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Problem 1

Problem 1a)

For UCC to be a $M/M/1$ queue these conditions must be satisfied:

1. Interarrival times need to be independent and identically distributed as $Exp(\lambda)$
2. Service times are independent and identically distributed as $Exp(\mu)$
3. There are only one server, and the service time are independent of the arrival process

The following points explain why these are satisfied for the UCC:

1. We know that the arrival times are Poisson-distributed with parameter λ, which of course means that the Interarrival times are exponential distributed with parameter λ. This satisfies the first condition.
2. We know that the UCC only has capacity to treat one patient at a time, thus μ is constant, and thus we have independent and identically distributed service times. This satisfies the second condition.
3. UCC has capacity for only one patient, and this satisfies the third condition.

We have for Poisson processes that for one increment and very samll time-intervalls h:

$$\begin{aligned} P_{i,i+1} &= \lambda h + o(h) \\ P_{i,i-1} &= \mu h + o(h) \\ P_{i,i} &= 1 - (\lambda + \mu)h + o(h) \end{aligned} \tag{1}$$

$X(t)$ being a Process for which the following are true

$$\begin{aligned} &P(X(t+h) - X(t)) = 1 \\ &\qquad = P_{i,i+1}(h) = \lambda_i h + o(h) \\ &P(X(t+h) - X(t)) = -1 \\ &\qquad = P_{i,i-1}(h) = \mu_i h + o(h) \\ &P(X(t+h) - X(t)) = 0 \\ &\qquad = P_{i,i}(h) = 1 - (\lambda_i + \mu)h + o(h) \end{aligned} \tag{2}$$

$X(t)$ can be defined as a Birth and Death process.

We can use Little's law, $L = \lambda W \longrightarrow W = \frac{L}{\lambda}$,to find the average amount of time a costumer spends in the UCC. We know from lecture 43:1 that $L = \frac{\lambda}{\mu - \lambda}$ so we get:

$$W = \frac{L}{\lambda} = \frac{E[X(t)]}{\lambda} = \frac{1}{\lambda}\frac{\lambda}{\mu-\lambda} = \frac{1}{\mu-\lambda}$$

Which gives us the average time a patient spends in the UCC as $\frac{1}{\mu-\lambda}$.

Problem 1b)

We have modelled the stochastic variable $X(t)$ using python, for precise implementation see attached code. The stepsize in time is chosen as minutes to account for the expected treatment time. As the treatment time of the patients are given by independent exponential distributions with the same expected value, in this case μ, we can assume that the number of patients that depart from the Urgent Care Center are distributed following a Poisson distribution with intensity μ. As we take the timesteps as minutes we have the reformulate our intensities λ and μ to suit our model. This is relatively simple as we only need to divide our coefficients by 60. We list the coefficients we use in table 1.

Intensities	Per hour	Per minute
λ	5	$\frac{1}{12}$
μ	6	$\frac{1}{10}$

Table 1: Calculated intensities for different timstep sizes.

Figure 1 shows a realization for $\{X(t) : t \geq 0\}$. We observe that there is always a whole number of patients at the Urgent Care Center.

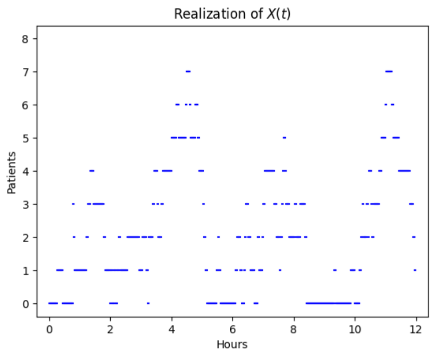

Figure 1: Plot of one realization for the first half day after $t = 0$.

We can also calculate how much the average time spent at the UCC is using multiple simulations of our stochastic variable $X(t)$. We do this by using the model described above, running it 30 times and storing the resulting long running mean of the number of patients at the UCC. We can then find a 95% confidence interval for the expected wait time using Little's law. We here simply use the first expression from **1a.3)**:

$$W = \frac{L}{\lambda}$$

where L is the long running mean of the number of patients at the UCC and λ is the intensity of arrivals. We then get the resulting confidence intervals in table 2.

Unit	Low	High	Exact **1a.3)**
Hours	0.9426	1.0356	1.0000
Minutes	56.56	62.13	60.00

Table 2: 95% confidence interval for wait- and treatment-time for each patient at the Urgent Care Center.

Problem 1c)

We must show that $U(t)$ satisfies the conditions from $1a.1$). The probability that a patient is urgent follows a binomial distribution where success (p) is denoted as a patient being urgent, and failure (1-p) denoted as a patient being normal. Since the arrival of patients follow a poisson-distribution, we can use Theorem 5.2 to say that

$$U(t) \sim Poisson(p\lambda t)$$

It follows that the service times, that are still $Exp(\mu)$, are independent. And there is still only one server, which concludes the conditions as fulfilled.

The arrival rate of $U(t)$ will be $p\lambda$.

