Lecture 5:
Fundamental Theorem of Asset Pricing

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FINM 36700: Portfolio Management

Notation

notation	description
$\tilde{r}$	excess return rate over the period
$\tilde{r}^i$	arbitrary asset i
$\tilde{r}^p$	arbitrary portfolio p
$\tilde{r}^{\mathrm{t}}$	tangency portfolio
$\tilde{r}^m$	market portfolio
$\beta^{i,j}$	regression beta of $\tilde{r}^i$ on $\tilde{r}^j$

Outline

Beta-Factor Representation

Fundamental Theorem

Fundamental Theorem

The First Fundamental Theorem of Asset Pricing guarantees...

- there exists a probability measure, $\mathbb{P}^*$,
- under which the expected return, $\mathbb{E}^*$,
- the expected return of any portfolio[1] r^p equals the risk-free rate:

$$
\begin{aligned}
r^f =& \mathbb{E}^* \left[r^p \right] \\
0 =& \mathbb{E}^* \left[\tilde{r}^p \right]
\end{aligned}
$$

FINM 33000 covers this in detail.

[1] Of course, this includes portfolios of a single asset, r^i.

Pricing kernel

The so-called risk-neutral probability measure defining $\mathbb{E}^*$, depends on a change of measure, q.[2]

$$\mathbb{E}^*[x] \equiv \mathbb{E}[xq]$$

Thus we can write the Fundamental Theorem as

$$r^f = \mathbb{E}[qr^p] \tag{1}$$

[2] From FINM 33000, you'll recall this is the Radon-Nikodym derivative.

A function of r^{t}

We state without proof that q can be written as a linear function of r^{t}:

$$q = a + b\, r^{\text{t}}$$

and that this results in[3]

$$\begin{aligned}\mathbb{E}\left[r^{p}\right] =& r^{f} + \frac{\text{cov}\left[r^{p}, r^{\text{t}}\right]}{\text{var}\left[r^{\text{t}}\right]}\mathbb{E}\left[r^{\text{t}} - r^{f}\right] \\ \mathbb{E}\left[\tilde{r}^{p}\right] =& \beta^{p,\text{t}}\mathbb{E}\left[\tilde{r}^{\text{t}}\right] \end{aligned} \tag{2}$$

[3]We can write q as,

$$q = 1 - \left(r^{\text{t}} - \mathbb{E}\left[r^{\text{t}}\right]\right)\frac{\mathbb{E}\left[r^{\text{t}} - r^{f}\right]}{\text{var}\left[r^{\text{t}}\right]}$$

Then, subbing this into equation (1), we get the desired equation, (2).

Sketch of proof

To see this from the MV mathematics, recall the following formula.[4]

$$\begin{aligned} \boldsymbol{w}^{\mathtt{t}} =& \Sigma^{-1}\tilde{\boldsymbol{\mu}}\frac{1}{\gamma} \\ \gamma \equiv& \mathbf{1}'\Sigma^{-1}\tilde{\boldsymbol{\mu}} \end{aligned} \tag{3}$$

where γ is just a scaling constant to ensure $\boldsymbol{w}^{\mathtt{t}}$ adds to one, $(\boldsymbol{w}^{\mathtt{t}})'\mathbf{1} = 1$.

[4]We could write this formula in terms of excess return space $\boldsymbol{w}$ or return space, $\boldsymbol{\omega}$. Recall that the tangency portfolio is the one portfolio on both MV and $\tilde{\text{M}}$V frontiers. That is, $\boldsymbol{w}^{\mathtt{t}} = \boldsymbol{\omega}^{\mathtt{t}}$.

