information at t such as S_t. In contrast, from (5) and (6), μ affects the distribution of D_{t+dt} and the distribution of dD_t. An increase in μ which leaves the stock price S_t unchanged, and which furthermore has no effect on N_0 and N_1, leaves the value D_t unchanged. In the case of a call option, an increase in μ holding S_t fixed increases the expected payoff $E_t D_{t+dt}$, and the expected call return $E_t dD_t$, but not the call value. If an increase in μ leaves the stock price S_t unchanged, it must be because risk aversion or the relevant risk has also increased to offset the effect exactly. Consequently, it is plausible that the call price has not changed, even though its expected payoff has risen.

The μ appearing in the $\triangle S_t$ term in (4) disappears for a reason completely different from the reason that the μ appearing in $(\triangle S_t)^2$ and higher powers disappears. The appearance of μ in the $\triangle S_t$ term is handled by the above linearity principle, which has nothing to do with continuous trading or a continuous price process. In other words, given that the future value of the derivative at $t + \triangle t$ is linear in the future stock price $S_{t+\triangle t}$ for any $\triangle t > 0$, the μ appearing in $\triangle S_t \equiv S_{t+\triangle t} - S_t$ disappears (as does volatility). In contrast, the appearance of μ in $(\triangle S_t)^2$ is handled by Itô's recognition that the distinction between variation about the mean and variation about zero vanishes as $\triangle t \downarrow 0$. Similarly, its appearance in terms like $(\triangle S_t)^3$ is handled by the result of stochastic (and ordinary) calculus that such terms are negligible in comparison to the leading terms. It is interesting to note that volatility only matters because stochastic calculus is being used rather than ordinary calculus.

For a call option, the linearity[1] of (7) is especially surprising just prior to maturity with the stock price at the strike price. Clearly, the payoff $\max[0, S_T - K]$ is nonlinear in S_T when S_T is allowed to take on all positive values. For $S_{T-dt} \neq K$, the continuity of the price process and the local linearity of the payoff makes a local linearity result plausible. However, for $S_{T-dt} = K$, the payoff is not locally linear. However, we argue that the relationship is effectively locally linear in the model as a consequence of both the locally perfect correlation and the local bivariate normality of S_T and D_T.

Since the claim's future value is locally linear in the stock's future value, the expected return on the claim is a *known* weighted average of μ and r, given the information at t such as S_t. The option price can thus alternatively be obtained by discounting its expected payoff at its expected return. If one uses an incorrect μ to compute the expected payoff, then the expected option return will also be incorrect. Nonetheless, since the claim's value is independent of μ, the correct answer is obtained by any choice of μ. It is computationally convenient to pretend that $\mu = r$ as this implies that the claim's expected return is also r. Under this false assumption, the claim's correct value is obtained by simply discounting its expected payoff at the riskfree rate r. For self-financing trading strategies, this trick works globally in time. Thus, the solution of the PDE (9) subject to a terminal condition $V(S, T) = f(S)$ is:

$$V(S, t) = e^{-r(T-t)} E^Q f(S_T),$$

where the measure Q indicates that one sets $\mu = r$ in (3) and interprets W as standard Brownian motion under Q. This risk-neutral valuation methodology turns out to be generalizeable beyond the Markov

[1] Note that to obtain this linearity result, it was crucial that the partial derivatives be evaluated at t rather than $t + dt$. This is in fact a requirement both mathematically for the Itô integral and financially for the replicating strategy.

setting and is now considered more fundamental[2].

There is a loose end in the above analysis. The argument *assumed* that the value of the derivative was a function V of only S and t. We now paraphrase an argument of Merton to show that the value of a path-independent contingent claim must be a function of only the stock price S and time t. By path-independent, we mean that the derivative has a final payoff $f(\cdot)$ at T which depends only on S_T. Itô's lemma applied to the product of a smooth function $\hat{V}(S_u, u)$ and the future value factor $e^{r(T-u)}$ gives:

$$\begin{aligned}\hat{V}(S_T, T) &= \hat{V}(S_t, t)e^{r(T-t)} + \int_t^T e^{r(T-u)} \frac{\partial \hat{V}}{\partial S}(S_u, u) dS_u \\ &+ \int_t^T e^{r(T-u)} \left[\frac{\sigma^2(S_u, u) S_u^2}{2} \frac{\partial^2 \hat{V}}{\partial S^2}(S_u, u) - r\hat{V}(S_u, u) + \frac{\partial \hat{V}}{\partial t}(S_u, u) \right] du.\end{aligned}$$

The first term is a constant, while the second is a stochastic integral. Thus, the first term can be created by depositing $\hat{V}(S_t, t)$ in the bank and the second term accumulates gains on $\frac{\partial \hat{V}}{\partial S}(S_u, u)$ shares held at each $u \in [t, T]$. If we borrow to finance the stock position, then gains from the stock are reduced by the carrying cost as follows:

$$\begin{aligned}\hat{V}(S_T, T) &= \hat{V}(S_t, t)e^{r(T-u)} + \int_t^T e^{r(T-u)} \frac{\partial \hat{V}}{\partial S}(S_u, u) \left[dS_u - (r-\delta) S_u du \right] \\ &+ \int_t^T e^{r(T-u)} \left[\frac{\sigma^2(S_u, u) S_u^2}{2} \frac{\partial^2 \hat{V}}{\partial S^2}(S_u, u) + (r-\delta) S_u \frac{\partial \hat{V}}{\partial S}(S_u, u) - r\hat{V}(S_u, u) \right. \\ &\left. + \frac{\partial \hat{V}}{\partial t}(S_u, u) \right] du. \qquad (10)\end{aligned}$$

Suppose we choose $\hat{V}(S, t)$ to solve the same p.d.e. (9) as $V(S, t)$, i.e.:

$$\frac{\sigma^2(S, t) S^2}{2} \frac{\partial^2 \hat{V}}{\partial S^2}(S, t) + (r-\delta) S \frac{\partial \hat{V}}{\partial S}(S, t) - r\hat{V}(S, t) + \frac{\partial \hat{V}}{\partial t}(S, t) = 0, \qquad (11)$$

and further set:

$$\hat{V}(S, T) = f(S). \qquad (12)$$

Substituting (11) and (12) in (10):

$$f(S_T) = \hat{V}(S_t, t)e^{r(T-t)} + \int_t^T e^{r(T-u)} \frac{\partial \hat{V}}{\partial S}(S_u, u) \left[dS_u - (r-\delta) S_u du \right].$$

[2]It is interesting to note that the reformulation of option pricing theory from PDE operators to risk-neutral measures parallels the reformulation in quantum mechanics from a Hamiltonian approach involving operators in Hilbert space to a Lagrangian formulation involving Feynman path integrals.

