Projection operators

Kristian Wichmann

July 26, 2017

1 Orthogonal complement

Let V be a finite-dimensional vector space with an inner product $\langle\cdot,\cdot\rangle$. Let U be a subspace of V. Then we define the *orthogonal complement* of U as:

$$U^\perp = \{v \in V | \forall u \in U : \langle u, v\rangle = 0\} \tag{1.1}$$

Theorem 1.1. *$U^\perp$ is a subspace of V.*

Proof. According to the subspace theorem, we need to show three things:

- $U^\perp$ is not empty: Clearly $0 \in U^\perp$.
- Closed under addition: If $v_1, v_2 \in U^\perp$, then for all $u \in U^\perp$:

$$\langle v_1 + v_2, u\rangle = \langle v_1, u\rangle + \langle v_2, u\rangle = 0 \tag{1.2}$$

- Closed under scalar multiplication: If $v \in U^\perp$ and $c \in \mathbb{R}$ then for all $u \in U^\perp$:

$$\langle cv, u\rangle = c\langle v, u\rangle = 0 \tag{1.3}$$

□

Since the only vector perpendicular to itself is 0, we further conclude that $U \cap U^\perp = \{0\}$.

Theorem 1.2. *If $e_1, e_2, \ldots, e_m$ is an orthonormal basis for U, then for any $v \in V$:*

$$v - \sum_{i=1}^{m} \langle v, e_i\rangle e_i \in U^\perp \tag{1.4}$$

Proof. Let $u \in U$. Then we can write $u = \sum_{j=1}^m \lambda_j e_j$ for some coefficients λ_j. Now calculate the inner product with the vector above:

$$\langle v - \sum_{i=1}^m \langle v, e_i \rangle e_i \ , \ \sum_{j=1}^m \lambda_j e_j \rangle = \sum_{i=j}^m \lambda_j \langle v, e_j \rangle - \sum_{i=1}^m \sum_{j=1}^m \lambda_j \langle v, e_i \rangle \langle e_i, e_j \rangle \quad (1.5)$$

Since $\langle e_i, e_j \rangle = \delta_{ij}$ this vanishes. □

This means that we may write any $v \in V$ as a sum of vectors from U and $U^\perp$ respectively:

$$v = \underbrace{\sum_{i=1}^m \langle v, e_i \rangle e_i}_{\in U} + \underbrace{v - \sum_{i=1}^m \langle v, e_i \rangle e_i}_{\in U^\perp} \quad (1.6)$$

Theorem 1.3. *The decomposition into elements from U and $U^\perp$ from equation 1.6 is unique.*

Proof. Let $v = u_1 + u_1^\perp$ and $v = u_2 + u_2^\perp$ be two such decompositions. Then $u_1 + u_1^\perp = u_2 + u_2^\perp$ and hence $u_1 - u_2 = u_2^\perp - u_1^\perp$. But this means that this vector is a member of both U and $U^\perp$, and hence it must be 0. This means $u_1 = u_2$ and $u_1^\perp = u_2^\perp$. □

2 The orthogonal projection

The previous section motivates the following:

Definition 2.1. *Let V be a finite-dimensional inner product vector space and U a subspace of V. The orthogonal projection from V onto U is the map $p : V \to V$ which satisfies:*

$$\forall v \in V : \quad p(v) \in U, \quad v - p(v) \in U^\perp \quad (2.1)$$

As we see, one could also define the co-domain of p to be U. Usually, the distinction will not matter much.

Theorem 2.1. *The orthogonal projection operator is linear.*

Proof. We need to show additivity and homogeneity:

- Additivity: Let $v_1, v_2 \in V$. Then $p(v_1) + p(v_2) \in U$ and:

$$v_1 - p(v_1) + v_2 - p(v_2) = v_1 + v_2 - (p(v_1) + p(v_2)) \in U^\perp \quad (2.2)$$

  Adding the two we get $v_1 + v_2$. So $p(v_1 + v_2) = p(v_1) + p(v_2)$.

- Homogeneity. Let $v \in V$ and $c \in \mathbb{R}$. Then $cp(v) \in U$ and $c(v - p(v)) = cv - cp(v) \in U^{\perp}$. Adding the two we get cv, so $p(cv) = cp(v)$.

