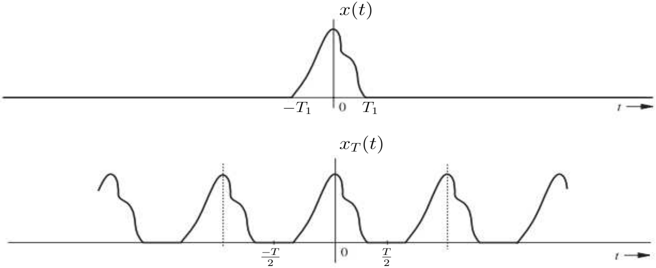
Lecture 15

- Continuous-time Fourier Transform
- Transforms of Some Useful Functions

From Fourier Series to Fourier Transform

- Objective: To have Fourier representation of continuous-time aperiodic signals. An aperiodic signal can be viewed as a periodic signal with an infinite period. Let $x(t)$ be a non-periodic signal of finite duration; that is,

$$x(t) = 0 \quad \text{for } |t_1| \geq T_1.$$



Let x_T be a periodic signal formed by repeating $x(t)$ with fundamental period T. We have

$$\lim_{T \to \infty} x_T(t) = x(t)$$

The complex exponential Fourier series of $x_T(t)$ is given by

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

where,

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt. \tag{1}$$

Let us define $X(j\omega)$ as

$$X\left(j\omega\right) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{2}$$

From (1) and (2), it can be seen that

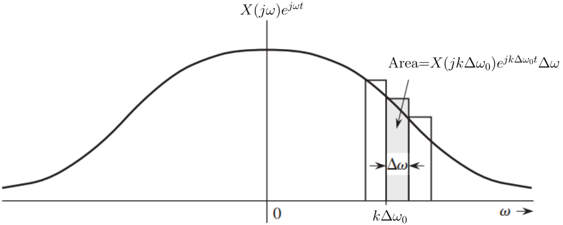
$$a_k = \frac{1}{T}X\left(jk\omega_0\right).$$

Thus we have,

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T}X\left(jk\omega_0\right)e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X\left(jk\omega_0\right)e^{jk\omega_0 t}\omega_0.$$

As $T \to \infty$, $\omega_0 = 2\pi/T$ becomes infinitesimal ($\omega_0 \to 0$). Thus, let $\omega_0 = \triangle\omega$. Then we have

$$x(t) = \lim_{T\to\infty} x_T(t) = \lim_{\Delta\omega\to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X\left(jk\Delta\omega\right) e^{jk\Delta\omega t} \Delta\omega$$



As $\Delta\omega \to 0$, $x(t)$ can be written as

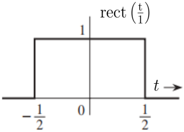
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X\left(j\omega\right) e^{j\omega t} d\omega$$

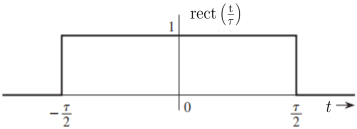
which is the Fourier representation of a nonperiodic signal $x(t)$

Fourier transform pair $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X\left(j\omega\right) e^{j\omega t} d\omega \qquad \mathcal{F}[x(t)] = X\left(j\omega\right) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- Rectangular Function



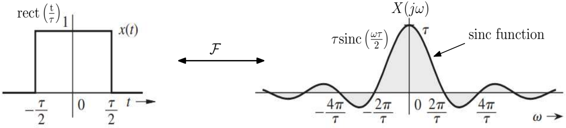


$$\mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 0 & \text{if } |t| > \tau/2 \\ \frac{1}{2} & \text{if } |t| = \tau/2 \\ 1 & \text{if } |t| < \tau/2 \end{cases}$$

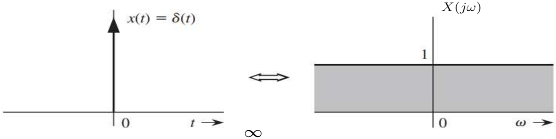
The Fourier transform of the rectangular function is given by:

$$X(j\omega) = \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega} = \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$$

$$= \tau \mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$



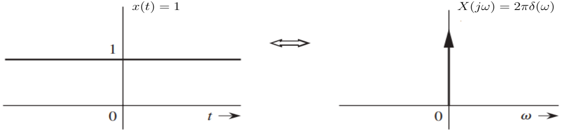
- Dirac Delta Function:



$$\mathcal{F}\left[\delta\left(t\right)\right] = \int_{-\infty}^{\infty} \delta\left(t\right) e^{-j\omega t} dt = 1$$