Thus the final payoff is the sum of the future value of the investment $\hat{V}(S_t, t)$ and the accumulated gains from holding $\frac{\partial \hat{V}}{\partial S}(S_u, u)$ shares, where all purchases are financed by borrowing and all sales are invested in the bank. Absence of arbitrage requires that the value of the derivative at t is $\hat{V}(S_t, t)$ for all $t \in [0, T]$, where recall $\hat{V}$ was just a smooth function of S and t satisfying (11) and (12). Since the solutions to such boundary value problems are uniquely determined, it follows that the value V is a smooth function of S and t as was to be shown. Furthermore, the value $V(S, t)$ of a claim paying $f(S_T)$ at T satisfies the same boundary value problem as $\hat{V}(S, t)$.

V Is the Hedging Argument Given in the Black-Scholes Paper Correct?

Several published[8, 1] and unpublished[2] papers have intimated that the derivation of the p.d.e. given in the Black-Scholes paper is incorrect. These papers generally take the view that the end result is correct, but that the proof is unsound. They then offer more complicated hedging arguments which arrive at the same result.

This section argues that the proof given in the Black Scholes paper can be made strictly correct, simply by replacing the mathematical operation of taking a total derivative with the financial operation of computing the gain on a portfolio. The hedging argument given in the Black Scholes paper fixed the number of shares held at one and varied the number of call options written. Since dynamic trading strategies in options are expensive in practice, the usual textbook derivation instead fixes the number of options written at one and then varies the number of shares held long. Without loss of generality, we will focus on the textbook derivation. Thus, consider the value of a hedge portfolio at t consisting of one written derivative security and $N_1(S, t)$ shares held long:

$$H(S_t, t) = -C(S_t, t) + N_1(S_t, t)S_t.$$

Following the Black Scholes derivation, textbooks generally offer that the total derivative of the portfolio is given by:

$$dH(S_t, t) = -dC(S_t, t) + N_1(S_t, t)dS_t.$$

Now, ordinary calculus requires that the total derivative of the function N_1 must also be computed and stochastic calculus further requires that this total derivative be multiplied by dS_t. The standard textbook argument for not including these terms is that the number of shares held is "instantaneously constant". To a mathematician, this argument must be perplexing since the total variation of the number of shares held in any finite time interval is in fact infinite. In fact, the number of shares is changing so fast that the ordinary rules of calculus do not apply.

The way out of this conundrum is to recognize that one does not want to compute the total derivative of the value of the hedge portfolio. Instead one is interested in the gain on the portfolio *defined* by:

$$gH(S_t, t) \equiv -dD(S_t, t) + N_1(S_t, t)[dS_t + \delta S_t dt], \quad t \in [0, T].$$

Substituting (5) and setting $N_1(S_t, t) = \frac{\partial V}{\partial S}(S_t, t)$ yields:

$$gH(S_t, t) = \left[\frac{\partial V}{\partial t}(S_t, t) + \frac{\sigma^2(S_t, t)S_t^2}{2}\frac{\partial^2 V}{\partial S^2}(S_t, t) + \delta S_t \frac{\partial V}{\partial S}(S_t, t)\right] dt, \quad t \in [0, T].$$

Since this gain is riskless, absence of arbitrage requires that it be the same as the interest gain on a dynamic position $B(S, t)$ in the riskless asset chosen so as to finance all trades in the option and the stock:

$$gB(S_t, t) = r\left[-V(S_t, t) + \frac{\partial V}{\partial S}(S_t, t)S_t\right] dt \quad t \in [0, T].$$

Equating the gains then leads to the Black Scholes p.d.e. (9).

It is worth noting that the portfolio consisting of the option and stock is *not* self-financing. Similarly, positions in the riskless asset are not self-financing. Nonetheless, by showing that the gains between two non-self-financing strategies are always equal under no arbitrage, the value of the derivative security can be determined. We note in passing that this result holds even if gains are stochastic. Thus, a nonself-financing strategy in a portfolio consisting of a static position in a derivative security and a dynamic strategy in the riskless asset can be made to have the same stochastic gains as a nonself-financing strategy in the stock (under no arbitrage). The derivative security value can again be obtained through this principle.

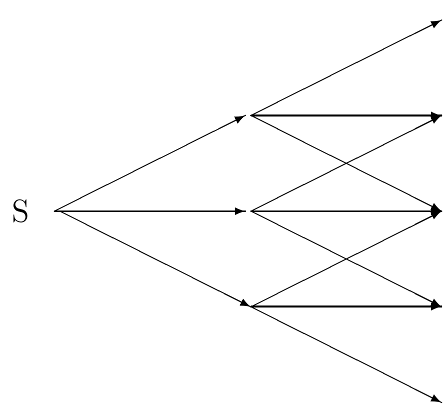
VI Can One Hedge Options in a Trinomial Model?

The binomial model of Rendleman and Bartter[11] and Cox, Ross, and Rubinstein[7] is a pedagogical marvel. To the uninitiated, an even more natural model is the trinomial model which captures the simple idea that the price of the asset can either move up, move down, or stay the same. Just as two assets (eg. stock and bond) can be used to hedge a derivative when there are two states, three assets can be used to hedge a derivative when there are three states. This is most obvious in a single period setting. Consider a stock which can double, halve, or remain at its initial price of \$2. To hedge a call struck at \$3, consider using a bond, a stock, and an at-the-money call. Let N^b, N^s, and N^a denote the number of bonds, shares, and at-the-money calls held in the hedge portfolio. Equating portfolio values to desired payoffs in the 3 states gives 3 linear equations in the 3 unknowns:

$$\begin{aligned}
\text{up} \quad N^b 1 + N^s 4 + N^a 2 &= \ 1 \\
\text{same} \quad N^b 1 + N^s 2 + N^a 0 &= \ 0 \\
\text{down} \quad N^b 1 + N^s 1 + N^a 0 &= \ 0.
\end{aligned}$$

The unique solution to this system is $N^b = 0$, $N^s = 0$, and $N^a = \frac{1}{2}$. Thus, given that the initial price of the at-the-money call is $\$\frac{2}{3}$, the arbitrage-free value of the call struck at \$3 is $\$\frac{1}{3}$.

Now consider a two period recombining trinomial tree as shown below:



As in the binomial model, we assume for simplicity that the gross periodic stock returns u, m, and d are constant over time and price. We also assume that interest rates and dividend yields are constant for simplicity. More specifically, let R be the periodic gross riskfree return (eg. 1.10 per period) and similarly let Y be the periodic gross dividend yield (eg. 1.03 per period). Thus an initial investment of one dollar in the riskless asset is worth R dollars after one period and R^2 after two. Similarly, the purchase of one share grows into Y shares after one period if the dividend is reinvested and this grows into Y^2 shares after reinvesting the dividend at period two.

If we mimic the backward recursion used in the binomial model, we quickly face a hurdle. Imagine placing ourselves at the upper node and the middle date, i.e. when the stock price has risen over the first period. Then since only 3 possible prices yawn before us, the situation appears to be identical to the single period problem. However, there is a subtle difference. In the single period problem, the price of the at-the-money was known to us and used to derive the value of a call with another strike. In the two period case, the time 1 price of any option is unknown because the only known prices are those at the root of the tree. At the middle date upper node, there are 3 possible future stock prices but only two known time 1 prices. In a sense, the future stock prices are like people in that two is company but three is a crowd.

