

Category theory

Kristian Wichmann

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1 Definition of a category

Definition 1.1. A category \mathcal{C} is a collection of objects called \mathcal{C} -objects and a collection of relations called \mathcal{C} -arrows. A \mathcal{C} -arrow f points from one \mathcal{C} -object called the source or $\text{src}(f)$, to another \mathcal{C} -object called the target or $\text{tar}(f)$. If $\text{src}(f) = A$ and $\text{tar}(f) = B$, this is written:

$$f : A \rightarrow B \quad \text{or} \quad A \xrightarrow{f} B \quad (1.1)$$

The collection of all arrows from A to B is called $\text{Hom}(A, B)$. The arrows must fulfil the following:

- *Composition:* Given that two \mathcal{C} -arrows $f : A \rightarrow B$ and $g : B \rightarrow C$ exists, then \mathcal{C} should also contain a composite arrow $g \circ f : A \rightarrow C$.
- *Associativity:* The composition rule should be associative, so that $(f \circ g) \circ h = f \circ (g \circ h)$ when defined.
- *Identity arrow:* For any \mathcal{C} -object A , there should be an identity arrow $1_A : A \rightarrow A$, such that for any \mathcal{C} -arrows $f : A \rightarrow B$ and $g : B \rightarrow A$:

$$f \circ 1_A = f \quad \text{and} \quad 1_A \circ g = g \quad (1.2)$$

Very often, the arrows will be *(homo)morphisms*, i.e. maps preserving the *structure* associated with the category. The Hom -notation is inspired by this fact. In this case, source and target are identified with domain and co-domain respectively. However in general, arrows need not be maps.

Note that \mathcal{C} is not necessarily a set - most categories we will be interested in are too large to be sets and are thus classes. In fact, we will naively work without necessarily worrying too much about set theory¹. Still, we do make the following definition.

¹Different set theories will each have categories of their own. See below

Definition 1.2. A category for which the collection of objects is a set is called *small*. A category for which all $\text{Hom}(A, B)$ are sets is called *locally small*.

Do note that some presentations of category theory have local smallness as an additional axiom.

1.1 Diagrams

To visualize objects and arrows, diagrams are often used. Of particular interest here are *commutative diagrams*. In such diagrams, following different compositions from objects A to D by following arrows, should result in the same combined arrow. For instance, the associative axiom can be expressed by demanding that the following diagrams commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ & \searrow g \circ f & \downarrow g & \searrow g \circ h & \\ & & C & \xrightarrow{h} & D \end{array}$$

No matter which arrow path we follow from A to D , we should get the same result. Similarly, the identity arrow axiom can be expressed by requiring that the following diagram commutes:

$$1_A \hookrightarrow A \xrightarrow{f} B \hookleftarrow 1_B$$

1.2 Uniqueness of identity arrows

This will be our very first baby step result:

Theorem 1.1. Let \mathcal{C} be a category, and let A be a \mathcal{C} -object. Then the corresponding identity arrow $1_A : A \rightarrow A$ is unique.

Proof. Let $1'_A : A \rightarrow A$ be another arrow that satisfies the identity arrow axioms. Then:

$$1'_A = 1'_A \circ 1_A = 1_A \tag{1.3}$$

□

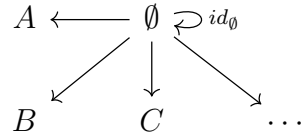
2 Examples of categories

2.1 The category of sets

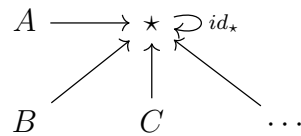
Here, \mathcal{C} is the collection of sets². This is, at least from a mathematical point of view, a category with minimal structure. So *any* map from one set to another will preserve set structure. Hence, the arrows consist of all (total) maps between sets. Composition is ordinary function composition, and the identity element corresponding to a set A is id_A . These clearly satisfy the category axioms.

