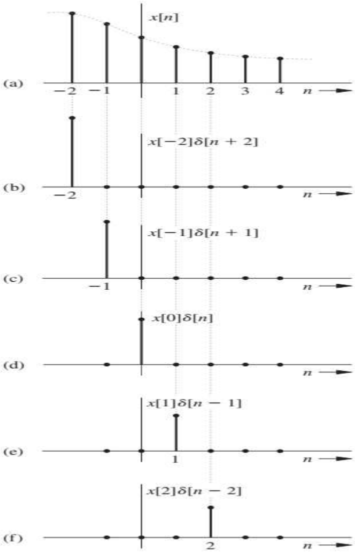
Lecture 10

- Response of Discrete-time System to External Input
- Convolutional Sum
- Properties of Convolutional Sum
- Convolution of Causal Signals
- Graphical Procedure for the Convolution

- An arbitrary input $x[n]$ as a sum of impulse components.
- The component of $x[n]$ at $n = m$ is $x[m]\delta[n-m]$.

$$
\begin{aligned}
x[n] =& x[0]\delta[n] + x[1]\delta\left[n-1\right] + x[2]\delta\left[n-2\right] + \cdots \\
&+ x[-1]\delta[n+1] + x[-2]\delta\left[n+2\right] + \cdots \\
=& \sum_{m=-\infty}^{\infty} x[m]\delta\left[n-m\right]
\end{aligned}
$$



System Response

- Let $h[n]$ be the system response to impulse input $\delta[n]$.
- We shall use the following notation to indicate the input and the corresponding response of the system

$$x[n] \Rightarrow y[n]$$

- Thus, if

$$\delta\left[n\right] \Rightarrow h[n]$$

then because of time invariance

$$\delta\left[n-m\right] \Rightarrow h[n-m]$$

- Because of linearity,

$$x[m]\delta\left[n-m\right] \Rightarrow x[m]h[n-m]$$

- Again because of linearity

$$\underbrace{\sum_{m=-\infty}^{\infty} x[m]\delta\left[n-m\right]}_{x[n]} \Rightarrow \underbrace{\sum_{m=-\infty}^{\infty} x[m]h[n-m]}_{y[n]}$$

Convolution

- The system response $y[n]$ to input $x[n]$ is given by

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

- The summation on the right-hand side is known as the *convolution sum* of $x[n]$ and $h[n]$, and is represented symbolically by $x[n] * h[n]$:

$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

PROPERTIES OF THE CONVOLUTION SUM

- The Commutative Property.

$$x_1[n] * x_2[n] = x_2[n] * x_1[n]$$

- The Distributive Property.

$$x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n]$$

- The Associative Property.

$$x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$$

- The Shifting Property.
  If

$$x_1[n] * x_2[n] = c[n]$$

  then

$$x_1[n-m]x_2[n-p] = c[n-m-p]$$

- The Convolution with an Impulse.

$$x[n] * \delta[n] = x[n]$$

PROPERTIES OF THE CONVOLUTION SUM *contd..*

- The Width Property.
  If $x_1[n]$ and $x_2[n]$ have finite widths of W_1 and W_2, respectively, then the width of $x_1[n] * x_2[n]$ is $W_1 + W_2$.
  The width of a signal is 1 less than the number of its elements (length).

CAUSALITY

- If the input $x[n]$ is causal, $x[m] = 0$ for $m < 0$.
- Similarly, if the system is causal (i.e., if $h[n]$ is causal), then $h[x] = 0$ for negative x so that $h[n-m] = 0$ when $m > n$.
- Therefore, if $x[n]$ and $h[n]$ are both causal, the product $x[m]h[n-m] = 0$ for $m < 0$ and for $m > n$, and it is nonzero only for the range $0 \leq m \leq n$.
- Thus, the output is given by

$$y[n] = \sum_{m=0}^{n} x[m]h[n-m]$$

Convolution of Causal Signals

Question Determine $c[n] = x[n] * g[n]$ for

$$x[n] = (0.8)^n u[n] \quad \text{and} \quad g[n] = (0.3)^n u[n]$$

Solution

$$c[n] = \sum_{m=0}^{n} x[m]g[n-m] = \sum_{m=0}^{n} (0.8)^m u[m](0.3)^{n-m} u\left[n-m\right]$$

Case 1 $n \geq 0$

$$c[n] = \sum_{m=0}^{n} (0.8)^m (0.3)^{n-m} = \sum_{m=0}^{n} \left(\tfrac{0.8}{0.3}\right)^m (0.3)^n$$
$$= (0.3)^n \tfrac{(0.8)^{n+1}-(0.3)^{n+1}}{(0.3)^n(0.8-0.3)}$$

