Assignment 1

Statistics 32950

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Due: 09:00 (CT) 2023-03-28

1.

(a)

```
##
##            1         3         5
##   0 0.0000000 0.0000000 0.3333333
##   2 0.3333333 0.0000000 0.0000000
##   4 0.0000000 0.3333333 0.0000000
```

- All the rows and columns each add up to $\frac{1}{3}$.

(b)

Given from the joint probability that all the B numbers are strictly larger than the R numbers, I should choose blue under Rule-I

(c)

If I choose blue, my wining probability is:

$$
\begin{aligned}
&\mathbf{P}(B=1, R<1)+\mathbf{P}(B=3, R<3)+\mathbf{P}(B=5, R<5) \\
=&\mathbf{P}(B=1)\times\mathbf{P}(R=0)+\mathbf{P}(B=3)\times\mathbf{P}(R\neq 4)+\mathbf{P}(B=5) \\
&(\because B \perp\!\!\!\perp R) \\
=&\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{3}+\frac{1}{3} \\
=&\frac{2}{3}
\end{aligned}
$$

Since there is no draw, my winning probability if I choose red would be:

$$1-\frac{2}{3}=\frac{1}{3}$$

Therefore, I should choose blue again under Rule-II

(d)

Similarly, if I choose blue:

$$
\begin{aligned}
&\mathbf{P}(B = 1, R < 1) + \mathbf{P}(B = 3, R < 3) + \mathbf{P}(B = 5, R < 5) \\
=&\mathbf{P}(B = 1) \times \mathbf{P}(R = 0 \mid B = 1) + \mathbf{P}(B = 3) \times \mathbf{P}(R \neq 4 \mid B = 3) + \mathbf{P}(B = 5) \\
=&\frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1 + \frac{1}{3} \\
=&\frac{5}{6}
\end{aligned}
$$

Since there is no draw, my winning probability if I choose red would be:

$$1 - \frac{5}{6} = \frac{1}{6}$$

Therefore, I should choose blue again under Rule-III

2.

```
ladyrun = read.table("ladyrun23.dat")
colnames(ladyrun)=c("Country","100m","200m","400m",
                    "800m","1500m","3000m","Marathon")
```

(a)

```
summary(ladyrun[-1])
```

```
##       100m            200m            400m            800m      
##  Min.   :10.49   Min.   :21.34   Min.   :45.14   Min.   :1.890  
##  1st Qu.:11.09   1st Qu.:22.48   1st Qu.:49.97   1st Qu.:1.962  
##  Median :11.28   Median :22.98   Median :51.56   Median :2.005  
##  Mean   :11.31   Mean   :23.07   Mean   :51.82   Mean   :2.021  
##  3rd Qu.:11.48   3rd Qu.:23.37   3rd Qu.:53.01   3rd Qu.:2.070  
##  Max.   :12.52   Max.   :25.91   Max.   :61.65   Max.   :2.290  
##      1500m           3000m            Marathon    
##  Min.   :3.840   Min.   : 8.100   Min.   :134.1  
##  1st Qu.:3.995   1st Qu.: 8.530   1st Qu.:143.4  
##  Median :4.100   Median : 8.845   Median :148.1  
##  Mean   :4.187   Mean   : 9.067   Mean   :153.3  
##  3rd Qu.:4.338   3rd Qu.: 9.325   3rd Qu.:157.7  
##  Max.   :5.420   Max.   :13.120   Max.   :221.1
```

The sample mean of "Country" is not meaningful since it is not a numeric variable, and serves more as an index

(b)

```
cov(ladyrun[-1])
```

```
##                100m       200m       400m        800m      1500m       3000m
## 100m     0.15160363 0.33398644  0.9114414 0.025373096 0.08181258  0.23144633
## 200m     0.33398644 0.82557460  2.2083772 0.060023305 0.19168805  0.53146111
## 400m     0.91144144 2.20837718  7.4956509 0.191735604 0.55504969  1.56931373
## 800m     0.02537310 0.06002331  0.1917356 0.007600175 0.02165912  0.06305461
## 1500m    0.08181258 0.19168805  0.5550497 0.021659119 0.07495849  0.21936604
## 3000m    0.23144633 0.53146111  1.5693137 0.063054612 0.21936604  0.68175629
## Marathon 3.72633326 8.57760297 30.7499534 1.243528686 3.63533836 11.01618166
##            Marathon
## 100m       3.726333
## 200m       8.577603
## 400m      30.749953
## 800m       1.243529
## 1500m      3.635338
## 3000m     11.016182
## Marathon 276.841762
```

```r
cor(ladyrun[-1], method = 'pearson')
```

```
##               100m      200m      400m      800m     1500m     3000m  Marathon
## 100m     1.0000000 0.9440519 0.8550062 0.7474925 0.7674584 0.7199145 0.5751897
## 200m     0.9440519 1.0000000 0.8877495 0.7577570 0.7705600 0.7084003 0.5673773
## 400m     0.8550062 0.8877495 1.0000000 0.8033158 0.7404859 0.6942095 0.6750314
## 800m     0.7474925 0.7577570 0.8033158 1.0000000 0.9074414 0.8759725 0.8572909
## 1500m    0.7674584 0.7705600 0.7404859 0.9074414 1.0000000 0.9703857 0.7980289
## 3000m    0.7199145 0.7084003 0.6942095 0.8759725 0.9703857 1.0000000 0.8018640
## Marathon 0.5751897 0.5673773 0.6750314 0.8572909 0.7980289 0.8018640 1.0000000
```

