Control Theory

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1 Control and error

Control theory deals with strategies for keeping a quantity at a constant level in a dynamic system. In mathematical terms we try to keep a quantity $y(t)$ at a constant level y_r over time t.

To achieve this goal, a *controller* will affect the system at all time. This will generally be based on the *error*, i.e. the deviation from the desired level:

$$e(t) = y_r - y(t) \tag{1.1}$$

2 P-control

P-control is the case where the controller correction u is proportional (hence the P) to the error:

$$u(t) = ke(t) = k(y_r - y(t)) \tag{2.1}$$

2.1 Example: Anaesthesia

Surgery is performed on a patient. During the procedure, it is desirable to keep the blood concentration of anaesthetic $y(t)$ at a constant level y_r. Without control, the concentration follows the following differential equation:

$$\frac{dy}{dt} = -ay \tag{2.2}$$

I.e. it will decay exponentially from a starting concentration $y_0 = y(0)$:

$$y(t) = y_0 \cdot e^{-at} \tag{2.3}$$

We now add the control term:

$$\frac{dy}{dt} = -ay + u = -ay + k(y_r - y(t)) = ky_r - (a+k)y \tag{2.4}$$

This is a differential equation of the form:

$$\frac{dy}{dy} = b + ay \tag{2.5}$$

Which has the general solution:

$$y(t) = -\frac{b}{a} + c \cdot e^{at} \tag{2.6}$$

Here, this means:

$$y(t) = \frac{ky_r}{a+k} + c \cdot e^{-(a+k)t} \tag{2.7}$$

With the boundary condition that $y(0) = 0$ we can determine c:

$$c = -\frac{ky_r}{a+k} \tag{2.8}$$

We can now write the solution as:

$$y(t) = \frac{ky_r}{a+k} - \frac{ky_r}{a+k} e^{-(a+k)t} \tag{2.9}$$

So the error is:

$$e(t) = y_r - y(t) = y_r - \frac{ky_r}{a+k} + \frac{ky_r}{a+k} e^{-(a+k)t} \tag{2.10}$$

Expand first term to get common denominator:

$$e(t) = \frac{y_r(a+k)}{a+k} - \frac{ky_r}{a+k} + \frac{ky_r}{a+k} e^{-(a+k)t} \tag{2.11}$$

$$= \frac{y_r a}{a+k} + \frac{y_r k}{a+k} e^{-(a+k)t} \tag{2.12}$$

The controller dose is then found by multiplying by k:

$$u(t) = \frac{y_r ak}{a+k} + \frac{y_r k^2}{a+k} e^{-(a+k)t} \tag{2.13}$$

However, we now see that in the limit $t \to \infty$ the error is actually not zero, as we would hope for, but instead:

$$\lim_{t \to \infty} e(t) = y_r \frac{a}{a+k} \tag{2.14}$$

3 Laplace transforms

Given a function $f = f(t)$ defined for all positive t. Then the Laplace transform of it is defined as:

$$\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-ts}\ dt \tag{3.1}$$

The notation $F(s)$ is often used as a shorthand, and similarly for other functions.

3.1 Properties of the Laplace transform

The Laplace transform is linear, since integration is:

$$\begin{align}
\mathcal{L}[af + bg](s) &= \int_0^\infty [af(t) + b(g(t)]\, e^{-ts}\ dt \tag{3.2}\\
&= a\int_0^\infty f(t)e^{-ts}\ dt + b\int_0^\infty g(t)e^{-ts}\ dt \tag{3.3}\\
&= a\mathcal{L}[f](s) + b\mathcal{L}[g](s) \tag{3.4}
\end{align}$$

Laplace transforming a derivative gives us:

$$\begin{align}
\mathcal{L}\left[\frac{df}{dt}\right](s) &= \int_0^\infty \frac{df(t)}{dt}e^{-ts}\ dt \tag{3.5}\\
&= \left[f(t)e^{-ts}\right]_0^\infty - \int_0^\infty f(t)\frac{d}{dt}e^{-ts}\ dt \tag{3.6}\\
&= -f(0) + s\int_0^\infty f(t)e^{-ts}\ dt \tag{3.7}\\
&= s\mathcal{L}[f](s) - f(0) \tag{3.8}
\end{align}$$

Here partial integration has been used. Note that we have assumed that $f(t)$ grows slower than an exponential for $t \to \infty$.

