

# Financial Mathematics 32000

## Lecture 1

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2023 March 22

# FINM 33000 and FINM 32000

This is the second course in the sequence.

- ▶ Fall 2022: Option pricing
- ▶ Spring 2023: Numerical methods [for option pricing]
- ▶ “Option pricing” is meant in a broad sense: the pricing and hedging of options and other financial *derivative contracts*
- ▶ A *derivative security* or *derivative contract* is a financial instrument whose payoff is defined in terms of an *underlying* (e.g.: A security such as a stock/bond. An index. An interest rate.)

# Why do we need numerical methods

Because we do not have simple exact formulas

- when the *dynamics* of the underlying risks are too complicated.

For example, when working outside the class of Gaussian models (such as BM and GBM), simple exact formulas are less common.

Why isn't GBM enough? Because it's not consistent with the **volatility skew**.

- ▶ when the *contract* to be priced/hedged is too complicated.

For example, path-dependent options are typically harder to price than European-style options. Path-dependent options include **American**-style options, which can be exercised before expiry.



## Realized Volatility

Realized variance of  $S$ , sampled at interval  $\Delta t$ , from time 0 to time  $T$  can be defined, using log-returns by letting  $t_n = n\Delta t$  and  $T = t_N$  and

$$RVar = \frac{1}{T} \sum_{n=0}^{N-1} \left( \log \frac{S_{t_{n+1}}}{S_{t_n}} \right)^2$$

Alternatively could use simple returns, letting  $\Delta S = S_{t_{n+1}} - S_{t_n}$  and

$$RVar = \frac{1}{T} \sum_{n=0}^{N-1} \left( \frac{\Delta S}{S_{t_n}} \right)^2$$

Realized volatility of  $S$  is

Instantaneous volatility

$$RVol = \sqrt{RVar}$$

If  $dS_t = \mu S_t dt + \sigma S_t dW_t$ , then  $RVol \rightarrow \sigma$  as  $\Delta t \rightarrow 0$ .

## Implied Volatility

Given a time- $t$  price  $C$  for a given call option  $(K, T)$  on an underlying  $S_t$  assuming interest rate  $r$ , the **implied volatility** is the  $\sigma$  such that

$$C = C^{BS}(S_t, t, K, T, r, \sigma)$$

where  $C^{BS}$  is the Black-Scholes formula.

$$f \left( \frac{S_t - K e^{-r(T-t)}}{\sigma} \right) \leq C < S_t$$

- ▶ This exists and is unique, if  $\sigma_{imp} < \sigma_t$
  - ▶ The bigger the dollar price  $C$ , the bigger the implied vol  $\sigma_{imp}$
  - ▶ Gives is another way quoting an option price. Instead of saying the option is trading at \$x.xx, can say it's trading at yy% vol.





## Volatility surface: SPX, by strike

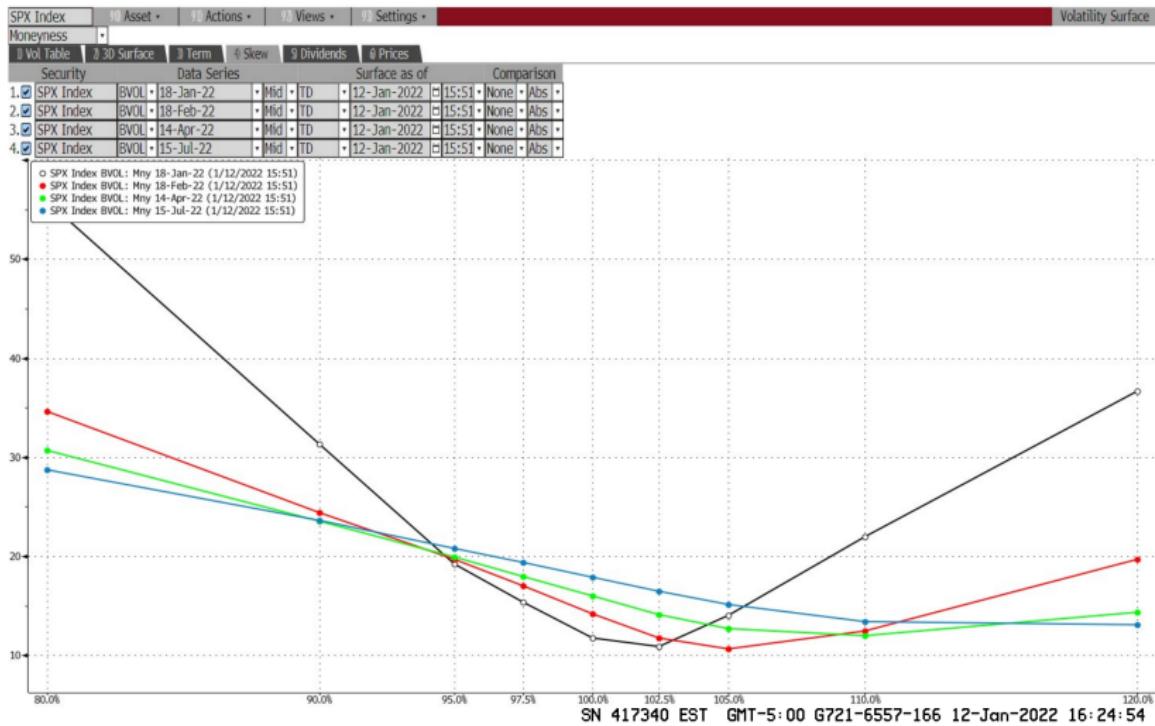
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## Volatility surface: SPX, by moneyness