(c)

```r
cor(ladyrun[-1], method = 'kendall')
```

```
##               100m      200m      400m      800m     1500m     3000m  Marathon
## 100m     1.0000000 0.7670175 0.6468121 0.5392341 0.5081088 0.4687719 0.4054631
## 200m     0.7670175 1.0000000 0.7224955 0.6111795 0.5588493 0.4877193 0.3970597
## 400m     0.6468121 0.7224955 1.0000000 0.6480241 0.5834012 0.5430979 0.4706295
## 800m     0.5392341 0.6111795 0.6480241 1.0000000 0.7139181 0.6745769 0.5779092
## 1500m    0.5081088 0.5588493 0.5834012 0.7139181 1.0000000 0.7843622 0.6448793
## 3000m    0.4687719 0.4877193 0.5430979 0.6745769 0.7843622 1.0000000 0.6757718
## Marathon 0.4054631 0.3970597 0.4706295 0.5779092 0.6448793 0.6757718 1.0000000
```

(d)

```r
cor(ladyrun[-1], method = 'spearman')
```

```
##               100m      200m      400m      800m     1500m     3000m  Marathon
## 100m     1.0000000 0.8990430 0.8227860 0.7050442 0.6666286 0.6368073 0.5641583
## 200m     0.8990430 1.0000000 0.8675232 0.7784637 0.7366474 0.6668700 0.5504727
## 400m     0.8227860 0.8675232 1.0000000 0.8096520 0.7783286 0.7379958 0.6562476
## 800m     0.7050442 0.7784637 0.8096520 1.0000000 0.8598986 0.8412661 0.7589243
## 1500m    0.6666286 0.7366474 0.7783286 0.8598986 1.0000000 0.9307044 0.8344106
## 3000m    0.6368073 0.6668700 0.7379958 0.8412661 0.9307044 1.0000000 0.8638469
## Marathon 0.5641583 0.5504727 0.6562476 0.7589243 0.8344106 0.8638469 1.0000000
```

(e)

```r
cor(log(ladyrun[-1]), method = 'pearson')
```

```
##               100m      200m      400m      800m     1500m     3000m  Marathon
## 100m     1.0000000 0.9424813 0.8498516 0.7440608 0.7659573 0.7217321 0.5856388
## 200m     0.9424813 1.0000000 0.8830559 0.7588229 0.7732850 0.7129298 0.5769940
## 400m     0.8498516 0.8830559 1.0000000 0.8005089 0.7498331 0.7115699 0.6730251
## 800m     0.7440608 0.7588229 0.8005089 1.0000000 0.9090472 0.8836627 0.8573069
```

```
## 1500m    0.7659573 0.7732850 0.7498331 0.9090472 1.0000000 0.9685678 0.8190800
## 3000m    0.7217321 0.7129298 0.7115699 0.8836627 0.9685678 1.0000000 0.8336751
## Marathon 0.5856388 0.5769940 0.6730251 0.8573069 0.8190800 0.8336751 1.0000000
```

```r
cor(log(ladyrun[-1]), method = 'kendall')
```

```
##               100m      200m      400m      800m     1500m     3000m  Marathon
## 100m     1.0000000 0.7670175 0.6468121 0.5392341 0.5081088 0.4687719 0.4054631
## 200m     0.7670175 1.0000000 0.7224955 0.6111795 0.5588493 0.4877193 0.3970597
## 400m     0.6468121 0.7224955 1.0000000 0.6480241 0.5834012 0.5430979 0.4706295
## 800m     0.5392341 0.6111795 0.6480241 1.0000000 0.7139181 0.6745769 0.5779092
## 1500m    0.5081088 0.5588493 0.5834012 0.7139181 1.0000000 0.7843622 0.6448793
## 3000m    0.4687719 0.4877193 0.5430979 0.6745769 0.7843622 1.0000000 0.6757718
## Marathon 0.4054631 0.3970597 0.4706295 0.5779092 0.6448793 0.6757718 1.0000000
```

```r
cor(log(ladyrun[-1]), method = 'spearman')
```

```
##               100m      200m      400m      800m     1500m     3000m  Marathon
## 100m     1.0000000 0.8990430 0.8227860 0.7050442 0.6666286 0.6368073 0.5641583
## 200m     0.8990430 1.0000000 0.8675232 0.7784637 0.7366474 0.6668700 0.5504727
## 400m     0.8227860 0.8675232 1.0000000 0.8096520 0.7783286 0.7379958 0.6562476
## 800m     0.7050442 0.7784637 0.8096520 1.0000000 0.8598986 0.8412661 0.7589243
## 1500m    0.6666286 0.7366474 0.7783286 0.8598986 1.0000000 0.9307044 0.8344106
## 3000m    0.6368073 0.6668700 0.7379958 0.8412661 0.9307044 1.0000000 0.8638469
## Marathon 0.5641583 0.5504727 0.6562476 0.7589243 0.8344106 0.8638469 1.0000000
```

