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Robust optimization of currency portfolios

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A currency investment strategy is used to maximize the return on a portfolio of foreign currencies relative to any appreciation of the corresponding foreign exchange rates. Given the uncertainty in the estimation of the future currency values, we employ robust optimization techniques in order to maximize the return on the portfolio for the worst-case foreign exchange rate scenario. Currency portfolios differ from stock-only portfolios in that a triangular relationship exists among foreign exchange rates to avoid arbitrage. Although the inclusion of such a constraint in the model would lead to a nonconvex problem, by choosing appropriate uncertainty sets for the exchange and the cross exchange rates we obtain a convex model that can be solved efficiently. Alongside robust optimization, an additional guarantee is explored by investing in currency options in order to cover the possibility that foreign exchange rates will materialize outside the specified uncertainty sets. We present numerical results that show the relationship between the size of the uncertainty sets and the distribution of the investment among currencies and options, and the overall performance of the model in a series of backtesting experiments.

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1 INTRODUCTION

Since Markowitz's (1952) seminal work on portfolio optimization and the benefits of diversification, much academic research into portfolio optimization has been carried out, and it has developed into a mature area of operations research. Recently, researchers have begun to investigate international investment and portfolios that comprise both national and international assets as a further way of increasing diversification and reducing risk. It is expected that international assets will have a lower correlation with national assets than national assets have amongst themselves.

Grubel (1968) was the first to describe and quantify the gains made from international diversification. He concludes that the international diversification of portfolios could bring a new source of gains and, at the same time, have an important impact on policy making, since international capital movements are a function not only of interest rate differentials, but also depend on the growth rates of asset holdings in both countries. A later study by Levy and Sarnat (1970) concludes that there are risk reduction gains to be made from international diversification, and that these can be measured by the variance of a portfolio. The authors of the above-mentioned paper suggest investing in developing countries as well as developed ones: although the risk associated with developing countries may be higher, their returns are also less closely correlated with the returns from developed countries and therefore allow a minimal overall portfolio variance.

The first results on gains made exclusively from foreign currency holdings were reported by Levy (1978). This work aimed at finding an alternative way of reducing the foreign exchange risk by using a portfolio balancing approach, because, following the collapse of the Bretton Woods system in the early 1970s, exchange rates had gained the ability to float freely. Levy shows that, in the period from January 1971 to July 1973, US investors could have made significant gains by holding only foreign currencies. In a mixed portfolio of currencies and stocks, although stocks generally yielded a higher return than currencies, optimal portfolios would still have a significant proportion of currencies, as these had lower standard deviations and lower correlation with stocks and therefore contributed to risk diversification. More recently, in the period following the introduction of the euro, the US dollar suffered constant depreciation against some of the major currencies (see Table 1 on the facing page), which created similar opportunities for US investors to profit from investment in these currencies.

However, international portfolios carry an additional risk related to unfavorable movements of foreign exchange rates. The issue of hedging the currency risk and, consequently, of determining the optimal hedge ratio and deciding on which financial instrument to use became more and more relevant. Black (1989) introduced the concept of "universal hedging", arguing that investors should always hedge their foreign

TABLE 1 Monthly average appreciation rate and standard deviation of major foreign currencies against the US dollar from January 2002 to December 2008.

Currency	Appreciation rate (%)	Standard deviation (%)
EUR	0.46	2.53
GBP	0.01	2.23
JPY	0.43	2.46
CHF	0.45	2.47
CAD	0.30	2.02
AUD	0.28	3.20

assets equally for all countries, but never 100%. His "universal hedging" formula has only three inputs based on averages across countries using the following parameters:

(1) excess expected return on the world market portfolio;

(2) volatility of the world market portfolio; and

(3) exchange rate volatility.

Eun and Resnick (1988) argue that the studies from Grubel (1968) and Levy and Sarnat (1970) overstate the actual gains made from international diversification as they do not account for parameter uncertainty, which affects the estimation of returns. They argue that the risk inherent to foreign exchange rates can eliminate or substantially reduce the gains made on an international portfolio due to their volatility and their positive correlation with the stock returns. Two methods are proposed to reduce this risk: firstly, diversification through the investment in several currencies and, secondly, a hedging strategy that sells the expected foreign currency returns at the forward rate. The effectiveness of this strategy depends on how accurate the investor's estimates are relative to the future returns. Eun and Resnick conclude that hedged portfolios dominated nonhedged ones. Similar results have also been reported by other authors (Glen and Jorion (1993); Larsen and Resnick (2000); and Topaloglou *et al* (2008)). Topaloglou *et al* (2008) implement a multistage stochastic programming model and jointly determine the asset weights and the corresponding hedge ratios for the international currencies. A survey of the subject can be found in Shawky *et al* (1997).

In all of the approaches mentioned, the hedging instrument is the forward rate, with little attention having been given to currency options. Giddy (1983) studied the application of foreign exchange options, the relationship between forward rates and currency options, and their pricing methodology. He concluded that options were a more adequate hedging instrument than forwards when the future revenues were uncertain. However, he does not test his hypothesis with real market data. In the same year, Garman and Kohlhagen (1983) developed a pricing model for currency options based on the Black and Scholes (1973) model for stock options. Steil (1993) argues that the "Giddy rule" is based on a false premise, as the underlying contingency (receiving or not receiving a future revenue) is not the same as the one underlying the option, which is the foreign exchange rate. He tests both hedging strategies using forwards and options for three different expected utility maximization functions and finds options perform poorly in comparison with forward rates. Similar conclusions were reached by Topaloglou *et al* (2002, 2007), where the conditional value-at-risk (VaR) is used as a risk measure.

In order to incorporate the uncertainty associated with the estimation of the relevant parameters, we propose to combine robust optimization with currency options to protect against a depreciation of the foreign currencies. We expand on the work of Rustem and Howe (2002), who present both a strategic and a tactical model for robust currency hedging. Robust optimization differs from other uncertainty reduction techniques in that it incorporates uncertainty directly into the model, as returns are assumed not to be deterministic but instead to be random variables which may be realized within a prespecified uncertainty set. Soyster (1973) discussed the optimization over a collection of sets, but the technique only gained widespread attention with simultaneous works by Ben-Tal and Nemirovski (1998) and El-Ghaoui (1997). The application of robust optimization to the particular problem of portfolio optimization and asset allocation has been studied by Goldfarb and Iyengar (2003). We refer the reader to Ben-Tal *et al* (2009) for a recent survey of robust optimization and its applications.

Although currencies are not commonly seen as investment assets, the added risk of an international portfolio has been thoroughly studied in the literature. The focus of these studies, however, has been on currencies from the perspective of an investor on assets, that is, an investor who manages a portfolio of foreign assets and who wishes to account for the currency risk and return. In contrast, this paper focuses on portfolios of currencies and, in particular, on the problem of hedging against a depreciation of foreign exchange rates. The main contributions of our work may be summarized as follows:

(1) We apply robust optimization to the problem of allocating investments between several currencies with different patterns of risk and return.

(2) We analyze the impact of the triangular relationship between foreign exchange rates in the model, particularly of the convexity issues that arise. Furthermore, we give solutions in order to overcome these same issues.

(3) We describe a hedging strategy that minimizes the currency risk by including currency options, and implement a model that combines currency options with robust optimization. We take on a portfolio perspective and simultaneously consider all currencies. That way, we aim to avoid being too conservative in our hedging strategy.

(4) We present numerical results that describe the relationship between the size of the uncertainty set and the total investment in options. We also present a series of backtesting experiments that assess the performance of both strategies, ie, robust optimization with and without currency options, relative to the Markowitz risk minimization approach.

The rest of the paper is organized as follows. Section 2 presents the robust portfolio optimization model, the distinguishing features of a currency portfolio and the approach followed in order to guarantee convexity of the model. In Section 3 we extend the model to include currency options and explain how the investor is further insured against depreciations of the foreign currencies. We also show how robust optimization can be used together with currency options as a global hedging strategy. Section 4 gives numerical results that compare the two models with and without currency options. Section 5 concludes.

