Assignment 3

Ki Hyun
FINM 36702: Portfolio Credit Risk: Modeling and Estimation

Due: 18:00 (CT) April 13th 2023

1: Default Rate and Loss Given Default

The below two statements were given in the question

$$pdf_{dr}[dr] = 2 - 2dr \tag{1 - 1}$$

$$lgd[dr] = dr^{\frac{1}{2}} \tag{1 - 2}$$

From the two, we may infer the probability density function of lgd:

$$\begin{aligned}
\mathbb{P}\{lgd \leq x\} &= \mathbb{P}\{dr^{\frac{1}{2}} \leq x\} \\
&(\because\ (1-2),\ 0 \leq dr) \\
&= \mathbb{P}\{dr \leq x^2\} \\
&= \begin{cases} \int_0^{x^2} (2-2dr)d(dr) & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases} \\
&(\because (1-1),\ 0 \leq lgd \leq 1)
\end{aligned}$$

Now focusing on the case where $0 \leq x \leq 1$:

$$\begin{aligned}
\mathbb{P}\{lgd \leq x\} &= \int_0^{x^2} (2-2dr)d(dr) \\
&= \left[2dr - (dr)^2\right]_0^{x^2} \\
&= (2x^2 - x^4) - 0 \\
&= 2x^2 - x^4
\end{aligned}$$

$$\begin{aligned}
\therefore pdf_{lgd}[x] &= \frac{\delta}{\delta x}\mathbb{P}\{lgd \leq x\} \\
&= \frac{\delta}{\delta x}(2x^2 - x^4) \\
&= 4x - 4x^3
\end{aligned}$$

Ultimately, for $0 \leq lgd \leq 1$:

$$pdf_{lgd}[lgd] = 4 \cdot lgd - 4 \cdot (lgd)^3 \tag{1 - 3}$$

Now if we plot the two pdfs in (1 - 1) and (1 - 3) for the range $(0, 1)$:

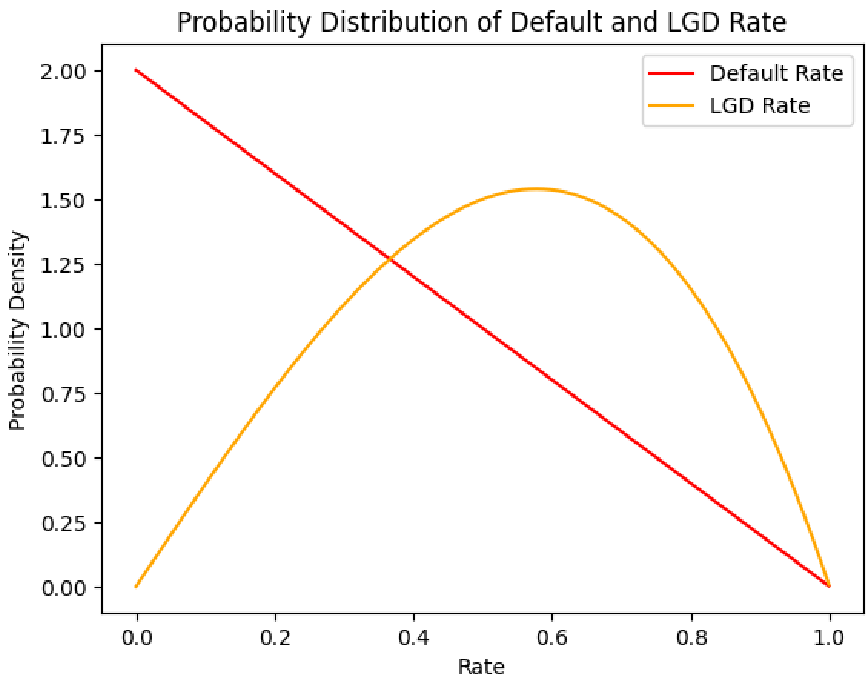

Figure 1: PDF of Default Rate and LGD

2: Loss from Default Rate and Loss Given Default

We know from definition that loss rate is the multiplication of the default rate and the loss given default rate.

$$loss[dr, lgd] = dr \times lgd \tag{2 - 1}$$

Now using the relationship given in (1 - 2), the loss function becomes:

$$loss[dr] = dr^{\frac{3}{2}} \tag{2 - 1*}$$

Similar to question 1, we can derive the probability density function of loss rate using (2 - 1*) and (1 - 1):

$$\begin{aligned}
\mathbb{P}\{loss \leq x\} &= \mathbb{P}\{dr^{\frac{3}{2}} \leq x\} \\
&(\because (2-1*),\ 0 \leq dr) \\
&= \mathbb{P}\{dr \leq x^{\frac{2}{3}}\} \\
&= \begin{cases} \int_0^{x^{\frac{2}{3}}} (2-2dr)d(dr) & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases} \\
&(\because (1-1),\ 0 \leq loss \leq 1)
\end{aligned}$$