Case 2 $n < 0 \qquad c[n] = 0$

Therefore, we have

$$c[n] = 2[(0.8)^{n+1} - (0.3)^{n+1}]u[n]$$

Graphical Procedure for the Convolution

The convolution sum of causal signals $x[n]$ and $g[n]$ is given by

$$c[n] = \sum_{m=0}^{n} x[m]g[n-m]$$

We first plot $x[m]$ and $g[n-m]$ as functions of m (not n), because the summation is over m. The convolution operation can be performed as follows:

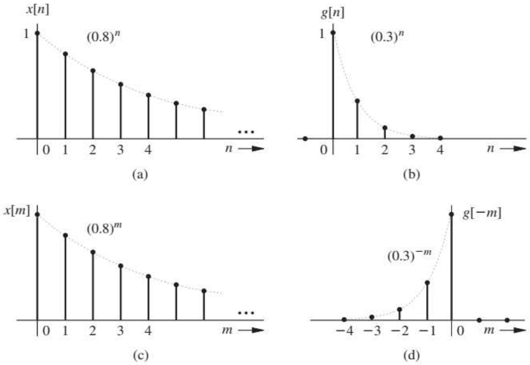
1. Invert $g[m]$ about the vertical axis $(m = 0)$ to obtain $g[-m]$
2. Shift $g[-m]$ by n units to obtain $g[n-m]$. For $n > 0$, the shift is to the right (delay); for $n < 0$, the shift is to the left (advance).
3. Next we multiply $x[m]$ and $g[n-m]$ and add all the products to obtain $c[n]$.
4. The procedure is repeated for each value of n over the range $-\infty$ to ∞.

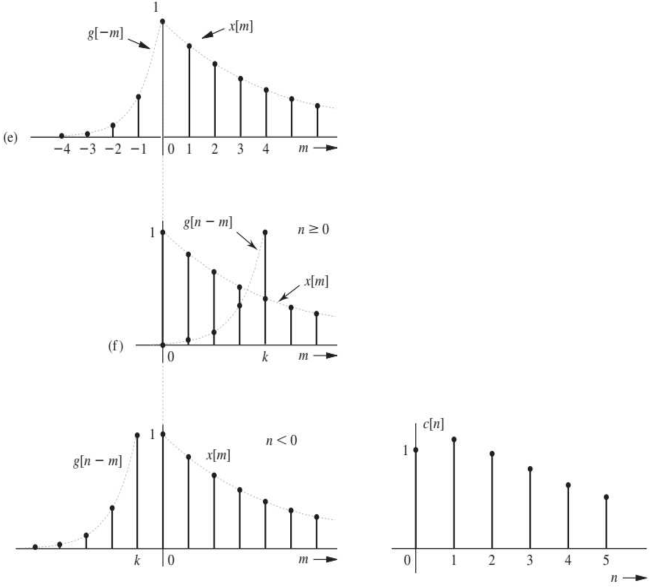
Example Find $c[n] = x[n] * g[n]$ where

$$x[n] = (0.8)^n u[n] \qquad g[n] = (0.3)^n u[n]$$

Therefore,

$$x[m] = (0.8)^n u[m] \qquad g[n-m] = (0.3)^{n-m} u[n-m]$$





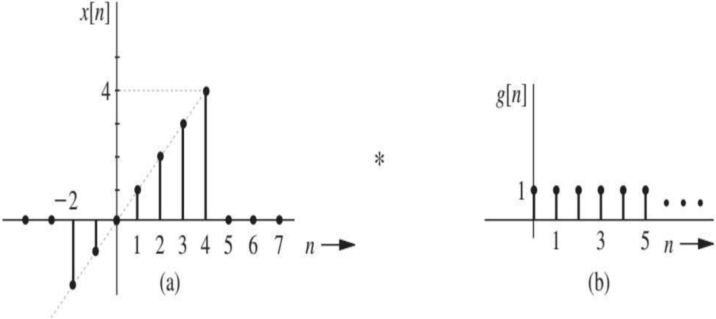
(g)

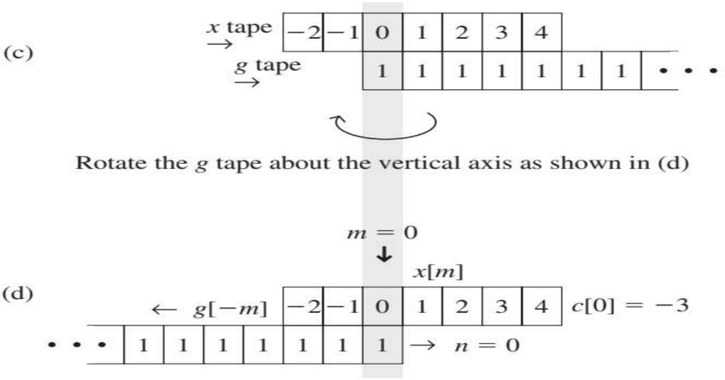
(h)

