THE UNIVERSITY OF CHICAGO

FINM 32000: Numerical Methods

TA Review Session – Chintan Singh

Spring 2023

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Topics

1. Tree Method – Review
   A. Non-Combining Binomial Tree
   B. Combining Binomial Tree
   C. Combining Binomial Tree Delta Xd = Delta Xu = Delta X
   D. Trinomial Tree
2. Option Types –
   A. European,
   B. American,
   C. Barrier Option (Up and Out European)

1. Tree Method – Review

Approximate the risk-neutral dynamics of a diffusion process (such as GBM) using a discrete model with a finite number of branches from each state.

L1.40

- Diffusion is

$$\mathrm{d}S_t = R_{grow} S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

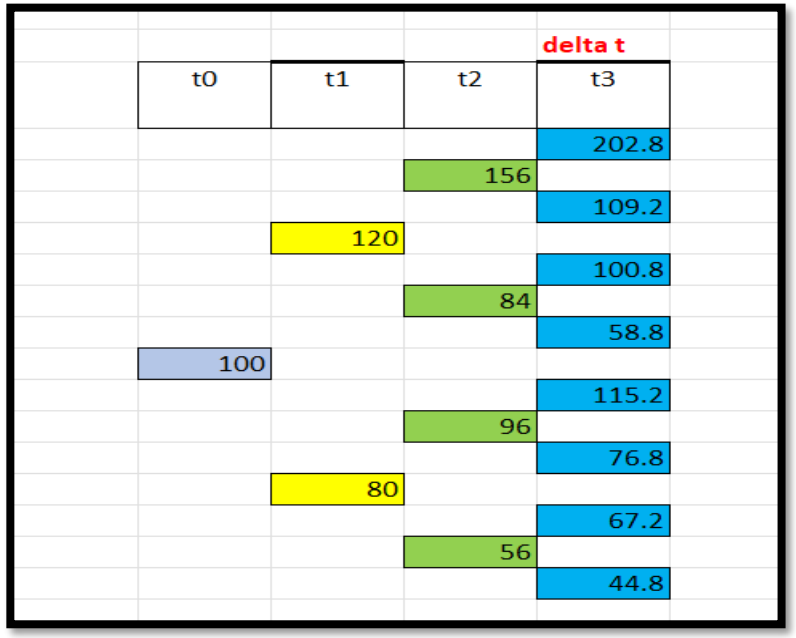
where W is $\mathbb{P}$-BM. So $X := \log S$ has dynamics

$$\mathrm{d}X_t = \nu \mathrm{d}t + \sigma \mathrm{d}W_t$$

where $\nu := R_{grow} - \sigma^2/2$.

1. Tree Method – Review (cont.)

A. **Non-Combining Binomial Tree**



S: Stock Price Model

Non-Combining Binomial Tree

- Different Xd and Xu at different time points
- Exponentially Growing Number of Price points

1. Tree Method – Review (cont.)

B. **Combining Binomial Tree Delta**

Xd != Delta Xu

			delta t							
t0	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10
										619.17
									515.98	
								429.98		412.78
							358.32		343.99	
						298.60		286.65		210.69
					248.83		238.88		175.58	
				207.36		199.07		146.31		122.90
			172.80		165.89		121.93		102.42	
		144.00		138.24		101.61		85.35		71.69
	120.00		115.20		84.67		71.12		59.74	
100.00		96.00		70.56		59.27		49.79		41.82
	70.00		58.80		49.39		41.49		34.85	
		49.00		41.16		34.57		29.04		24.40
			34.30		28.81		24.20		20.33	
				24.01		20.17		16.94		14.23
					16.81		14.12		11.86	
						11.76		9.88		8.30
							8.24		6.92	
								5.76		4.84
									4.04	
										2.82

S: Stock Price Model

Combining Binomial Tree

Xd != Xu

(Xd = 1.2, Xu = 0.7)

- Linearly Growing number of Price Points (Numerical Advantage)
- Still Need to maintain a different grid of Price points at each time point

1. Tree Method – Review (cont.)

B. **Combining Binomial Tree** **Delta Xd != Delta Xu**

L1.41

Match the means and variances

- Let $\Delta t := T/N$ where N is number of time steps.
- Now choose $p, \Delta x_u, \Delta x_d$ such that diffusion and tree agree on mean and variance of the increments. In the diffusion,

$$X_{t+\Delta t} - X_t = \int_t^{t+\Delta t} \nu \mathrm{d}s + \int_t^{t+\Delta t} \sigma \mathrm{d}W_s = \nu \Delta t + \sigma \Delta W.$$

So

$$\nu \Delta t = \mathbb{E}_t(X_{t+\Delta t} - X_t) = p\Delta x_u + (1-p)(-\Delta x_d)$$

$$\sigma^2 \Delta t = \mathrm{Var}_t(X_{t+\Delta t} - X_t) = p(\Delta x_u)^2 + (1-p)(-\Delta x_d)^2 - (\nu \Delta t)^2$$

Two equations and three unknowns: p, Δx_u, Δx_d. Can impose another condition (such as $\Delta x_u = \Delta x_d$ or $p = 1/2$) and solve. Then price the option in a tree with these parameters.

1. Tree Method – Review (cont.)

C. **Combining Binomial Tree Delta Xd = Delta Xu = Delta X**

	delta X	1.25	0.8									
				delta t								
	time >	t0	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10
	-10											931.32
	-9										745.06	
	-8									596.05		596.05
	-7								476.84		476.84	
	-6							381.47		381.47		381.47
	-5						305.18		305.18		305.18	
	-4					244.14		244.14		244.14		244.14
delta x	-3				195.31		195.31		195.31		195.31	
	-2			156.25		156.25		156.25		156.25		156.25
	-1		125.00		125.00		125.00		125.00		125.00	
	0	100.00		100.00		100.00		100.00		100.00		100.00
	1		80.00		80.00		80.00		80.00		80.00	
	2			64.00		64.00		64.00		64.00		64.00
	3				51.20		51.20		51.20		51.20	
	4					40.96		40.96		40.96		40.96
	5						32.77		32.77		32.77	
	6							26.21		26.21		26.21
	7								20.97		20.97	
	8									16.78		16.78
	9										13.42	
	10											10.74

S: Stock Price Model

Combining Binomial Tree

Delta Xd = Delta Xu = Delta X = 1.25

- Linearly Growing number of Price Points (Numerical Advantage)
- Repeating Price points (Numerical Advantage)

Not Flexible Enough – L1.42

1. Tree Method – Review (cont.)

D. Trinomial Trees (combining)

	delta X	1.25	0.8									
				delta t								
	time >	t0	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10
	-10											931.32
	-9										745.06	745.06
	-8									596.05	596.05	596.05
	-7								476.84	476.84	476.84	476.84
	-6							381.47	381.47	381.47	381.47	381.47
	-5						305.18	305.18	305.18	305.18	305.18	305.18
	-4					244.14	244.14	244.14	244.14	244.14	244.14	244.14
delta x	-3				195.31	195.31	195.31	195.31	195.31	195.31	195.31	195.31
	-2			156.25	156.25	156.25	156.25	156.25	156.25	156.25	156.25	156.25
	-1		125.00	125.00	125.00	125.00	125.00	125.00	125.00	125.00	125.00	125.00
	0	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	1		80.00	80.00	80.00	80.00	80.00	80.00	80.00	80.00	80.00	80.00
	2			64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00
	3				51.20	51.20	51.20	51.20	51.20	51.20	51.20	51.20
	4					40.96	40.96	40.96	40.96	40.96	40.96	40.96
	5						32.77	32.77	32.77	32.77	32.77	32.77
	6							26.21	26.21	26.21	26.21	26.21
	7								20.97	20.97	20.97	20.97
	8									16.78	16.78	16.78
	9										13.42	13.42
	10											10.74

S: Stock Price Model

- Linearly Growing number of Price Points (Numerical Advantage)
- Repeating Price points (Numerical Advantage)
  - **Notice the price points at t(n-1) is a subset of prices at t(n).**
  - **This would be used extensively while coding such trees in this course.**

