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REACHING GOALS BY A DEADLINE: DIGITAL OPTIONS AND CONTINUOUS-TIME ACTIVE PORTFOLIO MANAGEMENT

SID BROWNE,* *Columbia University*

Abstract

We study a variety of optimal investment problems for objectives related to attaining goals by a fixed terminal time. We start by finding the policy that *maximizes the probability of reaching a given wealth level by a given fixed terminal time*, for the case where an investor can allocate his wealth at any time between $n + 1$ investment opportunities: n risky stocks, as well as a risk-free asset that has a positive return. This generalizes results recently obtained by Kulldorff and Heath for the case of a single investment opportunity. We then use this to solve related problems for cases where the investor has an external source of *income*, and where the investor is interested solely in *beating the return of a given stochastic benchmark*, as is sometimes the case in institutional money management. One of the benchmarks we consider for this last problem is that of the return of the optimal growth policy, for which the resulting controlled process is a supermartingale. Nevertheless, we still find an optimal strategy. For the general case, we provide a thorough analysis of the optimal strategy, and obtain new insights into the behavior of the optimal policy. For one special case, namely that of a single stock with constant coefficients, the optimal policy is independent of the underlying drift. We explain this by exhibiting a correspondence between the probability maximizing results and the pricing and hedging of a particular derivative security, known as a digital or binary option. In fact, we show that for this case, the *optimal policy to maximize the probability of reaching a given value of wealth by a predetermined time is equivalent to simply buying a European digital option with a particular strike price and payoff.* A similar result holds for the general case, but with the stock replaced by a particular (index) portfolio, namely the optimal growth or log-optimal portfolio.

Keywords: Optimal gambling; stochastic control; portfolio theory; martingales; option pricing; hedging strategies; digital options

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1. Introduction

There are various approaches to the problem of determining optimal dynamic investment policies, depending on the objectives of the investor. In continuous time, which is the setting in this paper, optimal dynamic investment policies for the objective of maximizing expected *utility* derived from terminal wealth or consumption over a finite horizon as well as discounted utility of consumption over the infinite horizon are derived in the pioneering work of Merton [15]. Generalizations of the utility maximizing approach that incorporate bankruptcies as

Received 11 June 1997; revision received 16 March 1998.
* Postal address: 402 Uris Hall, Graduate School of Business, Columbia University, New York, NY 10027 USA.
Email address: sb30@columbia.edu
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well as more general price processes than those considered earlier are surveyed in [16] and [7]. These results also have substantial implication for the pricing and hedging of contingent claims, see for example [5].

However, there are many investment scenarios where approaches alternative to that of utility maximization might be preferable. In particular, many actual investment objectives are related solely to the achievement of specific *goals*. For example, in institutional money management, the practice of *benchmarking* is quite prevalent. In this scenario, a portfolio manager is judged solely by how his portfolio performs relative to that of another benchmark portfolio, or index. The Standard and Poor's (S&P) 500 index is a typical example of a benchmark. There is a distinction made between *passive portfolio management*, and *active portfolio management* (see, e.g., [20]). A *passive* portfolio manager is simply interested in tracking the index, while an *active* portfolio manager is interested in beating the return of the predetermined given benchmark or index. From the viewpoint taken here, the passive portfolio manager's investment decision is uninteresting, since we assume that for all intents and purposes, a passive portfolio manager can simply invest directly in the benchmark. The active portfolio manager faces an interesting problem however, since he is investing in order to beat a 'goal'. The goal the active portfolio manager is trying to beat is the *stochastic* return of the benchmark. In this paper we treat this among other goal problems. The active portfolio management problem is also relevant to the hedging of contingent claims, where in that setting, the benchmark portfolio would be taken to be the replicating strategy of the contingent claim.

Optimal investment policies for objectives relating solely to the achievement of goals have been studied previously, although perhaps not to the extent that utility maximization has. For illustration, suppose the investor starts off with initial capital $0 < X_0 < b$. Then, some classical problems include determining an investment policy that (if appropriate) maximizes the probability of reaching b before 0, or (if appropriate) minimizes the expected time to the upper goal b. We can refer to these, respectively, as the *survival problem*, and the *growth problem*. In discrete time, and over an infinite horizon, the survival problem is the centerpiece of the classical work of Dubins and Savage [6], and the growth problem was studied in [2] and [9]. In continuous time, the survival problem over an infinite horizon for diffusion processes was studied in [17, 18]. Survival problems related to various generalizations of the portfolio problem of Merton [15] were studied in [3] and [4] (see also [14]). The growth problem in continuous time was studied in a general framework in [11], [16, Chapter 6], and generalized to a model that incorporated liabilities in [4]. However, all these results are specific to the case of an *infinite horizon*. Since the performance of money managers is not judged over the infinite horizon, but rather over a finite (sometimes quite short) horizon, these studies are not directly applicable to the problem of active portfolio management. Similarly, in pension fund management, the horizon is typically finite. With a finite horizon, the distinction between a survival problem and a growth problem tends to blur, since in both cases they relate to maximizing probabilities: the survival problem would be to maximize the probability that the lower goal is not hit before the horizon, while the growth problem would be to maximize the probability that the upper goal *is* hit before the deadline.

A finite-horizon goal problem was studied recently by Kulldorff [13], for a model with a single risky favorable investment opportunity. Kulldorff obtained the optimal investment policy for the objective of *maximizing the probability that wealth attains a given constant goal by a fixed terminal time*. Both continuous and discrete time problems were considered there. For the continuous time version of the problem, the return of the single risky asset was modeled as a Brownian motion with a time-dependent drift coefficient and a constant diffusion coefficient. The goal as well as the constant diffusion coefficient were both normalized to 1. Heath, in [10], considered the same model, but with constant drift as well and took a somewhat different approach to establish the same results as in [13]. In both cases, there was no risk-free asset available other than cash, which had a zero return. One of the interesting features of their policy was the fact that for the case of a *constant* drift, the optimal policy was *independent of this underlying drift*. No explanation of this rather remarkable fact was given.

In this paper, we address a variety of more general goal problems, all with probability maximizing objectives. To that end we first generalize the important results of [10] and [13] in a few fundamental ways: first, we expand the investment opportunity set to include a risk-free asset that has a (positive) time-dependent rate of return, as well as multiple risky assets with time-varying covariance structure. We also obtain new representations and analysis of the resulting optimal policy. We then extend our general results to treat cases where the investor earns *income* from an exogenous source, and where the investor's objective is to beat the (stochastic) return of a given benchmark portfolio.

The explanation as to why the optimal policy is independent of the underlying drift for the single stock case with constant coefficients is provided as a byproduct of our analysis for the more general case. It turns out that when there are multiple risky stocks, the optimal policy, for maximizing the probability of attaining a preset level of wealth by a finite deadline, is no longer independent of the drift parameters, even for the case of constant coefficients. The resulting policy is quite interesting and we provide a new analysis that allows for a complete quantitative assessment of the risk-taking behavior of an investor following such an objective. Furthermore, we obtain a new representation of the optimal wealth process. This representation, together with the addition of a risk-free asset, allows us to exhibit a remarkable correspondence between the probability maximizing policy and the hedging strategy of a digital option for the single-stock constant coefficients case. In particular we show that the optimal dynamic investment strategy for the objective of maximizing the probability of reaching a given goal by a fixed terminal time is completely equivalent to the (static) investment strategy which simply purchases a European digital call option on the underlying stock, with a particular strike price and payoff. This result is of independent interest since it provides an example where a policy which is optimal for an objective stated on wealth, is equivalent to the purchase of an option on the underlying stock. Moreover, it also implicitly contains the explanation as to why the optimal policy is independent of the drift in the constant coefficients single-stock case. For the general case, we are able to show a similar result, however with the single stock replaced by the return of a particular portfolio policy: the *optimal growth*, or *log-optimal* portfolio. Specifically, we will show that the probability maximizing policy is completely equivalent to purchasing a European digital option on the return of the log-optimal portfolio.

A summary and outline of the remainder of the paper is as follows: In the next section, we introduce the basic model with multiple stocks and a risk-free asset with positive return. In Section 3 we provide the optimal policy for the problem of maximizing the probability of reaching the goal by terminal time $T < \infty$, as well as the new representation of the optimal wealth under this policy (Corollary 3.2 below). The proof is delayed until Section 6. In Section 4, we then use this representation to show the correspondence between the single stock case with constant coefficients and a digital option on the stock, as well as the correspondence with the digital option on the log-optimal portfolio in the general case.

In Section 5, we analyse the optimal policy for the general case. We first show that the optimal policy can be interpreted as a linear function of wealth, where the coefficient decomposes into the product of two distinct factors: (1) a purely time-dependent risk factor, which is determined solely by the risk premiums of the stocks and the time remaining until the deadline; and (2) a purely state-dependent function which is parameterized solely by the current percentage of the distance to the (in our case, time-dependent) goal achieved. As intuition would suggest, the former function typically increases as the horizon decreases, while the latter function decreases as the percentage increases. The optimal policy is therefore a dynamic portfolio strategy that continuously rebalances the portfolio weights depending upon how much time remains to the deadline as well as how close the current wealth is to the goal. The interplay between these two factors is analysed to a fairly explicit extent next when we analyse the region where borrowing takes place. It turns out that this region is determined by a single equation involving the 'risk-adjusted' remaining time, and the percentage of the goal achieved to that point.

In Section 7 we consider the case where the investor earns income from an external source other than trading gains. We show that contrary to utility maximizing strategies—where an investor uses the exogenous income to take a more risky position in stocks than he would otherwise—a 'probability maximizing' investor relies on this exogenous income to be more cautious. In particular, we show that external income causes the investor to incorporate a performance bound: if the performance of the stocks is such that wealth ever falls to the level that could have been achieved by simply investing all the previous income into the risk-free asset, then all investment in the risky stocks ceases.

In Section 8, we consider the case where the investor's goal is to beat the return of a given (stochastic) benchmark by a prespecified amount by a predetermined time. We also find the related policy that allows the investor to control for the downside risk. When the stochastic benchmark is given by the *optimal growth policy*, or equivalently, the policy that maximizes logarithmic utility, which is sometimes referred to as the *market portfolio* in continuous-time finance, then certain complications arise. Specifically, it is well known that the ratio of the return from any arbitrary portfolio strategy to the return generated by the optimal growth strategy is a nonnegative local martingale, hence a *supermartingale*. Nevertheless, we find a policy that does achieve the theoretical upper bound on the probability of beating the return of the optimal growth by a predetermined amount, and is hence an optimal policy.

2. The model

The model under consideration here is that of a complete market as in [7, 15, 16] and others, wherein there are n (correlated) risky assets generated by n independent Brownian motions. The prices of these stocks are assumed to evolve as

$$\mathrm{d}S_i(t) = S_i(t)\left[\mu_i(t)\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij}(t)\,\mathrm{d}W_t^{(j)}\right], \quad i = 1, \dots, n, \tag{2.1}$$

where $\mu_i(t), \sigma_{ij}(t)$ are deterministic functions, for $i, j, = 1, \dots, n$, and $t \geq 0$, and where $\boldsymbol{W}_t := (W_t^{(1)}, \dots, (W_t^{(n)})'$ is a standard n-dimensional Brownian motion, defined on the complete probability space $(\Omega, \mathcal{F}, P)$, where $\{\mathcal{F}_t, t \geq 0\}$ is the P-augmentation of the natural filtration $\mathcal{F}_t^{\boldsymbol{W}} = \sigma\{\boldsymbol{W}_s; 0 \leq s \leq t\}$. Thus the stock prices follow a *correlated* time-dependent geometric Brownian motion in $\mathbf{R}^n$.

There is also a riskless asset whose price, B_t, evolves according to

$$\mathrm{d}B_t = r(t) B_t\, \mathrm{d}t \tag{2.2}$$

where $r(t)$ is a deterministic function. It is assumed that $\mu_i(t)$, $\sigma_{ij}(t)$ and $r(t)$ are all uniformly bounded in $0 \le t < \infty$, for all i, j. We also assume that the matrix function $\boldsymbol{\sigma}(t) = (\sigma(t))_{ij}$ is invertible for all $0 \le t < \infty$, and we will write this inverse as $\boldsymbol{\sigma}^{-1}(t)$.

