Financial Mathematics 32000

Lecture 2

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Pricing: problems and solutions

A description of a pricing problem and solution consists of:

- a **contract** (what are you pricing)
- a **model** of the **dynamics** (what are you assuming about the underlying risks)
- a **pricer** / solution **method** (how are you pricing it)

For example:

- Last Fall: contract = European, model = GBM, pricer = analytic
- Last week: contract $\in$ {European, American, barrier} model = GBM or binomial tree; pricer = binomial/trinomial tree

(**contract=lookback**, dynamics=GBM, pricer=tree):

(contract=general, **dynamics=local volatility**, pricer=tree)

(contract=European call/put, dynamics=GBM, **pricer=Taylor**)

Another path-dependent option

A fixed-strike lookback with strike K, start date 0, and expiry T pays

$$\text{Call:} \qquad (\max_{t\in[0,T]\cap\mathcal{T}} S_t - K)^+$$

$$\text{Put:} \qquad (K - \min_{t\in[0,T]\cap\mathcal{T}} S_t)^+$$

where $\mathcal{T}$ is some set of monitoring times.

Pricing a fixed-strike lookback call

With risk-neutral dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

the level of S_t determines the time-t conditional distribution of

$$\max_{u \in [t,T] \cap \mathcal{T}} S_u.$$

But that's not enough information, if option is on $\max_{u \in [0,T] \cap \mathcal{T}} S_u$.

Need also to track the running maximum

$$M_t := \max_{u \in [0,t]} S_u.$$

So the time-t price of a fixed-strike lookback call is a function

$$C(t, S_t, M_t)$$

The path-dependence introduced the need for the extra state variable.

Pricing a fixed-strike lookback call in a tree

- Recall: the knockout option, also path-dependent, has price

  $$C(t, S_t, \mathbf{1}_{\text{knockout prior to time } t})$$

  Here the path-dependence is simple. The path-dependent state variable has only two states: 0/1 (live/dead).

  And the "1" state does not need to be tracked in the tree.

- Lookback is more complicated than the barrier. At each node (t, S_t) in the tree, need to store a table of values associating each possible M_t with $C(t, S_t, M_t)$.

Example

Assumes r=0	
nu	-0.042
sigma	0.29
deltaX	0.25
deltaT	0.25

running max	option price: fixed strike lookback call

pu	0.15
pm	0.66
pd	0.19

K	100

		n=1 time 0		n=2 time 0.25		n=3 time 0.5		n=4 time 0.75		n=5 time 1.00	
j=4	271.83									271.83	171.83
j=3	211.70							211.70	120.60	211.70	111.70
j=2	164.87					164.87	77.71	164.87	71.81	211.70 164.87	111.70 64.87
j=1	128.40			128.40	42.47	128.40	38.40	164.87 128.40	64.87 33.80	164.87 128.40	64.87 28.40
j=0	100	100.00	13.84	100.00	10.96	128.40 100.00	29.20 7.79	128.40 100.00	28.40 4.21	164.87 128.40 100.00	64.87 28.40 0.00
j=-1	77.88			100.00	1.57	100.00	0.62	128.40 100.00	28.40 0.00	128.40 100.00	28.40 0.00
j=-2	60.65					100.00	0.00	100.00	0.00	128.40 100.00	28.40 0.00
j=-3	47.24							100.00	0.00	100.00	0.00
j=-4	36.79									100.00	0.00

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Local volatility models

We want a model that is consistent with the observed non-constant implied vol skew. One approach: *local volatility* models specify the instantaneous volatility σ to be a function of (S_t, t).

- In the continuous-time setting,

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(S_t, t)S_t\mathrm{d}W_t$$

  where W is $\mathbb{P}$-BM.
- In the tree setting, we let σ depend on the node (S_t, t).

Note that the local volatility σ is not the same thing as σ_{imp}.

Analytic computation of option prices is difficult in the diffusion setting, even for European calls/puts (exception: $\sigma(S_t, t) = \sigma(t)$). But by approximating the diffusion in a tree, the computations are easy.

Option pricing in the tree setting

Suppose we are given the local volatility function σ.

