P-set0 **For your personal use in this course only. Do not circulate or post.**

1. (*Typing math formula*)

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}}$$

where $\bar{x} = \sum_{i=1}^n x_i/n$, $\bar{y} = \sum_{i=1}^n y_i/n$.

2. (*Producing R plot*) Omitted.

3. (*Matrix operation on rows and columns*)

(a) What happens to A when A is multiplied by B from the left?
For $i = 1, 2, 3$, the ith row of A is multiply by the ith diagonal element b_{ii} of the diagonal matrix B.

What happens to A when A is multiplied by B from the right?
For $i = 1, 2, 3$, the ith column of A is multiply by the ith diagonal element of the diagonal matrix B.

What happens to A when A is multiplied by E from the left?
The 2nd and 3rd rows of A are exchanged.
What happens to A when A is multiplied by E from the right?
The 2nd and 3rd columns of A are exchanged.

(b) $v = [7\ 3\ 24]^T$ is a column vector.
v as a linear combination of the a_i's: $v = 2a_1 - a_2 + 3a_3$. (Formally by solving $Ax = v$ for x.)
v as a linear combination of the b_i's. $v = 7b_1 + b_2 - 12b_3$.
v as a linear combination of the e_i's: $v = 7e_1 + 24e_2 + 3e_3$.

4. (*Mathematical induction, sigma summation notation, derivation type of proofs*)

(a) Use mathematical induction to prove the equation $\sum_{k=1}^n k^3 = (\sum_{k=1}^n k)^2$.
For $n = 1$: $\sum_{k=1}^n k^3 = 1^3 = 1^2 = (\sum_{k=1}^n k)^2$.
Assume the equation is true for an integer $n \geq 1$. For $n+1$, by the assumption for n and $\sum_{k=1}^n k = n(n+1)/2$,

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 = \left(\sum_{k=1}^n k\right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2 \frac{n^2 + 4(n+1)}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2 = \left(\sum_{k=1}^{n+1} k\right)^2\end{aligned}$$

The equation is true for $n + 1$. By the arbitrariness of n, this proves the equation by mathematical induction.

(b) Show $AB = BA$ not always true by constructing a counterexample (where A, B are 4×4 matrices).

5. (*Function of random variables*) Random variable U is uniform on $[-1, 1]$, $X = U^2$.

(a) By the definition of U, its density function is

$$f_U(u) = \frac{1}{2}\mathbf{1}_{[-1,1]}(u) = \begin{cases} \frac{1}{2}, & u \in [-1,1] \\ 0, & u \in (-\infty - 1) \cup (1, \infty) \end{cases}$$

For $x \in (-\infty, 0]$: $F_X(x) = P(X \leq x) = P(U^2 \leq x) = 0$, thus the density $f_X(x) = F'_X(x) = 0$ on $(-\infty, 0]$.
For $x \in [1, \infty)$: $1 - F_X(x) = P(X \geq x) = P(U^2 \geq x) = 0$. thus $F_X(x) = 1, f_X(x) = 0$ on $[1, \infty)$.

Note: Since $P(X = x) = P(U^2 = x) = 0$ for any $x \in \mathbb{R}$, therefore the end point $x = 0$ and $x = 1$ can be included in the region of $f_X(x) = 0$ in the above.

In the following we only need to derive the density $f_X(x)$ for $x \in (0, 1)$.

Method 1

For $x \in (0, 1)$,

$$F_X(x) = P(U^2 \leq x) = P(U \in [-\sqrt{x}, \sqrt{x}\,]) = \frac{1}{2}2\sqrt{x} = \sqrt{x}$$

Therefore, density

$$f_X(x) = F'_X(x) = \frac{1}{2\sqrt{x}}$$

for $x \in (0, 1)$, and $f_X(x) = 0$ otherwise as derived in the above.

Method 2

For $x \in (0, 1)$, $x = g(u) = u^2$, $g'(u) = 2u$. We only need to consider $u \in (0, 1)$.

$$u = u(x) = \begin{cases} \sqrt{x} & u \in (0,1) \\ -\sqrt{x} & u \in (-1, 0) \end{cases}, \qquad \frac{du}{dx} = \begin{cases} \frac{1}{2\sqrt{x}} & u \in (0,1) \\ -\frac{1}{2\sqrt{x}} & u \in (-1,0) \end{cases}$$

Use the change of variable formula for densities (on the region $1/g'(u)$ exists), then express in terms of x:

$$f_X(x) = \sum_{u^2=x} \frac{f_U(u)}{|g'(u)|} = \sum_{u^2=x} \frac{f_U(u)}{|2u|} = \frac{f_U(-\sqrt{x})}{|2(-\sqrt{x})|} + \frac{f_U(\sqrt{x})}{|2\sqrt{x}|} = \frac{1}{2\sqrt{x}}$$

for $x \in (0, 1)$, and we have derived that $f_X(x) = 0$ otherwise.

(b)

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{x=0}^{x=1} x\frac{1}{2\sqrt{x}}dx = \int_{x=0}^{x=1} \frac{\sqrt{x}}{2}\,dx = \frac{1}{2}\cdot\frac{2}{3}x^{3/2}\Big|_{x=0}^{x=1} = \frac{1}{3}$$