Portfolio Credit Risk

University of Chicago Masters in Financial Mathematics 36702

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Lecture 3

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Expected LGD, vended estimates of PD and ρ, statistical tools, search and significance

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Probability and Statistics

Probability

- Given a mechanism that produces random outputs,
  - what is the probability that a particular set of outcomes comes out?
- Example: The sum of spots on two six-sided dice.
  - We make assumptions (each surface appears exactly 1/6 of the time, independence, etc.) and use logic to arrive at answers. Very satisfying!

Statistics

- Given some data, find models that can generate it.
  - Simpler models have fewer things that can go wrong.
  - Richer models can match the data more closely.
  - We can test to see which model seems to work best, given the data sample.
  - We can never be sure that we have the best model. Very unsatisfying!

The portfolio default rate model

We imagine that a bank can accurately estimate PDs.

We do not know how the defaults of firms are ***connected***.

- This is as dispiriting to write as it is to read.

Statistical analysis offers few alternatives.

- The *t*-copula inhibits a derivation like the Vasicek Distribution.
- We use the Gauss copula, and so do most banks.
  - A 1,000 firm portfolio has 499,500 $\rho_{i,j}$.
  - "The bigger the matrix, the less likely it is positive definite."

We restrict the Gauss copula to the single factor model.

- The 499,500 values of $\rho_{i,j}$ are implied by 1,000 values of ρ_i.
  - Each Z_i is a linear function of Z and $\{X_i\}$. Therefore, $\{Z_i\}$ are jointly normal.
  - The correlation matrix of $\{Z_i\}$ is positive definite.

Questions or comments?

Questions or thoughts?

Week 3 topics

The *t*-copula: Slides 52-58 and 73-81 from Lecture 1

LGD and its expectation

Vended estimates of PD and ρ

Review and preview of statistical tools

How model search defeats indications of significance

LGD and its expectation

Definitions

Historical data

Expected LGD

Definitions

Loss *given* default

A firm declares bankruptcy to get protection from lawsuits.

- Otherwise the defaulting firm could get sued by every firm it owes.
- The court determines who gets what, usually honoring seniority.
  - It is rare for every creditor to gain 100% recovery.

If a firm is 90 days behind schedule on a bank loan, the bank has many options.

- The bank might continue to seek greater recovery for years.
  - Eventually the bank determines is has obtained all it can.
  - It discounts all cash flows back to the time of default and computes an LGD.

LGD is the lender's loss as a fraction of loan exposure.

Loans and bonds

Most forms of debt are loans or bonds.

- **Loans are private agreements between a firm and a "bank",**
  - **or other financial institution, hedge fund, etc.**
  - **Banks recognize a corporate default when the loan is 90 days past due.**
  - **LGDs on loans are private information.**
- **Bonds are publicly traded promises to repay on a schedule.**
  - **Bonds are considered to be in default if a payment is one day late.**
  - **LGD is usually calculated as LGD = 1 – post-default price / par.**

Each debt instrument has a defined <u>*seniority*</u>.

- **In bankruptcy, the seniority of a debt determines the likelihood that the lender will obtain full repayment.**
  - **The most-senior debt gets full repayment if possible.**
  - **If the firm has money is left over, the next-most-senior debt gets paid.**
  - **And so forth down the scale of seniority.**
  - **Net, more-senior debt tends to have lower LGD than junior debt.**

The scale of seniority

Loans are senior to bonds.

A firm probably has multiple bonds outstanding.

- The names of bonds usually reflect seniority.
  - "Senior Debentures" would be more senior.
  - "Junior Subordinated Notes" would be among the least senior.

A bankruptcy judge tends to follow "strict seniority."

- The banks recover the largest fraction of their exposure.
- If there is money left over after the banks are paid, it goes to the holders of the most senior bonds.
- And so on down the scale of seniority.

If there is no bankruptcy, loan recoveries remain private.

- But the firm must act as it has promised in debt documents.

Security

Some loans and some bonds have a “second way out”:

- **The debt is secured with *collateral*.**
  - **It is like a consumer auto loan or home loan.**
  - **In the event of default, the debt holder obtains ownership of the identified collateral asset and sells the collateral to obtain partial or full recovery.**
- **If the collateral does not provide a full recovery, the remaining exposure has a defined place on the scale of seniority.**

In a default,

- **LGD is likely to be least for a senior secured bank loan, and**
- **LGD is likely to be greatest for a junior unsecured bond.**

Looking across a large number of defaults,

- **Bank loans usually enjoy substantial recovery.**
- **A “sub” bond might a small fraction of par, on average.**

Strange things happen

- A loan secured by a gasoline station defaults.
  - The bank gets title to the station and the land under it.
  - The gasoline tanks had been leaking.
    - Land under the station must be removed and replaced.
    - The loss to the bank was several times the amount of the loan.
    - LGD is greater than 100%.
- A loan secured by a different station defaults.
  - The bank itself runs the gas station for months or years.
  - Surprisingly, the station becomes successful.
    - When the bank sells the station, total recovery is greater than the amount of the original loan. LGD is less than zero.

Historical data

Bonds and loans

For a publicly traded bond, a good measure of LGD is par (100%) minus the post-default price.

Publicly traded loans are scarce.

- Therefore, banks need a different measure for loans.

A bank estimates most LGDs by discounting cash flows to the date of the default.

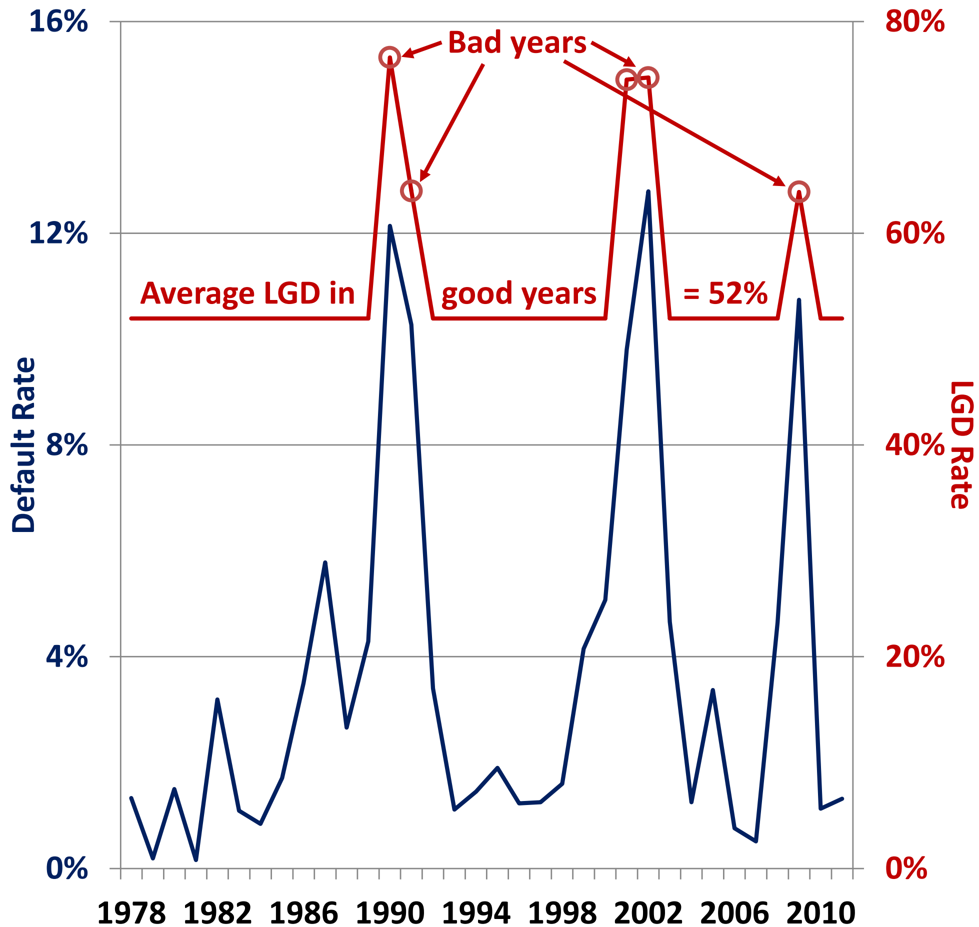
- A bank need to keep track of the costs of recovery.
- It can take months or years to fully resolve a default.
  - And no observable rate of discount is clearly appropriate.
  - No other asset has risks like those of a defaulted bank loan.

Historical bond LGDs

The history of bond LGDs shows three broad behaviors:

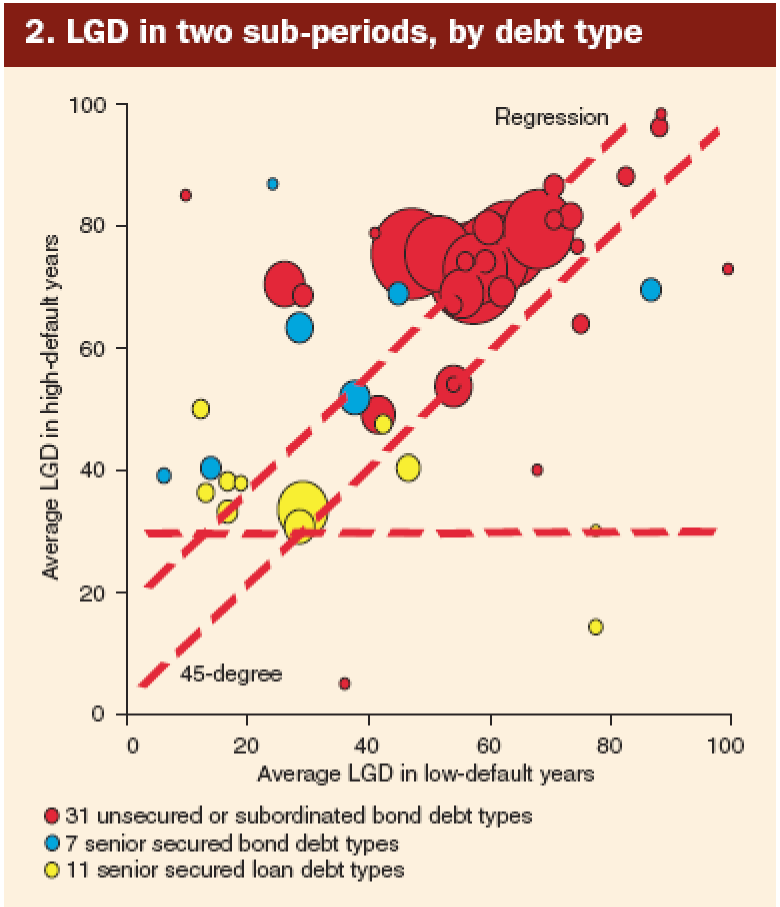
- **Average LGD is elevated when the default rate is elevated.**
- **The elevation is moderate.**
- **The elevation is similar across bonds with different seniorities and levels of security.**

Positive, moderate response



Altman's junk bond data by years

Similar response



Moody's loans and bonds

Questions or comments?