2.1.1 Characterizing sets in terms of arrows

We usually talk about sets in terms of elements, but we would like to characterize sets purely in terms of category terms, i.e. arrows. Let's start with the empty set \emptyset . A map with \emptyset as domain and B as co-domain is uniquely determined, since there's only one way to map zero elements to another set. Hence \emptyset can be characterized as the only object which has exactly one arrow to every other object:



Next step is singletons. Any function with a singleton \star as co-domain is uniquely defined, since all elements in the domain must map to the single element. Hence, we can characterize a singleton \star as an object for which there is only one arrow pointing to it from any other object:



We can now use singletons to describe elements of a set purely in categorical terms: Let \star be any singleton. Then consider an arrow $x : \star \rightarrow A$. Such a map picks out exactly one element from A . It is clear that there is one-to-one correspondence between the elements of A and such arrows. We may informally write $x \in A$.

$$\star \xrightarrow{x} A$$

It will also be practical to consider the image of an element $x \in A$ under the map $f : A \rightarrow B$. Since $f(x) \in B$, in category-speak it must be arrow

²Of a given set theory. Therefore the title should technically be 'categories of sets'.

$f(x) : \star \rightarrow B$. This naturally corresponds to a composition of the arrows x and f , such that $f(x) = f \circ x$.

$$\star \xrightarrow{x} A \xrightarrow{f} B$$

2.2 Monoids, and the category of monoids

Definition 2.1. *A monoid is a set M with a composition rule $M \times M \rightarrow M$ that satisfies:*

- *Associativity.*
- *Existence of an identity element.*

So one can informally think of monoids as "groups that does not necessarily have inverse elements".

We may take any monoid M and turn it into a category \mathcal{M} with just one element - let's call it \star . It's common to set $\star = \{M\}$, but any singleton will do. Now let each element f of M correspond to an arrow $f : \star \rightarrow \star$ with the composition rule being given by the monoid composition. Because of the monoid axioms, this is a category.

We can also think of the collection of monoids as a category. Here the arrows are morphisms between monoids. The structure to be preserved is the composition. In other words, $f(a_1 a_2) = f(a_1) f(a_2)$, where $f : A \rightarrow B$ is an arrow and $a_1, a_2 \in A$. This is analogous to group homomorphisms.

2.3 Groups, and Abelian groups

Groups are essentially monoids with invertible elements. So every group G is a monoid, and we may turn it into its own category using the method from the last subsection. Similarly, the collection of all groups is a category with group homomorphisms playing the role of arrows. The same is true for the collection of abelian groups.

2.4 Orderings

2.5 Equivalence relations

2.6 Programming language types

3 Duality: The opposite category

Given a category \mathcal{C} , it is always possible to construct the *opposite* or *dual* category \mathcal{C}^{op} :

Definition 3.1. *The opposite of the category \mathcal{C} is the category \mathcal{C}^{op} defined as:*

- *The objects of \mathcal{C}^{op} are the same as the objects of \mathcal{C} .*
- *For each \mathcal{C} -arrow $f : A \rightarrow B$ there is a corresponding \mathcal{C}^{op} -arrow with source and target reversed.*
- *Composition in \mathcal{C}^{op} is defined by letting $f \circ^{op} g$ correspond to $g \circ f$.*
- *The identity arrows stay the same.*

It is clear, that because \mathcal{C} obeys the category axioms, so must \mathcal{C}^{op} .

3.1 The duality principle

Consider \mathcal{L} , the elementary pure language of categories. This is loosely speaking the collection of statements that are possible to form about categories. Or rather, that's what *well-formed formulas* (wff's) are, loosely speaking: The subset of \mathcal{L} that makes syntactic sense.

Definition 3.2. *Given a wff φ from \mathcal{L} , the dual φ^{op} is the wff obtained by:*

- *Swapping src and tar.*
- *Reversing composition, such that $f \circ g$ becomes $g \circ f$ etc.*

The *duality principle* builds on the fact that \mathcal{C}^{op} is a category. It will often be a shortcut for proofs, basically giving us a bonus theorem each time one is proven.

