Lecture 13

- Properties of Continuous-time Fourier Series
- Examples

Properties of Continuous-time Fourier Series

Notations The pairing of a periodic signal with its Fourier series coefficients a_ks is denoted by the following notation:

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

- Linearity: If

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k,$$

then

$$z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k.$$

Proof: By direct application of the definition of Fourier series.

- Time Shifting: If

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k,$$

then

$$x(t-t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k.$$

Proof: The Fourier series coefficients b_k of $y(t) = x(t-t_0)$ may be expressed as

$$b_k = \frac{1}{T} \int_T x(t-t_0) e^{-jk\omega_0 t} dt$$

Putting $\tau = t - t_0$, we get

$$b_k = \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0(\tau+t_0)} d\tau = e^{-jk\omega_0 t_0} \underbrace{\left\{ \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0 \tau} d\tau \right\}}_{a_k} = e^{-jk\omega_0 t_0} a_k.$$

Note that $|b_k| = |a_k|$.

- Time Reversal: If

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$

then

$$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}.$$

Proof:

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T} = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T} \quad [\text{ Put } k = -m]$$

Therefore, the Fourier series coefficients b_k for $y(t)$ are given by

$$b_k = a_{-k}.$$

Interesting facts:

  - If $x(t)$ is even i.e., $x(-t) = x(t)$, then its Fourier series coefficients are also even, i.e. $a_{-k} = a_k$.
  - Similarly, if $x(t)$ is odd i.e., $x(-t) = -x(t)$, then its Fourier series coefficients are also odd, i.e. $a_{-k} = -a_k$.

- Time Scaling: If

$$\underbrace{x(t)}_{\text{Period } T} \xleftrightarrow{\mathcal{FS}} a_k,$$

then

$$\underbrace{x(\alpha t)}_{\text{Period } T/\alpha} \xleftrightarrow{\mathcal{FS}} a_k.$$

Proof:

$$y(t) = x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \alpha t} = \underbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}}_{\text{Fundamental frequency } \alpha\omega_0}$$

Therefore, the Fourier series coefficients b_k for $y(t)$ are given by

$$b_k = a_k.$$

While the Fourier series coefficients have not changed, the Fourier series representation has changed due to the change of the fundamental frequency.

- Multiplication: If

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$
$$y(t) \xleftrightarrow{\mathcal{FS}} b_k,$$

then

$$x(t)y(t) \xleftrightarrow{\mathcal{FS}} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Proof:

$$\begin{aligned} x(t)y(t) &= \sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t} \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t} = \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_l b_n e^{j(l+n)\omega_0 t} \\ &= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_l b_{k-l} e^{jk\omega_0 t} \quad [\text{ By putting } k = l+n] \\ &= \sum_{k=-\infty}^{\infty} \underbrace{\left(\sum_{l=-\infty}^{\infty} a_k b_{l-k} \right)}_{h_k} e^{jk\omega_0 t} \end{aligned}$$

The Fourier series coefficients h_k is the discrete-time convolution of the sequences representing Fourier coefficients of $x(t)$ and $y(t)$.

- **Conjugation and Conjugate Symmetry:** If

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

then

$$x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}$$

Proof:

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a^*_{-k} e^{jk\omega_0 t}$$

Special cases:

- If $x(t)$ is real i.e., $x(t) = x^*(t)$, then the Fourier series coefficients will be *conjugate symmetric*:

$$a_{-k} = a_k^*.$$

- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.
- If $x(t)$ is real and even, then the Fourier coefficients are also real and even.
- If $x(t)$ is real and odd, then its Fourier coefficients are purely imaginary and odd. In specific, $a_0 = 0$ if $x(t)$ is real and odd.

- Parseval's Relation for Continuous-Time Periodic Signals:

If

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

then

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

Proof:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)x^*(t)dt = \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} x^*(t)dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} a_k \int_T x^*(t)e^{jk\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \underbrace{\left\{\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\right\}^*}_{a_k} = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Note that $|a_k|^2$ is the average power in the kth harmonic since:

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2.$$

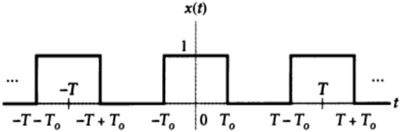
Thus, the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Example 1: Determine the Fourier series representation of the square wave depicted in the following figure:



Solution:

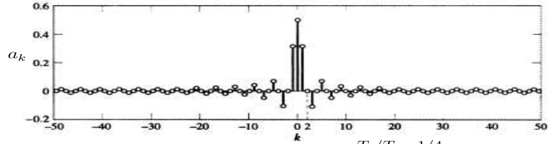
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\omega_0 t}dt = -\frac{1}{Tjk\omega_0}e^{-jk\omega_0 t}\Big|_{-T_0}^{T_0}$$

$$= \frac{2}{Tk\omega_0}\left(\frac{e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}}{2j}\right), \quad k \neq 0$$