2 ROBUST PORTFOLIO OPTIMIZATION

We consider a portfolio that comprises n different foreign currencies, taking the US dollar as our base currency. The return on a currency is measured by the ratio between the expected future spot exchange rate and the spot exchange rate today. We denote by E_i and $E_i^{\mathbf{0}}$ the expected future and the current spot exchange rates, respectively. Both quantities are expressed in terms of the base currency per unit of the foreign currency i. The expected return on a specific currency i is then described by $e_i = E_i/E_i^{\mathbf{0}}$. In the Markowitz (1992) mean–variance framework we would want to maximize our expected portfolio return given some risk measure, which, in this case, is the variance of the portfolio. The formulation of our problem would be:

$$\max_{\boldsymbol{w}\in\mathbb{R}^n} \{\boldsymbol{e}'\boldsymbol{w} - \lambda(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w})\} \tag{2.1}$$

subject to:

$$\mathbf{1}'\boldsymbol{w} = 1, \quad \boldsymbol{w} \geqslant 0$$

The variable $\boldsymbol{w}$ denotes the vector of currency weights in the portfolio, while the parameter Σ represents the covariance matrix of the currency returns. Parameter λ denotes the level of risk the investor is willing to take. Throughout this paper, variables or parameters in bold face denote vectors. We denote by **1** a vector of all 1s, the dimension of which is clear from the context.

Although the Markowitz model has stimulated a significant amount of research, the mean–variance framework has also been subject to criticism due to its lack of robustness. Model (2.1) is deterministic: it assumes that the expected returns are given, and it does not account for their random nature. Small changes in the value of the parameters, however, may pull the solution far from the optimum or even render it infeasible. Robust optimization assumes that there is a degree of uncertainty in these estimates: future returns are not certain, but random, and they may take any value within a predetermined uncertainty set. This uncertainty set represents the investor's expectations about the future currency returns and can be constructed according to some probabilistic measures.

Because we would like our solution to be robust to changes in the parameter values, we will maximize our portfolio return in view of the worst-case currency returns within the specified uncertainty set. We formulate our robust currency portfolio problem as:

$$\max_{\boldsymbol{w} \in \mathbb{R}^n} \min_{\boldsymbol{e} \in \Theta_{\boldsymbol{e}}} \boldsymbol{e}'\boldsymbol{w} \tag{2.2}$$

subject to:

$$\mathbf{1}'\boldsymbol{w} = 1, \quad \boldsymbol{w} \geqslant 0$$

Parameter $\boldsymbol{e}$ designates a random variable that represents the real currency returns, which we assume to be within the uncertainty set Θ. The subscript indicates which variable the uncertainty set refers to (in this case, $\boldsymbol{e}$). This uncertainty set can be described in several ways, the most widely used of which are range intervals and ellipsoids. In our models, we define $\Theta_{\boldsymbol{e}}$ as:

$$\Theta_{\boldsymbol{e}} = \{\boldsymbol{e} \geqslant 0\colon (\boldsymbol{e} - \bar{\boldsymbol{e}})'\Sigma^{-1}(\boldsymbol{e} - \bar{\boldsymbol{e}}) \leqslant \delta^2\} \tag{2.3}$$

which describes an ellipsoid that is centered at the expected returns $\bar{\boldsymbol{e}}$ and rotated and scaled by the covariance matrix of the returns. Ellipsoidal uncertainty sets were first described by Ben-Tal and Nemirovski (1998). They reflect the idea of a joint confidence region (the differences between the returns and their estimates are weighted by the covariance matrix), as opposed to an individual one as with hyper-rectangular sets. The problem is then how to choose the parameter δ in order to reflect the investor's expectations with regard to future currency returns. One approach is to choose δ^2 as the αth percentile of a χ^2 distribution with n degrees of freedom. Assuming the currency returns to be normally distributed, the probability of the future returns materializing inside the uncertainty set is at least α% (Ceria and Stubbs (2006)). Recent studies have focused on the close relationship between uncertainty sets and corresponding risk measures (see El-Ghaoui *et al* (2003); Natarajan *et al* (2009); and Bertsimas and Brown (2009)). In the case of ellipsoidal uncertainty sets, El-Ghaoui shows that:

$$\max_{w}\{-\boldsymbol{w}'\boldsymbol{e} \mid (\boldsymbol{e}-\bar{\boldsymbol{e}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{e}-\bar{\boldsymbol{e}}) \leqslant \delta_{\omega}^{2}\}$$

is equivalent to finding the worst-case ω-VaR over all exchange rate return distributions whose first two moments coincide with $\bar{\boldsymbol{e}}$ and Σ if δ_{ω} is set to $\sqrt{(1-\omega)/\omega}$.

Note that we have optimized our portfolio in view of the worst-possible outcome of the currency returns. As a result, we are bound to obtain at least the return exhibited by the objective value as long as the returns are realized within the uncertainty set. This is called the noninferiority property of robust optimization, and provides a guarantee to the investor with regard to future returns.

Although, in principle, the covariance matrix is also subject to uncertainty, its statistical estimation is much easier and hence more accurate than the estimation of the returns. Furthermore, mean–variance problems are much less sensitive to deviations from the estimate of the covariance matrix than to estimates of the returns (Fabozzi *et al* (2007)). Goldfarb and Iyengar (2003) consider an uncertainty structure for the covariance matrix based on a factor model and reformulate the problem as a second-order cone program. Halldorsson and Tutuncu (2003), on the other hand, define an uncertainty interval for each element in the covariance matrix and cast the problem as a saddle-point problem with semidefinite constraints. For simplicity we have not taken into account the uncertainty caused by the estimation of the covariance matrix. We have, however, included an extension to our model in Appendix A, where the covariance matrix is assumed not to be fixed. In our approach we have followed the work developed in Ye *et al* (2009). In the remainder of this paper, and particularly in the numerical results in Section 4, we have assumed a fixed covariance matrix. As suggested by Broadie (1993), the greatest reduction in errors in mean–variance frameworks arises from better estimation of the mean returns, whereas little or no significant impact can be made by improving the estimation of the standard deviations and correlations.

Foreign exchange rates have a particular feature that distinguishes them from other investment assets, such as stocks or bonds. If we define two exchange rates relative to a base currency, for example, the US dollar versus the euro and the US dollar versus the British pound, then we automatically define an exchange rate between the euro and the British pound as well. This triangular relationship between exchange rates must be observed at all times since, otherwise, arbitrage opportunities would arise and the market mechanisms would drive this relationship back to equilibrium. Robust optimization, on the other hand, takes into account all possible returns within the uncertainty set, and optimizes for the worst-case returns. While these are guaranteed by definition to be inside the uncertainty set considered, we need to ensure that the resulting cross exchange rates are also within appropriate boundaries, ie, that the worst cross exchange rates take on plausible values. Hence, we need to add a new constraint to the model that ensures that this triangular relationship is respected.

With n currencies in the model, the number of cross exchange rates is $\frac{1}{2}n(n-1)$. If we define X_{ij} as the cross exchange rate between E_i and E_j, that is, if X_{ij} is the number of units of currency i that equals one unit of currency j, then:

$$E_i \frac{1}{E_j} X_{ij} = 1 \tag{2.4}$$

By analogy to our previous notation, X_{ij}^0 denotes the current spot cross exchange rate, while x_{ij} is the return on the cross exchange rate. We may modify this equation to express the future exchange rates in terms of the currency returns and the spot exchange rates as follows:

$$\begin{aligned} E_i^0 e_i \frac{1}{E_j^0 e_j} X_{ij}^0 x_{ij} = 1 \quad &\iff \quad \left[E_i^0 \frac{1}{E_j^0} X_{ij}^0\right]\left[e_i \frac{1}{e_j} x_{ij}\right] = 1 \\ &\iff \quad e_i \frac{1}{e_j} x_{ij} = 1 \end{aligned}$$

The inclusion of this constraint, however, will make the problem nonconvex. Note that, although we need to model and estimate the future returns of the cross exchange rates, they do not have a direct impact on our objective function. In fact, the only effect of the cross exchange rates is to constrain further the uncertainty set originally defined for the exchange rates, that is, to render the model less conservative.