Let $\{\boldsymbol{f}_t, 0 \le t \le T\}$ denote a vector control process where $\boldsymbol{f}_t = (f_t^{(1)}, \dots, f_t^{(n)})'$. The interpretation of $f_t^{(i)}$ is the *total amount of money invested in the ith stock at time t*. It is assumed that $\boldsymbol{f}_t$ is admissible, in that $\boldsymbol{f}_t$ is a non-anticipating, $\mathcal{F}_t$-adapted process that satisfies $\int_0^T \boldsymbol{f}_s' \boldsymbol{f}_s \, \mathrm{d}s < \infty$ P-a.s. and keeps the wealth process (to be introduced directly) non-negative throughout. (This admissibility is a bit stronger than that of [13], but is a standard one in the theory of finance, see e.g. [7].)

Let X_t^f denote the *wealth* of the investor at time t, under an investment policy $\{\boldsymbol{f}_t\}$, with $X_0 = x$. Since any amount not invested in the risky stock is held in the bond, this wealth process then evolves as

$$\begin{aligned} \mathrm{d}X_t^f &= \sum_{i=1}^n f_t^{(i)} \frac{\mathrm{d}S_i(t)}{S_i(t)} + \left(X_t^f - \sum_{i=1}^n f_t^{(i)} \right) \frac{\mathrm{d}B_t}{B_t} \\ &= \left[r(t) X_t + \sum_{i=1}^n f_t^{(i)} (\mu_i(t) - r(t)) \right] \mathrm{d}t + \sum_{i=1}^n \sum_{j=1}^n f_t^{(i)} \sigma_{ij}(t) \, \mathrm{d}W_t^{(j)}, \end{aligned} \tag{2.3}$$

upon substituting from (2.1) and (2.2). Introducing the (column) vectors $\boldsymbol{\mu}(t) = (\mu_1(t), \dots, \mu_n(t))'$, $\mathbf{1} = (1, \dots, 1)'$, and using the matrix function $\boldsymbol{\sigma}(t)$, we can rewrite (2.3) as

$$\mathrm{d}X_t^f = [r(t) X_t^f + \boldsymbol{f}_t'(\boldsymbol{\mu}(t) - r(t)\mathbf{1})] \, \mathrm{d}t + \boldsymbol{f}_t' \boldsymbol{\sigma}(t) \, \mathrm{d}\boldsymbol{W}_t. \tag{2.4}$$

For Markov control processes $\{\boldsymbol{f}_t, t \ge 0\}$ and functions $\Psi(t, x) \in \mathcal{C}^{1,2}$, we may therefore write the generator of the (one-dimensional) wealth process as

$$\mathcal{A}^f \Psi(t, x) = \Psi_t + (r(t)x + \boldsymbol{f}_t'(\boldsymbol{\mu}(t) - r(t)\mathbf{1}))\Psi_x + \tfrac{1}{2} \boldsymbol{f}_t' \boldsymbol{\Sigma}(t) \boldsymbol{f}_t \Psi_{xx}, \tag{2.5}$$

where we have set $\boldsymbol{\Sigma}(t) = \boldsymbol{\sigma}(t)\boldsymbol{\sigma}(t)'$. The inverse of this matrix, $\boldsymbol{\Sigma}^{-1}(t)$, is assumed to exist since we assumed a complete market with $\boldsymbol{\sigma}(t)$ non-singular.

The fundamental vector $\boldsymbol{\theta}(t)$, defined by

$$\boldsymbol{\theta}(t) := \boldsymbol{\sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1}) \tag{2.6}$$

plays a pivotal role here since then according to the Girsanov theorem (cf. [19]) the vector process

$$\tilde{\boldsymbol{W}}_t = \boldsymbol{W}_t + \int_0^t \boldsymbol{\theta}(s) \, \mathrm{d}s \tag{2.7}$$

is an n dimensional standard Brownian motion under the measure $\tilde{P}$, where $\tilde{P}$ is the measure defined by

$$\begin{aligned} \frac{\mathrm{d}P(\omega)}{\mathrm{d}\tilde{P}(\omega)} &= \exp\left\{ \int_0^T \boldsymbol{\theta}(s)' \, \mathrm{d}\boldsymbol{W}_s + \frac{1}{2} \int_0^T \boldsymbol{\theta}(s)' \boldsymbol{\theta}(s) \, \mathrm{d}s \right\} \\ &\equiv \exp\left\{ \int_0^T \boldsymbol{\theta}(s)' \, \mathrm{d}\tilde{\boldsymbol{W}}_s - \frac{1}{2} \int_0^T \boldsymbol{\theta}(s)' \boldsymbol{\theta}(s) \, \mathrm{d}s \right\}. \end{aligned} \tag{2.8}$$

The measure $\tilde{P}$ is sometimes referred to as the *risk-neutral* (or equivalent martingale) measure, and the vector $\boldsymbol{\theta}(t)$ is also called the vector of *risk premiums*, or the market price of risk. For the sequel we will assume that $\boldsymbol{\theta}(s) \geq \mathbf{0}$, (equivalently $\boldsymbol{\mu}(s) \geq r(s)\mathbf{1}$) for all s.

In the next section, we will give the optimal policy for the problem of maximizing the probability that terminal wealth exceeds a predetermined level at the predetermined time T. The proof of this theorem will be deferred until later, in Section 6.

3. Maximizing the probability of reaching a goal in finite time

In this section, we present the optimal value function and optimal investment policy for maximizing the *probability* that terminal wealth at time T exceeds a given threshold b, with $X_0 < b$.

Let $P_{(t,x)}(\cdot) := P(\cdot \mid X_t^f = x)$, and let $V^f(t, x; b) := P_{(t,x)}(X_T^f \geq b)$ for any admissible policy $\boldsymbol{f}$. In the following theorem, we give $V(t, x; b) := \sup_{f \in \mathcal{G}} V^f(t, x; b)$, the optimal value function, as well as $\boldsymbol{f}_t^* = \arg\sup_{f \in \mathcal{G}} V^f(t, x; b)$, the associated optimal control policy; where $\mathcal{G}$ denotes the set of admissible controls for an investor whose wealth process evolves according to (2.3).

Remark on notation. For the sequel, $\phi(\cdot)$ denotes the density function (p.d.f.) of a standard normal variate, and $\Phi(\cdot)$ denotes the associated cumulative distribution function (c.d.f.).

Theorem 3.1. *For $\{X_t^f, 0 \leq t \leq T\}$ given by (2.3), let $V(t, x; b) = \sup_{f \in \mathcal{G}} P_{(t,x)}(X_T^f \geq b)$, with optimal control vector $\boldsymbol{f}_t^*$. Then*

$$V(t, x; b) = \Phi\left(\Phi^{-1}\left(\frac{x}{b\,R(t, T)}\right) + \sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\right) \tag{3.1}$$

and the optimal policy, for $t < T$, is

$$\boldsymbol{f}_t^*(x; b) = \left[\frac{\boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\right] bR(t, T)\ \phi\left(\Phi^{-1}\left(\frac{x}{b\,R(t, T)}\right)\right), \tag{3.2}$$

where $R(t, T) = \exp\{-\int_t^T r(s)\,\mathrm{d}s\}$.

Remark 3.1. Since $\Phi^{-1}(u) = \infty$ for $u \geq 1$, (3.1) shows that for $x > bR(t, T)$, we would have $V(t, x; b) = 1$, and correspondingly, (3.2) shows that we would in that case have $\boldsymbol{f}_t^* = \mathbf{0}$, where $\mathbf{0} = (0, 0, \ldots, 0)'$, whereby all the investor's wealth will be invested in the riskless asset.

This is due, of course, to the fact that when there is a risk-free asset available for investment that has a positive return of $r(t)$ per unit time, the problem of reaching the goal b at time T from the state (t, x) (i.e., from wealth $X_t = x$) is non-trivial and interesting *only for the case where $x < bR(t, T)$*, for if the converse holds, i.e., $x \geq bR(t, T)$, then the simple strategy of placing all the current (time t) wealth into the bond will yield a terminal wealth of $x/R(t, T)$, which would in turn beat the goal b with probability 1. Thus ensuring that terminal wealth at time T exceeds the fixed level b is equivalent to ensuring that for some $t < T$, wealth exceeds the time-dependent level $bR(t, T)$, i.e.,

$$P\Big(\sup_{0 \leq s \leq T} X_s^f \geq b\Big) \equiv P\big(X_t^f \geq bR(t, T),\ \text{for some } t \leq T\big).$$

If in Theorem 3.1 we take $n = 1$, $b = 1$, as well as $r(s) = 0$, and $\sigma(s) = 1$ for all s, then we recover the results of [13, Theorem 7].

Remark 3.2. Constant coefficients. For the case of constant coefficients, i.e., when $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}$, $\boldsymbol{\mu}(t) = \boldsymbol{\mu}$, and $r(t) = r$, for all t, the results of Theorem 3.1 reduce to

$$V(t,x;b) = \Phi\left(\Phi^{-1}\left(\frac{x}{b}\,\mathrm{e}^{r(T-t)}\right) + \sqrt{\boldsymbol{\theta}'\boldsymbol{\theta}(T-t)}\right) \tag{3.3}$$

and the optimal policy, for $t < T$, is

$$\boldsymbol{f}_t^*(x;b) = \left[\frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{\sqrt{\boldsymbol{\theta}'\boldsymbol{\theta}(T-t)}}\right] b\,\mathrm{e}^{-r(T-t)}\,\phi\left(\Phi^{-1}\left(\frac{x}{b}\,\mathrm{e}^{r(T-t)}\right)\right) \tag{3.4}$$

where $\boldsymbol{\theta} = \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})$ is a constant vector. Note that for this case, when $n = 1$ (and so there is only one risky stock), then $\theta = (\mu - r)/\sigma$, and the optimal control f_t^* reduces, for $t < T$, to

$$f_t^*(x;b) = \left[\frac{1}{\sigma\sqrt{T-t}}\right] b\,\mathrm{e}^{-r(T-t)}\phi\left(\Phi^{-1}\left(\frac{x}{b}\,\mathrm{e}^{r(T-t)}\right)\right). \tag{3.5}$$

It is important to note that it is only for this single-stock case that the optimal control is in fact independent of the underlying drift parameter μ, as is apparent from (3.5). In this case, the investor always invests less in the risky stock when there is a risk-free asset with $r > 0$ than he would in the corresponding case with $r = 0$, treated earlier in [13] and [10]. This follows directly from the fact that for any $a > 0$, $\mathrm{e}^{-a}\phi(z\,\mathrm{e}^{a}) < \phi(z)$, for any z. (In the general case, this may or may not be true, depending on the relationship between $r(s)$, $\boldsymbol{\mu}(s)$ and $\boldsymbol{\sigma}(s)$.) As we will show in the next section, the policy of (3.5) is intimately connected to the hedging strategy for a particular type of derivative security known as a *digital option.*

3.1. The optimal wealth process

When the control function $\boldsymbol{f}_t^*$ is placed back into the evolutionary wealth equation (2.4), we obtain an optimal wealth process, $\{X_t^*, 0 \le t < T\}$, that satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t^* = {} & \left[r(t)X_t^* + \frac{\boldsymbol{\theta}(t)'\boldsymbol{\theta}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\, bR(t,T)\,\phi\big(\Phi^{-1}\big(X_t^*[bR(t,T)]^{-1}\big)\big)\right]\mathrm{d}t \\ & + \frac{bR(t,T)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\,\phi\big(\Phi^{-1}\big(X_t^*[bR(t,T)]^{-1}\big)\big)\,\boldsymbol{\theta}(t)'\,\mathrm{d}\boldsymbol{W}_t, \quad \text{for } t < T. \end{aligned} \tag{3.6}$$

We will show later that the solution to this stochastic differential equation is given by the following corollary.