As is usually the case, the way out of this morass is to resurrect the linearity principle, which holds in discrete time models as well. We will consider successively more difficult situations to show how the linearity principle can be used to hedge and value derivatives. First suppose that the derivative's payoff at the end of the second period is linear in the stock price:

$$D_2 = N_0 + N_1 S_2.$$

Then the initial value must be linear in the initial stock price to avoid arbitrage:

$$D_0 = N_0 R^{-2} + N_1 S_0 Y^{-2}.$$

Next suppose that the derivative's payoff at the end of the second period is nonlinear in the stock price. For example, suppose that the payoff is quadratic:

$$Q_2(S_2) = (S_2 - 2)^2, \quad S_2 > 0.$$

The graph below shows the 5 possible payouts given that the stock price starts at $2 and over each period, it can double, halve, or remain the same.

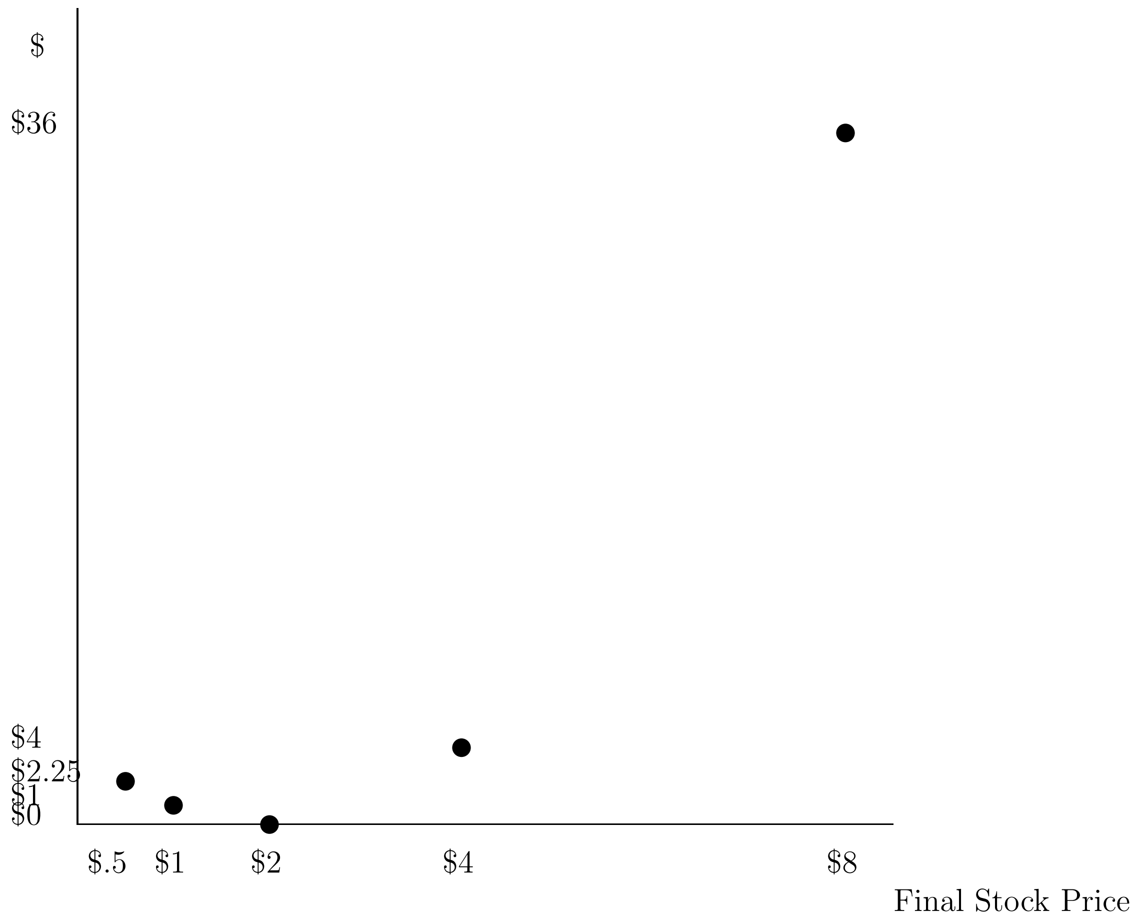


Figure 2: Payoff of Square of Stock Price Minus Two.

There are a couple of ways to linearize this payoff without resorting to passing to the continuous time limit. Both methods are made easier to understand by introducing the fundamental notion of a butterfly spread whose payoff is graphed below: Although the kinks can be placed at any option strike and the height is arbitrary, we will restrict attention to butterfly spreads with kinks only at possible stock prices and with heights of one.

In the first method, linearization is achieved by drawing a line through any 2 of the 5 possible payoffs and then using (static) positions in butterfly spreads centered at the other 3 to draw the payoff into line.

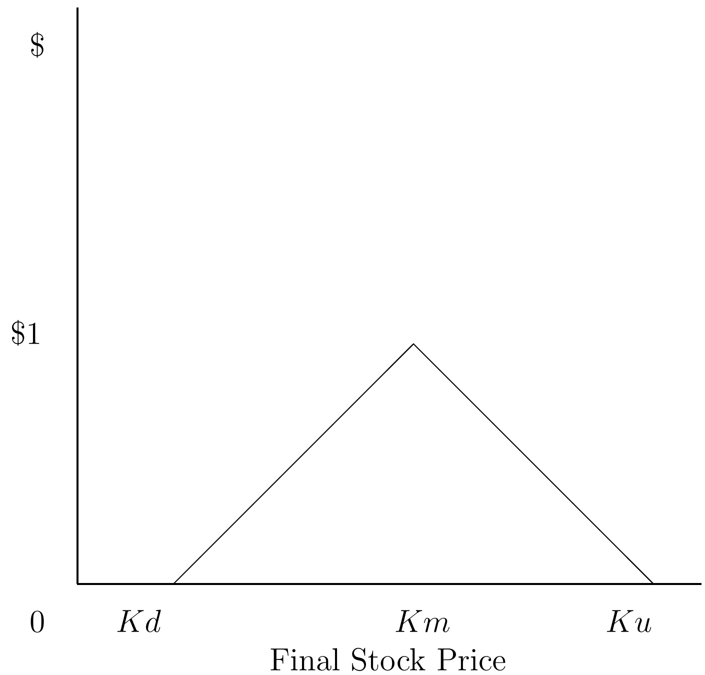


Figure 3: Payoff of a Butterfly Spread.

As before, let N_0 and N_1 denote the respective intercept and slope of the linearized payoff:

$$L_2(S_2) = N_0 + N_1 S_2, \quad S_2 > 0.$$

For example, if the second and third points from the left are used to draw a line, then the line goes through $(Sdm, D_{dm}) = (1, 1)$ and $(Smm, D_{mm}) = (2, 0)$ and so has equation:

$$L_2(S_2) = 2 - S_2, \quad S_2 > 0.$$

with intercept $N_0 = 2$ and slope $N_1 = -1$. The table below calculates the number of butterfly spreads to hold struck at each of the remaining three possible stock prices:

Stock Price	$L_2(S_2)$	$Q_2(S_2)$	# Butterfly Spreads
$\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{4}$	$-\frac{3}{4}$
4	-2	4	-6
8	- 6	36	-42

The second column computes the desired linear payoff at each of the 3 remaining possible stock prices, while the third column computes the quadratic payoff. The fourth column calculates the number of butterfly spreads to hold at each strike of interest as the difference between the second and third columns.

