1 The beta distribution

The beta distribution is useful in a number of applications. Specifically, it is often used in Bayesian inference.

1.1 Definition

The pdf of the beta distribution has the interval $[0,1]$ as closed support. The pdf f depends on two parameters, α and β:

$$f(x) \propto x^{\alpha-1}(1-x)^{\beta-1} \tag{1}$$

Here, the alpha indicates proportionality; there will also be a normalization constant that will depend on α and β:

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \tag{2}$$

Here, $B(\alpha,\beta)$ is known as the *beta function*.

1.2 The beta and gamma function relationship

It turns out that the beta function can be expressed in terms of the gamma function, which is defined as

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1}\, du \tag{3}$$

To see this connection, consider the following expression:

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= \int_0^\infty e^{-u} u^{x-1}\, du \int_0^\infty e^{-v} v^{y-1}\, dv && (4)\\
&= \int_0^\infty \int_0^\infty e^{-(u+v)} u^{x-1} v^{y-1}\, du\, dv && (5)
\end{aligned}$$

Now, consider the following change of variables:

$$(u,v) \mapsto (z,t), \quad u = zt, \quad v = z(1-t) \tag{6}$$

Adding the definitions of the two new variables, we get $z = u + v$. From the first definition, we now get $t = \frac{u}{z} = \frac{u}{u+v}$. So z can take on any value, but t must be between 0 and 1. The corresponding Jacobian matrix is:

$$J = \begin{pmatrix} \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} t & 1-t \\ z & -z \end{pmatrix} \tag{7}$$

The determinant of J is $t(-z) - z(1-t) = -z$, so $|\text{det}J| = z$. Now the integral reads:

$$\int_0^1 \int_0^\infty e^{-z}(zt)^{x-1}(z(1-t))^{y-1} \cdot \underbrace{z}_{|\text{det}J|} dz\ dt \tag{8}$$

Now rearrange to get:

$$\int_0^1 t^{x-1}(1-y)^{y-1}\ dt \int_0^\infty e^{-z}z^{x+y-1}\ dz = B(x,y)\Gamma(x+y) \tag{9}$$

This means that:

$$\Gamma(x)\Gamma(y) = B(x,y)\Gamma(x+y) \Leftrightarrow B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{10}$$

In other words, the pdf for a beta distribution with parameters α and β is:

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \tag{11}$$

A few of the possible pdf's are graphed below: