PGF operations for the Zipkin model

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1 Observation

Consider one time step. Define random variables:

- N_i = true abundance of stage i individuals
- Y_i = number of detected stage i individuals
- p_i = detection probability of stage i individuals

1.1 Unnormalized conditional PGF of $\mathbf{N}$, $\mathbf{Y} = \mathbf{y}$

Let $\mathbf{y} = (y_1, y_2)$ be a vector of observations at one time step, then:

$$F_{\mathbf{N},\mathbf{Y}=\mathbf{y}}(\mathbf{s}) = \frac{1}{y_1!y_2!}(s_1p_1)^{y_1}(s_2p_2)^{y_2}\left[\frac{\partial^{y_1+y_2}}{\partial u_1^{y_1}\partial u_2^{y_2}}F_{\mathbf{N}}(\mathbf{u})\right]_{u_i=s_i(1-p_i)}$$

Proof.

$$\begin{aligned}
F_{\mathbf{N},\mathbf{Y}=\mathbf{y}}(\mathbf{s}) &= \frac{1}{y_1!y_2!}\left[\frac{\partial^{y_1+y_2}}{\partial t_1^{y_1}\partial t_2^{y_2}}F_{\mathbf{N},\mathbf{Y}}(\mathbf{s},\mathbf{t})\right]_{\mathbf{t}=\mathbf{0}} \\
&= \frac{1}{y_1!y_2!}\left[\frac{\partial^{y_1+y_2}}{\partial t_1^{y_1}\partial t_2^{y_2}}F_{\mathbf{N}}(s_1(1+p_1t_1-p_1), s_2(1+p_2t_2-p_2))\right]_{\mathbf{t}=\mathbf{0}} \\
&= \frac{1}{y_1!y_2!}\left[\left[(s_1p_1)^{y_1}(s_2p_2)^{y_2}\frac{\partial^{y_1+y_2}}{\partial u_1^{y_1}\partial u_2^{y_2}}F_{\mathbf{N}}(u_1,u_2)\right]_{u_i=s_i(1+p_it_i-p_i)}\right]_{\mathbf{t}=\mathbf{0}} \\
&= \frac{1}{y_1!y_2!}(s_1p_1)^{y_1}(s_2p_2)^{y_2}\left[\frac{\partial^{y_1+y_2}}{\partial u_1^{y_1}\partial u_2^{y_2}}F_{\mathbf{N}}(u_1,u_2)\right]_{u_i=s_i(1-p_i)}
\end{aligned}$$

□

1.2 Invariance of the functional form

Suppose $F_{\mathbf{N}}(\mathbf{u}) = f(\mathbf{u})e^{\mathbf{a}^T\mathbf{u}+a_0}$, where $f(\mathbf{u})$ is a bivariate polynomial of degree d. Then, after observing evidence, the PGF maintains the same form $F_{\mathbf{N},\mathbf{Y}=\mathbf{y}}(\mathbf{s}) = g(\mathbf{s})e^{\mathbf{b}^T\mathbf{s}+b_0}$, where $g(\mathbf{s})$ is a bivariate polynomial of degree $d + y_1 + y_2$.

Proof.