Since the quadratic payoff has been linearized by shorting butterfly spreads, the linearity principle can be invoked. In our example, under zero interest rate and dividend yield (i.e. $R = Y = 1$), the linear payoff has zero initial value:

$$L_0 = 2 - 1(2) = 0,$$

since the initial stock price is $S_0 = 2$. To recover the value of the quadratic payoff, we need to add back the known premiums of the butterfly spreads:

$$Q_0 = L_0 + \frac{3}{4} BS_0 \left(\frac{1}{2}\right) + 6BS_0\left(4\right) + 42BS_0\left(8\right).$$

For example, if it so happens that all 3 butterfly spreads cost $\$\frac{1}{9}$, then the arbitrage-free value of the quadratic payoff is:

$$Q_0 = 0 + \frac{1}{9}\left[\frac{3}{4} + 6 + 42\right] = \frac{195}{36}.$$

To statically replicate the quadratic payoff, go long 2 unit bonds, short 1 stock, and long each of the 3 butterfly spreads using the absolute value of the quantities given by the last column in the above table. We note that the butterfly spreads struck at $\$\frac{1}{2}$ and \$8 are created simply by holding a put struck at $\$\frac{3}{2}$ and a call struck at \$7 respectively. In fact, static replication of the quadratic payoff can also be achieved by holding almost[3] any 3 options in place of the butterfly spreads.

The second method for linearizing involves combining dynamic trading in stock and bond with static positions in options. To obtain risk-neutral probabilities, we use forward induction. Specifically, suppose that we know that the initial price of a single period at-the-money call is \$4/7 and that the initial stock price is \$2 and finally that the single period unit bond price is \$6/7. Thus, the one period forward prices are 2/3, 7/3 and 1 for the call, stock, and bond respectively. The three risk neutral probabilities q_u, q_m, and q_d solve the system:

$$\begin{array}{rrl} \text{bond} & q_u + q_m + q_d = & 1 \\ \text{stock} & q_u 4 + q_m 2 + q_d 1 = & 7/3 \\ \text{call} & q_u 2 + q_m 0 + q_d 0 = & 2/3. \end{array}$$

[3]When combined with the payoffs from bond and stock, the payoffs from the 5 assets must span the quadratic payoff.

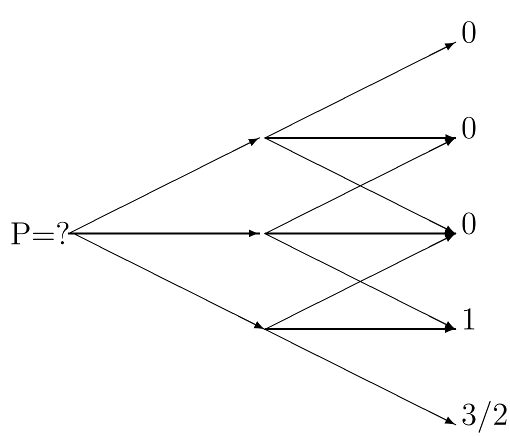
It is clear by inspection that the three risk-neutral probabilities are all 1/3.

Now place yourself at the middle date upper state. To get the risk-neutral probabilities of the 3 successor nodes, assume that we know the initial price of a 2 period call struck at 4 is 64/49. Since the call's only payoff of 4 occurs only if the stock goes up twice, we know:

$$\frac{1}{3}q_{uu}(\frac{6}{7})^2 4 = 64/49.$$

and hence $q_{uu} = 1/3$. By stationarity, the other two probabilities are $q_{um} = q_{ud} = 1/3$. Similarly, if we know the initial price of a two period European put struck at 1, we can back out q_{dd} and then q_{dm} and q_{du}. Finally, if we know the price of a European at-the-money call or put, the remaining three probabilities can be inferred. Since the risk-neutral price of each path is now known, it is straightforward to value the quadratic payoff

This second approach of combining dynamic trading in stock and bond with static replication in options can also be used to price path-dependent claims. To illustrate, consider valuing an at-the-money American put in our two period setup. If the stock goes down initially, then the put should be exercised since the final payoff if held alive is locally linear and thus the volatility value is zero. On all other paths, the put is either exercised only at maturity or not at all. The standard dynamic program begins by assigning final values to terminal nodes:



At this point, an FAQ is why are terminal values assigned to the lowest node when the only path to that node goes through an exercise point? The reason is that American options[4] display only binary path-dependence i.e. the only dependence on the path is through a dummy variable indicating whether the option has been previously exercised. The tree shows the values of only previously unexercised put values since previously exercised put values are not of interest.

Assuming that we know the price of 4 European options, the risk-neutral probabilities can again be determined. The American put is then valued in the standard manner.

In comparing the two approaches, we see that four European option prices were needed to determine the price of each path, while only three prices were needed to price the path-independent quadratic payoff. The market is said to be complete when the four prices are known and is said to be incomplete when only three are known. However, we can see that all European-style claims can still be priced in our incomplete market with only three option prices. In fact, the path dependent American put can be priced with only three option prices as well, since the local linearity of the payoff at the bottom three nodes implies that low strike options are not needed. The hedge for the American put would involve static positions in the three options along with dynamic trading in stock and bond. The positions can be determined by linearizing locally or even globally. In the latter case, the linear payoff across the 5 terminal nodes is propagated back one period and then the known[5] option values are subtracted off to get the continuation value of the American put. The larger of this value and the exercise value is computed and then linearized using the single period option. We can again progagate back and subtract off the known single period option value.

We note that in many options markets, there may not be enough options maturing at each date. Furthermore, as the number of periods grows beyond two, the number of options held grows exponentially. This shortage of available strikes can be addressed using dynamic replication with options as we now explore in the two period setup. Suppose that only the at-the-money European option price is available and we wish to price the quadratic payoff. To completely specify the price process for the at-the-money option, we need the three prices at the intermediate date. These can be obtained either directly or indirectly. In

[4] Barrier options also have binary path-dependence.

[5] Since all the relevant risk-neutral probabilities are known, the market is effectively complete.

the direct approach, we guess three values and hope that no arbitrage is introduced. Since this can be quite tricky, we advocate the indirect approach in which we specify the risk-neutral process directly and compute the values of the three hedging instruments at each node by discounting expected values. There are many ways to specify a risk-neutral process which in general trade off the competing requirements of tractability and realism.

Just as one need only dynamically trade a stock and bond in a binomial model, one *can* only dynamically trade in three assets in a trinomial model. However, in both the binomial and trinomial models, these strategies ignore useful information. A better approach in both cases is to use all of the information in the initial option prices and then to impose criteria such as smoothness or distance from a prior to select a single risk-neutral process. To minimize model risk, the hedge strategy should use all of the available instruments. Multivariate linear regression can be used recursively to choose the positions so that next period values of the target claim are as close to linear in hedging asset prices as possible.

