Chapter 8 - Graphical models

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1 Exercise 8.1

The parent node factorization for a Bayesian network is:

$$p(\mathbf{x}) = \prod_i p(x_i|\mathrm{pa}_i) \tag{1.1}$$

Show that if each of the conditional distributions are normalized, then so is the joint distribution.

1.1 Solution

The assumption is, that for any combination of values of pa_i, we have:

$$\int p(x_i|\mathrm{pa}_i)\ d\mu_i(x_i) = 1 \tag{1.2}$$

Here μ_i is the relevant dominating measure for the probability density function for x_i.

We wish to calculate:

$$\int p(\mathbf{x})\ d\mu_1(x_1)\cdots d\mu_K(x_K) \tag{1.3}$$

Using equation 1.1 this can be written:

$$\int p(x_K|\mathrm{pa}_K)\cdots p(x_1|\mathrm{pa}_1) d\mu_1(x_1)\cdots d\mu_K(x_K) \tag{1.4}$$

Now using the assumption from equation 1.2 for $i = 1$, the inner innermost integration turns to one. Doing so for $i = 2$ and so on up to $i = K$, we get that the entire integral is equal to 1 as desired.

2 Exercise 8.2

Show that the property of there being no directed cycles in a directed graphs follows from the statement that there exists an ordering of the nodes, such that there are no links to a lower-numbered node.

2.1 Solution

Let $x_1, x_2, \ldots, x_K$ be such an ordering. Assume that a directed cycle exists

$$x_{i_1} \to x_{i_2} \to \cdots \to x_{i_N} = x_{i_1} \tag{2.1}$$

By the ordering assumption we must have:

$$i_1 < i_2 < \cdots < i_n \tag{2.2}$$

But since $i_1 = i_n$ we have $i_1 < i_1$, which is a contradiction.

3 Exercise 8.3

The joint distribution of three binary random variables is summed up in table 1.

Show that the variables a and b are marginally dependent, but that they become independent when conditioning on c.

3.1 Solution

Margin dependence amounts to showing that $p(a)p(b) \neq p(a, b)$. The two first two distributions are found by summing over when each variable is 0

a	b	c	$p(a, b, c)$
0	0	0	0.192
0	0	1	0.144
0	1	0	0.048
0	1	1	0.216
1	0	0	0.192
1	0	1	0.064
1	1	0	0.048
1	1	1	0.096

Table 1: Joint distribution for exercise 8.3 and 8.4

and 1, respectively:

$$p(a=0) = 0.192 + 0.144 + 0.048 + 0.216 = 0.6 \tag{3.1}$$
$$p(a=1) = 0.192 + 0.064 + 0.048 + 0.096 = 0.4 \tag{3.2}$$
$$p(b=0) = 0.192 + 0.144 + 0.192 + 0.064 = 0.592 \tag{3.3}$$
$$p(b=1) = 0.048 + 0.216 + 0.048 + 0.096 = 0.408 \tag{3.4}$$

The distribution of $p(a, b)$ is found by summing over the four possibilities for the combinations of a and b:

$$p(a=0, b=0) = 0.192 + 0.144 = 0.336 \tag{3.5}$$
$$p(a=0, b=1) = 0.048 + 0.216 = 0.264 \tag{3.6}$$
$$p(a=1, b=0) = 0.192 + 0.064 = 0.256 \tag{3.7}$$
$$p(a=1, b=0) = 0.048 + 0.096 = 0.144 \tag{3.8}$$

It suffices to show the inequality for one case:

$$p(a=0)p(b=0) = 0.6 \cdot 0.592 = 0.3552 \neq 0.336 = p(a=0, b=0) \tag{3.9}$$

Now, we condition on c. I.e. we wish to show that $p(a|c)p(b|c) = p(a, b|c)$. To calculate the conditional probabilities, we need the distribution of c:

$$p(c=0) = 0.192 + 0.048 + 0.192 + 0.048 = 0.48 \tag{3.10}$$
$$p(c=1) = 0.144 + 0.216 + 0.064 + 0.096 = 0.52 \tag{3.11}$$

The conditional probablilities for a are:

$$p(a=0|c=0) = \frac{p(a=0, c=0)}{p(c=0)} = \frac{0.192 + 0.048}{0.48} = 0.5 \tag{3.12}$$
$$p(a=1|c=0) = \frac{p(a=1, c=0)}{p(c=0)} = \frac{0.192 + 0.048}{0.48} = 0.5 \tag{3.13}$$
$$p(a=0|c=1) = \frac{p(a=0, c=1)}{p(c=1)} = \frac{0.144 + 0.216}{0.52} = \frac{9}{13} \approx 0.692 \tag{3.14}$$
$$p(a=1|c=1) = \frac{p(a=1, c=1)}{p(c=1)} = \frac{0.064 + 0.096}{0.52} = \frac{4}{13} \approx 0.308 \tag{3.15}$$

And for b:

$$p(b=0|c=0) = \frac{p(b=0,c=0)}{p(c=0)} = \frac{0.192+0.192}{0.48} = 0.8 \tag{3.16}$$

$$p(b=1|c=0) = \frac{p(b=1,c=0)}{p(c=0)} = \frac{0.048+0.048}{0.48} = 0.2 \tag{3.17}$$

$$p(b=0|c=1) = \frac{p(b=0,c=1)}{p(c=1)} = \frac{0.144+0.064}{0.52} = 0.4 \tag{3.18}$$

$$p(b=1|c=1) = \frac{p(b=1,c=1)}{p(c=1)} = \frac{0.216+0.096}{0.52} = 0.6 \tag{3.19}$$

And the conditionals $p(a,b|c)$:

$$p(a=0,b=0|c=0) = \frac{p(a=0,b=0,c=0}{p(c=0)} = \frac{0.192}{0.48} = 0.4 \tag{3.20}$$

$$p(a=0,b=1|c=0) = \frac{p(a=0,b=1,c=0}{p(c=0)} = \frac{0.048}{0.48} = 0.1 \tag{3.21}$$

$$p(a=1,b=0|c=0) = \frac{p(a=1,b=0,c=0}{p(c=0)} = \frac{0.192}{0.48} = 0.4 \tag{3.22}$$

$$p(a=1,b=1|c=0) = \frac{p(a=1,b=1,c=0}{p(c=0)} = \frac{0.048}{0.48} = 0.1 \tag{3.23}$$

$$p(a=0,b=0|c=1) = \frac{p(a=0,b=0,c=1}{p(c=1)} = \frac{0.144}{0.52} = \frac{18}{65} \approx 0.277 \tag{3.24}$$

$$p(a=0,b=1|c=1) = \frac{p(a=0,b=1,c=1}{p(c=1)} = \frac{0.216}{0.52} = \frac{27}{65} \approx 0.415 \tag{3.25}$$

$$p(a=1,b=0|c=1) = \frac{p(a=1,b=0,c=1}{p(c=1)} = \frac{0.064}{0.52} = \frac{8}{65} \approx 0.123 \tag{3.26}$$

$$p(a=1,b=1|c=1) = \frac{p(a=1,b=1,c=1}{p(c=1)} = \frac{0.096}{0.52} = \frac{12}{65} \approx 0.185 \tag{3.27}$$

Now we can check if $p(a|c)p(b|c) = p(a,b|c)$. See table 2. Now the tables in each row contains the same probabilities, as desired.

Now, using this and the multiplication rule of probability we can write:

$$p(a,b,c) = p(a,b|c)p(c) = p(a|c)p(b|c)p(c) \tag{3.28}$$

The graph corresponding to this factorization is shown in figure 1.

4 Exercise 8.4

Show that

$$p(a,b,c) = p(a)p(c|a)p(b|c), \tag{4.1}$$

$p(a|c=0)p(b|c=0)$

	$b=0$	$b=1$
$a=0$	$0.5 \cdot 0.8$	$0.5 \cdot 0.2$
$a=1$	$0.5 \cdot 0.8$	$0.5 \cdot 0.2$

$p(a,b|c=0)$

	$b=0$	$b=1$
$a=0$	0.4	0.1
$a=1$	0.4	0.1

$p(a|c=1)p(b|c=1)$

	$b=0$	$b=1$
$a=0$	$\frac{9}{13} \cdot 0.4$	$\frac{9}{13} \cdot 0.6$
$a=1$	$\frac{4}{13} \cdot 0.4$	$\frac{4}{13} \cdot 0.6$

$p(a,b|c=1)$

	$b=0$	$b=1$
$a=0$	$\frac{18}{65}$	$\frac{27}{65}$
$a=1$	$\frac{8}{65}$	$\frac{12}{65}$

Table 2: Conditioning on c

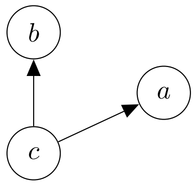

Figure 1: The graphical model for exercise 8.3

and draw the corresponding directed graph.

4.1 Solution

The only factor on the right hand side of equation 4.1 that we haven't yet calculated is $p(c|a)$. This can be found by Bayes' rule:

$$p(c|a) = \frac{p(a|c)p(c)}{p(a)} \tag{4.2}$$

Here, we've used the result from exercise 8.3. Now, let's evaluate the right hans side of equation 4.1:

$$p(a)p(c|a)p(b|c) = p(a)\frac{p(a|c)p(c)}{p(a)}p(b|c) = p(a|c)p(b|c)p(c) \tag{4.3}$$

Accordin to equation 3.28, we have the desired result.

The corresponding graph is shown in figure 2.

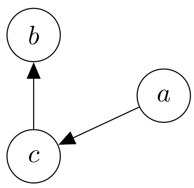

Figure 2: The graphical model for exercise 8.4