ELGD: Expected loss given default

Orientation

A lender wants to estimate ELGD, a loan's *expected* LGD.

- **Using data, take the average of historical LGD's of similar loans.**
  - **In the data there are many *more* defaults in 2009 than in most other years.**
  - **Each of these defaults produces an LGD.**
  - **On average, these LGDs are *greater* than in most other years.**

When calculating ELGD, one must build in the same effect.

- **The average of annual LGD rates would give the wrong answer,**
  - **because 2009 has many LGDs and it has elevated LGDs.**

Suppose that a loan defaults in one of two periods.

- **In a low-default period the loan is not likely to default, and it will tend to have a low LGD.**
- **In a high-default period the loan is more likely to default, and it will tend to have a high LGD.**
- **This affects the loan's ELGD.**

Roundup of symbols

LGD: The loss given default of a loan is a random variable.

cLGD: Conditionally expected LGD

- **What value of LGD should you expect in certain conditions?**

ELGD: Expected LGD

- **What value of LGD should you expect for a given loan?**

	Default	Loss given default
Observed random variable	D	LGD
Conditional expectation	cPD	cLGD
Unconditional expectation	PD	ELGD

Three identities for any loan

Loss = D * LGD

– If D = 0, then Loss = 0;   if D = 1, then Loss = LGD.

- Both Loss and LGD are expressed as fractions of the exposure amount.

cLoss = cPD * cLGD

– These quantities refer to particular conditions such as $Z = z$.

EL = PD * ELGD

– Proof: E [Loss] = E [D * LGD]

= E [D] * E [LGD] = PD * ELGD

- "A conditional variable is independent of whatever it is conditioned on."
- Suppose you know that if A happens, then B will happen. Then you find out that A has happened. It is still the case that B will happen, right?

Loan expectations

Suppose a loan has two states:

	Prob	cPD	cLGD	cLoss
State 1	2/3	0.1	0.3	0.03
State 2	1/3	0.4	0.6	0.24

PD = .1 (2/3) + .4 (1/3) = .2

EL = .03 (2/3) + .24 (1/3) = .1

ELGD = EL / PD = .1 / .2 = .5

An equivalent calculation weights cLGD by ***both*** probability ***and*** relative frequency:

ELGD = .3 (2/3) (.1 / PD) + .6 (1/3) (.4 / PD)

= .02 / PD + .08 / PD = .1 / .2 = .5

ELGD is also called "Default-weighted LGD."

Expected cLGD

Suppose the same loan as before:

	Prob	cPD	cLGD	cLoss
State 1	2/3	0.1	0.3	0.03
State 2	1/3	0.4	0.6	0.24

E [*c*LGD] = .3 (2/3) + .6 (1/3) = .4

- **This is sometimes called "time-weighted LGD."**
  - **Take average LGD each year, then take the average of averages.**

Note that E [cLGD] < ELGD. When cLGD is elevated, more defaulted loans are produced.

- **Averaging over *conditions* produces E [cLGD].**
- **Averaging over *loans* produces ELGD.**

Say it in math

Suppose you have a distribution of cPD, $\boldsymbol{f_{cPD}[r]}$.

Suppose that cLGD is a function of cPD, $\boldsymbol{cLGD = g[cPD]}$.

$$\boldsymbol{E[cLGD] = \int_0^1 g[r]\, f_{cPD}[r]\, dr}$$

$$\boldsymbol{E[LGD] = EL/PD = \frac{1}{PD}\int_0^1 r\, g[r]\, f_{cPD}[r]\, dr}$$

Introduction to LGD: Summary

- **In a default, a firm can't pay what it owes.**
  - **Usually, one or more of the lenders experiences loss.**
    - **Anticipating this, lenders want security and seniority.**

- **Each loan has specified Seniority (its place on the scale of seniority) and security ("second way out").**
  - **These influence a loan's expected LGD.**
    - **Still, the loss on each defaulted loan is highly random.**

- **Three identities for any given loan:**
  - **Loss = D * LGD**
  - **cLoss = cPD * cLGD**
  - **EL = PD * ELGD**

Questions? Comments?

* Frye-Jacobs Model

Vended estimates of PD and ρ

Vended estimates of PD and ρ

PD_i and $\rho_{i,j}$ are parameters in a credit portfolio model.

- **PD_i is the probability that firm *i* defaults.**
- **$\rho_{i,j}$ is a transformation of the probability that both *i* and *j* default.**

It would be strange and wonderful if these default probabilities could be estimated without using default data.

- **Therefore, this section is somewhat strange and wonderful.**

Asset returns and PD_i

Moody’s EDF estimate of PD_i

The Moody’s EDF is famous, so if you work in this area you will need to know something about it.

- EDF = Estimated Default Frequency = another name for PD.
  - Other vendors have similar methods.

Moody’s starts with an attractive intuition:

- A loan contains the option to default.
- A risk-neutral probability is part of option price theory.
- Therefore, ignoring risk preference, it might be possible to estimate the probability of default using option theory.

Moody's does not tell exactly how they calculate EDF.

- And their method changes over time.
- But here's my impression in stylized terms.

Applying option theory

The owners of a corporation have limited liability.

- **The most they can lose is their investment.**

Stylizing this, the owners of the firm have an option:

- **They can pay the debt holders in money.**
- **They can give debt holders the firm’s assets.**

The firm’s *equity* provides an *option* on its *assets*.

- **The "strike price" is the amount of the firm's liabilities.**
  - **Moody’s measures this as short-term debt plus half of long-term debt.**
- **The expiration of the option would be debt maturity.**
  - **Moody’s might assume one year in all cases.**

→ → → What is the probability that the equity holder exercises the option to default?

Prob [option exercise]

In the case of stocks, $d2 = \frac{Log[S/X] + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$;

(Risk neutral) Probability of call exercise = Φ [$d2$]

***<u>Analogous</u>* to $d2$ is the firm's "distance to default":**

$$DD = \frac{Log[A/D]}{Annual\ volatility\ of\ assets}$$

where A = Value of firm's assets

D = "default point": short-term debt plus half of long-term debt

Sorry to keep re-using the letter D, but that's what Moody's uses.

Probability of put exercise $\approx$ 1– Φ [DD] = Φ [- DD]

The two unknowns in a DD

$$DD = \frac{Log[A/D]}{Annual\ volatility\ of\ assets}$$

Two unknowns in the formula for DD:

- **A = the *__value__* of assets**
- **The *__volatility__* of assets.**
  - **The default point, D, is considered known:**
  - **short-term debt plus half of long-term debt.**

Neither asset value nor asset volatility is directly observable.

Moody's infers them from an iteration…

Inference by double iteration

Stock market capitalization depends on:

- **liabilities and the value and volatility of assets.**

The <u>*volatility*</u> of market cap. also depends on:

- **liabilities and the value and volatility of assets.**

So, some setting of (asset value, asset volatility) best produces the observed values of (market capitalization, market capitalization volatility).

Then, $PD = EDF = 1 - \Phi[DD] = \Phi[\frac{-Log[\frac{A}{D}]}{asset\ vol.}]$**.**

There is something you should know about this...

It gives a very poor answer!

1- Φ[DD] gives a very poor estimate of PD.

- **Consider a firm with DD = 6 = 6 SDs from the default point.**
  - **1- Φ [6] = 0.000000001**
  - **Moody's finds 42,000 instances of DD $\approx$ 6.0 in its data base.**
  - **Option theory says there should have been about 0.00004 defaults.**
  - **But there were 17 defaults. The forecast is wrong by a factor of 400,00.**

Something is exceedingly wrong here.

- **Maybe options theory does not apply well to default.**
- **Maybe quantities are being poorly measured.**
- **Maybe the assumptions of stability are wrong.**
  - **When firms get in trouble, the trouble compounds and vol goes up.**
- **Maybe expiration is not 1 year.**
- **Maybe D is not short term + half of long term.**
- **Maybe asset returns do not have a LogNormal Distribution.**

Questions? Comments?

The EDF work-around

Moody's does ***not*** take EDF = 1 - Φ [DD].

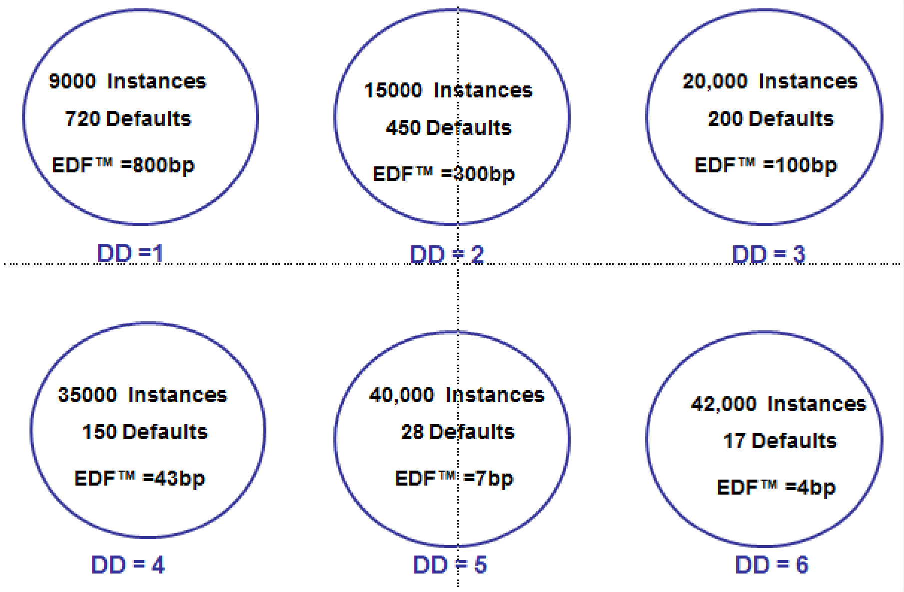
Instead, the EDF of a firm is set equal to the ***average historical default*** rate of firms having the same DD.