The results differ slightly from (b) since the actual value of each observation changes. However, since it is a monotonic transformation, the results do not differ by much.

On the other hand, since log is a monotonic transformation, the results from (c) and (d) does not change.

(f)

```r
q2f <- eigen(cor(ladyrun[-1], method = 'pearson'))
round(q2f$values, 2)
```

```
## [1] 5.70 0.74 0.29 0.11 0.09 0.05 0.02
```

```r
q2f$vectors
```

```
##            [,1]       [,2]        [,3]        [,4]        [,5]       [,6]
## [1,] -0.3720342 -0.4575195 -0.14870245  0.52629124 -0.15450205 -0.5677425
## [2,] -0.3738784 -0.4801563 -0.07423786  0.11131548 -0.09164471  0.7495258
## [3,] -0.3747904 -0.3314811  0.48724807 -0.50849863  0.45647911 -0.1996520
## [4,] -0.3949123  0.2210770  0.14789147 -0.37710528 -0.76947015 -0.1212119
## [5,] -0.3956582  0.2305757 -0.42485979 -0.13992068  0.08162078  0.1431547
## [6,] -0.3834289  0.3180749 -0.47659266 -0.07501674  0.37659087 -0.1401873
## [7,] -0.3490255  0.4970255  0.55267291  0.53351836  0.13707747  0.1455350
##             [,7]
## [1,]  0.08348107
```

```
## [2,] -0.20389904
## [3,]  0.07373480
## [4,] -0.15592393
## [5,]  0.75036695
## [6,] -0.59797109
## [7,]  0.03296996
```

```
sum(q2f$values)
```

```
## [1] 7
```

The sum of the eigenvalues is equal to the dimension of the variables (i.e., `100m`, `200m`, `400m`, `800m`, `1500m`, `3000m`, `Marathon`).

3.

(a)

$$\begin{aligned}
1 &= \mathbf{P}(U) \\
&= \mathbf{P}(X=1) + \mathbf{P}(X=2) + \mathbf{P}(X=3) \\
&= 2c + 3c + 4c = 9c \\
\therefore c &= \frac{1}{9}
\end{aligned}$$

Moreover,

$$f_X(x) = \mathbf{P}(X=x) = \begin{cases} \frac{2}{9} & x=1 \\ \frac{3}{9} & x=2 \\ \frac{4}{9} & x=3 \end{cases}$$

$$f_Y(y) = \mathbf{P}(Y=y) = \begin{cases} \frac{3}{9} & y=1 \\ \frac{3}{9} & y=2 \\ \frac{2}{9} & y=3 \\ \frac{1}{9} & y=4 \end{cases}$$

(b)