$$
\begin{aligned}
F_{\mathbf{N},\mathbf{Y}=\mathbf{y}}(\mathbf{s}) &= \frac{1}{y_1!y_2!}(s_1p_1)^{y_1}(s_2p_2)^{y_2}\left[\frac{\partial^{y_1+y_2}}{\partial u_1^{y_1}\partial u_2^{y_2}}F_{\mathbf{N}}(u_1,u_2)\right]_{u_i=s_i(1-p_i)} \\
&= \frac{1}{y_1!y_2!}(s_1p_1)^{y_1}(s_2p_2)^{y_2}\left[\frac{\partial^{y_1+y_2}}{\partial u_1^{y_1}\partial u_2^{y_2}}f(\mathbf{u})e^{\mathbf{a}^T\mathbf{u}+a_0}\right]_{u_i=s_i(1-p_i)} \\
&= \frac{1}{y_1!y_2!}(s_1p_1)^{y_1}(s_2p_2)^{y_2}\Bigg[e^{\mathbf{a}^T\mathbf{u}+a_0}\sum_{l=0}^{y_1}\binom{y_1}{l}a_1^{y_1-l}\sum_{m=0}^{y_2}\binom{y_2}{m}a_2^{y_2-m} \\
&\quad \frac{\partial^{l+m}}{\partial u_1^{l}\partial u_2^{m}}f(u_1,u_2)\Bigg]_{u_i=s_i(1-p_i)} \\
&= \frac{1}{y_1!y_2!}(s_1p_1)^{y_1}(s_2p_2)^{y_2}e^{a_1s_1(1-p_1)+a_2s_2(1-p_2)+a_0}\sum_{l=0}^{y_1}\frac{y_1!}{l!(y_1-l)!}a_1^{y_1}a_1^{-l} \\
&\quad \sum_{m=0}^{y_2}\frac{y_2!}{m!(y_2-m)!}a_2^{y_2}a_2^{-m}f_{u_1^l u_2^m}(s_1(1-p_1),s_2(1-p_2)) \\
&= e^{a_1(1-p_1)s_1+a_2(1-p_2)s_2+a_0}\sum_{l=0}^{y_1}\sum_{m=0}^{y_2}\frac{(a_1p_1)^{y_1}}{l!(y_1-l)!a_1^l}\frac{(a_2p_2)^{y_2}}{m!(y_2-m)!a_2^m} \\
&\quad s_1^{y_1}s_2^{y_2}f_{u_1^l u_2^m}(s_1(1-p_1),s_2(1-p_2))
\end{aligned}
$$

The first term has the form $e^{\mathbf{b}^T\mathbf{s}+b_0}$, where $\mathbf{b} = (a_1(1-p_1), a_2(1-p_2))$ and $b_0 = a_0$. Since the mixed partial derivative of a bivariate polynomial of degree d is also a bivariate polynomial of degree at most d, and the scalar $\frac{(a_1p_1)^{y_1}}{l!(y_1-l)!a_1^l}\frac{(a_2p_2)^{y_2}}{m!(y_2-m)!a_2^m}$ can be combined with the coefficient of each monomial of $f_{u_1^l u_2^m}(s_1(1-p_1), s_2(1-p_2))$, the term inside the summation is a bivariate polynomial of degree at most $d+y_1+y_2$. Furthermore, since the sum of polynomials is another polynomial of the same degree as its highest-degree term, the entire summation term must be a bivariate polynomial of degree $d+y_1+y_2$.

□

2 Transition

2.1 PGF of a multinomial rv

If $\mathbf{Z}|N \sim Multinomial(N, \mathbf{p})$, then $F_{\mathbf{Z}|N}(\mathbf{t}) = (\sum_k t_k p_k)^N$.

Proof.

$$
\begin{aligned}
F_{\mathbf{Z}|N}(\mathbf{t}) &= E_{\mathbf{Z}|N}(\prod_k t_k^{Z_k}|N) \\
&= \sum_{\mathbf{z}} \prod_k t_k^{z_k} p(\mathbf{z}|n) \\
&= \sum_{\mathbf{z}} \prod_k t_k^{z_k} \frac{n!}{\prod_k z_k!} \prod_k p_k^{z_k} \\
&= \sum_{\mathbf{z}} \frac{n!}{\prod_k z_k!} \prod_k (t_k p_k)^{z_k} \\
&= (\sum_k t_k p_k)^n, \text{ by the multinomial theorem}
\end{aligned}
$$

□

2.2 Joint PGF over $\{Z_{ij}\}$

If $\mathbf{Z}_1|N_1 \sim Multinomial(N_1, \mathbf{p}_1)$ and $\mathbf{Z}_2|N_2 \sim Multinomial(N_2, \mathbf{p}_2)$, then $F_{\mathbf{Z}_1,\mathbf{Z}_2}(\mathbf{t}_1, \mathbf{t}_2) = F_{\mathbf{N}_1,\mathbf{N}_2}(\sum_j t_{1j}p_{1j}, \sum_j t_{2j}p_{2j})$.