Now focusing on the case where $0 \leq x \leq 1$:

$$\begin{aligned}
\mathbb{P}\{loss \leq x\} &= \int_0^{x^{\frac{2}{3}}} (2-2dr)d(dr) \\
&= \left[2dr - (dr)^2\right]_0^{x^{\frac{2}{3}}} \\
&= (2x^{\frac{2}{3}} - x^{\frac{4}{3}}) - 0 \\
&= 2x^{\frac{2}{3}} - x^{\frac{4}{3}}
\end{aligned}$$

$$\begin{aligned}
\therefore pdf_{loss}[x] &= \frac{\delta}{\delta x}\mathbb{P}\{loss \leq x\} \\
&= \frac{\delta}{\delta x}(2x^{\frac{2}{3}} - x^{\frac{4}{3}}) \\
&= \frac{4}{3}x^{-\frac{1}{3}} - \frac{4}{3}x^{\frac{1}{3}}
\end{aligned}$$

Ultimately, for $0 \leq loss \leq 1$:

$$pdf_{loss}[loss] = \frac{4}{3} \cdot (loss)^{-\frac{1}{3}} - \frac{4}{3}(loss)^{\frac{1}{3}} \tag{2 - 2}$$

Now if we plot the two pdfs in (1 - 1), (1 - 3), and (2 - 2) for the range $(0, 1)$:

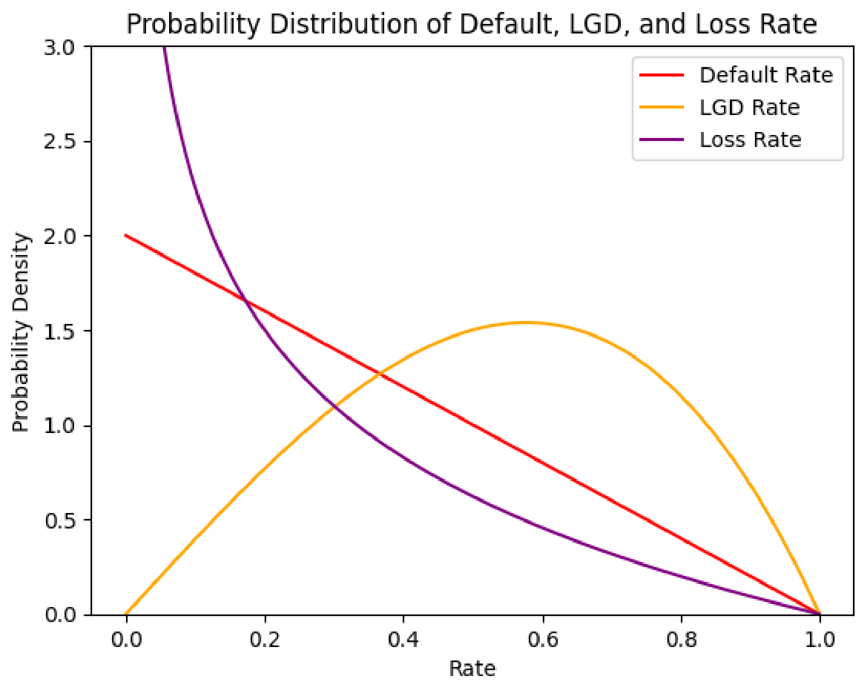

Figure 2: PDF of Default Rate, LGD, and Loss

- Expected Loss:

$$
\begin{aligned}
EL &= \mathbb{E}[loss] \\
&= \int_0^1 loss \cdot pdf_{loss}[loss] d(loss) \\
&= \int_0^1 loss \left(\frac{4}{3} \cdot (loss)^{-\frac{1}{3}} - \frac{4}{3} (loss)^{\frac{1}{3}} \right) d(loss) \\
&= \int_0^1 \left(\frac{4}{3} \cdot (loss)^{\frac{2}{3}} - \frac{4}{3} (loss)^{\frac{4}{3}} \right) d(loss) \\
&= \left[\frac{4}{5} \cdot (loss)^{\frac{5}{3}} - \frac{4}{7} (loss)^{\frac{7}{3}} \right]_0^1 \\
&= \left(\frac{4}{5} - \frac{4}{7} \right) - 0 \\
&= \frac{8}{35}
\end{aligned}
$$