1. Tree Method – Review (cont.)

D. Trinomial Trees (combining)

	delta X	1.25	0.8									
				delta t								
	time >	t0	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10
	-10											931.32
	-9										745.06	745.06
	-8									596.05	596.05	596.05
	-7								476.84	476.84	476.84	476.84
	-6							381.47	381.47	381.47	381.47	381.47
	-5						305.18	305.18	305.18	305.18	305.18	305.18
	-4					244.14	244.14	244.14	244.14	244.14	244.14	244.14
delta x	-3				195.31	195.31	195.31	195.31	195.31	195.31	195.31	195.31
	-2			156.25	156.25	156.25	156.25	156.25	156.25	156.25	156.25	156.25
	-1		125.00	125.00	125.00	125.00	125.00	125.00	125.00	125.00	125.00	125.00
	0	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	1		80.00	80.00	80.00	80.00	80.00	80.00	80.00	80.00	80.00	80.00
	2			64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00
	3				51.20	51.20	51.20	51.20	51.20	51.20	51.20	51.20
	4					40.96	40.96	40.96	40.96	40.96	40.96	40.96
	5						32.77	32.77	32.77	32.77	32.77	32.77
	6							26.21	26.21	26.21	26.21	26.21
	7								20.97	20.97	20.97	20.97
	8									16.78	16.78	16.78
	9										13.42	13.42
	10											10.74

S: Stock Price Model

- Closed solution to Pu, Pd, Pm for a fixed Delta x and Delta t **(L 1.45)**
- Delta t = T / N
- Choose Delta x using the following guideline (L 2.10)

Two guidelines: to make local discretization error small, we want

$$\Delta x \approx \sigma_{avg}\sqrt{3\Delta t}$$

but for stability reasons, we want

$$\Delta x \geq \sigma_{max}\sqrt{\Delta t}$$

So we can let $\Delta x = \max(\sigma_{avg}\sqrt{3\Delta t}, \sigma_{max}\sqrt{\Delta t})$

2. Option Pricing Using Trinomial Tree

A. European Put Option K = 110, Expiring at t10

Assumptions >	Pd =	0.4	Pm=	0.25	Pu =	0.35	K =	110	r =	0		
				delta t								
	time >	t0	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10
	-10											0.00
	-9										0.00	0.00
	-8									0.00	0.00	0.00
	-7								0.00	0.00	0.00	0.00
	-6							0.00	0.00	0.00	0.00	0.00
	-5						0.05	0.00	0.00	0.00	0.00	0.00
	-4					0.59	0.35	0.15	0.00	0.00	0.00	0.00
delta x	-3				2.32	1.85	1.37	0.88	0.43	0.00	0.00	0.00
	-2			5.87	5.29	4.64	3.92	3.12	2.21	1.23	0.00	0.00
	-1		11.44	10.92	10.34	9.68	8.90	7.98	6.85	5.43	3.50	0.00
	0	18.78	18.49	18.16	17.78	17.33	16.80	16.16	15.38	14.31	13.00	10.00
	1		27.39	27.38	27.37	27.36	27.34	27.35	27.34	27.53	27.60	30.00
	2			37.94	38.37	38.85	39.43	40.11	41.03	42.10	44.08	46.00
	3				49.72	50.61	51.60	52.76	54.05	55.68	57.26	58.80
	4					61.51	62.67	63.90	65.24	66.55	67.81	69.04
	5						72.01	73.12	74.19	75.24	76.25	77.23
	6							80.50	81.35	82.19	83.00	83.79
	7								87.08	87.75	88.40	89.03
	8									92.20	92.72	93.22
	9										96.18	96.58
	10											99.26

- Value at t10 is deterministic
- Discount back at each time step using Pd, Pm and Pu

C: Option Value Model

2. Option Pricing Using Trinomial Tree

B. American Put Option K = 110, Expiring at t10

Assumptions >	Pd = 0.4		Pm= 0.25		Pu = 0.35		K = 110		r = 0			
			delta t									
	time >	t0	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10
	-10											0.00
	-9										0.00	0.00
	-8									0.00	0.00	0.00
	-7								0.00	0.00	0.00	0.00
	-6							0.00	0.00	0.00	0.00	0.00
	-5						0.05	0.00	0.00	0.00	0.00	0.00
	-4					0.60	0.35	0.15	0.00	0.00	0.00	0.00
delta x	-3				2.44	1.93	1.41	0.88	0.43	0.00	0.00	0.00
	-2			6.33	5.65	4.92	4.11	3.22	2.21	1.23	0.00	0.00
	-1		12.71	12.01	11.26	10.43	9.51	8.43	7.15	5.43	3.50	0.00
	0	21.65	21.10	20.49	19.83	19.10	18.30	17.47	16.46	15.15	13.00	10.00
	1		32.27	31.91	31.51	31.08	30.64	30.18	30.00	30.00	30.00	30.00
	2			46.00	46.00	46.00	46.00	46.00	46.00	46.00	46.00	46.00
	3				58.80	58.80	58.80	58.80	58.80	58.80	58.80	58.80
	4					69.04	69.04	69.04	69.04	69.04	69.04	69.04
	5						77.23	77.23	77.23	77.23	77.23	77.23
	6							83.79	83.79	83.79	83.79	83.79
	7								89.03	89.03	89.03	89.03
	8									93.22	93.22	93.22
	9										96.58	96.58
	10											99.26

