Assignment 2

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FINM 36702: Portfolio Credit Risk: Modeling and Estimation

Due: 18:00 (CT) April 6th 2023

1: Std. Dev. of Defaults

Correlation Matrix	**Std. Dev**
Original	**1.2**
Identity Matrix*	**0.98**

Table 1: Correlation and Std. Dev. of Defaults

* The identity matrix mentioned above would be in the form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2: ρ vs Risk

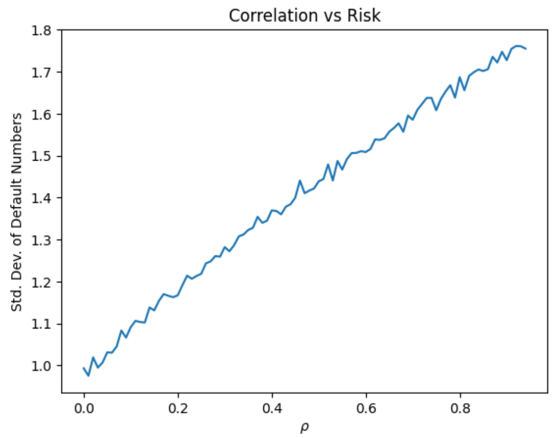

Figure 1: Std. Dev. of Defaults as a function of ρ

3: Exposures in Loan

(i: $\mathbb{P}[D_4 = 1, D_5 = 1]$**)**

$\mathbb{P}[D_4 = 1, D_5 = 1]$ can be calculated analytically using the multivariate normal distribution of the latent variables of all the 5 firms.

If the latent variable for firm i is d_i, the distribution can be represented as:

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{pmatrix} \sim MVN(\mu, \mathbf{\Sigma})$$

Here,

$$\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{\Sigma} = \begin{bmatrix} 1 & 0.15 & 0.2 & 0.25 & 0.3 \\ 0.15 & 1 & 0.25 & 0.3 & 0.35 \\ 0.2 & 0.25 & 1 & 0.35 & 0.4 \\ 0.25 & 0.3 & 0.35 & 1 & 0.45 \\ 0.3 & 0.35 & 0.4 & 0.45 & 1 \end{bmatrix}$$

Now the mathematical form for $\mathbb{P}[D_4 = 1, D_5 = 1]$ can be expressed as:

$$\begin{aligned}
&\mathbb{P}[D_4 = 1, D_5 = 1] \\
=&\int_{-\infty}^{\Phi^{-1}(PD_5)} \int_{-\infty}^{\Phi^{-1}(PD_4)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d} - \mu)^{\mathrm{T}} \mathbf{\Sigma}^{-1} (\mathbf{d} - \mu)\right) \\
&dd_1 dd_2 dd_3 dd_4 dd_5 \\
=&\int_{-\infty}^{\Phi^{-1}(0.5)} \int_{-\infty}^{\Phi^{-1}(0.4)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d} - \mu)^{\mathrm{T}} \mathbf{\Sigma}^{-1} (\mathbf{d} - \mu)\right) \\
&dd_1 dd_2 dd_3 dd_4 dd_5 \\
&(\because PD_4 = 0.4, PD_5 = 0.5) \\
\approx&0.27
\end{aligned}$$

$$\therefore \mathbb{P}[D_4 = 1, D_5 = 1] \approx 0.27$$

(ii: $\mathbb{P}[D_4 = 1, D_5 = 1 \mid D_3 = 1]$**)**

Using the definition of conditional probability:

$$\mathbb{P}[D_4 = 1, D_5 = 1 \mid D_3 = 1] = \frac{\mathbb{P}[D_4 = 1, D_5 = 1, D_3 = 1]}{\mathbb{P}[D_3 = 1]}$$

