Control Theory

Kristian Wichmann

July 15, 2018

1 Control and error

Control theory deals with strategies for keeping a quantity at a constant level in a dynamic system. In mathematical terms we try to keep a quantity $y(t)$ at a constant level y_r over time t.

To achieve this goal, a *controller* will affect the system at all time. This will generally be based on the *error*, i.e. the current deviation from the desired level:

$$e(t) = y_r - y(t) \tag{1.1}$$

2 P-control

P-control is the case where the controller correction u is proportional (hence the P) to the error:

$$u(t) = ke(t) = k(y_r - y(t)) \tag{2.1}$$

2.1 Example: Anaesthesia

Surgery is performed on a patient. During the procedure, it is desirable to keep the blood concentration of anaesthetic $y(t)$ at a constant level y_r. Without control, the concentration follows the following differential equation:

$$\frac{dy}{dt} = -ay \tag{2.2}$$

I.e. it will decay exponentially from a starting concentration $y_0 = y(0)$:

$$y(t) = y_0 \cdot e^{-at} \tag{2.3}$$

We now add the control term:

$$\frac{dy}{dt} = -ay + u = -ay + k(y_r - y(t)) = ky_r - (a + k)y \tag{2.4}$$

This is a differential equation of the form:

$$\frac{dy}{dy} = -b + ay \tag{2.5}$$

Which has the general solution:

$$y(t) = -\frac{b}{a} + c \cdot e^{-at} \tag{2.6}$$

Here, this means:

$$y(t) = \frac{ky_r}{a+k} + c \cdot e^{(a+k)t} \tag{2.7}$$

With the boundary condition that $y(0) = 0$ we can determine c:

$$c = -\frac{ky_r}{a+k} \tag{2.8}$$

We can now write the solution as:

$$y(t) = \frac{ky_r}{a+k} - \frac{ky_r}{a+k} e^{(a+k)t} \tag{2.9}$$

So the error is:

$$e(t) = y_r - y(t) = y_r - \frac{ky_r}{a+k} + \frac{ky_r}{a+k} e^{(a+k)t} \tag{2.10}$$

Expand first term to get common denominator:

$$\begin{aligned} e(t) &= \frac{y_r(a+k)}{a+k} - \frac{ky_r}{a+k} + \frac{ky_r}{a+k} e^{(a+k)t} && (2.11) \\ &= \frac{y_r}{a+k} \left[a + k(e^{(a+k)t} - 1) \right] && (2.12) \end{aligned}$$

The controller dose is then found by multiplying by k:

$$u(t) = \frac{y_r}{a+k} \left[ak + k^2(e^{(a+k)t} - 1) \right] \tag{2.13}$$

However, we now see that in the limit $t \to \infty$ the error is actually not zero, as we would hope for, but instead:

$$\lim_{t \to \infty} e(t) = y_r \frac{a}{a+k} \tag{2.14}$$

3 Laplace transforms

Given a function $f = f(t)$ defined for all positive t. Then the Laplace transform of it is defined as:

$$\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-ts}\ dt \tag{3.1}$$

The notation $F(s)$ is often used as a shorthand, and similarly for other functions.

3.1 Properties of the Laplace transform

The Laplace transform is linear, since integration is:

$$\begin{aligned}
\mathcal{L}[af + bg](s) &= \int_0^\infty [af(t) + b(g(t)]\, e^{-ts}\ dt && (3.2)\\
&= a\int_0^\infty f(t)e^{-ts}\ dt + b\int_0^\infty g(t)e^{-ts}\ dt && (3.3)\\
&= a\mathcal{L}[f](s) + b\mathcal{L}[g](s) && (3.4)
\end{aligned}$$

Laplace transforming a derivative gives us:

$$\begin{aligned}
\mathcal{L}\left[\frac{df}{dt}\right](s) &= \int_0^\infty \frac{df(t)}{dt}e^{-ts}\ dt && (3.5)\\
&= \left[f(t)e^{-ts}\right]_0^\infty - \int_0^\infty f(t)\frac{d}{dt}e^{-ts}\ dt && (3.6)\\
&= -f(0) + s\int_0^\infty f(t)e^{-ts}\ dt && (3.7)\\
&= s\mathcal{L}[f](s) - f(0) && (3.8)
\end{aligned}$$

Here partial integration has been used. Note that we have assumed that $f(t)$ grows slower than an exponential for $t \to \infty$.

Similarly, we can transform an integral:

$$\begin{aligned}
\mathcal{L}\left[\int_0^t f(x)\ dx\right](s) &= \int_0^\infty \int_0^t f(x)\ dx\ e^{-ts}\ dt && (3.9)\\
&= \left[\int_0^t f(x)\ dx \cdot \left(-\frac{1}{s}\right)e^{-ts}\right]_0^\infty - \int_0^\infty f(t)\left(-\frac{1}{s}\right)e^{-ts}\ dt && (3.10)\\
&= \frac{1}{s}\mathcal{L}[f](s) && (3.11)
\end{aligned}$$

Again, we have made assumptions on the growth speed of the integrand, i.e. this time of the integral of f.

3.2 A few select Laplace transforms

We consider two specific Laplace transforms in this section. First of a constant:

$$\mathcal{L}[k](s) = \int_0^\infty k \cdot e^{-st}\ dt \tag{3.12}$$

$$= k \int_0^\infty e^{-st}\ dt \tag{3.13}$$

$$= -\frac{k}{s}[e^{-st}]_0^\infty = \frac{k}{s} \tag{3.14}$$

And then of the function te^{at}:

$$\mathcal{L}[te^{at}](s) = \int_0^\infty te^{at} \cdot e^{-st}\ dt \tag{3.15}$$

$$= \int_0^\infty te^{(a-s)t}\ dt \tag{3.16}$$

$$= \left[t\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty - \int_0^\infty 1 \cdot \frac{1}{a-s}e^{(a-s)t}\ dt = \frac{1}{(s-a)^2} \tag{3.17}$$