SPX Index	Asset	Actions	Views	Settings	Volatility Surface					
S&P 500 INDEX	4719.19 USD	Bloomberg	Mid	As of	12-Jan-2022 15:40					
Vol Table	3D Surface	Term	Skew	Dividends	Prices					
Moneyness	Listed	Edit	Fwd	Strikes						
Exp Date	ImpFwd	80.0%	90.0%	95.0%	97.5%	100.0%	102.5%	105.0%	110.0%	120.0%
		3775.4	4247.3	4483.2	4601.2	4719.2	4837.2	4955.1	5191.1	5663.0
14 Jan 2022	4718.94	76.98	45.47	27.95	21.98	16.30	15.63	22.34	32.87	51.93
18 Jan 2022	4718.84	56.42	31.81	19.91	15.90	12.16	11.10	14.47	22.08	36.64
19 Jan 2022	4718.80	54.06	30.48	20.04	16.48	12.65	11.10	14.13	21.08	34.82
21 Jan 2022	4718.59	50.26	29.54	20.77	17.01	13.02	11.03	13.49	20.00	32.02
24 Jan 2022	4718.52	46.25	27.57	20.05	16.67	13.00	10.69	12.28	18.03	29.06
26 Jan 2022	4718.49	44.31	27.95	21.01	17.73	14.04	11.31	12.04	17.31	27.60
28 Jan 2022	4718.04	42.99	28.07	21.47	18.21	14.55	11.60	11.86	16.63	26.43
31 Jan 2022	4717.85	40.80	26.61	20.51	17.47	14.04	11.20	11.19	15.39	24.87
2 Feb 2022	4717.91	39.91	26.55	20.65	17.65	14.27	11.40	11.12	15.07	24.06
4 Feb 2022	4717.17	39.32	26.44	20.75	17.79	14.47	11.62	11.07	14.50	23.30
7 Feb 2022	4717.25	37.68	25.47	20.09	17.25	14.08	11.35	10.71	13.78	22.31
9 Feb 2022	4716.46	37.05	25.38	20.17	17.37	14.23	11.51	10.74	13.53	21.76
11 Feb 2022	4716.10	36.55	25.33	20.29	17.53	14.46	11.77	10.81	13.27	21.23
18 Feb 2022	4714.26	34.84	24.65	19.93	17.33	14.45	11.91	10.80	12.53	19.73
25 Feb 2022	4713.46	33.92	24.32	19.82	17.33	14.66	12.26	11.03	12.15	18.56
28 Feb 2022	4713.02	33.22	23.91	19.51	17.11	14.51	12.15	10.93	11.88	18.12
3 Mar 2022	4710.20	32.16	23.89	19.84	17.67	15.37	13.23	11.92	11.94	16.13
31 Mar 2022	4709.28	31.35	23.79	19.99	17.95	15.81	13.75	12.34	11.96	15.15
14 Apr 2022	4708.41	30.83	23.70	20.09	18.18	16.17	14.26	12.83	12.09	14.34
29 Apr 2022	4708.51	30.33	23.69	20.28	18.47	16.59	14.78	13.34	12.27	13.77
20 May 2022	4705.41	29.80	23.69	20.50	18.81	17.09	15.41	14.01	12.65	13.34
31 May 2022	4704.69	29.46	23.59	20.51	18.89	17.22	15.61	14.23	12.77	13.26
17 Jun 2022	4702.57	29.27	23.68	20.75	19.21	17.63	16.09	14.74	13.12	13.37
30 Jun 2022	4702.67	29.05	23.68	20.84	19.34	17.80	16.29	14.95	13.26	13.26
15 Jul 2022	4702.22	28.83	23.74	21.01	19.51	17.98	16.52	15.25	13.52	13.05
19 Aug 2022	4700.52	28.57	23.86	21.30	19.90	18.48	17.12	15.89	14.10	13.22
16 Sep 2022	4698.54	28.32	23.87	21.43	20.11	18.77	17.47	16.28	14.47	13.32
30 Sep 2022	4699.26	28.27	23.93	21.55	20.26	18.94	17.66	16.48	14.64	13.42
31 Oct 2022	4699.58	28.18	23.89	21.58	20.38	19.17	17.94	16.76	14.84	13.60

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## Volatility skews: SPX, by moneyness, 4 expirations



The implied volatility skew American options UNIT 1: Trees Binomial tree Tree pricing of Americans Trees as diffusions

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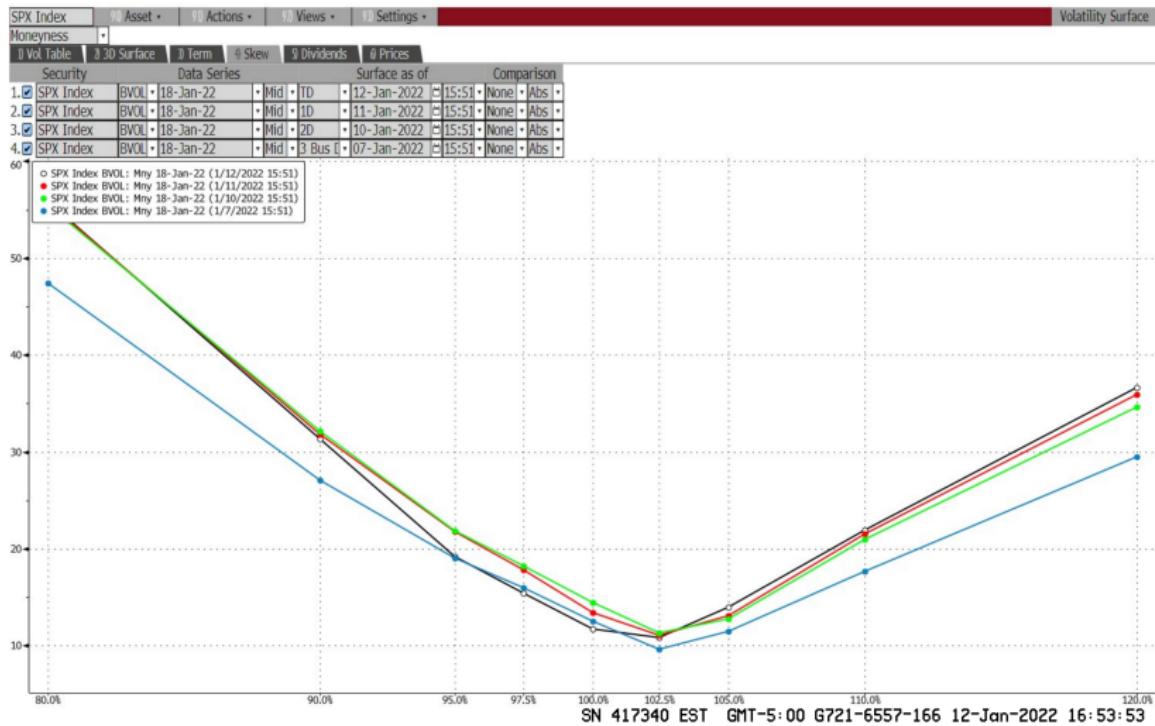
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## Volatility skews: SPX, by moneyness, 4 observations



The implied volatility skew American options UNIT 1: Trees Binomial tree Tree pricing of Americans Trees as diff

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# Volatility smile/skew: EBAY

