

Assignment 3

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FINM 36702: Portfolio Credit Risk: Modeling and Estimation

Due: 18:00 (CT) April 13th 2023

1: Default Rate and Loss Given Default

The below two statements were given in the question

$$pdf_{dr}[dr] = 2 - 2dr \quad (1 - 1)$$

$$lgd[dr] = dr^{\frac{1}{2}} \quad (1 - 2)$$

From the two, we may infer the probability density function of lgd :

$$\begin{aligned} \mathbb{P}\{lgd \leq x\} &= \mathbb{P}\{dr^{\frac{1}{2}} \leq x\} \\ &(\because (1 - 2), 0 \leq dr) \\ &= \mathbb{P}\{dr \leq x^2\} \\ &= \begin{cases} \int_0^{x^2} (2 - 2dr)d(dr) & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases} \\ &(\because (1 - 1), 0 \leq lgd \leq 1) \end{aligned}$$

Now focusing on the case where $0 \leq x \leq 1$:

$$\begin{aligned} \mathbb{P}\{lgd \leq x\} &= \int_0^{x^2} (2 - 2dr)d(dr) \\ &= [2dr - (dr)^2]_0^{x^2} \\ &= (2x^2 - x^4) - 0 \\ &= 2x^2 - x^4 \end{aligned}$$

$$\begin{aligned} \therefore pdf_{lgd}[x] &= \frac{\delta}{\delta x} \mathbb{P}\{lgd \leq x\} \\ &= \frac{\delta}{\delta x} (2x^2 - x^4) \\ &= 4x - 4x^3 \end{aligned}$$

Ultimately, for $0 \leq lgd \leq 1$:

$$pdf_{lgd}[lgd] = 4 \cdot lgd - 4 \cdot (lgd)^3 \quad (1 - 3)$$

Now if we plot the two pdfs in (1 - 1) and (1 - 3) for the range $(0, 1)$:

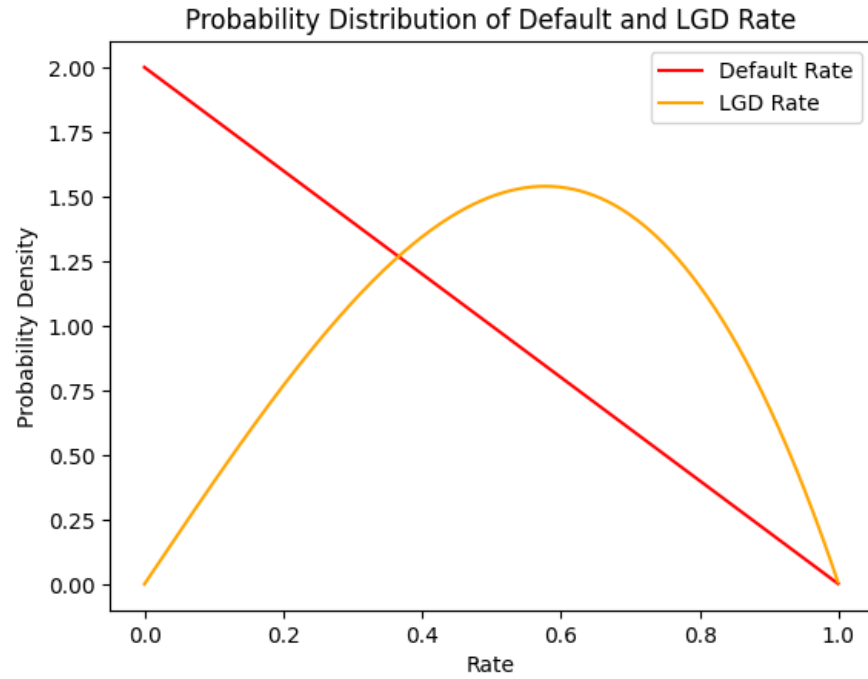


Figure 1: PDF of Default Rate and LGD

2: Loss from Default Rate and Loss Given Default

We know from definition that loss rate is the multiplication of the default rate and the loss given default rate.

$$loss[dr, lgd] = dr \times lgd \quad (2 - 1)$$

Now using the relationship given in (1 - 2), the loss function becomes:

$$loss[dr] = dr^{\frac{3}{2}} \quad (2 - 1^*)$$

Similar to question 1, we can derive the probability density function of loss rate using (2 - 1*) and (1 - 1):

$$\begin{aligned} \mathbb{P}\{loss \leq x\} &= \mathbb{P}\{dr^{\frac{3}{2}} \leq x\} \\ &(\because (2 - 1^*), 0 \leq dr) \\ &= \mathbb{P}\{dr \leq x^{\frac{2}{3}}\} \\ &= \begin{cases} \int_0^{x^{\frac{2}{3}}} (2 - 2dr)d(dr) & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases} \\ &(\because (1 - 1), 0 \leq loss \leq 1) \end{aligned}$$