We express the uncertainty associated with the returns of the cross exchange rates by the intersection of halfspaces of the following type:

$$\boldsymbol{a}'\boldsymbol{x} \leqslant b \iff \sum_{(i,j)} a_{ij} x_{ij} \leqslant b \iff \sum_{(i,j)} a_{ij} \frac{e_j}{e_i} \leqslant b \tag{2.5}$$

where the possible relationships between the cross exchange rates are taken into account. Replacing the variables x_{ij} by their equivalent ratio from the triangulation requirement e_j/e_i yields a nonconvex formulation in the variables e of expression (2.5). These constraints may be conservatively approximated by semidefinite programming (see Vandenberghe and Boyd (1996)). For the sake of exposition, in the remainder of the paper we will use uncertainty sets for the cross exchange rates of the following type:

$$\sum_{j} a_{ij} x_{ij} \leqslant b \iff \sum_{j} a_{ij} \frac{e_j}{e_i} \leqslant b \tag{2.6}$$

In (2.5) we consider any pair (i, j) of exchange rates, which means that, when replacing x_{ij} by the corresponding ratio e_j/e_i, we can have different foreign exchange rates

e_i. By using an uncertainty set of type (2.6), we restrict the relationship among cross exchange rates to those which have a common denominator e_i. In contrast with (2.5), Equation (2.6) has an explicit convex formulation:

$$\sum_j a_{ij} \frac{e_j}{e_i} \leqslant b \iff \sum_j a_{ij} e_j \leqslant b e_i \tag{2.7}$$

In our numerical experiments, we will use box uncertainty sets for the cross exchange rates, which can be regarded as a special case of type (2.6). We define the interval in which the future cross exchange rates may materialize as:

$$\begin{aligned} \boldsymbol{l} \leqslant \boldsymbol{x} \leqslant \boldsymbol{u} \quad &\iff \quad l_{ij} \leqslant x_{ij} \leqslant u_{ij} \quad \text{for all } i, j = 1, \dots, n, \ i \leqslant j \\ &\iff \quad l_{ij} \leqslant \frac{e_j}{e_i} \leqslant u_{ij} \\ &\iff \quad l_{ij} e_i \leqslant e_j \leqslant u_{ij} e_i \end{aligned} \tag{2.8}$$

Note that the relationship between various cross exchange rates is still present in the model, although indirectly, through the covariance matrix of the exchange rate returns. As in the case of the ellipsoidal uncertainty set, the upper and lower bounds can also be chosen to reflect certain confidence intervals around the estimate value. Parameters $\boldsymbol{l}$ and $\boldsymbol{u}$ may be calculated as $(\bar{\boldsymbol{x}} - f\sigma)$ and $(\bar{\boldsymbol{x}} + f\sigma)$, respectively, meaning that the future cross exchange rate returns are expected to be in the interval centered at their mean estimate $\bar{\boldsymbol{x}}$ plus or minus f times the respective standard deviation. A more conservative investor could, for example, simply take the historical upper and lower bounds of the returns over a certain period of time.

The simplification of (2.8) leads to the transformation of $\frac{1}{2}n(n-1)$ nonconvex inequalities into $n(n-1)$ linear ones. For two currencies, part (a) of Figure 1 on the next page represents different uncertainty sets for different values of the parameter δ. Including the triangulation constraint in the model restricts the size of the uncertainty set as shown in part (b) of Figure 1 on the next page. We define $\Theta_{\boldsymbol{x}}$ as the uncertainty set associated with the returns of the cross exchange rates, where:

$$\Theta_{\boldsymbol{x}} = \{\boldsymbol{x} \geqslant 0 \colon \boldsymbol{l} \leqslant \boldsymbol{x} \leqslant \boldsymbol{u}\} \tag{2.9}$$

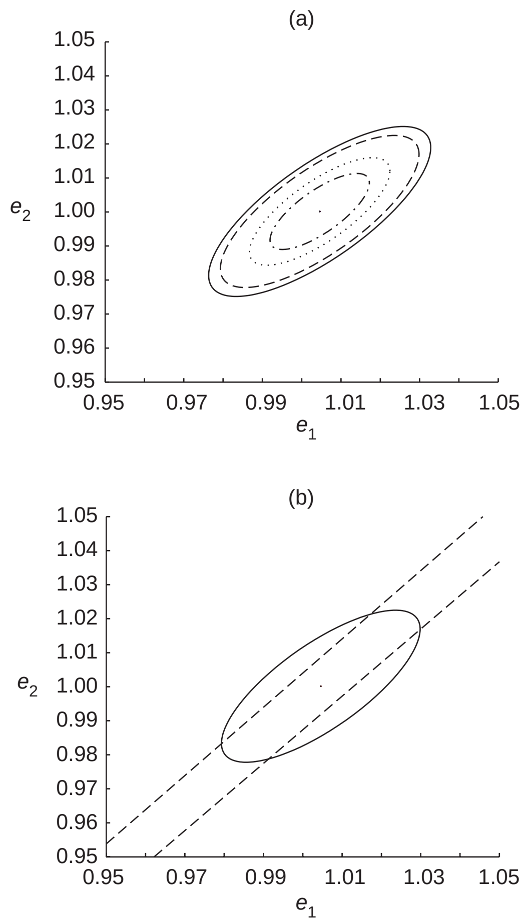
which, given the transformation (2.8), can be written as:

$$\Theta_{\boldsymbol{x}} = \{\boldsymbol{e} \geqslant 0 \colon \boldsymbol{A}\boldsymbol{e} \geqslant 0\} \tag{2.10}$$

The matrix $\boldsymbol{A}$ is the coefficient matrix reflecting all the triangular relationships between the foreign exchange rates, and consists of $n(n-1)$ rows and n columns. We redefine our uncertainty set $\Theta_{\boldsymbol{e}}$ to include this new constraint:

$$\Theta_{\boldsymbol{e}} = \{\boldsymbol{e} \geqslant 0 \colon (\boldsymbol{e} - \bar{\boldsymbol{e}})' \Sigma^{-1} (\boldsymbol{e} - \bar{\boldsymbol{e}}) \leqslant \delta^2 \wedge \boldsymbol{A}\boldsymbol{e} \geqslant 0\} \tag{2.11}$$

FIGURE 1 Uncertainty sets.



(a) Size of the uncertainty sets depending on the parameter δ. (b) Restriction of the ellipsoidal uncertainty set due to the triangulation requirement. Both figures were constructed with the ellipsoidal toolbox developed by Kurzhanskiy (2006).

Our approach has not included the transaction costs implicit in the bid–ask spread of the foreign exchange rates. The results could be easily extended to the situation where the investor buys foreign currency from a financial institution at a certain price (the ask) and sells currency at a lower price (the bid), thereby incurring a loss with such a transaction (the spread). When calculating the bid–ask prices for the cross exchange rates, we consider the respective bid–ask prices of the foreign exchange rates involved. So, for example, to obtain the bid price for the euro/British pound rate, we need the bid price for the euro/US dollar and US dollar/British pound rates. The same reasoning applies for the ask price.