$$\begin{aligned}
g(x) &= \mathbf{E}(Y \mid X=x) \\
&= 1 \times \mathbf{P}(Y=1 \mid X=x) + 2 \times \mathbf{P}(Y=2 \mid X=x) + 3 \times \mathbf{P}(Y=3 \mid X=x) + 4 \times \mathbf{P}(Y=4 \mid X=x) \\
&= \frac{\mathbf{P}(Y=1, X=x) + 2\mathbf{P}(Y=2, X=x) + 3\mathbf{P}(Y=3, X=x) + 4\mathbf{P}(Y=4, X=x)}{\mathbf{P}(X=x)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
g(1) &= \frac{\mathbf{P}(Y=1, X=1) + 2\mathbf{P}(Y=2, X=1) + 3\mathbf{P}(Y=3, X=1) + 4\mathbf{P}(Y=4, X=1)}{\mathbf{P}(X=1)} \\
&= \frac{\frac{1}{9} + \frac{2}{9}}{\frac{2}{9}} \\
&= \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
g(2) &= \frac{\mathbf{P}(Y=1, X=2) + 2\mathbf{P}(Y=2, X=2) + 3\mathbf{P}(Y=3, X=2) + 4\mathbf{P}(Y=4, X=2)}{\mathbf{P}(X=2)} \\
&= \frac{\frac{1}{9} + \frac{2}{9} + \frac{3}{9}}{\frac{3}{9}} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
g(3) &= \frac{\mathbf{P}(Y=1, X=3) + 2\mathbf{P}(Y=2, X=3) + 3\mathbf{P}(Y=3, X=3) + 4\mathbf{P}(Y=4, X=3)}{\mathbf{P}(X=3)} \\
&= \frac{\frac{1}{9} + \frac{2}{9} + \frac{3}{9} + \frac{4}{9}}{\frac{4}{9}} \\
&= \frac{5}{2}
\end{aligned}$$

(c)

$$
\begin{aligned}
&\mathbf{E}(Y^2 \mid X = x)\\
=&1 \times \mathbf{P}(Y = 1 \mid X = x) + 2^2 \times \mathbf{P}(Y = 2 \mid X = x) + 3^2 \times \mathbf{P}(Y = 3 \mid X = x) + 4^2 \times \mathbf{P}(Y = 4 \mid X = x)\\
=&\frac{\mathbf{P}(Y = 1, X = x) + 4\mathbf{P}(Y = 2, X = x) + 9\mathbf{P}(Y = 3, X = x) + 16\mathbf{P}(Y = 4, X = x)}{\mathbf{P}(X = x)}
\end{aligned}
$$

Therefore, for $X = 1$:

$$
\begin{aligned}
&\mathbf{E}(Y^2 \mid X = 1)\\
=&\frac{\mathbf{P}(Y = 1, X = 1) + 4\mathbf{P}(Y = 2, X = 1) + 9\mathbf{P}(Y = 3, X = 1) + 16\mathbf{P}(Y = 4, X = 1)}{\mathbf{P}(X = 1)}\\
=&\frac{\frac{1}{9} + \frac{4}{9}}{\frac{2}{9}}\\
=&\frac{5}{2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore Var(Y \mid X = 1) &= \mathbf{E}(Y^2 \mid X = 1) - (\mathbf{E}(Y \mid X = 1))^2\\
&= \frac{5}{2} - \left(\frac{3}{2}\right)^2\\
&= \frac{1}{4}
\end{aligned}
$$

For $X = 2$:

$$
\begin{aligned}
&\mathbf{E}(Y^2 \mid X = 2)\\
=&\frac{\mathbf{P}(Y = 1, X = 2) + 4\mathbf{P}(Y = 2, X = 2) + 9\mathbf{P}(Y = 3, X = 2) + 16\mathbf{P}(Y = 4, X = 2)}{\mathbf{P}(X = 2)}\\
=&\frac{\frac{1}{9} + \frac{4}{9} + \frac{9}{9}}{\frac{3}{9}}\\
=&\frac{14}{3}
\end{aligned}
$$

$$
\begin{aligned}
\therefore Var(Y \mid X = 2) &= \mathbf{E}(Y^2 \mid X = 2) - (\mathbf{E}(Y \mid X = 2))^2\\
&= \frac{14}{3} - (2)^2\\
&= \frac{2}{3}
\end{aligned}
$$

For $X = 3$:

$$
\begin{aligned}
&\mathbf{E}(Y^2 \mid X = 3)\\
=&\frac{\mathbf{P}(Y = 1, X = 3) + 4\mathbf{P}(Y = 2, X = 3) + 9\mathbf{P}(Y = 3, X = 3) + 16\mathbf{P}(Y = 4, X = 3)}{\mathbf{P}(X = 3)}\\
=&\frac{\frac{1}{9} + \frac{4}{9} + \frac{9}{9} + \frac{16}{9}}{\frac{4}{9}}\\
=&\frac{15}{2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore Var(Y \mid X=3) &= \mathbf{E}(Y^2 \mid X=3) - (\mathbf{E}(Y \mid X=3))^2 \\
&= \frac{15}{2} - \left(\frac{5}{2}\right)^2 \\
&= \frac{5}{4}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathbf{E}\left[\mathbf{E}(Y \mid X)\right] &= \mathbf{E}\left[g(X)\right] \\
&= g(1) \times \mathbf{P}(X=1) + g(2) \times \mathbf{P}(X=2) + g(3) \times \mathbf{P}(X=3) \\
&= \frac{3}{2} \times \frac{2}{9} + 2 \times \frac{3}{9} + \frac{5}{2} \times \frac{4}{9} \\
&= \frac{19}{9}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}(Y) =& \sum_y y f_Y(y) \\
=& 1 \times \frac{3}{9} + 2 \times \frac{3}{9} + 3 \times \frac{2}{9} + 4 \times \frac{1}{9} \\
=& \frac{19}{9}
\end{aligned}
$$