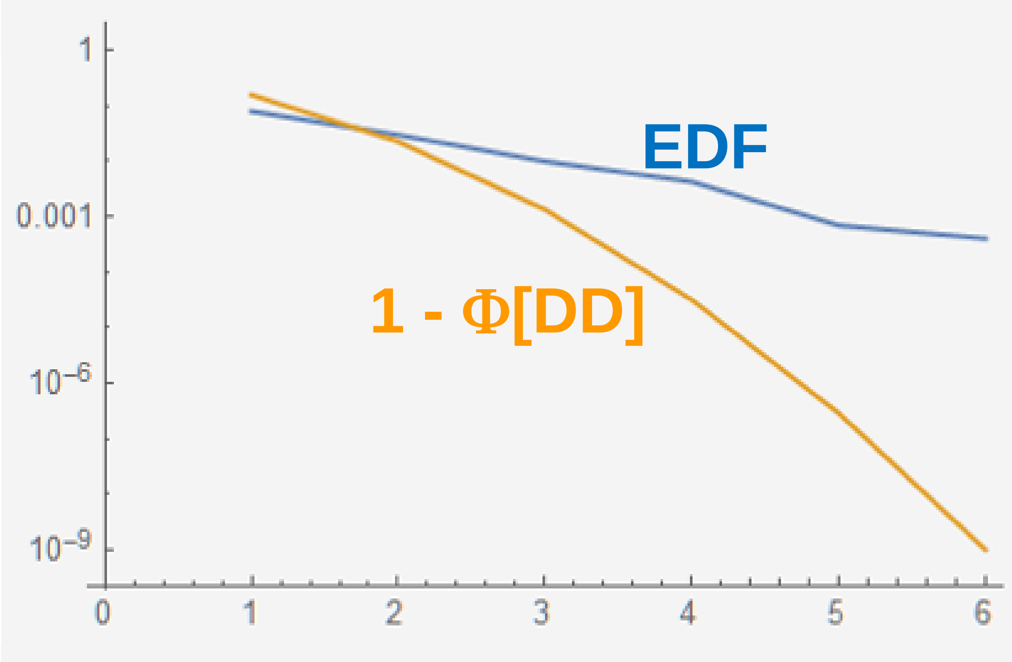
- If a firm has DD = 6, its EDF would be 17 / 42,000 = 0.04%
  - An EDF uses DD to find historical ***analogs*** of current firms.
  - Then it uses historical defaults to calibrate EDFs.

Years ago, I found this on the Moody's website….

Calibrating EDF to DD

Moody's quantifies EDFs with historical default rates, not with the theoretical value of 1 - Φ[DD].





EDF summary

Option theory gives very bad estimates of PD.

- A firm with low DD defaults about half of 1 - Φ[DD].
- A firm with high DD defaults much greater than 1 - Φ[DD].

Moody's recalibrates with historical data.

- An EDF of a firm today is the historical default rate among firms sharing the value of DD.
  - On average, EDFs tend to track the historical data. On average.

At least, that's what I think.

- Moody's changes its procedures over time.
- They do not say exactly what they do.
  - They sell EDFs; they don't sell how to calculate EDF's.
- So, I don't know for sure.

Questions? Comments?

Asset return correlation and $\rho_{i,j}$

Correlation in the credit model

"Correlation" refers to the correlation, $\rho_{i,j}$, between the latent variables $\{Z_i\}$ that are responsible for default.

- **We also call it "credit correlation."**
- **When we fit the Vasicek distribution of the default rate to data, we estimate the correlation between latent variables.**
  - **We can't estimate the correlation between two particular firms this way.**
  - **Instead, we estimate a *<u>uniform</u>* value for many pairs of firms.**

The result is the uniform value of ρ that would best explain the pattern of joint defaults in the data.

Asset return correlation

Many practitioners assume that correlation ($\rho_{i,j}$) is equal to the correlation between firm asset returns ($r_{i,j}$).

- **This requires historical estimates of firm asset values.**
  - **As with estimating PDs, this requires de-leveraging market cap.**

We saw that firm asset values and volatilities imply systematically wrong values of PD.

- **Why would the correlation between asset returns imply good values of $\rho_{i,j}$?**

When Moody's calculates EDF, it calibrates to default data.

- **There is *no* recalibration of correlations.**
  - **If there were, we show it should be *toward* *zero*.**

Why use asset return correlation

Robert Merton gave this argument years ago:

- **Today, the value of a firm's assets is known.**
- **Within the current period,**
  - **The rate of return of the firm's assets has a normal distribution.**
  - **At the end of the period, the firm owes a known amount to its lenders.**
  - **If the firm does not pay the debt in cash, the lenders gain ownership.**
  - **The firm will be able to get the cash if the value of its assets is greater than the value of its debt.**
  - **The firm will be unable to get the cash if its asset return is too low.**
- **Same things hold for a second firm.**
  - **Its asset returns are *jointly* normal with the first firm.**
- **The probability that both firms default depends on the correlation between their asset returns.**

It is a nice story

- **If the story were true, then we wouldn't have introduced the idea of latent variables responsible for default.**
  - **We would have said, "Asset returns are responsible for default."**
- **The less you know, the easier the story is to accept.**
  - **The value of a firm's assets is *<u>not</u>* known.**
  - **Asset rate of return is *<u>not</u>* normal.**
  - **There is *<u>no</u>* current period that ends.**
  - **Corporate debt *<u>never</u>* comes due all at once.**
  - **The lenders *<u>don't</u>* take the firm.**
  - **The value of a firm's assets will *<u>not</u>* be known.**
  - **RadioShack *<u>did</u>* get the money to pay its debts for a while.**
- **If you were estimating something that has consequences, you might like to add realism.**

Adding realism

Merton says that default and asset shortfall are the same.

- **Whenever a firm is in asset shortfall, it defaults on debt.**
- **Whenever a firm defaults on debt, it is in asset shortfall.**

But the link from asset return to default is not perfect.

- **Borrowers pay when they have enough money, not when their assets have returned enough.**
- **Bank lenders choose whether to declare an event of default.**
- **Nobody knows how much a firm is worth in any case.**

The connection between asset returns and defaults is *noisy*. The noise makes credit correlation less than asset return correlation.

What does the data say?

Source Study	Data Source	Results
Gordy (2002)	S&P	1.5% - 12.5%
Cespedes (2000)	Moody's	10%
Hamerle *et al.* (2003a)		max of 2.3%
Hamerle *et al.* (2003b)	S&P 1982 - 1999	0.4% - 6.04%
Frey *et al.* (2001)	UBS	2.6%, 3.8%, 9.21%
Frey & McNeil (2003)	S&P 1981 - 2000	3.4% - 6.4%
Dietsch & Petey (2004)	Coface 1994 - 2001	0.12% - 10.72%
	AK 1997 - 2001	
Jobst & de Servigny (2004)	S&P 1981 - 2003	intra 14.6%, inter 4.7%
Duellmann & Scheule (2003)	DB 1987 - 2000	0.5% - 6.4%
Jakubik (2006)	BF 1988 - 2003	5.7%
Duellmann *et al.* (2006)	KMV	10.1%
KMV (2001)	Undisclosed	9.46% - 19.98%
Fitch (2005)	Equity	intra 24.09%, inter 20.92%
Lopez (2002)	KMV Software	11.25%

Correla-tion from default data is less than correla-tion from asset data

Source: Chernih, Vanduffel, Henrad, 2006

Correlation and asset correlation

Years ago, Moody's subscribers got month-by-month estimates of the asset values of every rated firm.

I used them to imply the asset return correlation of every pair of firms over every 3-month period.

– I adjusted for small sample size to get unbiased estimates.

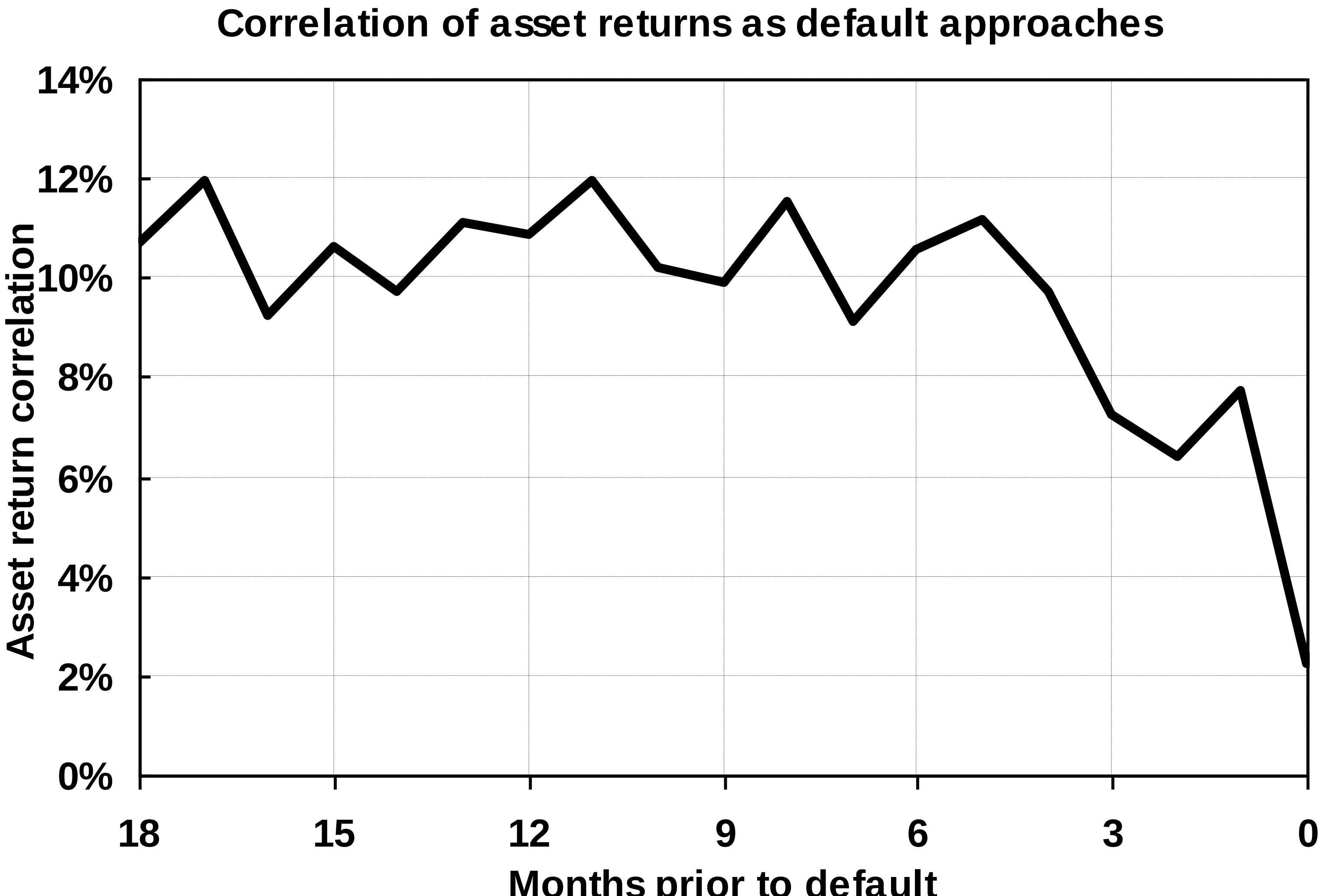
I calculated the "profile" ahead of each default:

– For each month prior to a default, I found the average asset return correlation of the defaulting firm with every other firm.