- Let $\Delta t = T/N$ (unless this fails to place important dates in the tree).
- To choose Δx, let σ_{avg} be some "representative" or "average" σ in the tree, and let σ_{max} be an upper bound on the σ in the tree. Two guidelines: to make local discretization error small, we want

$$\Delta x \approx \sigma_{avg}\sqrt{3\Delta t}$$

but for stability reasons, we want

$$\Delta x \geq \sigma_{max}\sqrt{\Delta t}$$

So we can let $\Delta x = \max(\sigma_{avg}\sqrt{3\Delta t}, \sigma_{max}\sqrt{\Delta t})$

Option pricing in the tree setting

- Then at each node, use the $\sigma(S_t, t)$ prevailing at that particular node to generate the ν and the probabilities for the branches out of that node. Same formulas as L1:

$$p_{u,d} = \frac{1}{2}\left[\frac{\sigma^2\Delta t + \nu^2(\Delta t)^2}{(\Delta x)^2} \pm \frac{\nu\Delta t}{\Delta x}\right], \quad p_m = 1 - \frac{\sigma^2\Delta t + \nu^2(\Delta t)^2}{(\Delta x)^2}$$

- Price options – including path-dependent and American-style options – as we did for GBM.

  The only change is that σ (hence ν, p_u, p_m, p_d) vary across nodes.

What if σ is not given

Then *calibrate* it to the prices of listed options.

The general idea of calibration:

- Observe prices of liquidly traded assets (e.g. listed options)
- Choose the model's parameters (e.g. the σ function) in such a way that the model generates theoretical prices that match closely the observed prices.

Then one can apply that model, with the calibrated parameters, to

- Compute hedges and risk sensitivities
- Price illiquid options
- Price and hedge complex deals (e.g. exotic options, structured products)

Calibration of local volatility $\sigma(t)$

If σ is a non-random function of t and

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}W_t$$

then $\mathrm{d}\log S_t = (r - \frac{1}{2}\sigma(t))\mathrm{d}t + \sigma(t)\mathrm{d}W_t$ so

$$\log S_T = \log S_0 + \Big(r - \frac{\bar{\sigma}_T^2}{2}\Big)T + \int_o^T \sigma(t)\mathrm{d}W_t$$

$$\sim \mathrm{Normal}\Big(\log S_0 + (r - \frac{\bar{\sigma}_T^2}{2})T,\ \bar{\sigma}_T^2 T\Big) \text{ where } \boxed{\bar{\sigma}_T := \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)\mathrm{d}t}}$$

(intuition: a nonrandomly weighted sum of indep normals is normal)

So time-0 call prices $C(K,T) = C^{BS}(\bar{\sigma}_T)$, thus $\sigma_{imp}(K,T) = \bar{\sigma}_T$.

Calibration: If given $C(K,T)$ at various expiries T, then obtain σ_{imp} and use boxed equation to (not uniquely) find $\sigma(t)$.

Calibration of local volatility $\sigma(S,t)$ in a trinomial tree

It's harder when σ depends on (S,t).

Here's the rough idea. Assume $r=0$ and a tree with S-levels s_j with equal spacing $\Delta S = s_{j+1}-s_j$. Let $\Delta K := \Delta S$.

Given: $C_0(K,\tau)$, the time-0 price of a strike-K expiry-τ call, for all (K,τ). Find at each node (s_j,t_n): Local volatility σ which generates risk-neutral probabilities consistent with call prices (and stock prices).

- The probability of reaching node (s_j,t_n) equals $(\Delta K)\mathrm{Fly}_0(s_j,t_n)$ where $\mathrm{Fly}_0(K,\tau)$ is the time-0 price of a *butterfly*:

$$\mathrm{Fly}_0(K,\tau) := \frac{C_0(K-\Delta K,\tau) - 2C_0(K,\tau) + C_0(K+\Delta K,\tau)}{(\Delta K)^2}$$

Calibration of local volatility in a trinomial tree

- The probability of reaching node (s_j, t_n) *and then going up* equals $(\Delta t/\Delta K)\mathrm{Cal}_0(s_j, t_n)$, where $\mathrm{Cal}_0(K, \tau)$ is the time-0 price of a *calendar spread*:

$$\mathrm{Cal}_0(K, \tau) := \frac{C_0(K, \tau + \Delta t) - C_0(K, \tau)}{\Delta t}$$