Theorem 3.1. *(The duality principle). Given a wff φ which contains no free variables - i.e. all variables must be category objects or arrows. Then if φ is provable from the axioms of category theory, then so is the dual wff φ^{op} .*

Proof. The proof of φ consists of a string of wff's derived by using the axioms. So by taking the dual of each of these, we obtain a proof of φ^{op} following from the dual axioms. But the dual axioms are themselves axioms, and so the proof is valid. \square

4 Monic and epic arrows

In this section we want to generalize certain characteristics of mappings to the language of arrows. Our starting point will be the category of sets, which will serve as a familiar territory from which we will then generalize.

4.1 Monomorphisms

Recall that a map $f : B \rightarrow C$ is injective or one-to-one iff:

$$\forall x, y \in B : f(x) = f(y) \Rightarrow x = y \quad (4.1)$$

Now, we need to translate this into a statement about arrows. Since an element of a set may be thought of as a map from a singleton \star into the set, this is equivalent to:

$$\forall x : \star \rightarrow B, y : \star \rightarrow A : f \circ x = f \circ y \Rightarrow x = y \quad (4.2)$$

So, if we assume the following stronger criterium, f must certainly be an injection:

$$\forall g : A \rightarrow B, h : A \rightarrow B : f \circ g = f \circ h \Rightarrow g = h \quad (4.3)$$

However, assume f to be an injection. Then $f(g(x)) = f(h(x))$ implies $g(x) = h(x)$, which means $g = h$. Hence injectivity is equivalent to equation (4.3). We now define:

Definition 4.1. *Let \mathcal{C} be a category. Then a \mathcal{C} -arrow $f : B \rightarrow C$ is said to be monomorphic (or monic for short) iff it is left-cancelable in the following sense:*

$$\forall g : A \rightarrow B, h : A \rightarrow B : f \circ g = f \circ h \Rightarrow g = h \quad (4.4)$$

So what we have just proved above is, that in the category of sets, the monic arrows are exactly the injective maps.

4.2 Epimorphisms

Recall that a map $f : B \rightarrow C$ is surjective or onto iff:

$$\forall c \in C \exists x \in B : f(x) = c \quad (4.5)$$

Assume this is true, and consider two functions $g : C \rightarrow D, h : C \rightarrow D$, so that $g \neq h$. This means that there must be a $c \in C$ such that $g(c) \neq h(c)$. Since f is surjective, there is an $x \in B$ such that $f(x) = c$, so $g(f(x)) = g(c) \neq h(c) = h(f(x))$. So $g \circ f \neq h \circ f$. To sum up:

$$g \neq h \Rightarrow g \circ f \neq h \circ f \quad (4.6)$$

But this is logically equivalent to:

$$g \circ f = h \circ f \Rightarrow g = h \quad (4.7)$$

Thus surjectivity implies *right-cancelability*. The converse is also true: Assume f is not surjective. Then there is some $c \in C$, so that no $x \in B$ satisfies $f(x) = c$. Let $g, h : C \rightarrow D$ be functions that take on equal values except for c . Then $g \neq h$, but still $g \circ f = h \circ f$. In other words non-surjectivity implies that equation (4.7) is false. This is logically equivalent to the desired result. We now define:

Definition 4.2. *Let \mathcal{C} be a category. Then a \mathcal{C} -arrow $f : B \rightarrow C$ is said to be epimorphic (or epic for short) iff it is right-cancelable in the following sense:*

$$\forall g : C \rightarrow D, h : C \rightarrow D : g \circ f = h \circ f \Rightarrow g = h \quad (4.8)$$

So in the category of sets, epic arrows are exactly the surjective maps. Note that the definitions of being monic and epic are each others duals.

4.3 Diagrammatic representation

Monomorphisms and epimorphisms are sometimes given special arrow notations in diagrams:

$$A \xrightarrow{\quad f \quad} B \qquad A \xrightarrow{\quad g \quad} \gg B$$

Here f is monic, and g is epic.