Corollary 3.2. *The optimal wealth process, X_t^*, for $0 \le t < T$, is given by*

$$\begin{aligned} X_t^* = {} & bR(t,T) \\ & \times \Phi\Bigg(\Bigg[\int_0^t \boldsymbol{\theta}(s)'\,\mathrm{d}\boldsymbol{W}_s + \int_0^t \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s + \sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\;\Phi^{-1}\big(X_0/(bR(t,T))\big)\Bigg] \\ & \times \Bigg(\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\Bigg)^{-1}\Bigg). \end{aligned} \tag{3.7}$$

This representation provides the link between the probability maximizing objective and the digital, or *binary* option, and contains the explanation as to why it is only in the single stock constant coefficient case that the policy is independent of the underlying drift. We discuss this directly in the next section. After that, we return to analyse the optimal policy in explicit detail. The proof of Theorem 3.1 will then be provided in the following section. After that we examine cases that include income and the problem of beating an index.

4. Connections with digital options

4.1. Constant coefficients

Consider a Black–Scholes [1] world with a single stock whose price, S_t, follows the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t \tag{4.1}$$

as well as a risk-free asset with constant return r.

A *digital (or binary) option* on this stock, with strike price K and payoff B, is a contract that pays \$$B$ at time T if $S_T \geq K$. Thus a digital option amounts to a straight bet on the terminal price of the underlying stock.

Let $C(t, S_t)$ denote the current rational price of such an option. Then, a standard Black–Scholes pricing argument shows that

$$C(t, S_t) = B\,\mathrm{e}^{-r(T-t)}\Phi\left(\frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right). \tag{4.2}$$

The underwriter of such an option (i.e., the party that agrees to pay \$$B$ at T if $S_T > K$) is interested in *hedging* its risk. A dynamic *hedging strategy* for the writer of such an option is a dynamic investment policy, say $\{\Delta_t, t \leq T\}$, which holds Δ_t shares of the underlying stock at time t so as to ensure that the underwriter's position is riskless at all times. It is also well known that this hedging, or replicating, strategy is given by $\Delta_t = C_2(t, S_t)$, where $C_2(t, x) = \partial C/\partial x$. It is easy to see that the hedging strategy for the digital option is simply

$$\Delta_t = B\,\mathrm{e}^{-r(T-t)}\phi\left(\frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\frac{1}{S_t\sigma\sqrt{T-t}}. \tag{4.3}$$

Observe that since Δ_t is the *number of shares* of the underlying stock the investor holds at time t, the actual amount of money invested in the stock at time t is $\Delta_t \cdot S_t$.

A general treatment of options that discusses pricing and hedging of various options, including the digital and the derivation of (4.2) and (4.3) can be found in such basic texts as [12] and [21]. A valuable source for more theoretical issues is [7].

To see the connection with our problem, consider an investor who at time t has sold this digital option for the Black–Scholes price of $C(t, S_t)$, and suppose the investor will then invest the proceeds in such a manner as to *maximize the probability that he can pay off the claim of this option at time* T, i.e., for all intents and purposes, the investor's 'wealth' at time t is $C(t, S_t)$, and the investor will then invest this wealth so as to *maximize the probability that the terminal fortune from this strategy is equal to* B. Our previous results show that the optimal policy is at time t is given by f_t^* of (3.5) with $x = C(t, S_t)$ and $b = B$, i.e., by $f_t^*(C(t, S_t); B)$.

But placing $C(t, S_t)$ of (4.2) into (3.5) with $b = B$ and simplifying gives

$$f_t^*(C(t, S_t); B) = B\,\mathrm{e}^{-r(T-t)}\phi\left(\Phi^{-1}\left(\frac{C(t, S_t)}{B}\,\mathrm{e}^{r(T-t)}\right)\right)\frac{1}{\sigma\sqrt{T-t}}$$
$$= B\,\mathrm{e}^{-r(T-t)}\phi\left(\frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\frac{1}{\sigma\sqrt{T-t}}$$
$$\equiv \Delta_t \cdot S_t \tag{4.4}$$

where Δ_t is given by (4.3). Thus, in this case, f_t^* is equivalent to the hedging strategy of the digital option.

Moreover, if we specialize the representation of the optimal wealth process given in Corollary 3.2, i.e. X_t^* of (3.7), to the single stock case with constant coefficients, we find that

$$X_t^* = B\,\mathrm{e}^{-r(T-t)}\Phi\left(\frac{\sigma W_t + (\mu - r)t + \sigma\sqrt{T}\Phi^{-1}(X_0\,\mathrm{e}^{rT}/B)}{\sigma\sqrt{T-t}}\right) \tag{4.5}$$

is the wealth of an investor at time t who started off with initial wealth X_0 and is investing so as to maximize the probability of his terminal time T wealth being equal to B.

Observe now that by (4.1) we have

$$S_t = S_0 \cdot \exp\{(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\}$$

from which we may infer that

$$\sigma W_t + (\mu - r)t \equiv \ln(S_t/S_0) - (r - \tfrac{1}{2}\sigma^2)t.$$

When this is placed back into (4.5), it yields the following representation of the optimal wealth process in terms of the underlying stock price

$$X_t^* = B\,\mathrm{e}^{-r(T-t)}\Phi\left(\frac{\ln(S_t/S_0) - (r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{T}\Phi^{-1}\left(X_0\,\mathrm{e}^{rT}/B\right)}{\sigma\sqrt{T-t}}\right). \tag{4.6}$$

Note that there is no explicit dependence on μ in (4.6). If the investor's initial wealth is taken to be the Black–Scholes price of the digital option at time 0, i.e., take $X_0 \equiv C(0, S_0)$, where $C(\cdot, \cdot)$ is given by (4.2), then it is seen that (4.6) reduces to

$$X_t^* \equiv C(t, S_t)$$

where $C(t, S_t)$ is given by (4.2), i.e., the optimal wealth process under policy f_t^* is just the Black–Scholes price for a digital option with payoff B!

It is interesting to note that the Black–Scholes value (4.2) and its resulting hedging strategy (4.3) are both calculated and determined by the risk-neutral probability measure (under which μ is replaced by r), while the optimal strategy for maximizing the probability of terminal wealth being greater than B was determined under the regular measure.

The analysis above can be inverted to show the following rather interesting fact:

Proposition 4.1. *Consider an investor, with initial wealth X_0, whose objective is to maximize the probability that terminal wealth at time T exceed some fixed level B, and who has the following two investment opportunities—the risky stock S_t of (4.1) and a bond with constant return r. Then investing according to the* dynamically *optimal policy* $\{f_t^*, 0 \le t < T\}$ *of (3.5) is equivalent to the* static *policy that purchases (at time 0) one European call digital option with payoff B and strike price K^*, where K^* is given by*

$$K^* = S_0 \cdot \exp\{(r - \tfrac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(X_0 \mathrm{e}^{rT}/B)\}. \tag{4.7}$$

Proof. The representation of the optimal wealth process under f_t^*, i.e., X_t^* of (4.6), shows that, for a fixed X_0, the event $\{X_T^* \ge B\}$ is equivalent to the event $\{S_T \ge K^*\}$.

Note that (4.7) is equivalent to

$$X_0 = B\,\mathrm{e}^{-rT}\,\Phi\left(\frac{\ln(S_0/K^*) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right),$$

which when compared with (4.2) shows that X_0 is simply the Black–Scholes price at time $t = 0$ of a digital option with payoff B and strike price K^*.

Remark 4.1. It is important to note that the results of this section are specific to the constant coefficient case in one dimension and do not generalize to the time-dependent case in even one dimension, nor to the multi-dimensional constant-coefficient case. The reason for this is that it is only for the constant coefficients one-dimensional case that the representation of X_t^* of (3.7) reduces to the form in (4.6), whereby we see that X_t^* is determined completely by the current value of the underlying stock. As such, we can represent the event $\{X_T^* \ge B\}$ in terms of a corresponding simple event determined by the terminal value S_T alone, which leads to the equivalence with the digital option, whose price must be independent of μ by risk neutral valuation arguments. For more general cases, this does not occur. However, as we will show directly, the general case is intimately connected to a digital option on a particular portfolio, namely the *growth-optimal portfolio*.

4.2. The general case: optimal growth and digital options

For the general model considered earlier, let $\boldsymbol{\pi}_t^*$ denote the vector

$$\boldsymbol{\pi}_t^* := \boldsymbol{\Sigma}^{-1}(t)\,[\boldsymbol{\mu}(t) - r(t)\mathbf{1}] \equiv \boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t). \tag{4.8}$$

The corresponding specific portfolio policy $\{\boldsymbol{f}_t^{\pi^*}\}$, where $\boldsymbol{f}_t^{\pi^*}(x) = \boldsymbol{\pi}_t^* \cdot x$, is the policy that maximizes logarithmic utility of wealth and also maximizes the growth rate of wealth, see the discussion in [16, Chapter 6]. For the case of constant coefficients, it is also optimal for minimizing the expected time to reach any specific level of wealth (in an infinite horizon setting) (cf. [4], [11], [16]). As such, $\{\boldsymbol{\pi}_t^*\}$ is referred to as the *optimal growth policy*. It is also sometimes referred to as the *market portfolio*.

Let Π_t denote the corresponding 'wealth' under policy $\boldsymbol{\pi}_t^*$, with initial wealth Π_0. Then taking $\boldsymbol{f}_t = \boldsymbol{\pi}_t \cdot X_t^f$ and $X_t^f = \Pi_t$ in (2.4) shows that Π_t satisfies the linear stochastic differential equation

$$\mathrm{d}\Pi_t = \Pi_t[r(t) + \boldsymbol{\theta}(t)'\boldsymbol{\theta}(t)]\,\mathrm{d}t + \Pi_t\boldsymbol{\theta}(t)'\,\mathrm{d}\boldsymbol{W}_t$$

and as such we have

$$\Pi_t = \Pi_0 \exp\left\{ \int_0^t [r(s) + \tfrac{1}{2}\boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)]\,\mathrm{d}s + \int_0^t \boldsymbol{\theta}(s)'\,\mathrm{d}\boldsymbol{W}_s \right\}. \tag{4.9}$$

We can also represent the wealth process under the optimal growth policy in terms of the change of measure of (2.8) by

$$\Pi_t = \Pi_0 \frac{1}{R(t,T)} E\left(\frac{\mathrm{d}P}{\mathrm{d}\tilde{P}} \middle| \mathcal{F}_t \right).$$

The main result of this section is that the optimal policy of Theorem 3.2 is equivalent to the hedging strategy for a digital option on Π_T, and so the optimal wealth process of Corollary 3.2 is therefore equivalent to the Black–Scholes price on this option. Before we state this formally, recognize that for every $t \geq 0$, the optimal growth portfolio, Π_t, is equivalent in distribution to the process $\hat{\Pi}_t$, where

$$\mathrm{d}\hat{\Pi}_t = \hat{\Pi}_t[r(t) + \sigma_*^2(t)]\,\mathrm{d}t + \hat{\Pi}_t\sqrt{\sigma_*^2(t)}\,\mathrm{d}\hat{W}_t \tag{4.10}$$

where $\hat{W}_t$ is an independent Brownian motion (in $\mathbf{R}^1$), and $\sigma_*^2(t)$ is defined by

$$\sigma_*^2(t) := \boldsymbol{\theta}(t)'\boldsymbol{\theta}(t) \equiv \boldsymbol{\pi}_t^{*\prime}\boldsymbol{\Sigma}(t)\boldsymbol{\pi}_t^*. \tag{4.11}$$

As such, $\sigma_*^2(t)$ is the volatility of the optimal growth portfolio.