Robust optimization uses duality theory to reformulate the inner minimization problem of model (2.2) as a maximization problem for a fixed vector $\boldsymbol{w}$ of weights. The inner minimization problem determines the worst-possible outcome of the currency returns and may be formulated as:

$$\min_{\boldsymbol{e} \in \mathbb{R}^n} \boldsymbol{e}'\boldsymbol{w} \tag{2.12}$$

subject to:

$$\|\Sigma^{-1/2}(\boldsymbol{e} - \bar{\boldsymbol{e}})\| \leqslant \delta, \quad \boldsymbol{A}\boldsymbol{e} \geqslant 0, \ \boldsymbol{e} \geqslant 0$$

where the operator $\|\cdot\|$ denotes the Euclidean two-norm. Problem (2.12) is a second-order cone program (see Boyd and Vandenberghe (2004)), and its dual may be written as (see Boyd and Vandenberghe (2004) and Lobo *et al* (1998)):

$$\max_{\boldsymbol{k},\boldsymbol{y},v} \bar{\boldsymbol{e}}'(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y}) - \delta v \tag{2.13}$$

subject to:

$$\|\Sigma^{1/2}(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y})\| \leqslant v, \qquad \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{v} \geqslant 0$$

In the case of second-order cone programs, strong duality holds, that is, as long as both problems are feasible, the value of the objective function of the dual problem is equal to the value of the objective function in the primal problem. Our problem now becomes:

$$\max_{\boldsymbol{w}} \max_{\boldsymbol{k},\boldsymbol{y},v} \bar{\boldsymbol{e}}'(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y}) - \delta v \tag{2.14}$$

subject to:

$$\begin{gathered} \|\Sigma^{1/2}(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y})\| \leqslant v \\ \mathbf{1}'\boldsymbol{w} = 1 \\ \boldsymbol{w}, \boldsymbol{k}, \boldsymbol{y}, v \geqslant 0 \end{gathered}$$

which simplifies to:

$$\max_{\boldsymbol{w},\boldsymbol{k},\boldsymbol{y}} \bar{\boldsymbol{e}}'(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y}) - \delta\|\Sigma^{1/2}(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y})\| \tag{2.15}$$

subject to:

$$\mathbf{1}'\boldsymbol{w} = 1, \qquad \boldsymbol{w}, \boldsymbol{k}, \boldsymbol{y} \geqslant 0$$

Problems (2.2) and (2.15) are equivalent, but (2.15) constitutes a tractable formulation that can be easily computed with modern conic optimization software.

Note how this formulation is similar to the original Markowitz mean–variance problem (2.1). In problem (2.15), however, we penalize the expected returns objective via the standard deviation of the portfolio returns instead of the variance. The advantage of the robust approach is that the parameters determining the size of the uncertainty sets, as we have seen in the cases of polyhedral and ellipsoidal uncertainty sets, may be chosen to reflect some probabilistic measures.

An investor may wish to minimize the risk while at the same time demanding a minimum expected return. In that case, we may include a further constraint in the problem:

$$\mathbb{E}[e] = \boldsymbol{w}'\bar{\boldsymbol{e}} \geqslant e_{\text{target}} \tag{2.16}$$

Maximizing in view of the worst-possible outcome of the future returns gives the investor a guarantee that the portfolio value at maturity date will always be at least as high as the objective value of (2.15). The investor is protected against any depreciation of the foreign exchange rates that materializes within the uncertainty set and, hence, robust optimization provides guarantees against the currency risk without the need to enter into any hedging agreement. The main disadvantage of this approach is that it only protects the portfolio value for fluctuations inside the uncertainty set. If the future spot exchange rates fall outside this set, robust optimization does not provide any guarantees. In the next section we present an additional strategy which includes investing in currency options in order to hedge against the possibility of the foreign exchange rates falling outside the uncertainty set.

3 HEDGING AND ROBUST OPTIMIZATION

Although robust optimization insures the investor against exchange rate fluctuations within the uncertainty set, the investor is left without any guarantees if these materialize outside the uncertainty set. Insurance against the latter case can be obtained by using currency options, which allow *a priori* chosen exchange rates to be locked in.

Although the focus of our work is on the use of currency options, forwards and currency futures are also popular hedging instruments. The latter, however, are binding agreements and do not offer the investor the flexibility to move away from it. They are therefore more appropriate for situations when the amount to be paid or received in the future is known with certainty (see Giddy (1983)).

Options entitle the investor to a right, and not to an obligation, to buy (call) or sell (put) a particular asset at a specified strike price at a certain point in the future (see Hull (2006)). Currency options are similar to other options, but the strike price considered here is a foreign exchange rate. Buying a put option on euros versus US dollars with a strike price of US$1.25 gives the right to transform euros into US dollars at the rate of US$1.25 at the maturity date. Whether or not the investor chooses to exercise the option will depend on the spot exchange rate at maturity. We consider only European options, and therefore options may only be exercised at maturity.

We assume that, for each currency, the investor has a set of m available put and call options with different premiums and strike prices. By E_i we denote the future spot exchange rate and by K_{il} we denote the strike price of the lth option on the ith currency. We can compute the payout V_{il} of the lth option on currency i versus the US dollar as:

$$V_{il}^{\text{call}} = \max\{0, E_i - K_{il}\} \tag{3.1}$$

$$V_{il}^{\text{put}} = \max\{0, K_{il} - E_i\} \tag{3.2}$$

Assume now that a portfolio is comprised of one unit of currency i and one put option on currency i with a strike price K_{il}. At maturity date, the payout of the portfolio would be:

$$\begin{aligned} V_{\text{port}} &= E_i + \max\{0, K_{il} - E_i\} \\ &= \max\{E_i, K_{il}\} \end{aligned} \tag{3.3}$$

Hence, by including a put option that corresponds with currency i in the portfolio, we are able to lock the foreign exchange rate at K_{il}. The aim of including currency options in the portfolio is therefore to guarantee a minimum return for the extreme cases where the exchange rates materialize outside the uncertainty set. If the realized exchange rate is higher than the strike price, the option will not be exercised and the investor may still benefit from the corresponding appreciation. This flexibility is a differentiating characteristic of options relative to other instruments such as forward contracts and futures: the latter instruments are binding agreements that lock the investor into a predefined exchange rate.

The price of this increased flexibility is the premium of the option, which must be paid upfront and which is incurred independently from the exercise of the option. Currency options are priced by the model of Garman and Kohlhagen (1983), which can be derived from the Black and Scholes (1973) model by assuming that currencies are equivalent to stocks with a known dividend yield, namely, the risk-free rate prevailing in the foreign country.

In this analysis we follow the notation of Lutgens (2006) and the approach of Zymler *et al* (2009). We define the vector of returns by $\boldsymbol{e}^d$ and the vector of weights of the options by $\boldsymbol{w}^d$. If p_{il} is the price of the lth put option on currency i, then its return can be calculated as:

$$e_{il}^d = \max\left\{0, \frac{K_{il} - E_i}{p_{il}}\right\} \tag{3.4}$$

The value of the future spot exchange rate may be rewritten as a function of the return on the ith currency e_i by taking into account the relationship $E_i = E_i^0 e_i$:

$$e_{il}^d = f(e_i) = \max\left\{0, \frac{K_{il} - E_i^0 e_i}{p_{il}}\right\} \tag{3.5}$$

which leads to a simplified expression that we will be using in the following formulations of our model:

$$e_{il}^d = f(e_i) = \max\{0, a^{il} + b^{il} e_i\} \quad \text{with } a^{il} = \frac{K_{il}}{p_{il}} \text{ and } b^{il} = -\frac{E_i^0}{p_{il}} \tag{3.6}$$

Similarly, if c_{il} is the price of the lth call option on currency i, its return may be expressed as:

$$e_{il}^d = f(e_i) = \max\{0, a^{il} + b^{il} e_i\} \quad \text{with } a^{il} = -\frac{K_{il}}{c_{il}} \text{ and } b^{il} = \frac{E_i^0}{c_{il}} \tag{3.7}$$

As in the previous section, our investor wishes to maximize the portfolio return in view of the worst-case currency returns, while assuming that these will materialize within the uncertainty set $\Theta_{\boldsymbol{e}}$ as defined in (2.11):

$$\max_{\boldsymbol{w}, \boldsymbol{w}^d \in \mathbb{R}^n} \min_{\substack{\boldsymbol{e} \in \Theta_{\boldsymbol{e}} \\ \boldsymbol{e}^d = f(\boldsymbol{e})}} \boldsymbol{e}'\boldsymbol{w} + \boldsymbol{e}^{d\prime}\boldsymbol{w}^d \tag{3.8}$$

subject to:

$$\mathbf{1}'(\boldsymbol{w} + \boldsymbol{w}^d) = 1, \quad \boldsymbol{w}, \boldsymbol{w}^d \geqslant 0$$

