

# 1 The beta distribution

The beta distribution is useful in a number of applications. Specifically, it is often used in Bayesian inference.

## 1.1 Definition

The pdf of the beta distribution has the interval  $[0, 1]$  as closed support. The pdf  $f$  depends on two parameters,  $\alpha$  and  $\beta$ :

$$f(x) \propto x^{\alpha-1}(1-x)^{\beta-1} \quad (1)$$

Here, the alpha indicates proportionality; there will also be a normalization constant that will depend on  $\alpha$  and  $\beta$ :

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad (2)$$

Here,  $B(\alpha, \beta)$  is known as the *beta function*.

## 1.2 The beta and gamma function relationship

It turns out that the beta function can be expressed in terms of the gamma function, which is defined as

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du \quad (3)$$

To see this connection, consider the following expression:

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-u} u^{x-1} du \int_0^\infty e^{-v} v^{y-1} dv \quad (4)$$

$$= \int_0^\infty \int_0^\infty e^{-(u+v)} u^{x-1} v^{y-1} du dv \quad (5)$$

Now, consider the following change of variables:

$$(u, v) \mapsto (z, t), \quad u = zt, \quad v = z(1-t) \quad (6)$$

Adding the definitions of the two new variables, we get  $z = u + v$ . From the first definition, we now get  $t = \frac{u}{z} = \frac{u}{u+v}$ . So  $z$  can take on any value, but  $t$  must be between 0 and 1. The corresponding Jacobian matrix is:

$$J = \begin{pmatrix} \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} t & 1-t \\ z & -z \end{pmatrix} \quad (7)$$

The determinant of  $J$  is  $t(-z) - z(1 - t) = -z$ , so  $|\det J| = z$ . Now the integral reads:

$$\int_0^1 \int_0^\infty e^{-z} (zt)^{x-1} (z(1-t))^{y-1} \cdot \underbrace{z}_{|\det J|} dz dt \quad (8)$$

Now rearrange to get:

$$\int_0^1 t^{x-1} (1-t)^{y-1} dt \int_0^\infty e^{-z} z^{x+y-1} dz = B(x, y) \Gamma(x+y) \quad (9)$$

This means that:

$$\Gamma(x)\Gamma(y) = B(x, y)\Gamma(x+y) \Leftrightarrow B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (10)$$

In other words, the pdf for a beta distribution with parameters  $\alpha$  and  $\beta$  is:

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad (11)$$

A few of the possible pdf's are graphed below:

