Time series

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1 Stochastic processes and time series

Definition 1.1. *Let T be a set, called the index set, and $(\Omega, \mathcal{F}, P)$ be a probability space. Then a stochastic process is a set of random variables $\{X_t | t \in T\}$, i.e. $\mathcal{F}$-measurable functions $X_t : \Omega \to \mathbf{R}$.*

Definition 1.2. *For each $\omega \in \Omega^T$ we can define a function $x : T \to \mathbf{R}$ by:*

$$x(t) = x_t = X_t(\omega(t)) \tag{1.1}$$

These are known as realizations or sample-paths of the stochastic process.

Definition 1.3. *The distribution function for a stochastic process is a function $F : \mathbf{R}^T \to [0, 1]$ defined by:*

$$F(x) = P(\forall\ t \in T : X_t \leq x_t) \tag{1.2}$$

Here x may be any function $T \to \mathbf{R}$.

Definition 1.4. *A stochastic process for which the index set $T \subseteq \mathbf{Z}$ is called a time series.*

As long as there's no chance of confusion, we will use the term 'time series' interchangeably for the stochastic process itself, and any relevant realizations of it.

2 Simple random walk

Let the random variables $Y_1, Y_2, Y_3, \ldots$ be i.i.d. with the distribution:

$$P(Y_i = 1) = 1/2, \quad P(Y_i = -1) = 1/2 \tag{2.1}$$

Now let a time series be defined as:

$$X_0 = 0, \quad X_n = \sum_{i=1}^{n} Y_i \tag{2.2}$$

This is known as the *simple random walk*.

2.1 Asymptotic behaviour

For each of the Y's we have:

$$E[Y_i] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0, \quad \text{var}[Y_i] = \frac{1}{2}1^2 + \frac{1}{2}(-1)^2 = 1 \tag{2.3}$$

So, according to the central limit theorem, for large n, X_n will be approximately normally distributed:

$$X_n \sim N(0, n), \quad n \gg 1 \tag{2.4}$$

This means that the standard deviation for large n is $\sqrt{n}$.

3 Markov chains

A *Markov chain* is a time series, in which the conditional distribution of X_{n+1} given the realizations of $X_0, X_1, \ldots, X_n$ only depends on the realization of X_n. Formally:

$$P(X_{n+1} = s | X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = P(X_{n+1} = s | X_n = x_n) \tag{3.1}$$

3.1 Markov chains with a finite number of states

In the case where each X only has a finite number of realizations n, the Markov chain can be conveniently specified in matrix form. Assume the realization of X_n is state i, then we might ask what to probability of X_{n+1} being realized as state j. This probability is called p_{ij}. These probabilities can be neatly organized in matrix form:

$$A = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix} \tag{3.2}$$

Since the n states exhaust the possibilities, each column must sum to 1:

$$\sum_{j=1}^{n} p_{ij} = 1, \quad i \in 1, 2, \ldots n \tag{3.3}$$

4 Martingales