The service-time distribution from $1a$) is unchanged because $U(t)$ is always prioritized in the queue, and thus new arrivals does not effect the treatment time of the Urgent patients.

From chapter 9.2.1 in **[1]** we know that the long run mean number of patients ca be written as $L = \frac{\lambda}{\mu - \lambda}$, but since we have $\lambda = p\lambda$ our final expression will be:

$$L_U = \frac{p\lambda}{\mu - p\lambda}$$

Problem 1d)

We now take a closer look at $N(t)$.

As it was for the $U(t)$ queue, whether a patient is Normal or Urgent has a binomial distribution, and $N(t)$ then has a Poisson-distribution $N(t) \sim Poisson((1-p)\lambda)$. If $N(t)$ then is a $M/M/1$ queue, it also need to satisfy the conditions from a). But since the treatment of a Normal patient is prioritized below an Urgent patient, and the Normal person are immediately set aside if a Urgent person should arrive, $N(t)$ does not satisfy the second condition in $1a.1$). We can see this because the service time is not independent of arrivals, as explained above.

If L is the total expected average waiting-time for patients, then $L - L_U$ must be the expected average waiting-time for all patients that are not urgent.

$$\begin{aligned} L - L_U &= \frac{\lambda}{\mu-\lambda} - \frac{p\lambda}{\mu-p\lambda} = \frac{\lambda(\mu-p\lambda)}{(\mu-\lambda)(\mu-p\lambda)} - \frac{(\mu-\lambda)p\lambda}{(\mu-\lambda)(\mu-p\lambda)} \\ &= \frac{\lambda(\mu-p\lambda)-(\mu-\lambda)p\lambda}{(\mu-\lambda)(\mu-p\lambda)} = \frac{\lambda\mu(1-p)}{(\mu-\lambda)(\mu-p\lambda)} \end{aligned} \tag{3}$$

The long-run mean number of normal patients in the UCC is $L_N = \frac{\lambda\mu(1-p)}{(\mu-\lambda)(\mu-p\lambda)}$

Problem 1e)

$$L = \lambda W \longrightarrow W = \frac{L}{\lambda} \tag{4}$$

$$W_U = \frac{1}{p\lambda} L_U = \frac{1}{p\lambda}\frac{p\lambda}{\mu-p\lambda} = \frac{1}{\mu-p\lambda} \tag{5}$$

$$W_N = \frac{1}{(1-p)\lambda} L_N = \frac{1}{(1-p)\lambda}\frac{\lambda\mu(1-p)}{(\mu-\lambda)(\mu-p\lambda)} = \frac{\mu}{(\mu-\lambda)(\mu-p\lambda)} \tag{6}$$

Problem 1f)

Taking λ and μ from table 1 we can plot the functions W_U and W_N from **1e)**. These functions can be found in figure 2. To interpret these values we need to know what p measures, which we know from **1c)** to be the probability that a newly arrived patient is an urgent patient. From this we can interpret that $p \approx 0$ means that close to none of the patients that arrive are urgent. This means that we are left with the same situation as in **1b)** where no patients are urgent. In the opposite case where $p \approx 1$ almost all the patients will be urgent. This will again almost reduce to the same situation as in **1b)** where no patients are treated preferentially. We have calculated the value of W_N in the extreme cases, giving $p \approx 0 \Rightarrow W_N = 1.0$ and $p \approx 1 \Rightarrow W_N = 6.0$.

We want to find the urgency probability, p, where the expected time spent, W_N, is equal to two hours. We calculate this by hand:

$$W_N = \frac{\mu}{(\mu-\lambda)(\mu-p\lambda)} \Rightarrow (\mu-p\lambda) = \frac{\mu}{W_N(\mu-\lambda)} \Rightarrow p = \frac{\mu}{\lambda}\left(1 - \frac{1}{W_N(\mu-\lambda)}\right)$$

We can now exchange for what we know which gives us

$$p = \frac{6}{5}\left(1 - \frac{1}{2\cdot(6-5)}\right) = \frac{6}{5}\cdot\frac{1}{2} = 0.6$$

We see from this calculation that if the urgency probability $p = 0.6$ we get an expected wait time for normal patients that equals 2 hours.

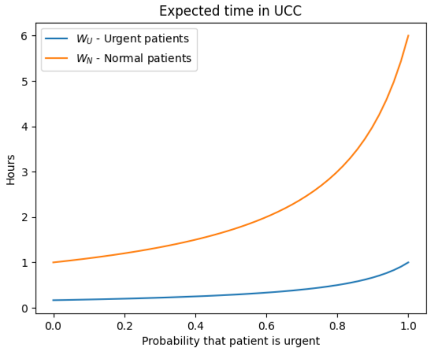

Figure 2: Plot of W_U and W_U as functions of the probability that the patient is an urgent patient.

