Differential Geometry

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1 Regular curves in Euclidean spaces

Let V be a finite-dimensional vector space over $\mathbb{R}$. Then a C^k *regular curve* in V is a mapping:

$$x : I \to V \tag{1.1}$$

Here I is an interval, and we require that x is C^k and that $x'(t) \neq 0$ for any $t \in I$ (this last requirement is what makes the curve regular).

Given a C^k regular curve $x : I \to V$. Then if φ is a C^k bijection from the interval J to I having $\varphi' > 0$, we consider x equivalent to the regular curve $y : J \to V, y = x \circ \varphi$.

We now assume V has an inner product $(\cdot, \cdot)$ and an induced norm $||v|| = \sqrt{(v, v)}$. Then for a regular curve x we define an arc length function $s(t)$ by:

$$\frac{ds}{dt} = ||x'(t)|| \tag{1.2}$$

Or equivalently:

$$s(t) = \int ||x'(t)||\ dt \tag{1.3}$$

This means that two arc length functions differ only by a constant. Since x' is never zero, $||x'(t)|| > 0$, and so $s(t)$ is strictly increasing. As a integral of a C^{k-1} function it is itself C^k.

We now definte the function $X(s(t)) = x(t)$. Being a composition of two C^k functions, so is X. So we can always parametrize a regular curve by its arc length. The derivative of X is found by the chain rule:

$$\frac{dX}{ds} = \frac{dx}{dt}\frac{dt}{ds} = x'(t(s))\frac{1}{ds/dt} = \frac{x'(t(s))}{||x'(t(s))||} \tag{1.4}$$

Here we've used, that since there is a bijection between I and $s(t)$ we can regard t as a function of s. Taking the norm, we get $||X'(s)|| = 1$. Squaring this gives $||X'(s)||^2 = (X'(s), X'(s)) = 1$. Differentiate with respect to s to get:

$$2(X''(s), X'(s)) = 0 \tag{1.5}$$

So the second derivate is orthogonal to the tangent $X'(s)$.

1.1 Example: Circular curve

Let $x_0 \in V$ and let $r_1, r_2 \in V$ be perpendicular vectors, each of length R. Then we can make a circular curve centered in x_0, radius R, and in the plane spanned by r_1 and r_2 by:

$$x(s) = x_0 + r_1 \cos(s/R) + r_2 \sin(s/R) \tag{1.6}$$

The tangent, i.e. the first derivative is:

$$x'(s) = \frac{1}{R}[-r_1 \sin(s/R) + r_2 \cos(s/R)] \tag{1.7}$$

The second derivative is:

$$x''(s) = -\frac{1}{R^2}[r_1 \sin(s/R) + r_2 \cos(s/R)] = -\frac{1}{R^2}(x(s) - x_0) \tag{1.8}$$

This is pointed at the center of the circle and has a magnitude of $||x''(s)|| = 1/R$.

1.2 Curvature of regular Euclidean curves

Inspired by the circular curve example, we make the following definition:

Definition 1.1. *Given a regular C^2 curve in V parametrized by arc length: $x(s)$ and a point on the curve where $x''(s) \neq 0$.*

Then $n = x''(s)/||x''(s)||$ is called the principal normal to the curve at $x(s)$.

$1/||x''(s)||$ is known as the radius of curvature at $x(s)$. The circle with center at $x(s) + x''(s)/||x''(s)||$, radius equal to the radius of curvature, in the plane spanned by $x'(s)$ and n is known as the osculating circle.

$\kappa = ||x''(s)||$ is called the curvature of the curve at $x(s)$, even if equal to zero. So when $\kappa \neq 0$ we have $x''(s) = n\kappa$

What if out curve $x(t)$ is not parametrized by arc length s? Then we can still calculate curvature by using the X function: $x(t) = X(s(t))$. The first dervative of x can be found using the chain rule:

$$x'(t) = \frac{dX}{dt} = \frac{dX}{ds}\frac{ds}{dt} = X'(s)\frac{ds}{dt} \tag{1.9}$$

And the second derivative by the multiplication rule:

$$x''(t) = X''(s)\left(\frac{ds}{dt}\right)^2 + X'(s)\frac{d^2s}{dt^2} \tag{1.10}$$