Here,

$$
\begin{aligned}
&\mathbb{P}[D_4 = 1, D_5 = 1, D_3 = 1] \\
&= \int_{-\infty}^{\Phi^{-1}(PD_5)} \int_{-\infty}^{\Phi^{-1}(PD_4)} \int_{-\infty}^{\Phi^{-1}(PD_3)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d}-\mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{d}-\mu)\right) \\
&\quad dd_1 dd_2 dd_3 dd_4 dd_5 \\
&= \int_{-\infty}^{\Phi^{-1}(0.5)} \int_{-\infty}^{\Phi^{-1}(0.4)} \int_{-\infty}^{\Phi^{-1}(0.3)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d}-\mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{d}-\mu)\right) \\
&\quad dd_1 dd_2 dd_3 dd_4 dd_5 \\
&\quad (\because PD_4 = 0.4, PD_5 = 0.5, PD_3 = 0.3)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\mathbb{P}[D_3 = 1] \\
&= \int_{-\infty}^{\Phi^{-1}(PD_3)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d}-\mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{d}-\mu)\right) dd_1 dd_2 dd_4 dd_5 dd_3 \\
&= \int_{-\infty}^{\Phi^{-1}(0.3)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d}-\mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{d}-\mu)\right) dd_1 dd_2 dd_4 dd_5 dd_3 \\
&(\because PD_3 = 0.3)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{P}[D_4 = 1, D_5 = 1 \mid D_3 = 1] \\
=&\frac{\mathbb{P}[D_4 = 1, D_5 = 1, D_3 = 1]}{\mathbb{P}[D_3 = 1]} \\
=&\frac{\int_{-\infty}^{\Phi^{-1}(0.5)} \int_{-\infty}^{\Phi^{-1}(0.4)} \int_{-\infty}^{\Phi^{-1}(0.3)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d}-\mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{d}-\mu)\right) dd_1 dd_2 dd_3 dd_4 dd_5}{\int_{-\infty}^{\Phi^{-1}(0.3)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d}-\mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{d}-\mu)\right) dd_1 dd_2 dd_4 dd_5 dd_3} \\
\approx&0.44
\end{aligned}
$$

$$\therefore \mathbb{P}[D_4 = 1, D_5 = 1 \mid D_3 = 1] \approx 0.44$$

(iii: Portfolio Expected Loss Rate)

Given the ELGD and Exposure of the portfolio across the 7 loans, the expected loss at default for each firm could be summarized to:

Firm	Expected Loss at Default
Firm 1	70
Firm 2	120
Firm 3	150
Firm 4	280
Firm 5	220

Now using 1,000,000 simulations from the $MVN(\mu, \mathbf{\Sigma})$ and comparing each result with the Probabilities of Default $(PD_1, \ldots, PD_5)$ to get 1,000,000 vectors in the form:

$$\mathbf{D}_{sim=j} = \begin{pmatrix} D_{1,sim=j} \\ D_{2,sim=j} \\ D_{3,sim=j} \\ D_{4,sim=j} \\ D_{5,sim=j} \end{pmatrix}$$

Here, $D_{i,sim=j}$ will represent if firm i has defaulted in the j^{th} simulation:

$$D_{i,sim=j} = \begin{cases} 1 & \text{Firm } i \text{ Defaulted in the jth simulation} \\ 0 & \text{Otherwise in the jth simulation} \end{cases}$$

Now, if we let $\mathbb{E}[\text{Loss}]$ be the vector:

$$\mathbb{E}[\text{Loss}] = \begin{pmatrix} 70 \\ 120 \\ 150 \\ 280 \\ 220 \end{pmatrix}$$

Our expected **portfolio** loss from the simulation will be:

$$\sum_{j=1}^{1{,}000{,}000} \mathbf{D}_{sim=j} \cdot \mathbb{E}[\text{Loss}]$$

Moreover, the expected loss rate can be estimated as:

$$\mathbb{E}[\text{Loss Rate}] = \frac{2800}{\sum_{j=1}^{1{,}000{,}000} \mathbf{D}_{sim=j} \cdot \mathbb{E}[\text{Loss}]} \approx 0.11$$

(iv: Dcorr)

First, similar to (part i), we can get $PDJ_{3,4}$ as:

$$\begin{aligned}
&\mathbb{P}[D_3 = 1, D_4 = 1] \\
&= \int_{-\infty}^{\Phi^{-1}(PD_4)} \int_{-\infty}^{\Phi^{-1}(PD_3)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d} - \mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{d} - \mu)\right) \\
&dd_1 dd_2 dd_5 dd_3 dd_4 \\
&= \int_{-\infty}^{\Phi^{-1}(0.4)} \int_{-\infty}^{\Phi^{-1}(0.3)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{d} - \mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{d} - \mu)\right) \\
&dd_1 dd_2 dd_5 dd_3 dd_4 \\
&(\because PD_3 = 0.3, PD_4 = 0.4)
\end{aligned}$$