Inverse Fourier Transform of the Dirac Delta Function:

$$\mathcal{F}^{-1}\left[\delta\left(\omega\right)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta\left(\omega\right) e^{j\omega t} d\omega = \frac{1}{2\pi}$$

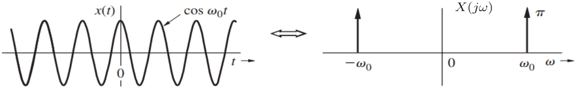


Inverse Fourier Transform of a Shifted Dirac Delta Function

$$\mathcal{F}^{-1}\left[\delta\left(\omega-\omega_0\right)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta\left(\omega-\omega_0\right) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

$$\mathcal{F}\left[e^{j\omega_0 t}\right] = 2\pi\delta\left(\omega-\omega_0\right)$$

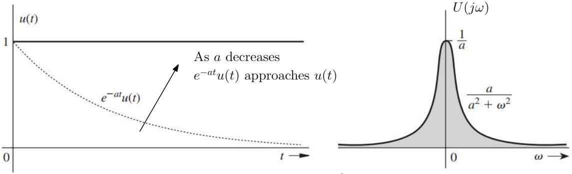
- Sinusoid



$$X(j\omega) = \mathcal{F}\left[\cos \omega_0 t\right] = \int_{-\infty}^{\infty} \cos \omega_0 t \; e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{e^{j\omega_0 t} + e^{-\omega_0 t}}{2} \; e^{-j\omega t} dt$$

$$= \frac{1}{2}\left[\mathcal{F}\left[e^{j\omega_0 t}\right] + \mathcal{F}\left[e^{-j\omega_0 t}\right]\right] = \pi \left\{\delta\left(\omega - \omega_0\right) + \delta\left(\omega + \omega_0\right)\right\}$$

- Unit Step Function:



$$u\left(t\right) = \lim_{a \to 0} e^{-at} u(t)$$

$$U\left(j\omega\right) = \mathcal{F}\left[u(t)\right] = \lim_{a \to 0} \left[\mathcal{F}\left[e^{-at} u(t)\right]\right] = \lim_{a \to 0} \left[\int_{0}^{\infty} e^{-at} e^{-j\omega t} dt\right] = \lim_{a \to 0} \left[\frac{1}{a + j\omega}\right]$$

$$= \lim_{a \to 0} \left[\frac{a - j\omega}{a^2 + \omega^2}\right] = \lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2}\right] + \frac{1}{j\omega}$$

The function $a/(a^2+\omega^2)$ has interesting properties:

1. The area under this function is π regardless of the value of a:

$$\int_{-\infty}^{\infty} \frac{a}{a^2+\omega^2} d\omega = \tan^{-1}\frac{\omega}{a}\Big|_{-\infty}^{\infty} = \pi$$

2. When $a \to 0$, this function approaches zero for all $\omega \neq 0$ and all its area (π) is concentrated at a single point $\omega = 0$.

Clearly, as $a \to 0$, this function approaches an impulse of strength π. Thus,

$$U(j\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

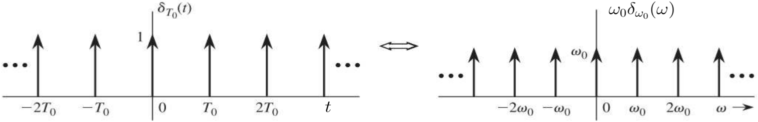
- **Fourier Transform of a Periodic Signal:** The Fourier series of a periodic signal $x(t)$ with period T_0 is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

  Taking the Fourier transform of both sides, we obtain

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) \tag{3}$$

- Dirac Delta Train:



Fourier coefficients a_k for $\delta_{T_0}(t)$ are constant:

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta_{T_0}(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}$$

Thus from (3), we have

$$\mathcal{F}\left[\delta_{T_0}(t)\right] = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - k\omega_0\right) = \frac{2\pi}{T_0} \underbrace{\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\omega_0\right)}_{\text{Pulse train with freq.}\omega_0} = \omega_0\delta_{\omega_0}\left(\omega\right)$$

Important Fourier Transform Pairs

$x(t)$	$X(j\omega)$	
$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
$\delta(t)$	1	
1	$2\pi\delta(\omega)$	
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	

$x(t)$	$X(j\omega)$
$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$
$\cos\omega_0 t\,u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$
$\sin\omega_0 t\,u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$
$e^{-at}\sin\omega_0 t\,u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
$e^{-at}\cos\omega_0 t\,u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$
$\operatorname{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\,\operatorname{sinc}\left(\dfrac{\omega\tau}{2}\right)$
$\dfrac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\dfrac{\omega}{2W}\right)$
$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$
$\dfrac{W}{2\pi}\operatorname{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$
$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$