CAN YOU DO BETTER THAN KELLY IN THE SHORT RUN ?

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Abstract

The Kelly criterion, also known as the optimal growth or logarithmic utility strategy, is optimal, or near optimal, in a wide variety of settings when wealth grows purely multiplicatively. However, most of these optimality properties are over an infinite horizon or are asymptotic in nature. In this paper, we analyze some of the short-run properties of the Kelly strategy, and compare its performance to the dynamic state and time dependent policy that is optimal for maximizing the probability of achieving a given value, or outperforming another strategy, by a given fixed deadline.

1 Introduction

The Kelly criterion plays a key role in the mathematical analysis of favorable games. Arguably, the most important example of repeated sequences of favorable games is the standard portfolio problem of financial mathematics, for which the Kelly criterion is more formally known as the optimal growth portfolio strategy, or the logarithmic utility strategy.

In particular, for a wealth or gains process with continuously reinvested winnings, the Kelly criterion has a variety optimality properties, surveyed in the next section. However, the majority of these optimality properties hold over an infinite horizon, or only asymptotically. For example, one well known optimality property of the Kelly criterion is that with probability 1, it will eventually outperform any competing strategy. While this is of course is true, perhaps the attractiveness of this property might fade somewhat in the face of some finite-time calculations. For example, consider an economy with two assets: one risky with annual returns forming a sequence of independent and identically distributed normal random variables with a mean of 15% and a standard deviation of 30%. The other asset is riskless with a fixed annual return of 7%. Under this economic scenario, the Kelly criterion will place 89% of its wealth into the risky asset with the remaining 11% placed into

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the risk-free asset. As we will see, it takes 157 years before we can have a 95% probability that the Kelly strategy will outperform, by 10%, the strategy that keeps all its wealth in the riskless asset, which we will call cash for the sequel. Moreover, it will take 10,286 years for the Kelly strategy to outperform the stock itself by 10%, with 95% confidence, or probability.

While these numbers are somewhat disheartening, they are meaningless without a comparative benchmark. In this paper we will analyze a policy - the *probability maximizing* strategy of Browne [7] - for which these 95% limits fall to 1.3 years and 85 years, respectively. While these numbers are orders of magnitude better than those of the Kelly strategy, it of course cannot serve as an indictment of, or prove any dominance over, the Kelly criterion, since the probability maximizing strategy does take significantly more risk than the Kelly criterion. In particular, the expected time for the Kelly strategy to outperform cash by 10% is 2.8 years, and to outperform the stock by 10% will take an expected time of 184 years. However, for the probability maximizing strategy, the expected time to outperform any of these by 10% is *infinite*. Nevertheless, the strategy does have many attractive features relative to the Kelly criterion, as we discuss below.

The mathematical treatment here will be informal. Rigorous expositions of all claimed results can be found in the appropriate research papers cited in the bibliography.

2 Gambling/Investing on favorable games: Portfolio theory

The problem of optimal gambling in repeated favorable games is intimately related to the optimal multi-period portfolio investment problem of financial economics. The only difference in fact is the option of investing in a riskless security that pays a non-stochastic interest rate per unit.

Consider a sequence of risky gambles, $\{Z_i, i \geq 1\}$. In the portfolio problem, the interpretation of Z_i is the *return* on a risky security over period i. For the sequel, we will assume that these returns form an iid sequence, with $E(Z_i) = \mu$, and $Var(Z_i) = \sigma^2$. We will assume as well that there is also a riskless asset available for investment, with a constant return R per unit time. We will consider here only favorable games, with $\mu > R \geq 0$.

Let π_n denote the fraction of wealth invested in the risky security over the nth investment period, and let $\{X_n, n \geq 1\}$ denote the associated wealth process, i.e., X_n will denote the wealth immediately after the nth investment period.

