Single value decomposition and pseudo-inverses

Kristian Wichmann

December 12, 2016

1 Gramian matrices

Given a set of vectors $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$, the Gramian matrix is the traditionally matrix of inner products $\langle a_i, a_j \rangle$. If these vectors are collected into a $m \times n$ matrix A, this matrix can be expressed as $A^t A$. Here, we will use the term for any matrix in this form. By starting out with the transpose instead, this means that AA^t is also a Gramian, with dual results.

Theorem 1.1. *If $A \in \mathbb{R}^{m \times n}$, then $A^t A$ is symmetric and positive semi-definite. Iff A has rank m, $A^t A$ is positive definite.*

Proof. $(A^t A)^t = A^t (A^t)^t = A^t A$ shows symmetry. positive semi-definiteness, let $x \in \mathbb{R}^n$. Then:

$$x^t A^t A x = \langle Ax, Ax \rangle = ||Ax||^2 \tag{1.1}$$

As a norm, this is greater than or equal to zero. Hence $A^t A$ is positive semi-definite. If A has rank m the map $x \mapsto Ax$ has a trivial kernel by the rank-kernel theorem. Which means only the zero vector is mapped to zero, and hence $A^t A$ is positive definite. If the rank is less than m, the kernel is non-trivial and positive definiteness cannot be true. □

2 Single value decomposition

Let $A \in \mathbb{R}^{m \times n}$. Since $A^t A$ is symmetric, it is diagonalizable. So there is an orthogonal $n \times n$ matrix O such that $A^t A = ODO^t$, where D is a diagonal matrix of eigenvalues.