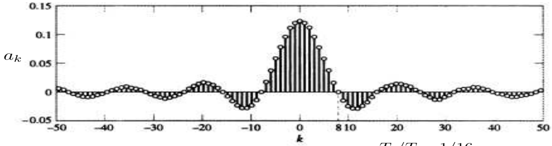
$$= \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0} = \frac{2T_0}{T}\mathrm{sinc}\left(k\frac{2T_0}{T}\right), \quad k \neq 0 \quad \left[\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}\right]$$

We have $a_0 = \frac{1}{T}\int_{-T_0}^{T_0} 1dt = \frac{2T_0}{T}$. This can also verified from

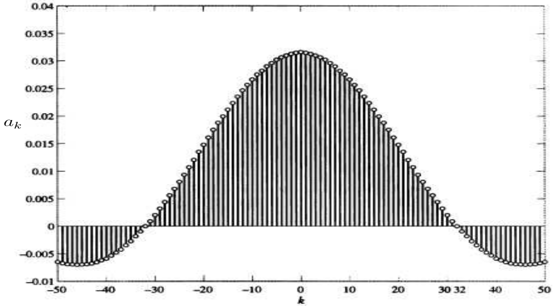
$$a_0 = \lim_{k\to 0} a_k = \lim_{k\to 0}\frac{2\sin(k\omega_0 T_0)}{Tk\omega_0} = \frac{2T_0}{T}$$



$T_0/T = 1/4$

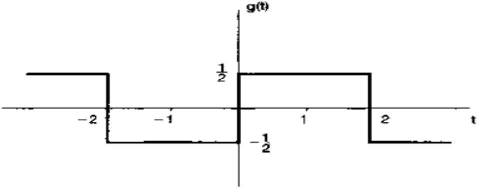


$T_0/T = 1/16$



$T_0/T = 1/64$

Example 2: Determine the Fourier series representation of the square wave depicted in the following figure:



Solution: Compare this square wave with that of Example 1, i.e., $x(t)$. Note that here we have period $T = 4$ and $T_0 = 1$. The signal $g(t)$ is given by $g(t) = x(t-1) - 1/2$. The Fourier coefficients of $x(t)$ are given by

$$a_k = \frac{2T_0}{T}\mathrm{sinc}\left(k\frac{2T_0}{T}\right) = \frac{1}{2}\mathrm{sinc(k/2)}$$

By the time-shifting property, the Fourier coefficients of $x(t-1)$ are given by

$$b_k = a_k e^{-jk\pi/2}.$$

The Fourier coefficients of the dc offset of $g(t)$, i.e., $-1/2$ is given by

$$c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}$$

Invoking the linearity property, the Fourier coefficients of $g(t)$ are given by

$$d_k = \begin{cases} \frac{1}{2}\mathrm{sinc}(k/2)e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

Example 3: Evaluate the Fourier series representation of the following signal:

$$x(t) = \sin(3\pi t) + \cos(4\pi t).$$

Solution: The time period of $\sin(3\pi t)$ is $T_1 = 2/3$ and the time period of $\cos(4\pi t)$ is $T_2 = 1/2$. Therefore, the time period of $x(t)$ is $T = \mathrm{lcm}(\mathrm{T}_1, \mathrm{T}_2) = 2$,
$\omega_0 = 2\pi/T = \pi$.

$$\begin{aligned} x(t) &= \sin(3\pi t) + \cos(4\pi t) \\ &= \frac{1}{2j}e^{j(3)\pi t} - \frac{1}{2j}e^{j(-3)\pi t} + \frac{1}{2}e^{j(4)\pi t} + \frac{1}{2}e^{j(-4)\pi t} \end{aligned}$$

Therefore, the Fourier series coefficients are given by:

$$a_k = \begin{cases} \frac{1}{2} & k = \pm 4 \\ \frac{1}{2j} & k = 3 \\ -\frac{1}{2j} & k = -3 \\ 0 & \text{otherwise} \end{cases}$$

Example 4: Determine the time-domain signal represented by the following Fourier series coefficients:

$$a_k = \left(-\frac{1}{3}\right)^{|k|}, \omega_0 = 1.$$

Solution:

$$\begin{aligned} x(t) &= \sum_{m=-\infty}^{\infty} \left(-\frac{1}{3}\right)^{|k|} e^{jkt} = \sum_{m=0}^{\infty} \left(-\frac{1}{3}e^{jt}\right)^{k} + \sum_{m=1}^{\infty} \left(-\frac{1}{3}e^{-jt}\right)^{k} \\ &= \frac{1}{1+\frac{1}{3}e^{jt}} - \frac{\frac{1}{3}e^{-jt}}{1+\frac{1}{3}e^{-jt}} \\ &= \frac{8}{10+6\cos(t)} \end{aligned}$$