Similarly, we can transform an integral:

$$\begin{align}
\mathcal{L}\left[\int_0^t f(x)\ dx\right](s) &= \int_0^\infty \int_0^t f(x)\ dx\ e^{-ts}\ dt \tag{3.9}\\
&= \left[\int_0^t f(x)\ dx \cdot \left(-\frac{1}{s}\right)e^{-ts}\right]_0^\infty - \int_0^\infty f(t)\left(-\frac{1}{s}\right)e^{-ts}\ dt \tag{3.10}\\
&= \frac{1}{s}\mathcal{L}[f](s) \tag{3.11}
\end{align}$$

Again, we have made assumptions on the growth speed of the integrand, i.e. this time of the integral of f.

3.2 A few select Laplace transforms

We consider three specific Laplace transforms in this section. First of a constant:

$$\begin{aligned}
\mathcal{L}[k](s) &= \int_0^\infty k \cdot e^{-st}\ dt && (3.12)\\
&= k\int_0^\infty e^{-st}\ dt && (3.13)\\
&= -\frac{k}{s}[e^{-st}]_0^\infty = \frac{k}{s} && (3.14)
\end{aligned}$$

Then of an exponential:

$$\begin{aligned}
\mathcal{L}[k](e^{at}) &= \int_0^\infty e^{at} \cdot e^{-st}\ dt && (3.15)\\
&= \int_0^\infty e^{(a-s)t}\ dt && (3.16)\\
&= \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty = \frac{1}{s-a} && (3.17)
\end{aligned}$$

And finally of the function te^{at}:

$$\begin{aligned}
\mathcal{L}[te^{at}](s) &= \int_0^\infty te^{at} \cdot e^{-st}\ dt && (3.18)\\
&= \int_0^\infty te^{(a-s)t}\ dt && (3.19)\\
&= \left[t\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty - \int_0^\infty 1 \cdot \frac{1}{a-s}e^{(a-s)t}\ dt = \frac{1}{(s-a)^2} && (3.20)
\end{aligned}$$

3.3 Anaesthesia revisited

Consider now the example from the previous section. Here, there's three equations governing the behaviour of the system:

$$\begin{aligned}
e(t) &= y_r - y(t) && (3.21)\\
u(t) &= k \cdot e(t) && (3.22)\\
\frac{d}{dt}y(t) &= a \cdot y(t) + u(t) && (3.23)
\end{aligned}$$

Now Laplace transform all three equations, using the properties derived above:

$$E(s) = \frac{y_r}{s} - Y(s) \tag{3.24}$$

$$U(s) = kE(s) \tag{3.25}$$

$$sY(s) - y(0) = -aY(s) + U(s) \tag{3.26}$$

Again, we use the boundary condition $y(0) = 0$ to simplify. We wish to isolate $E(t)$. First isolate $Y(t)$ in 3.26:

$$(s+a)Y(s) = U(s) \Leftrightarrow Y(s) = \frac{U(s)}{s+a} \tag{3.27}$$

Combined with 3.25 this gives:

$$Y(s) = \frac{kE(s)}{s+a} \tag{3.28}$$

And finally inserting into 3.24:

$$E(s) = \frac{y_r}{s} - \frac{kE(s)}{s+a} \Leftrightarrow \tag{3.29}$$

$$(s+a)E(s) = \frac{y_r(s+a)}{s} - kE(s) \Leftrightarrow \tag{3.30}$$

$$(s+a+k)E(s) = \frac{y_r(s+a)}{s} \Leftrightarrow \tag{3.31}$$

$$E(s) = \frac{y_r(s+a)}{s(s+a+k)} \tag{3.32}$$

We now use a partial fraction expansion on the right side:

$$\frac{y_r(s+a)}{s(s+a+k)} = y_r\left(\frac{A_1}{s} + \frac{A_2}{s+a+k}\right) \tag{3.33}$$

For this to hold, we must have:

$$\frac{s+a}{s(s+a+k)} = \frac{(s+a+k)A_1}{s(s+a+k)} + \frac{sA_2}{s(s+a+k)} \tag{3.34}$$