Proof. First, find $F_{\mathbf{Z}_1,\mathbf{Z}_2|N_1,N_2}(\mathbf{t}_1, \mathbf{t}_2)$:

$$
\begin{aligned}
F_{\mathbf{Z}_1,\mathbf{Z}_2|N_1,N_2}(\mathbf{t}_1, \mathbf{t}_2) &= \sum_{\mathbf{z}_1,\mathbf{z}_2} \prod_{ij} t_{ij}^{z_{ij}} p(\mathbf{z}_1, \mathbf{z}_2|n_1, n_2) \\
&= \sum_{\mathbf{z}_1,\mathbf{z}_2} \prod_{ij} t_{ij}^{z_{ij}} p(\mathbf{z}_1|n_1)p(\mathbf{z}_2|n_2) \\
&= \sum_{\mathbf{z}_1} \prod_j t_{1j}^{z_{1j}} p(\mathbf{z}_1|n_1) \sum_{\mathbf{z}_2} \prod_j t_{2j}^{z_{2j}} p(\mathbf{z}_2|n_2) \\
&= (\sum_j t_{1j}p_{1j})^{n_1} (\sum_j t_{2j}p_{2j})^{n_2}
\end{aligned}
$$

Find the joint PGF over $\mathbf{Z}_1, \mathbf{Z}_2, N_1, N_2$:

$$\begin{aligned}
F_{N_1,N_2,\mathbf{Z}_1,\mathbf{Z}_2}(s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) &= \sum_{n_1,n_2,\mathbf{z}_1,\mathbf{z}_2} \prod_i s_i^{n_i} \prod_j t_{ij}^{z_{ij}} p(n_1, n_2, \mathbf{z}_1, \mathbf{z}_2) \\
&= \sum_{n_1,n_2,\mathbf{z}_1,\mathbf{z}_2} \prod_i s_i^{n_i} \prod_j t_{ij}^{z_{ij}} p(\mathbf{z}_1, \mathbf{z}_2 | n_1, n_2) p(n_1, n_2) \\
&= \sum_{n_1,n_2} \prod_i s_i^{n_i} p(n_1, n_2) \sum_{\mathbf{z}_1,\mathbf{z}_2} \prod_{ij} t_{ij}^{z_{ij}} p(\mathbf{z}_1, \mathbf{z}_2 | n_1, n_2) \\
&= \sum_{n_1,n_2} \prod_i s_i^{n_i} p(n_1, n_2) (\sum_j t_{1j} p_{1j})^{n_1} (\sum_j t_{2j} p_{2j})^{n_2} \\
&= \sum_{n_1,n_2} \prod_i (s_i \sum_j t_{ij} p_{ij})^{n_i} p(n_1, n_2) \\
&= F_{N_1,N_2}(s_1 \sum_j t_{1j} p_{1j}, s_2 \sum_j t_{2j} p_{2j})
\end{aligned}$$

Finally, marginalize to obtain:

$$\begin{aligned}
F_{\mathbf{Z}_1,\mathbf{Z}_2}(\mathbf{t}_1, \mathbf{t}_2) &= F_{N_1,N_2,\mathbf{Z}_1,\mathbf{Z}_2}(1, 1, \mathbf{t}_1, \mathbf{t}_2) \\
&= F_{N_1,N_2}(\sum_j t_{1j} p_{1j}, \sum_j t_{2j} p_{2j})
\end{aligned}$$

□

2.3 Joint PGF over M_1, M_2

If $m_1 = z_{11} + z_{21}$ and $m_2 = z_{12} + z_{22}$, then $F_{M_1,M_2}(u_1, u_2) = F_{N_1,N_2}(\sum_j u_j p_{1j}, \sum_j u_j p_{2j})$

Proof. Since $m_j = z_{1j} + z_{2j}$:

$$\begin{aligned}
F_{M_j|Z_{1j},Z_{2j}}(u_j) &= \sum_{m_j} u_j^{m_j} p(m_j | z_{1j}, z_{2j}) \\
&= u_j^{z_{1j}+z_{2j}}
\end{aligned}$$

Find the conditional PGF:

$$\begin{aligned}
F_{M_1,M_2|\mathbf{Z}_1,\mathbf{Z}_2}(u_1, u_2) &= \sum_{m_1,m_2} \prod_j u_j^{m_j} p(m_1, m_2 | \mathbf{z}_1, \mathbf{z}_2) \\
&= \sum_{m_1,m_2} \prod_j u_j^{m_j} p(m_1 | z_{11}, z21) p(m_2 | z_{12}, z_{22}) \\
&= \sum_{m_1} u_1^{m_1} p(m_1 | z_{11}, z21) \sum_{m_2} u_2^{m_2} p(m_2 | z_{12}, z_{22}) \\
&= \prod_j u_j^{z_{1j}+z_{2j}}
\end{aligned}$$