Since we assume that all wealth not invested in the risky security is invested in the riskless, the wealth process evolves according to

$$X_n = X_{n-1} + (\pi_n X_{n-1}) Z_n + X_{n-1}(1 - \pi_n)R = X_{n-1} (1 + R + \pi_n [Z_n - R]) .$$

The solution to this first-order stochastic difference equation is

$$X_n = X_0 \prod_{i=1}^{n} (1 + R + \pi_i [Z_i - R]) . \tag{1}$$

The problem of interest now is to determine an *optimal* investment strategy for the process X. Optimality is defined relative to a particular objective function. Economic theory might postulate, for example, that in the absence of any other expenses (such as consumption), an investor would choose a strategy to maximize utility from terminal wealth: for a given utility function $U(\cdot)$, and a given time N, the investor *dynamically* chooses a policy $\{\pi_1^*, \pi_2^*, \ldots, \pi_N^*\}$ to maximize $E\left[U(X_N)\right]$. If we let $F(x,n)$ denote the maximal possible expected utility from an initial wealth of \$$x$, and with n investment periods to go, i.e., $F(x,n) = \max E\left[U(X_N) \mid X_{N-n} = x\right]$, then the optimal policy is determined from the dynamic programming equation

$$F(x,n) = \max_{\pi} E\left[F\left(x\left[1+R+\pi\left(Z-R\right)\right], \, n-1\right)\right]$$

with the boundary condition $F(x,0) = U(x)$. Here Z is a random variable having the same distribution as the Z_i's.

Of particular interest is the case where the optimal dynamic policy is *constant*: i.e., where $\pi_1^* = \pi_2^* = \cdots =$ constant. A characterization theorem for when this occurs was provided already by Bellman and Kalaba in 1957, and rediscovered decades later by various economists.

Theorem 1 (Bellman and Kalaba 1957, Hakansson 1970) *For the expected utility problem outlined above, the optimal dynamic policy is constant if and only if the utility function is of the form $U(x) = x^\alpha$, for some $\alpha < 1$, or $U(x) = \ln(x)$.*

Before we proceed, observe that a constant proportions policy is somewhat of a contrarion policy; to keep the proportions of wealth held in the risky stock and the riskless asset constant, one needs to *sell* shares of the stock when its price rises, and to *buy* shares of the stock when its price falls.

Bellman and Kalaba's result was precipitated by Kelly's influential paper [19] published the year before. Although Kelly (and Bellman and Kalaba as well) only treated the case where the Z_i's are increments in a simple random walk, his basic idea generalizes to the following (see e.g., Breiman [5], Thorp [26]): consider a constant proportions policy, $\pi_i = \pi$ for all $i \geq 1$, and then rewrite (1) as $X_n = X_0 e^{nG_n(\pi)}$, where $G_n(\pi)$ is the n-step compounding growth rate

$$G_n(\pi) = \frac{1}{n}\sum_{i=1}^{n} \ln\left(1+R+\pi\left[Z_i - R\right]\right) .$$

By the law of large numbers, $G_n(\pi) \longrightarrow G(\pi)$ with probability one as $n \to \infty$, where $G(\pi) = E\left[\ln\left(1+R+\pi\left[Z_1 - R\right]\right)\right]$.

The Kelly criterion is to choose π to maximize this (asymptotic) growth rate, i.e., for the sequel, π^* will denote the optimal Kelly fraction $\pi^* = \arg\left\{\sup_\pi G(\pi)\right\}$, and the resulting optimal wealth process will be denoted by X_n^*, i.e., $X_n^* = X_0 \prod_{i=1}^{n}\left(1+R+\pi^*\left[Z_i - R\right]\right)$.

It is obvious that this policy is optimal as well for maximizing *logarithmic* utility of terminal wealth. Observe that if $\pi^* > 1$, then the investor is leveraged in that he is borrowing money (at the riskless rate R) to invest in the stock.

For general distributions, π^* might be difficult to compute. One simple case is where Z_i form the increments of a simple random walk: specifically, if

$$Z_i = \begin{cases} +\delta & \text{with probability } \theta \\ -\delta & \text{with probability } 1-\theta \end{cases}$$

For this case, the optimal policy reduces to

$$\pi^* = \frac{(1+R)}{\delta^2 - R^2}\left[\delta(2\theta - 1) - R\right] \tag{2}$$

In the pure gambling case, where $R = 0$, this reduces further to $\pi^* = (2\theta - 1)/\delta$.