So:

$$s + a = (s+a+k)A_1 + sA_2 = s(A_1 + A_2) + (a+k)A_1 \tag{3.35}$$

This is one equation with two unknowns, so we can set a boundary condition ourselves. Setting $A_1 + A_2 = 1$ simplifies this s part. It also means that $A_1 = 1 - A_2$:

$$s + a = s + (a + k)(1 - A_2) \Leftrightarrow \tag{3.36}$$

$$a = (a + k)(1 - A_2) \Leftrightarrow \tag{3.37}$$

$$A_2 = 1 - \frac{a}{a + k} = \frac{k}{a + k} \tag{3.38}$$

Similarly:

$$A_1 = 1 - A_2 = 1 - \frac{k}{a + k} = \frac{a}{a + k} \tag{3.39}$$

Putting it all together the error can be written:

$$E(s) = y_r \frac{a}{a + k} \underbrace{\frac{1}{s}}_{\mathcal{L}[1]} + y_r \frac{k}{a + k} \underbrace{\frac{1}{s + a + k}}_{\mathcal{L}[e^{-(a+k)t}]} \tag{3.40}$$

As the braces show, we recognize two of the Laplace transforms from the previous section. Therefore, we find the original error $e(t)$ to be:

$$e(t) = y_r \frac{a}{a + k} + y_r \frac{k}{a + k} e^{-(a+k)t} \tag{3.41}$$

This is the same as the result from 2.12.

This may seem like a way more complicated way to get the same result as solving a standard differential equation. But is shows how Laplace transforms can be useful to solve problems like this.

4 PI-control

PI-control is an extension of P-control, which also includes a term proportional to the integral of the error function. Hence the I is short for integration. In mathematical terms:

$$u(t) = k_1 e(t) + k_2 \int_0^t e(t')\ dt' \tag{4.1}$$

Otherwise, the problem stays the same, so the three Laplace transformed equations become:

$$E(s) = \frac{y_r}{s} - Y(s) \tag{4.2}$$

$$U(s) = k_1 E(s) + k_2 \frac{E(s)}{s} \tag{4.3}$$

$$sY(s) = -aY(s) + U(s) \tag{4.4}$$

Here the transformation rule for an integral has come in handy, and we have once again assumed $y(0) = 0$. Once again, 4.4 can be written:

$$U(s) = (s + a)Y(s) \tag{4.5}$$

Insert this in 4.3:

$$(s + a)Y(s) = \left[k_1 + \frac{k_2}{s}\right] E(s) \tag{4.6}$$

From 4.2 we get:

$$Y(s) = \frac{y_r}{s} - E(s) \tag{4.7}$$

Insert in 4.6:

$$(s + a)\left[\frac{y_r}{s} - E(s)\right] = \left[k_1 + \frac{k_2}{s}\right] E(s) \Leftrightarrow \tag{4.8}$$

$$(s + a)\frac{y_r}{s} - (s + a)E(s) = \left[k_1 + \frac{k_2}{s}\right] E(s) \Leftrightarrow \tag{4.9}$$

$$(s + a)\frac{y_r}{s} = \left[k_1 + \frac{k_2}{s} + s + a\right] E(s) \tag{4.10}$$

Now isolate $E(s)$ to get:

$$E(s) = \frac{s + a}{k_1 + \frac{k_2}{s} + s + a}\frac{y_r}{s} = \frac{s + a}{s^2 + (k_1 + a)s + k_2} y_r \tag{4.11}$$

4.1 Rewriting the error

To rewrite the error consider an expression of the more general form:

$$\frac{s + a}{s^2 + bs + c} \tag{4.12}$$

Theorem 4.1. *If the polynomial $p(s) = s^2 + bs + c$ has two distinct roots ω_1 and ω_2, then:*

$$\frac{s + a}{s^2 + bs + c} = \frac{a + \omega_1}{\omega_2 - \omega_1}\frac{1}{s - \omega_1} + \frac{a + \omega_2}{\omega_1 - \omega_2}\frac{1}{s - \omega_2} \tag{4.13}$$

