Chapter 8 - Graphical models

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1 Exercise 8.1

The parent node factorization for a Bayesian network is:

$$p(\mathbf{x}) = \prod_i p(x_i|\mathrm{pa}_i) \tag{1.1}$$

Show that if each of the conditional distributions are normalized, then so is the joint distribution.

1.1 Solution

The assumption is, that for any combination of values of pa_i, we have:

$$\int p(x_i|\mathrm{pa}_i)\ d\mu_i(x_i) = 1 \tag{1.2}$$

Here μ_i is the relevant dominating measure for the probability density function for x_i.

We wish to calculate:

$$\int p(\mathbf{x})\ d\mu_1(x_1)\cdots d\mu_K(x_K) \tag{1.3}$$

Using equation 1.1 this can be written:

$$\int p(x_K|\mathrm{pa}_K)\cdots p(x_1|\mathrm{pa}_1) d\mu_1(x_1)\cdots d\mu_K(x_K) \tag{1.4}$$

Now using the assumption from equation 1.2 for $i = 1$, the inner innermost integration turns to one. Doing so for $i = 2$ and so on up to $i = K$, we get that the entire integral is equal to 1 as desired.

2 Exercise 8.2

Show that the property of there being no directed cycles in a directed graphs follows from the statement that there exists an ordering of the nodes, such that there are no links to a lower-numbered node.

2.1 Solution

Let $x_1, x_2, \ldots, x_K$ be such an ordering. Assume that a directed cycle exists

$$x_{i_1} \to x_{i_2} \to \cdots \to x_{i_N} = x_{i_1} \tag{2.1}$$

By the ordering assumption we must have:

$$i_1 < i_2 < \cdots < i_n \tag{2.2}$$

But since $i_1 = i_n$ we have $i_1 < i_1$, which is a contradiction.

3 Exercise 8.3

The joint distribution of three binary random variables is summed up in table 1.

Show that the variables a and b are marginally dependent, but that they become independent when conditioning on c.

3.1 Solution

Margin dependence amounts to showing that $p(a)p(b) \neq p(a, b)$. The two first two distributions are found by summing over when each variable is 0

a	b	c	$p(a, b, c)$
0	0	0	0.192
0	0	1	0.144
0	1	0	0.048
0	1	1	0.216
1	0	0	0.192
1	0	1	0.064
1	1	0	0.048
1	1	1	0.096

Table 1: Joint distribution for exercise 8.3 and 8.4

and 1, respectively:

$$p(a=0) = 0.192 + 0.144 + 0.048 + 0.216 = 0.6 \tag{3.1}$$
$$p(a=1) = 0.192 + 0.064 + 0.048 + 0.096 = 0.4 \tag{3.2}$$
$$p(b=0) = 0.192 + 0.144 + 0.192 + 0.064 = 0.592 \tag{3.3}$$
$$p(b=1) = 0.048 + 0.216 + 0.048 + 0.096 = 0.408 \tag{3.4}$$

The distribution of $p(a, b)$ is found by summing over the four possibilities for the combinations of a and b:

$$p(a=0, b=0) = 0.192 + 0.144 = 0.336 \tag{3.5}$$
$$p(a=0, b=1) = 0.048 + 0.216 = 0.264 \tag{3.6}$$
$$p(a=1, b=0) = 0.192 + 0.064 = 0.256 \tag{3.7}$$
$$p(a=1, b=0) = 0.048 + 0.096 = 0.144 \tag{3.8}$$

It suffices to show the inequality for one case:

$$p(a=0)p(b=0) = 0.6 \cdot 0.592 = 0.3552 \neq 0.336 = p(a=0, b=0) \tag{3.9}$$

Now, we condition on c. I.e. we wish to show that $p(a|c)p(b|c) = p(a, b|c)$. To calculate the conditional probabilities, we need the distribution of c:

$$p(c=0) = 0.192 + 0.048 + 0.192 + 0.048 = 0.48 \tag{3.10}$$
$$p(c=1) = 0.144 + 0.216 + 0.064 + 0.096 = 0.52 \tag{3.11}$$

The conditional probablilities for a are:

$$p(a=0|c=0) = \frac{p(a=0, c=0)}{p(c=0)} = \frac{0.192 + 0.048}{0.48} = 0.5 \tag{3.12}$$
$$p(a=1|c=0) = \frac{p(a=1, c=0)}{p(c=0)} = \frac{0.192 + 0.048}{0.48} = 0.5 \tag{3.13}$$
$$p(a=0|c=1) = \frac{p(a=0, c=1)}{p(c=1)} = \frac{0.144 + 0.216}{0.52} = \frac{9}{13} \approx 0.692 \tag{3.14}$$
$$p(a=1|c=1) = \frac{p(a=1, c=1)}{p(c=1)} = \frac{0.064 + 0.096}{0.52} = \frac{4}{13} \approx 0.308 \tag{3.15}$$