(e)

$$
\begin{aligned}
Var\left[\mathbf{E}(Y \mid X)\right] &= Var\left[g(X)\right] \\
&= (g(1) - \mathbf{E}(g(X)))^2 \times \mathbf{P}(X=1) + (g(2) - \mathbf{E}(g(X)))^2 \times \mathbf{P}(X=2) + (g(3) - \mathbf{E}(g(X)))^2 \times \mathbf{P}(X=3) \\
&= \left(\frac{3}{2} - \frac{19}{9}\right)^2 \times \frac{2}{9} + \left(2 - \frac{19}{9}\right)^2 \times \frac{3}{9} + \left(\frac{5}{2} - \frac{19}{9}\right)^2 \times \frac{4}{9} \\
&= \frac{25}{162}
\end{aligned}
$$

$$
\begin{aligned}
Var(Y) =& Var\left[\mathbf{E}(Y \mid X)\right] + \mathbf{E}\left[Var(Y \mid X)\right] \\
=& \frac{25}{162} + \frac{1}{4} \times \frac{2}{9} + \frac{2}{3} \times \frac{3}{9} + \frac{5}{4} \times \frac{4}{9} \\
=& \frac{80}{81}
\end{aligned}
$$

4.

(a)

i.

$$
\begin{aligned}
&|\mathbf{A} - \lambda \mathbf{I}| = 0 \\
\Leftrightarrow &\left| \begin{bmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{bmatrix} \right| = 0 \\
\Leftrightarrow &(1-\lambda)^2 - \rho^2 = 0 \\
\Leftrightarrow &(1-\rho-\lambda)(1+\rho-\lambda) = 0
\end{aligned}
$$

Therefore, the two eigenvalues are $\lambda_1 = 1+\rho$ and $\lambda_2 = 1-\rho$.

ii. For λ_1, if we let $\vec{v_1} = \begin{pmatrix} x \\ y \end{pmatrix}$:

$$
\begin{aligned}
\mathbf{A}\vec{v_1} &= \lambda_1 \vec{v_1} \\
\Leftrightarrow \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} (1+\rho)x \\ (1+\rho)y \end{pmatrix} \\
\Leftrightarrow \begin{pmatrix} x+\rho y \\ \rho x + y \end{pmatrix} &= \begin{pmatrix} (1+\rho)x \\ (1+\rho)y \end{pmatrix} \\
\Leftrightarrow \begin{pmatrix} \rho y \\ \rho x \end{pmatrix} &= \begin{pmatrix} \rho x \\ \rho y \end{pmatrix}
\end{aligned}
$$

The relationship revealed from above is $x = y$. Therefore, a unit-length eigenvector($\vec{v_1}$) woud be:

$$
\vec{v_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}
$$

For λ_2, if we let $\vec{v_2} = \begin{pmatrix} x \\ y \end{pmatrix}$:

$$
\begin{aligned}
\mathbf{A}\vec{v_2} &= \lambda_2 \vec{v_2} \\
\Leftrightarrow \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} (1-\rho)x \\ (1-\rho)y \end{pmatrix} \\
\Leftrightarrow \begin{pmatrix} x+\rho y \\ \rho x + y \end{pmatrix} &= \begin{pmatrix} (1-\rho)x \\ (1-\rho)y \end{pmatrix} \\
\Leftrightarrow \begin{pmatrix} \rho y \\ \rho x \end{pmatrix} &= \begin{pmatrix} -\rho x \\ -\rho y \end{pmatrix}
\end{aligned}
$$

The relationship revealed from above is $y = -x$. Therefore, a unit-length eigenvector($\vec{v_2}$) woud be:

$$
\vec{v_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}
$$

Now to check $\vec{v_1}$ and $\vec{v_2}$ are orthogonal:

$$
\vec{v_1} \cdot \vec{v_2} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \left(-\frac{1}{\sqrt{2}}\right) = 0
$$

iii.