EBAY US \$ * ! 74.61 -1.59 B 1s										Equity OCM					
DELAY 12:26 Vol 5,752,860 Op 76.6 Q Hi 76.68 Q Lo 74.07 Q ValTrd 433.120m															
OPTION MONITOR 1 COMP Center: 74.12					1 <GO> to Edit Spreadsheet										
BID	dASK	IVBd	IVAS	IVMD	OPIN	d	BID	dASK	IVBd	IVAS	IVMD	OPIN	d		
Bid	Ask	Imp	Imp	Imp			EBAY		Imp	Imp	Imp				
Price	Price	Volat	Volat	Volat	Open		<-CALLS	Bid	Ask	Volat	Volat	Volat	Open		
Price	Price	Bid	Ask	Mid	Intrst		PUTS->	Price	Price	Bid	Ask	Mid	Intrst		
74.61074.620							QXB JUL4	74.61074.620							
44.60	44.80	N.A.	69.07	N.A.	335	130	16		.05	N.A.	74.86	69.02	478		
39.60	39.80	N.A.	70.07	54.16	305	235	-17		.05	N.A.	62.87	57.97	453		
34.70	34.80	N.A.	57.73	36.43	378	340	18		.05	N.A.	52.53	48.44	1,169		
29.70	29.90	N.A.	52.91	46.90	355	445	19	.05	.10	43.44	47.80	45.89	735		
24.80	25.00	37.23	46.40	42.65	514	550	20	.15	.20	41.55	43.68	42.61	1,559		
20.00	20.20	36.56	41.46	39.18	889	655	21	.30	.35	38.88	40.05	39.50	2,205		
					XBA JUL4										
15.40	15.50	34.85	37.76	36.34	1,931	760	22	.65	.75	35.56	37.06	36.31	4,757		
11.10	11.30	33.40	34.36	33.88	3,851	865	23	1.40	1.45	33.46	34.45	33.95	9,285		
7.40	7.60	31.33	32.79	32.06	3,313	970	24	2.65	2.75	31.84	32.56	32.20	4,905		
4.50	4.60	29.70	31.03	30.37	7,000	1075	25	4.70	4.80	30.15	30.81	30.48	2,415		
2.45	2.55	29.16	29.88	29.52	2,166	1180	26	7.60	7.80	28.98	30.39	29.68	619		
1.20	1.30	28.78	29.25	29.02	1,429	1285	27	11.40	11.60	27.94	29.89	28.92	209		
.55	.65	28.82	29.50	29.16	1,115	1390	28	15.70	15.90	27.27	30.24	28.80			
.20	.30	27.69	30.02	28.92	16	1495	29	20.40	20.70	N.A.	32.77	29.60			
.10	.15	28.66	30.60	29.69	305	15100	30	25.30	25.50	N.A.	32.83	30.88			

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The implied volatility skew American options UNIT 1: Trees Binomial tree Tree pricing of Americans Trees as diffu  
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## Volatility smile/skew: GOOG

GOOG US \$ ↑ 555.40 + .50 Q555.20/555.37Q 1x1  
At 12:19 d Vol 1,878.854 0 559.620 H 562.00P L 552.95P Val 1.048B

GOOG US Equity		95) Templates		96) Actions		97) Expiry		Option Monitor: Option Monitor								
.... GOOGLE INC-C	1555.4001	.5001	.0901%	555.20 / 555.3701	Hi 562.00	Lo 552.95	Volm 1878854	HV.00								
Calc Mode		Center		555.00		Strikes 19		Exch US Composit		92) Next Earnings(EM)						
81) Center	Strike	82) Calls/Puts	83) Calls	84) Puts	85) Term Structure											
Calls						Strike										
Ticker	Bid	Ask	Last	IVM	DM	Volm	OInt	Ticker	Bid	Ask	Last	IVM	DM	Volm	OInt	
19 Apr 14 (10d); CSIZE 100; IDIV .77; R .12; IFWD 556.55								19 Apr 14 (10d); CSIZE 100; IDIV .77; R .12; IFWD 556.55								
1) GOOG 4 C510	48.20	50.80	50.00	58.97	.84	2	34	510.00	58) GOOG 4 P510	3.20	3.60	3.08	55.30	-.14	69	203
2) GOOG 4 C515	43.70	46.80	43.00	57.94	.82	37	35	515.00	59) GOOG 4 P515	4.10	4.40	4.24	55.20	-.17	28	304
3) GOOG 4 C520	39.90	40.60	44.40	53.44	.81	11	35	520.00	60) GOOG 4 P520	4.90	5.30	4.90	54.18	-.20	13	485
4) GOOG 4 C525	36.20	37.70	40.00	55.38	.76	1	58	525.00	61) GOOG 4 P525	6.00	6.40	6.14	53.70	-.23	27	163
5) GOOG 4 C530	32.40	33.50	34.00	54.21	.73	1	76	530.00	62) GOOG 4 P530	7.10	7.70	7.05	52.96	-.27	14	183
6) GOOG 4 C535	28.90	29.80	30.56	53.31	.69	45	73	535.00	63) GOOG 4 P535	8.70	9.10	7.80	52.84	-.30	45	348
7) GOOG 4 C540	25.60	26.50	27.52	53.30	.65	12	214	540.00	64) GOOG 4 P540	10.40	10.80	9.91	52.60	-.35	26	249
8) GOOG 4 C545	22.50	23.30	23.70	52.81	.61	18	168	545.00	65) GOOG 4 P545	12.10	12.70	12.30	51.82	-.39	17	148
9) GOOG 4 C550	19.60	20.20	20.00	52.13	.56	168	358	550.00	66) GOOG 4 P550	14.30	14.80	13.80	51.63	-.44	84	155
10) GOOG 4 C555	17.00	17.50	17.95	51.84	.52	192	256	555.00	67) GOOG 4 P555	16.60	17.10	16.50	51.18	-.48	41	159
11) GOOG 4 C560	14.60	15.00	15.41	51.29	.47	472	694	560.00	68) GOOG 4 P560	19.20	19.70	18.61	51.09	-.53	101	113
12) GOOG 4 C565	12.40	13.00	13.14	51.28	.43	115	321	565.00	69) GOOG 4 P565	21.80	22.50	20.50	50.64	-.58	1	29
13) GOOG 4 C570	10.50	10.70	10.51	50.74	.38	154	652	570.00	70) GOOG 4 P570	24.90	25.70	23.82	50.26	-.62	1	51
14) GOOG 4 C575	8.80	9.20	9.20	50.26	.34	244	317	575.00	71) GOOG 4 P575	28.00	28.80	28.10	51.60y	-.67	8	8
15) GOOG 4 C580	7.30	7.80	7.50	50.28	.30	124	431	580.00	72) GOOG 4 P580	31.40	32.30	29.10	49.19	-.71	11	47
16) GOOG 4 C585	5.90	6.30	6.43	50.02	.25	28	211	585.00	73) GOOG 4 P585	34.80	36.00	38.00y	48.78	-.75	11	11
17) GOOG 4 C590	4.80	5.30	5.07	50.20	.22	71	333	590.00	74) GOOG 4 P590	37.70	40.20	36.23	46.97	-.80	8	21
18) GOOG 4 C595	3.90	4.40	4.15	50.05	.19	91	136	595.00	75) GOOG 4 P595	41.60	44.10	40.26	44.85	-.84	8	13
19) GOOG 4 C600	3.20	3.60	3.39	49.94	.16	80	837	600.00	76) GOOG 4 P600	45.90	48.30	63.95y	44.80	-.87	35	35
17 May 14 (38d); CSIZE 100; IDIV .23; R .16; IFWD 556.62								17 May 14 (38d); CSIZE 100; IDIV .23; R .16; IFWD 556.62								
20) GOOG 5 C510	52.10	50.80	51.30	36.10	.79	15	510.00	77) GOOG 5 P510	6.70	7.20	5.80	33.61	-.20	6	323	