And for b:

$$p(b=0|c=0) = \frac{p(b=0,c=0)}{p(c=0)} = \frac{0.192+0.192}{0.48} =0.8 \tag{3.16}$$

$$p(b=1|c=0) = \frac{p(b=1,c=0)}{p(c=0)} = \frac{0.048+0.048}{0.48} =0.2 \tag{3.17}$$

$$p(b=0|c=1) = \frac{p(b=0,c=1)}{p(c=1)} = \frac{0.144+0.064}{0.52} =0.4 \tag{3.18}$$

$$p(b=1|c=1) = \frac{p(b=1,c=1)}{p(c=1)} = \frac{0.216+0.096}{0.52} =0.6 \tag{3.19}$$

And the conditionals $p(a,b|c)$:

$$p(a=0,b=0|c=0) = \frac{p(a=0,b=0,c=0}{p(c=0)} = \frac{0.192}{0.48} =0.4 \tag{3.20}$$

$$p(a=0,b=1|c=0) = \frac{p(a=0,b=1,c=0}{p(c=0)} = \frac{0.048}{0.48} =0.1 \tag{3.21}$$

$$p(a=1,b=0|c=0) = \frac{p(a=1,b=0,c=0}{p(c=0)} = \frac{0.192}{0.48} =0.4 \tag{3.22}$$

$$p(a=1,b=1|c=0) = \frac{p(a=1,b=1,c=0}{p(c=0)} = \frac{0.048}{0.48} =0.1 \tag{3.23}$$

$$p(a=0,b=0|c=1) = \frac{p(a=0,b=0,c=1}{p(c=1)} = \frac{0.144}{0.52} =\frac{18}{65} \approx 0.277 \tag{3.24}$$

$$p(a=0,b=1|c=1) = \frac{p(a=0,b=1,c=1}{p(c=1)} = \frac{0.216}{0.52} =\frac{27}{65} \approx 0.415 \tag{3.25}$$

$$p(a=1,b=0|c=1) = \frac{p(a=1,b=0,c=1}{p(c=1)} = \frac{0.064}{0.52} =\frac{8}{65} \approx 0.123 \tag{3.26}$$

$$p(a=1,b=1|c=1) = \frac{p(a=1,b=1,c=1}{p(c=1)} = \frac{0.096}{0.52} =\frac{12}{65} \approx 0.185 \tag{3.27}$$

Now we can check if $p(a|c)p(b|c) = p(a,b|c)$. See table 2. Now the tables in each row contains the same probabilities, as desired.

4 Exercise 8.4

Show that

$$p(a,b,c) = p(a)p(c|a)p(b|c), \tag{4.1}$$

and draw the corresponding directed graph.

$p(a|c=0)p(b|c=0)$

	$b=0$	$b=1$
$a=0$	$0.5 \cdot 0.8$	$0.5 \cdot 0.2$
$a=1$	$0.5 \cdot 0.8$	$0.5 \cdot 0.2$

$p(a,b|c=0)$

	$b=0$	$b=1$
$a=0$	0.4	0.1
$a=1$	0.4	0.1

$p(a|c=1)p(b|c=1)$

	$b=0$	$b=1$
$a=0$	$\frac{9}{13} \cdot 0.4$	$\frac{9}{13} \cdot 0.6$
$a=1$	$\frac{4}{13} \cdot 0.4$	$\frac{4}{13} \cdot 0.6$

$p(a,b|c=1)$

	$b=0$	$b=1$
$a=0$	$\frac{18}{65}$	$\frac{27}{65}$
$a=1$	$\frac{8}{65}$	$\frac{12}{65}$

Table 2: Conditioning on c

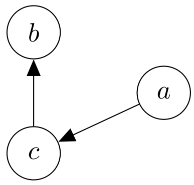

Figure 1: The graph for exercise 8.4

4.1 Solution

The only factor on the right hand side of equation 4.1 that we haven't yet calculated is $p(c|a)$. This can be found by Bayes' rule:

$$p(c|a) = \frac{p(a|c)p(c)}{p(a)} \tag{4.2}$$

Now, according to the multiplication rule of probability:

$$p(a,b,c) = p(a,b|c)p(c) = p(a|c)p(b|c)p(c) \tag{4.3}$$

Here, we've used the result from exercise 8.3. Now, let's evaluate the right hans side of equation 4.1:

$$p(a)p(c|a)p(b|c) = p(a)\frac{p(a|c)p(c)}{p(a)}p(b|c) = p(a|c)p(b|c)p(c) \tag{4.4}$$

Accordin to equation 4.3, we have the desired result.

The corresponding graph is shown in figure 1.