□

Theorem 2.2. *The orthogonal projection operator $p : V \to V$ is idempotent. I.e. $p \circ p = p$.*

Proof. Let $v \in V$. Then $p(v) \in U$. But this means that the decomposition of $p(v)$ is $p(v) + 0$. So $p \circ p(v) = p(v)$. □

3 Orthogonal subspaces

Two subspaces $L_1, L_2 \subseteq V$ of an inner product space are said to be *orthogonal* if every vector from L_1 is orthogonal to every vector from L_2. We may characterize this property through the orthogonal projection operators of the spaces:

Theorem 3.1. *Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let L_1 and L_2 be subspaces of V with associated orthoogonal projection operators p_1 and p_2. Then, the following are equivalent:*

- *L_1 and L_2 are orthogonal.*
- $p_1 \circ p_2 = 0$
- $p_2 \circ p_1 = 0$

Proof. First, assume L_1 and L_2 to be orthogonal, and let $v \in V$. First, $p_2(v) \in L_2$. But because of the orthogonality we also have $p_2(v) \in L_1^{\perp}$. So $p_1(p_2(v)) = 0$. By a totally symmetric argument, $p_2(p_1(v)) = 0$.

Conversely, assume $p_1 \circ p_2 = 0$. Let $v_1 \in L_1$ and $v_2 \in L_2$. This means that $p_1(v_1) = v_1$, and $p_2(v_2) = v_2$. Then:

$$\langle v_1, v_2 \rangle = \langle p_1(v_1), p_2(v_2) \rangle \tag{3.1}$$

But p_1 is symmetric, so:

$$\langle p_1(v_1), p_2(v_2) \rangle = \langle v_1, p_1(p_2(v_2)) \rangle = \langle v_1, 0 \rangle = 0 \tag{3.2}$$

So L_1 and L_2 are orthogonal. Again, a symmetrical proof can be made starting from $p_2 \circ p_1 = 0$. □

We can also use it to tell when subspaces are contained in each other:

Theorem 3.2. *Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let L_1 and L_2 be subspaces of V with associated orthogonal projection operators p_1 and p_2. Then, the following are equivalent:*

- $L_1 \subseteq L_2$
- $p_1 \circ p_2 = p_2$
- $p_2 \circ p_1 = p_2$

Proof. Assume $L_1 \subseteq L_2$. Then L_2 and $L_1^\perp$ must be orthogonal. From theorem 3.1 we know this means p_2 and the orthogonal projection on $L_1^\perp$, $1 - p_1$, must be zero when combined either way:

$$p_2 \circ (1 - p_1) = p_2 - p_2 \circ p_1 = 0 \Leftrightarrow p_2 \circ p_1 = p_2 \tag{3.3}$$

$$(1 - p_1) \circ p_2 = p_2 - p_1 \circ p_2 = 0 \Leftrightarrow p_1 \circ p_2 = p_2 \tag{3.4}$$

Conversely, assume $p_1 \circ p_2 = p_2$. Then $(1 - p_1) \circ p_2 = p_2$. According to theorem 3.1 $L_1^\perp$ and L_2 are orthogonal. But then $L_2 \subseteq L_1$. A completely analogous argument can be made for $p_2 \circ p_1 = p_2$. □

4 Relative, orthogonal complement

Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let L_1 and L_2 be subspaces of V, such that $L_2 \subseteq L_1$. Then we define the *orthogonal complement of L_2 relative to L_1* as:

$$L_1 \ominus L_2 = \{v \in V | v \in L_1 \text{ and } \forall w \in L_2 : \langle v, w \rangle = 0\} \tag{4.1}$$

In other words, $L_1 \ominus L_2 = L_1 \cap L_2^\perp$. But we still view this as a subspace of V rather than L_1.