Then I averaged the profiles.

– This gave the average asset return correlation of a firm that soon defaults with other firms. Correlation is not stable.

Before default, *r* <u>*changes*</u>



Interpretation

A year before a firm defaults, its average asset return correlation with other firms is in the range 10%-12%.

- This is consistent with published asset correlation estimates.

As default approaches, average asset return correlation falls into the range of 6%-10%.

- This is broadly consistent with the ***credit*** correlation estimates calibrated to rates of joint default.

To predict future joint defaults over the one-year time frame, the value of $\rho_{i,j}$ must be closer to zero than $r_{i,j}$.

The values of $\rho_{i,j}$ could be calibrated to history by adjusting the $r_{i,j}$ to reflect historical data,

- just as data adjusts 1 - Φ [DD] to EDF.

Questions? Comments?

Review and preview of tools

Review and preview of tools

Maximum likelihood estimation

Hypothesis testing

p-value

How model search defeats indications of significance

Maximum likelihood estimation

MLE works like this:

1. Write a symbolic PDF for each data record.

- **The symbols represent parameters to be estimated.**

2. Suppose that the data records are independent.

- **The PDF of the data sample equals the product of the PDFs of the data records.**

3. Find parameter values that maximize the PDF.

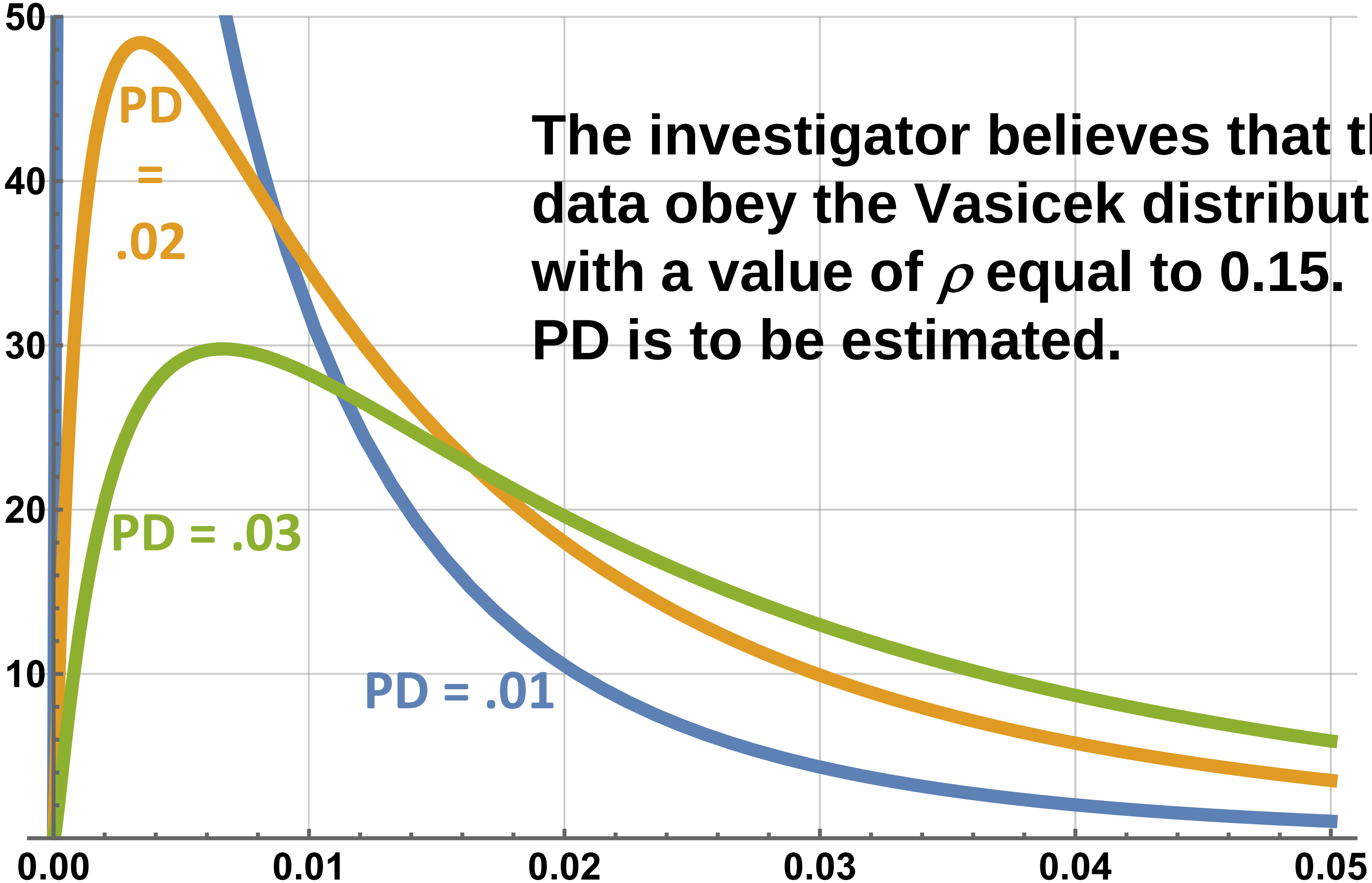
- **Compared to other sets of parameter values, this one places greatest probability on the data set.**
- **The function being maximized is called the likelihood.**
- **The maximizing values are estimates called MLE's.**

An example of MLE

- The investigator believes that a variable R obeys the Vasicek distribution with a value of ρ equal to 0.15.
  – Only the value of PD remains to be estimated.
- The data set consists of a single record: r = 0.01.
  – A single data record makes easy illustrations with diagrams.
- The next slide shows three possible PDFs.
  – Each PDF is Vasicek with $\rho = 0.15$.
  – The PDFs have differing values of PD: 0.01, 0.02, 0.03.

3 possible PDFs

R ~ Vasicek[PD, ρ = 0.15]

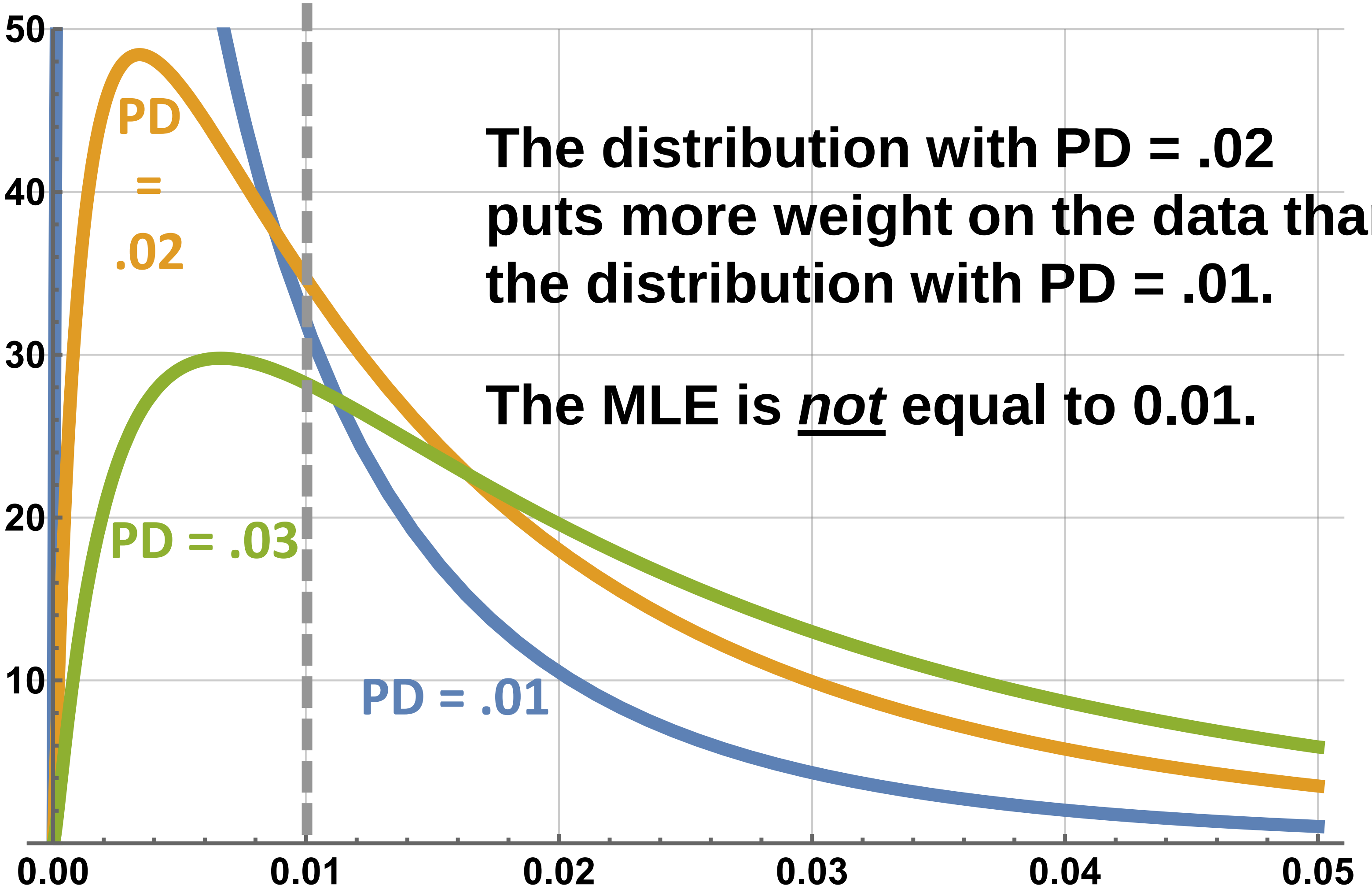


The investigator believes that the data obey the Vasicek distribution with a value of ρ equal to 0.15. PD is to be estimated.