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^T$$

iv.

$$\begin{aligned} \mathbf{A}^{-1} &= (V\Lambda V^T)^{-1} \\ &= (V^T)^{-1}(V\Lambda)^{-1} \\ &= V\Lambda^{-1}V^{-1} \\ &(\because V^T = V^{-1}) \\ &= V\Lambda^{-1}V^T \end{aligned}$$

v. If we let $\mathbf{X} = \Lambda^{\frac{1}{2}}$ such that $\mathbf{XX} = \Lambda$:

$$\begin{aligned} (V\mathbf{X}V^T)(V\mathbf{X}V^T) &= V\mathbf{X}V^TV\mathbf{X}V^T \\ &= V\mathbf{X}(V^TV)\mathbf{X}V^T \\ &= V\mathbf{X}I\mathbf{X}V^T \\ &(\because V^T = V^{-1}) \\ &= V(\mathbf{XX})V^T \\ &= V\Lambda V^T \\ &= \mathbf{A} \end{aligned}$$

Therefore,

$$\mathbf{R} = \mathbf{A}^{\frac{1}{2}} = V\Lambda^{\frac{1}{2}}V^T$$

(b)

Proof by contradiction)

Let's assume that there is a negative eigenvalue λ^* of the covariance matrix Σ. Then, by definition, there would be a corresponding eigenvector $\vec{v}^*$ where

$$\Sigma\ \vec{v}^* = \lambda^*\ \vec{v}^*$$

Here, we also know that the covariance matrix is positive semi-definite.

Therefore since $\vec{v}^*$ is a p vector, by definition of a positive semi-definite matrix:

$$(\vec{v}^*)^T\Sigma\vec{v}^* \geq 0$$

However, also using the relationship between eigenvector and eigenvalue:

$$\begin{aligned} (\vec{v}^*)^T\Sigma\vec{v}^* &= (\vec{v}^*)^T\lambda^*\vec{v}^* \\ &= \lambda^*(\vec{v}^*)^T\vec{v}^* \\ &(\because \lambda^* \text{ is scalar}) \end{aligned}$$

Here, $(\vec{v}^*)^T\vec{v}^*$ is the inner-product $\vec{v}^* \cdot \vec{v}^*$.

Knowing that $\vec{v}^*$ is an eigenvector and therefore cannot be $\vec{0}$, $(\vec{v}^*)^T\vec{v}^*$ would be strictly positive.

Moreover,

$$\lambda^*(\vec{v}^*)^T\vec{v}^* < 0$$

since $(\vec{v}^*)^T\vec{v}^* > 0$ and $\lambda^* < 0$.

Therefore,

$$(\vec{v}^*)^T\Sigma\vec{v}^* < 0$$

and we have reached a contradiction with the definition of positive semi-definiteness of Σ.

Thus, all eigenvalues of a covariance matrix has to be non-negative.

$$Q.E.D.$$

(c)

First, since (X_1, Y_1) and (X_2, Y_2) are continuous and independent:

$$\mathbf{P}[X_1 = X_2] = 0$$

$$\mathbf{P}[Y_1 = Y_2] = 0$$

and therefore,

$$\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) = 0] = 0$$

Knowing from common sense that:

$$\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] + \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) = 0] + \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) < 0] = 1$$

we may also say that

$$\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) < 0] = 1 - \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0]$$

since $\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) = 0] = 0$.

Therefore,

$$\begin{aligned}
\tau &= \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) < 0] \\
&= \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - (1 - \mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0]) \\
&= 2\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - 1
\end{aligned}$$

Now let's look further into $\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0]$.

$$
\begin{aligned}
\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] =& \mathbf{P}[(X_1 - X_2) > 0, (Y_1 - Y_2) > 0] + \mathbf{P}[(X_1 - X_2) < 0, (Y_1 - Y_2) < 0] \\
=& \mathbf{P}[X_1 > X_2, Y_1 > Y_2] + \mathbf{P}[X_1 < X_2, Y_1 < Y_2]
\end{aligned}
$$

Here, we also know that (X_1, Y_1) and (X_2, Y_2) follow the same distribution. Therefore, without loss of generality:

$$\mathbf{P}[X_1 > X_2, Y_1 > Y_2] = \mathbf{P}[X_1 < X_2, Y_1 < Y_2]$$

Moreover,

$$\mathbf{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] = 2\mathbf{P}[X_1 < X_2, Y_1 < Y_2]$$

and

$$\tau = 4\mathbf{P}[X_1 < X_2, Y_1 < Y_2] - 1$$

Now to express $\mathbf{P}[X_1 < X_2, Y_1 < Y_2]$ in integral form, if we let $f_{XY}(x, y)$ be the joint pdf of (X_i, Y_i), and $F(x, y)$ as given in the question,

$$
\begin{aligned}
\mathbf{P}[X_1 < X_2, Y_1 < Y_2 \mid X_2 = x_2, Y_2 = y_2] &= \int_{-\infty}^{x_2} \int_{-\infty}^{y_2} f_{XY}(x, y) dy dx \\
&= F(x_2, y_2)
\end{aligned}
$$

