Lecture 12

- Convergence of Fourier Series
- Gibbs phenomenon
- Computation of fundamental frequency

Fourier series pair

Synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

Analysis equation:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

Convergence of Fourier Series

- Questions
  - Can any periodic signal (possibly discontinuous at some points) be represented as Fourier series?
  - If yes, how many complex sinusoids are sufficient?
- Suppose $x(t)$ is the actual signal and

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

  is Fourier representation with the middle $2N+1$ complex exponentials.
- Does $x_N(t)$ converge to $x(t)$ as $N \to \infty$?

- To check that the criteria of energy of the error signal

$$e_N(t) = x(t) - x_N(t)$$

over one time period is considered:
The Fourier series $x_N(t)$ converges (in mean square mode) to $x(t)$ in the interval of one time period if

$$\int_T |e_N(t)|^2 dt = \int_T |x(t) - x_N(t)|^2 dt = 0 \quad \text{as} \quad N \to \infty. \tag{1}$$

- All practically encountered periodic signals have finite energy over one time period, i.e., $\int_T |x(t)|^2 dt < \infty$. Consequently the condition in (1) is satisfied and the Fourier series converges.
- Point-wise convergence is not considered.

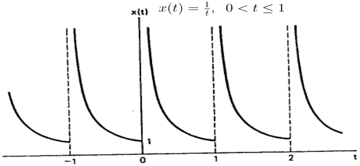
Dirichlet Conditions: (Point-wise convergence)

If $x(t)$ satisfies three conditions, then

$$\lim_{N\to\infty} x_N(t) = \begin{cases} x(t) & \text{if } x(t) \text{ is continuous at } t \\ \frac{x(t^+)+x(t^-)}{2} & \text{if } x(t) \text{ is discontinuous at } t \end{cases}.$$

1. $x(t)$ is absolutely integrable over one period, that is,

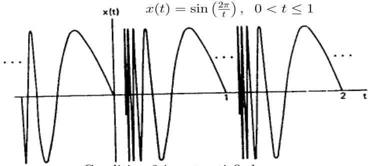
$$\int_T |x(t)| dt < \infty.$$



Condition 1 is not satisfied

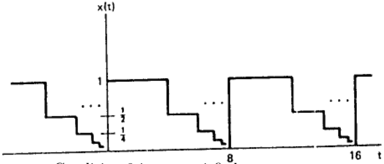
$$|a_k| \leq \frac{1}{T} \int_T \left| x(t) e^{-jk\omega_0 t} \right| dt = \frac{1}{T} \int_T |x(t)|\, dt < \infty$$

2. $x(t)$ contains a finite number of maxima and minima in one period.



Condition 2 is not satisfied

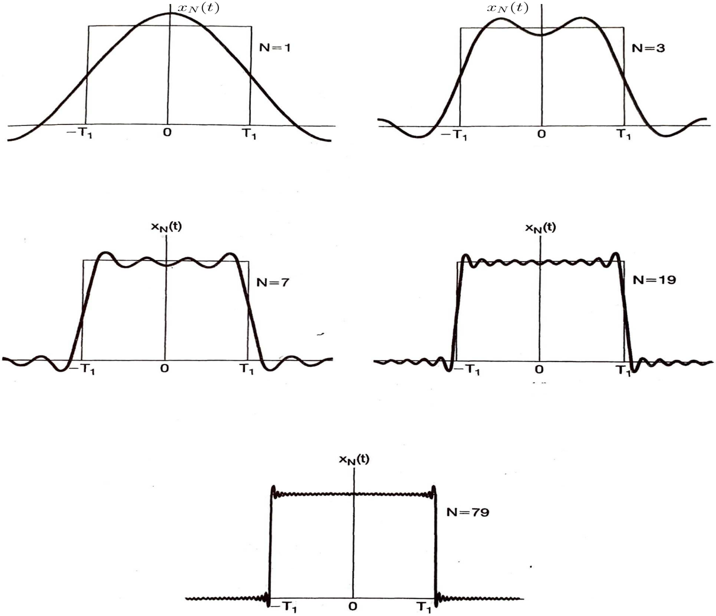
3. $x(t)$ has a finite number of finite discontinuities in one period.



Condition 3 is not satisfied

$$
\begin{aligned}
x(t) &= 1, \;\; 0 \leq t < 4 \\
x(t) &= \tfrac{1}{2}, \;\; 4 \leq t < 6 \\
x(t) &= \tfrac{1}{4}, \;\; 6 \leq t < 7 \\
x(t) &= \tfrac{1}{8}, \;\; 7 \leq t < 7.5
\end{aligned}
$$

Fourier series representation of a square wave and Gibbs phenomenon:



Gibbs Phenomenon

Main Observations

- Even for large N, the truncated series exhibits an oscillatory behavior.
- An overshoot approaching a value of about 9% in the vicinity of the discontinuity at the nearest peak of oscillation.
- As N increases the ripples get concentrated near the point of discontinuities.
- However, the amount of overshoot remains constant at 9% regardless of the value of N.
- At the point of discontinuity, they converge to one-half of the sum of the values of the signal on either side of the discontinuity.
- The Gibbs phenomenon is present only when there is a jump discontinuity in $x(t)$, like sawtooth signals, square waves etc.

Determining the Fundamental Frequency and Period

1.
$$x_1(t) = 2 + 7\cos\left(\frac{1}{2}t + \theta_1\right) + 3\cos\left(\frac{2}{3}t + \theta_2\right) + 5\cos\left(\frac{7}{6}t + \theta_3\right)$$

2.
$$x_2(t) = 2\cos\left(2t + \theta_1\right) + 5\sin\left(\pi t + \theta_2\right)$$

3.
$$x_3(t) = 3\sin\left(3\sqrt{2}t + \theta\right) + 7\cos\left(6\sqrt{2}t + \phi\right)$$

1.

$$T_1 = 4\pi,\ T_2 = 3\pi, T_3 = 12\pi$$

So the effective time period is l.c.m$\{4\pi, 3\pi, 12\pi\} = 12\pi$
The fundamental frequency is $\frac{1}{6}$

2.

$$T_1 = \pi,\ T_2 = 2$$

The signal is not periodic.

3.

$$T_1 = \frac{\sqrt{2}\pi}{3},\ T_2 = \frac{\sqrt{2}\pi}{6}$$

So the effective time period is

$$\text{l.c.m}\left\{\frac{\sqrt{2}\pi}{3}, \frac{\sqrt{2}\pi}{6}\right\} = \frac{\text{l.c.m. of Numerators}}{\text{g.c.f. of Denominators}} = \frac{\sqrt{2}\pi}{3}$$

The fundamental frequency is $3\sqrt{2}$.

DRILL 6.2 Determining Periodicity, Fundamental Frequency, and Harmonic Content

Determine whether the signal

$$x(t) = \cos\left(\tfrac{2}{3}t + 30^\circ\right) + \sin\left(\tfrac{4}{5}t + 45^\circ\right)$$

is periodic. If it is periodic, find the fundamental frequency and the period. What harmonics are present in $x(t)$?

ANSWERS

Periodic with $\omega_0 = 2/15$ and period $T_0 = 15\pi$. Signal $x(t)$ contains the fifth and sixth harmonics.