Remember that $ds/dt = ||x'(t)||$:

$$x''(t) = X''(s)||x'(t)||^2 + X'(s)\frac{d}{dt}||x'(t)|| \tag{1.11}$$

Consider the last derivative:

$$\frac{d}{dt}||x'(t)|| = \frac{d}{dt}\sqrt{(x'(t), x'(t))} = \frac{1}{2\sqrt{(x'(t), x'(t))}}2(x'(t), x'') = \frac{(x'(t), x''(t))}{||x'(t)||} \tag{1.12}$$

Rearranging terms and using equation 1.4 we get:

$$X''(s)||x'(t)||^2 = x''(t) - \frac{(x'(t), x''(t))}{||x'(t)||^2}x'(t) \tag{1.13}$$

Finally we can find the second derivative of X:

$$X''(s) = \frac{x''(t)}{||x'(t)||^2} - \frac{(x'(t), x''(t))}{||x'(t)||^4}x'(t) \tag{1.14}$$

$$= \frac{1}{||x'(t)||^2}\left(x''(t) - \frac{(x'(t), x''(t))}{||x'(t)||^2}x'(t)\right) \tag{1.15}$$

This means that the curvature is:

$$\kappa = ||X''(s)|| = \frac{1}{||x'(t)||^2}\left\|x''(t) - \frac{(x'(t), x''(t))}{||x'(t)||^2}x'(t)\right\| \tag{1.16}$$

1.3 Regular curves in $\mathbb{R}^2$

Let $x(t)$ be a regular curve in $\mathbb{R}^2$. Then we may attach a sign to the curvature κ. Let $x(s)$ be a parametrization of the curve by arc length. Then if $||x''(s)|| \neq 0$, we know that $x'(s)$ and $x''(s)$ are perpendicular. Because we're in two dimensions, $x''(s)$ is either rotated 90 degrees in positive of negative direction (conter-clockwise or clockwise). This determines the sign of the curvature.

Now consider a regular curve $x(s)$ in $\mathbb{R}^2$ that is also closed and simple, i.e. one that is periodic, and has no self-intersections apart from this. If the arc length is L we will still let the parametrization be defined for all $s \in \mathbb{R}$, so that:

$$x(s) = x(s') \Leftrightarrow s' - s = nL, n \in \mathbb{Z} \tag{1.17}$$

Since $||x'(s)|| = 1$ we can parametrize:

$$x'(s) = (\cos(\theta(s)), \sin(\theta(s)), 0 \leq s \leq L \tag{1.18}$$

Now differentiate to find $x''(s)$:

$$x''(s) = (-\sin(\theta(s)), \cos(\theta(s)))\frac{d\theta}{ds} = \widehat{x'(s)}\frac{d\theta}{ds} \tag{1.19}$$

Which means that the signed curvature is $\kappa(s) = d\theta/ds$. So we can integrate:

$$\int_0^L \kappa(s)\ ds = \theta(L) - \theta(0) \tag{1.20}$$

This means that the integral must be a multiple of 2π.

Theorem 1.1. *The integral 1.20 is equal to $\pm\pi$, with the positive sign if the curve lies entirely to the left of some tangent line and the overall motion is counter-clockwise.*

Proof. Without loss of generality we can assume that $x(0)$ is the point with the lowest second coordinate value. This implies that $x'(0) = (1, 0)$, i.e. in the direction of the first axis because of the counter-clockwise motion.

Define the relative cord direction:

$$x(t, s) = \frac{x(t) - x(s)}{||x(t) - x(s)||}, 0 \leq s < t \leq L \tag{1.21}$$

This is well-defined since the curve is closed, so the denominator never becomes zero. When $s = t$ we set $x(s, s) = x'(s)$, which makes the function continuous on the triangle $T = \{(t, s) \in \mathbb{R}^2 | 0 \leq s \leq t \leq T\}$.

T is simply connected and by construction $||x(t, s)|| = 1$, so there exists a continuous function $\varphi : T \to \mathbb{R}^2$ so that:

$$x(t, s) = (\cos(\varphi(t, s)), \sin(\varphi(t, s))), \quad \varphi(s, s) = \theta(s) \tag{1.22}$$

We wish to evaluate:

$$\theta(L) - \theta(0) = \varphi(L, L) - \varphi(0, 0) \tag{1.23}$$

Because of the chosen geometry, we have $x(0, 0) = x'(0) = (1, 0)$. So $\varphi(0, 0)$ must be a multiple of 2π. We choose 0. So we only need to consider the value of $\varphi(L, L)$.