VII Can Jumps be Hedged in a Continuous Time Model?

It is a wide-spread fallacy on Wall Street that jumps cannot be hedged. The binomial and trinomial models can be viewed as continuous time jump models with known jump times. At the opposite end of the spectrum are Poisson processes with completely unanticipated jump times. Cox and Ross showed that options on stocks following such processes can be hedged if the jump size is known. The intuition is identical to the binomial model in that over each trading interval, only two prices are possible. Thus, the value at the next trading interval of the derivative security is locally linear in the stock price. Just as the statistical probabilities are irrelevant in the binomial model, the statistical arrival rate is irrelevant in this Poisson model. The statistical drift does matter and thus observed option prices can be used to imply out this important and otherwise difficult to estimate parameter. Note that the jump size does not have to be the same over time and in general can depend on the path.

By analogy, a jump process with two possible jump sizes can be hedged by dynamic trading in a bond, stock, and call. To avoid arbitrage in the process specification, the risk-neutral arrival rates for the two jump sizes would be specified in a manner consistent with observed prices, while the statistical arrival rates are again irrelevant. By extension, a continuous jump size distribution as in Merton's lognormal jump model requires continuous trading in a continuum of assets in order to make markets complete. However, as in the discrete time case, many interesting contracts may still be spanned in incomplete markets. Furthermore, not all options are used up in the hedge/calibration, so that for example it is still theoretically possible to price options of one maturity in terms of options of another maturity.

VIII Why does the "market price of risk" appear in many stochastic interest rate model or volatility models? Why doesn't it appear in the HJM model

Since hedge portfolios are riskless only locally, the spot rate of interest appears in every model, unless it so happens as in Merton[9] that the hedge portfolio is also costless. Recognizing this, the early literature on interest rate derivatives modelled the spot rate process exogenously. The seminal Vasicek paper showed that the values of interest rate derivatives appear to depend on the market price of interest rate risk. Yet no such concept appears in the Black model for a bond option or in the HJM model. Why? The reason is that the Black model relates option prices to bond *prices*, while the HJM model relates option prices to a known mathematical function of bond prices, namely forward rates. In the Vasicek model, if one writes down the PDE for the function relating the option price to the bond price (rather than to the spot rate), then the market price of interest rate risk drops out.

We now demonstrate this phenomenon in the more general context of single factor spot rate models. If the statistical process for the spot rate r is given by:

$$dr_t = b(r_t, t)dt + a(r_t, t)dW_t,$$

then the absence of arbitrage implies that the price $C(r, t)$ of any non-coupon paying claim obeys:

$$\frac{\partial C}{\partial t} + \frac{a^2}{2}\frac{\partial^2 C}{\partial r^2} + [b + \lambda a]\frac{\partial C}{\partial r} = rC, \tag{13}$$

where $\lambda(r,t)$ is the market price of interest rate risk. It follows that the price of a pure discount bond, $P(r,t)$ obeys the same PDE:

$$\frac{\partial P}{\partial t} + \frac{a^2}{2}\frac{\partial^2 P}{\partial r^2} + [b + \lambda a]\frac{\partial P}{\partial r} = rP, \tag{14}$$

Suppose that we relate the claim price to the bond price by defining the function $V(P,t)$ by:

$$C(r,t) = V(P(r,t),t). \tag{15}$$

Differentiating w.r.t. t and r implies:

$$\begin{aligned}
\frac{\partial C}{\partial t} &= \frac{\partial V}{\partial P}\frac{\partial P}{\partial t} + \frac{\partial V}{\partial t} \\
\frac{\partial C}{\partial r} &= \frac{\partial V}{\partial P}\frac{\partial P}{\partial r} \\
\frac{\partial^2 C}{\partial r^2} &= \frac{\partial^2 V}{\partial P^2}\left(\frac{\partial P}{\partial r}\right)^2 + \frac{\partial V}{\partial P}\frac{\partial^2 P}{\partial r^2}.
\end{aligned} \tag{16}$$

Substituting (15) to (16) in (13) yields:

$$\frac{\partial V}{\partial P}\frac{\partial P}{\partial t} + \frac{\partial V}{\partial t} + \frac{a^2}{2}\left[\frac{\partial^2 V}{\partial P^2}\left(\frac{\partial P}{\partial r}\right)^2 + \frac{\partial V}{\partial P}\frac{\partial^2 P}{\partial r^2}\right] + [b + \lambda a]\frac{\partial V}{\partial P}\frac{\partial P}{\partial r} = rV, \tag{17}$$

Substituting (14) in (17) yields the following generalization of the Black model PDE:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 P^2}{2}\frac{\partial^2 V}{\partial P^2} + r\frac{\partial V}{\partial P} = rV, \tag{18}$$

where:

$$\sigma(r,t) \equiv \frac{a(r,t)}{P(r,t)}\frac{\partial P}{\partial r}(r,t)$$

is the volatility of bond returns, which is being modelled here as a known function of the spot rate and time. The market price of risk λ does not appear in (18). The Feynman Kac theorem implies that the values of claims paying off $f(P_T)$ at T have the form:

$$V(P,t) = E^Q[e^{-\int_t^T r_u du} f(P_T)|P_t = P],$$

where the measure Q is defined so that:

$$\frac{dP_t}{P_t} = r_t dt + \sigma_t dW_t, t \in [0,T]. \tag{19}$$

The market price of risk also does not appear whenever option prices are related to known functions of the bond price. Brennan and Schwartz[5] showed that the market price of long rate risk does not appear in their valuation PDE when the long rate is used as a state variable, since it is known to be the reciprocal of the perpetuity value. Similarly, since the term structure of forward rates $f_t(T) \equiv -\frac{\partial \ln P_t(T)}{\partial T}$, is a known function of the term structure of bond prices, the market price of risk does not appear in the HJM model. The risk-neutral dynamics of the forward rate in the single factor[6] version of this model are easily obtained as follows. Applying Itô's lemma to (19) implies that:

$$d[\ln P_t(T)] = \left[r_t - \frac{\sigma_t^2(T)}{2}\right] dt + \sigma_t(T) dW_t, \quad t \in [0, T]. \tag{20}$$

Differentiating w.r.t. T and negating implies:

$$df_t(T) = \sigma_t(T)\sigma_t'(T)dt - \sigma_t'(T)dW_t, \quad t \in [0, T]. \tag{21}$$

Defining $v_t(T) \equiv -\sigma_t'(T)$ as the forward rate volatility, the risk-neutral dynamics of the forward rate are:

$$df_t(T) = \int_t^T v_t(u)du v_t(T)dt + v_t(T)dW_t, t \in [0, T]. \tag{22}$$

As is well-known, the HJM approach specifies forward rate volatility $v_t(T)$ directly, and determines the bond volatility $\sigma_t(T)$ and risk-neutral dynamics for the spot rate $r_t \equiv f_t(t)$ endogenously.