93) Default color legend Zoom 90%  
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The implied volatility skew American options UNIT 1: Trees Binomial tree Tree pricing of Americans Trees as diffu

# Volatility surface: GOOG

GOOG US \$ ↓ 556.80 +1.90 P556.65/556.84Q 2x1  
... At 12:32 d Vol 1,926,365 0 559.62Q H 562.00P L 552.95P Val 1.075B

GOOG US Equity		95) Templates		96) Actions		97) Expiry		Option Monitor: Option Monitor	
.... GOOGLE INC-C		1556.80 1.90 .3424% 556.6501 / 556.8401		Hi 562.00		Lo 552.95 Volm 1926365 HV.00			
□ Calc Mode		Center 555.00		Strikes 19		Exch US Composit		92) Next Earnings(EM) 04/16/14 C	
81) Center Strike		82) Calls/Puts		83) Calls		84) Puts		85) Term Structure	
Expiry		19 Apr 2014		17 May 2014		21 Jun 2014		20 Sep 2014	
Calls/Puts		Calls	Puts	Calls	Puts	Calls	Puts	Calls	Puts
Strike	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM
505.00	.91	51.46	-.11	56.24	-.83	33.50	-.17	34.07	.80
510.00	.88	50.89	-.14	55.57	-.81	33.13	-.20	34.25	.77
515.00	.85	51.33	-.16	55.34	.79	32.84	-.22	33.86	.75
520.00	.82	51.42	-.19	54.24	.76	33.23	-.24	33.63	.73
525.00	.79	51.53	-.22	54.26	.73	33.14	-.27	33.50	.70
530.00	.75	51.68	-.26	53.59	.70	33.56	-.30	33.37	.67
535.00	.71	52.68	-.29	53.02	.67	33.28	-.33	33.16	.65
540.00	.67	52.50	-.33	52.64	.63	33.20	-.37	32.79	.62
545.00	.62	52.27	-.38	52.45	.60	32.95	-.40	32.84	.59
550.00	.58	51.82	-.42	51.92	.57	32.76	-.43	32.58	.56
555.00	.53	51.92	-.47	51.67	.53	32.61	-.47	32.44	.53
560.00	.48	50.99	-.52	51.18	.50	32.35	-.50	32.40	.51
565.00	.44	50.88	-.56	50.90	.46	32.34	-.54	32.05	.48
570.00	.39	50.26	-.61	50.53	.43	32.02	-.57	32.02	.45
575.00	.35	49.95	-.65	50.23	.39	31.85	-.61	31.75	.42
580.00	.30	49.76	-.69	50.05	.36	31.95	-.64	31.76	.39
585.00	.26	49.38	-.73	50.06	.33	31.86	-.67	31.66	.36
590.00	.23	49.41	-.76	51.33	.30	31.71	-.70	31.47	.34
595.00	.19	49.62	-.79	51.70	.27	31.30	-.73	31.21	.31
600.00	.16	49.15	-.82	51.71	.24	31.34	-.76	31.59	.29
605.00	.14	49.64	-.85	51.82	.22	31.06	-.78	31.40	.27
610.00	.12	49.72	-.88	50.09	.19	31.02	-.80	31.89	.24

93) Default color legend

Zoom

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Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2014 Bloomberg Finance L.P.  
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## Volatility surface: AMZN

AMZN US \$ **↓ 326.48** - .59 P 326.36 / 326.51P 2x3  
At 13:08 d Vol 2,908,935 O 328.470 H 328.49D L 322.500 Val 947.826M

