Please do not share or post. (Derivations were required, some are not shown here.)

1. (a) Using $\boldsymbol{e}_1'\boldsymbol{e}_1 = 1, \boldsymbol{e}_1'\boldsymbol{e}_3 = 0$ etc. $\Rightarrow$ $e_{11} = 3/13,\ e_{23} = -12/13,\ e_{33} = 3/13$.
   (b) i. The maximum variance is achieved by the linear combination $e_1'X = \frac{3}{13}X_1 + \frac{4}{13}X_2 + \frac{12}{13}X_3$
   ii. $\max_a Var(a'X) = e_1'\Sigma e_1 = \lambda_1 = 5.0$.
   iii. The component variable X_3 has the most weight $\frac{12}{13}$.

2. (a) To test the hypothesis H_o: $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ vs H_a: $\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$, under $\Sigma_i = \Sigma$, Hotelling's T^2 statistic is

$$T^2 = [(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)]' \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pool}\right]^{-1} [(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)] = [-3\ \ -3]\ \left(\frac{1}{11} + \frac{1}{12}\right)^{-1} \times \frac{1}{19}\begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

   (b) Under H_o: $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$, $T^2 \sim \frac{(n_1+n_2-2)p}{n_1+n_2-p-1}F_{2,n_1+n_2-p-1} = \frac{21}{10}F_{2,20}$

3. (a)

Source of variation	Matrix of sum of squares and cross products	Degrees of freedom
Groups	$B = \sum_{i=1}^{3} 50(\bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}})(\bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}})'$	$g - 1 = 3 - 1 = 2$
Residuals	$W = \sum_{i=1}^{3}\sum_{j=1}^{50}(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)'$	$\sum_{i=1}^{3} n_i - g = 150 - 3 = 147$
Total	$B + W = \sum_{i=1}^{3}\sum_{j=1}^{50}(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}})(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}})'$	$n - 1 = 150 - 1 = 149$

   (b) i. $\Lambda^* = \dfrac{det(W)}{det(B+W)} = \dfrac{22096.88}{942754.6}$
   ii. $H_0 : \mu_1 = \mu_2 = \mu_3$, i.e., all three groups' population mean vectors are the same.
   iii. A small value of Λ^* leads to rejection of the H_0.

4. (a) $\rho_1^* = \sqrt{0.62} = 0.79,\ \rho_2^* = \sqrt{0.0052} = 0.07$.

   (b) $a_1 = \frac{1}{\sqrt{1.73}}\begin{bmatrix} 0.72 \\ 0.69 \end{bmatrix} = \begin{bmatrix} 0.54 \\ 0.53 \end{bmatrix}$, $Var(U_1) = a_1'S_{11}a_1 = 1$, where $U_1 = a_1'[X_1\ X_2]' = 0.54X_1 + 0.53X_2$.
   $b_1 = \frac{1}{\sqrt{1.84}}\begin{bmatrix} -0.68 \\ -0.73 \end{bmatrix} = \begin{bmatrix} -0.50 \\ -0.54 \end{bmatrix}$, $Var(V_1) = b_1'S_{22}b_1 = 1$, $V_1 = b_1[X_3\ X_4]' = -0.50X_3 - 0.54X_4$.

   (c) $\begin{bmatrix} 1 & 0.79 & 0 & 0 \\ 0.79 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.07 \\ 0 & 0 & 0.07 & 1 \end{bmatrix}$ (i.e. used $-V_1$) OR $\begin{bmatrix} 1 & -0.79 & 0 & 0 \\ -0.79 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.07 \\ 0 & 0 & 0.07 & 1 \end{bmatrix}$

   (d) Here $\rho_1^* = 0.79 >> \rho_2^* = 0.07 \approx 0$. (U_1, V_1) has already captured most correlation between $[X_1\ X_2]'$ and $[X_3\ X_4]'$, the palm measurement vectors of the two daughters, respectively. The leftover correlation captured by the second pair (U_2, V_2) is 0.07, negligibly small.

