Control Theory

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1 Control and error

Control theory deals with strategies for keeping a quantity at a constant level in a dynamic system. In mathematical terms we try to keep a quantity $y(t)$ at a constant level y_r over time t.

To achieve this goal, a *controller* will affect the system at all time. This will generally be based on the *error*, i.e. the current deviation from the desired level:

$$e(t) = y_r - y(t) \tag{1.1}$$

2 P-control

P-control is the case where the controller correction u is proportional (hence the P) to the error:

$$u(t) = ke(t) = k(y_r - y(t)) \tag{2.1}$$

2.1 Example: Anaesthesia

Surgery is performed on a patient. During the procedure, it is desirable to keep the blood concentration of anaesthetic $y(t)$ at a constant level y_r. Without control, the concentration follows the following differential equation:

$$\frac{dy}{dt} = -ay \tag{2.2}$$

I.e. it will decay exponentially from a starting concentration $y_0 = y(0)$:

$$y(t) = y_0 \cdot e^{-at} \tag{2.3}$$

We now add the control term:

$$\frac{dy}{dt} = -ay + u = -ay + k(y_r - y(t)) = ky_r - (a+k)y \tag{2.4}$$

This is a differential equation of the form:

$$\frac{dy}{dy} = -b + ay \tag{2.5}$$

Which has the general solution:

$$y(t) = -\frac{b}{a} + c \cdot e^{-at} \tag{2.6}$$

Here, this means:

$$y(t) = \frac{ky_r}{a+k} + c \cdot e^{(a+k)t} \tag{2.7}$$

With the boundary condition that $y(0) = 0$ we can determine c:

$$c = -\frac{ky_r}{a+k} \tag{2.8}$$

We can now write the solution as:

$$y(t) = \frac{ky_r}{a+k} - \frac{ky_r}{a+k} e^{(a+k)t} \tag{2.9}$$

So the error is:

$$e(t) = y_r - y(t) = y_r - \frac{ky_r}{a+k} + \frac{ky_r}{a+k} e^{(a+k)t} \tag{2.10}$$

Expand first term to get common denominator:

$$e(t) = \frac{y_r(a+k)}{a+k} - \frac{ky_r}{a+k} + \frac{ky_r}{a+k} e^{(a+k)t} \tag{2.11}$$

$$= \frac{y_r}{a+k} \left[a + k(e^{(a+k)t} - 1)\right] \tag{2.12}$$

The controller dose is then found by multiplying by k:

$$u(t) = \frac{y_r}{a+k} \left[ak + k^2(e^{(a+k)t} - 1)\right] \tag{2.13}$$

However, we now see that in the limit $t \to \infty$ the error is actually not zero, as we would hope for, but instead:

$$\lim_{t \to \infty} e(t) = y_r \frac{a}{a+k} \tag{2.14}$$