Now focusing on the case where $0 \leq x \leq 1$:

$$\begin{aligned} \mathbb{P}\{loss \leq x\} &= \int_0^{x^{\frac{2}{3}}} (2 - 2dr)d(dr) \\ &= [2dr - (dr)^2]_0^{x^{\frac{2}{3}}} \\ &= (2x^{\frac{2}{3}} - x^{\frac{4}{3}}) - 0 \\ &= 2x^{\frac{2}{3}} - x^{\frac{4}{3}} \end{aligned}$$

$$\begin{aligned} \therefore pdf_{loss}[x] &= \frac{\delta}{\delta x} \mathbb{P}\{loss \leq x\} \\ &= \frac{\delta}{\delta x} (2x^{\frac{2}{3}} - x^{\frac{4}{3}}) \\ &= \frac{4}{3}x^{-\frac{1}{3}} - \frac{4}{3}x^{\frac{1}{3}} \end{aligned}$$

Ultimately, for $0 \leq loss \leq 1$:

$$pdf_{loss}[loss] = \frac{4}{3} \cdot (loss)^{-\frac{1}{3}} - \frac{4}{3}(loss)^{\frac{1}{3}} \quad (2 - 2)$$

Now if we plot the two pdfs in (1 - 1), (1 - 3), and (2 - 2) for the range (0, 1):

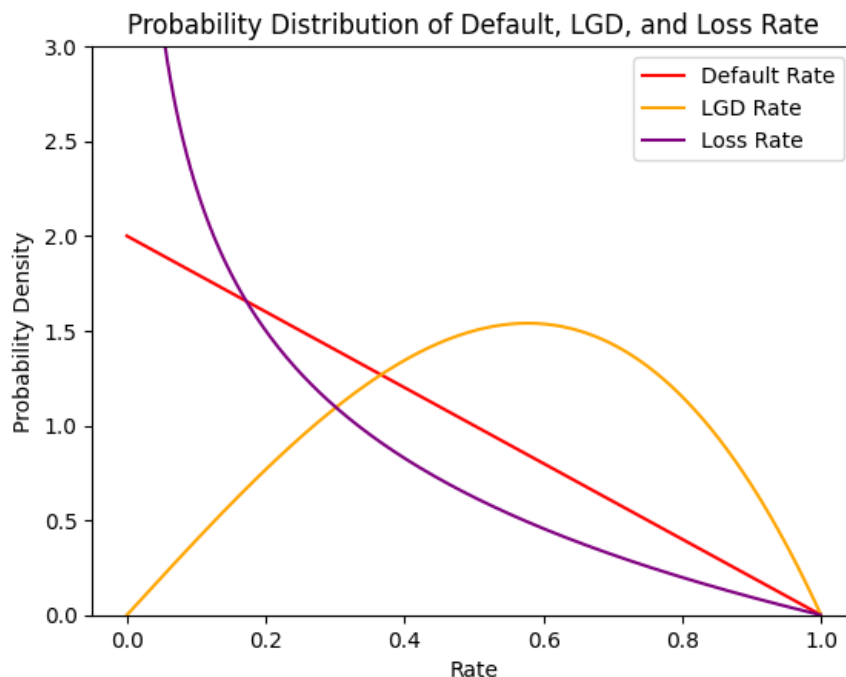


Figure 2: PDF of Default Rate, LGD, and Loss

- Expected Loss:

$$\begin{aligned}
 EL &= \mathbb{E}[loss] \\
 &= \int_0^1 loss \cdot pdf_{loss}[loss] d(loss) \\
 &= \int_0^1 loss \left(\frac{4}{3} \cdot (loss)^{-\frac{1}{3}} - \frac{4}{3} (loss)^{\frac{1}{3}} \right) d(loss) \\
 &= \int_0^1 \left(\frac{4}{3} \cdot (loss)^{\frac{2}{3}} - \frac{4}{3} (loss)^{\frac{4}{3}} \right) d(loss) \\
 &= \left[\frac{4}{5} \cdot (loss)^{\frac{5}{3}} - \frac{4}{7} (loss)^{\frac{7}{3}} \right]_0^1 \\
 &= \left(\frac{4}{5} - \frac{4}{7} \right) - 0 \\
 &= \frac{8}{35}
 \end{aligned}$$