C: Option Value Model

- Value at t10 is deterministic
- At each point before t10, the value is max (European Call value, Intrinsic Value)

This Row Corresponds to Price = 80

2. Option Pricing Using Trinomial Tree

C. Up and Out European Put Option K = 110, H = 200 (monitored here at each time point), Expiring at t10

	delta X	1.25	0.8									
				delta t								
	time >	t0	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10	
	-10											931.32	
	-9										745.06	745.06	
	-8									596.05	596.05	596.05	
	-7								476.84	476.84	476.84	476.84	
	-6							381.47	381.47	381.47	381.47	381.47	
	-5						305.18	305.18	305.18	305.18	305.18	305.18	
	-4					244.14	244.14	244.14	244.14	244.14	244.14	244.14	H = 200
delta x	-3				195.31	195.31	195.31	195.31	195.31	195.31	195.31	195.31	
	-2			156.25	156.25	156.25	156.25	156.25	156.25	156.25	156.25	156.25	
	-1		125.00	125.00	125.00	125.00	125.00	125.00	125.00	125.00	125.00	125.00	
	0	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	
	1		80.00	80.00	80.00	80.00	80.00	80.00	80.00	80.00	80.00	80.00	
	2			64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	64.00	
	3				51.20	51.20	51.20	51.20	51.20	51.20	51.20	51.20	
	4					40.96	40.96	40.96	40.96	40.96	40.96	40.96	
	5						32.77	32.77	32.77	32.77	32.77	32.77	
	6							26.21	26.21	26.21	26.21	26.21	
	7								20.97	20.97	20.97	20.97	
	8									16.78	16.78	16.78	
	9										13.42	13.42	
	10											10.74	

S: Stock Price Model

- Same Price Model as Before

2. Option Pricing Using Trinomial Tree

C. Up and Out European Put Option K = 110, H = 200 (monitored here at each time point), Expiring at t10

Assumptions >	Pd = 0.4		Pm= 0.25		Pu = 0.35		K = 110		r = 0				
				delta t									
	time >	t0	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10	
	-10											0.00	
	-9										0.00	0.00	
	-8									0.00	0.00	0.00	
	-7								0.00	0.00	0.00	0.00	
	-6							0.00	0.00	0.00	0.00	0.00	
	-5						0.00	0.00	0.00	0.00	0.00	0.00	
	-4					**0.00**	**0.00**	**0.00**	0.00	0.00	0.00	0.00	H = 200
delta x	-3				2.04	1.70	1.31	0.88	0.43	0.00	0.00	0.00	
	-2			5.74	5.22	4.62	3.92	3.12	2.21	1.23	0.00	0.00	
	-1		11.37	10.89	10.33	9.68	8.90	7.98	6.85	5.43	3.50	0.00	
	0	18.76	18.48	18.16	17.78	17.33	16.80	16.16	15.38	14.31	13.00	10.00	
	1		27.39	27.38	27.37	27.36	27.34	27.35	27.34	27.53	27.60	30.00	
	2			37.94	38.37	38.85	39.43	40.11	41.03	42.10	44.08	46.00	
	3				49.72	50.61	51.60	52.76	54.05	55.68	57.26	58.80	
	4					61.51	62.67	63.90	65.24	66.55	67.81	69.04	
	5						72.01	73.12	74.19	75.24	76.25	77.23	
	6							80.50	81.35	82.19	83.00	83.79	
	7								87.08	87.75	88.40	89.03	
	8									92.20	92.72	93.22	
	9										96.18	96.58	
	10											99.26	

Any Value from nodes above the Knockout Threshold goes to Zero

C: Option Value Model