Theorem 4.1. *Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let $L_2 \subseteq L_1$ be subspaces of V with associated orthogonal projection operators p_2 and p_1. Then $p_1 - p_2$ is the orthogonal projection operator of $L_1 \ominus L_2$.*

Proof. Start by noting, that since $L_2 \subseteq L_1$, for any $v \in V$, $p_1(v), p_2(v) \in L_1$ and hence $(p_1 - p_2)v \in L_1$. Now let $w \in L_2$. Then, using the symmetry of p_1 and p_2:

$$\langle (p_1 - p_2)(v), w \rangle = \langle p_1(v), w \rangle - \langle p_2(v), w \rangle = \tag{4.2}$$

$$\langle v, p_1(w) \rangle - \langle v, p_2(w) \rangle = \tag{4.3}$$

$$\langle v, w \rangle - \langle v, w \rangle = 0 \tag{4.4}$$

So $(p_1 - p_2)(v) \in L_2^\perp$ as well, and so we conclude $(p_1 - p_2)(v) \in L_1 \ominus L_2$.

Now, let $v \in V$ and $w \in L_1 \ominus L_2$. Then we calculate:

$$\langle v - (p_1 - p_2)(v), w \rangle = \langle v, w \rangle - \langle p_1(v), w \rangle + \langle p_2(v), w \rangle = \tag{4.5}$$

$$\langle v, w \rangle - \langle v, p_1(w) \rangle + \langle v, p_2(w) \rangle = \tag{4.6}$$

$$\langle v, w \rangle - \langle v, w \rangle + \langle v, 0 \rangle = 0 \tag{4.7}$$

This is exactly the condition an orthogonal projection operator must satisfy. □

5 Geometric orthogonality

Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let L_1 and L_2 be subspaces of V, and let $L_0 = L_1 \cap L_2$. We now call L_1 and L_2 *geometrically orthogonal* iff:

$$L_1 \ominus L_0 \perp L_2 \ominus L_0 \tag{5.1}$$

We will use the following notation:

$$L_1 \perp_G L_2 \tag{5.2}$$

Theorem 5.1. *Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let L_1 and L_2 be subspaces of V, and let $L_0 = L_1 \cap L_2$. et p_0, p_1, and p_2 be the orthogonal projections of L_0, L_1, and L_2. The following three statements are then equivalent:*

- $L_1 \perp_G L_2$
- $p_1 \circ p_2 = p_2 \circ p_1$
- $p_1 \circ p_2 = p_0$

Proof. From theorem 4.1 we know that $p_1 - p_0$ and $p_2 - p_0$ are the orthogonal projections into $L_1 \ominus L_0$ and $L_2 \ominus L_0$ respectively. According to theorem 3.1 $L_1 \ominus L_0$ and $L_2 \ominus L_0$ are orthogonal if and only if:

$$(p_1 - p_0) \circ (p_2 - p_0) = 0 \tag{5.3}$$

Expanding the left side:

$$p_1 \circ p_2 - p_1 \circ p_0 - p_0 \circ p_2 + p_0 \circ p_0 \tag{5.4}$$

Since $L_0 \subseteq L_1$, according to theorem 3.2 $p_1 \circ p_0 = 0_0$. And likewise, because $L_0 \subseteq L_2$, we must have $p_0 \circ p_2 = p_0$. Finally, because of idempotence, $p_0 \circ p_0 = p_0$. So:

$$(p_1 - p_0) \circ (p_2 - p_0) = p_1 \circ p_2 - p_0 - p_0 + p_0 = p_1 \circ p_2 - p_0 \tag{5.5}$$

This is zero exactly when $p_1 \circ p_2 = p_0$. So we've shown that the first and third statements are equivalent.

Next, assume $p_1 \circ p_2 = p_0$, and let $v, w \in V$. Then:

$$\begin{aligned}
\langle (p_1 \circ p_2)(v), w \rangle =& \langle p_0(v), w \rangle = && (5.6)\\
& \langle v, p_0(w) \rangle = && (5.7)\\
& \langle v, (p_1 \circ p_2)(w) \rangle = && (5.8)\\
& \langle v, p_1(p_2(w)) \rangle = && (5.9)\\
& \langle p_1(v), p_2(w), \rangle = \langle (p_2 \circ p_1)(v), w \rangle && (5.10)
\end{aligned}$$

Here, we've repeated used the symmetry of projection operators. Since this is true for arbitrary $v, w \in V$ we must have $p_1 \circ p_2 = p_2 \circ p_1$.