In general, π^* has many optimality properties associated with it, most notably, π^* *asymptotically minimizes the expected time to reach any wealth level*, as the wealth level increases without bound (see Breiman [5]). Furthermore, for any other strategy (not necessarily of constant proportion type), the ratio of the resulting wealth processes forms a supermartingale sequence, i.e., letting H_n denote the history[1] of the returns process up to and including time n, we have

$$E\left(\frac{X_{n+m}(\pi')}{X^*_{n+m}} \,\middle|\, H_n\right) \leq \frac{X_n(\pi')}{X^*_n}\,, \quad \text{for all} \quad m \geq 0\,,$$

as well as the converse

$$E\left(\frac{X^*_{n+m}}{X_{n+m}(\pi')} \,\middle|\, H_n\right) \geq \frac{X^*_n}{X_n(\pi')}\,, \quad \text{for all} \quad m \geq 0\,.$$

Furthermore, it can be shown that the latter (submartingale) sequence is divergent, i.e., that $X^*_n/X_n(\pi') \to \infty$, for any policy π' that is not asymptotically equivalent to π^*. Given all this, it is easy to understand the results of Bell and Cover [1], who showed that a randomized version of π^* is game theoretically optimal for the problem of maximizing the probability of beating an opponent in a single play.

It is in the sense of these optimality result that you *cannot* do better than Kelly.

3 Short run properties from the continuous-time approximation

While it is easy to provide the asymptotic optimality properties of π^* just discussed, the short-run properties of π^* are quite difficult to analyze for the general case. As such, we will move to an approximating process involving Brownian motion (Ethier [11], Browne and Whitt [6]).

[1] Formally, H_n is the completed sigma-field generated by $\{Z_i, i = 1, \ldots, n\}$

Formally, the approximating process is constructed from a sequence of random walks

$$\{Z_i^{(1)}\}_{i\ge1}, \{Z_i^{(2)}\}_{i\ge1}, \ldots, \{Z_i^{(n)}\}_{i\ge1}, \ldots$$

where the increments in the nth random walk are

$$Z_i^{(n)} = \begin{cases} +\delta_n & \text{w.p. } \theta_n \\ -\delta_n & \text{w.p. } 1-\theta_n \end{cases}$$

where the step size in the nth random walk is $\delta_n := \sigma/\sqrt{n}$, the probability of taking a positive step is $\theta_n := 1/2 + \mu/(2\sigma\sqrt{n})$, and the risk-free rate is $R_n := r/n$. Here $\mu > r \ge 0$.

Observe that for the nth random walk, the Kelly fraction is given by

$$\pi_n^* = \frac{(1+R_n)}{\delta_n^2 - R_n^2}\left[\delta_n(2\theta_n - 1) - R_n\right] \equiv \frac{(1+r/n)(\mu-r)}{\sigma^2 - r/n}$$

which converges *pointwise*, as $n \to \infty$, to the value $(\mu - r)/\sigma^2$. Since this parameter plays an important role in the sequel, we will for the remainder denote it by π^*, i.e., for the sequel,

$$\pi^* := \frac{\mu - r}{\sigma^2}\,. \tag{3}$$

The theory of weak convergence (Billingsley [3]) provides that the sequence of returns processes (the random walks), converges *weakly* to a Brownian motion with drift: for all $t \ge 0$, we have

$$\sum_{i=1}^{[nt]} Z_i^{(n)} \xrightarrow{\text{w}} \mu t + \sigma W_t$$

where $\{W_t, t \ge 0\}$ is a standard Brownian motion, and $\xrightarrow{\text{w}}$ denotes weak convergence.

A further application then shows that the controlled wealth process converges weakly to a *geometric* Brownian motion: specifically, for any constant π,

$$X_{[nt]} \xrightarrow{\text{w}} X_t^\pi := X_0 \cdot \exp\left\{\left(r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}\right)t + \pi\sigma W_t\right\}\,. \tag{4}$$

Remarks:

(i) Observe that for this approximating wealth process, ruin is impossible in finite time. (Although for π such that $r + \pi(\mu - r) - \pi^2\sigma^2/2 < 0$, we do have $X_t^\pi \to 0$ a.s.)

(ii) From Ito's formula, we observe that X_t^π follows the stochastic differential equation

$$dX_t^\pi = \left[r + \pi(\mu - r)\right] X_t^\pi dt + \pi\sigma X_t^\pi dW_t\,.$$

For the special case $\pi = 1$, this reduces to the stock price in the Black-Scholes model,

$$dX_t = \mu X_t dt + \sigma X_t dW_t\,.$$

(iii) Observe further that for any π, we have, by (4)

$$\ln\left(X_t^{\pi}/X_0\right) = \left(r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}\right)t + \pi\sigma W_t$$

and since $W_t \sim N(0,t)$, upon taking expectations we find

$$E\ln\left(X_t^{\pi}/X_0\right) = \left(r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}\right)t$$

which is indeed maximized by π^* of (3).

