FINM 32000: More Practice Final Exam Questions

May 2023

Problem 5

Let f be a smooth function mapping the underlying x to the price $f(x)$ of some contract.

(a) For general $x \in \mathbb{R}$ and $h \to 0$, find the second-order (so it includes zeroth, first, and second order terms) Taylor approximation of $f(x+h)$ in terms of $f(x)$, $f'(x)$, $f''(x)$, and h. Find the second order Taylor appoximation of $f(x+2h)$ in terms of $f(x)$, $f'(x)$, $f''(x)$, and h.

(b) Find the linear combination of $f(x)$, $f(x+h)$, and $f(x+2h)$ that produces a second-order accurate approximation of $f'(x)$. Specifically, find a, b, c such that

$$\frac{af(x) + bf(x+h) + cf(x+2h)}{h} \approx f'(x) \qquad (*)$$

where the error is (you don't need to prove this:) bounded by h^2 times a constant as $h \to 0$.

Hint:

$$f'(x) = 0 \times f(x) + 1 \times f'(x) + 0 \times f''(x)$$

Now compare coefficients.

(c) Let $C(X,t)$ be the time-t value of some contract, given time-t underlying price X. Here is a table of values of $C(X,t)$.

X=20	1.62	1.39
X=15	1.40	1.20
X=10	1.24	1.03
	t=0	t=0.1

Given underlying $X_0 = 10$, find the time-0 delta of the contract, using a second-order accurate finite difference approximation.

Given underlying $X_0 = 15$, find the time-0 gamma of the contract, the time-0 theta of the contract, and the time-0 *color* of the contract, where

$$\text{Color} = \frac{\partial^3 C}{\partial t \partial x^2} = \frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{\partial}{\partial x}C$$

(Your t-derivative calculations do not need to be second-order accurate.)

Problem 6

Let $Y = W \times X \times Z$ where W and X and Z are independently and identically distributed random variables, each one having 50% probability of being 0, and 50% probability of being 1.

(This is a model of the Presidential election contracts[1] from Fall quarter, but the price data from Fall quarter are not consistent with the probability assumptions here, so I advise you to treat this as a separate problem.)

Suppose that someone (who fails to see that $\mathbb{E}Y$ can be directly calculated) decides to calculate $\mathbb{E}Y$ by Monte Carlo, by simulating (W, X, Z) and multiplying them together to simulate Y. Suppose he runs M simulations of Y generated in this way, in all parts (a,b,c).

(a) What is the variance of the ordinary Monte Carlo estimate?

(b) Suppose that he uses X as a control variate. What is the optimal coefficient β to multiply the control variate? What is the variance of the resulting control variate estimate?

(c) Instead of using a control variate, suppose that you use antithetic variates to estimate $\mathbb{E}Y$, by simulating pairs $(Y_m, \tilde{Y}_m)$ for $m = 1, \ldots, M$, where

$$Y_m := W_m X_m Z_m$$

and the antithetic variate is defined by

$$\tilde{Y}_m := (1 - W_m)(1 - X_m)(1 - Z_m)$$

What is the variance of your antithetic variate estimate of $\mathbb{E}Y$?

Problem 7

Assume a constant interest rate r. All options in this problem are European-style. Let $C(K)$ denote the time-0 price of a T-expiry K-strike vanilla call on a non-dividend-paying stock S. Assume that $S_0 = 100$, and that for all $K \geqslant 0$, we observe

$$C(K) = \frac{12500}{125 + K}$$

In the "exact" parts of the problem, exact values should not be replaced by decimal approximations. In the "approximate" parts of the problem, you are free to choose whether to use decimals or not.

Assume that $e^{-rT} = 0.8$ in all parts.

(a) Find the exact time-0 price of a T-expiry 250-strike binary put on S.

(b) Find exactly or approximately the risk-neutral probability

$$\mathbb{P}(124.5 < S_T < 125.5).$$

(To have a definitive standard, let us say that to get full credit, the absolute approximation error should be less than 10^{-7}. No need to prove that your approximation is sufficiently accurate.)

[1] where $Y = \texttt{Trump.US}$, and $(W, X, Z) = (\texttt{Trump.GA}, \texttt{Trump.PA}, \texttt{Trump.AZ})$

Solutions

5a. $f(x+h) \approx f(x) + hf'(x) + \frac{1}{2}h^2 f''(x)$ and $f(x+2h) \approx f(x) + 2hf'(x) + \frac{1}{2}(2h)^2 f''(x)$

5b.

$$\frac{af(x) + bf(x+h) + cf(x+2h)}{h} \approx \frac{(a+b+c)f(x) + (b+2c)hf'(x) + (\frac{1}{2}b + 2c)h^2 f''(x)}{h}$$

so solve the system of three equations

$$a+b+c=0, \qquad b+2c=1, \qquad \frac{1}{2}b+2c=0$$

to find $a=-1.5, b=2, c=-0.5$

5c. By part b, the time-0 delta is $(-1.5 \times 1.24 + 2 \times 1.40 - 0.5 \times 1.62)/5 = 0.026$

Given $X_0 = 10$, the time-0 gamma is $(1.24 - 2 \times 1.40 + 1.62)/5^2 = 0.0024$

Given $X_0 = 15$:

The time-0.1 gamma is $(1.39 - 2 \times 1.20 + 1.03)/5^2 = 0.0008$

The time-0 theta is $\frac{1.2-1.4}{0.1} = -2$

The time-0 color is $\frac{0.0008-0.0024}{0.1} = -0.016$

6a. $\operatorname{Var} Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = (1/8) - (1/8)^2 = 7/64$ so the MC estimate has variance $\boxed{7/(64M)}$.

6b. Optimal β by L6.5 is

$$\frac{\operatorname{Cov}(X,Y)}{\operatorname{Var} X} = \frac{\mathbb{E}(XY) - (\mathbb{E}X)(\mathbb{E}Y)}{\mathbb{E}X^2 - (\mathbb{E}X)^2} = \frac{1/8 - (1/2)(1/8)}{(1/2) - (1/2)^2} = \frac{1/16}{1/4} = \boxed{\frac{1}{4}}$$

The optimized variance by L6.5 is

$$\frac{7}{64M}\left(1 - \frac{\operatorname{Cov}^2(X,Y)}{\operatorname{Var} X \operatorname{Var} Y}\right) = \frac{7}{64M}\left(1 - \frac{1/256}{(1/4)(7/64)}\right) = \boxed{\frac{3}{32M}}$$

6c. We have $\operatorname{Cov}(Y,\tilde{Y}) = \mathbb{E}(Y\tilde{Y}) - (\mathbb{E}Y)(\mathbb{E}\tilde{Y}) = 0 - (1/8)^2 = -1/64$, so variance of the AV estimate is $\frac{1}{2M}(\operatorname{Var} Y + \operatorname{Cov}(Y,\tilde{Y})) = (7/64 - 1/64)/(2M) = \boxed{3/(64M)}$

Comment: So antithetics do better than controls in this case, mainly because X isn't a very good control variate for Y, with a squared correlation of only 1/7.

7a. $-C'(K) = \frac{12500}{(125+K)^2}$ so binary call price 4/45 and put price $4/5 - 4/45 = 32/45$.

7b. Density $e^{rT}C''(125) = \frac{1}{0.8} \times \frac{2 \times 12500}{(125+125)^3} = \frac{1}{500}$, so probability $\approx (125.5 - 124.5) \times \frac{1}{500} = \frac{1}{500}$