Finally, assume $p_1 \circ p_2 = p_2 \circ p_1$. For any $v \in V$, $p_1(v) \in L_1$ and $p_2(v) \in L_2$. So $p_2(p_1(v)) \in L_1$ and $p_1(p_2(v)) \in L_2$. But since the two are equal, they must lie in $L_1 \cap L_2$. Now assume $v \in V, w \in L_1 \cap L_2$. Then we calculate:

$$\langle v - p_1(p_2(v)), w \rangle = \langle v, w \rangle - \langle p_1(p_2((v)), w \rangle = \langle v, w \rangle - \langle v, p_2(p_1(w)) \rangle \tag{5.11}$$

In the last step, we repeated used the symmetry of the p's. Now, since $p_2(p_1(w)) \in L_1 \cap L_2$ it is simply w and we get:

$$\langle v - p_1(p_2(v)), w \rangle = \langle v, w \rangle - \langle v, w \rangle = 0 \tag{5.12}$$

This shows that $p_1 \circ p_2$ is the orthogonal projection operator on $L_1 \cap L_2$, and therefore equal to p_0. □

6 Projections on sums of subspaces

Recall, that if L_1 and L_2 are both subspaces of the vector space V, we can form a new subspace as follows:

$$L_1 + L_2 = \{v_1 + v_2 | v_1 \in L_1, v_2 \in L_2\} \tag{6.1}$$

We now wish to consider orthogonal projections on such spaces. It will turn out that we can only express it in terms of the orthogonal projection operators of L_1 and L_2 in certain cases.

Theorem 6.1. *Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let L_1 and L_2 be subspaces of V with associated orthogonal projection operators p_1 and p_2. If L_1 and L_2 are orthogonal then:*

$$dim(L_1 + l_2) = dim(L_1) + dim(L_2) \tag{6.2}$$

And the orthogonal projection on $L_1 + L_2$ is:

$$p_{1+2} = p_1 + p_2 \tag{6.3}$$

Furthermore:

$$\forall v \in V : \; ||p_{1+2}v||^2 = ||p_1 v||^2 + ||p_2 v||^2 \tag{6.4}$$

Proof. The dimensionality follows trivially from the orthogonality, since the union of a basis for L_1 and L_2 respectively will be a basis for the sum.

Let $v \in V$. Since $p_1 v \in L_1$ and $p_2 v \in L_2$, it follows that $(p_1 + p_2)(v) \in L_1 + L_2$. If $w \in L_1 + L_2$ we can decompose it as $w = w_1 + w_2$, where $w_1 \in L_1$ and $w_2 \in L_2$. Now calculate:

$$\begin{aligned}
\langle v - (p_1 + p_2)(v), w\rangle =& \langle v - p_1 v - p_2 v, w_1 + w_2\rangle = && (6.5)\\
& \langle v - p_1 v - p_2 v, w_1\rangle + \langle v - p_1 v - p_2 v, w_2\rangle = && (6.6)\\
& \langle v - p_1 v, w_1\rangle - \langle p_2 v, w_1\rangle + && (6.7)\\
& \langle v - p_2 v, w_2\rangle - \langle p_1 v, w_2\rangle && (6.8)
\end{aligned}$$

Because p_1 and p_2 are projection operators, the first term on each line is zero by definition. And because L_1 and L_2 are orthogonal, so are the other two terms. So the total is zero, which proves that $p_1 + p_2$ is the orthogonal projection operator of $L_1 + L_2$.

Finally, the norm relation follows directly from Pythagoras' theorem. □

We will now extend this result to the case where L_1 and L_2 are geometrically orthogonal rather than strictly so. To do so we need a result about sums of such subspaces:

Theorem 6.2. *Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let L_1 and L_2 be geometrically orthogonal subspaces of V. Then $L_0 = L_1 \cap L_2$ is orthogonal to both $L_1 \ominus L_0$ and $L_1 \ominus L_0$. In addition:*

$$L_1 + L_2 = L_1 \ominus L_0 + L_2 \ominus L_0 + L_0 \tag{6.9}$$

Proof. Remember that $L_1 \ominus L_0 = L_1 \cap L_0^{\perp}$. Hence any vector $v \in L_1 \ominus L_0$ is also in $L_0^{\perp}$. And so is orthogonal to L_0. Similarly for $L_2 \ominus L_0$.