Sketch continued

Using this formula, note that the covariance of any portfolio return, r^p, with r^{t} is,

$$\begin{aligned}
\mathrm{cov}\left[\tilde{r}^p, \tilde{r}^{\mathrm{t}}\right] =& \left(\boldsymbol{w}^p\right)' \Sigma \boldsymbol{w}^{\mathrm{t}} \\
=& \left(\boldsymbol{w}^p\right)' \Sigma\Sigma^{-1} \tilde{\boldsymbol{\mu}} \frac{1}{\gamma} \\
=& \left(\boldsymbol{w}^p\right)' \tilde{\boldsymbol{\mu}} \frac{1}{\gamma} \\
=& \mu^p \frac{1}{\gamma}
\end{aligned}$$

Thus,

$$\tilde{\mu}^p = \mathrm{cov}\left[\tilde{r}^p, \tilde{r}^{\mathrm{t}}\right] \gamma \tag{4}$$

Sketch finished

It is easy to show[5]

$$\gamma = \frac{\mathbb{E}\left[\tilde{r}^{\mathrm{t}}\right]}{\mathrm{var}\left[\tilde{r}^{\mathrm{t}}\right]}$$

Thus Equation (4) is equivalent to (2).

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[5] Using the formula for $\boldsymbol{w}^{\mathrm{t}}$,

$$\mathbb{E}\left[\tilde{r}^{\mathrm{t}}\right] = \tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\gamma$$

And

$$\begin{aligned}\mathrm{var}\left[r^{\mathrm{t}}\right] =& \left(\boldsymbol{w}^{\mathrm{t}}\right)'\boldsymbol{\Sigma}\boldsymbol{w}^{\mathrm{t}} \\ =&\gamma\tilde{\boldsymbol{\mu}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\gamma \\ =&\gamma\tilde{\boldsymbol{\mu}}\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\gamma \\ =&\gamma\mathbb{E}\left[\tilde{r}^{\mathrm{t}}\right]\end{aligned}$$

Factor pricing

We then conclude that all asset returns have a factor-beta notation,

$$\mathbb{E}\left[\tilde{r}^{i}\right] = \beta^{i,\mathrm{t}}\,\mathbb{E}\left[\tilde{r}^{\mathrm{t}}\right] \tag{5}$$

$$\beta^{i,\mathrm{t}} \equiv \frac{\mathrm{cov}\left(\tilde{r}^{i},\tilde{r}^{\mathrm{t}}\right)}{\mathrm{var}\left(\tilde{r}^{\mathrm{t}}\right)}$$

This will hold in sample exactly, by mathematical identity.

Multiple factors

Suppose we have a set of (excess) factor returns, $\tilde{\boldsymbol{r}}^z$, such that the tangency portfolio is a linear combination of them: (no need for linear combination to sum to one.)

$$\tilde{r}^{\mathrm{t}} = \boldsymbol{\omega}' \tilde{\boldsymbol{r}}^z$$

Then,

$$\mathbb{E}\left[\tilde{r}^p\right] = \left(\boldsymbol{\beta}^{p,z}\right)' \mathbb{E}\left[\tilde{\boldsymbol{r}}^z\right]$$

where $\boldsymbol{\beta}^{p,z}$ is the vector of betas from a multivariate regression of $\tilde{r}^p$ on $\tilde{\boldsymbol{r}}^z$.

Outline

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Generalization

This relationship is not special to the tangency portfolio; it holds for any arbitrary MV portfolio, $\boldsymbol{\omega}^{\subset}$.

$$\mathbb{E}\left[\tilde{r}^{p}\right] = \beta^{p,\subset}\mathbb{E}\left[\tilde{r}^{\subset}\right]$$

Though this relationship works for any MV portfolio, we focus on the tangency portfolio, without loss of generality.

Optimization Conditions

The First Fundamental Theorem of Asset Pricing holds if—and only if—the mean-variance efficient portfolios are well defined. In our MV optimization language, the theorem relies on...

- Convexity of the control space.
- Convexity of the constraint and objective.
- Well-defined solution.

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Payoff conditions

Portfolio Formation

For any two security payoffs Γ^i and Γ^j, the payoff $a\Gamma^i + b\Gamma^j$ is also an available security.