Then the joint PGF:

$$\begin{aligned}
F_{\mathbf{Z}_1,\mathbf{Z}_2,M_1,M_2}(\mathbf{t}_1,\mathbf{t}_2,u_1,u_2) &= \sum_{\mathbf{z}_1,\mathbf{z}_2,m_1,m_2} \prod_i u_i^{m_i} \prod_j t_{ij}^{z_{ij}} p(\mathbf{z}_1,\mathbf{z}_2,m_1,m_2) \\
&= \sum_{\mathbf{z}_1,\mathbf{z}_2,m_1,m_2} \prod_i u_i^{m_i} \prod_j t_{ij}^{z_{ij}} p(m_1,m_2|\mathbf{z}_1,\mathbf{z}_2) p(\mathbf{z}_1,\mathbf{z}_2) \\
&= \sum_{\mathbf{z}_1,\mathbf{z}_2} \prod_{ij} t_{ij}^{z_{ij}} p(\mathbf{z}_1,\mathbf{z}_2) \sum_{m_1,m_2} \prod_i u_i^{m_i} p(m_1,m_2|\mathbf{z}_1,\mathbf{z}_2) \\
&= \sum_{\mathbf{z}_1,\mathbf{z}_2} \prod_{ij} t_{ij}^{z_{ij}} p(\mathbf{z}_1,\mathbf{z}_2) \prod_j u_j^{z_{1j}+z_{2j}} \\
&= \sum_{\mathbf{z}_1,\mathbf{z}_2} \prod_{ij} (u_j t_{ij})^{z_{ij}} p(\mathbf{z}_1,\mathbf{z}_2) \\
&= F_{\mathbf{Z}_1,\mathbf{Z}_2}(u_1t_{11}, u_2t_{12}, u_1t_{21}, u_2t_{22}) \\
&= F_{N_1,N_2}(u_1t_{11}p11 + u_2t_{12}p_{12}, u_1t_{21}p21 + u_2t_{22}p_{22})
\end{aligned}$$

So the marginal PGF over M_1, M_2 is:

$$\begin{aligned}
F_{M_1,M_2}(u_1,u_2) &= F_{1,1,M_1,M_2}(\mathbf{t}_1,\mathbf{t}_2,u_1,u_2) \\
&= F_{N_1,N_2}(u_1p11 + u_2p_{12}, u_1p21 + u_2p_{22})
\end{aligned}$$

Therefore, $F_{\mathbf{M}}(\mathbf{u}) = F_{\mathbf{N}}(\mathbf{P}\mathbf{u})$.

□

2.4 Invariance of the functional form

Suppose $F_{\mathbf{N},\mathbf{Y}(\mathbf{u})=\mathbf{y}}(\mathbf{s}) = f(\mathbf{u})e^{\mathbf{a}^T\mathbf{u}+a_0}$, where $f(\mathbf{u})$ is a bivariate polynomial of degree d. Then, after the transition operation, the PGF still has the same form $F_{\mathbf{N}}(\mathbf{s}) = g(\mathbf{s})e^{\mathbf{b}^T\mathbf{s}+b_0}$, where $g(\mathbf{s})$ is also a bivariate polynomial of degree d.

Proof.

$$\begin{aligned}
F_{\mathbf{N}}(\mathbf{s}) &= F_{\mathbf{N},\mathbf{Y}}(\mathbf{\Delta}\mathbf{s}) \\
&= f(\mathbf{\Delta}\mathbf{s})e^{\mathbf{a}^T\mathbf{\Delta}\mathbf{s}+a_0} \\
&= g(\mathbf{s})e^{(\mathbf{a}^T\mathbf{\Delta})\mathbf{s}+a_0}
\end{aligned}$$

Let $h(\mathbf{s}) = \mathbf{\Delta}\mathbf{s}$, then the first term is the composition of a bivariate polynomial of degree d and a linear transformation of a vector, $g(\mathbf{s}) = foh(\mathbf{s})$, which is another bivariate polynomial of degree d by ???. The second term has the form $e^{\mathbf{b}^T\mathbf{s}+b_0}$, where $\mathbf{b}^T = \mathbf{a}^T\mathbf{\Delta}$ and $b_0 = a_0$.

□

3 Reproduction