Call the resulting *optimal* wealth process $X_t^{\pi^*} = X_t^*$, whereby we have

$$X_t^* := X_0 \exp\left\{\left[r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right]t + \left(\frac{\mu - r}{\sigma}\right)W_t\right\} \equiv X_0 \exp\left\{\left(r + \frac{1}{2}\sigma^2\pi^{*2}\right)t + \sigma\pi^* W_t\right\}. \quad (5)$$

Observe that from (4) and (5) we find that for any other (constant) policy π we have

$$X_t^*/X_t^{\pi} = \exp\left\{\frac{1}{2}\sigma^2\left(\pi^* - \pi\right)^2 t + \sigma\left(\pi^* - \pi\right)W_t\right\} \quad (6)$$

from which it follows from basic properties of Brownian motion[2] that the process $\{X_t^*/X_t^{\pi}, t \geq 0\}$ is a submartingale, while the process $\{X_t^{\pi}/X_t^*, t \geq 0\}$ is a martingale.

Thus in a *martingale* sense, you clearly cannot do better than Kelly.

Two cases of alternative strategies are of particular interest for the sequel: $\pi = 0$ and $\pi = 1$. In the first case, wealth is simply the portfolio associated with an all cash strategy, and in the second case, wealth is the value of a portfolio fully invested in the stock itself.

3.1 Growth rates and the expected time to reach goals

Elsewhere (e.g., [18, 24, 8]) it has been established that for the continuous-time model treated here, π^* is in fact the optimal dynamic policy to minimize the expected time to reach a goal. To get some feeling though for the magnitude of this optimality result, observe that if W_t is a standard Brownian motion, and Z_t is the geometric Brownian motion

$$e^{\gamma t + \beta W_t}$$

then for any $\gamma > 0$, and any β, the expected time for Z to "grow $100 \cdot \lambda\%$" (i.e., the time until $Z_t = 1 + \lambda$) is given by

$$\frac{1}{\gamma}\ln\left(1 + \lambda\right). \quad (7)$$

[2]Recall that for any α, the process $e^{\alpha W_t - \alpha^2 t/2}$ is a non-negative martingale, and that a convex function (such as $1/x$, on $x > 0$) of a non-negative martingale is a submartingale.

From (4) we observe that the relevant growth rate γ is $\gamma = r+\pi(\mu-r)+\sigma^2\pi^2/2$. In the following table we present the expected time (in years) until initial wealth is *doubled* (i.e., $\lambda = 1$) for the three strategies of interest (all cash, all stock, and the Kelly strategy). To obtain the numerical values, we will assume, here and throughout, the values

$$\mu = .15, \sigma = .3, r = .07 \tag{8}$$

for which the associated Kelly strategy is then $\pi^* = (.15 - .07)/.3^2 = .89$, i.e. the optimal growth strategy for this case invests 89% of its wealth in the risky stock and the remaining 11% in the riskless asset.

Strategy	Growth rate (γ)	E(time to double)
Cash ($\pi = 0$)	r	9.9 years
Stock ($\pi = 1$)	$\mu - \sigma^2/2$	6.6 years
Kelly ($\pi = \pi^*$)	$r + \sigma^2\pi^{*2}/2$	6.5 years

Table 1: Expected doubling times for various strategies.

From (7) and (6) it also follows that for any $\epsilon > 0$, the expected time for the Kelly wealth to beat the wealth associated with any other strategy π, by ϵ% is given by

$$\frac{2}{\sigma^2\left(\pi^* - \pi\right)^2} \ln\left(1+\epsilon\right) . \tag{9}$$

Evaluating (9) for $\epsilon = .1$ with the illustrative numbers of (8) above, gives the following:

Strategy	E(time to beat by 10%)
All cash ($\pi = 0$)	2.8 years
All stock ($\pi = 1$)	184 years

Table 2: Expected time for Kelly to beat competing strategies by 10%.