Remark 4.2. Note that in terms of the optimal growth policy, we may write the optimal policy, $\boldsymbol{f}_t^*$ of (3.2) in Theorem 3.1 as

$$\boldsymbol{f}_t^*(x;b) = \boldsymbol{\pi}_t^* \frac{1}{\sqrt{\int_t^T \sigma_*^2(s)\,\mathrm{d}s}}\, b\,R(t,T)\,\phi\big(\Phi^{-1}(x/[b\,R(t,T)])\big). \tag{4.12}$$

Proposition 4.2. *Consider an investor whose wealth, $\{X_t^f\}$, satisfies (2.4), and whose objective is to maximize $P(X_T^f \geq b)$. Then the optimal policy for this objective, given by $\boldsymbol{f}_t^*$ of (3.2) in Theorem 3.1, is completely equivalent to the* static *policy which purchases a European digital option on Π_T with payoff b and strike price K^{**}, where*

$$K^{**} = \Pi_0 \exp\left\{ \int_0^T \left[r(s) - \frac{\sigma_*^2(s)}{2} \right] \mathrm{d}s - \sqrt{\int_0^T \sigma_*^2(s)\,\mathrm{d}s}\;\Phi^{-1}\big(X_0/[b\,R(t,T)]\big) \right\}. \tag{4.13}$$

Equivalently, under the optimal policy $\boldsymbol{f}_t^*$, the optimal wealth $\{X_t^*\}$ is equivalent to the (no arbitrage) Black–Scholes price of this digital option on Π_T, i.e.,

$$X_t^* = bR(t,T)\,\Phi\left(\frac{\ln(\Pi_t/K^{**}) + \int_t^T [r(s) - \frac{1}{2}\sigma_*^2(s)]\,\mathrm{d}s}{\sqrt{\int_t^T \sigma_*^2(s)\,\mathrm{d}s}} \right), \quad \textit{for all } t < T. \tag{4.14}$$

Remark 4.3. Observe from (4.14) that terminal wealth, X_T^* is either 0 or b, with

$$\{X_T^* = b\} \Leftrightarrow \{\Pi_T \geq K^{**}\}.$$

Proof. We can use (4.9) to write the representation of the optimal wealth X_t^* of (3.7) in terms of the optimal growth portfolio, Π_t, (using $\sigma_*^2(t)$) as

$$X_t^* = bR(t,T)\,\Phi\Bigg(\Bigg[\ln(\Pi_t/\Pi_0) - \int_0^t [r(s) - \frac{1}{2}\sigma_*^2(s)]\,\mathrm{d}s + \sqrt{\int_0^T \sigma_*^2(s)\,\mathrm{d}s}\;\Phi^{-1}\big(X_0/[b\,R(t,T)]\big)\Bigg]\Bigg(\sqrt{\int_t^T \sigma_*^2(s)\,\mathrm{d}s}\Bigg)^{-1}\Bigg).$$

When we substitute K^{**} of (4.13) back into this, we obtain (4.14).

It remains to show that (4.14) is indeed the Black–Scholes price of a digital option on Π_T. To see this, recall that if $\psi(\Pi_T)$ is the payoff on any contingent claim written on Π_T, then the (no arbitrage) Black–Scholes price of this claim at time t is given by $C(t, \Pi_t)$, where

$$C(t,\Pi_t) = \tilde{E}\big[R(t,T)\,\psi(\Pi_T)|\mathcal{F}_t\big], \tag{4.15}$$

where $\tilde{E}$ denotes the expectation taken under the risk-neutral measure $\tilde{P}$ of (2.8).

Observe now that in terms of the Brownian motion $\{\tilde{\boldsymbol{W}}_t\}$ of (2.7), we may write Π_t of (4.9) as

$$\Pi_t = \Pi_0 \exp\left\{\int_0^t [r(s) - \tfrac{1}{2}\boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)]\,\mathrm{d}s + \int_0^t \boldsymbol{\theta}(s)'\,\mathrm{d}\tilde{\boldsymbol{W}}_s\right\}, \tag{4.16}$$

and so if, for an arbitrary strike price K, we take $\psi(\Pi_T) = b \cdot 1\{\Pi_T \geq K\}$, then a simple computation shows that for this case we have

$$C(t,\Pi_t) = bR(t,T)\,\Phi\left(\frac{\ln(\Pi_t/K) + \int_t^T [r(s) - \frac{1}{2}\sigma_*^2(s)]\,\mathrm{d}s}{\sqrt{\int_t^T \sigma_*^2(s)\,\mathrm{d}s}}\right), \tag{4.17}$$

from which it is apparent that (4.14) is indeed the (time t) Black–Scholes price of a digital option on Π_T, with payoff b and strike price K^{**}.

We now move on to analyse the optimal investment policy, $\boldsymbol{f}_t^*$ of Theorem 3.1 for the general case in explicit detail.

5. Analysis of the optimal policy in the general case

In this section we examine the investment policy $\boldsymbol{f}_t^*$ of (3.2) for the general case in greater detail. Note that for a fixed $t < T$, as wealth, x, gets closer to the 'goal' $bR(t,T)$, we have $\boldsymbol{f}_t^* \to \boldsymbol{0}$. Similarly, as the wealth gets closer to the barrier 0, we also have $\boldsymbol{f}_t^* \to \boldsymbol{0}$. Thus if wealth is close to 0 with enough time remaining on the clock, the investor does not 'panic' and start taking aggressive positions in the stocks to get away from 0, but rather the investor should be patient and wait for his wealth to grow a bit before taking active positions in the risky stocks. (It is interesting to note that if there is enough time remaining until the deadline, the investor does not even borrow an excessive amount when his wealth is close to 0, even though he does have the ability to borrow an unlimited amount. See the discussion below on 'the borrowing region'.) However, should the investor get near bankruptcy, i.e., $x \downarrow 0$, with little time remaining, i.e., $t \uparrow T$, then the investor must take a fairly active position in the stocks since then the denominator is also going to 0.

Let $z(x,t;T,b)$ be defined by

$$z(x,t;T,b) := \frac{x}{bR(t,T)}, \quad \text{for } x \leq b\,R(t,T) \tag{5.1}$$

whereby $0 \leq z \leq 1$, and the interpretation of z is the *percentage of the goal reached by time t*. (Note that this is not the percentage of b reached, but rather the *time-dependent* goal $bR(t,T)$.) Then we may write the optimal policy as

$$f_t^*(x;b) = \left[\frac{\boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\right]\left[\frac{\phi(\Phi^{-1}(z))}{z}\right]x.$$

Introducing now the variable

$$\nu := \Phi^{-1}(z)$$

allows us to rewrite this in terms of ν as

$$f_t^*(x;b) = \left[\frac{\boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\right]\left[\frac{\phi(\nu)}{\Phi(\nu)}\right]x. \tag{5.2}$$

Note that ν is a monotonically increasing function of z, and that as z goes from 0 to 1, ν goes from $-\infty$ to ∞. As wealth, x, gets closer to the (effective) goal, $bR(t,T)$, $z \uparrow 1$ and hence $\nu \uparrow +\infty$, while as wealth gets closer to the bankruptcy barrier, 0, we have $z \downarrow 0$ and accordingly $\nu \downarrow -\infty$. Notice too that the function $\phi(\nu)/\Phi(\nu)$ is positive and decreasing in ν.

Equation (5.2) represents the optimal investment strategy as a linear function of wealth (x) where the linear multiple decomposes into a product of two distinct components: the first component is the vector of purely time-dependent elements, $\boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t)\big/\sqrt{\left(\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s\right)}$; and the second is a scalar determined solely by the percentage of the (effective) goal currently achieved, $\phi(\nu)/\Phi(\nu)$. It is easy to see that for a fixed t, as wealth gets closer to the effective goal, and so z increases as does ν, this latter quantity *decreases*. The effect of increasing t on the former quantity is not as clear to check since we have allowed all parameters to be time dependent. For the special case of *constant coefficients*, this time-dependent vector is simply $\boldsymbol{\sigma}^{-1\prime}\boldsymbol{\theta}/\sqrt{\boldsymbol{\theta}'\boldsymbol{\theta}(T-t)}$, which is clearly increasing as $t \uparrow T$. We will examine the tradeoff between these two effects directly.

5.1. The borrowing region

It is interesting to examine the nature of risk-taking dictated by the policy described above. One manifestation of risk-taking is in the amount of borrowing required by a policy. Here, we examine this dimension of the behavior required by $\{f_t^*\}$ of Theorem 3.1.

Notice first that borrowing takes place only in the region $\Gamma(x,t) := \{x : \mathbf{1}'f_t^*(x;b) \geq x\}$, since $\mathbf{1}'f_t^*(x;b)$ is the total amount of money invested in the risky stocks. This region can be characterized more explicitly by using (5.2). In particular, (5.2) shows that the borrowing region, $\Gamma(x,t)$ is equivalent to the region

$$\Gamma := \left\{\nu : \frac{\phi(\nu)}{\Phi(\nu)} \geq q(t,T)\right\},$$

where the (scalar) function q is defined by $q(t,T) := \sqrt{\left(\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s\right)}\big/\mathbf{1}'\boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t)$, and where the variable $\nu = \nu(x,t)$ is defined by $\nu := \Phi^{-1}(z)$.

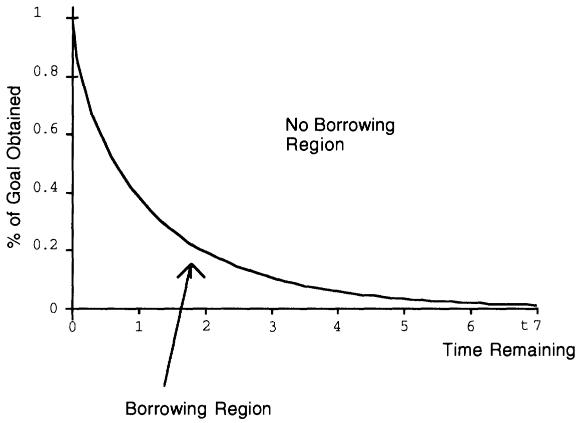

FIGURE 1: $z^*(\tau)$ plotted against τ.

For the case of *constant coefficients*, $q(t, T)$ reduces to $q(T-t) := \xi \cdot \sqrt{(T-t)}$ where the constant ξ is defined by

$$\xi := \frac{\sqrt{(\boldsymbol{\mu} - r\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})} \equiv \frac{\sqrt{\boldsymbol{\pi}^{*\prime}\boldsymbol{\Sigma}\boldsymbol{\pi}^*}}{\mathbf{1}'\boldsymbol{\pi}^*},$$

which in the single-stock case reduces to $\xi = \sigma$.

If we set $\sqrt{\tau} := q(T-t) \equiv \xi\sqrt{T-t}$, then we can consider τ to be the *risk adjusted time to play*, since it is in fact just the time to play $(T-t)$ multiplied by a risk factor (ξ^2). The borrowing region is then equivalent to

$$\Gamma(\nu, \tau) := \{\nu : \phi(\nu)/\Phi(\nu) \geq \sqrt{\tau}\}.$$

In order to analyse the boundary of this region, let $\nu^*(\tau)$ denote the root to the equation $\phi(\nu)/\Phi(\nu) - \sqrt{\tau} = 0$, i.e., $\nu^*(\tau)$ is the unique number such that

$$\phi(\nu^*)/\Phi(\nu^*) = \sqrt{\tau}.$$

(Since the left hand side is decreasing in ν while the right hand side is increasing in τ, it is clear that there is a unique root.) Furthermore, it is easy to establish that these roots are decreasing in the remaining time, τ, i.e., $\nu^*(\tau) > \nu^*(\tau + \delta)$, for all $\delta > 0$.

The borrowing region is thus as follows: borrowing occurs when $\nu(\tau) < \nu^*(\tau)$, but not if $\nu(\tau) > \nu^*(\tau)$. Equivalently, since $\Phi(\cdot)$ is an increasing invertible function, we see that borrowing occurs only when $\Phi(\nu(\tau)) < \Phi(\nu^*(\tau))$, i.e., when $z(\tau) < z^*(\tau)$, where $z(\tau)$ has the interpretation of being the proportion of the goal obtained with τ (risk-adjusted) time units left to go. A graph of $z^*(\tau)$ (calculated and drawn in MAPLE V) is given in Figure 1.

Some select values of τ, $\nu^*(\tau)$ and $z^*(\tau)$ are given in Table 1. As Table 1 shows, if there is 0.05 time units left until the deadline, the investor must borrow under policy $\boldsymbol{f}^*$ unless the investor's wealth is already 88% of the distance to the goal. As the time to go increases, the investor needs to borrow only at lower percentages. For example, if there is $\tau = 1$ unit of time left to go, then the investor will need to borrow unless his wealth at that time is at least 38% of the way to the effective goal.