- Expected Loss Given Default:

$$
\begin{aligned}
ELGD &= \frac{EL}{PD} \\
&= \frac{EL}{\int_0^1 dr \cdot pdf_{dr}[dr]d(dr)} \\
&= \frac{EL}{\int_0^1 dr \cdot (2-2dr)d(dr)} \\
&= \frac{EL}{\int_0^1 \left(2 \cdot dr - 2 \cdot (dr)^2\right) d(dr)} \\
&= \frac{EL}{\left[(dr)^2 - \frac{2}{3}(dr)^3\right]_0^1} \\
&= \frac{EL}{\left(1-\frac{2}{3}\right) - 0} \\
&= \frac{EL}{\frac{1}{3}} \\
&= \frac{24}{35}
\end{aligned}
$$

- "Time-weighted" LGD:

$$
\begin{aligned}
EcLGD &= \mathbb{E}[cLGD] \\
&= \int_0^1 lgd \cdot pdf_{lgd}[lgd]d(lgd) \\
&= \int_0^1 lgd \left(4 \cdot lgd - 4 \cdot (lgd)^3\right) d(lgd) \\
&= \int_0^1 \left(4 \cdot (lgd)^2 - 4 \cdot (lgd)^4\right) d(lgd) \\
&= \left[\frac{4}{3}(lgd)^3 - \frac{4}{5}(lgd)^5\right]_0^1 \\
&= \left(\frac{4}{3} - \frac{4}{5}\right) - 0 \\
&= \frac{8}{15}
\end{aligned}
$$

3: Standard Deviation of a Vasicek Distribution

Let X follow a Vasicek distribution. Then, the standard deviation becomes:

$$\begin{aligned}
\sigma_X &= \sqrt{Var(X)} \\
&= \sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2]} \\
&= \sqrt{\int_X (x-p)^2 \cdot pdf_X(x)dx} \\
&(\because \mathbb{E}[X] = p) \\
&= \sqrt{\int_0^1 \sqrt{\frac{1-\rho}{\rho}}(x-p)^2 e^{-\frac{1}{2\rho}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(p)\right)^2 + \frac{1}{2}(\Phi^{-1}(x))^2} dx}
\end{aligned}$$

Using the integration form above, we may plot the standard deviation of the Vasicek distribution across $0.05 < \rho < 0.95$ for a given $p = 0.10$:

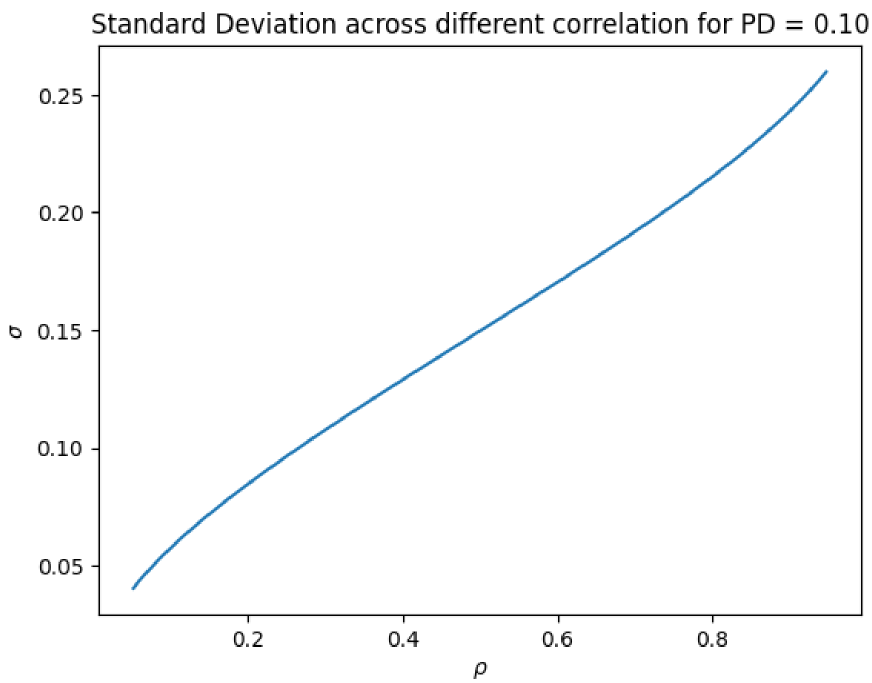

Figure 3: Std. Dev. of Vasicek vs ρ for $PD = 0.10$

The two Vasicek distributions ($Vasicek(p = 0.10, \rho = 0.05)$ and $Vasicek(p = 0.10, \rho = 0.95)$) can be plotted as:

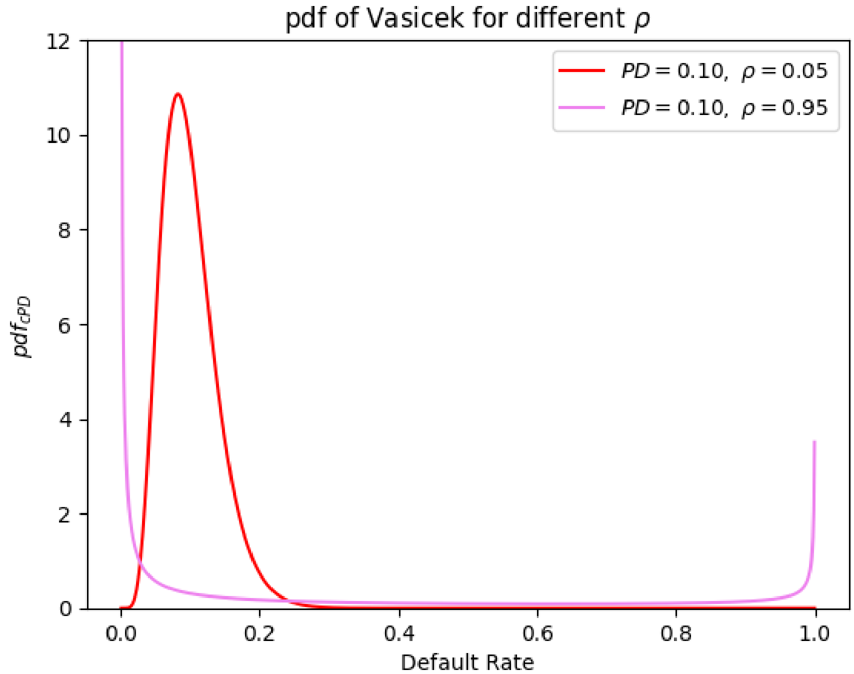

Figure 4: PDF of the two Vasicek Distribution

4: Inverse CDF of Vasicek distribution

Using the CDF of the Vasicek distribution:

$$F(x \mid p, \rho) = \Phi\left(\frac{\sqrt{1-\rho}\,\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\rho}}\right) \tag{4 - 0}$$

From the above CDF (4 - 0), the inverse CDF can be derived as:

$$F^{-1}(x \mid p, \rho) = \Phi\left(\sqrt{\frac{\rho}{1-\rho}}\Phi^{-1}(x) + \frac{1}{\sqrt{1-\rho}}\Phi^{-1}(p)\right) \tag{4 - 1}$$

The two inverse CDFs may be plotted as below:

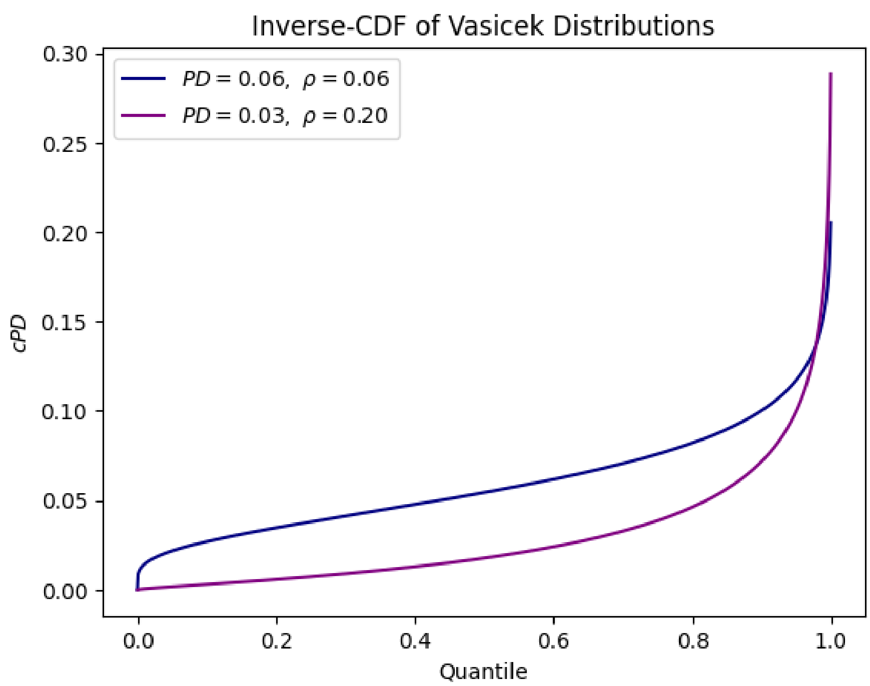

Figure 5: Inverse CDF Plot

Using optimization, the quantile at which both loans have the same value of cPD can be estimated as ≈ 0.98