3.2 Probability calculations

Equation (6) enables comparisons that allow us to understand the short run properties of the Kelly strategy relative to others. In particular, (6) implies that for any ϵ, the probability that the Kelly strategy outperforms any other (fixed) strategy π by ϵ% by a *fixed* time T is

$$P\left(X_T^* > (1+\epsilon)X_T^\pi\right) = P\left(\sigma\left(\pi^* - \pi\right)W_T > -\frac{1}{2}\sigma^2\left(\pi^* - \pi\right)^2 T + \ln(1+\epsilon)\right) .$$

Since $W_T \sim N(0,T)$, if we let $\Phi(z)$ denote the cumulative distribution function of a standard normal, some simple manipulations show that we can write this probability as

$$P\left(X_T^* > (1+\epsilon)X_T^\pi\right) = \Phi\left(\frac{1}{2}M - \frac{\ln(1+\epsilon)}{M}\right) \quad \text{where} \quad M = \sqrt{\sigma^2\left(\pi^* - \pi\right)^2 T} . \tag{10}$$

Observe that M is a monotonically increasing function of the time horizon T, and so as $T \nearrow \infty$, so does M. Therefore, from (10) it is easy to conclude that as $T \nearrow \infty$, the probability that the Kelly wealth outperforms any other (fixed) strategy π by $\epsilon\%$ tends to 1, for any ϵ. This of course is one of the well known "*long-run*" optimality properties of the Kelly strategy, but a relevant question is "how long is the long-run ?"

To answer this somewhat, observe that for any given π and ϵ, and for any fixed given probability level $1-\alpha$, we can invert (10) for the associated time T. Specifically, if for any $0 \le \alpha \le 1$, we use the standard normal quantile notation $\Phi(z_\alpha) = 1-\alpha$, then by setting the probability in (10) equal to $1-\alpha$, we see that for a fixed ϵ the associated M that achieves this probability of $1-\alpha$ is given as the positive root to the quadratic equation $\frac{1}{2}M^2 - z_\alpha M - \ln(1+\epsilon) = 0$, i.e., $M = z_\alpha + \sqrt{z_\alpha^2 + 2\ln(1+\epsilon)}$. We may solve this for T to get

$$T = \left(\frac{z_\alpha + \sqrt{z_\alpha^2 + 2\ln(1+\epsilon)}}{\sigma\left(\pi^* - \pi\right)} \right)^2 . \tag{11}$$

To obtain some insight into this, here is a table that shows the value of T, using $\epsilon = .1$, for various values of α and for the case of an all cash strategy and an all stock strategy:

Probability $1-\alpha$	z_α	Time (in years) to beat π by 10%: All cash ($\pi = 0$)	All stock ($\pi = 1$)
.95	1.645	157	10,286
.99	2.33	310	20,289
.999	3.08	538	35,193
.9999	3.62	741	48,483

Table 3: Years for Kelly strategy to beat competing strategies by 10%.

These results are somewhat unsettling. As we can see, it would take 157 years for the Kelly strategy to have a 95% probability of beating an all cash strategy, while it would take 10,286 years for the Kelly strategy to be 95% certain of beating an all stock strategy.

In the next section we will compare these results with the *optimal* policy for a probability maximizing objective.

4 Probability Maximizing Strategies

Since it was with respect to the finite-time probabilities of beating competing strategies that the performance of the Kelly strategy was less than satisfactory, it is only right to compare its performance with the strategy that is indeed optimal for this objective.

As a preliminary step, we must first provide the answer to the following question: For a given wealth level, say \$$b$, and a given deadline T, what (dynamic) strategy *maximizes the probability* that the goal b will be reached by the deadline ? Formally the problem is to determine the control policy $\{\pi_t, 0 \leq t \leq T\}$ that optimizes $P\left(\sup_{0\leq s\leq T} X_s^\pi \geq b\right)$. Browne [7] solved this problem for a much more general model than the one considered here. The results in [7] generalized the important earlier work of Kulldorff [20] and Heath [17] on this problem.

Observe first that the probability that terminal wealth reaches $\$b$ is equivalent to the probability that for some $t \leq T$, the wealth process hit the time-dependent boundary $be^{-r(T-t)}$. (Once this boundary is hit, one need only invest in the riskless asset to assure a terminal wealth of $\$b$.)