TABLE 1: Borrowing region. $z^*(\tau)$ = critical percentage of the distance to the goal attained with τ risk-adjusted time units remaining. $\nu^*(\tau)$ is the root to the equation $\phi(\nu)/\Phi(\nu) - \tau^{1/2} = 0$, and $z^* = \Phi^{-1}(\nu^*)$.

τ	$\nu^*(\tau)$	$z^*(\tau)$	τ	$\nu^*(\tau)$	$z^*(\tau)$	τ	$\nu^*(\tau)$	$z^*(\tau)$
0.001	2.26	0.99	0.50	0.15	0.56	1.00	−0.30	0.38
0.05	1.19	0.88	0.55	0.09	0.54	1.5	−0.61	0.27
0.10	0.93	0.82	0.60	0.04	0.51	2.0	−0.86	0.19
0.15	0.75	0.77	0.65	−0.01	0.49	2.5	−1.07	0.14
0.20	0.63	0.73	0.70	−0.06	0.48	3.0	−1.25	0.10
0.25	0.52	0.70	0.75	−0.11	0.46	3.5	−1.42	0.08
0.30	0.43	0.67	0.80	−0.15	0.44	4.0	−1.57	0.06
0.35	0.35	0.64	0.85	−0.19	0.43	4.5	−1.71	0.04
0.40	0.27	0.61	0.90	−0.23	0.41	5.0	−1.84	0.03
0.45	0.21	0.58	0.95	−0.27	0.39			

It is important to note that increasing the risk factor, ξ^2, has the same effect as increasing the actual time left to play, $T - t$. Therefore, for a higher risk factor, one would borrow less, in the hopes of reaching the goal later.

Remark 5.1. Asymptotics, near the barriers. When wealth, x, is close to the barriers, 0 or $bR(t, T)$, then z is correspondingly near 0 or 1, and ν is correspondingly near $-\infty$ or $+\infty$. It is interesting to examine what happens to the state-dependent factor, $\phi(\nu)/\Phi(\nu)$, near these boundaries. Note that since ϕ is symmetric, the behavior of $\phi(\nu)/\Phi(\nu)$ as $\nu \downarrow -\infty$ is equivalent to the behavior of $\phi(\nu)/(1 - \Phi(\nu))$ as $\nu \uparrow +\infty$.

Therefore the following two asymptotic results are immediate:

$$\lim_{z\to 1} \frac{\phi(\nu)}{\Phi(\nu)} = \lim_{\nu\uparrow+\infty} \frac{\phi(\nu)}{\Phi(\nu)} = 0$$
$$\lim_{z\to 0} \frac{\phi(\nu)}{\Phi(\nu)} = \lim_{\nu\downarrow-\infty} \frac{\phi(\nu)}{\Phi(\nu)} = \lim_{\nu\uparrow+\infty} \frac{\phi(\nu)}{1 - \Phi(\nu)} = +\infty.$$

As expected, we see that as wealth approaches the goal, the state dependent factor goes to 0, but as wealth approaches 0, the state dependent factor increases without bound even though, as we observed previously, total investment in the risky stocks actually decreases to 0 in this case.

5.2. Comparison with utility maximizing policies

It is interesting to compare this behavior with that of an investor whose objective is to maximize terminal utility from wealth. For example, consider the case where the investor wants to maximize $E[u(X_T^f)]$, where

$$u(x) = \frac{\delta}{\delta - 1} x^{1-1/d}, \quad \text{for } x > 0,\ \delta > 0.$$

(Note that this includes logarithmic utility, when $\delta = 1$.) This power utility function has constant relative risk aversion $1/\delta$. The optimal policy for this case, call it $\{f_t^\delta, 0 \le t \le T\}$, is the vector (cf. [7], [15]) $f_t^\delta(x) = \delta \boldsymbol{\pi}_t^* \cdot x$, where $\boldsymbol{\pi}_t^*$ is the optimal growth policy discussed earlier. The utility maximizing investor invests more heavily in the risky stocks, relative to the probability maximizing investor, when $f_t^{\delta}(x) > f_t^*(x; b)$, and vice versa. It is easily seen that this occurs for values (x, t) for which

$$\frac{\phi(\nu)}{\Phi(\nu)} \leq \delta \sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}$$

and vice versa. Thus the dynamics of this comparison reduce essentially to that described above by the borrowing region, modified by the risk aversion parameter δ.

6. Proof of Theorem 3.1

In this section, we provide the proof of Theorem 3.1. We will first show that the function V satisfies the appropriate Hamilton–Jacobi–Bellman (HJB) equations of stochastic control theory and then employ a martingale argument to verify optimality. This will prove the Theorem as well as Corollary 3.2. We then show how we obtained the candidate value function by extending the elegant argument of [10] to our case.

6.1. Verification of optimality

Standard arguments in control theory (see e.g. [8, Example 2, p. 161]) show that the appropriate HJB optimality equation for V is

$$\sup_f \{\mathcal{A}^f V(t, x)\} = 0 \tag{6.1}$$

subject to the boundary conditions

$$V(t, x; b) = \begin{cases} 1 & \text{for } x \geq bR(t, T),\ t \leq T \\ 0 & \text{for } x = 0,\ t \leq T \\ I_{\{b\}} & \text{for } t = T. \end{cases} \tag{6.2}$$

The generator of (2.5) shows that the HJB optimality equation (6.1) is

$$\sup_f \{V_t + (f_t'(\boldsymbol{\mu}(t) - r(t)\mathbf{1}) + r(t)x)V_x + \tfrac{1}{2} f_t' \boldsymbol{\Sigma}(t) f_t V_{xx}\} = 0. \tag{6.3}$$

Assuming that a classical solution to (6.3), say V, exists and satisfies $V_x > 0$, $V_{xx} < 0$ for $0 < x < bR(t, T)$, we may then optimize with respect to f_t in (6.3) to obtain the maximizer

$$f_t^*(x; b) = -\boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1})\frac{V_x}{V_{xx}} \equiv -\boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t)\frac{V_x}{V_{xx}}. \tag{6.4}$$

When (6.4) is then placed back into (6.3) and the resulting equation simplified, we find that (6.3) is equivalent to the nonlinear partial differential equation

$$V_t + r(t)xV_x - \tfrac{1}{2}\boldsymbol{\theta}(t)'\boldsymbol{\theta}(t)\frac{V_x^2}{V_{xx}} = 0, \quad \text{for } t < T, \text{ and } 0 < x < b\,R(t, T) \tag{6.5}$$

subject to the (discontinuous) boundary condition (6.2).

Recalling now the basic facts about the normal p.d.f. and c.d.f.:

$$\frac{\mathrm{d}\Phi(y)}{\mathrm{d}y} = \phi(y); \quad \frac{\mathrm{d}\Phi^{-1}(y)}{\mathrm{d}y} = \frac{1}{\phi(\Phi^{-1}(y))}; \quad \frac{\mathrm{d}\phi(y)}{\mathrm{d}u} = -y\phi(y); \tag{6.6}$$

it can be verified that for the function $V(t, x; b)$ of (3.1), we have

$$V_x = \phi\left(\Phi^{-1}\left(\frac{x}{b\,R(t,T)}\right) + \sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\right)\frac{1}{b\,R(t,T)}\left[\phi\left(\Phi^{-1}\left(\frac{x}{b\,R(t,T)}\right)\right)\right]^{-1} \tag{6.7}$$

$$V_{xx} = -V_x\left[\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\,\frac{1}{b\,R(t,T)}\left[\phi\left(\Phi^{-1}\left(\frac{x}{b\,R(t,T)}\right)\right)\right]^{-1}\right] \tag{6.8}$$

$$V_t = -V_x\left[r(t)x + \frac{1}{2}\frac{\boldsymbol{\theta}(t)'\boldsymbol{\theta}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\,\phi\left(\Phi^{-1}\left(\frac{x}{b\,R(t,T)}\right)bR(t,T)\right)\right]. \tag{6.9}$$

When these derivatives are placed back into (6.5), it is seen that in fact V solves (6.5), and so the optimal control $\boldsymbol{f}_t^*(x)$ of (3.2) is obtained by placing (6.7) and (6.8) into (6.4).

Moreover, it is also seen that the first two boundary conditions in (6.2) are satisfied for V of (3.1). This follows from the fact that Φ is a c.d.f., and hence $\Phi^{-1}(u) = -\infty$ for $u \le 0$, $\Phi^{-1}(u) = \infty$ for $u \ge 1$. The third boundary condition of (6.2) causes a *discontinuity*. This problem (for 'probability maximizing objectives') is discussed in [8, Example 2, p. 161], where it is shown that such a discontinuity is acceptable, provided that the optimal wealth process, $\{X_t^*, t \le T\}$ (i.e., the wealth under the control $\boldsymbol{f}_t^*$), satisfies the condition $P(0 < X_T^* < b) = 0$. (That is, the optimal terminal wealth must be equal to one of the two barriers, 0 or b.) We will show directly that this condition is in fact met here.

To proceed, let $H_t := z(X_t^*, t; T, b)$, where the function z was defined earlier in (5.1), and where X_t^* denotes the optimal wealth process; i.e., $H_t = X_t^*/[bR(t,T)]$, and X_t^* is determined by the stochastic differential equation in (3.6). An application of Itô's formula, valid for $t < T$, to (3.6) then shows that

$$\mathrm{d}H_t = \frac{\boldsymbol{\theta}(t)'\boldsymbol{\theta}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\phi(\Phi^{-1}(H_t))\,\mathrm{d}t + \frac{1}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\phi(\Phi^{-1}(H_t))\boldsymbol{\theta}(t)'\,\mathrm{d}\boldsymbol{W}_t. \tag{6.10}$$

Another application of Itô's formula (with respect to the process $\int_0^t \boldsymbol{\theta}(s)'\,\mathrm{d}\boldsymbol{W}_s$) will verify that the solution to (6.10) is, for $t < T$

$$H_t = \Phi\left(\frac{\int_0^t \boldsymbol{\theta}(s)'\,\mathrm{d}\boldsymbol{W}_s + \int_0^t \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s + \sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\,\Phi^{-1}(H_0)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\right), \tag{6.11}$$

from which the representation of X_t^* given in (3.7) is immediate, proving Corollary 3.2.

However, all this is valid only for $t < T$. By continuity we may extend H_t of (6.11) to include the values 0 and 1, but it still remains to show that H_T is either 0 or 1. To proceed, we follow essentially the steps suggested in [10].

Observe first that H_t is bounded with $0 \le H_t \le 1$, for all $0 \le t \le T$. Also note that (6.10) can be rewritten using $\tilde{\boldsymbol{W}}_t$ as

$$\mathrm{d}H_t = \frac{\phi(\Phi^{-1}(H_t))}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\boldsymbol{\theta}(t)'\,\mathrm{d}\tilde{\boldsymbol{W}}_t \equiv \left[bR(t,T)\right]^{-1}\boldsymbol{f}_t^{*\prime}\boldsymbol{\sigma}(t)\,\mathrm{d}\tilde{\boldsymbol{W}}_t \tag{6.12}$$

and since $\boldsymbol{f}_t^*$ has a continuous extension to $[0, T]$, we have

$$H_T = H_0 + \int_0^T [bR(t,T)]^{-1} \boldsymbol{f}_t^{*\prime} \boldsymbol{\sigma}(t)\, \mathrm{d}\tilde{\boldsymbol{W}}_t$$

almost surely, because the equality holds for all $t < T$ and the stochastic integral is almost surely path continuous. Observe now that by properties of stochastic integrals, H_t is a $\tilde{P}$ local-martingale. Since H_t is indeed bounded it is in fact a $\tilde{P}$-martingale as well for all $t < T$, and therefore converges at T. Furthermore, the martingale representation theorem (e.g. [19, Section V.3]) now provides the representation $H_T = H_0 + \int_0^T \boldsymbol{\xi}_s' \, \mathrm{d}\tilde{\boldsymbol{W}}_s$ with $\boldsymbol{\xi}$ a measurable, adapted process that is square-integrable, i.e., $\int_0^T \boldsymbol{\xi}_s' \boldsymbol{\xi}_s \, \mathrm{d}s < \infty$ $\tilde{P}$-a.s. But comparison with (6.12) shows that $\boldsymbol{\xi}_t = [bR(t,T)]^{-1} \boldsymbol{f}_t^*$, and as such $\int_0^T \boldsymbol{f}_t^{*\prime} \boldsymbol{f}_t^* \, \mathrm{d}t < \infty$, $\tilde{P}$-a.s, and since P and $\tilde{P}$ are equivalent, we must also have $\int_0^T \boldsymbol{f}_t^{*\prime} \boldsymbol{f}_t^* \, \mathrm{d}t < \infty$, P-a.s. Inspection of $\boldsymbol{f}_t^*$ now shows that this requirement in turn implies $H_T = 0$ or $H_T = 1$, equivalently $X_T = 0$ or $X_T = b$.