The sole data record is r = 0.01

- **Therefore, an unbiased estimator of PD is PD = 0.01.**
  - **This is unbiased because the expected value of a data point is the expected value of the distribution.**
    - **And the expected value of a Vasicek distribution is PD.**
- **The next diagram shows that the MLE of PD is not 0.01.**
  - **The distribution with PD = 0.02 places greater probability density on r = 0.01 than does the distribution with PD = 0.01.**
    - **A distribution with a mean different from 0.01 places more weight on the data set than does the distribution with mean equal to 0.01.**
    - **A distribution with a biased estimate of PD produces the data more frequently than a distribution with an unbiased estimate of PD.**

The sole data record is $r = .01$



The distribution with PD = .02 puts more weight on the data than the distribution with PD = .01.

The MLE is *not* equal to 0.01.

Economists and risk people

Economists care about parameter estimates.

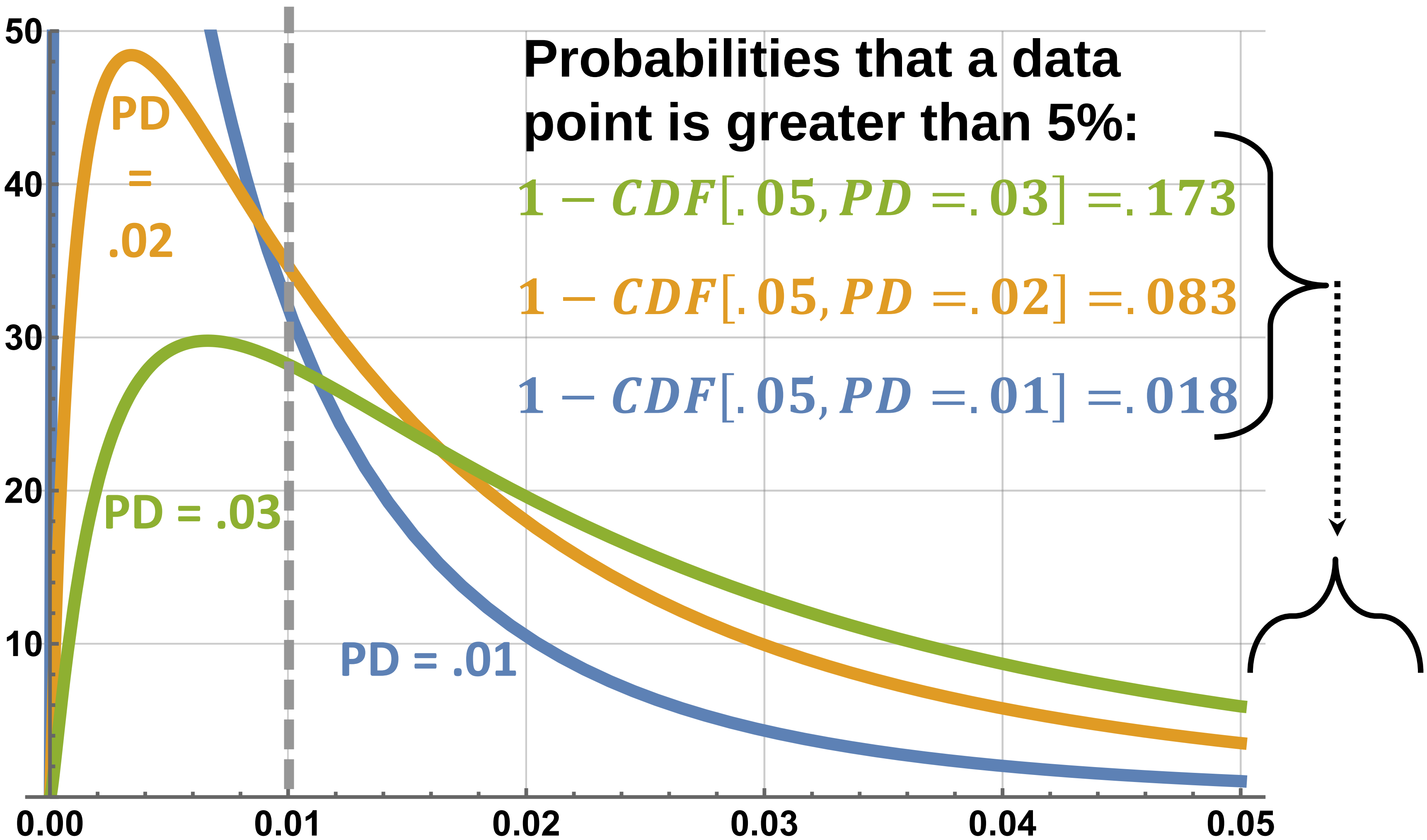
- **Their focus is the value of a parameter.**

Risk people care about distributions.

- **A distribution with mean 2% is more likely to produce the historical data than a distribution with mean equal to 1%.**
  - **Risk people prefer the distribution that produces the data most readily.**
  - **This distribution has a mean parameter that is different from the data average. So what?**
  - **Risk people probably don't care much about the mean anyway.**

It is worth noting that it really matters (to risk people, anyway) which distribution is chosen…

Distributions **are much different**



Questions or comments

The likelihood function

The likelihood function compares the 3 distributions to each other and to many other distributions.

Until now, we've plotted probability density functions.

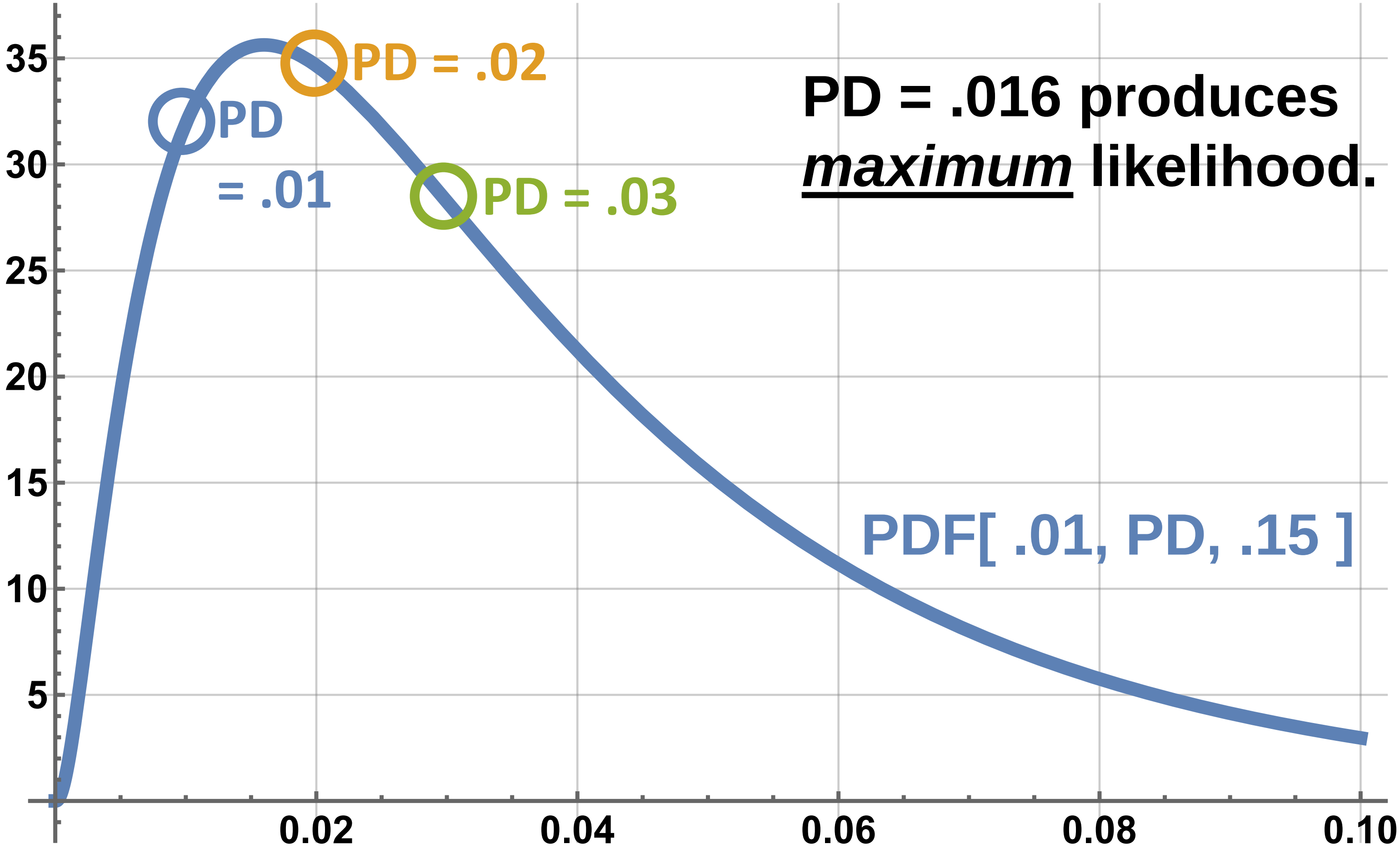
- **For example, the Vasicek: $PDF_{vas}[R, PD, \rho]$.**
- **We assume values for PD and ρ.**
- **We plot "probability density" as a function of R.**
- **The area underneath is 1.0.**

The arguments of a likelihood function are parameters.

- **In this case, it is $PDF_{Vasicek}[r = .01, PD, \rho = 0.15]$. and given the data point, $r = 0.1$, we plot likelihood as a function of the parameter PD.**
- **The area underneath is *not* 1.0…**

The likelihood function

The three distributions are different; the likelihoods have quite similar values. Lack of data!



The MLE

Every point on the likelihood function corresponds to a different Vasicek distribution with $\rho = 0.15$.

Among them, the one with $PD = 0.016$ places greatest probability density on the data sample.

– If the population has $PD = 0.016$, the sample would be produced more often than if PD had a different value.

The data might be the rare product of a distribution with its PD far from 0.016.

– We don't know and we can't tell.
  - PD is what it is with probability 1.0 and 0.0 otherwise.

– Nothing can tell us $Prob[PD = 0.016]$.