Therefore, the joint probability $\mathbf{P}[X_1 < X_2, Y_1 < Y_2]$ becomes

$$\mathbf{P}[X_1 < X_2, Y_1 < Y_2] = \iint_{\mathbf{R}^2} F(x, y) dF(x, y)$$

Moreover,

$$\tau = 4\mathbf{P}[X_1 < X_2, Y_1 < Y_2] - 1 = 4 \iint_{\mathbf{R}^2} F(x, y) dF(x, y) - 1$$

Q.E.D.

(d)

i.

- $k \times r$

ii.

$$c_{i,j} = \sum_{m=1}^{p} \sum_{n=1}^{p} a_{i,n} x_{n,m} b_{m,j}$$

iii. If we first look at $\mathbf{E}(c_{i,j})$, since A and B are scalar matrices:

$$
\begin{aligned}
\mathbf{E}(c_{i,j}) &= \mathbf{E}\left[\sum_{m=1}^{p}\sum_{n=1}^{p} a_{i,n}x_{n,m}b_{m,j}\right] \\
&= \sum_{m=1}^{p}\sum_{n=1}^{p} \mathbf{E}\left[a_{i,n}x_{n,m}b_{m,j}\right] \\
&(\because \text{Linearity of Expectation}) \\
&= \sum_{m=1}^{p}\sum_{n=1}^{p} a_{i,n}\mathbf{E}(x_{n,m})b_{m,j} \\
&(\because A \text{ and } B \text{ are scalar matrices})
\end{aligned}
$$

Here, $\mathbf{E}(x_{n,m})$ is the (n, m)th entry of $\mathbf{E}(\mathbf{X})$ and $\mathbf{E}(c_{i,j})$ is the is the (i, j)th entry of $\mathbf{E}(\mathbf{C})$.

Therefore, we may conclude that the three matrices would have the relationship:

$$\mathbf{E}(\mathbf{C}) = A\mathbf{E}(\mathbf{X})B$$

$$Q.E.D.$$

5.

```r
sigma=matrix(c(2,-1,1,-1,4,0,1,0,3),3,3)
eigen(sigma)
```

```
## eigen() decomposition
## $values
## [1] 4.532089 3.347296 1.120615
##
## $vectors
##            [,1]      [,2]       [,3]
## [1,]  0.4490988 0.2931284 -0.8440296
## [2,] -0.8440296 0.4490988 -0.2931284
## [3,]  0.2931284 0.8440296  0.4490988
```

(a)

$\lambda_1 \approx 4.53$, $\lambda_2 \approx 3.35$, $\lambda_3 \approx 1.12$

(b)

$$Y_1 = \mathbf{a}_1' X = 0.4490988X_1 + -0.8440296X_2 + 0.2931284X_3$$

$$Y_2 = \mathbf{a}_2' X = 0.2931284X_1 + 0.4490988X_2 + 0.8440296X_3$$

$$Y_3 = \mathbf{a}_3' X = -0.8440296X_1 + -0.2931284X_2 + 0.4490988X_3$$

(c)

$$\begin{aligned} Var(Y_1) &= (0.4490988)^2 \cdot Var(X_1) + (-0.8440296)^2 \cdot Var(X_2) + (0.2931284)^2 \cdot Var(X_3) \\ &+ 2 \cdot (0.4490988) \cdot (-0.8440296) \cdot Cov(X_1, X_2) + 2 \cdot (0.4490988) \cdot (0.2931284) \cdot Cov(X_1, X_3) \\ &+ 2 \cdot (-0.8440296) \cdot (0.2931284) \cdot Cov(X_2, X_3) \\ &= 4.5320889 \end{aligned}$$

$Var(Y_1)$ has the same value as λ_1.

6.