Note that the option returns are written as a function of the currency returns. Following the same procedure as in the previous section, we will reformulate the inner minimization problem as a maximization problem by using duality theory. The minimization problem is concerned with finding the worst-case currency returns:

$$\min_{\boldsymbol{e}, \boldsymbol{e}^d} \boldsymbol{e}'\boldsymbol{w} + \boldsymbol{e}^{d\prime}\boldsymbol{w}^d \tag{3.9}$$

subject to:

$$\begin{gathered} \|\Sigma^{-1/2}(\boldsymbol{e} - \bar{\boldsymbol{e}})\| \leqslant \delta \\ \boldsymbol{A}\boldsymbol{e} \geqslant 0 \\ \boldsymbol{e}^d \geqslant \boldsymbol{a} + \boldsymbol{b}\boldsymbol{e} \\ \boldsymbol{e}, \boldsymbol{e}^d \geqslant 0 \end{gathered}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are as given in (3.6) and (3.7). The dual of problem (3.9) may be formulated as:

$$\max_{\boldsymbol{k}, \boldsymbol{y}, \boldsymbol{u}, v} \bar{\boldsymbol{e}}'(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y} + \boldsymbol{b}'\boldsymbol{u}) - \delta v + \boldsymbol{a}'\boldsymbol{u} \tag{3.10}$$

subject to:

$$\|\Sigma^{1/2}(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y} + \boldsymbol{b}'\boldsymbol{u})\| \leqslant v$$
$$\boldsymbol{u} \leqslant \boldsymbol{w}^d$$
$$\boldsymbol{k}, \boldsymbol{y}, \boldsymbol{u}, v \geqslant 0$$

Strong duality holds as problem (2.12) is a second-order cone program, which means that as long as they are feasible, the primal and dual problems have the same objective function values. Hence, we can replace the inner minimization problem in problem (3.8) by:

$$\max_{\boldsymbol{w},\boldsymbol{w}^d,\boldsymbol{k},\boldsymbol{y},\boldsymbol{u}} \bar{\boldsymbol{e}}'(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y} + \boldsymbol{b}'\boldsymbol{u}) - \delta\|\Sigma^{1/2}(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y} + \boldsymbol{b}'\boldsymbol{u})\| + \boldsymbol{a}'\boldsymbol{u} \tag{3.11}$$

subject to:

$$\mathbf{1}'(\boldsymbol{w} + \boldsymbol{w}^d) = 1$$
$$\boldsymbol{u} \leqslant \boldsymbol{w}^d$$
$$\boldsymbol{w}, \boldsymbol{w}^d, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{u} \geqslant 0$$

We have described a model based on two different hedging strategies, where we see hedging as a policy to reduce or to eliminate risk by making what is uncertain (such as future returns) more certain. By using robust optimization, the investor is protected against any depreciation of the foreign exchange rates within the uncertainty set. Adding currency options to the model provides a cap on the value of the future foreign exchange rates. Our hedging strategy has a portfolio point of view, in that it is not concerned with the individual depreciation of any particular currency, but instead looks at the portfolio return as a whole and provides guarantees to total return. In the event of the foreign exchange rates materializing outside the uncertainty set, robust optimization provides no guarantees. However, the put options held in the portfolio guarantee a minimum return given by their strike price as described in (3.3). Depending on the respective holdings of the put options, however, this guaranteed return may not be enough for the investor.

We would like to insure our portfolio further, even against the occurrence of a sharp depreciation of foreign exchange rates, that is, against their materializing outside the uncertainty set. We reformulate our model in order to include an additional constraint guaranteeing a minimum return, expressed as a percentage of the worst-case portfolio return for all the possible values of the currency returns subject to $\boldsymbol{e} \geqslant 0$. We change the formulation of our problem in order to include this new constraint:

$$\max_{\boldsymbol{w},\boldsymbol{w}^d} \phi \tag{3.12a}$$

subject to:

$$\boldsymbol{e}'\boldsymbol{w} + \boldsymbol{e}^{d\prime}\boldsymbol{w}^d \geqslant \phi \quad \text{for all } \boldsymbol{e} \in \Theta_{\boldsymbol{e}},\ \boldsymbol{e}^d = f(\boldsymbol{e}) \tag{3.12b}$$

$$\boldsymbol{e}'\boldsymbol{w} + \boldsymbol{e}^{d\prime}\boldsymbol{w}^d \geqslant \rho\phi \quad \text{for all } \boldsymbol{e} \geqslant 0,\ \boldsymbol{e}^d = f(\boldsymbol{e}) \tag{3.12c}$$

$$\mathbf{1}'(\boldsymbol{w} + \boldsymbol{w}^d) = 1 \tag{3.12d}$$

$$\boldsymbol{w}, \boldsymbol{w}^d \geqslant 0 \tag{3.12e}$$

We have already shown how to reformulate the inner minimization problem to correspond to constraint (3.12b) as a maximization problem. We use the same approach for constraint (3.12c):

$$\min_{\boldsymbol{e},\boldsymbol{e}^d} \boldsymbol{e}'\boldsymbol{w} + \boldsymbol{e}^{d\prime}\boldsymbol{w}^d \tag{3.13}$$

subject to:

$$\boldsymbol{e}^d \geqslant \boldsymbol{a} + \boldsymbol{b}\boldsymbol{e}$$
$$\boldsymbol{e}, \boldsymbol{e}^d \geqslant 0$$

The dual of this linear problem can be formulated as:

$$\max_{t} \boldsymbol{a}'\boldsymbol{t} \tag{3.14}$$

subject to:

$$\boldsymbol{w} + \boldsymbol{b}'\boldsymbol{t} \geqslant 0$$
$$\boldsymbol{t} \leqslant \boldsymbol{w}^d$$
$$\boldsymbol{t} \geqslant 0$$

Since strong duality also holds for linear problems, we may replace the inner minimization in our original problem (3.12a) by the objective function of problem (3.14) and the reformulation of constraint (3.12b):

$$\max_{\boldsymbol{w},\boldsymbol{w}^d,\boldsymbol{k},\boldsymbol{y},\boldsymbol{u},\boldsymbol{t}} \phi \tag{3.15}$$

subject to:

$$\bar{\boldsymbol{e}}'(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y} + \boldsymbol{b}'\boldsymbol{u}) - \delta\|\Sigma^{1/2}(\boldsymbol{w} - \boldsymbol{A}'\boldsymbol{k} - \boldsymbol{y} + \boldsymbol{b}'\boldsymbol{u})\| + \boldsymbol{a}'\boldsymbol{u} \geqslant \phi$$
$$\boldsymbol{a}'\boldsymbol{t} \geqslant \rho\phi$$
$$\boldsymbol{w} + \boldsymbol{b}'\boldsymbol{t} \geqslant 0$$
$$\mathbf{1}'(\boldsymbol{w} + \boldsymbol{w}^d x) = 1$$
$$\boldsymbol{u} \leqslant \boldsymbol{w}^d$$
$$\boldsymbol{t} \leqslant \boldsymbol{w}^d$$
$$\boldsymbol{w}, \boldsymbol{w}^d, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{t} \geqslant 0$$

Note that neither the currency returns nor the currency option returns appear in the final formulation (3.15). This is a tractable problem which can be solved efficiently by any second-order cone optimization software.

As before, if the investor wishes to move away from the minimum risk solution, a constraint on the expected return may be added to the model:

$$\mathbb{E}[e] = \boldsymbol{w}'\bar{\boldsymbol{e}} \geqslant e_{\text{target}}$$

We have chosen not to include the options return in this constraint as this would distort our solution. On the one hand, our goal when including currency options is from a hedging strategy point of view, that is, we want to protect the portfolio return from depreciations in the foreign exchange rates and not to speculate on options. On the other hand, because options are leveraged assets and we are optimizing in view of the worst-possible outcome of the currency returns, the optimal solution would be to invest the full budget on in-the-money options and not on currencies. Note how the hedging strategy presented has a portfolio point of view and does not focus on any individual currency. The investor does not limit the weights of the currency options to the weights of the respective currency holdings. The guaranteed portfolio when the foreign exchange rates materialize outside the uncertainty set is defined by the investor and does not depend on the individual depreciation of any currency.

