Assignment 0

Statistics 32950 (Spring 2023)

Ki Hyun

Due: 23:00 (CT) 2023-03-24

1.

$$r = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2 \sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2}}$$

Here,

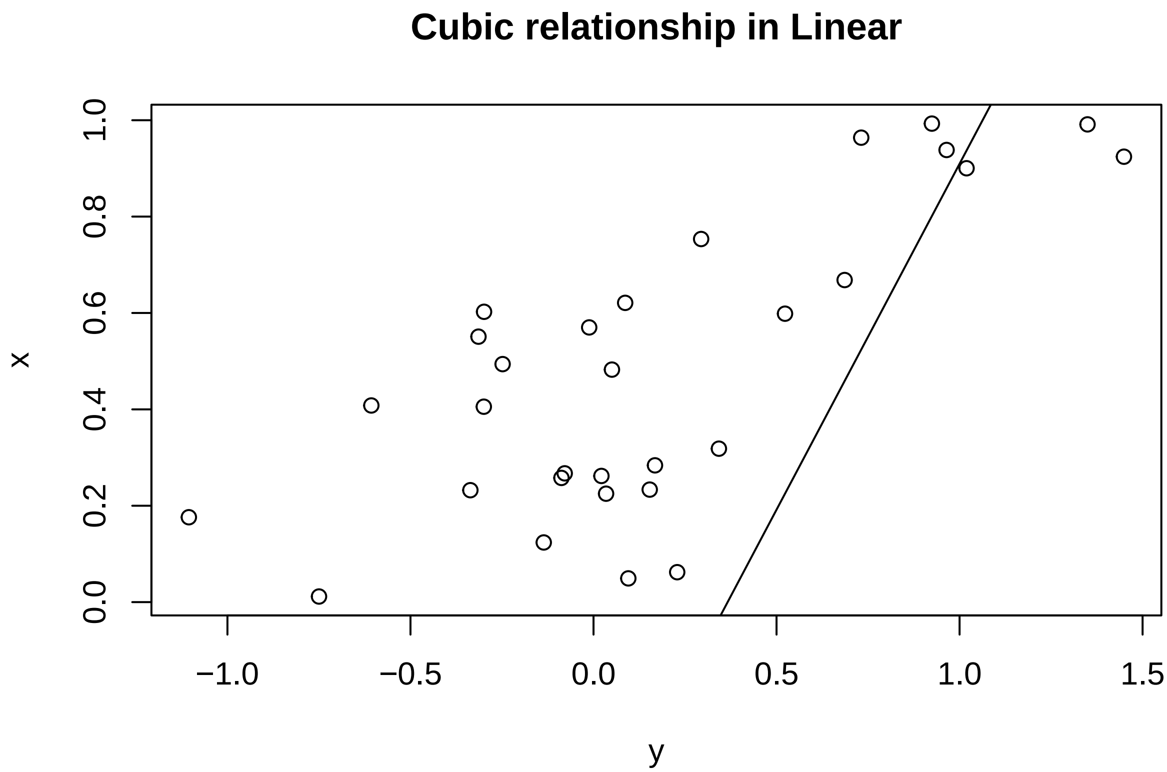
$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Moreover,

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

2.

```
x = runif(30); y = x^3 + rnorm(30)/3
plot(y, x); abline(lm(y ~ x))
title("Cubic relationship in Linear")
```



3.

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -1 \\ -6 & 0 & 5 \\ 4 & 5 & 7 \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$E = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(a)

i) When A is multiplied by B from the left (i.e., BA), the following row operations on A occurs:

- The first elements of each column remain the same.
- The second elements of each column gets multiplied by 3.
- The third elements of each column gets multiplied by -2.

ii) When A is multiplied by B from the right (i.e., AB), the following column operations on A occurs:

- The first column remains the same. (a_1)
- The second column gets multiplied by 3. $(3a_2)$
- The third column gets multiplied by -2. $(-2a_3)$

iii) When A is multiplied by E from the left (i.e., EA), the following row operations on A occurs:

- The first row remains the same.
- The second row becomes the third row.
- The third row becomes the second row.

iv) When A is multiplied by E from the right (i.e., AE), the following column operations on A occurs:

- The first column remains the same.
- The second column becomes the third column. (a_3)
- The third column becomes the second column. (a_2)

(b)

$$v = \begin{bmatrix} 7 \\ 3 \\ 24 \end{bmatrix}$$

```r
A = matrix(c(3, -4, -1, -6, 0, 5, 4, 5, 7), nrow = 3, byrow = T)
v = matrix(c(7, 3, 24), nrow = 3, byrow = T)
solve(A) %*% v
```

i)

```
##      [,1]
## [1,]    2
## [2,]   -1
## [3,]    3
```

$$v = 2a_1 - a_2 + 3a_3$$

ii)

$$v = 7b_1 + b_2 - 12b_3$$

iii)

$$v = 7e_1 + 24e_2 + 3e_3$$

4.

(a)

$$\sum_{i=1}^{n} k^3 = \left(\sum_{i=1}^{n} k\right)^2$$

Proof by induction)

i) Let's first check that the statement is true for $n = 1$.

$$1^3 = (1)^2$$

ii) Let's assume that the statement is true for m, (i.e., $\sum_{i=1}^{m} k^3 = \left(\sum_{i=1}^{m} k\right)^2$).
Then for $m+1$:

$$\begin{aligned}
\sum_{i=1}^{m+1} k^3 &= \sum_{i=1}^{m} k^3 + (m+1)^3 \\
&= \left(\sum_{i=1}^{m} k\right)^2 + m^3 + 3m^2 + 3m + 1 \\
&= \left(\frac{m(m+1)}{2}\right)^2 + m^3 + 3m^2 + 3m + 1 \\
&= \frac{m^4}{4} + \frac{m^3}{2} + \frac{m^2}{4} + m^3 + 3m^2 + 3m + 1 \\
&= \frac{1}{4}\left(m^4 + 6m^3 + 13m^2 + 12m + 4\right) \\
&= \frac{1}{4}\left(m^2 + 3m + 2\right)^2 \\
&= \left(\frac{(m+1)(m+2)}{2}\right)^2 \\
&= \left(\sum_{i=1}^{m+1} k\right)^2
\end{aligned}$$

$$\therefore \sum_{i=1}^{n} k^3 = \left(\sum_{i=1}^{n} k\right)^2$$

$$Q.E.D.$$

(b)

Let,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

However,

$$BA = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore AB \neq BA$$

$$Q.E.D.$$

5.

$$U \sim Uniform[-1, 1]$$

$$X = U^2$$

(a)

For $0 < x \leq 1$

$$\mathbf{P}\{X \leq x\} = \mathbf{P}\{-\sqrt{x} \leq U \leq \sqrt{x}\} = \frac{2\sqrt{x}}{2} = \sqrt{x}$$

Here,

$$\frac{\delta}{\delta x}\mathbf{P}\{X \leq x\} = \frac{\delta}{\delta x}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

Therefore, the p.d.f. of X ($f(x)$) can be described as:

$$f(x) = \begin{cases} \frac{1}{2\sqrt{x}} & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} \mathbf{E}\left(X\right) &= \int_0^1 x f(x) dx \\ &= \int_0^1 \frac{x}{2\sqrt{x}} dx \\ &= \int_0^1 \frac{\sqrt{x}}{2} dx \\ &= \left[\frac{1}{3} x^{\frac{3}{2}}\right]_0^1 \\ &= \frac{1}{3} \end{aligned}$$