If the polynomial has a double root ω, then:

$$\frac{s + a}{s^2 + bs + c} = \frac{1}{s - \omega} + \frac{a + \omega}{(s - \omega)^2} \tag{4.14}$$

Proof. For distict roots: The polynomial can be factored as:

$$p(s) = (s - \omega_1)(s - \omega_2) \tag{4.15}$$

We now do a partial fraction expansion:

$$\frac{s+a}{(s-\omega_1)(s-\omega_2)} = \frac{A_1}{s-\omega_1} + \frac{A_2}{s-\omega_2} = \frac{A_1(s-\omega_2) + A_2(s-\omega_1)}{(s-\omega_1)(s-\omega_2)} \tag{4.16}$$

So:

$$s + a = A_1(s-\omega_2) + A_2(s-\omega_1) = (A_1 + A_2)s - A_1\omega_2 - \omega_1 A_2 \tag{4.17}$$

Choose $A_1 + A_2 = 1$. This means $A_2 = 1 - A_1$:

$$s + a = s - A_1\omega_2 - \omega_1(1 - A_1) \Leftrightarrow \tag{4.18}$$
$$a = -A_1\omega_2 - \omega_1 + A_1\omega_1 \Leftrightarrow \tag{4.19}$$
$$a = (\omega_1 - \omega_2)A_1 - \omega_1 \Leftrightarrow \tag{4.20}$$
$$A_1 = \frac{a + \omega_1}{\omega_1 - \omega_2} \tag{4.21}$$

Now A_2 can be found:

$$A_2 = 1 - \frac{a+\omega_1}{\omega_1 - \omega_2} \tag{4.22}$$
$$= 1 + \frac{a+\omega_1}{\omega_2 - \omega_1} \tag{4.23}$$
$$= \frac{\omega_2 - \omega_1 + a + \omega_1}{\omega_2 - \omega_1} \tag{4.24}$$
$$= \frac{a + \omega_2}{\omega_2 - \omega_1} \tag{4.25}$$

This proves the distinct case.

When there's a double root, the factorization is:

$$p(s) = (s - \omega)^2 \tag{4.26}$$

Reduce the proposed rewrite of the fraction:

$$\frac{1}{s-\omega} + \frac{a+\omega}{(s-\omega)^2} = \frac{s - \omega + a + \omega}{(s-\omega)^2} = \frac{s+a}{(s-\omega)^2} \tag{4.27}$$

The denominator is now equal to $p(s)$, and we're done. □

4.2 Solving the PI-control equations

Wanting to apply theorem 4.1 to 4.11 we look for roots of the denominator. The discriminant is:

$$d = (k_1 + a)^2 - 4k_2 \tag{4.28}$$

So when $(k_1+a)^2 > 4k_2$ there's two distinct roots, found by the usual formula:

$$\omega_1 = \frac{-(k_1 + a) + \sqrt{d}}{2}, \quad \omega_2 = \frac{-(k_1 + a) - \sqrt{d}}{2} \tag{4.29}$$

We now get the error function:

$$E(s) = \frac{a + \omega_1}{\omega_2 - \omega_1} \underbrace{\frac{1}{s - \omega_1}}_{\mathcal{L}[e^{\omega_1 t}]} + \frac{a + \omega_2}{\omega_1 - \omega_2} \underbrace{\frac{1}{s - \omega_2}}_{\mathcal{L}[e^{\omega_2 t}]} \tag{4.30}$$

As shown by the braces, in this case the solution in the time domain is:

$$e(t) = \frac{a + \omega_1}{\omega_2 - \omega_1} e^{\omega_1 t} + \frac{a + \omega_2}{\omega_1 - \omega_2} e^{\omega_2 t} \tag{4.31}$$

If $(k_1 + a)^2 = 4k_2$ there's a double root:

$$\omega = -\frac{k_1 + a}{2} \tag{4.32}$$

In this case the error function is:

$$E(s) = \underbrace{\frac{1}{s - \omega}}_{\mathcal{L}[e^{\omega t}]} + (a + \omega) \underbrace{\frac{1}{(s - \omega)^2}}_{\mathcal{L}[te^{\omega t}]} \tag{4.33}$$

So in the time domain:

$$e(t) = e^{\omega t} + (a + \omega)te^{\omega t} \tag{4.34}$$