Thus, we may now conclude from (6.11) that

$$\begin{aligned} P(H_T = 1) &= P\left(\int_0^T \boldsymbol{\theta}(s)' \, \mathrm{d}\boldsymbol{W}_s \geq - \int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\, \mathrm{d}s - \sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\, \mathrm{d}s}\ \Phi^{-1}(H_0) \right) \\ &\equiv \Phi\left(\Phi^{-1}(H_0) + \sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\, \mathrm{d}s} \right), \end{aligned}$$

the last equality following from the fact that $\int_0^T \boldsymbol{\theta}(s)' \, \mathrm{d}\boldsymbol{W}_s$ has a normal distribution with mean 0 and variance $\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\, \mathrm{d}s$. Substituting now for X_T^* shows that

$$P(X_T^* = b) = V(0, X_0; b) = 1 - P(X_T^* = 0)$$

which is the desired conclusion.

6.2. Obtaining the candidate value function

While we have already shown that V does indeed solve the HJB equation, we have not actually solved the resulting nonlinear partial differential equation (6.5) directly for V. Rather, we obtained a candidate solution to (6.5) by reducing the problem to a form whose optimal value function we were able to guess by extending an argument of Heath [10], as we now show.

To proceed, set $Y_t^f \equiv \exp\{-\int_0^t r(s)\, \mathrm{d}s\} X_t^f$, and then apply Itô's formula to get

$$\mathrm{d}Y_t^f = \exp\left\{ -\int_0^t r(s)\, \mathrm{d}s \right\} \left[\boldsymbol{f}_t'(\boldsymbol{\mu}(t) - r(t)\mathbf{1})\, \mathrm{d}t + \sum_{i=1}^n \sum_{j=1}^n f_t^{(i)} \sigma_{ij}(t)\, \mathrm{d}W_t^{(j)} \right]. \tag{6.13}$$

Setting now $\boldsymbol{g}_t \equiv \boldsymbol{f}_t \exp\{-\int_0^t r(s)\, \mathrm{d}s\}$, let us first consider the following problem: find $U(t, y; c) := \sup_g P_{(t,y)}(Y_T^g \geq c)$ and its associated optimal control vector, $\{\boldsymbol{g}_t^*, t \leq T\}$, where

$$\mathrm{d}Y_t^g = \boldsymbol{g}_t'(\boldsymbol{\mu}(t) - r(t)\mathbf{1})\, \mathrm{d}t + \sum_{i=1}^n \sum_{j=1}^n g_t^{(i)} \sigma_{ij}(t)\, \mathrm{d}W_t^{(j)}. \tag{6.14}$$

This is now essentially (aside from scaling factors) the multivariate generalization of the problem studied in [13] and [10]. Following Heath [10], we note that we can rewrite (6.14) in terms of the $\tilde{P}$ Brownian motion $\tilde{\boldsymbol{W}}_t = \boldsymbol{W}_t + \int_0^t \boldsymbol{\theta}(t)\,\mathrm{d}t$, as

$$\mathrm{d}Y_t^g = \sum_{i=1}^n \sum_{j=1}^n g_t^{(i)} \sigma_{ij}(t)\,\mathrm{d}\tilde{W}_t^{(j)}. \tag{6.15}$$

It is clear, by the admissibility assumption on $\boldsymbol{f}_t$—and hence on $\boldsymbol{g}_t$—that the local martingale Y_t^g is in fact a *martingale* under the measure $\tilde{P}$. Hence, letting $A^g := \{\omega : Y_T^g \geq c\}$, it is clear by the martingale inequality that we have $P(A^g) \leq Y_0/c$, and since the right hand side of this is independent of the policy g, we may conclude that $\sup\{\boldsymbol{g} : P(A^g)\} \leq Y_0/c$. If we assume this upper bound can be hit, then a straightforward generalization of an argument in [10], reproduced here for the sake of completeness in the Appendix, then shows that

$$U(t, y; c) = \Phi\left(\Phi^{-1}\left(\frac{y}{c}\right) + \sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\right). \tag{6.16}$$

An HJB argument then shows that $\boldsymbol{g}_t^* = -\boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1})U_y/U_{yy}$, and by differentiating (6.16) appropriately we obtain

$$\boldsymbol{g}_t^*(y; c) = \left[\frac{\boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1})}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\right] c \cdot \phi\left(\Phi^{-1}\left(\frac{y}{c}\right)\right). \tag{6.17}$$

All we need do now is recognize that

$$\sup_{\boldsymbol{f}} P_{(t,x)}(X_T^f \geq b) \equiv \sup_{\boldsymbol{g}} P_{(t,y)}\left(Y_T^g \geq \exp\left\{-\int_0^T r(s)\,\mathrm{d}s\right\} b\right) \tag{6.18}$$

with

$$y = x \exp\left\{-\int_0^t r(s)\,\mathrm{d}s\right\}, \quad \text{and} \quad \boldsymbol{f}_t^* \equiv \exp\left\{\int_0^t r(s)\,\mathrm{d}s\right\} \boldsymbol{g}_t^*.$$

And indeed taking $y = \exp\{-\int_0^t r(s)\,\mathrm{d}s\}x$ and $c = b\exp\{\int_0^T r(s)\,\mathrm{d}s\}$ in (6.16) and (6.17) gives

$$V(t, x; b) = U\left(t, x\exp\left\{-\int_0^t r(s)\,\mathrm{d}s\right\}; b\exp\left\{-\int_0^T r(s)\,\mathrm{d}s\right\}\right)$$

where $U(t, \bullet; \bullet)$ is the function given by (6.16), and

$$\boldsymbol{f}_t^*(x; b) = \exp\left\{\int_0^t r(s)\,\mathrm{d}s\right\} \boldsymbol{g}_t^*\left(x\exp\left\{-\int_0^t r(s)\,\mathrm{d}s\right\}; b\exp\left\{-\int_0^T r(s)\,\mathrm{d}s\right\}\right),$$

where $\boldsymbol{g}_t^*(\bullet; \bullet)$ is defined by (6.17).

Of course, this argument still needs a rigorous verification, which is the content of the previous proof. The functions U and $\boldsymbol{g}_t^*$ in (6.16) and (6.17) are central to the remainder of the paper.

7. Exogenous income

In this section we analyse the case where the investor has an exogenous (deterministic) income stream. Specifically, suppose that income is earned at rate $\{l(t), 0 \leq t \leq T\}$, where $l(s) \geq 0$ for all s. The investor's wealth then evolves according to

$$\begin{aligned} \mathrm{d}X_t^f &= \sum_{i=1}^{n} f_t^{(i)} \frac{\mathrm{d}S_i(t)}{S_i(t)} + \left(X_t^f - \sum_{i=1}^{n} f_t^{(i)}\right) \frac{\mathrm{d}B_t}{B_t} + l(t)\,\mathrm{d}t \\ &= \left[r(t)X_t + \sum_{i=1}^{n} f_t^{(i)}(\mu_i(t) - r(t)) + l(t)\right] \mathrm{d}t + \sum_{i=1}^{n}\sum_{j=1}^{n} f_t^{(i)} \sigma_{ij}(t)\,\mathrm{d}W_t^{(j)}. \end{aligned} \tag{7.1}$$

This case can be treated by modifying our previous analysis, as we now show. To that end, define first the function $\kappa(x, t)$ and the constant Λ by

$$\kappa(x,t) := \exp\left\{-\int_0^t r(u)\,\mathrm{d}u\right\} x - \int_0^t \exp\left\{-\int_0^s r(u)\,\mathrm{d}u\right\} l(s)\,\mathrm{d}s. \tag{7.2}$$

$$\Lambda := b\exp\left\{-\int_0^T r(u)\,\mathrm{d}u\right\} - \int_0^T \exp\left\{-\int_0^s r(u)\,\mathrm{d}u\right\} l(s)\,\mathrm{d}s. \tag{7.3}$$

The optimal value function and optimal policy for maximizing the probability that the investor achieves the wealth level b by the terminal time T are given in the following theorem.

Theorem 7.1. *For an investor whose wealth process follows (7.1),*

$$\Psi(t, x; b, T) := \sup_f P_{(t,x)}(X_T^f \geq b) = \Phi\left(\Phi^{-1}\left(\frac{\kappa(x,t)}{\Lambda}\right) + \sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\right) \tag{7.4}$$

and the associated optimal investment policy is

$$\begin{aligned} f_t^{(l)}(x; b) &:= \arg\sup_f P_{(t,x)}(X_T^f \geq b) \\ &= \left[\frac{\boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t) - r(t)\mathbf{1})}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\right]\left[\exp\left\{\int_0^t r(u)\,\mathrm{d}u\right\}\Lambda\right]\phi\left(\Phi^{-1}\left(\frac{\kappa(x,t)}{\Lambda}\right)\right). \end{aligned} \tag{7.5}$$

Remark 7.1. An analysis similar to that done after Theorem 3.1 shows that under this policy, the investor stops investing in the risky stocks altogether, i.e., $\boldsymbol{f}_t^{(l)}(x; b) = \mathbf{0}$, when $\kappa(x, t) \geq \Lambda$ as well as when $\kappa(x, t) \leq 0$. The former case is in effect for wealth levels, x, that satisfy

$$x \geq b\exp\left\{-\int_t^T r(u)\,\mathrm{d}u\right\} - \int_t^T \exp\left\{\int_t^s r(u)\,\mathrm{d}u\right\} l(s)\,\mathrm{d}s, \tag{7.6}$$

while the latter case occurs for wealth levels that satisfy

$$x \leq \int_0^t \exp\left\{\int_s^t r(u)\,\mathrm{d}u\right\} l(s)\,\mathrm{d}s. \tag{7.7}$$

The economic interpretation of (7.6) is clear since we may rewrite it as

$$x\exp\left\{\int_t^T r(u)\,\mathrm{d}u\right\} + \int_t^T \exp\left\{\int_s^T r(u)\,\mathrm{d}u\right\} l(s)\,\mathrm{d}s \geq b$$

which shows directly that by investing current wealth (x) and all future income in the riskless asset, the goal b can be reached with certainty at time T.

The interpretation of (7.7) is perhaps more interesting, since the right hand side of (7.7) is the level of wealth that would have been achieved at time t by simply investing all the income until that time directly into the risk-free asset. Thus if wealth ever falls to that level, the investor following policy $\boldsymbol{f}_t^{(l)}$ would stop investing in the risky stocks and just put all wealth into the risk-free asset. Therefore, as opposed to the case without income treated earlier, wealth is effectively bounded away from 0, and so the investor with positive income can never go bankrupt under policy $\boldsymbol{f}_t^{(l)}$. Moreover, this lower bound on wealth acts as a sort of 'performance bound' on the investor: if the performance of the risky stocks is such that wealth drops to the value of the income stream invested in the risk-free asset, investment in the risky stocks stop.