(a)

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{c}{(1+x+y)^3} dxdy = 1 \\
\Leftrightarrow &\int_{0}^{\infty}\int_{0}^{\infty} \frac{c}{(1+x+y)^3} dxdy = 1 \\
\Leftrightarrow &\int_{0}^{\infty} \left[-\frac{c}{2(1+x+y)^2}\right]_0^{\infty} dy = 1 \\
\Leftrightarrow &\int_{0}^{\infty} \frac{c}{2(1+y)^2} dy = 1 \\
\Leftrightarrow &\left[-\frac{c}{2(1+y)}\right]_0^{\infty} = 1 \\
\Leftrightarrow &\frac{c}{2} = 1 \\
&\therefore c = 2
\end{aligned}
$$

(b)

For $x > 0$:

$$
\begin{aligned}
\mathbf{P}(X \leq x) &= \int_0^x \int_0^\infty \frac{2}{(1+x+y)^3} dydx \\
&= \int_0^x \left[-\frac{1}{(1+x+y)^2}\right]_0^\infty dx \\
&= \int_0^x \frac{1}{(1+x)^2} dx
\end{aligned}
$$

Therefore, the marginal density of X can be written as:

$$
f_X(x) = \begin{cases} \frac{1}{(1+x)^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}
$$

(c)

For $x, y > 0$:

$$
\begin{aligned}
f_{Y|X}(y \mid x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\
&= \frac{\frac{2}{(1+x+y)^3}}{\frac{1}{(1+x)^2}} \\
&= \frac{2(1+x)^2}{(1+x+y)^3}
\end{aligned}
$$

Therefore, the conditional density of Y for $x > 0$ can be written as:

$$
f_{Y|X}(y \mid x) = \begin{cases} \frac{2(1+x)^2}{(1+x+y)^3} & y > 0 \\ 0 & \text{otherwise} \end{cases}
$$

(d)

i. Similar to (b), for $y > 0$:

$$\begin{aligned}
\mathbf{P}(Y \leq y) &= \int_0^y \int_0^\infty \frac{2}{(1+x+y)^3} dx dy \\
&= \int_0^y \left[-\frac{1}{(1+x+y)^2} \right]_0^\infty dy \\
&= \int_0^y \frac{1}{(1+y)^2} dy
\end{aligned}$$

Therefore, the marginal density of Y can be written as:

$$f_Y(y) = \begin{cases} \frac{1}{(1+y)^2} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

From this, we can derive the expectation as

$$\begin{aligned}
\mathbf{E}(Y) &= \int_Y y f_Y(y) dy \\
&= \int_0^\infty \frac{y}{(1+y)^2} dy \\
&= \int_0^\infty \left(\frac{1}{1+y} - \frac{1}{(1+y)^2} \right) dy \\
&= \int_0^\infty \frac{1}{1+y} dy - \int_0^\infty \frac{1}{(1+y)^2} dy \\
&= [\log(1+y)]_0^\infty - \left[-\frac{1}{1+y} \right]_0^\infty \\
&= \lim_{y \to \infty} \log(1+y) - 1 \\
&\to \infty
\end{aligned}$$

The expected value of Y is unbounded as goes to infinity.

ii.

$$\begin{aligned}
g(x) &= \mathbf{E}(Y \mid X = x) \\
&= \int_Y y f_{Y|X}(y \mid x) dy \\
&= \int_0^\infty \frac{2y(1+x)^2}{(1+x+y)^3} dy \\
&= 2(1+x)^2 \int_0^\infty \frac{y}{(1+x+y)^3} dy
\end{aligned}$$

Here, let

$$h(y) = -\frac{y}{2(1+x+y)^2}$$

then,

$$h'(y) = \frac{y}{(1+x+y)^3} - \frac{1}{2(1+x+y)^2}$$

which would be equivalent to saying:

$$\frac{y}{(1+x+y)^3} = h'(y) + \frac{1}{2(1+x+y)^2}$$

and in differential equation form:

$$\frac{y}{(1+x+y)^3}dy = h(y) + \frac{1}{2(1+x+y)^2}dy$$

Therefore, in other words,

$$\int_Y \frac{y}{(1+x+y)^3}dy = [h(y)]_Y + \int_Y \frac{1}{2(1+x+y)^2}dy$$

Moreover, specifically to our case,

$$\begin{aligned}
&\int_0^\infty \frac{y}{(1+x+y)^3}dy \\
=&\left[-\frac{y}{2(1+x+y)^2}\right]_0^\infty + \int_0^\infty \frac{1}{2(1+x+y)^2}dy \\
=&\int_0^\infty \frac{1}{2(1+x+y)^2}dy \\
&\left(\because \lim_{y\to\infty} -\frac{y}{2(1+x+y)^2} = 0\right) \\
=&\left[-\frac{1}{2(1+x+y)}\right]_0^\infty \\
=&\frac{1}{2(1+x)}
\end{aligned}$$

We can use the above finding above to derive the conditional expectation as below:

$$\begin{aligned}
g(x) &= 2(1+x)^2 \int_0^\infty \frac{y}{(1+x+y)^3}dy \\
&= 2(1+x)^2 \frac{1}{2(1+x)} \\
&= 1+x
\end{aligned}$$