Now using the properties of a Bernoulli distribution:

$$
\begin{aligned}
&Dcorr_{3,4}\\
=&\frac{PDJ_{3,4}-PD_3\cdot PD_4}{\sqrt{PD_3(1-PD_3)}\sqrt{PD_4(1-PD_4)}}\\
=&\frac{\int_{-\infty}^{\Phi^{-1}(0.4)}\int_{-\infty}^{\Phi^{-1}(0.3)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(2\pi)^{5/2}|\mathbf{\Sigma}|^{1/2}}\exp\left(-\frac{1}{2}(\mathbf{d}-\mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{d}-\mu)\right)dd_1dd_2dd_5dd_3dd_4}{\sqrt{0.3(1-0.3)\cdot 0.4(1-0.4)}}\\
&-\frac{0.3\cdot 0.4}{\sqrt{0.3(1-0.3)\cdot 0.4(1-0.4)}}\\
\approx&0.22
\end{aligned}
$$

$$\therefore Dcorr_{3,4}\approx 0.22$$

4: 36702 Distribution

The joint probability density of x and y was given as:

$$f_{x,y}[x,y]=\frac{1+3x-y}{2} \tag{4 - 0}$$

From this, we can derive the marginal density as:

$$f_x[x]=\int_0^1\frac{1+3x-y}{2}dy=\frac{3}{2}x+\frac{1}{4}$$

and

$$f_y[y]=\int_0^1\frac{1+3x-y}{2}dx=-\frac{1}{2}y+\frac{5}{4}$$

The marginal density now gives way to marginal CDF:

$$F_x[z]=\int_0^z\frac{3}{2}x+\frac{1}{4}dx=\frac{3}{4}z^2+\frac{1}{4}z$$

and

$$F_y[z]=\int_0^z-\frac{1}{2}y+\frac{5}{4}dy=-\frac{1}{4}z^2+\frac{5}{4}z$$

Now from the marginal CDF, we get get the inverse CDF (quantile function) as:

$$F_x^{-1}[p]:p=\frac{3}{4}z^2+\frac{1}{4}z,\ 0\le p,z\le 1$$

$$\Leftrightarrow F_x^{-1}[p]=\frac{-1+\sqrt{1+48p}}{6} \tag{4 - 1}$$

Moreover,

$$F_y^{-1}[p] : p = -\frac{1}{4}z^2 + \frac{5}{4}z,\ 0 \leq p, z \leq 1$$

$$\Leftrightarrow F_y^{-1}[p] = \frac{5 - \sqrt{25 - 16p}}{2} \tag{4 - 2}$$

(i: PDJ)

The probability of default for x and y were given as $PD_x = 0.1$ and $PD_y = 0.2$.

Using this knowledge and the above, we can express the PDJ as:

$$\begin{aligned}
&PDJ_{x,y} \\
&= \int_0^{F_y^{-1}[0.2]} \int_0^{F_x^{-1}[0.1]} f_{x,y}[x, y] dxdy \\
&= \int_0^{F_y^{-1}[0.2]} \int_0^{F_x^{-1}[0.1]} \frac{1 + 3x - y}{2} dxdy \\
&(\because (4 - 0)) \\
&= \int_0^{\frac{5-\sqrt{25-16\cdot 0.2}}{2}} \int_0^{\frac{-1+\sqrt{1+48\cdot 0.1}}{6}} \frac{1 + 3x - y}{2} dxdy \\
&(\because (4 - 1),\ (4 - 2)) \\
&\approx 0.025
\end{aligned}$$

(ii: Dcorr)

Using the PDJ above, the Dcorr can be calculated using the characteristics of a Bernoulli distribution as:

$$\begin{aligned}
&Dcorr_{x,y} \\
=&\frac{PDJ_{x,y} - PD_x \cdot PD_y}{\sqrt{PD_x(1 - PD_x)}\sqrt{PD_y(1 - PD_y)}} \\
\approx&0.039
\end{aligned}$$

(iii: ρ)

In order to find the ρ we need to declare another set of latent variables that are Gaussian copula. If we let those latent variables be n_x and n_y:

$$\mathbf{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix} \sim MVN(\mu, \mathbf{\Sigma})$$

where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

$PDJ_{x,y}$ can be expressed using the above Gaussian distribution as:

$$PDJ_{x,y} = \int_{-\infty}^{\Phi^{-1}[PD_y]} \int_{-\infty}^{\Phi^{-1}[PD_x]} \frac{1}{(2\pi)|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{n}-\mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{n}-\mu)\right) dxdy$$

Now using the calculated $PDJ_{x,y}$ value and the given PD_x and PD_y values, the equation above becomes:

$$0.025 = \int_{-\infty}^{\Phi^{-1}[0.2]} \int_{-\infty}^{\Phi^{-1}[0.1]} \frac{1}{(2\pi)|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{n}-\mu)^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{n}-\mu)\right) dxdy$$

Using python, the ρ that satisfies the above equation turns out to be approximately 0.090.

$$\therefore \rho \approx 0.090$$