Product of normal pdf's

Kristian Wichmann

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While the product of two normal distributions is not a normal, it turns out that a distribution whose pdf is the product of two normal pdf's is proportional to a normal.

1 The normal pdf

The probability density function of a normally distributed variable with mean μ and variance σ^2 is:

$$\phi(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{1}$$

2 Product of normals

Consider the product of two different normals:

$$\phi(x;\mu_1,\sigma_1^2)\phi(x;\mu_2,\sigma_2^2) = \tag{2}$$

$$\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right) = \tag{3}$$

$$\frac{1}{2\pi\sigma_1\sigma_2} \exp\left[-\frac{1}{2}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(x-\mu_2)^2}{\sigma_2^2}\right)\right] \tag{4}$$

Consider the contents of the inner parenthesis:

$$\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(x-\mu_2)^2}{\sigma_2^2} = \frac{\sigma_2^2(x-\mu_1)^2 + \sigma_1^2(x-\mu_2)^2}{\sigma_1^2\sigma_2^2} \tag{5}$$

Expand the numerator:

$$\sigma_2^2(x^2 - 2\mu_1 x + \mu_1^2) + \sigma_1^2(x^2 - 2\mu_2 x + \mu_2^2) = \tag{6}$$

$$(\sigma_1^2 + \sigma_2^2)x^2 - 2(\sigma_1^2\mu_2 + \sigma_2\mu_1)x + \sigma_1^2\mu_2^2 + \sigma_2^2\mu_1^2 \tag{7}$$

If we reduce the fraction from equation (5) by $\sigma_1^2 + \sigma_2^2$, the denominator becomes $\sigma_1^2\sigma_2^2/(\sigma_1^2 + \sigma_2^2)$ and the numerator:

$$x^2 - 2\underbrace{\frac{\sigma_1^2\mu_2 + \sigma_2^2\mu_1}{\sigma_1^2 + \sigma_2^2}}_{\alpha} x + \underbrace{\frac{\sigma_1^2\mu_2^2 + \sigma_2^2\mu_1^2}{\sigma_1^2 + \sigma_2^2}}_{\beta} \tag{8}$$

Completing the square, this can be rewritten:

$$(x - \alpha)^2 + \beta - \alpha^2 \tag{9}$$

This means that the product is proportional to:

$$\exp\left(-\frac{(x-\alpha)^2}{2\left(\sigma_1^2\sigma_2^2/(\sigma_1^2 + \sigma_2^2)\right)}\right) \tag{10}$$

In other words, the new distribution is proportional to a normal with mean and standard deviation:

$$\mu^* = \frac{\sigma_1^2\mu_2 + \sigma_2^2\mu_1}{\sigma_1^2 + \sigma_2^2}, \quad \sigma^* = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} \tag{11}$$

3 Proportionality constant

Sometimes, the proportionality constant from the example above may be of interest as a normalization constant. We have:

$$\phi(x; \mu_1, \sigma_1^2)\phi(x; \mu_2, \sigma_2^2) = C \cdot \phi\left(x; \mu^*, (\sigma^*)^2\right) \tag{12}$$

From the previous section, we know this implies:

$$\frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{(x-\mu^*)^2}{(\sigma^*)^2}\right) \exp\left(-\frac{\beta - \alpha^2}{2\sigma_1\sigma_2(\sigma_1^2 + \sigma_2^2)}\right) = \tag{13}$$

$$C \cdot \frac{1}{\sqrt{2\pi \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}} \exp\left(-\frac{(x-\mu^*)^2}{(\sigma^*)^2}\right) \tag{14}$$

From this C can be isolated:

$$C = \frac{\sqrt{2\pi}\sigma_1\sigma_2}{2\pi\sigma_1\sigma_2\sqrt{\sigma_1^2 + \sigma_2^2}} \exp\left(-\frac{\beta - \alpha^2}{2\sigma_1\sigma_2(\sigma_1^2 + \sigma_2^2)}\right) \tag{15}$$

The fraction simplifies to $\frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}$. The hard part is simplifying $\beta - \alpha^2$:

$$\beta - \alpha^2 = \frac{\sigma_1^2\mu_2^2 + \sigma_2^2\mu_1^2}{\sigma_1^2 + \sigma_2^2} - \left(\frac{\sigma_1^2\mu_2 + \sigma_2^2\mu_1}{\sigma_1^2 + \sigma_2^2}\right)^2 \tag{16}$$

Common denominator:

$$\frac{(\sigma_1^2+\sigma_2^2)(\sigma_1^2\mu_2^2+\sigma_2^2\mu_1^2)-(\sigma_1^2\mu_2+\sigma_2^2\mu_1)^2}{(\sigma_1^2+\sigma_2^2)^2} \tag{17}$$

Expand the numerator:

$$\sigma_1^4\mu_2^2+\sigma_1^2\sigma_2^2\mu_1^2+\sigma_1^2\sigma_2^2\mu_2^2+\sigma_2^4\mu_1^2-(\sigma_1^4\mu_2^2+\sigma_2^4\mu_1^2+2\sigma_1^2\sigma_2^2\mu_1\mu_2)= \tag{18}$$

$$\sigma_1^2\sigma_2^2(\mu_1^2+\mu_2^2-2\mu_1\mu_2)=\sigma_1^2\sigma_2^2(\mu_1-\mu_2)^2 \tag{19}$$

We now have:

$$\frac{\beta-\alpha^2}{2\sigma_1\sigma_2(\sigma_1^2+\sigma_2^2)}=\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)} \tag{20}$$

This means that C can be expressed as:

$$C=\frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\exp\left(-\frac{(\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right)=\phi(\mu_1;\mu_2,\sigma_1^2+\sigma_2^2) \tag{21}$$