This behavior differs significantly from the behavior of a utility maximizer with an income stream. For example, in the constant coefficients case with $l(s) = l$, it can be shown that the optimal policy for an investor who wishes to maximize expected logarithmic utility of terminal wealth at time T, is to invest

$$\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})\left(x + \frac{l}{r}(1 - \mathrm{e}^{-r(T-t)})\right)$$

in the risky stocks (cf. [15]). Thus a utility maximizer essentially capitalizes future earnings (till T) at the risk-free rate and then invests as if his current fortune already included this amount. Therefore, even if wealth goes *negative*, so long as $x > -(l/r)(1 - \mathrm{e}^{-r(T-t)})$, the investor still invests a positive amount in the risky stocks. Not so for the probability maximizer who behaves in the manner described above. In essence, a utility maximizer uses income to take extra risks, by borrowing against future earnings, while our Theorem 7.1 shows that a probability maximizer relies on income to become more cautious.

Proof. Let $\boldsymbol{g}_t := \exp\{-\int_0^t r(s)\,\mathrm{d}s\}\,\boldsymbol{f}_t$, and for the remainder of this section only, let Y_t^g be defined now by

$$Y_t^g := \exp\left\{-\int_0^t r(u)\,\mathrm{d}u\right\} X_t^f - \int_0^t \exp\left\{-\int_0^s r(u)\,\mathrm{d}u\right\} l(s)\,\mathrm{d}s \tag{7.8}$$

where X_t^f is as in (7.1). Then a simple application of Itô's formula shows that

$$\mathrm{d}Y_t^g = \boldsymbol{g}_t'(\boldsymbol{\mu}(t) - r(t)\mathbf{1})\,\mathrm{d}t + \sum_{i=1}^n\sum_{j=1}^n g_t^{(i)}\sigma_{ij}(t)\,\mathrm{d}W_t^{(j)} \equiv \sum_{i=1}^n\sum_{j=1}^n g_t^{(i)}\sigma_{ij}(t)\,\mathrm{d}\tilde{W}_t^{(j)} \tag{7.9}$$

which is the same as (6.13) and (6.14). Thus we observe immediately that if

$$U(t, y; c) = \sup_{\boldsymbol{g}} P(Y_T^g \geq c \mid Y_t = y),$$

with optimizer $\boldsymbol{g}_t^*$, then $U(t, y; c)$ and $\boldsymbol{g}_t^*$ are given by (6.16) and (6.17). Now, by (7.8), we see that

$$\sup_{\boldsymbol{g}} P(Y_T^g \geq c \mid Y_t = y) \equiv \sup_{\boldsymbol{f}} P\Bigg(X_T^f \geq c\exp\left\{\int_0^T r(u)\,\mathrm{d}u\right\} + \int_0^T \exp\left\{\int_s^T r(u)\,\mathrm{d}u\right\} l(s)\,\mathrm{d}s \mid Y_t = \kappa(x, t)\Bigg) \tag{7.10}$$

where $\kappa(x,t)$ is given by (7.2). Thus we may invert this to deduce that in terms of a goal problem for X_t^f given by (7.1), with arbitrary b, we have

$$\sup_f P(X_T^f \geq b \mid X_t = x) = U(t, \kappa(x,t); \Lambda),$$

as well as

$$f_t^{(l)}(x; b) = \exp\left\{\int_0^t r(u)\,\mathrm{d}u\right\} g_t^*(\kappa(x,t); \Lambda),$$

where the function $U(t, \bullet; \bullet)$ is defined by (6.16), $g_t^*(\bullet; \bullet)$ is defined by (6.17), $\kappa(x,t)$ by (7.2) and Λ by (7.3). Direct substitution then shows that $U(t, \kappa(x,t); \Lambda) = \Psi(t, x; b, T)$ of (7.4) and that the policy given in (7.5) is indeed optimal for the problem of maximizing the probability of attaining the goal b by time T.

Remark 7.2. It is tempting to extend the analysis here to include the case of liabilities as well, however for this case the optimal policy allows wealth to become negative. This raises a variety of new problems and issues which will be discussed elsewhere.

8. Beating another portfolio

In this section we apply our earlier results to derive optimal portfolio strategies for an investor, such as a fund manager, who is interested solely in beating a given stochastic *benchmark*. The benchmark is most typically an index, such as the S&P 500. In particular, it is just one specific portfolio strategy. We consider first the problem of beating the benchmark portfolio by a given percentage, which is the focal point of active portfolio management. We then find the related policy which corrects for the downside risk. When the benchmark portfolio strategy is the optimal growth policy, then the ratio of the wealth from any arbitrary portfolio strategy to the benchmark is in fact a supermartingale, and as such problems arise. Nevertheless, we are able to find a strategy which does achieve the maximal possible probability of beating the optimal growth policy by a prespecified percentage by a fixed deadline.

8.1. Beating the benchmark portfolio by a given percentage

Here we consider the objective of ensuring that X_T^f exceeds the terminal value of a benchmark portfolio, Q_T, by a preset percentage. To that end, we will find it more convenient to use a geometric, or proportional parameterization of the model, i.e., instead of parameterizing the problem by taking $\boldsymbol{f}_t$ to be the vector of absolute amounts invested in the stocks, we instead—for the remainder of this section—let $f_t^{(i)}$ denote the proportion of wealth invested in stock i at time t, with $\boldsymbol{f}_t := (f_t^{(1)}, \ldots, f_t^{(n)})'$ now denoting the corresponding column vector of proportions. Then under this parameterization, it is clear that the wealth process evolves as (compare with (2.3))

$$\begin{aligned}\mathrm{d}X_t^f &= X_t^f \sum_{i=1}^n f_t^{(i)} \frac{\mathrm{d}S_i(t)}{S_i(t)} + X_t^f\left(1 - \sum_{i=1}^n f_t^{(i)}\right)\frac{\mathrm{d}B_t}{B_t} \\ &= X_t^f\left[r(t) + \sum_{i=1}^n f_t^{(i)}(t)(\mu_i(t) - r(t))\right]\mathrm{d}t + X_t^f \sum_{i=1}^n \sum_{j=1}^n f_t^{(i)}(t)\sigma_{ij}(t)\,\mathrm{d}W_t^{(j)}, \qquad (8.1)\end{aligned}$$

upon substituting from (2.1) and (2.2).

Similarly let the benchmark portfolio, $\{Q_t, 0 \le t \le T\}$ be defined by

$$\mathrm{d}Q_t = Q_t\left[r(t) + \sum_{i=1}^{n} \pi_t^{(i)}(\mu_i(t) - r(t))\right]\mathrm{d}t + Q_t \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_t^{(i)}\sigma_{ij}(t)\,\mathrm{d}W_t^{(j)}, \tag{8.2}$$

where $\boldsymbol{\pi}_t = (\pi_t^{(1)}, \dots, \pi_t^{(n)})'$ is the (column) vector of portfolio weights in the benchmark process.

An interesting objective related to active portfolio management is then to choose a policy that maximizes the probability that for a fixed T, X_T^f exceeds Q_t by a predetermined percentage, say λ. Denote the resulting optimal policy by f^π, i.e.,

$$\{f_t^\pi\} = \arg\sup_f P(X_T^f > (1+\lambda)\cdot Q_T).$$

Let $Z_t^f(\pi) = X_t^f/Q_t$, with $Z_0^f = X_0/Q_0$. A simple application of Itô's formula then gives

$$\mathrm{d}Z_t^f = Z_t^f(\boldsymbol{f}_t - \boldsymbol{\pi}_t)'(\boldsymbol{\mu}(t) - r(t)\mathbf{1} - \boldsymbol{\Sigma}(t)\boldsymbol{\pi}_t)\,\mathrm{d}t + Z_t^f \sum_{i=1}^{n}\sum_{j=1}^{n}(f_t^{(i)} - \pi_t^{(i)})\sigma_{ij}(t)\,\mathrm{d}W_t^{(j)}. \tag{8.3}$$

In the next theorem, we provide the optimal policy and the optimal value function. It depends solely on the current value of the ratio of the two portfolios, Z_t, as it is obvious that

$$\sup_f P(X_T^f > (1+\lambda)Q_T \mid X_t^f = x, Q_t = y) \equiv \sup_f P\left(Z_T^f > 1+\lambda \mid Z_t^f = \frac{x}{y}\right).$$

Theorem 8.1. *Let $Z_s^f = X_s^f/Q_s$ be defined by (8.3). Then*

$$F(t, z; \lambda, T) := \sup_f P(Z_T^f > 1+\lambda \mid Z_t = z) = \Phi\left(\Phi^{-1}\left(\frac{z}{1+\lambda}\right) + \sqrt{\int_t^T \gamma(s)\,\mathrm{d}s}\right) \tag{8.4}$$

and the optimal (proportional) control f_t^π, defined as

$$f_t^\pi := \arg\{\sup_f P(Z_T^f > 1+\lambda \mid Z_t = z)\},$$

is

$$f_t^\pi(z; \lambda, \boldsymbol{\pi}_t) = \boldsymbol{\pi}_t + \left[\frac{\boldsymbol{\pi}_t^* - \boldsymbol{\pi}_t}{\sqrt{\int_t^T \gamma(s; \boldsymbol{\pi}_s)\,\mathrm{d}s}}\right]\left(\frac{1+\lambda}{z}\right)\phi\left(\Phi^{-1}\left(\frac{z}{1+\lambda}\right)\right) \tag{8.5}$$

where the function $\gamma(\cdot;\cdot)$ is defined by

$$\gamma(t; \boldsymbol{\pi}_t) := (\boldsymbol{\pi}_t^* - \boldsymbol{\pi}_t)'\boldsymbol{\Sigma}(t)(\boldsymbol{\pi}_t^* - \boldsymbol{\pi}_t) \tag{8.6}$$

where $\boldsymbol{\pi}_t^* = \boldsymbol{\Sigma}^{-1}(t)\,(\boldsymbol{\mu}(t) - r(t)\mathbf{1})$.

Proof. Let $\hat{\mathbf{g}}_t = \mathbf{f}_t - \boldsymbol{\pi}_t$, with components $\hat{g}_t^{(i)} = f_t^{(i)} - \pi_t^{(i)}$, and let $\hat{\boldsymbol{\mu}}(t)$ be defined by

$$\hat{\boldsymbol{\mu}}(t) = \boldsymbol{\mu}(t) - \boldsymbol{\Sigma}(t)\boldsymbol{\pi}_t. \tag{8.7}$$

Then it is clear that we have

$$\mathrm{d}Z_t^f = Z_t^f \left(\hat{\mathbf{g}}_t(\hat{\boldsymbol{\mu}}(t) - r(t)\mathbf{1})\,\mathrm{d}t + \sum_{i=1}^n \sum_{j=1}^n \hat{g}_t^{(i)} \sigma_{ij}(t)\,\mathrm{d}W_t^{(j)} \right)$$

which, with $\hat{\boldsymbol{\mu}}(t)$ replacing $\boldsymbol{\mu}(t)$, is equivalent to the 'proportional control' parameterization of Y_t^g of (6.14) considered previously, with $\mathbf{g}_t = Z_t^f \hat{\mathbf{g}}_t$. As such its value function is $U(t, z : 1+\lambda)$ of (6.16), with $\hat{\boldsymbol{\mu}}(t)$ replacing $\boldsymbol{\mu}(t)$, which is equivalent to (8.4). Similarly, the optimal control is then given by $\hat{\mathbf{g}}_t^* = \mathbf{g}_t^*(z; 1+\lambda)/z$, where $\mathbf{g}_t^*(\cdot;\cdot)$ is given by (6.17), with $\hat{\boldsymbol{\mu}}(t)$ replacing $\boldsymbol{\mu}(t)$. Recalling now that $f_t^\pi = \boldsymbol{\pi}_t + \hat{\mathbf{g}}_t^*$, and substituting back for $\hat{\boldsymbol{\mu}}(t)$ gives the control in (8.5).