- Expected Loss Given Default:

$$\begin{aligned}
 ELGD &= \frac{EL}{PD} \\
 &= \frac{EL}{\int_0^1 dr \cdot pdf_{dr}[dr]d(dr)} \\
 &= \frac{EL}{\int_0^1 dr \cdot (2 - 2dr)d(dr)} \\
 &= \frac{EL}{\int_0^1 (2 \cdot dr - 2 \cdot (dr)^2) d(dr)} \\
 &= \frac{EL}{\left[(dr)^2 - \frac{2}{3}(dr)^3\right]_0^1} \\
 &= \frac{EL}{\left(1 - \frac{2}{3}\right) - 0} \\
 &= \frac{EL}{\frac{1}{3}} \\
 &= \frac{24}{35}
 \end{aligned}$$

- "Time-weighted" LGD:

$$\begin{aligned}
 EcLGD &= \mathbb{E}[cLGD] \\
 &= \int_0^1 lgd \cdot pdf_{lgd}[lgd]d(lgd) \\
 &= \int_0^1 lgd (4 \cdot lgd - 4 \cdot (lgd)^3) d(lgd) \\
 &= \int_0^1 (4 \cdot (lgd)^2 - 4 \cdot (lgd)^4) d(lgd) \\
 &= \left[\frac{4}{3}(lgd)^3 - \frac{4}{5}(lgd)^5\right]_0^1 \\
 &= \left(\frac{4}{3} - \frac{4}{5}\right) - 0 \\
 &= \frac{8}{15}
 \end{aligned}$$

3: Standard Deviation of a Vasicek Distribution

Let X follow a Vasicek distribution. Then, the standard deviation becomes:

$$\begin{aligned}
 \sigma_X &= \sqrt{\text{Var}(X)} \\
 &= \sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2]} \\
 &= \sqrt{\int_X (x - p)^2 \cdot \text{pdf}_X(x) dx} \\
 &(\because \mathbb{E}[X] = p) \\
 &= \sqrt{\int_0^1 \sqrt{\frac{1-\rho}{\rho}} (x - p)^2 e^{-\frac{1}{2\rho} (\sqrt{1-\rho} \Phi^{-1}(x) - \Phi^{-1}(p))^2 + \frac{1}{2} (\Phi^{-1}(x))^2} dx}
 \end{aligned}$$

Using the integration form above, we may plot the standard deviation of the Vasicek distribution across $0.05 < \rho < 0.95$ for a given $p = 0.10$:

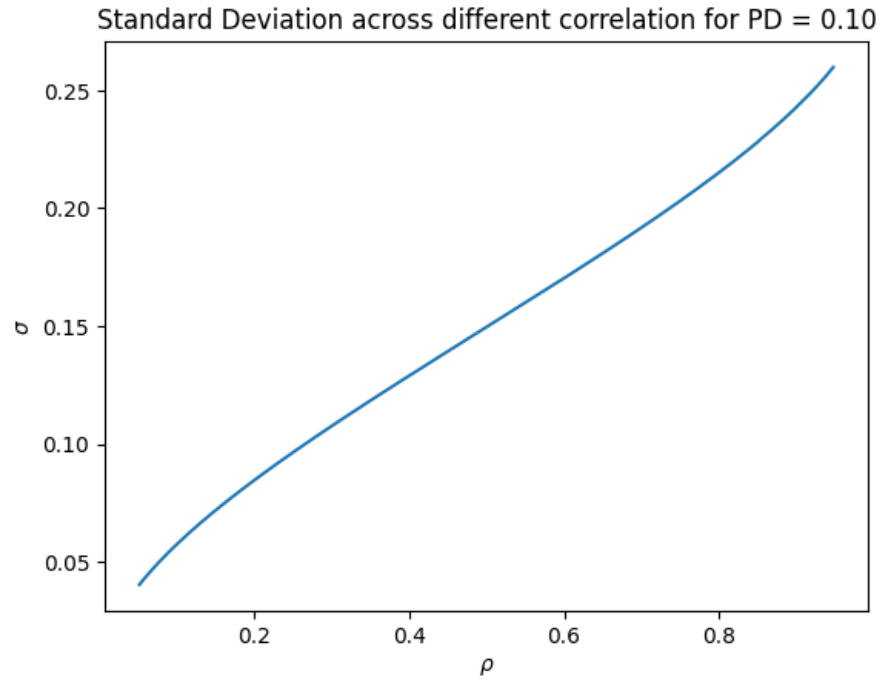


Figure 3: Std. Dev. of Vasicek vs ρ for $PD = 0.10$

The two Vasicek distributions ($Vasicek(p = 0.10, \rho = 0.05)$ and $Vasicek(p = 0.10, \rho = 0.95)$) can be plotted as:

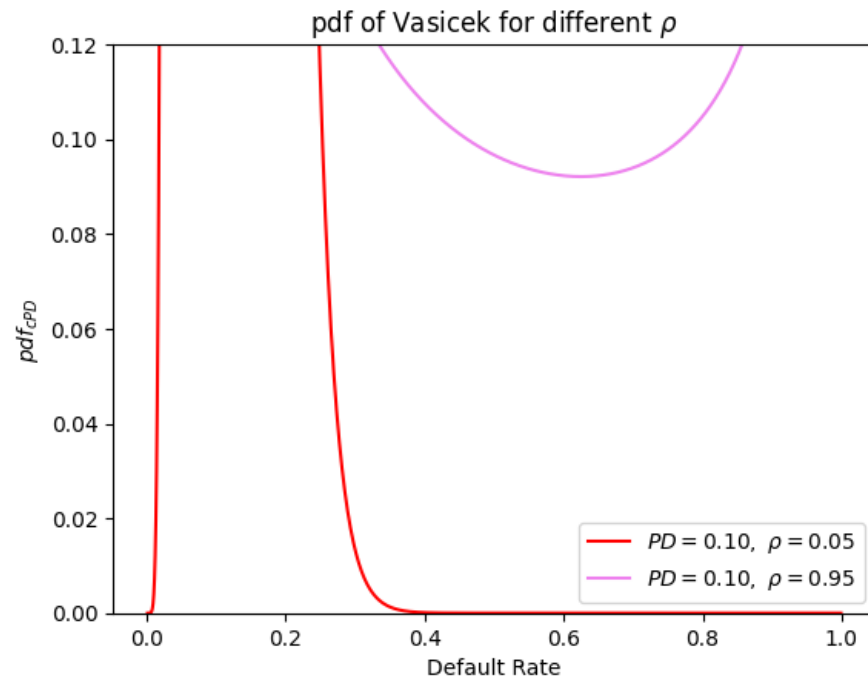


Figure 4: PDF of the two Vasicek Distribution

4: Inverse CDF of Vasicek distribution

Using the CDF of the Vasicek distribution:

$$F(x | p, \rho) = \Phi \left(\frac{\sqrt{1-\rho} \Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}} \right) \quad (4 - 0)$$

From the above CDF (4 - 0), the inverse CDF can be derived as:

$$F^{-1}(x | p, \rho) = \Phi \left(\sqrt{\frac{\rho}{1-\rho}} \Phi^{-1}(x) + \frac{1}{\sqrt{1-\rho}} \Phi^{-1}(p) \right) \quad (4 - 1)$$

The two inverse CDFs may be plotted as below:

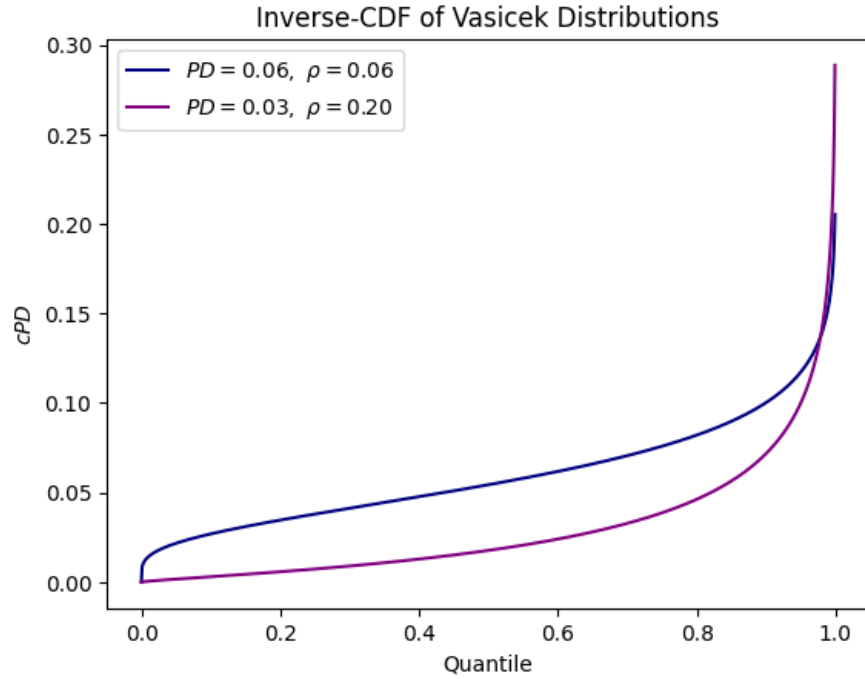


Figure 5: Inverse CDF Plot

Using optimization, the quantile at which both loans have the same value of cPD can be estimated as ≈ 0.98