Ridge Lasso Elastic-net - Demo

Comparisons of model coefficient estimate

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Regularized linear regressions

Comparisons of **Ridge, Lasso, and Elastic-net**

Goal: Parameter estimation in linear regression

$$Y = \beta_o + \mathbf{X}\beta + \epsilon$$

```
library(MASS)    # use lm.ridge
library(glmnet)
```


Estimators

$\hat{\beta}_{LS} = \operatorname{argmin}_\beta \{\|Y - \beta_0 - \mathbf{X}\beta\|_2^2\}$

$\hat{\beta}_{Ridge} = \operatorname{argmin}_\beta \{\|Y - \beta_0 - \mathbf{X}\beta\|_2^2 + \lambda_{ridge}\|\beta\|_2^2\}$

$\hat{\beta}_{Lasso} = \operatorname{argmin}_\beta \{\|Y - \beta_0 - \mathbf{X}\beta\|_2^2 + \lambda_{lasso}\|\beta\|_1\}$

$\hat{\beta}_{E-net} = \operatorname{argmin}_\beta \{\|Y - \beta_0 - \mathbf{X}\beta\|_2^2 + \lambda\left[\alpha\|\beta\|_2^2 + (1-\alpha)\|\beta\|_1\right]\}$

Example 1: Explanatory variables X

$p = 30$ input X variables

$n = 50$ observations

```
set.seed(246329)
n=50
p1=10   # "signals"
p2=20   # "noise"
X = matrix(rnorm(n*(p1+p2)),n, p1+p2)
dim(X)
```

```
## [1] 50 30
```

Example 1: Response variable Y

10 signal, 20 uniform noise input variables

```
c1 = 0.5 + (runif(10))/2  # "signals" 0.5 + U(0,1)/2
c2 = (runif(20))*3/10      # "noise"  U(0,1)*3/10

Y = X[,1:10]%*%c1 + X[,11:30]%*%c2  + rnorm(n)
```

To fit the model

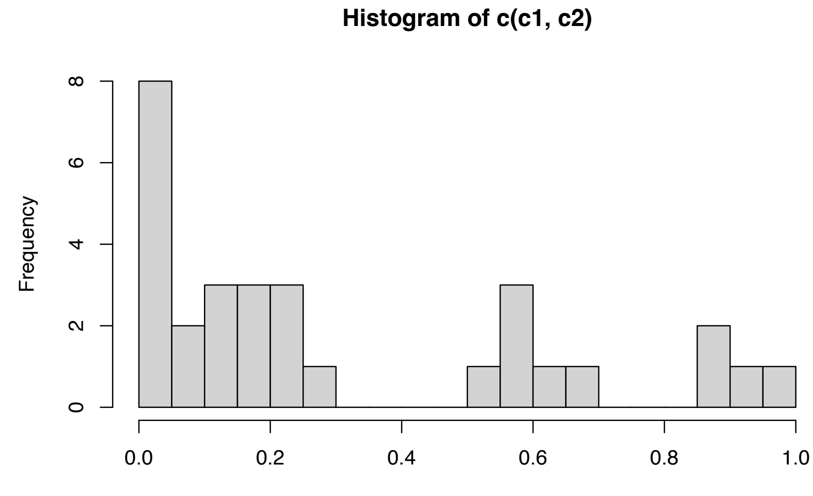
$$Y = \beta_o + \mathbf{X}\beta + \epsilon$$

In expanded form,

$$Y = \beta_o + \beta_1 X_1 + \cdots + \beta_{10} X_{10} + \beta_{11} X_{11} + \cdots + \beta_{30} X_{30} + \epsilon$$

True β values in $Y = \beta_o + \mathbf{X}\beta + \epsilon$

```
#range(c1)
#range(c2)
hist(c(c1,c2),nclass=15)
```



Least Squares for $Y = \beta_o + \mathbf{X}\beta + \epsilon$

```
Y = X[,1:10]%*%c1 + X[,11:30]%*%c2  + rnorm(n)
LSfit = lm(Y~X)
round(LSfit$coeff,2)
```

