Financial Mathematics 32000

Lecture 9

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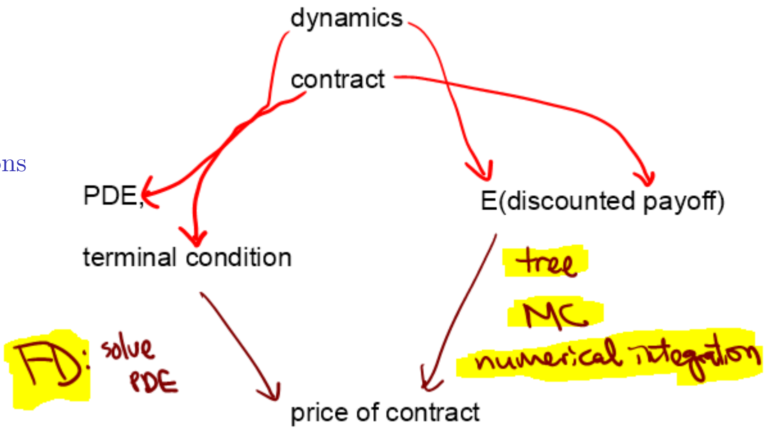
Conclusions



Numerical integration (quadrature)

- Given g the density of X, and payoff $f(X)$, numerical integration calculates the expectation $\int f(x)g(x)\mathrm{d}x$.
- Midpoint quadrature (let $h = fg$, restrict to domain of length L, divided into N intervals of length $\Delta x = L/N$ and midpoints x_n. Numerical result: $\sum_n h(x_n)\Delta x$), is equivalent to doing first-order Taylor approximation of h at each midpoint.
  Max error in approximating h on each interval is $O(\Delta x)^2$. Error in approximating $\int h$ on each interval is $O(\Delta x)^3$. Total error in approximating $\int h$ on N intervals is $N \times O(\Delta x)^3 = O(\Delta x)^2$.
- Thus one-dimensional numerical integration of payoff×density is a preferred approach, provided that the density is available.

Numerical integration in Fourier space

- Given the CF of X, another way to calculate expectations of payoffs $f(X)$ is by numerical integration in Fourier space. Instead of integrating Payoff $\times$ Density, the Fourier transform approach integrates Payoff transform $\times$ Density transform (CF)
- One-dimensional numerical integration of payoff transform $\times$ CF is a preferred approach – provided that the CF is available.
- Often useful for pricing vanilla contracts quickly – for example in *calibration* applications.

Trees and FD

Trees

- Trees can be regarded as *explicit* finite difference methods. But FD have greater flexibility, because FD can also be done by implicit, C-N, etc.

Finite Differences

- Explicit: Simple. Equivalent to trinomial tree. But only **first**-order accurate in Δt, and have stability restrictions.
- Implicit and C-N: Unconditionally stable and (in C-N case) **second**-order accurate in Δt. But requires solution of linear system at each time step.

Monte Carlo

- Typically easy to code, even for complex dynamics and contracts.
- Estimates have random noise, which goes to zero as $O(1/\sqrt{M})$.
- Advantages on multidimensional problems (Multi-asset contracts. Or multi-factor dynamics. Or some path-dependent contracts.) For a 1-dimensional problem, FD methods typically more efficient than MC. But FD computational burden grows exponentially as the number of dimensions grows. ("Curse of Dimensionality").
- FD / quadrature: If error is constant$/N^2$ and work is const $\times N^D$ then work to achieve ε error is const $\times$ (constant$/\sqrt{\varepsilon})^D$. MC: standard error $\sqrt{\operatorname{Var} Y}/\sqrt{M}$. If work is constant $\times DM$ then work to achieve ε standard error is constant $\times D \operatorname{Var}(Y)/\varepsilon^2$. FD better for $D \leq 3$, but this may overstate FD advantage.

Conclusions

Fastest to slowest execution:

- Explicit formula, such as Black-Scholes (using what vol? Maybe no model needed, if implied vols of related contracts available).
- 1-dimensional numerical integration
- Low-dimensional PDE solution, or numerical integration, or trees
- Monte Carlo
- High-dimensional PDE solution, or numerical integration, or trees

But remember:

- Rapid coding may be more important than rapid execution.
- Finance rewards those who see relevant relationships/similarities between A and B (which may denote assets/risks/situations).