Law of One Price (LOOP)

The pricing function, $\mathcal{P}(\cdot)$ is linear:

$$\mathcal{P}(w^i\,\Gamma^i + w^j\,\Gamma^j) = w^i\,\mathcal{P}(\Gamma^i) + w^j\,\mathcal{P}(\Gamma^j)$$

No Arbitrage

For every payoff Γ, if

$$Pr(\Gamma \geq 0) = 1 \text{ and } Pr(\Gamma > 0) > 0 \Longrightarrow \mathcal{P}(\Gamma) > 0.$$

Portfolio implications

Portfolio Formation

- if $\boldsymbol{w}^i, \boldsymbol{w}^j$ are permissible,
- $\boldsymbol{w}^k = \tilde{\delta}\boldsymbol{w}^i + (1-\tilde{\delta})\boldsymbol{w}^j$ is permissible for any $\tilde{\delta} \in (-\infty, \infty)$.

LOOP

- $\tilde{r}^k = \tilde{\delta}\tilde{r}^i + (1-\tilde{\delta})\tilde{r}^j$
- More generally, *Suppose* $\boldsymbol{w}$ *is a vector of portfolio weights. If* $\boldsymbol{w}'\Gamma = 0$ *for every state, then* $\boldsymbol{w}'\mathcal{P}(\Gamma) = 0$.

No Arbitrage

- Non-trivial, limited-liability, portfolios have well-defined returns.

Meaning

Portfolio Formation

- Short-selling and leverage are allowed.

LOOP

- Prices and returns equal the sum of their parts.

No arbitrage

- If a portfolio has cash-flow in any contingency, (without incurring liabilities in any contingencies,) then the portfolio must have a positive price.

Necessity of risk-free rate?

We derived (2) in terms of excess returns, which assumes the existence of a risk-free rate.

- Without a risk-free rate, the same arguments apply, simply adjusting the equations to have 0 in place of r^f.
- In deriving (4), the argument holds simply changing $(\boldsymbol{w}^{\mathrm{t}}, \tilde{\boldsymbol{\mu}}, \tilde{r})$ to $(\boldsymbol{\omega}^{\mathrm{t}}, \boldsymbol{\mu}, r)$.

Modeling vs Estimation

This factor-beta notation seems to give us a model for all mean returns. But it depends on knowing $\tilde{r}^{\mathrm{t}}$ via

- a theoretical model for $\tilde{r}^{\mathrm{t}}$.
- direct empirical estimation of $\tilde{r}^{\mathrm{t}}$.

One might consider using equation (3), for direct estimation, but this does not work in practice.

Circularity in direct estimation

Suppose we want to use the factor pricing model to estimate μ^i.

- The estimation of $\tilde{r}^{\mathrm{t}}$ via Equation (3) requires $\boldsymbol{\mu}$.

$$\boldsymbol{w}^{\mathrm{t}} = \Sigma^{-1}\boldsymbol{\mu}\frac{1}{\phi}$$

- But $\boldsymbol{w}^{\mathrm{t}}$ should consider all available assets, meaning $\boldsymbol{\mu}$ should includes μ^i itself!

- So this cannot be a way to estimate μ^i.

Imprecision in direct estimaiton

Suppose we instead estimate $\tilde{r}^{\mathrm{t}}$ from one set of assets and then use it to estimate the mean return for some other asset, i.

- Still, direct estimation does not work well.
- We will be ignoring the weight of $\boldsymbol{w}^{\mathrm{t}}$ that should be in i.
- Worse, the poor conditioning of Σ means that inverting it will greatly magnify the (substantial!) estimation errors in $\boldsymbol{\mu}$.

Thus, it is not practical to statistically extract an $\tilde{\text{MV}}$ portfolio to use in the linear pricing formula above.

Linear Factor Pricing Models

Linear factor pricing models (LFPM) are assertions about the identity of the tangency portfolio.

- This avoids the problems of direct estimation.
- But it relies on the assumption about the identity of the tangency portfolio (or some other mean-variance portfolio.)

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Allocation vs Pricing

The theory does not assume investors allocate to this assumed MV portfolio.

- It assumes the portfolio is MV for the purposes of pricing expected returns.
- If we additionally assume investors prefer MV portfolios, then this portfolio will both price securities and be the equilibrium allocation.