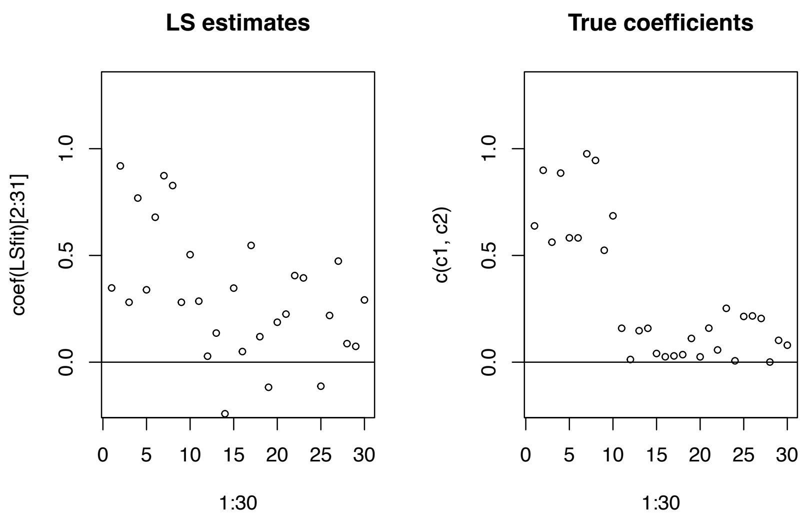
```
## (Intercept)          X1          X2          X3          (
##       -0.18        0.35        0.92        0.28          (
##          X6          X7          X8          X9
##        0.68        0.87        0.83        0.28          (
##         X12         X13         X14         X15
##        0.03        0.14       -0.24        0.35          (
##         X18         X19         X20         X21
##        0.12       -0.12        0.19        0.23          (
##         X24         X25         X26         X27
##       -0.35       -0.11        0.22        0.47          (
##         X30
##        0.29
```


Compare model fitted $\hat{\beta}$ with true β (code)

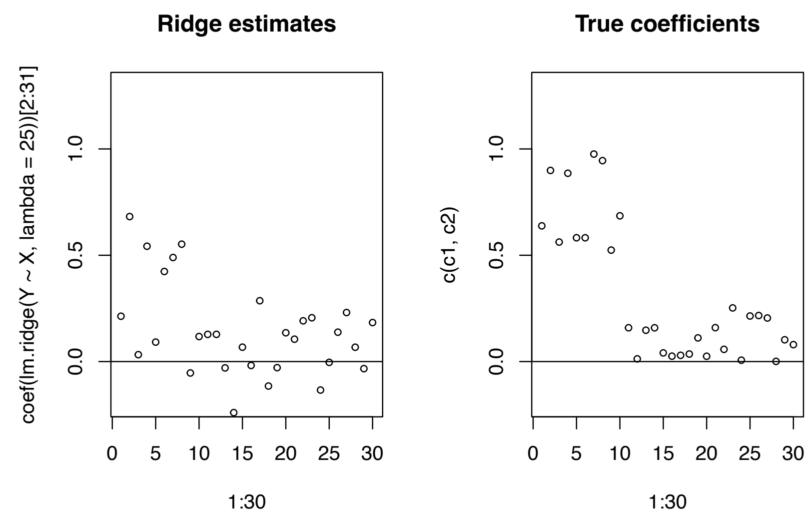
```
par(mfrow=c(1,2))  # Get LS beta
plot(1:30,coef(LSfit)[2:31],cex=.7,ylim=c(-.2,1.3),
     main="LS estimates");abline(h=0)
plot(1:30,c(c1,c2),cex=.7,ylim=c(-.2,1.3),
     main="True coefficients");abline(h=0)
```

```
par(mfrow=c(1,2)) # Get Ridge beta
plot(1:30,coef(lm.ridge(Y~X,lambda=25))[2:31],cex=.7,
     ylim=c(-.2,1.3),main="Ridge estimates")
abline(h=0)
plot(1:30,c(c1,c2),cex=.7,ylim=c(-.2,1.3),
     main="True coefficients");abline(h=0)
```