To do this, start by noticing, that since we chose $\varphi(0, 0) = 0$, we can never have $\varphi(L, L) > 2\pi$, because of the location of $x(0)$. Now consider $\varphi(L, t)$ in the limit $t \to L$. Again, because of the geometry, we must have $\varphi(L, L) \leq 2\pi$. This leaves 2π as the only option. $\square$

1.4 Torsion and binormal

Back to the general case. Consider a C^k curve with arc length as parameter $x(k)$. If $k \geq 3$ we can differentiate the principal normal. Since $||n(s)|| = 1$, we have $||n(s)||^2 = (n(s), n(s)) = 1$. Differentiation gives $2(n'(s), n(s)) = 0$, so $n'(s)$ is normal to $n(s)$.

Now define a new quantity:

$$\tau(s) = ||n'(s) + \kappa(s)x'(s)|| \tag{1.24}$$

This is known as the *torsion* of the curve. The vector inside the norm turns out to have scalar product with both $x'(s)$ and $n(s)$ equal to zero.

For $x'(s)$:

$$(n'(s) + \kappa(s)x'(s), x'(s)) = (n'(s), x'(s)) + \kappa(x'(s), x'(s)) = (n'(s), x'(s)) + \kappa \tag{1.25}$$

The inner product can be found by differentiation of $(n(s), x'(s))$:

$$\frac{d}{ds}(n(s), x'(s)) = (n'(s), x'(s)) + (n(s), x''(s)) \tag{1.26}$$

But since $n(s)$ and $x'(s)$ are normal, this means:

$$\begin{aligned} (n'(s), x'(s)) &= -(n(s), x''(s)) && (1.27) \\ &= -\left(\frac{x''(s)}{||x''(s)||}, x''(s)\right) && (1.28) \\ &= -||x''(s)|| = -\kappa && (1.29) \end{aligned}$$

This shows the desired result.

$n(s)$ is a bit easier:

$$(n'(s) + \kappa(s)x'(s), n(s)) = (n'(s), n(s)) + \kappa(x'(s), n(s)) = 0 + 0 = 0 \tag{1.30}$$

So it seems like $n'(s) + \kappa(s)x'(s)$ is normal to both $x'(s)$ and $n(s)$. Unless it is equal to zero. If the dimensionality of V is less than three, this must be the case as there's not room for another perpendicular vector.

For $\tau(s) \neq 0$ we define the *binormal* of the curve to be the normalized version of this vector:

$$b(s) = \frac{n'(s) + \kappa(s)x'(s)}{\tau(s)} \tag{1.31}$$

We may now continue the process and differentiate the following equations:

$$(b(s), x'(s)) = 0, \quad (b(s), n(s)) = 0, \quad (b(s), b(s)) = 1 \tag{1.32}$$

First equation:

$$(b'(s), x'(s)) + (b(s), x''(s)) = 0 \Leftrightarrow (b'(s), x'(s)) = -(b(s), x''(s)) = 0 \tag{1.33}$$

Second equation:

$$(b'(s), n(s)) + (b(s), n'(s)) = 0 \Leftrightarrow (b'(s), n(s)) = -(b(s), n'(s)) \tag{1.34}$$

The right hand side can be calculated as follows:

$$(b(s), n'(s)) = (b(s), n'(s)) + \kappa \underbrace{(b(s), x'(s))}_{0} \tag{1.35}$$

$$= (b(s), n'(s) + \kappa x'(s)) \tag{1.36}$$

$$= \left(b(s), \frac{1}{\tau} b(s) \right) = \frac{\tau^2}{\tau} = \tau(s) \tag{1.37}$$

So $(b'(s), n(s)) = -\tau(s)$.

Third equation:

$$2(b'(s), b(s)) = 0 \Leftrightarrow (b'(s), b(s)) = 0 \tag{1.38}$$

1.5 The Frenet formulas

1.5.1 In 3 dimensions

If $\dim V = 3$ we know that the three mutually perpendicular unit vectors $x'(s), n(s)$, and $b(s)$ form an orthonormal basis for V. Their dynamics can be summed up as follows:

$$x''(s) = \qquad +\kappa(s)n(s) \tag{1.39}$$

$$n'(s) = -\kappa(s)x'(s) \qquad +\tau(s)b(s) \tag{1.40}$$

$$b'(s) = \qquad -\tau(s)n(s) \tag{1.41}$$

Or in matrix form:

$$\frac{d}{ds} \begin{pmatrix} x'(s) \\ n(s) \\ b(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} x'(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{1.42}$$

These are the *Frenet formulas* in three dimensions. But how can we generalize to an arbitrary number of dimensions?