AMZN US Equity				95 Templates				90 Actions				97 Expiry				Option Monitor: Option Monitor							
AMAZON.COM INC				↓326.48	.59	-1.80%	4%	326.360	/ 326.51	Hi 328.49		Lo 322.50	Volm 2908935	HV 24.76									
Calc Mode				Center	326.66	Strikes	5	Exch	US Composit														
81) Center Strike				82) Calls/Puts	83) Calls	84) Puts	85) Term Structure									92) Next Earnings(EM) 04/25/14 E							
Expiry				19 Apr 2014				17 May 2014				21 Jun 2014				19 Jul 2014				18 Oct 2014			
Calls/Puts				Calls	Puts				Calls	Puts				Calls	Puts				Calls	Puts			
Strike				DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM
280.00	.98	55.66	-.02	53.40	.86	48.25	-.14	46.67	.83	40.35	-.16	38.61	.81	36.77	-.18	35.63	.77	34.48	-.22	33.08			
285.00	.98	47.38	-.03	50.74	.83	48.31	-.16	45.71	.81	39.40	-.19	38.04	.79	36.61	-.20	35.10	.75	34.06	-.24	32.75			
290.00	.94	53.36	-.05	48.05	.81	47.17	-.19	44.94	.78	39.05	-.21	37.48	.76	36.34	-.23	34.44	.73	33.43	-.27	32.50			
295.00	.93	47.70	-.06	45.69	.78	45.17	-.22	44.18	.75	38.71	-.24	37.02	.74	35.31	-.26	34.08	.71	32.87	-.29	32.11			
300.00	.89	47.88	-.09	43.23	.75	44.56	-.25	43.42	.72	38.21	-.27	36.51	.71	34.87	-.29	34.00	.68	32.81	-.31	31.80			
305.00	.85	45.62	-.13	41.09	.71	43.74	-.29	42.74	.69	37.40	-.31	36.07	.68	34.48	-.32	33.54	.66	32.30	-.34	31.68			
310.00	.82	39.98	-.18	39.65	.67	43.80	-.33	42.16	.65	37.05	-.34	35.56	.65	33.86	-.35	33.14	.63	32.05	-.37	31.33			
315.00	.74	38.38	-.26	38.31	.63	42.99	-.37	41.56	.61	36.59	-.38	35.35	.62	33.54	-.38	32.92	.61	32.05	-.39	31.12			
320.00	.65	37.84	-.35	37.21	.58	42.54	-.42	41.19	.58	36.16	-.42	34.67	.58	33.56	-.42	32.66	.58	31.64	-.42	30.93			
325.00	.54	36.78	-.46	36.31	.54	41.87	-.46	40.53	.54	35.78	-.46	34.46	.54	33.18	-.46	32.43	.55	31.49	-.45	30.69			
330.00	.43	35.57	-.57	34.99	.49	41.35	-.51	40.00	.50	35.45	-.50	33.89	.51	33.00	-.49	32.16	.53	31.53	-.47	30.65			
335.00	.32	34.82	-.69	34.19	.44	40.81	-.56	39.45	.46	35.01	-.54	33.67	.47	32.67	-.53	31.98	.50	31.29	-.50	30.59			
340.00	.22	34.59	-.78	33.80	.40	40.58	-.60	39.57	.42	34.77	-.58	33.26	.44	32.63	-.57	31.62	.47	31.21	-.53	30.29			
345.00	.15	34.47	-.85	34.14	.35	40.18	-.65	38.82	.39	34.63	-.61	33.73	.40	32.40	-.60	31.50	.45	31.09	-.56	30.05			
350.00	.09	34.92	-.91	33.55	.31	40.07	-.70	38.37	.35	34.56	-.65	33.51	.37	32.30	-.63	31.43	.42	30.80	-.58	30.03			
355.00	.06	35.90	-.97	28.83	.27	39.71	-.73	38.86	.31	34.42	-.70	32.51	.34	32.17	-.67	31.27	.40	30.93	-.61	30.04			
360.00	.04	37.29	-.99	29.45	.24	39.64	-.77	38.67	.28	34.01	-.72	33.19	.31	32.09	-.70	31.04	.37	30.75	-.63	29.83			
365.00	.03	38.73			.20	39.53	-.80	38.60	.25	33.81	-.75	32.76	.28	31.82	-.73	30.85	.35	30.49	-.66	29.60			
370.00	.02	40.89	-1.0		.17	39.55	-.84	37.03	.22	33.96	-.79	31.97	.25	31.75	-.76	30.72	.32	30.37	-.68	29.49			

### 93) Default color legend

Zoom

100%

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Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2014 Bloomberg Finance L.P.  
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The implied volatility skew American options UNIT 1: Trees Binomial tree Tree pricing of Americans Trees as diffu  
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## Volatility smile: possible explanations

Why does the smile exist?

- ▶ The market prices options using a risk-neutral distribution of log-returns with *fatter tails* than the Normal (*excess kurtosis*).

What contributes to those fat tails?

- ▶ Clustering of volatility.
  - ▶ Jumps introduce extreme outcomes into return distributions, especially short-term.



# Volatility skew: possible explanations in equity markets

Why's the smile skewed? (Why fatter tails on *left*?) Physical causes:

- ▶ Negative correlation between price and instantaneous volatility.  
Empirically, instantaneous volatility increases as price decreases.  
This fattens the left tail of the distribution (negative skewness).
- ▶ Possibility of big down-jumps (crashes) also fattens the left tail.
- ▶ “The bull walks up the stairs, the bear jumps out the window.”

Combined with risk preferences:

- ▶ “Supply and demand”: Fear of downside leads to demand for protection/insurance, driving up the prices (hence implied vols) of OTM puts. And willingness to sell part of the upside leads (covered-) call writers to supply OTM calls, driving prices down.

# Volatility skew: possible explanations

SKP

Why is the smile skewed? Another way to think about it:

- ▶ Implied volatility at strike  $K$  depends on the option price, which depends on expected future volatility – specifically, on expected future volatility *along paths near  $K$* . Why? Future volatility along paths away from  $K$  does not help the option holder, because linear payoffs (e.g. zero, forward) are insensitive to vol. Only the *convex* part of the payoff brings benefits from volatility.
- ▶ Future volatility along paths near downside strikes is likely higher than future volatility along paths near upside strikes.  
So by the above logic, downside-strike options should trade higher above their intrinsic lower bound than upside-strike options do.

## Volatility skew: implications

Presence of the volatility skew is inconsistent with GBM dynamics.

- ▶ Black-Scholes not adequate for pricing derivatives such as forward-starting options, volatility derivatives, etc, which are highly path-dependent and require a proper dynamic model.
- ▶ For simple purposes, using Black-Scholes may be adequate (even if logically inconsistent):
  - ▶ Interpolating/extrapolating call or put prices at liquid strikes, to price a call or put at an illiquid strike
  - ▶ Detecting strikes at a given expiry that seem under/overpriced

These purposes may just require fitting an implied vol curve.

The implied volatility skew

American options

UNIT 1: Trees

Binomial tree

Tree pricing of Americans

Trees as diffusion approximations

# Pricing an American

- ▶ An *American* option on  $S$  with strike  $K$  and expiry  $T$  pays

$$\text{Put: } (K - S_\tau)^+ \quad \text{Call: } (S_\tau - K)^+$$

at the time of exercise  $\tau$ , where  $\tau \leq T$  is a *stopping time* chosen by the holder. Not a stopping time:  $\tau = \operatorname{argmin}_{t \in [0, T]} S_t$

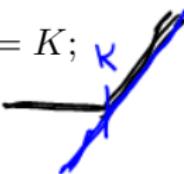
- ▶ By definition the *stopping time*  $\tau$  can be random, but for each  $t$  it must satisfy  $\{\tau = t\} \in \mathcal{F}_t$ ; thus the decision on whether to exercise at time  $t$  must depend only on information that has been revealed up to and including time  $t$ .
- ▶ The ability to exercise early makes the contract path-dependent; its payout no longer depends only on the underlying at a particular time  $T$ .

## Pricing an American: call options

In general, pricing early-exercisable contracts requires numerics.

An exception: if  $r > 0$ , then it is *never optimal to exercise early an American call on a non-dividend-paying stock.*

Model-independent reason: (with all expiries =  $T$ ; all strikes =  $K$ ; discount bond  $Z$ ), the time- $t$  (where  $t < T$ ) values of



$$\text{American call} \geq \text{European call} \geq \text{Forward} = S_t - KZ_t > S_t - K$$

American is worth  $>$  the exercise payoff  $S_t - K$ . So *don't exercise*.

- ▶ But what if you think  $S$  is going down? Shorting the stock beats exercising your call. By shorting you still get  $S_t$ , but you *delay* paying  $K$ , and if stock dips, you may pay *less* than  $K$  to cover.

Therefore, *American call price = European call price*

# Pricing an American call: arbitrage argument

Arbitrage argument to show that American = European call:

Let  $r \geq 0$  and no dividends. If American > European then go short the American and long the European for an immediate profit.

- ▶ If the counterparty never exercises, then no further liabilities.
- ▶ If the counterparty exercises, then you are short a share and long a value of  $K$  dollars in the bank account, which becomes  $\geq K$  by expiry. Covering the short at expiration requires  $\leq K$  dollars, due to the European.

Since American > European leads to arbitrage, we must have  
*American call price = European call price.*

And *early exercise of the American cannot be strictly optimal.*

## The implied volatility skew

## American options

## UNIT 1: Trees

## Binomial tree

## Tree pricing of Americans

The implied volatility skew

American options

UNIT 1: Trees

Binomial tree

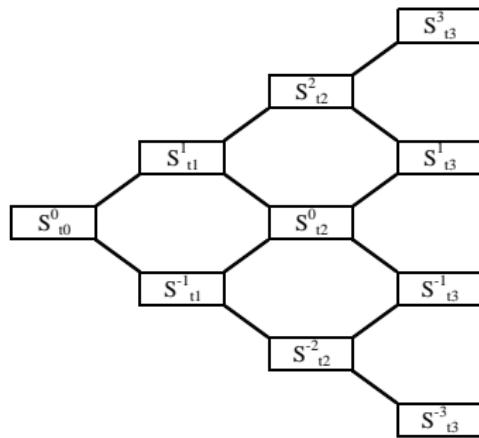
Tree pricing of Americans

Trees as diffusion approximations

# Binomial tree

Consider an  $N$ -period model (example:  $N = 3$ ) with a bank account  $B_t = e^{rt}$  and a stock  $S$  with dynamics

time  $t_0$       time  $t_1$       time  $t_2$       time  $t_3$



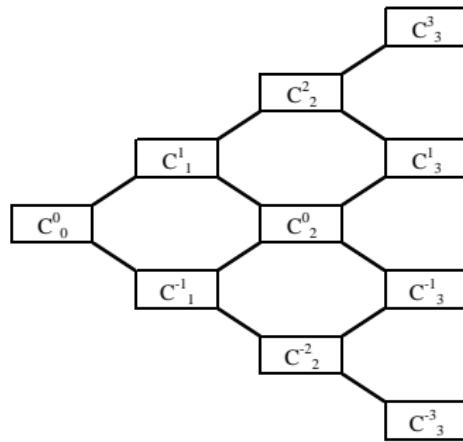
Superscripts are not powers. Subscripts  $t_n = n\Delta t$  for  $n = 0, \dots, N$ .

# Binomial tree

Want to find time-0 price  $C_0^0$  of some option.

To simplify notation, we will write  $C_n^j$  instead of  $C_{t_n}^j$ .

time  $t_0$               time  $t_1$               time  $t_2$               time  $t_3$



Consider three contract types: European, barrier, and American.

## Pricing a European

If the expiry date is at time  $T = t_N$  and the payoff function is  $f(S)$ , then for all  $j$ , the option price is

$$C_N^j = f(S_{t_N}^j).$$

Now induct backwards. If  $C_{n+1}^j$  has been computed for all  $j$ , then apply the one-period no-arbitrage results in the one-period subtree rooted at each  $(j, n)$  to find

$$C_n^j = e^{-r\Delta t} [p_n^j C_{n+1}^{j+1} + (1 - p_n^j) C_{n+1}^{j-1}],$$

where the up-probability is

$$p_n^j := \frac{S_{t_n}^j e^{r\Delta t} - S_{t_{n+1}}^{j-1}}{S_{t_{n+1}}^{j+1} - S_{t_{n+1}}^{j-1}}.$$

## Pricing an up-and-out barrier option

An *up-and-out put* on  $S$  with strike  $K$ , expiry  $T$ , and barrier  $H$  pays

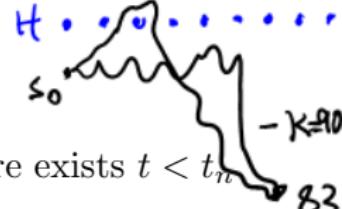
$$(K - S_T)^+ \mathbf{1}_{\max_{t \in \mathcal{T}} S_t < H}.$$

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } A \\ 0 & \text{if not } A \end{cases}$$

where  $\mathcal{T} \subseteq [0, T]$  are the observation times, specified in the contract.

Continuous monitoring:  $\mathcal{T} = [0, T]$ . Discrete monitoring:  $\mathcal{T}$  is some finite set. The option price is a function of *three* state variables:

$$C(S_{t_n}^j, t_n, \mathbf{1}_{\text{knockout prior to time } t_n}).$$



where “knockout prior to time  $t_n$ ” is the event that there exists  $t < t_n$  with  $t \in \mathcal{T}$  and  $S_t \geq H$ . Computationally:  $C(S, t, 1)$  does not need to be tracked; it's always zero. So what we track in the tree is

$C_n^j := C(S_{t_n}^j, t_n, 0)$ , the option price *in the case of no prior knockout*.

## Pricing an up-and-out barrier option

Let  $t_N = T$ . At terminal nodes,  $C_N^j = (K - S_{t_N}^j)^+ \mathbf{1}_{S_{t_N}^j < H \text{ or } t_N \notin \mathcal{T}}$ .

Inducting backwards, if  $C_{n+1}^j$  has been computed for all  $j$ , then at node  $(j, n)$  we have two cases:

$$t_n \in \mathcal{T} \text{ and } S_{t_n}^j \geq H \Rightarrow C_n^j = 0 \quad (1)$$

$$t_n \notin \mathcal{T} \text{ or } S_{t_n}^j < H \Rightarrow C_n^j = e^{-r\Delta t} [p_n^j C_{n+1}^{j+1} + (1 - p_n^j) C_{n+1}^{j-1}] \quad (2)$$

So everywhere at or beyond the barrier at monitoring times, set the price to 0; elsewhere, take the usual discounted average of the next-time-step values.

(And in building a tree, you try to have the dates important in the contract – such as expiry and barrier monitoring dates – be among the dates represented in the tree.)

## The implied volatility skew

American options

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## Tree pricing of Americans

# Pricing an American

Notation:  $x \wedge y := \min(x, y)$ .

On a time interval  $[t_1, t_2 \wedge \tau]$ , consider a portfolio with price process  $V$ , that includes a static position in an American option, to be exercised at time  $\tau \geq t_1$ . We say it is an *arbitrage* if  $V_{t_1} = 0$  and

$$\mathbb{P}(V_{t_2 \wedge \tau} < 0) = 0 \text{ and } \mathbb{P}(V_{t_2 \wedge \tau} > 0) > 0$$

for *some* stopping time  $\tau \geq t_1$ , if the portfolio is long the option;  
 for *all* stopping times  $\tau \geq t_1$ , if the portfolio is short the option.

# Pricing an American put

At expiry  $T = t_N$ , we have  $C_N^j = (K - S_{t_N}^j)^+$ . Inducting backwards, if  $C_{n+1}^j$  is the no-arb price that has been computed for each  $j$  then

$$C_n^j = \max((K - S_{t_n}^j)^+, e^{-r\Delta t}[p_n^j C_{n+1}^{j+1} + (1 - p_n^j) C_{n+1}^{j-1}])$$

This is because the holder at node  $(j, n)$  can either exercise and receive  $(K - S_{t_n}^j)^+$ , or hold on to the option which tomorrow will have no-arbitrage price of  $C_{n+1}^{j+1}$  (if up) or  $C_{n+1}^{j-1}$  (if down), which implies

- ▶ If  $C_n^j < \max$ , then there is arbitrage: go long the option.
- ▶ If  $C_n^j > \max$ , then there is arbitrage: short the option; if holder exercises, you close out with a profit; if not, then use the funds to buy a portfolio that superreplicates the time- $t_{n+1}$  option value.

The time-0 option price is  $C_0^0$ .

## Pricing an American: another formulation

Math fact: For any adapted process  $Z_{t_n}$ , define  $Y_{t_N} = Z_{t_N}$  and

$$Y_{t_n} = \max(Z_{t_n}, \mathbb{E}_{t_n} Y_{t_{n+1}}), \quad n = N-1, N-2, \dots, 0.$$

(E.g.:  $Z_t = e^{-rt}(K - S_t)^+$  and  $Y_t = e^{-rt} \times$  time- $t$  value of American).

Then the **optimality principle of dynamic programming** says that

$$Y_0 = \max\{\mathbb{E}Z_\tau : \tau \text{ is a stopping time, } 0 \leq \tau \leq T\}$$

and  $Y_0 = \mathbb{E}Z_{\tau^*}$  where  $\tau^* := \min\{t_n : Y_{t_n} = Z_{t_n}\}$ .

This leads to another formulation of the American option price:

$$C_0 = \max\{\mathbb{E}e^{-r\tau}(K - S_\tau)^+ : \tau \text{ is a stopping time, } 0 \leq \tau \leq T\}$$

(which is also valid in continuous time for continuous processes  $S$ ).

# Another application of dynamic programming

Interview (?!?) question:

Take a 52 card deck with 26 red and 26 black cards, in random order.

Reveal cards sequentially without replacement.

With each black card you get +1 dollar.

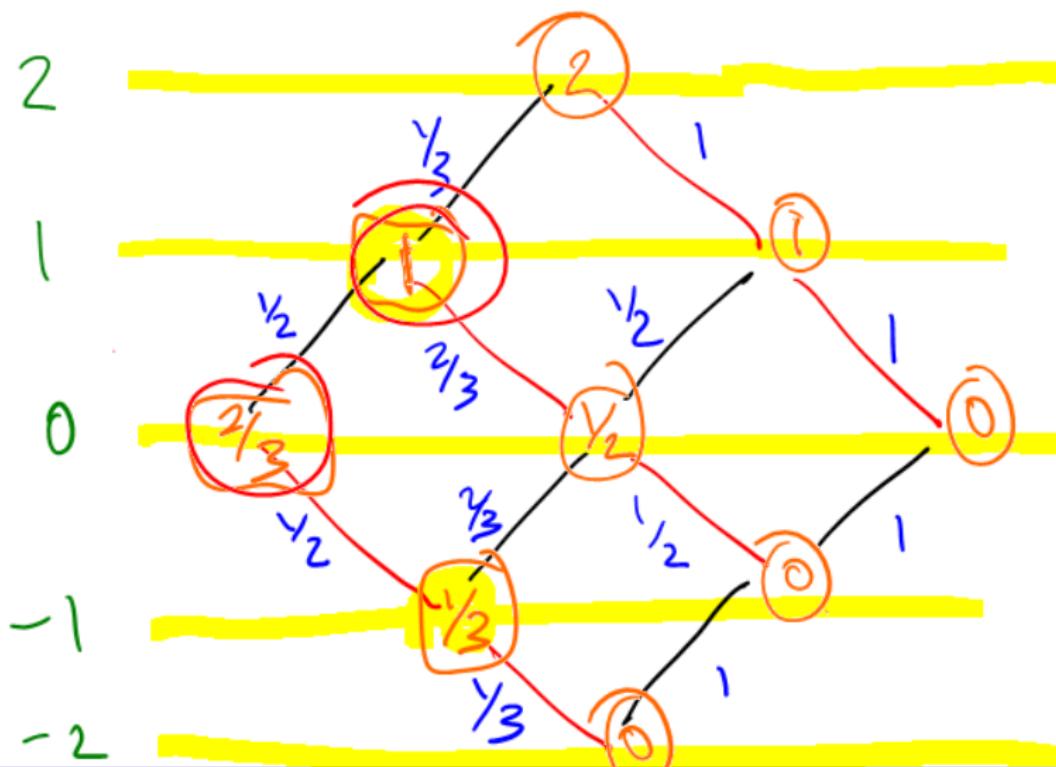
With each red card you get -1 dollar.

You can stop playing the game at any time.

What's the optimal stopping time, and what's your expected profit from the optimal strategy?

(Easier interview question: 4 cards – 2 red, 2 black)

# Another application of dynamic programming



The implied volatility skew

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Trees as diffusion approximations

## The tree method

- ▶ Approximate the risk-neutral dynamics of a diffusion process (such as GBM) using a discrete model with a finite number of branches from each state.
- ▶ The expectations (hence option prices) that we find in the tree approximate the expectations (hence option prices) under the diffusion process.
- ▶ For example, let us use a tree to approximate the diffusion

$$dS_t = R_{grow} S_t dt + \sigma S_t dW_t$$

# Use binomial tree to approximate diffusion dynamics

- ▶ Diffusion is

$$dS_t = R_{grow} S_t dt + \sigma S_t dW_t$$

where  $W$  is  $\mathbb{P}$ -BM. So  $X := \log S$  has dynamics

$$dX_t = \nu dt + \sigma dW_t$$

where  $\nu := R_{grow} - \sigma^2/2$ .