MLE—five data points

- **Suppose R has a Vasicek PDF:**

$$f[R, PD, \rho] = \frac{\sqrt{1-\rho}}{\sqrt{\rho}\,\phi[\Phi^{-1}[R]]}\,\phi\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}[R] - \Phi^{-1}[PD]}{\sqrt{\rho}}\right]$$

- **Suppose there are five data records: 1%,2%,3%,4%,5%.**
  - **If the records are *independent* draws,**
    - **then the PDF of the data set is the product of five Vasicek PDFs.**
- **The likelihood function for these data is L [PD, ρ] =**

  f[.01, PD, ρ] f[.02, PD, ρ] f[.03, PD, ρ] f[.04, PD, ρ] f[.05, PD, ρ]

  As a practical matter, one often maximizes Log[L [PD, ρ]].
- **Simultaneous MLEs are PD = 3.02%, ρ = 5.45%.**
  - **Check for yourself and see.**

Questions? Comments?

Hypothesis testing

Two models are each fit to a data set.

- **A simple "Null" model fits the data to a certain degree.**
- **A richer "Alternative" model has more parameters.**
  - **Setting the parameters to certain values reproduces the Null model.**
  - **Setting them to their MLE's provides a better fit than the Null.**

Only if the Alternative fits *<u>significantly</u>* better is the Null Hypothesis rejected.

- **The model being tested is the Null Hypothesis.**
  - **If a Null Hypothesis survives numerous tests, it can become a modeling standard.**
  - **If it is rejected numerous times, it should be improved.**

We prefer the simpler hypothesis

We prefer the Null until we are convinced that it is wrong.

- **William of Occam said the simpler idea is more likely to be *<u>true</u>*.**
  - **Today, no one depends on "truth". We wouldn't know it if we saw it.**
- **A modern person prefers the Null because it is *<u>simple</u>*.**
  - **A more complicated model has more things that can go wrong.**
  - **It is more likely to contain false effects. This is called Type 1 Error.**
  - **This is like superstition in daily life. It is best to get rid of it.**
- **It is very likely that the Null *<u>fails</u>* to contain something it *<u>should</u>*.**
  - **This is called Type 2 error.**
  - **This is less serious because we don't expect to know everything right now.**

The test knows that the Alternative always fits better.

- **It prefers the Null unless the Alternative fits *<u>significantly</u>* better.**

Putting in a little math

Each hypothesis places a certain probability on the data set.

- MLE maximizes this probability for each hypothesis in turn.
  - Because it is more flexible, the alternative puts ***more*** probability.

The "likelihood ratio" is the ratio of these two probabilities.

Twice the log of the likelihood ratio has a known distribution!

- This is "Wilks' Theorem." It depends on two conditions:
  - There is an asymptotic amount of data. (There isn't! Finite data set.)
  - The Null Hypothesis is true. (It isn't! All models are false.)

If twice the log likelihood ratio is in the tail of its distribution, then the test rejects the Null.

- Tests assume that nothing rare ever happens.
- The test says nothing about the Alternative. It wasn't tested.

Wilks' Theorem

Define L_0 **= Max [PDF [Data | Null hypothesis]]**

L_1 **= Max [PDF [Data | Alternative hypothesis]]**

k **= number of extra parameters in the Alternative**

If the Null hypothesis is true, and if the number of independent records rises without limit, then the distribution of D = -2 Log [L_0 **/** L_1 **] approaches** $\chi^2{}_k$**.**

- **Too bad for us, people symbolize this statistic by "D".**
  - **D is positive and equals 2 (Log [**L_1/L_0**]).**

Decision criterion

- D = 2 Log [L_1 / L_0] is asymptotically distributed $\chi^2{}_k$.
  - In practice, D is ***assumed*** distributed $\chi^2{}_k$ for the data at hand.
    - The variance in the finite sample is greater than the asymptotic variance.
- Suppose that the D statistic is a ***tail*** observation. Either:
  - The Null Hypothesis is true and something unlikely happened.
  - The Null Hypothesis is not true.
- Statisticians make the second choice:
  - If D is a tail observation, then the test rejects the Null.
    - The data sample is ***not*** infinite, and this leads to fatter tails as always.
    - So there are more rejections than the stated "size" of the test.
- A common criterion is the 95th percentile.
  - Size = 5%: Critical values of D are 1df = 3.84; 2df = 5.99.
    - If 2 Log [L_1 / L_0] = 3.84, then L_1 / L_0 = Exp[1.92] = 6.82.
    - The alternative must place 6.82 times more probability on the data.

Testing the three distributions

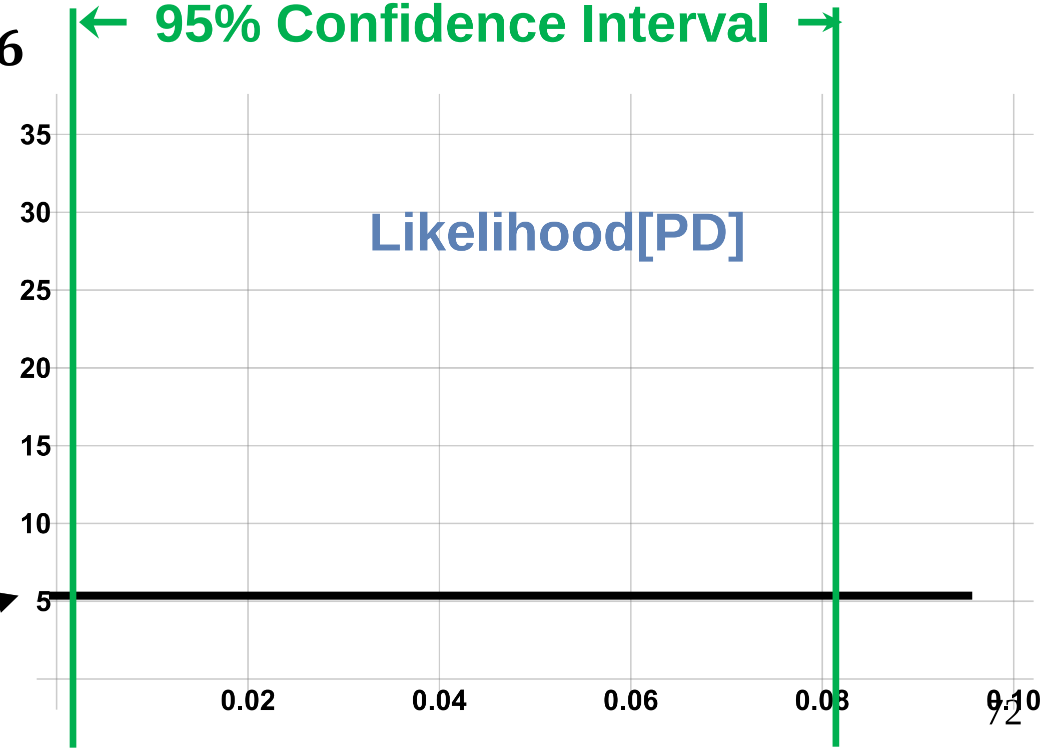
From the likelihood function,

- $L_1[PD = MLE = .016] = 35.63$
- $L_0[PD = .01] = 31.9; D = .22$
- $L_0[PD = .02] = 34.7; D = .06$
- $L_0[PD = .03] = 28.2; D = .46$

No hypothesis would be rejected.

For a test of size 5%, the critical value of L_0 is

$$\frac{L_1}{Exp[1.92]} = \frac{35.63}{6.82} = 5.22$$



Questions? Comments?

The 5-data-record sample

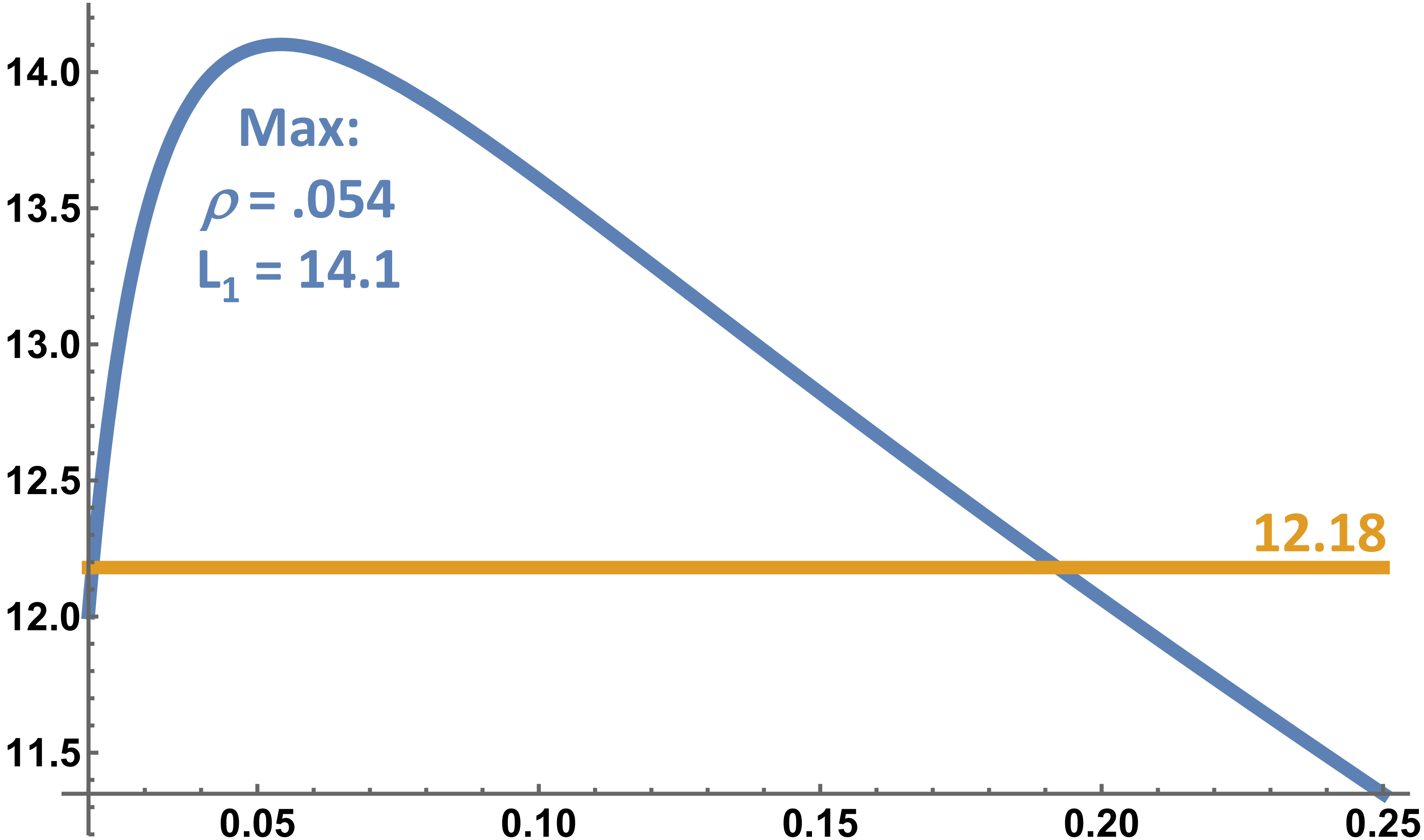
Suppose there are five data records: 1%,2%,3%,4%,5%.