Now, let $B_0 = \{e_1, e_2, \cdots, e_{n_0}\}$ be as basis for L_0. Since $L_1 \subseteq L_0$ we can expand this into a basis for L_1: $B_1 = B_0 \cup \{e_{n_0+1}, \cdots, e_{n_1}\}$. Now, the basis for $L_1 \ominus L_0$ must then be $\{e_{n_0+1}, \cdots, e_{n_1}\}$. Similarly, we can construct a basis $B_2 = B_0 \cup \{f_{n_0+1}, \cdots, f_{n_2}\}$ for L_2, which leads to the basis $\{f_{n_0+1}, \cdots, f_{n_1}\}$ for $L_1 \ominus L_0$. Now consider the sum:

$$L_1 \ominus L_0 + L_2 \ominus L_0 + L_0 \tag{6.10}$$

This is spanned by $\{e_{n_0+1}, \cdots, e_{n_1}\} \cup \{f_{n_0+1}, \cdots, f_{n_2}\} \cup B_0 = B_1 \cup B_2$. Which means that it is exactly equal to $L_1 + L_2$. $\square$

Theorem 6.3. *Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let L_1 and L_2 be subspaces of V with associated orthogonal projection operators p_1 and p_2. If L_1 and L_2 are geometrically orthogonal the orthogonal projection on the sum space is:*

$$p_{1+2} = p_1 + p_2 - p_1 \circ p_2 \tag{6.11}$$

Furthermore:

$$\forall v \in V: \ ||p_{1+2}v||^2 = ||p_1 v||^2 + ||p_2 v||^2 - ||p_1 \circ p_2 v||^2 \tag{6.12}$$

Note: With the notation from theorem 5.1 we could also write these equations as:

$$p_{1+2} = p_1 + p_2 - p_0, \quad ||p_{1+2}v||^2 = ||p_1 v||^2 + ||p_2 v||^2 - ||p_0 v||^2 \tag{6.13}$$

Proof. Since $L_1 \perp_G L_2$, we know from theorem 5.1 that $L_1 \ominus L_0$ and $L_2 \ominus L_0$, where $L_0 = L_1 \cap L_2$, are (properly) orthogonal. And L_0 is orthogonal to both according to theorem 6.2, which also states that:

$$L_1 + L_2 = (L_1 \ominus L_0) + (L_2 \ominus L_0) + L_0 \tag{6.14}$$

According to theorem 6.1 we can now find the projection operator as:

$$p_{1+2} = (p_1 - p_0) + (p_2 - p_0) + p_0 = p_1 + p_2 - p_0 \tag{6.15}$$

The norm identity from the same theorem now states:

$$||p_{1+2}v||^2 = ||(p_1 - p_0)v||^2 + ||(p_2 - p_0)v||^2 + ||p_0 v||^2 \tag{6.16}$$

Consider the first term:

$$||p_1 v - p_0 v||^2 = ||p_1 v||^2 + ||p_0 v||^2 - 2\langle p_1 v, p_0 v \rangle \tag{6.17}$$

We can rewrite the inner product using the symmetry of projections, theorem 3.2, and the idempotency of p_0:

$$\begin{aligned}\langle p_1 v, p_0 v\rangle =&\langle v, p_1 \circ p_0 v\rangle = && (6.18)\\ &\langle v, p_0 v\rangle = && (6.19)\\ &\langle v, p_0 \circ p_0 v\rangle = && (6.20)\\ &\langle p_0 v, p_0 v\rangle = ||p_0 v|| && (6.21)\\ & && (6.22)\end{aligned}$$

This means that:

$$||p_1 v - p_0 v||^2 = ||p_1 v||^2 - ||p_0 v||^2 \tag{6.23}$$

Similarly:

$$||p_2 v - p_0 v||^2 = ||p_2 v||^2 - ||p_0 v||^2 \tag{6.24}$$

Now equation 6.16 becomes:

$$\begin{aligned}||p_{1+2} v||^2 =&||p_1 v||^2 - ||p_0 v||^2 + ||p_2 v||^2 - ||p_0 v||^2 + ||p_0 v||^2 && (6.25)\\ &||p_1 v||^2 + ||p_2 v||^2 - ||p_0 v||^2 && (6.26)\end{aligned}$$

□