$\hat{\beta}_{LS}$ values for $Y = \beta_o + \mathbf{X}\beta + \epsilon$



$\hat{\beta}_{Ridge}$ values for $Y = \beta_o + \mathbf{X}\beta + \epsilon$

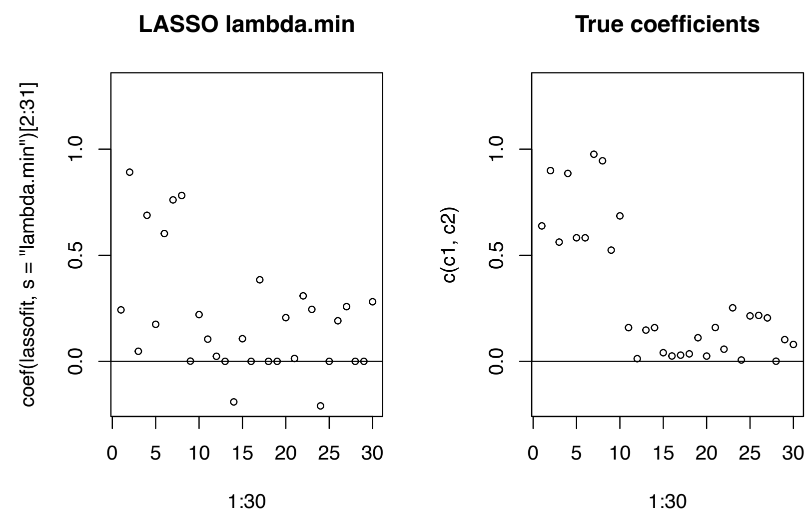


$\hat{\beta}_{Lasso}$ values for $Y = \beta_o + \mathbf{X}\beta + \epsilon$

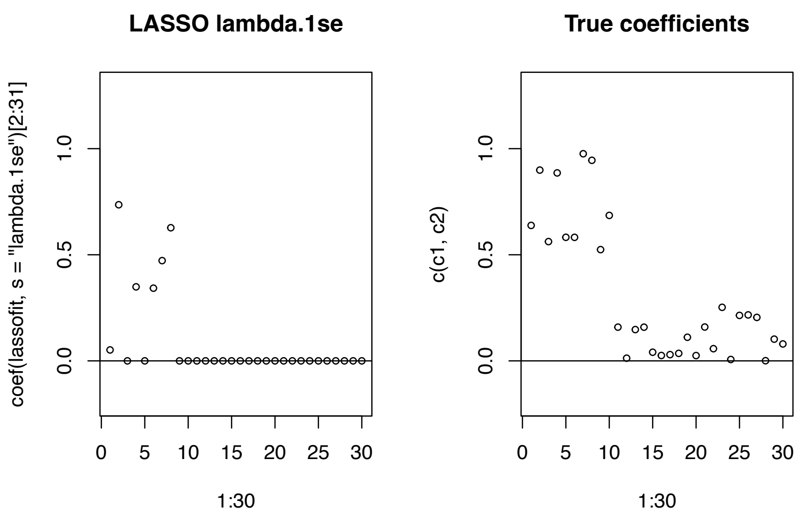
```
lassofit = cv.glmnet(X,Y)
par(mfrow=c(1,2))  # Get LASSO beta, lambda-min
plot(1:30,coef(lassofit,s="lambda.min")[2:31],cex=.7,
     ylim=c(-.2,1.3),main="LASSO lambda.min");
abline(h=0)
plot(1:30,c(c1,c2),cex=.7,ylim=c(-.2,1.3),
     main="True coefficients");abline(h=0)
```

```
lassofit = cv.glmnet(X,Y)
par(mfrow=c(1,2))   # Get LASSO beta, lambda-1se
plot(1:30,coef(lassofit,s="lambda.1se")[2:31],cex=.7,
     ylim=c(-.2,1.3),main="LASSO lambda.1se");
abline(h=0)
plot(1:30,c(c1,c2),cex=.7,ylim=c(-.2,1.3),
     main="True coefficients");abline(h=0)
```


$\hat{\beta}_{Lasso}$ values when $\lambda = \texttt{lambda.min}$



$\hat{\beta}_{Lasso}$ values when $\lambda =$ `lambda.1se`



Model parameter estimates comparison

$Y = \beta_o + \mathbf{X}\beta + \epsilon$

We compare value and range of the true coefficients

$$\beta = (\beta_1, \cdots, \beta_{30})$$

with estimated $\hat{\beta}$ values

$$\hat{\beta} = (\hat{\beta}_1, \cdots, \hat{\beta}_{30})$$

Using the method

- Least Squares
- Ridge
- Lasso
- Elastic-net

Viusualize model parameter estimates $\hat{\beta}$

Fit linear regression model $Y = \beta_o + \mathbf{X}\beta + \epsilon$ by

- Least Squares
- Ridge
- Lasso
- Elastic-net

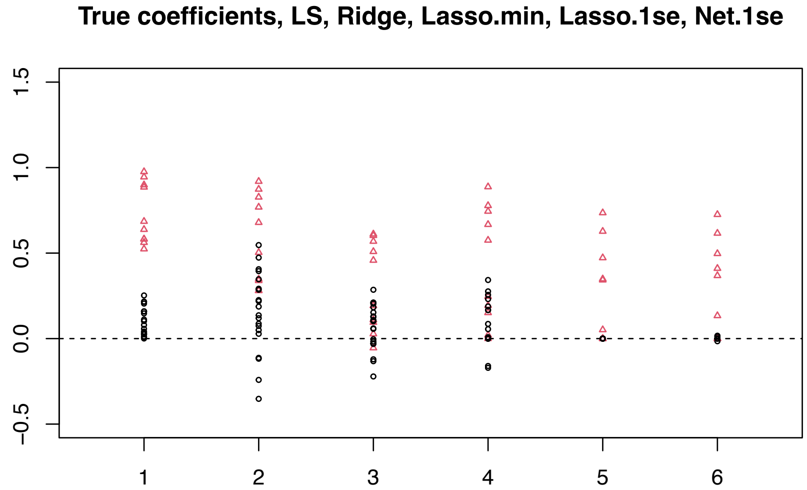
```
netfit = cv.glmnet(X,Y,alpha=0.5)
plot(rep(1,30),c(c1,c2),cex=.5,col=c(rep(2,10),rep(1,20)),
     xlim=c(0.5,6.5),ylim=c(-.5,1.5),xlab="",ylab="")
abline(h=0,lty=2)
```


(Code)

```
points(rep(2,30),lm(Y~X)$coef[2:31],cex=.5,
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
points(rep(3,30),lm.ridge(Y~X,lambda=25)$coef,cex=.5,
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
points(rep(4,30),coef(lassofit,s="lambda.min")[2:31],cex=.5
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
points(rep(5,30),coef(lassofit,s="lambda.1se")[2:31],cex=.5
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
points(rep(6,30),coef(netfit,s="lambda.1se")[2:31],cex=.5,
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
title("True coefficients, LS, Ridge,
      Lasso.min, Lasso.1se, Net.1se")
```

Input variables: **Signal** (triangles) + uniform **Noise**

β, $\hat{\beta}_{LS}$ $\hat{\beta}_{Ridge}$ $\hat{\beta}_{Lasso.min}$ $\hat{\beta}_{Lasso.1se}$ $\hat{\beta}_{E-net}$



Example 2: Signal + white noise

X: 10 Singal + 20 normal noise input variables

```
n=50; p1=10; p2=20
set.seed(19)
X = matrix(rnorm(n*(p1+p2)),n, p1+p2)
dim(X)
```

```
## [1] 50 30
```

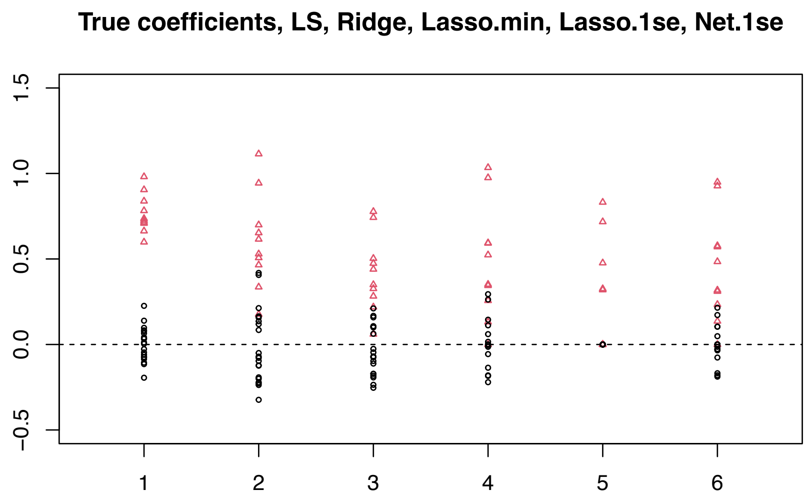
```
c1 = 0.5 + runif(10)/2          # "signals"
c2 = (rnorm(20))*3/40          # "noise" ~ N(0,sigma^2)
Y = X[,1:10]%*%c1 + X[,11:30]%*%c2  + rnorm(n)
lassofit = cv.glmnet(X,Y); par(mfrow=c(1,2))
```

$Y = \beta_o + \mathbf{X}\beta + \epsilon$

Example 2: Parameter estimates $\hat{\beta}$ comparison

Input variables: **Signal** (triangles) + **Noise** (normal)

β, $\hat{\beta}_{LS}$ $\hat{\beta}_{Ridge}$ $\hat{\beta}_{Lasso.min}$ $\hat{\beta}_{Lasso.1se}$ $\hat{\beta}_{E-net}$



(Code)

```
netfit = cv.glmnet(X,Y,alpha=0.5)
plot(rep(1,30),c(c1,c2),cex=.5,col=c(rep(2,10),rep(1,20)),
     xlim=c(0.5,6.5),ylim=c(-.5,1.5),xlab="",ylab="")
abline(h=0,lty=2)
points(rep(2,30),lm(Y~X)$coef[2:31],cex=.5,
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
points(rep(3,30),lm.ridge(Y~X,lambda=25)$coef,cex=.5,
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
points(rep(4,30),coef(lassofit,s="lambda.min")[2:31],cex=.5
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
points(rep(5,30),coef(lassofit,s="lambda.1se")[2:31],cex=.5
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
points(rep(6,30),coef(netfit,s="lambda.1se")[2:31],cex=.5,
  col=c(rep(2,10),rep(1,20)), pch=c(rep(2,10),rep(1,20)))
title("True coefficients, LS, Ridge,
     Lasso.min, Lasso.1se, Net.1se")
```