8.2. Controlling for downside risk

The previous development gives a policy under which the maximal probability of beating the benchmark by $\lambda\%$ is $F(0, z_0; \lambda, T) = \Phi\big(\Phi^{-1}(z_0/(1+\lambda)) + \sqrt{\big(\int_0^T \gamma(s; \boldsymbol{\pi}_s)\,\mathrm{d}s\big)}\big)$. However, as noted earlier, the nature of this policy is such that we also have the downside risk that the ratio at the terminal time might end up at 0, with $P(Z_T^* = 0) = 1 - F(0, z_0; \lambda, T)$, and so bankruptcy is possible. This probably entails too much risk-taking for most applications. Instead, consider then the following objective: the portfolio manager's goal is to beat the prescribed benchmark by $\lambda\%$, with the additional proviso that the portfolio never underperform the benchmark by another given percentage, say $\delta\%$. Formally, the portfolio manager's objective then is to determine the optimal policy $\mathbf{f}^\delta$ that solves

$$\sup_f P\left(Z_T^f \geq 1+\lambda, \inf_{0 \leq s \leq T} Z_s^f \geq 1-\delta \right). \tag{8.8}$$

Denote this optimal value function by $\hat{F}(t, z; \lambda, \delta, T)$. The change of variable $\tilde{Z}_t^f := Z_t^f - (1-\delta)$, then shows that

$$\begin{aligned}\hat{F}(t, z; \lambda, \delta, T) &:= \sup_f P\left(Z_T^f \geq 1+\lambda, \inf_{t \leq s \leq T} Z_s^f \geq 1-\delta \mid Z_t = z \right) \\ &\equiv \sup_f P\left(\tilde{Z}_T^f > \lambda+\delta, \inf_{t \leq s \leq T} \tilde{Z}_s^f \geq 0 \mid \tilde{Z}_t = z - (1-\delta) \right).\end{aligned}$$

But this last term is precisely the problem considered earlier in Theorem 9.1, where the non-negativity condition was implicit. As such, we know that $\hat{F}(t, z; \lambda, \delta, T) = F(t, z - (1-\delta); \lambda + \delta - 1, T)$, where $F(t, \bullet; \bullet; T)$ was defined in (8.4). Explicitly, we have

$$\hat{F}(t, z; \lambda, \delta, T) = \Phi\left(\Phi^{-1}\left(\frac{z-(1-\delta)}{\lambda+\delta} \right) + \sqrt{\int_t^T \gamma(s; \boldsymbol{\pi}_s)\,\mathrm{d}s} \right). \tag{8.9}$$

Similarly, the optimal policy for this problem is therefore given by

$$\boldsymbol{f}_t^{\delta}(z;\lambda,\delta,\boldsymbol{\pi}_t)=\boldsymbol{\pi}_t+\left[\frac{\boldsymbol{\pi}_t^*-\boldsymbol{\pi}_t}{\sqrt{\int_t^T\gamma(s;\boldsymbol{\pi}_s)\,\mathrm{d}s}}\right]\left(\frac{\lambda+\delta}{z-(1-\delta)}\right)\phi\left(\Phi^{-1}\left(\frac{z-(1-\delta)}{\lambda+\delta}\right)\right). \tag{8.10}$$

It is interesting to note that this policy invests *more* (for a fixed z) in the risky stocks than does $\boldsymbol{f}_t^{\pi}$ of (8.5). This follows from the fact that $\phi\left(\Phi^{-1}(u)\right)/u$ is decreasing in u, and for $z\le 1+\lambda$, we have $(z-(1-\delta))/(\lambda+\delta)<z/(1+\lambda)$, for all δ.

8.3. Can we beat the optimal growth policy?

Suppose now that $\boldsymbol{\pi}_t$ is the *optimal growth policy*, $\boldsymbol{\pi}_t^*$ given by

$$\boldsymbol{\pi}_t^*:=\boldsymbol{\Sigma}^{-1}(t)(\boldsymbol{\mu}(t)-r(t)\mathbf{1})\equiv(\boldsymbol{\sigma}(t)')^{-1}\boldsymbol{\theta}(t). \tag{8.11}$$

As noted earlier, this policy maximizes the 'growth rate from investment', see e.g. [16], and is also sometimes referred to as the market portfolio.

When we place $\boldsymbol{\pi}_t^*$ back into the evolutionary stochastic differential equation for Z^f, and simplify, we find that $\{Z^f(\pi^*)_t\}$ satisfies

$$\mathrm{d}Z_t^f\equiv Z_t^f(\boldsymbol{f}_t'-\boldsymbol{\pi}_t^*)'\boldsymbol{\sigma}(t)\,\mathrm{d}\boldsymbol{W}_t \tag{8.12}$$

implying that Z_t^f is a non-negative local martingale and hence a supermartingale for every admissible policy $\boldsymbol{f}_t$. In fact, since Z^f is bounded above by $1+\lambda$ in our application, it is indeed a martingale, although in general we would need to restrict our attention to policies that satisfy the Novikov condition (i.e., $E(\exp\{(1/2)\int_0^t\boldsymbol{f}_s'\boldsymbol{f}_s\,\mathrm{d}s\})<\infty$, for all t) to ensure that Z is a martingale.

Since Z^f is a martingale we have $E(Z_t^f\mid\mathcal{F}_s)=Z_s^f$ for all policies $\boldsymbol{f}_t$. Thus, since Z^f is now a 'fair game', it is not clear if we can find an optimal strategy to beat it in finite time.

Note that for $\boldsymbol{\pi}_t=\boldsymbol{\pi}_t^*$, we have $\gamma(t;\boldsymbol{\pi}_t^*)=0$, for all t, and the value function of (8.4) evaluated at $\gamma=0$ gives $F(t,z;\lambda,T)=z/(1+\lambda)$, which is of course consistent with the martingale inequality $P(Z_T^f>1+\lambda\mid Z_t^f=z)\le z/(1+\lambda)$, which holds for all admissible policies $\boldsymbol{f}$. Since the right hand side of this inequality is independent of the policy, we can extend it to

$$\sup_{\boldsymbol{f}}P(Z_T^f>1+\lambda\mid Z_t^f=z)\le\frac{z}{1+\lambda}.$$

However, the value function F of (8.4)—treated as a function of γ—is not continuous at $\gamma=0$. Moreover, the policy, $\boldsymbol{f}_t^{\pi}$ of (8.5) is indeterminate when $\boldsymbol{\pi}_t=\boldsymbol{\pi}_t^*$, since we also have $\gamma(t;\boldsymbol{\pi}_t^*)=0$. However, it turns out that this policy does have a limit as $\boldsymbol{\pi}_t\to\boldsymbol{\pi}_t^*$.

To see this, take $\boldsymbol{\pi}_t=(1-\epsilon)\boldsymbol{\pi}_t^*$ in (8.5). When we do this we find that it reduces to

$$\boldsymbol{f}_t^{\pi}(z;\lambda,(1-\epsilon)\boldsymbol{\pi}_t^*)=(1-\epsilon)\boldsymbol{\pi}_t^*+\left[\frac{\boldsymbol{\pi}_t^*}{\sqrt{\int_t^T\boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\right]\left(\frac{1+\lambda}{z}\right)\phi\left(\Phi^{-1}\left(\frac{z}{1+\lambda}\right)\right).$$

This control is a continuous function of ϵ, and so we may evaluate it at $\epsilon=0$ to obtain

$$\boldsymbol{f}_t^{\pi}(z;\lambda,\boldsymbol{\pi}_t^*)=\boldsymbol{\pi}_t^*+\left[\frac{\boldsymbol{\pi}_t^*}{\sqrt{\int_t^T\boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\right]\left(\frac{1+\lambda}{z}\right)\phi\left(\Phi^{-1}\left(\frac{z}{1+\lambda}\right)\right). \tag{8.13}$$

The control function $f_t^\pi(z, \lambda, \boldsymbol{\pi}_t^*)$ in fact does achieve the optimal possible value, $z/(1+\lambda)$. We formalize this in the following.

Theorem 8.2. *Suppose that the benchmark is the optimal growth policy given by $\boldsymbol{\pi}_t^*$ of (8.11). Then the optimal value function is*

$$\sup_f P_{(t,z)}(Z_T^f > 1+\lambda) = \frac{z}{1+\lambda} \tag{8.14}$$

and this optimal value can be achieved by the policy $f_t^\pi(z, \lambda, \boldsymbol{\pi}_t^*)$ in (8.13).

Remark 8.1. Note that the linear form of the value function in (8.14) precludes us from using any HJB methods.

Proof. As noted above, since $t\{Z_t^f, t \le T\}$ is a martingale, the martingale inequality provides the upper bound $z/(1+\lambda)$. Thus the equality in (8.14) will follow if we can find a policy that achieves this upper bound. To see that $f_t^\pi(z, \lambda, \boldsymbol{\pi}_t^*)$ is such a policy, replace f_t in the stochastic differential equation (8.12) with f_t^π of (8.13), and denote the resulting process by Z_t^*, and let $H_t^* := Z_t^*/(1+\lambda)$, to find

$$\mathrm{d}H_t^* = \frac{-\boldsymbol{\pi}_t^{*\prime}\boldsymbol{\sigma}(t)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\phi(\Phi^{-1}(H_t^*))\,\mathrm{d}\boldsymbol{W}_t \equiv \frac{-1}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\phi(\Phi^{-1}(H_t^*))\boldsymbol{\theta}(t)'\,\mathrm{d}\boldsymbol{W}_t \tag{8.15}$$

for $t < T$, where the last equality follows from the fact that $\boldsymbol{\pi}_t^* = \boldsymbol{\sigma}^{-1}(t)'\boldsymbol{\theta}(t)$.

The solution to this stochastic differential equation is

$$H_t^* = \Phi\left(\frac{\int_0^t \boldsymbol{\theta}(s)'\,\mathrm{d}\boldsymbol{W}_s + \sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\,\Phi^{-1}(H_0)}{\sqrt{\int_t^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}}\right), \quad \text{for } 0 \le t < T \tag{8.16}$$

as can be verified by an application of Itô's formula. An argument identical to that used in the proof earlier shows that (8.16) is valid as well for $t = T$ with H_T^* is equal to 1 or 0. As such, examination of (8.16) shows that

$$P(H_T^* = 1) = P\left(\int_0^T \boldsymbol{\theta}(s)'\,\mathrm{d}\boldsymbol{W}_s > -\sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\;\Phi^{-1}(H_0)\right) \equiv H_0$$

where the last equality follows from the fact that $\int_0^T \boldsymbol{\theta}(s)'\,\mathrm{d}\boldsymbol{W}_s \sim N(0, \int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s)$ under P. The statement of the theorem now follows since $H_t^* \equiv Z_t^*/(1+\lambda)$.

Appendix. The argument in Heath [11]

The modification of Heath's argument in [10] to the case treated in Section 6.2 is as follows: for Y_t^g defined by (6.14) and (6.15), let $A := \{\omega : Y_T^g \ge c\}$, and so $P(A) \le Y_0/c$. But $P(A) \le \sup\{P(B) : \tilde{P}(B) \le Y_0/c\}$, and the 'sup' can be evaluated via the Neyman–Pearson Lemma. Hence there exists a unique number λ such that $\tilde{P}\{\mathrm{d}P/\mathrm{d}\tilde{P} \ge \lambda\} = Y_0/c$. But this probability is, by (2.8), simply

$$\tilde{P}\left(\exp\left\{\int_0^T \boldsymbol{\theta}(s)'\,\mathrm{d}\tilde{\boldsymbol{W}}_s - \frac{1}{2}\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s\right\} \ge \lambda\right) = Y_0/c,$$

and since under $\tilde{P}$, $\int_0^T \boldsymbol{\theta}(s)'\,\mathrm{d}\tilde{\mathbf{W}}_s \sim N(0, \int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s)$, we find

$$\ln(\lambda) = -\sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\ \Phi^{-1}(Y_0/c) - \frac{1}{2}\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s$$

and hence the corresponding P probability is

$$P\bigg(\int_0^T \boldsymbol{\theta}(s)'\,\mathrm{d}\mathbf{W}_s + \frac{1}{2}\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s \geq -\sqrt{\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s}\ \Phi^{-1}(Y_0/c) - \frac{1}{2}\int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s\bigg)$$

which, by virtue of the fact that $\int_0^T \boldsymbol{\theta}(s)'\,\mathrm{d}\mathbf{W}_s \sim N(0, \int_0^T \boldsymbol{\theta}(s)'\boldsymbol{\theta}(s)\,\mathrm{d}s)$, is simply $U(0, Y_0; c)$ of (6.16). Dynamic programming then applies for arbitrary (t, y) to give $U(t, y; c)$ of (6.16).

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