The same observations apply for stochastic volatility models. Thus, one can develop stochastic volatility models which do not depend on the market price of volatility risk by relating option prices to either other option prices or to known functions of option prices such as "forward local volatility" or Black Scholes implied volatility. To illustrate these points, we show how the market price of volatility risk drops out when an option price is related to another option price. Using standard arguments, one can derive the following PDE governing the function $C^{(1)}(t, X, Y)$ relating the price of an option to time t, the price of the underlying stock X, and the price of a state variable governing volatility Y:

$$\frac{\partial C^{(1)}}{\partial t} + r\left[X\frac{\partial C^{(1)}}{\partial X} - C^{(1)}\right] + [\alpha(m-Y) - \lambda\beta]\frac{\partial C^{(1)}}{\partial Y} + \frac{f^2(Y)}{2}\frac{\partial^2 C^{(1)}}{\partial X^2} + \rho f(Y)\beta\frac{\partial^2 C^{(1)}}{\partial X \partial Y} + \frac{\beta^2}{2}\frac{\partial^2 C^{(1)}}{\partial Y^2} = 0, \tag{23}$$

[6]See Musiela and Rutkowski, page 310 for the multi-factor result.

where λ is the market price of volatility risk. The above PDE also holds for the price of a second option:

$$\frac{\partial C^{(2)}}{\partial t}+r\left[X\frac{\partial C^{(2)}}{\partial X}-C^{(2)}\right]+[\alpha(m-Y)-\lambda\beta]\frac{\partial C^{(2)}}{\partial Y}+\frac{f^2(Y)}{2}\frac{\partial^2 C^{(2)}}{\partial X^2}+\rho f(Y)\beta\frac{\partial^2 C^{(2)}}{\partial X\partial Y}+\frac{\beta^2}{2}\frac{\partial^2 C^{(2)}}{\partial Y^2}=0. \quad (24)$$

Let $\gamma(t, X, C^{(2)})$ be the function relating the price of the first option to time t, the underlying stock price X, and the price of the second option:

$$\gamma(t, X, C^{(2)}) \equiv C^{(1)}(t, X, Y),$$

where $C^{(2)}$ solves (24). Equivalently:

$$C^{(1)}(t, X, Y) = \gamma(t, X, C^{(2)}(t, X, Y)) \quad (25)$$

Differentiating once:

$$\frac{\partial C^{(1)}}{\partial t} = \gamma_1 + \gamma_3\frac{\partial C^{(2)}}{\partial t} \quad (26)$$

$$\frac{\partial C^{(1)}}{\partial X} = \gamma_2 + \gamma_3\frac{\partial C^{(2)}}{\partial X} \quad (27)$$

$$\frac{\partial C^{(1)}}{\partial Y} = \gamma_3\frac{\partial C^{(2)}}{\partial Y} \quad (28)$$

Differentiating one more time:

$$\frac{\partial^2 C^{(1)}}{\partial X^2} = \gamma_{22} + 2\gamma_{23}\frac{\partial C^{(2)}}{\partial X} + \gamma_{33}\left(\frac{\partial C^{(2)}}{\partial X}\right)^2 + \gamma_3\frac{\partial^2 C^{(2)}}{\partial X^2} \quad (29)$$

$$\frac{\partial^2 C^{(1)}}{\partial X\partial Y} = \gamma_{23}\frac{\partial C^{(2)}}{\partial Y} + \gamma_{33}\frac{\partial C^{(2)}}{\partial X}\frac{\partial C^{(2)}}{\partial Y} + \gamma_3\frac{\partial^2 C^{(2)}}{\partial X\partial Y} \quad (30)$$

$$\frac{\partial^2 C^{(1)}}{\partial Y^2} = \gamma_{33}\left(\frac{\partial C^{(2)}}{\partial Y}\right)^2 + \gamma_3\frac{\partial^2 C^{(2)}}{\partial Y^2} \quad (31)$$

Substituting (26) to (31) in (23):

$$\gamma_1 + \gamma_3\tfrac{\partial C^{(2)}}{\partial t} + r[X[\gamma_2 + \gamma_3\tfrac{\partial C^{(2)}}{\partial X}] - \gamma] + [\alpha(m-Y) - \lambda\beta]\gamma_3\tfrac{\partial C^{(2)}}{\partial Y}$$
$$+\tfrac{f^2(Y)}{2}[\gamma_{22} + 2\gamma_{23}\tfrac{\partial C^{(2)}}{\partial X} + \gamma_{33}\left(\tfrac{\partial C^{(2)}}{\partial X}\right)^2 + \gamma_3\tfrac{\partial^2 C^{(2)}}{\partial X^2}] + \rho f(Y)\beta[\gamma_{23}\tfrac{\partial C^{(2)}}{\partial Y} + \gamma_{33}\tfrac{\partial C^{(2)}}{\partial X}\tfrac{\partial C^{(2)}}{\partial Y} + \gamma_3\tfrac{\partial^2 C^{(2)}}{\partial X\partial Y}]$$
$$+\tfrac{\beta^2}{2}[\gamma_{33}\left(\tfrac{\partial C^{(2)}}{\partial Y}\right)^2 + \gamma_3\tfrac{\partial^2 C^{(2)}}{\partial Y^2}] = 0.$$

Re-arranging terms:

$$\gamma_1 + r[X\gamma_2 - \gamma]$$
$$+\gamma_3\left[\frac{\partial C^{(2)}}{\partial t} + rX\frac{\partial C^{(2)}}{\partial X} + [\alpha(m-Y) - \lambda\beta]\frac{\partial C^{(2)}}{\partial Y} + \frac{f^2(Y)}{2}\frac{\partial^2 C^{(2)}}{\partial X^2} + \rho f(Y)\beta\frac{\partial C^{(2)}}{\partial X}\frac{\partial C^{(2)}}{\partial Y} + \frac{\beta^2}{2}\frac{\partial^2 C^{(2)}}{\partial Y^2}\right]$$
$$+\frac{f^2(Y)}{2}\gamma_{22} + \gamma_{23}\left[f^2(Y)\frac{\partial C^{(2)}}{\partial X} + \rho f(Y)\beta\frac{\partial C^{(2)}}{\partial X}\frac{\partial C^{(2)}}{\partial Y}\right] + \gamma_{33}\left[\frac{f^2(Y)}{2}\left(\frac{\partial C^{(2)}}{\partial X}\right)^2 + \rho f(Y)\beta\frac{\partial C^{(2)}}{\partial X}\frac{\partial C^{(2)}}{\partial Y} + \frac{\beta^2}{2}\left(\frac{\partial C^{(2)}}{\partial Y}\right)^2\right] = 0.$$