In the following theorem, which is a special case of Theorem 3.1 of Browne [7], $\phi(u)$ denotes the density function of a standard normal variate, $\phi(u) = e^{-u^2/2}/\sqrt{2\pi}$; $\Phi(u)$ denotes the associated distribution function $\Phi(u) = \int_{-\infty}^{u} \phi(x)dx$; and $\Phi^{-1}(u)$ denotes the latter's inverse.

Theorem 2 *Let $V(t, x : b)$ denote the optimal value function with $T - t$ time units left to go and a current wealth level of x, i.e., $V(t, x; b) \equiv \sup_\pi P\left(X_T^\pi \geq b \,\middle|\, X_t = x\right)$. Furthermore, let $f_t^*(x : b)$ denote the associated optimal strategy. Then*

$$V(t, x; b) = \Phi\left(\Phi^{-1}\left(\frac{x}{b}e^{r(T-t)}\right) + \frac{\mu - r}{\sigma}\sqrt{(T-t)}\right) \tag{12}$$

$$f_t^*(x; b) = \left[\frac{1}{\sigma\sqrt{T-t}}\right]\left[\frac{be^{-r(T-t)}}{x}\right]\phi\left(\Phi^{-1}\left(\frac{x}{b}e^{r(T-t)}\right)\right). \tag{13}$$

Observe that for this case the optimal policy, f^* of (13), is *independent* of the drift parameter μ. In fact, it is only for this case that this is true, i.e., if the mean drift return (and/or the volatility as well) of the risky stock is time dependent, or if there are multiple stocks, then the probability maximizing strategy *will* depend on the function $\mu(t)$ – but only through its role in the Kelly strategy. See Browne [7] for further details. In particular, Browne [7, Section 4] showed that the reason for this independence is that in a Black-Scholes world[3], the probability maximizing f_t^* of (13) is completely *equivalent to the hedging strategy of a binary-digital option*[4]. In the general case, it turns out that the optimal strategy is the hedging strategy for a digital option on the wealth

[3]I.e., an economy with a constant risk-free rate of return and a single risky stock whose price follows geometric Brownian motion with constant coefficients.

[4]Specifically, if S_t denotes the stock price in the Black-Scholes economy, then a digital-binary option with strike price K and payoff b is an option that pays $\$b$ to the bearer at time T if and only if the stock price exceeds the strike price at T, i.e., if $S_T \geq K$. Standard results on option pricing show that the (fair) price at time t for this option is

$$C(t, S_t) = be^{-r(T-t)}\Phi\left(\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

The associated hedging, or replicating, strategy is the strategy that holds $\partial C/\partial S_t$ units of the stock at time t, with the remaining wealth held in the riskless asset.

process associated with the Kelly strategy ([7, Section 4]). Thus, this strategy is such that the terminal wealth is b with probability V and 0 with probability $1 - V$, i.e., under this probability maximizing strategy, there is *positive probability of bankruptcy at* T (see [7] for all the details). Bankruptcy in finite-time is not possible under the Kelly strategy (for the continuous-time model). Nevertheless, we will see later that this strategy does have some interesting domination features over Kelly.

Since $\Phi^{-1}(u) \nearrow \infty$ as $u \nearrow 1$, with $\Phi^{-1}(u) = \infty$ for $u \geq 1$, we observe from (12) and (13) that indeed for $x \geq be^{-r(T-t)}$, we have $V = 1$, and $f^* = 0$, i.e., as x gets closer to the goal, the probability maximizing policy invests less and less. The effect of wealth going to 0 is more delicate. To analyze all this in greater detail, observe that if we define $z = x/be^{-r(T-t)}$, i.e., z is the percentage of the goal reached by time t, we may rewrite f_t^* as

$$f_t^* = \frac{1}{\sigma\sqrt{T-t}}\left[\frac{\phi(\nu)}{\Phi(\nu)}\right] \quad \text{where } \nu := \Phi^{-1}(z). \tag{14}$$