- **If the records are *independent* draws,**
  - **then the PDF of the data set is the product of five Vasicek PDFs.**
- **Suppose that $PD = 0.03$.**
- **Which values of ρ would not be rejected?**

The next slide plots the value of L_1. It also has a horizontal line 1.92 below the peak.

- **1.92 = 3.84 / 2. Any Null Hypothesis that produces likelihood above the horizontal line is *not* rejected.**
- **Big enough likelihoods stem from 0.021 < ρ < 0.192.**
  - **The data reject that ρ = 0. (BTW, what would the PDF look like if ρ = 0?)**
  - **Otherwise, just about anything is OK.**
- **This is one way to find a confidence interval for ρ.**

Log[L_1] of the 5-record sample



The p-value

There would be no rejection of a Null Hypothesis that ρ is a number on the interval 0.021 to 0.192.

Suppose a Null hypothesis that names a value, ρ_0.

- **Under this Null hypothesis, we can simulate**
  - **drawing data samples from the population, and**
  - **finding the associated MLEs of ρ.**
- **Some fraction of the simulated estimates of ρ are further from ρ_0 than the MLE based on the real data.**
  - **This fraction is the p-value of the test.**
  - **If it is low, then the MLE is not likely to have been produced by the Null.**

The usual standard is to reject the Null if p-value < 0.05.

What the p-value is *not*

The p-value is *not* the probability that the Null is true.

Listen: *Of course* the Null is not true.

Models are not reality.

A map is not a territory.

All models are wrong

From Wikipedia, the free encyclopedia

All models are wrong is a common aphorism in statistics; it is often expanded as “All models are wrong, but some are useful”.

Hypothesis testing summary

Define two “nested” hypotheses.

- **With certain parameter values, the Alternative equals the Null.**

Estimate the parameters of each hypothesis using MLE.

Find $D = 2 \operatorname{Log} [L_1 / L_0]$.

If D > the 95th percentile of the χ^2 distribution with k degrees of freedom, then reject the null hypothesis.

- **k is the number of extra parameters in the alternative.**

If the Alternative does “better enough” on the *calibration* data, then the Null hypothesis is judged to be not a good guide to *future* data.

More perspectives

The purpose of a hypothesis test is to improve forecasts.

- **We want a model that fits the data we *haven't* seen yet.**

A wrong, simple model can be useful.

- **Example: Newtons' inverse square law of gravity.**

Even though the data say that people die younger if they carry a cigarette lighter, people should quit smoking.

- **A rejection can tell you that *something* is wrong with the Null, but it doesn't tell you what is wrong or how to fix it.**

Progress does *not* occur by rejecting the same model that has been rejected countless times before.

- **If we know that the Null is wrong, why test it?**
  - **"This is how you get your work published," is not a good enough answer.**
- **Instead, find a Null Hypothesis that is *difficult* to reject.**

Questions? Comments?

There are two kinds of model:

Those that have been rejected by the data, and those that have not been rejected, yet.

How ***search*** defeats ***significance***

How *search* defeats *significance*

This section shows how something modelers do

(search among many possible model specifications)

invalidates something modelers want

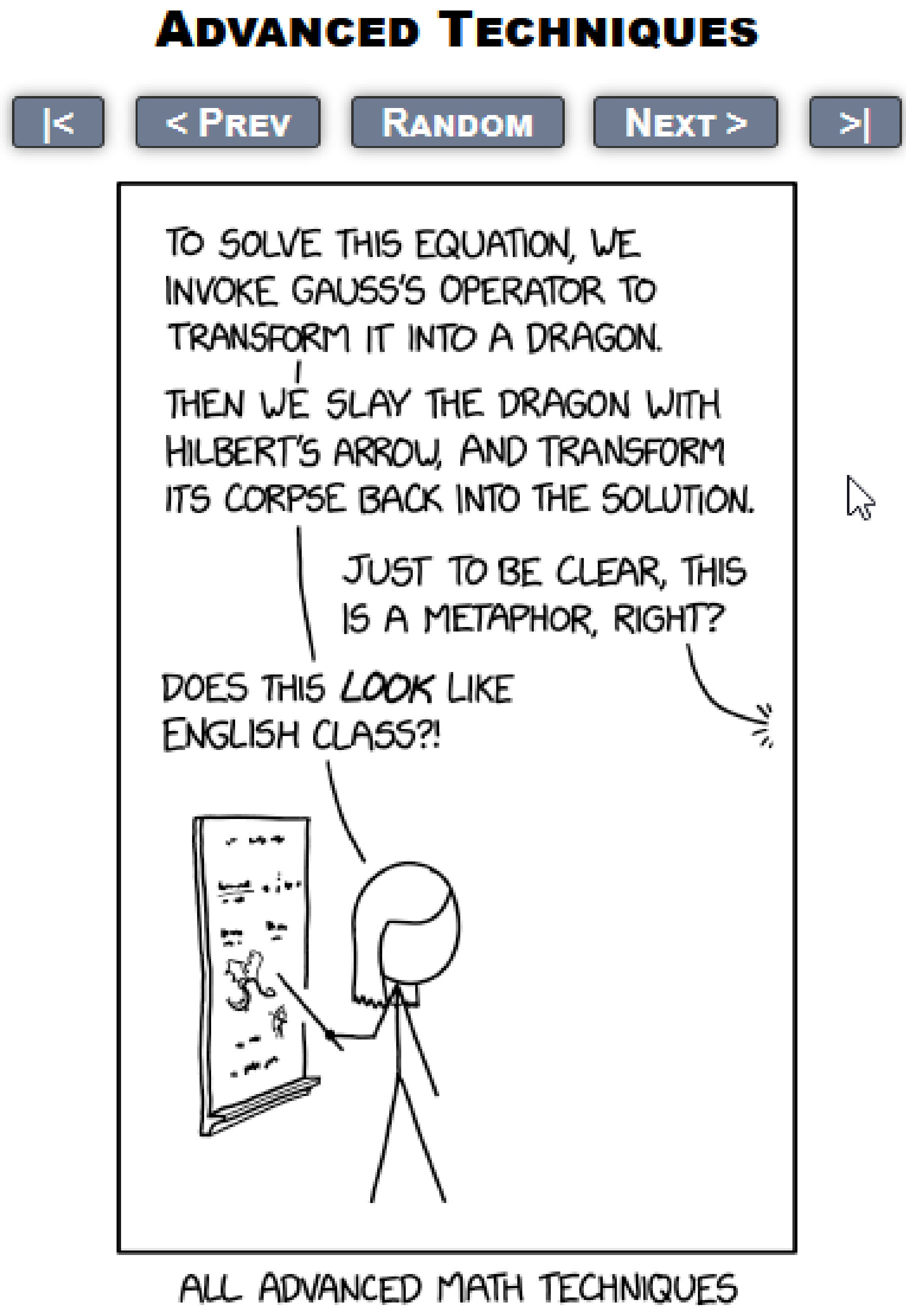
(a valid measure of statistical significance).

When a modeler performs ad hoc model search, the model is chosen based on the data.

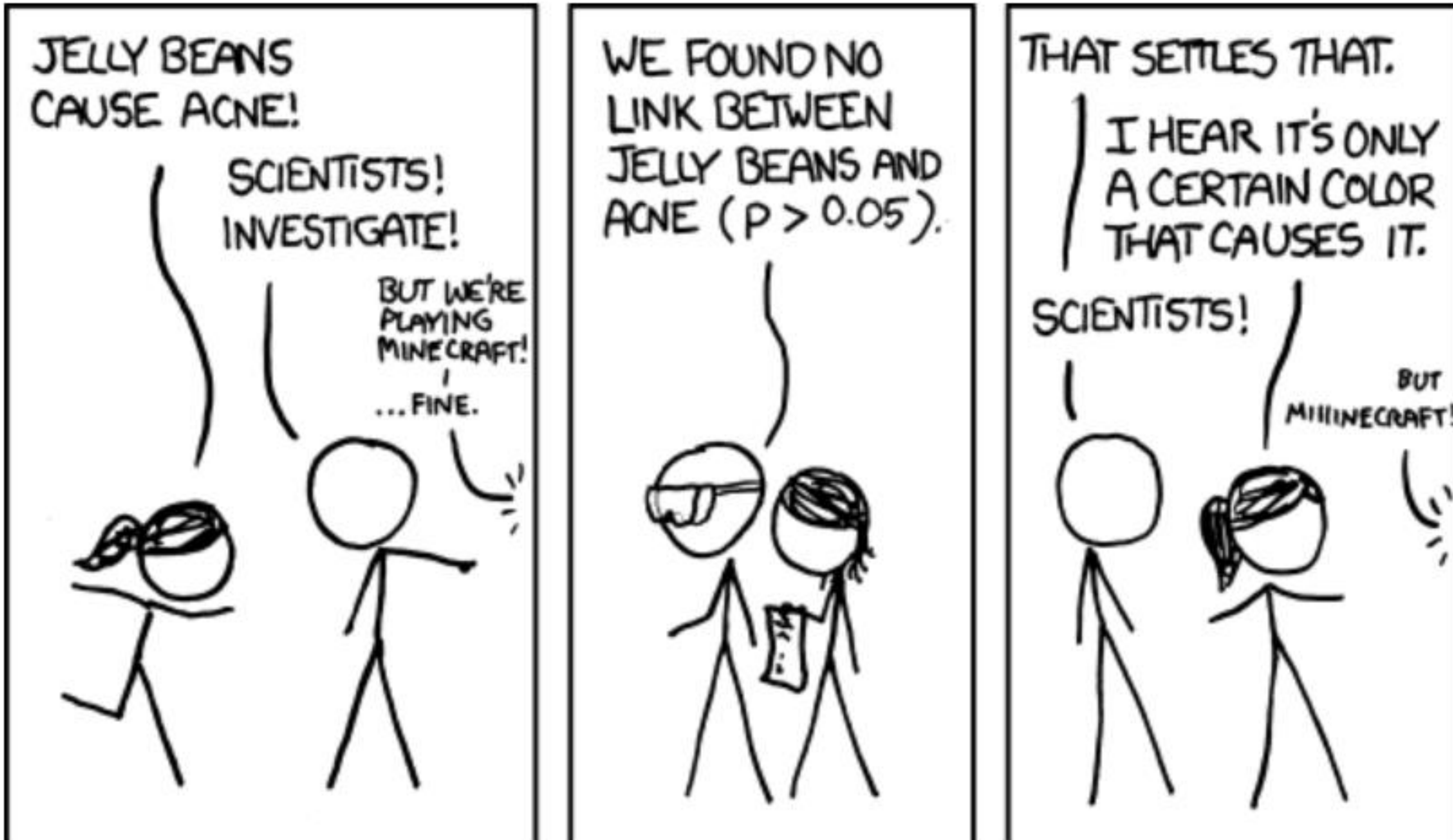
- It is not valid to say that the data support the model when the model has been chosen to fit the data.
  - Model search is often called “building the model.”
  - That sounds better than “shameless data fitting.”