THE SLIDING-TAPE METHOD

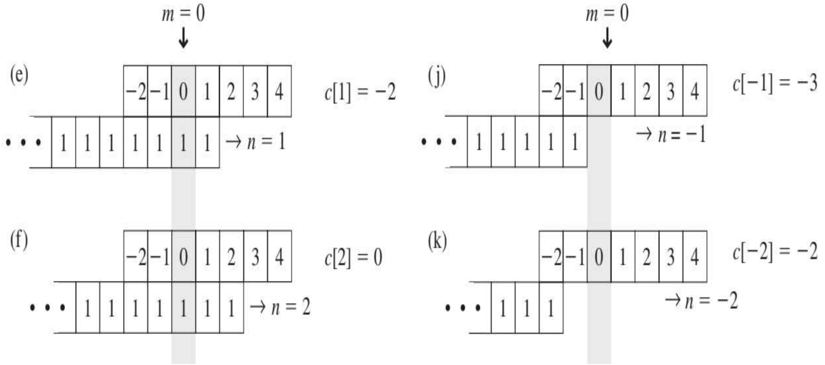
This algorithm is convenient when the sequences $x[n]$ and $g[n]$ are short or when they are available only in graphical form.

Example Use the sliding-tape method to convolve the two sequences $x[n]$ and $g[n]$ depicted below:

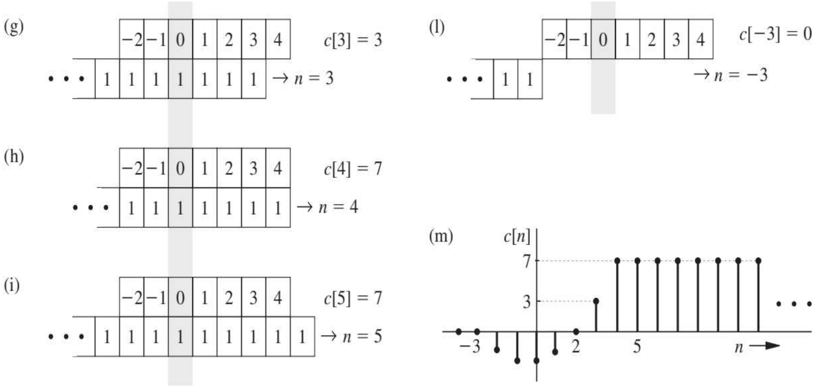




(c)
x tape
−2
−1
0
1
2
3
4
g tape
1
1
1
1
1
1
1
Rotate the g tape about the vertical axis as shown in (d)
m = 0
x[m]
(d)
← g[−m]
−2
−1
0
1
2
3
4
c[0] = −3
1
1
1
1
1
1
1
→ n = 0



m = 0
m = 0
(e)
c[1] = −2
→ n = 1
(j)
c[−1] = −3
→ n = −1
(f)
c[2] = 0
→ n = 2
(k)
c[−2] = −2
→ n = −2


(g)
−2 −1 0 1 2 3 4
c[3] = 3
• • • 1 1 1 1 1 1 1 → n = 3
(h)
−2 −1 0 1 2 3 4
c[4] = 7
• • • 1 1 1 1 1 1 1 → n = 4
(i)
−2 −1 0 1 2 3 4
c[5] = 7
• • • 1 1 1 1 1 1 1 1 1 → n = 5
(l)
−2 −1 0 1 2 3 4
c[−3] = 0
• • • 1 1
→ n = −3
(m)
c[n]
7
3
−3
2
5
n
• • •

DRILL 3.19 Sliding-Tape Method for the Convolution Sum

Use the graphical procedure of Ex. 3.24 (sliding-tape technique) to show that $x[n] * g[n] = c[n]$ in Fig. 3.24. Verify the width property of convolution.

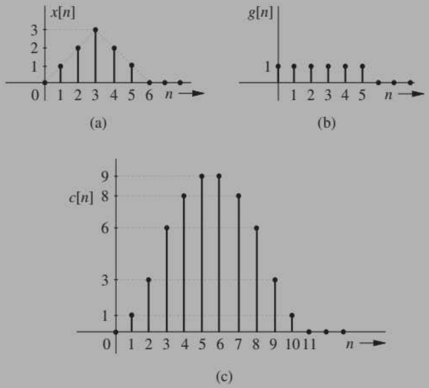


Figure 3.24 Signals for Drill 3.19.