Equation (14) represents the optimal investment strategy as a state an time dependent proportional strategy where the proportion decomposes into a product of two distinct components: the first component is a purely time-dependent elements, $1/\sqrt{\sigma^2(T-t)}$; and the second is a scalar determined solely by the percentage of the (effective) goal currently achieved, $\phi(\nu)/\Phi(\nu)$. The first quantity is clearly increasing as $t \uparrow T$. To examine the behavior of the second, observe that as wealth, x, gets closer to the (effective) goal, $be^{-r(T-t)}$, $z \uparrow 1$ and hence $\nu \uparrow +\infty$, while as wealth gets closer to the bankruptcy barrier, 0, we have $z \downarrow 0$ and accordingly $\nu \downarrow -\infty$. Notice too that the function $\phi(\nu)/\Phi(\nu)$ is positive and decreasing in ν with terminal limits (in terms of z)

$$\lim_{z\to 0}\frac{\phi(\nu)}{\Phi(\nu)} = +\infty \ \text{ and } \ \lim_{z\to 1}\frac{\phi(\nu)}{\Phi(\nu)} = 0\,.$$

Thus we see that for a fixed t, as wealth gets closer to the effective goal, and so z increases as does ν, this latter quantity *decreases*. We will examine the tradeoff between these two effects directly.

4.1 Risk Taking and Borrowing

It is interesting to examine the nature of risk-taking dictated by the policy described above. One manifestation of risk-taking is in the amount of borrowing required by an investment strategy. Here, borrowing takes place when $f_t^* \geq 1$, i.e., when $\phi(\nu)/\Phi(\nu) \geq \sigma\sqrt{T-t}$. Let $\tau = \sigma^2(T-t)$ denote the risk-adjusted remaining time, and let $\nu^*(\tau)$ denote the unique root to the equation $\phi(\nu)/\Phi(\nu) = \sqrt{\tau}$. It is easy to establish that these roots are decreasing in τ. Observe that borrowing occurs when $\nu(\tau) < \nu^*(\tau)$ – equivalently, when $z(\tau) < z^*(\tau)$, where $z(\tau)$ is the proportion of the goal attained with τ-risk adjusted time units to go. A graph of the function $z^*(\tau)$, as well as a table of some select values of τ, $\nu^*(\tau)$ and $z^*(\tau)$, is given below. As we can see from there, if .05 risk-adjusted time units remain until the deadline, borrowing occurs unless the investor is already 88% of the way to the goal. As the time to go increases, the investor needs to borrow only at lower percentages. For example, if there is $\tau = 1$ unit of time left to go, then the investor will borrow only if his wealth at that time is less than 38% of the way there.

It is important to note that *increasing the risk factor, σ^2, has the same effect as increasing the actual time left to play, $T - t$.* Therefore, for a higher risk factor, one would borrow less, in the hopes of reaching the goal later.

Figure 1. $z^*(\tau)$ plotted against τ.

τ	$\nu^*(\tau)$	$z^*(\tau)$	τ	$\nu^*(\tau)$	$z^*(\tau)$
.001	2.26	.99			
.05	1.19	.88	.55	.09	.54
.10	.93	.82	.60	.04	.51
.15	.75	.77	.65	-.01	.49
.20	.63	.73	.70	-.06	.48
.25	.52	.70	.75	-.11	.46
.30	.43	.67	.80	-.15	.44
.35	.35	.64	.85	-.19	.43
.40	.27	.61	.90	-.23	.41
.45	.21	.58	.95	-.27	.39
.50	.15	.56	1.00	-.30	.38
1.5	-.613	.27	3.5	-1.42	.08
2.0	-.86	.19	4.0	-1.57	.06
2.5	-1.07	.14	4.5	-1.71	.04
3.0	-1.25	.10	5.0	-1.84	.03

Table 4: Borrowing Region: τ=(risk-adjusted) time to go, $z^*(\tau)$=% of distance to goal by τ. $\nu^*(\tau)$ is the root to the equation $\phi(\nu)/\Phi(\nu) - \tau^{1/2} = 0$.

4.2 Maximizing the probability of beating another strategy

A consequence of the previous results to our problem at hand is the following result, which is again a special case of the more general results in Browne [7, Section 9]:

Corollary 1 *For any given constant proportion strategy, π, let $V(t,w;\epsilon,\pi)$ denote the maximal probability at time t of beating strategy π by ϵ% by time T i.e.,*

$$V(t,w;\epsilon,\pi) = \sup_f P\left(X_T^f > (1+\epsilon) X_T^\pi \;\middle|\; \left(X_t^f / X_t^\pi\right) = w \right)$$

and let $f_t^*(w;\epsilon,\pi)$ denote the associated optimal strategy.