In the next section we present numerical results for the models with and without considering currency options and assess their performance.

4 NUMERICAL RESULTS

The theoretical framework presented in Sections 2 and 3 will now be used to compute optimal currency portfolios based on real market data. We assume a US investor who wishes to invest in six foreign currencies: euro, British pound, Japanese yen, Swiss franc, Canadian dollar and Australian dollar. The models were implemented using the modeling language YALMIP (Lofberg (2004)) together with the second-order cone solver SDPT3 (Toh (1999) and Tutuncu (2003)). Both the expected returns on the foreign exchange rates and the covariance matrix are constructed from seven years of monthly data between January 2002 and December 2008 (see Table 2 on the next page).

We start by studying the composition of the portfolio and the distribution of weights between currencies and options for different levels of risk, defined by ω, and different levels of hedging, defined by ρ.

4.1 Portfolio composition

In the following analysis we will designate problem (2.15) as the robust problem and problem (3.15) as the hedging problem. We start by comparing the robust model with

TABLE 2 Distributional parameters of monthly currency returns against the US dollar (January 2002–December 2008).

	Annual return (%)	Standard deviation (%)	Correlations					
EUR	5.64	8.75	1.00					
GBP	0.18	7.74	0.77	1.00				
JPY	5.32	8.51	0.42	0.16	1.00			
CHF	5.52	8.55	0.91	0.69	0.62	1.00		
CAD	3.61	7.00	0.56	0.51	0.01	0.41	1.00	
AUD	3.43	11.09	0.69	0.65	0.04	0.53	0.78	1.00

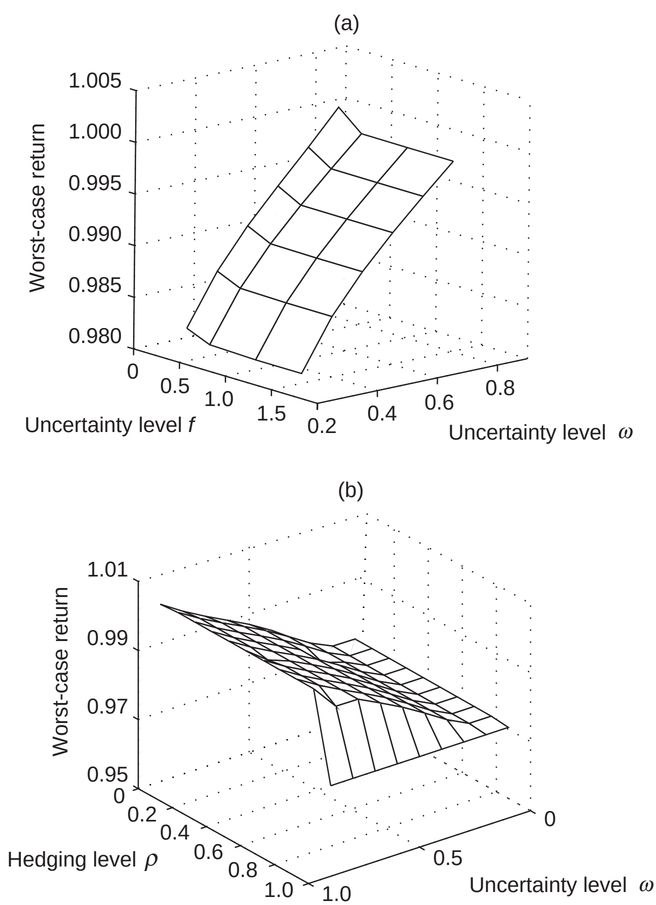
the Markowitz minimum risk model, where portfolio variance is minimized subject to some lower bound on the portfolio expected return. In our robust model, the size of the uncertainty set defined by δ_ω, $\delta = \sqrt{(1-\omega)/\omega}$, can be interpreted as a risk measure, namely, the worst-case VaR (El Ghaoui *et al* (2003)). It is expected that as ω increases, the risk associated with the portfolio will increase as well. If we measure the risk of the portfolio as its variance, we are able to conclude that, for higher values of ω, there is an increased value of the variance of the portfolio. The portfolio composition of problem (2.15) reflects this increase: for higher levels of ω the optimizer concentrates its investment on a single currency. This behavior is similar to the Markowitz model, the difference being that, in this case, the focus is not on the currency with the highest estimated return rate, but on the one with the highest worst-case return rate.

One of the key differences between our robust model and the Markowitz framework is the triangulation requirement, which restricts the size of the uncertainty set and leads to less conservative models. In order to assess the impact of this constraint, we build our matrix $\boldsymbol{A}$ based on the historical means and covariance matrix of the cross exchange rates over the same period. The cross exchange rates uncertainty set is defined as in (2.9):

$$\Theta_{\boldsymbol{x}} = \{\boldsymbol{x} \geqslant 0 \colon \bar{\boldsymbol{x}} - f\sigma_{\boldsymbol{x}} \leqslant \boldsymbol{x} \leqslant \bar{\boldsymbol{x}} + f\sigma_{\boldsymbol{x}}\} \tag{4.1}$$

where f is a scalar which defines the size of the interval in terms of the mean $\bar{\boldsymbol{x}}$ and the standard deviation $\sigma_{\boldsymbol{x}}$ of the cross exchange rates. We tested the parameter $f \in \{0.25; 0.5; 1.0; 1.5\}$ for different sizes of the uncertainty set as defined by ω in the robust model (2.15) and computed the worst-case portfolio return. The results are presented in part (a) of Figure 2 on the facing page. The impact of the triangulation constraint on the portfolio return is closely related to the size of the uncertainty set defined by the ellipsoid. A tighter constraint for $f = 0.25$ implies a less conservative

FIGURE 2 Worst-case returns.



(a) Worst-case return for different values of f and ω. (b) Worst-case return for different values of ω and ρ.

model and a higher portfolio return. For example, when $\omega = 0.6$, the worst-case portfolio return ranges from 0.9940 to 0.9927 for $f = 0.25$ and $f = 1.5$, respectively.

We would like to assess the impact of adding currency options to the portfolio, and how the insurance provided by the options relates to the guarantees provided by robust optimization. We consider fifty put options and fifty call options available in the market, with strike prices ranging between 75% and 125% of the current spot prices. In the experiments described below, we include a budget constraint and we do not allow short selling. Compared with the robust model, there is a change in the weights allocation between the different currencies, in favor of the currencies with the highest worst-possible returns. In our first set of experiments we have not considered a minimum expected return, and we have studied how the worst-case return and the total investment in options changes relative to the size of the uncertainty set defined by ω. For higher values of ω (that is, smaller uncertainty sets), the optimal portfolio is comprised mainly of currencies and not of options. As the uncertainty set increases in size, the percentage allocated to put options reaches almost 20%, with the remaining budget distributed among the currencies. Protection against the currency risk in this situation is made through the acquisition of deep-in-the-money options, while, for small uncertainty sets, this is done by currency diversification. The worst-case return is constant at 1.0025 (annual rate of 3%) for $\omega \leqslant 80\%$. Investment in options is actively "capping" the maximum portfolio loss.

We now add an expected return constraint of an annual average return of 5%. Because this constraint does not include options, a larger percentage must be allocated to foreign currency holdings to meet this constraint. In this situation, not only is the weight of the options in the portfolio considerably lower, but the options on which to invest are at-the-money options. In contrast with the previous case, the worst-case return degrades to values below zero, that is, the worst case implies a loss for the investor of about 3%.

Part (b) of Figure 2 on the preceding page shows the trade-off between the two different sets of guarantees provided by robust optimization and by the currency options. For the same level of desired hedging of the currency risk (expressed by parameter ρ), a higher value of ω (ie, a smaller uncertainty set) leads to an increase of the worst-case returns. For smaller values of ω, the uncertainty set converges to the full support of the currency returns, which leads to overly conservative portfolios. We may then conclude that the worst-case return monotonically increases with ω. In contrast, for the same size of the uncertainty set ω, a higher level of hedging (given by ρ) leads to a decrease in the worst-case return. This is because options are expensive assets, and a higher hedging demand may only be satisfied if the worst case is smaller simultaneously. Therefore, the worst-case portfolio return has an inverse relationship with ρ.