Hence the conditional up-probability from node (s_j, t_n) is the ratio

$$p_u = \frac{\mathbb{P}(\text{reach } (s_j, t_n) \text{ then up})}{\mathbb{P}(\text{reach } (s_j, t_n))} = \frac{(\Delta t/\Delta K)\mathrm{Cal}_0}{(\Delta K)\mathrm{Fly}_0}$$

Calibration of local volatility in a trinomial tree

On the other hand, the up-probability is also obtained by solving

$$\begin{aligned} p_u(\Delta K) + p_m(0) + p_d(-\Delta K) &= 0 \\ p_u(\Delta K)^2 + p_m(0) + p_d(-\Delta K)^2 &= \sigma^2 s_j^2(\Delta t) \end{aligned} \quad \Rightarrow p_u = \frac{\sigma^2 s_j^2 \Delta t}{2(\Delta K)^2}$$

where σ, p_u, p_m, p_d all depend on (s_j, t_n). Therefore

$$\frac{\sigma^2 s_j^2 \Delta t}{2(\Delta K)^2} = \frac{(\Delta t/\Delta K)\mathrm{Cal}_0}{(\Delta K)\mathrm{Fly}_0}$$

Conclusion: at node (s_j, t_n), the local volatility calibrated at time 0 is

$$\sigma(s_j, t_n) = \sqrt{\frac{2}{s_j^2} \times \frac{\mathrm{Cal}_0(s_j, t_n)}{\mathrm{Fly}_0(s_j, t_n)}}$$

Conceptually, the local volatility $\sigma(s, t)$ can be inferred from prices of options with strikes near s and expiries near t.

Calibration of local volatility in a trinomial tree

Comments:

- We have derived a discrete version of the "Dupire equation." We'll return to this formula in the PDE context.
- If you try to implement this formula directly, the calibrated σ will be sensitive to noise in the option price observations. In practice some regularization/smoothing procedure is advisable.
- Calibration is much easier in the case where σ depends only on t instead of jointly on (S, t).

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Taylor approximation

Postpone error analysis until we do finite difference methods.

A key tool in that error analysis will be Taylor approximation:

If f has $n+1$ continuous derivatives in a neighborhood of x_0 then

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 \\ + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + O(x-x_0)^{n+1} \end{aligned}$$

as $x \to x_0$.

We may use Taylor approximation for two different purposes:

- To approximate prices and sensitivities – example on next page.
- To analyze the error in tree or finite difference calculation of prices and sensitivities

Interview question

No calculators allowed. Your interviewer says to you:

> *Spot is* 100. *No dividends. What's the price of a European-style* 1*-year at-the-money-forward (ATMF) vanilla option with* 20% *implied volatility?*

(ATMF at time 0 means $K = F_0$. Recall $F_0 = S_0 e^{rT}$ if no divs.)

Answer: Black-Scholes call price is

$$S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_{1,2} := d_{+,-} := \frac{\log(S_0 e^{rT}/K)}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$$

ATMF option prices are almost linear in vol

ATMF option price is

$$S_0 N(d_1) - Ke^{-rT} N(d_2) = S_0(N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2))$$

For small $|x|$,

$$N(x) = N(0) + N'(0)x + \frac{1}{2}N''(0)x^2 + O(x^3) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x + 0 + O(x^3).$$

So option price is approximately

$$S_0\left(\frac{1}{2} + \frac{\sigma\sqrt{T}/2}{\sqrt{2\pi}} - \frac{1}{2} + \frac{\sigma\sqrt{T}/2}{\sqrt{2\pi}}\right) = \frac{S_0\sigma\sqrt{T}}{\sqrt{2\pi}} \approx 0.4 \times S_0\sigma\sqrt{T}.$$

Your answer: 8 dollars

(True answer: 7.97 dollars)

Follow-up question

Same assumptions. What's the delta of the ATMF call?

Differentiating the approximate option price $S_0\sigma\sqrt{T}/\sqrt{2\pi}$ with respect to S_0, we have a delta of

$$\frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \approx 0.08$$

Or do we? I thought the delta of an ATMF call should be close to 0.5.