Incidentally

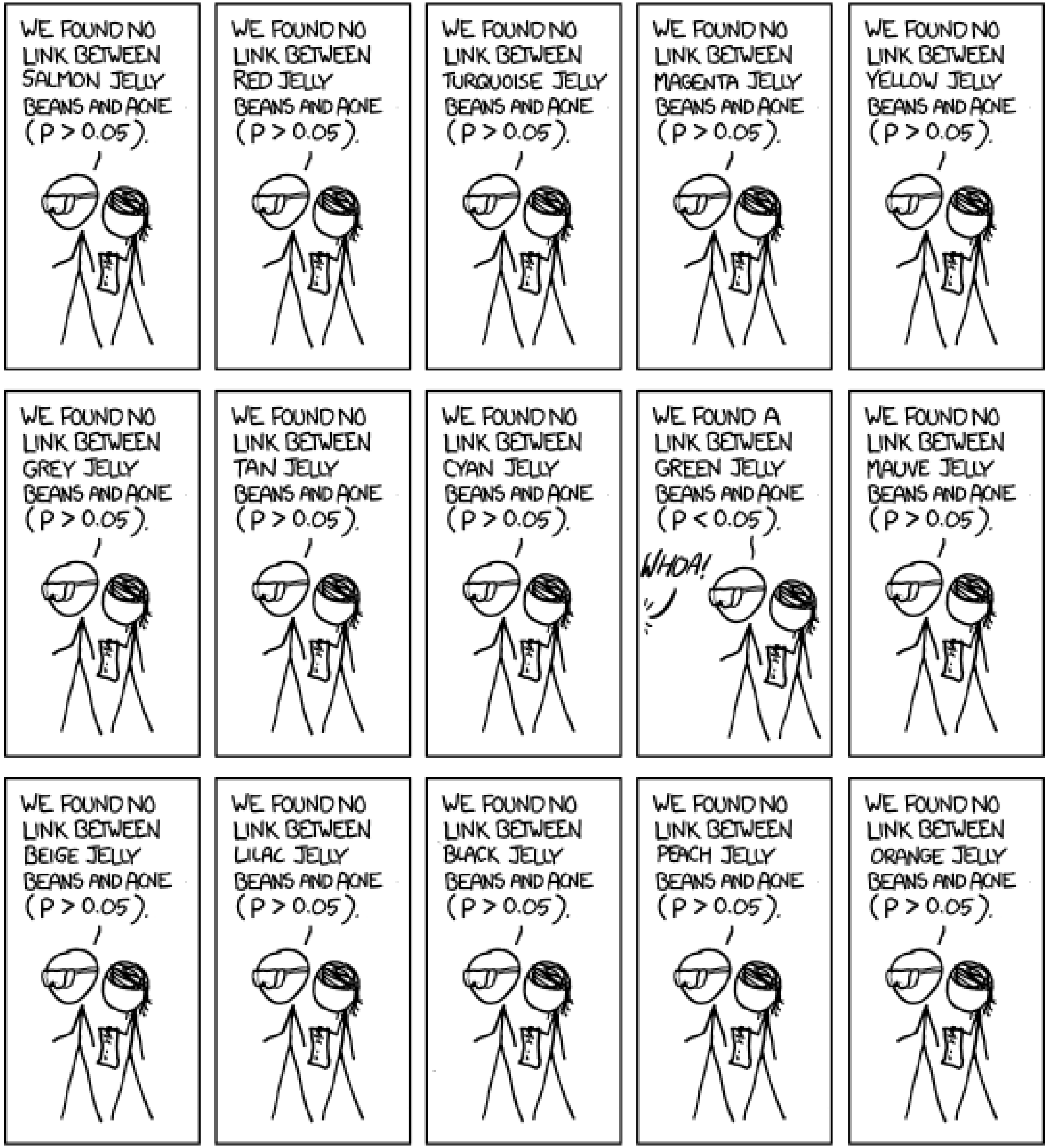
Sometime when you aren’t listening to me, put “XKCD” into a browser. Insight!



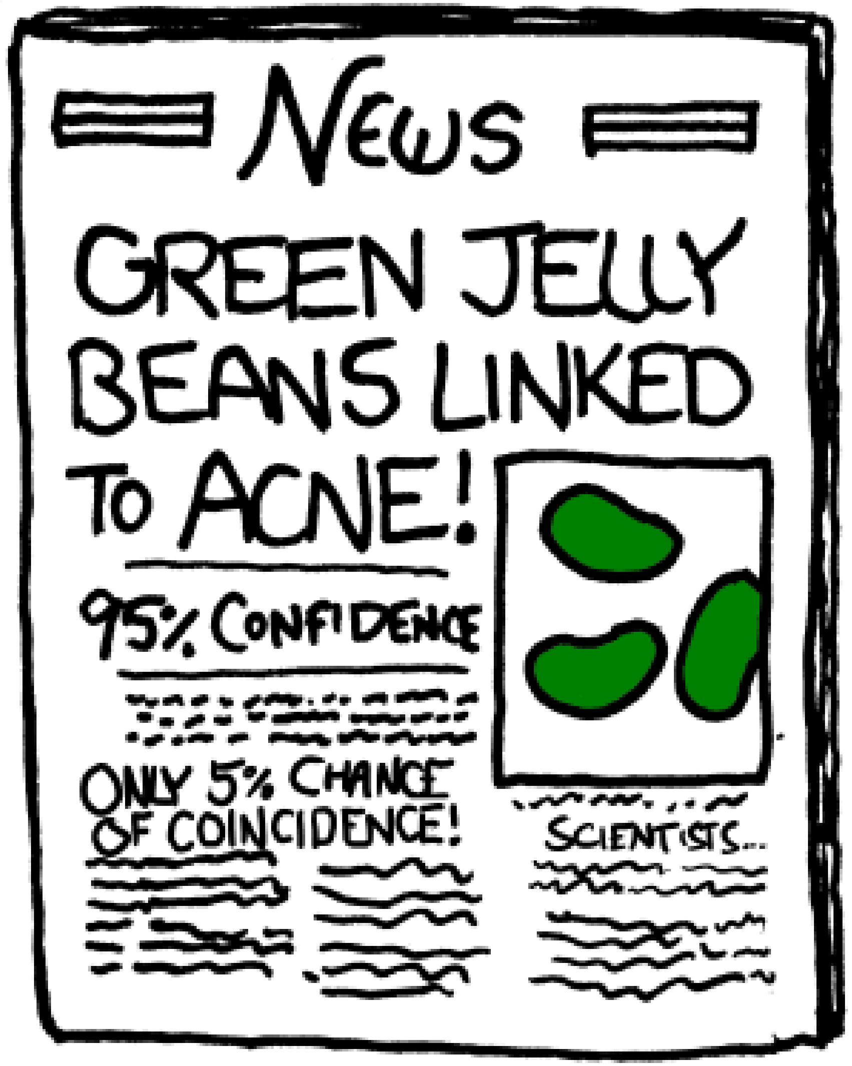
Specification search, part 1



Spec … part 2


WE FOUND NO LINK BETWEEN SALMON JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN RED JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN TURQUOISE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN MAGENTA JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN YELLOW JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN GREY JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN TAN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN CYAN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND A LINK BETWEEN GREEN JELLY BEANS AND ACNE (P < 0.05).
WHOA!
WE FOUND NO LINK BETWEEN MAUVE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BEIGE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN LILAC JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BLACK JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN PEACH JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN ORANGE JELLY BEANS AND ACNE (P > 0.05).

Specification search, part 3



How ***search*** defeats ***significance***

Note that the scientists made 20 different studies, one for each color, and the summary reported about green ones.

- ***Of course*** **one of the 20 colors will show the greatest association with acne. You can sort anything!**
- ***Of course*** **one color is likely to show "significance."**
  - **A well-designed test should show significance in 5% of data samples when the variables have no relation whatever, and 20 things were tried.**

Outsiders might not know that 20 things were tried.

- **If they did, they could compute that this is an *expected* result.**
  - **But a bad scientist is careful to conceal ad hoc *model search*.**

Hey, let's do big-time research on corporate mergers!

- **What's the effect of location, CEO academic degrees, and mentions in top newspapers, when these factors interact!**

Bad forecasts

The green jellybean model makes bad forecasts.

– It predicts that green jellybeans cause acne.
– It predicts that only green jelly beans cause acne.
  - The search among colors ***biases*** the results.

This prediction can shape behaviors.

– If people stop eating green jellybeans, what will they eat?
  - Will they eat other colors of jellybean?
  - Will they eat grilled salmon and broccoli?
  - Will they avoid all jellybeans and eat hot fudge sundaes?

The point: A bad risk model ***causes*** risky behavior.

– If a risk model understates the risk of a certain trade, that trade will be done too frequently.

Questions? Comments?

Common practices reduce model validity:

- **Model "specification search"**
- **Model "building"**
- **Preliminary data regressions**
- **"Correlograms" to specify serial dependence**

Don’t forget

Homework 3 is due next Thursday at 6 Chicago time.

Lisheng’s TA session will be online Sunday at 6.