Then we have

$$V(t,w;\epsilon,\pi) = \Phi\left(\Phi^{-1}\left(\frac{w}{1+\epsilon}\right) + \sqrt{\sigma^2(\pi^*-\pi)^2(T-t)}\right) \tag{15}$$

$$f_t^*(w;\epsilon,\pi) = \pi + \frac{1}{\sqrt{\sigma^2(T-t)}}\left(\frac{1+\epsilon}{w}\right)\phi\left(\Phi^{-1}\left(\frac{w}{1+\epsilon}\right)\right) \tag{16}$$

where π^* is the Kelly strategy of (3).

Observe that (15) gives us the proper tool for comparing the performance of alternative strategies. In particular, by evaluating (15) at $t=0$ with $w=1$ we find that the best you can do against any other strategy (with respect to maximizing the probability of beating it by ϵ%) is given by $V(0,1;\epsilon,\pi)$. For a given strategy π, a given excess level ϵ, and a given probability level, $1-\alpha$, we may then invert for the corresponding time, T, that is needed to achieve the probability level $1-\alpha$; i.e., set $V(0,1;\epsilon,\pi) = 1-\alpha$ and invert for the corresponding time T. Specifically, from (15) we have

$$\Phi\left(\Phi^{-1}\left(\frac{1}{1+\epsilon}\right) + \sqrt{\sigma^2\left(\pi^*-\pi\right)^2 T}\right) = 1-\alpha$$

or equivalently,

$$\Phi^{-1}\left(\frac{1}{1+\epsilon}\right) + \sqrt{\sigma^2\left(\pi^*-\pi\right)^2 T} = \Phi^{-1}\left(1-\alpha\right) .$$

This relationship in turn implies

$$T = \left[\frac{\Phi^{-1}\left(1-\alpha\right) - \Phi^{-1}\left(\frac{1}{1+\epsilon}\right)}{\sigma\left(\pi^*-\pi\right)}\right]^2 . \tag{17}$$

We may now evaluate this for various values of π,ϵ and α. In particular, by taking $\pi=0$ we get the relevant time to beat an all cash strategy, and by taking $\pi=1$ we get the relevant time to beat an all stock strategy.

For illustrative purposes, here is a table using the numerical values of (8) and using ϵ=.1 (observe that $1/1.1 = .91$, and $\Phi^{-1}(.91) = 1.34$).

Probability	Time (in years) to beat π by 10%	
$1-\alpha$	All cash ($\pi = 0$)	All stock ($\pi = 1$)
.95	1.3	85
.99	14	900
.999	43	2,780
.9999	73	4,774

Table 5: Years for the probability maximizing strategy to beat competing strategies by 10%.

As we can see, this strategy gives results that are order of magnitudes better than the comparative results for the Kelly strategy. The downside of course, is that under this strategy, the terminal wealth at time T has positive probability of being 0. However, as Table 5 shows for example, if we employ this strategy for 1.3 years, we have only a 5% of going bankrupt with a 95% chance of beating an all cash strategy by 10%.

We will leave further inferences to be drawn by the reader.

4.3 Beating Kelly

When we take $\pi = \pi^*$ in (15) we see that for any time t and $w > 0$ we always have $V(t, w : \epsilon, \pi^*) = w/(1+\epsilon)$. This is of course consistent with the previously mentioned fact for any strategy f, the ratio $\left\{X_t^f / X_t^*, t \geq 0\right\}$ is a supermartingale[5], and with Kolmogorov's inequality for supermartingales. The fact that the inequality is met at equality is proved in [7], where it is shown that the policy of (16) evaluated at π^*, i.e., $f_t^*(w; \epsilon, \pi^*)$, does in fact achieve this optimal value. Thus, we have the rather interesting result that for *any* fixed time T, we have found a strategy that will beat the Kelly strategy by any arbitrary ϵ% ($\epsilon > 0$) with initial *probability* $1/(1+\epsilon)$.

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[5] So long as $\{f_t, t \geq 0\}$ satisfies suitable integrability conditions, the ratio is indeed a martingale. In general though, the most we can say about the ratio is that it is a (positive) local martingale, which is hence a supermartingale.

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