Substituting (24) into the above simplifies the result to:

$$\gamma_1 + r[X\gamma_2 - \gamma] + rC^{(2)}\gamma_3$$
$$+\frac{f^2(Y)}{2}\gamma_{22} + \gamma_{23}\left[f^2(Y)\frac{\partial C^{(2)}}{\partial X} + \rho f(Y)\beta\frac{\partial C^{(2)}}{\partial X}\frac{\partial C^{(2)}}{\partial Y}\right] + \gamma_{33}\left[\frac{f^2(Y)}{2}\left(\frac{\partial C^{(2)}}{\partial X}\right)^2 + \rho f(Y)\beta\frac{\partial C^{(2)}}{\partial X}\frac{\partial C^{(2)}}{\partial Y} + \frac{\beta^2}{2}\left(\frac{\partial C^{(2)}}{\partial Y}\right)^2\right] = 0.$$

Now the risk-neutral dynamics of X are:

$$dX_t = rX_t dt + f(Y_t)dW_{1t}.$$

Note that the variance rate of X is:

$$\mathrm{Var}(dX) = f^2(Y_t)dt.$$

The risk-neutral dynamics of $C^{(2)}$ are given by:

$$dC^{(2)} = rC^{(2)}dt + \frac{\partial C^{(2)}}{\partial X}f(Y_t)dW_{1t} + \frac{\partial C^{(2)}}{\partial Y}\beta dW_{2t},$$

where $dW_{1t}dW_{2t} = \rho dt$. Note that the covariance rate of $C^{(2)}$ with X is:

$$\mathrm{Cov}(dC^{(2)}, dX) = \left[f^2(Y)\frac{\partial C^{(2)}}{\partial X} + \rho f(Y)\beta\frac{\partial C^{(2)}}{\partial X}\frac{\partial C^{(2)}}{\partial Y}\right]dt,$$

while the variance rate of $C^{(2)}$ is:

$$\mathrm{Var}(dC^{(2)}) = \left[f^2(Y)\left(\frac{\partial C^{(2)}}{\partial X}\right)^2 + 2\rho f(Y)\beta\frac{\partial C^{(2)}}{\partial X}\frac{\partial C^{(2)}}{\partial Y} + \beta^2\left(\frac{\partial C^{(2)}}{\partial Y}\right)^2\right]dt.$$

Substituting into the above PDE gives:

$$\gamma_1 + r[X\gamma_2 - \gamma] + rC^{(2)}\gamma_3 + \frac{\mathrm{Var}(df)}{2}\gamma_{22} + \mathrm{Cov}(dC^{(2)}, dX)\gamma_{23} + \frac{\mathrm{Var}(dC^{(2)})}{2}\gamma_{33} = 0. \qquad (32)$$

If one can exogenously model the volatility structure of $C^{(2)}$ without reference to λ, then one does not need to specify λ. There are many ways to do this.

IX Which volatility should one hedge at - historical or implied?

It is difficult to answer this question in complete generality, so we will instead attempt an answer to this FAQ, while ensconsed safely within the confines of an overly familiar model. Thus, we will assume that all of the standard assumptions of the Black Scholes model hold except for the critical one that the volatility rate is known. We assume frictionless markets, no arbitrage, a constant riskfree rate $r \geq 0$, and a constant continuous dividend yield $\delta \geq 0$ over a time period $[0, T]$. Included within the rubric of the frictionless markets assumption is the highly assailable proposition that the option writer is able to borrow money or short sell stock in arbitrarily large amounts, notwithstanding the embarrassing possibility that the writer will have sustained arbitrarily large paper losses when marked to market. The stock price process $\{S_t, t \in [0, T]\}$ is assumed to be continuous and of the form:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \qquad t \in [0, T],$$

where S_0 is given and the drift rate μ_t and the diffusion rate σ_t are arbitrary adapted stochastic processes. For any positive constant σ, let $V(S, t; \sigma)$ be the unique $C^{2,1}$ function solving the Black Scholes p.d.e.:

$$\frac{\partial}{\partial t} V(S, t; \sigma) + \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} V(S, t; \sigma) + (r - \delta) S \frac{\partial}{\partial S} V(S, t; \sigma) - rV(S, t; \sigma) = 0, \tag{33}$$

subject to the terminal condition:

$$\lim_{t \uparrow T} V(S, t; \sigma) = f(S), \tag{34}$$

where f should be continuous, but need not be differentiable everywhere.

Suppose that at time 0, a trader sells a European-style claim with terminal payoff $f(S_T)$ for the Black Scholes model value $V(S_0, 0; \sigma_i)$, where σ_i is the initial implied volatility. Let N_t denote the number of shares held by the trader at time $t \in [0, T]$ and let σ_h denote the assumed constant hedge volatility. To hedge the sale of the claim, the trader initially buys $N_0 = \frac{\partial}{\partial S} V(S_0, 0; \sigma_h)$ shares, each at price S_0. Let $\beta_t \geq 0$ denote cumulative borrowing at time $t \in [0, T]$. The initial borrowing is the difference between the cost of setting up the initial stock hedge and the proceeds from the sale of the claim:

$$\beta_0 \;=\; \frac{\partial}{\partial S} V(S_0, 0; \sigma_h) S_0 - V(S_0, 0; \sigma_i)$$

$$= \left[\frac{\partial}{\partial S}V(S_0,0;\sigma_h)S_0 - V(S_0,0;\sigma_h)\right] + [V(S_0,0;\sigma_h) - V(S_0,0;\sigma_i)]. \quad (35)$$

We now assume that the trader follows an equity trading strategy where all stock purchases are financed by borrowing and all stock sales are used to reduce cumulative borrowing. We next note that cumulative borrowing is also affected by carrying costs. The cumulative borrowings at t, denoted β_t will grow at the riskfree rate r over time. Furthermore, the stock position pays dividends which reduces cumulative borrowings. Thus at each t, the change in cumulative borrowings can be expressed as:

$$d\beta_t = \underbrace{dN_t(S_t + dS_t)}_{\text{dollar cost of buying shares}} + \underbrace{r\beta_t dt}_{\text{additional interest}} - \underbrace{N_t \delta S_t dt}_{\text{dividends received}}.$$

The first term can be represented as the change in the dollar value of the stock position, *less* the portion of that change due to capital gains on the shares held:

$$d\beta_t = \underbrace{d(N_t S_t)}_{\text{change in \$ value of stock position}} - \underbrace{N_t dS_t}_{\text{capital gains in stock}} - \underbrace{N_t \delta S_t dt}_{\text{dividends received}} + \underbrace{r\beta_t dt}_{\text{additional interest}}. \quad (36)$$