4.4 Some basic properties

Theorem 4.1. *Identity arrows are always monic as well as epic.*

Proof. Assume $id_A \circ g = id_A \circ h$. Then by definition of the identity arrow $g = h$, so id_A is monic. Similarly for epic. It also follows as it is the dual statement. \square

Theorem 4.2. *Assume f and g are monic (epic). Then $f \circ g$, if defined, is monic (epic) as well.*

Proof. Assume f and g to be monic. Further assume $(f \circ g) \circ j = (f \circ g) \circ k$. By associativity $f \circ (g \circ j) = f \circ (g \circ k)$. Since f is monic, this means $g \circ j = g \circ k$. And since g is also monic $j = k$. The epic case follows from duality. \square

Theorem 4.3. *If $f \circ g$ is monic, so is g . If $f \circ g$ is epic, so is f .*

Proof. Assume $f \circ g$ is monic. Further assume $g \circ j = g \circ k$. Apply f from the left on both sides and use associativity to get $(f \circ g) \circ j = (f \circ g) \circ k$. Because $f \circ g$ is monic $j = k$ and so g is monic. The epic case follows by duality. \square

5 Inverses

Inverses of arrows are defined much like matrix inverses:

Definition 5.1. *Let \mathcal{C} be a category and $f : A \rightarrow B$ be a \mathcal{C} -arrow. Then $g : B \rightarrow A$ is...*

- ... a right-inverse of f iff $f \circ g = id_B$.
- ... a left-inverse of f iff $g \circ f = id_A$.
- ... an inverse of f iff it is both a right-inverse and a left-inverse of f .

Note that if $f \circ g = 1_B$ in \mathcal{C} , then $g \circ^{\text{op}} f = 1_B$ in \mathcal{C}^{op} . Hence right- and left-inverses are dual concepts of each other, and inverses are dual to themselves. Right- and left-inverses need not be unique. However:

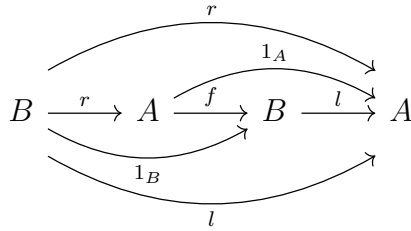
Theorem 5.1. *If the arrow f has both a right- and a left-inverse, then these are equal. Hence inverses are unique.*

Proof. Assume $f : A \rightarrow B$ has a right-inverse $r : B \rightarrow A$ and a left-inverse $l : B \rightarrow A$. Then:

$$r = 1_A \circ r = (l \circ f) \circ r = l \circ (f \circ r) = l \circ 1_B = l \quad (5.1)$$

□

This is equivalent to requiring the following diagram to commute:



So monomorphisms are natural generalizations of injective mappings, and epimorphisms for surjective ones. Maps are left-invertible iff they are injective, and right-invertible iff they are surjective. How does this generalize to categories? The following theorem sheds some light on the situation:

Theorem 5.2. *Not every monomorphism has a left-inverse. But every left-invertible arrow is a monomorphism. Dually, not every epimorphism has a right-inverse, but every right-invertible arrow is an epimorphism.*

Proof. Consider the following two-element category:

$$1_A \hookrightarrow A \xrightarrow{f} B \rightrightarrows 1_B$$

If $f \circ g = f \circ h$, the only option is $g = h = 1_A$, and so f is monic. Similarly, if $g \circ f = h \circ f$ the only option is $g = h = 1_B$ and so f is also epic. However, since there are no arrows from B to A in the category, f cannot have any kind of inverse. So f proves both "not every ..." statements.

As for the "converses", assume $f : A \rightarrow B$ to have a left inverse $l : B \rightarrow A$. Further assume $f \circ g = f \circ h$. Now apply l to both sides of the equation:

$$l \circ (f \circ g) = l \circ (f \circ h) \Leftrightarrow (l \circ f) \circ g = (l \circ f) \circ h \Rightarrow g = h \quad (5.2)$$

This proves that f is monic. Dually for right-inverse/epic. □