The results obtained from our experiments lead us to conclude that the constraint on the minimum guaranteed return outside the uncertainty (3.12c) is not a binding constraint. This conclusion, however, may be flawed due to problems related to estimating the option prices. We have used the Garman and Kohlhagen (1983) model to obtain the option prices. The model assumes that the implied volatility is constant and depends neither on the strike price of the option nor on its time to maturity. In reality, however, the volatility depends on the strike price of the option and exhibits what is known as a "smile", that is, it is higher for out-of-the-money and in-the-money options, while it is lower for at-the-money options (Hull (2006)). By considering the same volatility for all of the fifty strike prices tested, we underestimate the option prices. The model may therefore choose either to invest in deep-in-the-money options or to generally overinvest in options given their low prices. This would make the minimum guaranteed return constraint (3.12c) redundant. Given the reasons described above, however, we have chosen to retain this constraint in the model when performing the historical backtesting, which is described next.

4.2 Model evaluation with historical market prices

We want to assess the performance of our model under real market conditions by computing the portfolio returns over a long period of time. To this end, we consider the real currency returns in the period from January 2002 to March 2009 and conduct a backtest with a rolling horizon of twelve months. Every month, we compute the estimated average returns $\bar{e}$ based on the historical returns from the previous twelve months, and calculate the optimal portfolio weights. The triangulation matrix $\boldsymbol{A}$ is constructed monthly based on the historical means of the cross exchange rates from the previous twelve months. A factor of $f = 1$ is used to weigh the standard deviation, which was assumed to be constant. The covariance matrix Σ is also assumed to remain the same throughout the time series. An expected portfolio return constraint of 5% per year was added. At the end of each month, the portfolio return is computed based on the materialized returns, and the options are exercised or left to expiry depending on the spot rate. This procedure is repeated until March 2009 and the accumulated returns are calculated.

We note that, although our backtesting experiment is performed out-of-sample, the covariance matrix is calculated in-sample. Estimating a covariance matrix within a rolling horizon of twelve months would be very difficult considering that, in the case of the cross exchange rates, we have fifteen different parameters. While in-sample estimation could lead to problems of data mining or overly optimistic results, Rapach and Woharb (2006) show that out-of-sample tests are also subject to data mining issues. In fact, there is not a great difference between in-sample and out-of-sample predictability tests. Moreover, as suggested by Broadie (1993), the impact of errors in the estimation of the covariance matrix in the portfolio return is not significant.

Given that currency options are traded mainly over-the-counter, there are no records of historical prices, but only of three different volatilities that may be used to construct the volatility smile and compute the option price. Contrary to the assumptions of the Black–Scholes and the Garman–Kohlhagen models, volatility is not constant throughout the spectrum of the strike prices, but is higher for out-of-the-money and for in-the-money options, while it is lower for at-the-money options. Moreover, it has also been verified empirically that options with the same exercise price but with different maturities exhibit different implied volatilities, designated as the term structure (Hull (2006)). The probability distribution of the currency returns, consequently, is not lognormal, but has heavier tails, making it more likely for extreme variations of the returns. The volatility associated to a given strike price may be calculated from the volatility smile, for which there is an approximate expression (see Malz (2001)):

$$\sigma(\delta, T) = \sigma_{\mathrm{ATM},T} - 2\,\mathrm{rr}_T(\delta - \tfrac{1}{2}) + 16\,\mathrm{str}_T(\delta - \tfrac{1}{2})^2 \tag{4.2}$$

where:

$$\delta = e^{-r_{\mathrm{d}}T}\,\Phi\left[\frac{\ln(S/K) + (r_{\mathrm{d}} - r_{\mathrm{f}} + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right] \tag{4.3}$$

Expression (4.3) corresponds to the delta of a call option and is used in the Garman–Kohlhagen model. The quadratic approximation of the volatility smile (4.2) includes three different volatilities: σ, corresponding to the implied volatility of an at-the-money option ($\delta = 50$); risk reversal (rr), the difference in volatilities between a long out-of-the-money call option and a short out-of-the-money put option ($\delta = 25$); and strangle (str), the average of the volatility of two long out-of-the-money call and put options ($\delta = 25$) minus the volatility of the at-the-money option. The volatility obtained by this expression can then be used in the Garman–Kohlhagen model to calculate the option price. We considered an annual risk-free rate of 3.32% for the US investor (based on LIBOR annual rates for the same period).

We have run the robust (Equation (2.15)) and the hedging (Equation (3.15)) models over the period considered, rebalancing the portfolio every month and measuring the cumulative gains for different values of the parameters ω and ρ. As a benchmark, we have also run the Markowitz models with and without the consideration of currency options. The Markowitz model without options minimizes the variance of the portfolio, while holding an expected return constraint. When considering options, we maximize the return with both types of assets, while constraining the expected portfolio return with currencies only and imposing an upper bound on the variance of the portfolio. This upper bound corresponds to the minimum variance achieved with the Markowitz model without options. This way, in both models, options are only used as a downside risk protection.

While the minimum risk model yields an average annual return of 2.8%, the robust model consistently yields a higher return, from 5.7% ($\omega = 80\%$) to 4.1% ($\omega = 30\%$). As the uncertainty set increases, the average returns move closer to the values exhibited by the minimum risk model. Table 3 on the facing page presents the average annual returns obtained for different sizes of the ellipsoidal uncertainty set combined with different sizes of the box uncertainty set regarding the triangulation requirement. Note how the inclusion of the triangulation constraint may render the model less conservative, thereby yielding a higher average annual return. For all the values of ω considered, a tighter constraint with $f = 0.25$ always provides a higher average annual return.

TABLE 3 Average annual return rates for different values of the parameters ω and ρ.

f	ω (%)	Annual return (%)	f	ω (%)	Annual return (%)
0.25	30	5.1	1	30	4.1
0.25	40	5.2	1	40	4.2
0.25	50	5.3	1	50	4.3
0.25	60	5.3	1	60	4.3
0.25	70	5.4	1	70	4.8
0.25	80	5.5	1	80	5.7
0.5	30	4.3	1.5	30	4.1
0.5	40	4.4	1.5	40	4.3
0.5	50	4.4	1.5	50	4.3
0.5	60	4.4	1.5	60	4.3
0.5	70	4.8	1.5	70	4.8
0.5	80	5.7	1.5	80	5.7

FIGURE 3 Accumulated wealth over the period from January 2002 to March 2009.

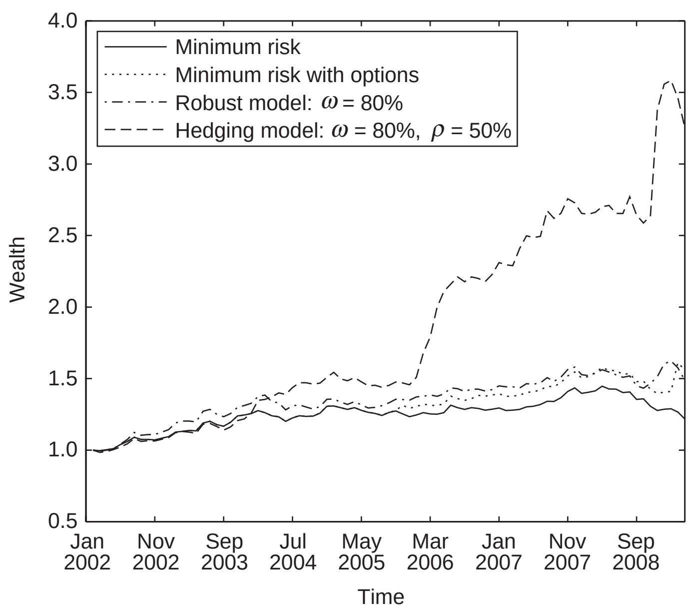


TABLE 4 Average annual return rates for different values of the parameters ω and ρ.