- ▶ Approximate dynamics of  $X := \log S$  using a binomial tree.

$$\mathbb{P}(X_{t+\Delta t} = X_t + \Delta x_u) = p$$

$$\mathbb{P}(X_{t+\Delta t} = X_t - \Delta x_d) = 1 - p$$

where  $\Delta x_u$  and  $\Delta x_d$  and  $p$  will be nonrandom parameters.

## Match the means and variances

- ▶ Let  $\Delta t := T/N$  where  $N$  is number of time steps.
- ▶ Now choose  $p, \Delta x_u, \Delta x_d$  such that diffusion and tree agree on mean and variance of the increments. In the diffusion,

$$X_{t+\Delta t} - X_t = \int_t^{t+\Delta t} \nu ds + \int_t^{t+\Delta t} \sigma dW_s = \nu \Delta t + \sigma \Delta W.$$

So

$$\nu \Delta t = \mathbb{E}_t(X_{t+\Delta t} - X_t) = p \Delta x_u + (1-p)(-\Delta x_d)$$

$$\sigma^2 \Delta t = \text{Var}_t(X_{t+\Delta t} - X_t) = p(\Delta x_u)^2 + (1-p)(-\Delta x_d)^2 - (\nu \Delta t)^2$$

Two equations and three unknowns:  $p, \Delta x_u, \Delta x_d$ . Can impose another condition (such as  $\Delta x_u = \Delta x_d$  or  $p = 1/2$ ) and solve.

Then price the option in a tree with these parameters.

## Not flexible enough

- ▶ However, often we want to specify  $\Delta x_u$  and  $\Delta x_d$  in advance.  
For example, we may want  $\Delta x_u = \Delta x_d = \Delta x$ , so that the *same* price levels are represented at every time point in the tree.  
Maybe, furthermore, we want to specify the size of  $\Delta x$  to optimize convergence or to choose *which* price levels are represented.
- ▶ This leaves only one free parameter  $p$ .  
Not enough to match both the mean and the variance.

Solution: trinomial trees.

# Use trinomial tree to approximate diffusion dynamics

- ▶ Diffusion is

$$dS_t = R_{grow} S_t dt + \sigma S_t dW_t$$

where  $W$  is  $\mathbb{P}$ -BM. So  $X := \log S$  has dynamics

$$dX_t = \nu dt + \sigma dW_t$$

where  $\nu := R_{grow} - \sigma^2/2$ .

- ▶ Approximate dynamics of  $X := \log S$  using a trinomial tree.

$$\mathbb{P}(X_{t+\Delta t} = X_t + \Delta x) = p_u$$

$$\mathbb{P}(X_{t+\Delta t} = X_t) = p_m$$

$$\mathbb{P}(X_{t+\Delta t} = X_t - \Delta x) = p_d$$

## Match the means and variances

- ▶ Let  $\Delta t := T/N$  where  $N$  is number of time steps.

Choose  $\Delta x \approx \sigma\sqrt{3\Delta t}$  for accuracy reasons.

Can modify these suggestions to make sure certain price levels or times are represented in the tree.

- ▶ Now choose  $p_u, p_m, p_d$  such that **diffusion** and **tree** agree on mean and variance of the increments. In the diffusion,

$$X_{t+\Delta t} - X_t = \int_t^{t+\Delta t} \nu ds + \int_t^{t+\Delta t} \sigma dW_s = \nu \Delta t + \sigma \Delta W.$$

So

$$\nu \Delta t = \mathbb{E}_t(X_{t+\Delta t} - X_t) = p_u \Delta x + p_m 0 + p_d (-\Delta x)$$

$$\sigma^2 \Delta t = \text{Var}_t(X_{t+\Delta t} - X_t) = p_u (\Delta x)^2 + p_m (0)^2 + p_d (-\Delta x)^2 - (\nu \Delta t)^2$$

$$p_u + p_m + p_d = 1$$

# Solution

Solve system of three equations in three unknowns.

$$p_u = \frac{1}{2} \left[ \frac{\sigma^2 \Delta t + \nu^2 (\Delta t)^2}{(\Delta x)^2} + \frac{\nu \Delta t}{\Delta x} \right]$$

$$p_m = 1 - \frac{\sigma^2 \Delta t + \nu^2 (\Delta t)^2}{(\Delta x)^2}$$

$$p_d = \frac{1}{2} \left[ \frac{\sigma^2 \Delta t + \nu^2 (\Delta t)^2}{(\Delta x)^2} - \frac{\nu \Delta t}{\Delta x} \right]$$

Intuition: For fixed  $\Delta t$  and  $\Delta x$ ,

- ▶ The bigger the  $\sigma$ , the more probability mass in the wings, the less in the middle.
- ▶ The bigger the  $\nu$ , the more mass in the up-branch, the less in the down-branch.

# Option pricing

To find option prices, induct backwards in tree with time points

$$t_n = n\Delta t \quad n = 0, \dots, N,$$

and log-stock-price points

$$x_j = \log S_0 + j\Delta x \quad j = -N, \dots, N.$$

Option price in tree for node at time  $t_n$  and log-price  $x_j$  is

$$C_n^j = e^{-r\Delta t} [p_u C_{n+1}^{j+1} + p_m C_{n+1}^j + p_d C_{n+1}^{j-1}].$$

With large enough  $N$ , the option price in the tree  $\approx$  the option price for diffusion, because in the  $N \rightarrow \infty$  (or equivalently  $\Delta t \rightarrow 0$ ) limit, the tree's option price  $\rightarrow$  option price under diffusion dynamics.

For Europeans, the proof is by a form of the CLT.

# Can't replicate general options in 3-state 2-assets model

- ▶ Recall that in the 3-state model, general options cannot be replicated using stock and bank acct. There is not a unique arbitrage-free price for the option. So how can we talk about finding “the” price of the option?
- ▶ Answer: The trinomial tree is not the model of the market. The model is GBM in continuous time. In that model, replication using  $\{B, S\}$  is possible. So unique price exists.  
The tree is a *computational device* that we use in order to approximate the risk-neutral expectations (hence the prices) of the continuous-time model.

# Option Pricing: Knock-outs and Americans

Pricing of knock-outs

$$C_n^j = e^{-r\Delta t} [p_u C_{n+1}^{j+1} + p_m C_{n+1}^j + p_d C_{n+1}^{j-1}] \mathbf{1}_{\text{no knock-out at time } t_n}$$

Pricing of American put

$$C_n^j = \max((K - S_{t_n}^j)^+, e^{-r\Delta t} [p_u C_{n+1}^{j+1} + p_m C_{n+1}^j + p_d C_{n+1}^{j-1}])$$

Formulas are similar to binomial tree.