Normal distributions on vector spaces

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1 Affine transformations of euclidean spaces

Let $s : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. This means that the is an $m \times n$ matrix A so $s(x) = Ax$.

An *affine* transformation t is formed by following this linear map by a translation:

$$t : \mathbb{R}^n \to \mathbb{R}^m, t(x) = Ax + v \tag{1.1}$$

Here, $v \in \mathbb{R}^m$. Since translations are always bijective, we note that t is bijective iff A is invertible.

Each component of an affine transformation is composed from measurable function - is is understood that we mean with respect to the Borel algebras of each space) - so the affine transformation itself is measurable as well.

1.1 Transformation properties of the Lebesgue measure

Recall that the Lebesgue measure in n dimensions m_n is invariant under translation: If t is a translation $t : \mathbb{R}^n \to \mathbb{R}^n, t(x) = x + x_0$, where $x_0 \in \mathbb{R}^n$ then:

$$t(m_n) = m_n \tag{1.2}$$

Also, if $s : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax$ is an isomorphism, then:

$$s(m_n) = m_n |\det A^{-1}| \tag{1.3}$$

Combining the two, the formula for affine transformation is the same as for linear ones.

2 Orthogonal complement

Let V be a finite-dimensional vector space with an inner product $\langle \cdot, \cdot \rangle$. Let U be a subspace of V. Then we define the *orthogonal complement* of U as:

$$U^\perp = \{v \in V | \forall u \in U : \langle u, v \rangle = 0\} \tag{2.1}$$

Theorem 2.1. *$U^\perp$ is a subspace of V.*

Proof. According to the subspace theorem, we need to show three things:

- $U^\perp$ is not empty: Clearly $0 \in U^\perp$.
- Closed under addition: If $v_1, v_2 \in U^\perp$, then for all $u \in U^\perp$:

$$\langle v_1 + v_2, u \rangle = \langle v_1, u \rangle + \langle v_2, u \rangle = 0 \tag{2.2}$$

- Closed under scalar multiplication: If $v \in U^\perp$ and $c \in \mathbb{R}$ then for all $u \in U^\perp$:

$$\langle cv, u \rangle = c\langle v, u \rangle = 0 \tag{2.3}$$

□

Since the only vector perpendicular to itself is 0, we further conclude that $U \cap U^\perp = \{0\}$.

Theorem 2.2. *If $e_1, e_2, \ldots, e_m$ is an orthonormal basis for U, then for any $v \in V$:*

$$v - \sum_{i=1}^{m} \langle v, e_i \rangle e_i \in U^\perp \tag{2.4}$$

Proof. Let $u \in U$. Then we can write $u = \sum_{j=1}^{m} \lambda_j e_j$ for some coefficients λ_j. Now calculate the inner product with the vector above:

$$\langle v - \sum_{i=1}^{m} \langle v, e_i \rangle e_i \ , \ \sum_{j=1}^{m} \lambda_j e_j \rangle = \sum_{i=j}^{m} \lambda_j \langle v, e_j \rangle - \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_j \langle v, e_i \rangle \langle e_i, e_j \rangle \tag{2.5}$$

Since $\langle e_i, e_j \rangle = \delta_{ij}$ this vanishes. □

This means that we may write any $v \in V$ as a sum of vectors from U and $U^\perp$ respectively:

$$v = \underbrace{\sum_{i=1}^{m} \langle v, e_i \rangle e_i}_{\in U} \ + \ \underbrace{v - \sum_{i=1}^{m} \langle v, e_i \rangle e_i}_{\in U^\perp} \tag{2.6}$$

Theorem 2.3. *The decomposition into elements from U and $U^\perp$ from equation 2.6 is unique.*

Proof. Let $v = u_1 + u_1^\perp$ and $v = u_2 + u_2^\perp$ be two such decompositions. Then $u_1 + u_1^\perp = u_2 + u_2^\perp$ and hence $u_1 - u_2 = u_2^\perp - u_1^\perp$. But this means that this vector is a member of both U and $U^\perp$, and hence it must be 0. This means $u_1 = u_2$ and $u_1^\perp = u_2^\perp$. $\square$

2.1 The orthogonal projection

The previous section motivates the following:

Definition 2.1. *Let V be a finite-dimensional inner product vector space and U a subspace of V. The orthogonal projection from V onto U is the map $p : V \to V$ which satisfies:*

$$\forall v \in V : \quad p(v) \in U, \quad v - p(v) \in U^\perp \tag{2.7}$$

As we see, one could also define the co-domain of p to be U. Usually, the distinction will not matter much.

Theorem 2.4. *The orthogonal projection operator is linear.*

Proof. We need to show additivity and homogeneity:

- Additivity: Let $v_1, v_2 \in V$. Then $p(v_1) + p(v_2) \in U$ and:

  $$v_1 - p(v_1) + v_2 - p(v_2) = v_1 + v_2 - (p(v_1) + p(v_2)) \in U^\perp \tag{2.8}$$

  Adding the two we get $v_1 + v_2$. So $p(v_1 + v_2) = p(v_1) + p(v_2)$.

- Homogeneity. Let $v \in V$ and $c \in \mathbb{R}$. Then $cp(v) \in U$ and $c(v - p(v)) = cv - cp(v) \in U^\perp$. Adding the two we get cv, so $p(cv) = cp(v)$.

$\square$

Theorem 2.5. *The orthogonal projection operator $p : V \to V$ is idempotent. I.e. $p \circ p = p$.*

Proof. Let $v \in V$. Then $p(v) \in U$. But this means that the decomposition of $p(v)$ is $p(v) + 0$. So $p \circ p(v) = p(v)$. $\square$

3 Lebesgue measures on vector spaces

3.1 Coordinate maps

Let V be a finite-dimensional vector space of dimension n. Our question is, if we can turn V into a measure space in a natural way. Since we know that V is isomorphic to $\mathbb{R}^n$, it makes sense to tweak the usual Lebesgue measure in N dimensions:

Let $e_1, e_2, \ldots, e_n$ be a basis for V. Then we can define the *coordinate map* as follows:

$$\phi : \mathbb{R}^n \to V, \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sum_{i=1}^{n} x_i e_i \tag{3.1}$$

This is obviously an isomorphism. Specifically, it is invertible with inverse $\phi^{-1} : V \to \mathbb{R}^n$.

The coordinate map depends on the chosen basis. If we had chosen another basis $e_1^*, e_2^*, \ldots, e_n^*$ we would get another isomorphism ϕ^*.

3.2 Borel algebra on V

We can now use ϕ^{-1} to induce a σ-algebra on V. Set $\mathbb{B}_V$ to the smallest σ-algebra that makes ϕ^{-1} measurable when $\mathbb{R}^n$ is equipped with the Borel algebra $\mathbb{B}_n$. We call $\mathbb{B}_V$ the *Borel algebra on* V.

At first this object seems to depend of the choice of basis for V. But it turns out that the use of definite article in the definition is justified:

Theorem 3.1. *If $e_1, e_2, \ldots, e_n$ and $e_1^*, e_2^*, \ldots, e_n^*$ are bases for V, then the induced σ-algebra $\mathbb{B}_V$ and $\mathbb{B}_V^*$ is the same thing.*

Proof. We know that ϕ^{-1} is $\mathbb{B}_V - \mathbb{B}_n$ measurable by definition. □