ω (%)	ρ (%)	Annual return (%)	ω (%)	ρ (%)	Annual return (%)
50	0	17.1	70	0	11.7
50	10	33.8	70	10	19.7
50	20	19.4	70	20	17.8
50	30	16.9	70	30	24.4
50	40	13.2	70	40	23.3
50	50	6.7	70	50	14.1
50	60	6.8	70	60	10.9
50	70	6.4	70	70	3.9
60	0	13.9	80	0	14.8
60	10	24.4	80	10	10.9
60	20	29.1	80	20	14.2
60	30	22.2	80	30	27.3
60	40	15.8	80	40	26.0
60	50	8.3	80	50	17.9
60	60	9.0	80	60	7.9
60	70	7.6	80	70	7.2

Figure 3 on the preceding page depicts the accumulated wealth when optimizing the portfolio with the different models, taking $\omega = 80\%$ and $\rho = 50\%$. For this particular parameter choice, the minimum risk model is dominated by both the robust and the hedging model, while the hedging model clearly outperforms the robust model, with average annual returns of 17.9% and 5.7%, respectively. We note that the minimum risk model with options and the robust model have a very similar growth pattern with an average annual return of 6.4% and 5.7%, respectively. This illustrates the risk protection feature of robust optimization by providing a guaranteed worst-case return, and of currency put options which effectively cap the value of the foreign exchange rate at the respective strike price.

The hedging model (3.15) outperforms the robust model (2.15) for smaller values of the parameter ρ. Without any restriction on the minimum return guarantees outside the uncertainty set (3.12c), we may choose expensive, deep-in-the-money options, although only a small number of units. These options will be exercised with high probability and will yield a high return per unit as well. In contrast, as we impose a higher restriction on the minimum return, that, is, as ρ increases (3.12c), we also choose cheaper options (ie, more at-the-money) in order to be able to buy the necessary number of units to satisfy the constraint. These options will have a 50% chance of being exercised and therefore returns are potentially lower. Table 4 illustrates this relationship. Note that the high returns yielded by some of the models are mainly in the same period where most of the currencies suffered severe losses, that is, from March 2006 onward. Options may have played an important role in this period in protecting the portfolio from depreciations of the foreign exchange rates.

However, the conclusion that the hedging and the robust models perform better than the Markowitz models should be made with caution. Factors such as transaction costs and the risk of default from the writer of the options have not been taken into consideration and may have an impact on the accumulated portfolio wealth. Moreover, we have also not considered uncertainty in the covariance matrix, which, though possibly insignificant, may also have a negative impact.

5 CONCLUSION

In this paper we apply robust optimization techniques to a currency-only portfolio. We show that, due to the triangular relationship between foreign exchange rates, a new nonarbitrage constraint must be added to the model, which seems to render the model nonconvex. Given that the cross exchange rates do not have an impact on portfolio return, we may simplify the triangulation constraint by eliminating the variables referring to the cross exchange rates and obtain a set of linear constraints. We further extend the robust model to include currency options as a hedging instrument. We rely on put options to guarantee a minimum value of the foreign exchange rates, thereby providing a cap to the worst-case portfolio return. The resulting model provides the investor with two different sets of complementary guarantees: firstly, robust optimization provides a noninferiority guarantee as long as the realized currency returns are within the uncertainty set; and secondly, put options limit the portfolio losses by stopping the depreciation of the foreign exchange up to the value of the strike price.

The approach to the hedging problem that we have suggested has the advantage of being more flexible than the standard hedging strategies, as it relies on options and robust optimization and not on forwards or futures. The backtesting experiment conducted with real market data seems to point toward a better overall performance of the robust and of the hedging model when compared with the Markowitz minimum risk model. Moreover, we observe that, when the imposition on the guaranteed portfolio return for the entire support of the currency returns is not too restrictive, the hedging model outperforms the robust model.

APPENDIX A: UNCERTAIN COVARIANCE MATRICES

Consideration of separate uncertainty sets for the mean returns and the covariance matrix often leads to inconsistent probability measures. To overcome this, we follow the approach suggested in Ye *et al* (2009) and, instead of defining an uncertainty set over the covariance matrix directly, we define a componentwise bound on the mean returns $\bar{\boldsymbol{e}}$ and on the second moment matrix of returns Γ. Within this framework, we are interested in minimizing the portfolio variance for the worst case of the covariance matrix:

$$\min_{\boldsymbol{w}\in\mathbb{R}^n} \max_{\Gamma\in\Xi_\Gamma,\,\bar{\boldsymbol{e}}\in\Xi_{\bar{\boldsymbol{e}}}} \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} = \boldsymbol{w}'(\boldsymbol{\Gamma} - \bar{\boldsymbol{e}}\bar{\boldsymbol{e}}')\boldsymbol{w} \tag{A.1}$$

subject to $\mathbf{1}'\boldsymbol{w} = 1$, $\boldsymbol{w} \geqslant 0$, where the covariance matrix is written as $\Sigma = \Gamma - \bar{\boldsymbol{e}}\bar{\boldsymbol{e}}'$, and where Ξ_Γ and $\Xi_{\bar{\boldsymbol{e}}}$ define the uncertainty regions for Γ and $\bar{\boldsymbol{e}}$, respectively. As our final model will be cast as a semidefinite program, we rewrite the variables in the form of positive semidefinite symmetric matrices. We define:

$$\hat{\Gamma} \equiv \begin{bmatrix} \Gamma & \bar{\boldsymbol{e}} \\ \bar{\boldsymbol{e}}' & 1 \end{bmatrix}, \quad \hat{\boldsymbol{E}} \equiv \begin{bmatrix} \bar{\boldsymbol{e}}\bar{\boldsymbol{e}}' & \bar{\boldsymbol{e}} \\ \bar{\boldsymbol{e}}' & 1 \end{bmatrix} \quad \text{and} \quad \hat{\boldsymbol{W}} \equiv \begin{bmatrix} \boldsymbol{w}\boldsymbol{w}' & \boldsymbol{w} \\ \boldsymbol{w}' & 1 \end{bmatrix} \tag{A.2}$$

For any given fixed set of portfolio weights $\boldsymbol{w}$, we formulate the inner maximization problem in (A.1) as:

$$\max_{\Gamma,\bar{\boldsymbol{e}}} \boldsymbol{w}'\boldsymbol{\Gamma}\boldsymbol{w} - \boldsymbol{w}'\bar{\boldsymbol{e}}\bar{\boldsymbol{e}}'\boldsymbol{w} \tag{A.3}$$

subject to:

$$\bar{\boldsymbol{e}} \in [\bar{\boldsymbol{e}}_{\mathrm{l}}, \bar{\boldsymbol{e}}_{\mathrm{u}}] \tag{A.4}$$

$$\boldsymbol{A}\bar{\boldsymbol{e}} \geqslant 0 \tag{A.5}$$

$$\Gamma \in [\Gamma_{\mathrm{l}}, \Gamma_{\mathrm{u}}] \tag{A.6}$$

$$\hat{\boldsymbol{E}} \succeq 0 \tag{A.7}$$

$$\hat{\Gamma} \succeq 0 \tag{A.8}$$

Constraint (A.4) represents the componentwise upper and lower bounds of the mean vector defined in $\Xi_{\bar{\boldsymbol{e}}}$, while constraint (A.5) corresponds to the triangulation requirement as before. The uncertainty set for the second moments of the returns as explicitly defined in Ξ_Γ is represented by the constraint (A.6).

Semidefinite programs are convex optimization problems. Therefore, assuming that strong duality holds, we can determine the dual of problem (A.3) and replace it in our original problem (A.1) (see Section (2)). We obtain a single minimization problem without the uncertain first and second moments of the returns. We refer the reader to Ye *et al* (2009) for a detailed consideration of the covariance matrix uncertainty in portfolio allocation problems, and to Boyd and Vanderberghe (2004) for semidefinite programming duality.

This approach can easily be extended to the consideration of a joint uncertainty set for the first and the second moments of the returns, currency options and ellipsoidal uncertainty sets. Since these extensions require no new ideas, we omit them for the sake of brevity.

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