Suppose that the trader holds the number of shares warranted by the Black Scholes model:

$$N_t = \frac{\partial}{\partial S}V(S_t,t;\sigma_h). \quad (37)$$

Now, by Itô's Lemma:

$$\begin{aligned} dV_t &= \left[\frac{\partial}{\partial t}V(S_t,t;\sigma_h) + \frac{\sigma_t^2 S_t^2}{2}\frac{\partial^2}{\partial S^2}V(S_t,t;\sigma_h)\right]dt + \frac{\partial}{\partial S}V(S_t,t;\sigma_h)dS_t \\ &= (\sigma_t^2 - \sigma_h^2)\frac{S_t^2}{2}\frac{\partial^2}{\partial S^2}V(S_t,t;\sigma_h)dt + r\left[V(S_t,t;\sigma_h) - S_t\frac{\partial}{\partial S}V(S_t,t;\sigma_h)\right]dt + \delta S_t\frac{\partial}{\partial S}V(S_t,t;\sigma_h)dt \\ &\quad + \frac{\partial}{\partial S}V(S_t,t;\sigma_h)dS_t \end{aligned}$$

from the Black Scholes p.d.e. (33). Solving for the last two terms and substituting this result and (37) in (36) gives:

$$\begin{aligned} d\beta_t &= d\left[S_t\frac{\partial}{\partial S}V(S_t,t;\sigma_h)\right] - dV(S_t,t;\sigma_h) + r\left[V(S_t,t;\sigma_h) - S_t\frac{\partial}{\partial S}V(S_t,t;\sigma_h)\right]dt + r\beta_t dt \\ &+ (\sigma_t^2 - \sigma_h^2)\frac{S_t^2}{2}\frac{\partial^2}{\partial S^2}V(S_t,t;\sigma_h)dt. \end{aligned}$$

The solution to this equation is:

$$\begin{aligned}\beta_t &= \beta_0 e^{rt} + S_t \frac{\partial}{\partial S} V(S_t, t; \sigma_h) - V(S_t, t; \sigma_h) - e^{rt}\left[S_0 \frac{\partial}{\partial S} V(S_0, 0; \sigma_h) - V(S_0, 0; \sigma_h)\right] \\ &+ \int_0^t e^{-rs}(\sigma_s^2 - \sigma_h^2)\frac{S_s^2}{2}\frac{\partial^2}{\partial S^2} V(S_s, s; \sigma_h) ds.\end{aligned}$$

Substituting in the initial borrowings from (35), by maturity (i.e.$t = T$), the borrowings will have accumulated to:

$$\begin{aligned}\beta_T &= [V(S_0, 0; \sigma_i) - V(S_0, 0; \sigma_h)]e^{rT} + S_T f'(S_T) - f(S_T) \\ &+ \int_0^T e^{r(T-t)}(\sigma_t^2 - \sigma_h^2)\frac{S_t^2}{2}\frac{\partial^2}{\partial S^2} V(S_t, t; \sigma_h) dt,\end{aligned} \tag{38}$$

from (37) and (34), where $f'(S)$ may be a generalized function. Equations (37) and (34) also imply that the terminal number of shares held is:

$$N_T = f'(S_T), \tag{39}$$

with each share having value S_T. Thus, the terminal P&L is:

$$\begin{aligned}P\&L_T &\equiv N_T S_T - \beta_T - f(S_T) \\ &= [V(S_0, 0; \sigma_i) - V(S_0, 0; \sigma_h)]e^{rT} + \int_0^T e^{r(T-t)}(\sigma_h^2 - \sigma_t^2)\frac{S_t^2}{2}\frac{\partial^2}{\partial S^2} V(S_t, t; \sigma_h) dt,\end{aligned} \tag{40}$$

from (39) and (38). Notice that if $\sigma_t = \sigma_h$ for all $t \in [0, T]$, then $\beta_T = [V(S_0, 0; \sigma_t) - V(S_0, 0; \sigma_i)]e^{rT} + S_T f'(S_T) - f(S_T)$ from (38) and terminal $P\&L_T = [V(S_0, 0; \sigma_i) - V(S_0, 0; \sigma_t)]e^{rT}$ from (40). In words, if the true vol realizes to the constant hedge vol everywhere along the path, then the value of the leveraged stock position exceeds the payoff by the terminal P&L. If $f'' \geq 0$ as is the case for options, then V will be increasing in σ and so the P&L will be positive if the option writer is able to sell the option for a higher implied vol than is realized subsequently. Furthermore, if $f'' \geq 0$, then $\frac{\partial^2}{\partial S^2} V(S_t, t; \sigma_h) \geq 0$ for all $t \in [0, T]$. For such claims, if $\sigma_t > \sigma_h = \sigma_i$ for all $t \in [0, T]$, then (40) implies that $P\&L_T \leq 0$ regardless of the path. Conversely, if $\sigma_t < \sigma_h = \sigma_i$ for all $t \in [0, T]$, then (40) implies that $P\&L_T \geq 0$ regardless of the path. Also note that if $f(S) = 2 \ln S$, then the Black Scholes value is $V(S_t, t; \sigma_h) = e^{-r(T-t)} 2 \ln S_t$ and the Black Scholes gamma is $\frac{\partial^2}{\partial S^2} V(S_t, t; \sigma_h) = -e^{-r(T-t)}\frac{2}{S_t^2}$. Substituting into (40) and assuming $\sigma_i = \sigma_h$ implies that the terminal P&L is the payoff from a variance swap:

$$P\&L_T = \int_0^T (\sigma_t^2 - \sigma_h^2) dt.$$

X Summary

We did our best to answer several FAQ's. Here are a few more to try on your own:

1. Does risk-neutral valuation work over infinite horizons? Bear in mind that shorting a stock appears to be an arbitrage opportunity since the probability is one in many models (eg. Black Scholes) that a short investor can cover for a dollar lower later.

2. To what extent does expected return matter in incomplete markets? For example, consider a pure jump process and suppose an investor chooses an equivalent martingale measure and delta-hedges in the standard way. While the P&L distribution depends on expected return, Monte Carlo simulation suggests that there are many paths over which this dependence is mild.

3. How does the inability to withstand arbitrarily large losses impact standard valuation procedures? Why would anyone lend at the riskfree rate to imperfect credits engaging in risky strategies? If the lending rate varies with the credit risk, should the risk-neutral drift reflect this credit spread?

References

[1] Beck, T., 1993, "Black Scholes Revisited: Some Important Details", *The Financial Review*, **28**, 1, 77-90.

[2] Bergman, Y, 1992, "A Characterization of Self-Financing Portfolio Strategies" first essay in UC Berkeley doctoral dissertation entitled "Pricing of Contingent Claims in Perfect and in Imperfect Markets".

[3] Black, F., 1976, "The Pricing of Commodity Contracts", *Journal of Financial Economics*, **3**, 167-79.

[4] Black, F. and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities", *The Journal of Political Economy*, **81**, 637–659.

[5] Brennan M., and E. Schwartz, 1979, "A Continuous Time Approach to the Pricing of Bonds", *Journal of Banking and Finance.*

[6] Cox, J. and S. Ross, 1976, "The Valuation of Options for Alternative Stochastic Processes", *Journal of Financial Economics*, **3**, 145-166.

[7] Cox, J., S. Ross, and M. Rubinstein, 1979, "'Option Pricing: A Simplified Approach", *Journal of Financial Economics*, **7**, 229-63.

[8] Long, J.B., 1974, "Discussion", *Journal of Finance*, **29** May, 485-8.

[9] Merton, R., 1973, "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, **4**, 141–183.

[10] Musiela M., and M. Rutkowski, 1997, Martingale Methods in Financial Modelling, Springer-Verlag, Berlin, Heidelberg New York.

[11] Rendleman R., and B. Bartter, 1979, "Two-State Option Pricing", *Journal of Finance*, **34**, 1093-1110.