5. (a) $$\mathbb{E}(\boldsymbol{X}_3 \mid \boldsymbol{X}_1 = \boldsymbol{x}_1, \boldsymbol{X}_2 = \boldsymbol{x}_2) = \boldsymbol{\mu}_3 + [\Sigma_{31}\ \Sigma_{32}]\begin{bmatrix} \Sigma_{11} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22} \end{bmatrix}^{-1}\begin{bmatrix} \boldsymbol{x}_1 - \boldsymbol{\mu}_1 \\ \boldsymbol{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix} = \boldsymbol{\mu}_3 + [\Sigma_{31}\ \Sigma_{32}]\begin{bmatrix} \Sigma_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22}^{-1} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 - \boldsymbol{\mu}_1 \\ \boldsymbol{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix}$$
$$= \boldsymbol{\mu}_3 + \Sigma_{31}\Sigma_{11}^{-1}(\boldsymbol{x}_1 - \boldsymbol{\mu}_1) + \Sigma_{32}\Sigma_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)$$

   (b) $$Var(\boldsymbol{X}_3 \mid \boldsymbol{X}_1 = \boldsymbol{x}_1, \boldsymbol{X}_2 = \boldsymbol{x}_2) = \Sigma_{33} - [\Sigma_{31}\ \Sigma_{32}]\begin{bmatrix} \Sigma_{11} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22} \end{bmatrix}^{-1}\begin{bmatrix} \Sigma_{13} \\ \Sigma_{23} \end{bmatrix} = \Sigma_{33} - \Sigma_{31}\Sigma_{11}^{-1}\Sigma_{13} - \Sigma_{32}\Sigma_{22}^{-1}\Sigma_{23}$$

6. (a) Two maps are the same. Classical MDS produces Euclidean distances. Thus in the same dimensions, Euclidean distance input will result in the same output. Consequently, Stress = 0.
   (b) i. Let dissimilarity between individuals be the number of different answers. The pairwise distance matrix for items A, B, C and D would look like $\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 6 & 2 & 2 \\ 6 & 0 & 4 & 4 \\ 2 & 4 & 0 & 2 \\ 2 & 4 & 2 & 0 \end{bmatrix} \end{array}$ or equivalent.
   ii. The relative MDS configuration should be consistent with the above distance matrix (omitted).

7. (a) i.
$$\boldsymbol{X}_{3\times1} = \boldsymbol{\mu}_{3\times1} + L_{3\times1}\boldsymbol{F}_{1\times1} + \boldsymbol{\varepsilon}_{3\times1}$$
   ii.
$$\begin{cases} \ell_1^2 + \psi_1 = 5 \\ \ell_1\ell_2 = 1 \\ \ell_1\ell_3 = 2 \\ \ell_2^2 + \psi_2 = 5 \\ \ell_2\ell_3 = 2 \\ \ell_3^2 + \psi_3 = 6 \end{cases} \Rightarrow\ \ \ell_1 = 1,\ \ell_2 = 1,\ \ell_3 = 2,\ \psi_1 = 4,\ \psi_2 = 4,\ \psi_3 = 2$$
   iii. $\%Var(X_i)$ explained by the common factor is $\frac{\ell_i^2}{Var(X_i)} = \frac{\ell_i^2}{\ell_i^2 + \psi_i}$.
   For $i = 1, 2, 3$, they are $\left(\frac{1}{5}, \frac{1}{5}, \frac{4}{6}\right) = \left(\frac{1}{5}, \frac{1}{5}, \frac{2}{3}\right) = (20\%, 20\%, 67\%)$

   (b) i. Write $\Sigma = [\sigma_{ij}]$, then $cov(Y_i, Y_j) = \sigma_{ij} = \ell_i\ell_j$ for $i \neq j$, $var(Y_i) = \sigma_{ii} = \ell_i^2 + \psi_i$. Then $Corr(Y_1, Y_2) = \frac{cov(Y_1, Y_2)}{\sqrt{var(Y_1)}\sqrt{var(Y_2)}} = \frac{\ell_1\ell_2}{\sqrt{\ell_1^2+\psi_1}\sqrt{\ell_2^2+\psi_2}}$
   ii. (Required for 32950, optional for 24620 with up to 3 bonus pts)
$$R = \frac{Corr(Y_i, Y_k)}{Corr(Y_j, Y_k)} = \frac{\sqrt{var(Y_j)}}{\sqrt{var(Y_i)}}\frac{Cov(Y_i, Y_k)}{Cov(Y_j, Y_k)} = \frac{\sqrt{\ell_j^2 + \psi_j}}{\sqrt{\ell_i^2 + \psi_i}}\frac{\ell_i\ell_k + \psi_{ik}}{\ell_j\ell_k + \psi_{jk}},\quad where\quad \psi_{ik} = \begin{cases} \psi_k, & i = k; \\ 0 & i \neq k. \end{cases}$$
   Then you should derive and check for each and all cases (the final expressions are omitted here):
   - Case $i = j$
   - Case $i \neq j$
     - Case $k \neq i, k \neq j$
     - Case $k = i$ (so $k \neq j$)
     - Case $k = j$ (so $k \neq i$