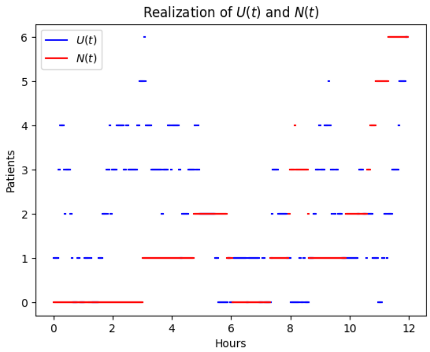

Figure 3: Realization of $U(t)$ and $N(t)$ in the same figure.

Problem 1g)

We can plot a realization of $U(t)$ and $N(t)$ in the same figure showing how the treatment of urgent patients affect the treatment of normal patients. This is done in figure 3.

We estimate the two quantities using a simulation similar to the one used in **1b**. The model simulates using minutes as timesteps and checking when the next arrival/departure is for each step. We then figure out which of $U(t)$ or $N(t)$ is changed and if it is incremented or decremented using the intensities $\lambda_U = p\lambda$, $\lambda_N = (1-p)\lambda$, and mu. The intensities λ and μ we refer to here are the same as in **1b** and can be found in table 1. The simulation gives us confidence intervals for the wait times for urgent and normal patients, which can be found in table 3.

Unit	Low	High	Exact **1e)**
$U(t)$ (hours)	0.4975	0.5290	0.5000
$U(t)$ (minutes)	29.85	31.74	30.00
$N(t)$ (hours)	2.6540	3.1011	3.0000
$N(t)$ (minutes)	159.24	186.07	180.00

Table 3: 95% confidence interval for wait- and treatment-time for each patient at the UCC simulating $U(t)$ and $N(t)$.

Problem 2

Problem 2a)

We can see that the confidence-interval is thinner where the θ-distance are shorter between the points. This is an expected result.

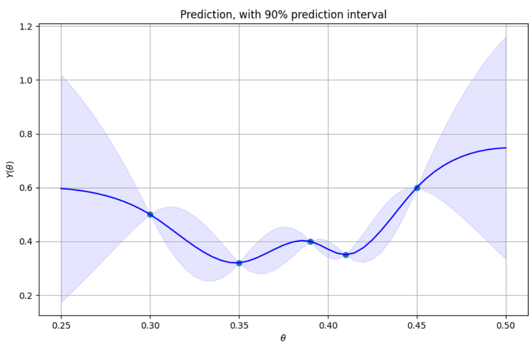

Figure 4: Prediction of $Y(\theta)$, measuring the effect of the albedo of sea ice θ, with a 90% prediction interval.

Problem 2b)

We can see that the probability of $Y(\theta) < 0.30$ is largest around $\theta = 0.34$.

Problem 2c)

Getting the maximal value of the function in figure 7, we would recommend the next value to be tested to be $\theta = 0.36$ as this is where the error is the most pronounced.

Sources

Pinsky, M.A. and Karlin, S., 2011, An Introduction to stochastic modeling, Academic Press/Elsevier. (4th Ed) [1]

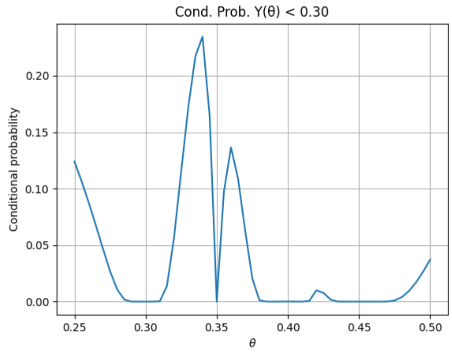

Figure 5: Conditional probability for getting $Y(\theta) < 0.30$ as a function of θ.

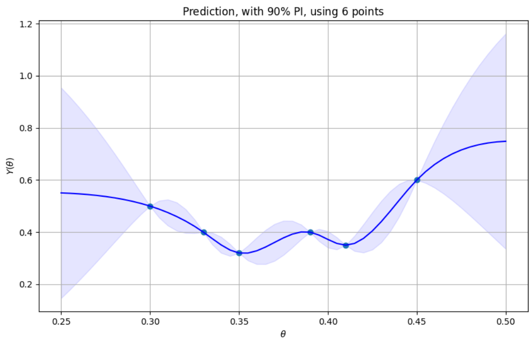

Figure 6: Prediction of $Y(\theta)$, measuring the effect of the albedo of sea ice θ, with a 90% prediction interval using 6 points.

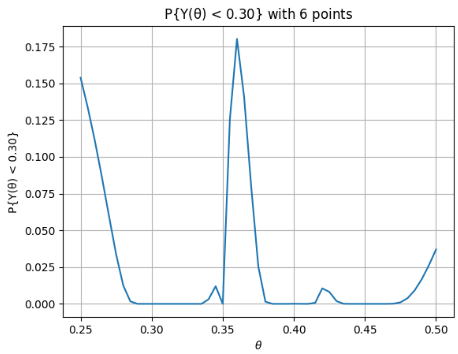

Figure 7: Conditional probability for getting $Y(\theta) < 0.30$ as a function of θ using 6 points.